Some existence and uniqueness results for wave (weak) turbulence kinetic equations

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Irene’s birthday, Austin, Sep 21, 2017

Joint work with Irene M. Gamba (Austin), Leslie M. Smith (Madison)
Joint work with Pierre Germain (Courant), Alexandru D. Ionescu (Princeton)
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The 4-wave equation (joint work with Germain, Ionescu)

The 3-wave turbulence kinetic equation (joint work with Gamba and Smith)

Conclusion

My mom always said life was like a box of chocolates. You never know what you’re gonna get... (Forrest Gump)
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Having the lucky chance to work with you is one of the nicest pieces of chocolates I have ever got.

HAPPY BIRTHDAY IRENE!
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Classical Boltzmann collision operator

\[ Q[f](p) = \int\int\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} B_{p,p_1,p_2,p_3} [f_3 f_2 - f_1 f] \times \]
\[ \times \delta(|p_3|^2 + |p_2|^2 - |p_1|^2 - |p|^2) \delta(p_3 + p_2 - p_1 - p) dp_1 dp_2 dp_3 \]

where \( f = f(t, p), f_1 = f(t, p_1), f_2 = f(t, p_2), f_3 = f(t, p_3) \).

\[ p + p_1 = p_2 + p_3, \quad |p|^2 + |p_1|^2 = |p_2|^2 + |p_3|^2. \]

\( p, p_1, p_2, p_3 \in S^2 \left( \frac{p+p_1}{2}, \frac{|p-p_1|}{2} \right) \rightarrow \text{integral on a sphere.} \)
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Dispersion relation - Particle energy

\[ p + p_1 = p_2 + p_3, \quad \omega(p) + \omega(p_1) = \omega(p_2) + \omega(p_3) \]

\[ p = p_1 + p_2, \quad \omega(p) = \omega(p_1) + \omega(p_2). \]

Bohm-Pines dispersion law in quantum plasma

\[ \omega(p) = \sqrt{\theta_0 + \theta_1 |p|^2 + \theta_2 |p|^4}, \]

\[ \rightarrow \text{Collision operators are integrals on complicated manifolds.} \]

\[ \rightarrow \text{Resonance Manifold Problem}^{1,2} \]

\[ \rightarrow \text{This is one of the questions we would like to answer in our talk.} \]

\[ \rightarrow \text{Another question is to replace the nonlinearity } f_2 f_3 - f_1 f \text{ by, for instance, } f_2 f_3 (f_1 + f) - f_1 f (f_2 + f_3). \]

\[ ^1 \text{Zakharov, Lvov, Falkovich. } \textit{Kolmogorov spectra of turbulence: Wave turbulence.} \textit{ 2012.} \]

\[ ^2 \text{Nazarenko. } \textit{Wave turbulence.} \textit{ 2011.} \]
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1. The 4-wave equation (joint work with Germain, Ionescu)

2. The 3-wave turbulence kinetic equation (joint work with Gamba and Smith)

3. Conclusion
Defocusing cubic NLS on a torus

\[- i \partial_t v + \frac{1}{2\pi} \Delta v = \epsilon^2 |v|^2 v, \quad v(t = 0) = v_0, \quad x \in \mathbb{T}_L^d.\]

- **Energy cascade phenomenon:** solution transfers energy to higher and higher Fourier modes

  \[\limsup_{t \to \infty} \|v(t)\|_{H^s} = \infty.\]

Partial results: Colliander, Keel, Staffilani, Takaoka, Tao’10, Hani’14, Guardia, Kaloshin’15, Haus, Procesi’15

- **A good introduction:** Terry Tao’s blog “Weak turbulence solutions for the cubic defocusing nonlinear Schrödinger equation”
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4-wave turbulence kinetic equation

\[ v(t, x) = \sum_{p \in \mathbb{Z}^d / L} a_p e^{2\pi i p \cdot x}, \quad \tilde{a}_p(t) = e^{-4\pi |p|^2 t} a_p(t) \]

\[ -i \partial_t \tilde{a}_p(t) = \frac{\epsilon^2}{L^4} \sum_{p_1 - p_2 + p_3 = p} \tilde{a}_{p_1}(t) \tilde{a}_{p_2}(t) \tilde{a}_{p_3}(t) \]

