Analysis of PDE and the geometry of the sphere

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Outline

Notation and conventions

We will refer with the letters $u, v, ...$ to **differentiable functions**

$$u : \Omega \to \mathbb{R}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded.
Pretty much all of the time (for today) $n = 2$ or $n = 3$.
Gradient of a function. If u is differentiable, we will write

$$Du = \nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n} \right)$$

$$(\text{ if say } n = 2, \;\; \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$

Hessian of a function. If the derivatives of u are themselves differentiable, we will also write

$$D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{ij}$$

1. Surfaces and their curvature
2. Elliptic partial differential equations
3. Aleksandrov's moving plane method

Surfaces

(Surfaces are neat. They even have higher dimensional cousins)

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Definition. A (smooth) **surface** is a set of points $\Sigma \subset \mathbb{R}^3$ such that for each $p \in \Sigma$ we can find a small enough radius r and $u : \mathbb{R}^2 \to \mathbb{R}$ such that in some Cartesian system of coordinates we have

$$\Sigma \cap B_r(p) = \{(x, y, z) \ : z = u(x, y)\} \cap B_r(p)$$

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Homework 0. Suppose $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $\nabla f \neq 0$ for all (x, y, z), for each fixed $c \in \mathbb{R}$ the level set

$$\{(x, y, z) : \quad f(x, y, z) = c\}$$

is a surface in the sense of the definition above. **Prove** this.

The Gauss map

Given any oriented surface Σ, the **Gauss map** is nothing but the map that associates to each $p \in \Sigma$ the normal vector to Σ at p, understood as lying in S^2. In particlar, if we represent a portion of Σ by a function u, the Gauss map is given by

$$\begin{aligned}(x, y, u(x, y)) \to \quad & \left(\tfrac{u_x}{\sqrt{1+|\nabla u|^2}}, \tfrac{u_y}{\sqrt{1+|\nabla u|^2}}, \tfrac{1}{\sqrt{1+|\nabla u|^2}} \right) \\ = & \left(\tfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \tfrac{1}{\sqrt{1+|\nabla u|^2}} \right)\end{aligned}$$

The derivative of the Gauss map at a point p, defines a symmetric matrix in two variables, known as the **Weingarten** operator, its eigenvalues give a way of measuring how "curved" the surface is, and they are known as the **principal curvatures** of Σ at p. (for instance, for a plane, the eigenvalues are zero for every p, for a sphere, the eigenvalues are both equal to the inverse of its radius).

The Gauss map

The Gauss map records the way Σ bends around in space. It is not surprising then, that the *derivative* of this map at p has something to say about the bending of the Σ near this point.

The Gauss and Mean curvatures

Motivated by this discussion, we introduce the Gauss and Mean curvatures. Denoting by $\lambda_1(p)$ and $\lambda_2(p)$ the principal curvatures at $p \in \Sigma$, then we define

Gauss curvature. $\kappa(p) = \lambda_1(p)\lambda_2(p)$.
Mean curvature. $H(p) = \lambda_1(p) + \lambda_2(p)$.

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Homework 1. For a surface given locally at p by u, carry out the necessary computations to show that

$$\begin{aligned} \kappa(p) &= \det\left(\frac{1}{\sqrt{1+|\nabla u|^2}} D^2u + \frac{\nabla u \otimes \nabla u}{(1+|\nabla u|^2)^{\frac{3}{2}}}\right) \\ H(p) &= \mathrm{Tr}\left(\frac{1}{\sqrt{1+|\nabla u|^2}} D^2u + \frac{\nabla u \otimes \nabla u}{(1+|\nabla u|^2)^{\frac{3}{2}}}\right) \quad = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \end{aligned}$$

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Homework 2. In case you don't know, find out what $\otimes$ means.

The Gauss and Mean curvatures

Some history: the Mean curvature was studied for the first time by the 19th century mathematician Sophie Germain, who came across it in her studies on elasticity. Carlitos Gauss, around a similar time, was studying the problem of how to make "good maps" of the earth, which led him to the curvature that nowadays bears his name.

