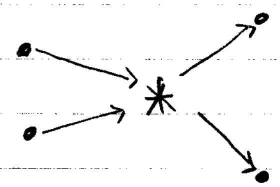
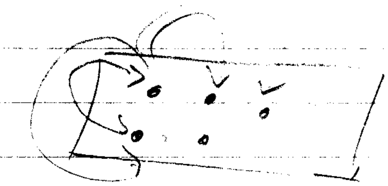
Symmetry Groups in Classical Physics



"The space of all classical solution of Newtonian physics," $\mathcal{C}$ is invariant under:

- reflections of space
- uniform translations in space
- rotations in space
- boosts



$\mathcal{C}$

These operations form a group, called a "Galilei group" Gal.

Defn. A group G is a set equipped w/ a product

$$\times : G \times G \longrightarrow G$$
$$(g,h) \longrightarrow gh$$

which is i) associative: $(gh)k = g(hk)$

ii) has identity 1: $g1 = g = 1g$

iii) has inverses g^{-1}: $gg^{-1} = 1 = g^{-1}g$

A simple system single particle <u>at rest</u>

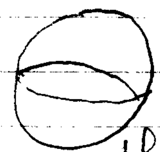
$$\mathcal{C} = \bullet$$

$\vec{x}_0 \bullet$

$\mathcal{C}$ is acted on by group of rotations around the fixed $\vec{x}_0 \in \mathbb{R}^3$.

(This group is called $O(3)$, group of 3×3 orthogonal matrices.)

If the particle has some internal angular momentum,

a two sphere — then $\mathcal{C} =$  , i.e. $\mathcal{C} = S^2$ with size given by the magnitude of the angular momentum.

Remark: Passage from Newtonian physics to Special Relativity replaces the Galilei group with the group of Lorentz transformations, $O(3,1)$.

[this means matrices $\mathcal{L}$ obeying $\mathcal{L}\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}\mathcal{L}^T = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}$]

Quantization

"Art not science"

What we expect: Replace the space of classical solutions $\mathcal{C}$ by a vector space (actually Hilbert space) $\mathcal{H}$.

If $\mathcal{C}$ is acted on on by symm group G, then $\mathcal{H}$ is a representation of G.

Defn: A representation of group G is a vector space $\mathcal{H}$ together with a map $\rho : G \to End(\mathcal{H})$

obeying $\rho(g)\rho(h) = \rho(gh)$.

Think of G - representation as a vector space w/ additional structure.

You know that any n-dimensional vector space V can be decomposed $V \simeq \mathbb{C} \oplus \mathbb{C} \oplus \ldots \oplus \mathbb{C}$

For G representation, the building blocks are more interesting:

Def: A G-rep $\mathcal{H}$ is reducible if $\exists \mathcal{H}' \subsetneq \mathcal{H}$ s.t. $\rho(g)\mathcal{H}' \subset \mathcal{H}'$

Basic building blocks: irreducible reps. (irreps)

What are the irreps. of G?

Answer, in case $G = SO(3)$

There's one irreducible rep $\mathcal{H}_n$ for every $n \in \mathbb{Z}_{\geq 0}$

$$\dim_{\mathbb{C}} \mathcal{H}_n = 2n+1.$$

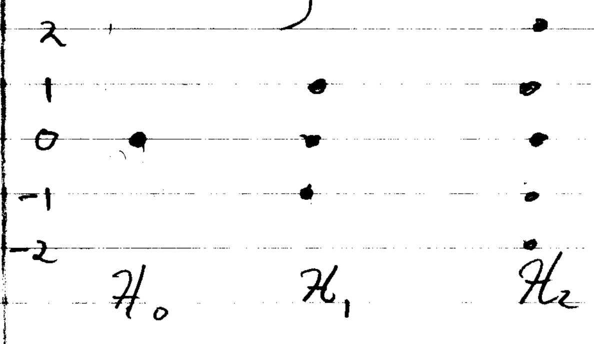
To understand the structure of $\mathcal{H}_n$:
Fix an axis (say X^3) and look at subgroup $SO(2) \subset SO(3)$ of rotations around this axis.

Irreps of $SO(2)$ are all 1-dimensional.