Wave turbulence theory\(^1,2,3\): on \(0 \leq t \leq T_{kin} = \frac{L^d}{\epsilon^2}\), \(|\tilde{a}_p|^2\) can be described by \(f(t, p)\) satisfying

\[ \partial_t f = C_{4W}[f] \]

\[ C_{4W}[f](p) = \int_{\mathbb{R}^{3d}} \delta(p_3 + p_2 - p_1 - p)\delta(|p_3|^2 + |p_2|^2 - |p_1|^2 - |p|^2) \times \]

\[ \times [f_3 f_2(f_1 + f) - f_1 f(f_3 + f_2)] dp_1 dp_2 dp_3. \]


4-wave turbulence kinetic equation

- The dispersion relation can be of various types\(^1\)
  \[
  \Psi_t = -\Psi_{xx} + a\Psi_{xxx} - i|\Psi|^2\Psi + 3a|\Psi|^2\Psi_x
  \]
  \[
  C_{4W}[f](p) = \int_{\mathbb{R}^{3d}} \delta(p_3 + p_2 - p_1 - p)\delta(\omega(p_3) + \omega(p_2) - \omega(p_1) - \omega(p))
  \times [f_3 f_2 (f_1 + f) - f_1 f (f_3 + f_2)] dp_1 dp_2 dp_3, \quad \omega(p) = |p|^2 + a|p|^3.
  \]
- Escobedo-Velazquez's local theory\(^2,3\): 
  \[
  f(t, p) = g(t, \omega_p), \quad \omega_p = |p|^2, \quad \text{one gets}
  \]
  \[
  \partial_t g = \int_{\mathbb{R}^2_+} K[g_3 g_4 (g_1 + g_2) - g_1 g_2 (g_3 + g_4)] d\omega_3 d\omega_4,
  \]
  where \(g_i = g(t, \omega_i), \omega_2 = \omega_3 + \omega_4 - \omega_1, K = \frac{\min\{\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4}\}}{\sqrt{\omega_1}}.\]
- The nonradial, with general dispersion relation case is still open!

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\(^3\)Escobedo-Velazquez. *Finite time blow-up and condensation for the bosonic Nordheim equation*. Invent. Math. 2015
The 4-wave equation (joint work with Germain, Ionescu)

\[ C_{4w}[f] := \int_{\mathbb{R}^9} \delta(p + p_1 - p_2 - p_3) \delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \]
\[ \times [f_1 f_2 (f_3 + f) - ff_1 (f_2 + f_3)] dp_1 dp_2 dp_3 \]
\[ = \int_{\mathbb{R}^9} \delta(p + p_1 - p_2 - p_3) \delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \]
\[ \times [f_1 f_2 f_3 + ff_2 f_3 - 2ff_1 f_2] dp_1 dp_2 dp_3 \]
\[ = T_1[f] + T_2[f] + T_3[f]. \]

We will estimate each operator separately.
Estimating the collision operator in the general case

\[ \frac{1}{2\pi} \int_{\mathbb{R}} e^{x\xi} d\xi = \delta_{x=0}, \]

we obtain the following formula, well-known by physicists\(^1\)

\[ \delta(p + p_1 - p_2 - p_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \]

\[ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3_x} e^{ix(p+p_1-p_2-p_3)} e^{-is(\omega+\omega_1-\omega_2-\omega_3)} \, dxds \]

\[ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3_x} e^{ixp-ias\omega} e^{ixp_1-ias\omega_1} e^{-ixp_2+ias\omega_2} e^{-ixp_3+ias\omega_3} \, dxds. \]

\[ \rightarrow \int_{\mathbb{R}^9} \delta(p + p_1 - p_2 - p_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \]

\[ \times G(p_1)H(p_2)K(p_3) \, dp_1 \, dp_2 \, dp_3 \]

\[ = (2\pi)^2 \mathcal{F} \left[ \int_{\mathbb{R}_s} e^{i\omega(D)} \left( e^{i\omega(D)} \hat{G}(x) e^{i\omega(D)} \hat{H}(x) e^{i\omega(D)} \hat{K}(x) \right) \, ds \right]. \]