Curvature of the Sphere

Now, we can check that if $\Sigma = S_r^2$, then both κ and H are constant!, in fact, in this case we have

$$\kappa \equiv \frac{1}{r^2}, \quad H \equiv \frac{2}{r}$$

Homework 3. Prove this!

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Are there any other surfaces of constant Gauss or constant mean curvature? Actually, there are lots of them, and they are observed all over nature.

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...his answer relies on the analysis of partial differential equations.

(good luck with this one, Euclid)
(you can borrow my ruler and compass, let me know how it goes)

1. Surfaces and their curvature
2. Elliptic partial differential equations
3. Aleksandrov's moving plane method

What is a Partial Differential Equation?

A partial differential equation is an equation that relates the pointwise values of a differentiable function and its derivatives. Some important examples are

$$\begin{aligned} u_{xx} - u_{tt} &= 0 \\ u_{xx} + u_{yy} &= f(u(x)) \\ \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= h \end{aligned}$$

Partial differential equations are the equations that arise all over geometry, science and engineering (Navier-Stokes, Maxwell's equations, Schrödinger's equation in quantum mechanics and Einstein's field equations in general relativity! all these are really hard to study, but specially Navier Stokes and Einstein's, which have not been understood even in low dimensions).

Reductionism FAIL

(Hamlet probably knew about PDEs)
There are as many PDEs as physical models and geometrical problems, in fact, there are so many different PDEs that it is widely agreed that there is no theory or “big machine” that allows you to understand all of them at once. There are nonetheless wide classes of PDE that can be dealt with using similar techniques, and try to subdivide the field of PDEs into types of equations, for instance, all equations that model diffusion-reaction type phenomena are all more or less similar, but they are completely different from those that govern shock waves. The main point is that one may ask the same question about two different PDEs, but the answers and the tools to answer it in each case might be completely different.

Today, we are only dealing with **elliptic** PDE.

Elliptic PDE

A linear **elliptic** PDE is an equation of the type

$$Lu = \sum_{ij} a_{ij}(x)u_{ij}(x) + \sum_{i} b_i(x)u_i(x) + c(x)u(x) = f(x)$$

where for each x the matrix $a_{ij}(x)$ is symmetric its eigenvalues all lie in between two fixed positive numbers. Moreover, we will assume that the functions $a_{ij}(x), b_i(x), c(x)$ are all nice (say continuous), L above, is what is known as an "elliptic differential operator".

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Homework 4. Show that if u satisfies

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h$$

then it solves a linear elliptic equation, which one?!. Hint: the coefficients depend on u.

Linearity breeds contempt -Luis Caffarelli

To be fair, I should really be talking about quasilinear elliptic equations, those are the ones which are linear in the second order derivatives but that might be nonlinear in all the other terms, namely

$$Lu = \sum_{ij} a_{ij}(x, u, \nabla u) u_{ij}(x) + b(x, u, \nabla u) = 0$$

most equations in physics are quasilinear (for instance, **any** equation that came from a Lagrangian is quasilinear), there are other equations carrying the adjective "Fully non Linear", those appear most often in geometry and probability.

Homework 4'. Actually, show that the expression for the mean curvature **is** a quasilinear elliptic PDE.

The maximum principle

The distinctive feature of elliptic PDE is *the maximum principle*.

Theorem. Let u and v satisfy $Lu \geq Lv$ in Ω for some elliptic (possibly quasilinear) operator L. Suppose also that $u \leq v$ in $\partial\Omega$, then actually $u \leq v$ in all of Ω.

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there is a reinforcement of this theorem, known as the Strong maximum principle (also Hopf's Lemma).

Theorem Let u and v be as above, then
1) If $u(x_0) = v(x_0)$ at some boudary point $x_0 \in \partial\Omega$ then either $u_\nu(x_0) < v_\nu(x_0)$ or $u \equiv v$.
2) If $u(x_0) = v(x_0)$ in some *interior* point of Ω then actually $u \equiv v$ everywhere.