They're given by (rotation by θ)

$$R_\theta \longmapsto e^{ik\theta}, \ k \in \mathbb{Z}$$

Decomposing $\mathcal{H}_n$ into irreps of $SO(2)$:



etc...

Tensor Products

Recall that for vector spaces given V, W we can define $V \otimes W$.

Suppose $\mathbb{C}^n \otimes \mathbb{C}^m \simeq \mathbb{C}^{mn}$ for trivial case,

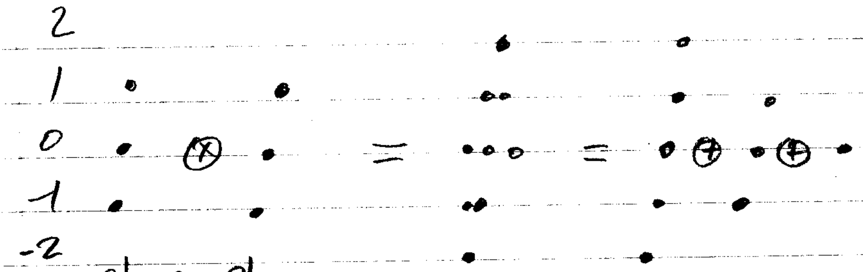
$[\text{cf. } \mathbb{C}^n \oplus \mathbb{C}^m \simeq \mathbb{C}^{m+n}]$ $\quad [\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}]$

If $\mathcal{H}_1, \mathcal{H}_2$ are G reps. then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is also naturally a G-rep.

So which one is it?

e.g. for $G = SO(3)$, $\underset{\dim=3}{\mathcal{H}_1} \otimes \underset{\dim=3}{\mathcal{H}_1} = \underset{1}{\mathcal{H}_0} \oplus \underset{3}{\mathcal{H}_1} \oplus \underset{5}{\mathcal{H}_2}$

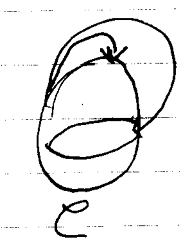
How to get this answer?



$\mathcal{H}_1 \otimes \mathcal{H}_1 = \mathcal{H}_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_0$

Physics of this:

Question of the spinning particle



Should give an irrep. of $SO(3)$.

Indeed, if we take a particle with n units of angular momentums, its quantization gives $\mathcal{H}_n$.

<u>Internal Symmetries</u>

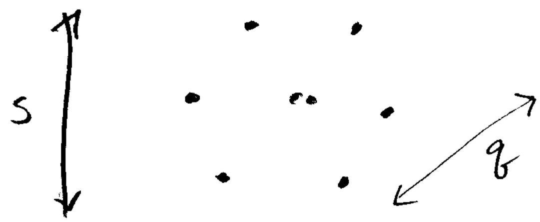
The Standard Model is a good quantum theory.
It has $SO(3,1)$ mentioned before.
Also has more...

SM has an approximate symmetry similar to this.

$SU(3) = \{$ unitary 3×3 matrices with $\det M = 1\}$

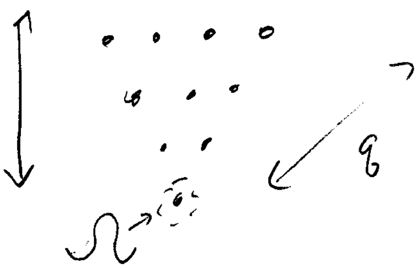
If we look at the Hilbert Space describing a single particle at rest, it should form a representation $SU(3)$.

Discovered by Gell-Mann: observed that the spin-1/2 mesons could be naturally organized into an "octet"



This picture represents the 8-dimensional irrep. of $SU(3)$.

Look at spin-3/2 baryons



The 8-dim repr A can be constructed by taking the fund 3-d rep. $\mathcal{F}$ and the anti-fundamental 3-d rep $\bar{\mathcal{F}}$

$$\mathcal{F} \otimes \bar{\mathcal{F}} \simeq A \oplus (\text{trivial})$$

A symmetry that enters the SM in more fund^{l} way: "gauge" symmetry.

$$G = SU(3) \times SU(2) \times U(1)$$

strong

electroweak

$U(n)$ = $n \times n$ unitary matrices