Strichartz estimate

Lemma

(Germain-Ionescu-MBT’2017) The following standard Strichartz estimate holds under some assumptions on $\omega$

$$\| e^{it\omega(D)} f \|_{L_t^p L_x^q} \leq C \| f \|_{L^2} \quad \text{if} \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad p, q \geq 2.$$

Lemma

(Germain-Ionescu-MBT’2017) For $s \geq 3 - \frac{5}{r}$, and $2 \leq r \leq \infty$.

$$\| T_1[f] \|_{L^r_s} = \left\| \int_{\mathbb{R}^9} \delta(p + p_1 - p_2 - p_3) \delta(\omega + \omega_1 - \omega_2 - \omega_3) \times f_1 f_2 f_3 dp_1 dp_2 dp_3 \right\|_{L^r_s} \leq C \| f \|_{L^3_s}^3.$$
The general case: Technique 2

\[ T_2[f] = \int_{\mathbb{R}^9} \delta(p + p_1 - p_2 - p_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \]
\[ \times f_2 f_3 dp_1 dp_2 dp_3 \]
\[ = f Q^{gain}[f]. \]

\[ Q^{gain}[f] := \int_{\mathbb{R}^9} \delta(p + p_1 - p_2 - p_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \]
\[ \times f_2 f_3 dp_1 dp_2 dp_3. \]
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The general case: Technique 2

\[ I = \int_{\mathbb{R}^{12}} \delta(p + p_1 - p_2 - p_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) f_2 f_3 f dp dp_1 dp_2 dp_3 \]

By the definition of the Delta function

\[ I = \int_{\mathbb{R}^9} \delta(\omega(p) + \omega(p_2 + p_3 - p) - \omega(p_2) - \omega(p_3)) f_2 f_3 f dp dp_2 dp_3 \]

The resonance manifold \( S_{p_2, p_3} \): all of the points \( z \) such that

\[ \mathcal{G}(z) := \omega(p_2 + p_3 - z) + \omega(z) - \omega(p_2) - \omega(p_3) = 0, \]

which leads to the following representation of \( I \),

\[ I = \int_{\mathbb{R}^6} f_2 f_3 \left( \int_{S_{p_2, p_3}} \frac{f(z)}{|\nabla_z \mathcal{G}(z)|} d\mu(z) \right) dp_2 dp_3. \]
The general case: Technique 2

Averaging operators of the type

\[ J(p, p') = \int_{S_{p,p'}} K(z) f(z) d\mu(z), \]

\[ \varphi(z) := \omega(p + p' - z) + \omega(z) - \omega(p) - \omega(p') = 0, \]

has been studied before by several authors, under the condition that \( K \) is compactly supported\(^1,2,3\).

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\(^3\) Ionescu. *Fourier integral operators on noncompact symmetric spaces of real rank one*. JFA, 2000.
The general case: Technique 2

- Suppose we need to estimate

\[ \| J(\cdot, p') \|_{L^1} = \int_{\mathbb{R}^3} \int_{S_{p,p'}} \frac{f(z)}{|\nabla z \mathcal{G}(z)|} d\mu(z) dp \]

\[ \leq \left\| \int_{S_{p,p'}} d\mu(z) \right\|_{\infty} \int_{\mathbb{R}^3} \frac{f(z)}{|\nabla z \mathcal{G}(z)|} dp \]

- One of our techniques: the change of variable \( z \rightarrow p \).
Set $p + p_1 = \rho$, then

$$\nabla_z \mathcal{G} = \frac{Z - \rho}{|\rho - Z|} \omega'(|\rho - Z|) + \frac{Z}{|Z|} \omega'(|Z|).$$