Elliptic regularization

The other important aspect of elliptic PDE, is that they are very rigid in some sense. In fact, if u is a twice-differentiable function that happens to be the solution of an elliptic PDE (with good coefficients), then u has to be **analytic**!.

The first time a mathematician comes across this phenomenon is usually when studying complex analysis and harmonic functions (remember? if a function $f(z)$ solves the Cauchy-Riemann equations then it has derivatives of all orders and can be written as a converging power series).

OK. That ends the background discussion, now I can tell you about Aleksandrov's proof.

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Moving plane method

With the preliminaries aside, Aleksandrov's proof (circa 1955) is remarkable in its simplicty and elegance. Nowadays, his technique is known as the moving plane method.

Main idea: to show Σ has to be a sphere, it is enough to prove that it has the following property:

"Given any direction e, there is some plane Π_e perpendicular to this direction and such that Σ is symmetric with respect to Π_e"

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Main idea: to show Σ has to be a sphere, it is enough to prove that it has the following property:

"Given any direction e, there is some plane Π_e perpendicular to this direction and such that Σ is symmetric with respect to Π_e"

Homework 5. Prove that Σ has the above property if and only if Σ is a sphere.

Moving plane method

Now we prove, following Aleksandrov:

If Σ has constant mean curvature and e is some direction, there exists some plane Π_e perpendicular to e which is a plane of symmetry of Σ.

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Step 1. Pick a plane Π perpendicular to e and that does not intersect Σ.

Step 2. Slide the plane in the direction of e until it touches Σ for the first time

Moving plane method

Step 3. Slide the plane a bit further, say a distance ϵ, and call it Π_ϵ, this plane “cuts” through Σ. Call Σ_ϵ the reflection of Σ by Π_ϵ

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Step 4. Now, we increase ϵ until at least one of the following two things happens:

i) The side of Σ_ϵ that lies (say) to the left of Π_ϵ touches Σ at some point away from Π_ϵ
ii) At some point $p \in \Sigma \cap \Pi_\epsilon$, there is a tangent vector to Σ at p parallel to e

Moving plane method

Step 5. We are going to show that there is some $p \in \Sigma$ and $r > 0$ such that $\Sigma_\epsilon \cap B_r(p) = \Sigma \cap B_r(p)$ in each case. Here is where we are going to use that both Σ and Σ_ϵ have the same mean curvature.

Case i. Say the point of contact is p, then near p we may represent both surfaces by functions u and $v$such that

$$u \leq v \in \partial\Omega \ \text{ for some domain of the plane}$$

$$Lu = Lv \in \Omega$$

and $u = v$ at some interior point. Then the second part of the Strong maximum principle says that $\Sigma \cap B_r(p) = \Sigma_\epsilon \cap B_r(p)$ for some very small r.

Moving plane method

Case ii. Given the point p, we write both surfaces locally again as the graph of functions u and v, but now p corresponds to a point on the boundary of the patch Ω, the u and v then satisfy:

$$\begin{aligned} u &\leq v && \text{on } \partial\Omega \\ Lu &= Lv && \text{in } \Omega \\ u &= v && \text{at some point of } \partial\Omega \\ u_\nu &= v_\nu && \text{at the same point} \end{aligned}$$

by the first part of the Strong Maximum principle, we conclude that $u = v$ in all of Ω and thus $\Sigma \cap B_r(p') = \Sigma_\epsilon \cap B_r(p')$ for some small r and some p' near p.

Step 6. To finish, note that if u and v represent locally any patch of Σ and Σ_ϵ, then since they solve the elliptic PDE “mean curvature $=$ c”, then they are actually analytic functions. A surface which can be given locally by the graph of *analytic* functions is called an **analytic function**.

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Homework 6. Suppose that S and S' are two analytic surfaces that actually agree in a neighborhood of some p, then $S = S'$ (this is known as "analytic continuation").

By Homework 6 and Step 5. We conclude that $\Sigma = \Sigma_\epsilon$, and that finishes the proof.

Thanks for your attention!
(Now you may have some Pizza)

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