In particular, let $q$ be any vector orthogonal to $\rho$ i.e. $\rho \cdot q = 0$. The directional derivative of $G$ in the direction of $q$, with $z = \alpha \rho + q$, $\alpha \in \mathbb{R}$, satisfies

$$q \cdot \nabla_z \mathcal{G} = |q|^2 \left[ \frac{\omega'(|\rho - Z|)}{|\rho - Z|} + \frac{\omega'(|Z|)}{|Z|} \right] > 0,$$

that means, $\mathcal{G}(z)$ is strictly increasing in any direction that is orthogonal to $\rho$. This proves that the intersection between the surface $S_{p,p_1}$ and the plane

$$\mathcal{P}_\alpha = \left\{ \alpha \rho + q, \rho \cdot q = 0 \right\}$$

is either empty or the circle centered at $\alpha \rho$ and of a finite radius $|q_\alpha|$, for $\alpha \in \mathbb{R}$. 

**Parametrize $S_{p,p_1}$**
We can parametrize $S_{p,p_1}$ as follows. Fix a vector $\rho^\perp$ be in $P_0 = \{ \rho \cdot q = 0 \}$ and let $e_\theta$ be the unit vector in $P_0$ such that the angle between $\rho^\perp$ and $e_\theta$ is $\theta$. We parametrize $S_{p,p_1}$ by

$$\left\{ z = \alpha \rho + q_\alpha = \alpha \rho + |q_\alpha| e_\theta : \theta \in [0, 2\pi], \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}] \right\},$$

where $\alpha_{\text{min}}, \alpha_{\text{max}}$ are the smallest and biggest values that $\alpha$ can take.
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The change of variable z to ρ

Notice that $z = \alpha \rho + q_\alpha$ and we use the standard notation $z = (z^1, z^2, z^3)$, $\rho = (\rho^1, \rho^2, \rho^3)$, $q_\alpha = (q^1_\alpha, q^2_\alpha, q^3_\alpha)$, $p = (p^1, p^2, p^3)$, $p_1 = (p^1_1, p^2_1, p^3_1)$. We compute the derivative of each component of $z$ with respect to each component of $p$

$$\partial p_i z^i = \partial p_i (\alpha p^i + \alpha p^i_1 + q^i_\alpha) = \partial p_i q^i_\alpha = \partial_{|\rho|} q^i_\alpha \frac{\rho_j}{|\rho|}, i, j \in \{1, 2, 3\}, i \neq j,$$

$$\partial p_i z^i = \partial p_i (\alpha p^i + \alpha p^i_1 + q^i_\alpha) = \alpha + \partial p_i q^i_\alpha = \alpha + \partial_{|\rho|} q^i_\alpha \frac{\rho_i}{|\rho|}, i \in \{1, 2, 3\}.$$

The Jacobian matrix can be then written as

$$\text{Jac} = \begin{bmatrix}
\alpha + \partial_{|\rho|} q^1_\alpha \frac{\rho_1}{|\rho|} & \partial_{|\rho|} q^1_\alpha \frac{\rho_2}{|\rho|} & \partial_{|\rho|} q^1_\alpha \frac{\rho_3}{|\rho|} \\
\partial_{|\rho|} q^2_\alpha \frac{\rho_1}{|\rho|} & \alpha + \partial_{|\rho|} q^2_\alpha \frac{\rho_2}{|\rho|} & \partial_{|\rho|} q^2_\alpha \frac{\rho_3}{|\rho|} \\
\partial_{|\rho|} q^3_\alpha \frac{\rho_1}{|\rho|} & \partial_{|\rho|} q^3_\alpha \frac{\rho_2}{|\rho|} & \alpha + \partial_{|\rho|} q^3_\alpha \frac{\rho_3}{|\rho|}
\end{bmatrix},$$

whose determinant can be computed explicitly

$$|\text{Jac}| = \alpha^3 + \alpha^2 \partial_{|\rho|} q_\alpha \cdot \frac{\rho}{|\rho|}.$$
The change of variable $z$ to $\rho$

We can therefore do the change of variable $z$ to $\rho$ with the cost $\frac{1}{\alpha^3}$.

Notice that $S_{\rho,p_1}$ is symmetric on $S_{\rho,p_1}$ with respect to $z$ and $\rho - z$. Let us split $S_{\rho,p_1}$ into two surfaces

$$S_{\rho,p_1}^+ := \{ z_\alpha \in S_{\rho,p_1} \mid |z_\alpha| \geq |\rho - z_\alpha| \} ,$$

$$S_{\rho,p_1}^- := \{ z_\alpha \in S_{\rho,p_1} \mid |z_\alpha| < |\rho - z_\alpha| \} ,$$

that lead to

$$S_{\rho,p_1}^+ := \left\{ z_\alpha \in S_{\rho,p_1} \mid \alpha \geq \frac{1}{2} \right\} ,$$

$$S_{\rho,p_1}^- := \left\{ z_\alpha \in S_{\rho,p_1} \mid \alpha < \frac{1}{2} \right\} .$$

We can assume $\alpha \geq \frac{1}{2}$ and $\frac{1}{\alpha^3} \leq 8$. 
Estimates on $T_2$

**Lemma**

*(Germain, Ionescu, MBT’2017)* For $r \in [1, \infty]$

$$\| T_2[f]\|_{L^r_s} \leq C\| f\|_{L^r_s}^2 \| f\|_{L^\infty_1}.$$

*(3)*

for $s > 4 - \frac{3}{r}$.

**Theorem**

*(Germain, Ionescu, MBT’2017)* Suppose that the dispersion relation $\omega$ is of the general abstract form and satisfies some assumptions. For any initial condition $f_0 \in \mathbb{X} := L^\infty_s \cap L^r_{s_0}$, $(s > 3, r \geq 2, s_0 > 5 - \frac{3}{r}), f_0 \geq 0$, there exists a time interval $[0, T)$ such that the equation has a unique, strong, positive solution $f$ in $C^1([0, T), \mathbb{X})$.
The 3-wave turbulence kinetic equation

\[ \partial_t f(t, p) = C_{3W}[f](t, p), \]

\[
C_{3W}[f](p) = \int_{\mathbb{R}^6} \delta(p - p_1 - p_2) \delta(\omega_p - \omega_{p_1} - \omega_{p_2}) \times \\
\times [f(p_2)f(p_1) - f(p)f(p_1) - f(p)f(p_2)] dp_1 dp_2 \\
- 2 \int_{\mathbb{R}^6} \delta(p_2 - p - p_1) \delta(\omega_{p_1} - \omega_p - \omega_{p_2}) \times \\
\times [f(p)f(p_1) - f(p_2)f(p_1) - f(p_2)f(p)] dp_1 dp_2
\]
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Other dispersive equations

- Water wave equation

\[
\Delta \Phi(t, x, y, z) = 0, \text{ for } z < \zeta(t, x, y), (t, x, y, z) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\zeta_t - \Phi_z = -\zeta_x \Phi_x - \zeta_y \Phi_y \bigg|_{z=\zeta}, \\
\Phi_t - \alpha(\zeta_{xx} + \zeta_{yy}) = \frac{|\nabla \Phi|^2}{2} \bigg|_{z=\zeta}, \Phi \bigg|_{z=-\infty} = 0,
\]

where \( \Phi(t, x, y, z) \) is the velocity potential, \( \zeta(t, x, y) \) is the deviation of the surface from equilibrium. The \( z \) axis is directed away from the liquid and the pressure is 0.

Fourier analysis\(^1\) for \( \Phi(t, x, y, \zeta(t, x, y)) \) \( \rightarrow \) the 3 wave turbulence kinetic equation

- The primitive equation\(^2\).


Resonance Broadening

Exact resonances do not capture some important physical effects, such as energy transfer to non-propagating wave modes with zero frequency. A natural way to include more physics is to allow near-resonant interactions $^{1,2}$ defined as

$$ p = p_1 + p_2, \quad |\omega_p - \omega_{p_1} - \omega_{p_2}| < \theta, $$

where $\theta$ accounts for broadening of the resonant surfaces.

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One loop approximation

Exact resonance equation for stratified flows in the ocean

\[ C_{3w}^{\text{exact}}[f] \sim \int \int_{\mathbb{R}^2} \delta(p - p_1 - p_2) \delta(\omega_p - \omega_{p_1} - \omega_{p_2})(f_1 f_2 - ff_1 - ff_2) dp_1 dp_2 \]

Near resonance equation \(^1,^2,^3\) for stratified flows in the ocean

\[ C_{3w}^{\text{broaden}}[f] \sim \int \int_{\mathbb{R}^2} \delta(p - p_1 - p_2) \mathcal{L}_f(\omega_p - \omega_{p_1} - \omega_{p_2})(f_1 f_2 - ff_1 - ff_2) dp_1 dp_2 \]

\[ \mathcal{L}_f(\zeta) = \frac{\Gamma_{p,p_1,p_2}}{\zeta^2 + \Gamma_{p,p_1,p_2}^2}, \quad \Gamma_{p,p_1,p_2} = \gamma_p + \gamma_{p_1} + \gamma_{p_2}, \quad \gamma_p = c|p|^2 \int_{\mathbb{R}^3} f(t, p) dp. \]
For any $R^* > 0$, $R_1^* > 1$, and for $N, t > 0$, we introduce $S_t$ to be consisting of functions $f \in L^{1}_{N+3}(\mathbb{R}^d)$ so that

(S1) Positivity of the set $S_t$: $f \geq 0$;
(S2) Upper bound of the set $S_t$: $\|f\|_{L^{1}_{N+3}} \leq c_0(t)$;
(S3) Lower bound of the set $S_t$: $\|f\|_{L^{1}} \geq c_1(t)$;

$$c_0(t) := (2R^*_1 + 1)e^{C^*_1 t}, \quad c_1(t) := \frac{R^* e^{-C^*t}}{2}.$$ 

**Theorem**

(Gamba, Smith, MBT’2017) Let $N > 0$, and let $f_0(k) \in S_0$. Then the 3-wave weak turbulence equation, with resonance broadening and initial data $f(0, k) = f_0(k) \geq 0$, $\omega(p) = \sqrt{C_{Coriolis} + |p|^2}$ has a unique strong solution $f(t, k) \in C([0, T); L^{1}_{N}(\mathbb{R}^d)) \cap C^1((0, T); L^{1}_{N}(\mathbb{R}^d))$. Moreover, $f(t, k) \in S_T$ for all $t \in [0, T)$.

1 Alonso, Gamba, Tran. *The Cauchy problem for the quantum Boltzmann equation for bosons at very low temperature.* arXiv:1609.07467
Our results on weak turbulence-quantum Boltzmann

- **Kinetic equations point of view:**
  - Convergence to equilibrium: M. Escobedo, M.-B. Tran
  - Hydrodynamics limits: S. Jin, M.-B. Tran
  - 3-wave equation: R. Alonso, I. M. Gamba, L. M. Smith, M.-B. Tran
  - 4-wave equation: P. Germain, A. Ionescu, M.-B. Tran
  - Positivity of the solutions: T. Nguyen, M.-B. Tran

- **Dispersive equations point of view:**
  - Normal form transformation: A. Soffer, M.-B. Tran
  - Uncertainty principle: G. Ponce, A. Soffer, L. Vega, M.-B. Tran

- **Dynamical systems point of view:**
  - The global attractor conjecture: G. Craciun, M.-B. Tran

- **Physics point of view:**
  - Derivation: L. Reichl, M.-B. Tran
  - Applications in plasma physics: S. Boldyrev, G. Craciun, M.-B. Tran
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