SCREENING EFFECT DUE TO HEAVY LOWER TAILS IN ONE-DIMENSIONAL PARABOLIC ANDERSON MODEL

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Abstract: We consider the large-time behavior of the solution $u\colon [0,\infty)\times\mathbb{Z}\to[0,\infty)$ to the parabolic Anderson problem $\partial_t u = \kappa\Delta u + \xi u$ with initial data $u(0,\cdot)=1$ and non-positive finite i.i.d. potentials $(\xi(z))_{z\in\mathbb{Z}}$. Unlike in dimensions $d\geq 2$, the almost-sure decay rate of $u(t,0)$ as $t\to\infty$ is not determined solely by the upper tails of $\xi(0)$; too heavy lower tails of $\xi(0)$ accelerate the decay. The interpretation is that sites x with large negative $\xi(x)$ hamper the mass flow and hence screen off the influence of more favorable regions of the potential. The phenomenon is unique to $d=1$. The result answers an open question from our previous study [BK00] of this model in general dimension.

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1. Introduction

1.1 Model and main aim

In a recent paper [BK00], we have studied the asymptotic behavior of the solution $u(t, z)$ to the so-called parabolic Anderson model for non-positive i.i.d. potentials. Here we answer an open question concerning the almost-sure asymptotics of $u(t, 0)$ as $t \to \infty$ in dimension one for potentials lacking the first logarithmic moment. Interestingly, a new phenomenon arises: too heavy tails of the potential at $-\infty$ hamper the mass flow to remote areas, thus rendering the more favorable regions inaccessible. This effect is unique to $d = 1$, since only in one-dimensional topology particles are not able to bypass deep broad valleys in the potential landscape.

The general model is defined as follows. Let $u\colon [0, \infty) \times \mathbb{Z}^d \to [0, \infty)$ be the solution to the parabolic problem

$$\begin{aligned} \partial_t\, u(t, z) &= \kappa \Delta^{\mathrm{d}} u(t, z) + \xi(z) u(t, z), \qquad && (t, z) \in (0, \infty) \times \mathbb{Z}^d, \\ u(0, z) &= 1, && z \in \mathbb{Z}^d, \end{aligned} \tag{1.1}$$

where ∂_t is the time derivative, $\kappa > 0$ is the diffusion constant, Δ^{d} is the discrete Laplacian on $\mathbb{Z}^d$, $[\Delta^{\mathrm{d}} f](z) = \sum_{x \sim z} [f(x) - f(z)]$, and $\xi = (\xi(z))_{z \in \mathbb{Z}^d}$ is an i.i.d. field. We use $\langle \cdot \rangle$ to denote the expectation with respect to ξ and $\mathrm{Prob}(\cdot)$ to denote the underlying probability measure. One interpretation of the quantity $u(t, z)$ is the total expected mass accumulated at time t by a particle starting at z at time 0 and diffusing through a random field of sources (sites x with $\xi(x) > 0$) and sinks (sites x with $\xi(x) < 0$). The references [GM90], [CM94] and [K00] provide more explanation and other interpretations.

Besides [BK00], the large-t behavior of the solution to (1.1) has extensively been studied (in general dimension) for various other classes of distributions: see [GM90, GM98, GH99] for ξ having the so-called double-exponential upper tail, and [GK98, GKM99] for a continuous variant of (1.1) with ξ either Gaussian or (smeared) Poissonian field. The techniques used in these studies go back to the pioneering work of Donsker and Varadhan [DV75, DV79]; however, there is also an intimate relation to Sznitman's method of enlargement of obstacles [S98]. We refer to [K00] for a comprehensive discussion of these relations and a unified presentation of the above results. Henceforth, we shall focus on the almost-sure behavior of $u(t, 0)$ in the non-positive case, i.e., $\xi \in [-\infty, 0]^{\mathbb{Z}^d}$.

In dimensions $d \geq 2$, the analysis in [BK00] produced a fairly complete picture. Indeed, interesting behavior occurs only when $p = \mathrm{Prob}(\xi(0) > -\infty) > p_{\mathrm{c}}(d)$, the threshold for site percolation on $\mathbb{Z}^d$, and when conditioned on the event that the origin lies in the infinite cluster of sites x with $\xi(x) > -\infty$. Below and, provided there is no critical percolation (which is rigorously known for $d = 2$ [R78] and $d \geq 19$ [HS90]), also at $p_{\mathrm{c}}(d)$, and also when the origin lies in a finite cluster for $p > p_{\mathrm{c}}(d)$, the quantity $u(t, 0)$ decays exponentially in t with a ξ-dependent rate.

In dimension $d = 1$, we have $p_{\mathrm{c}}(d) = 1$, which necessitated setting $\mathrm{Prob}(\xi(0) = -\infty) = 0$ in [BK00]. However, the latter condition was not sufficient because the existence of the first logarithmic moment, i.e., $\langle \log(-\xi(0) \vee 1) \rangle < \infty$, also had to be assumed in order to establish an asymptotics analogous to the supercritical case in $d \geq 2$. In particular, two intriguing questions remained unanswered:

- Is $\langle \log(-\xi(0) \vee 1) \rangle < \infty$ optimal in the sense that $\langle \log(-\xi(0) \vee 1) \rangle = \infty$ implies a strictly different asymptotic behavior of $u(t, 0)$?
- What is the precise decay rate when the finiteness of $\langle \log(-\xi(0) \vee 1) \rangle$ is robustly violated (keeping however the restriction to "no atom at $-\infty$")?

In this paper we give answers to these questions under mild regularity conditions on the lower tail of the distribution of ξ. In particular, we show that $\langle \log(-\xi(0) \vee 1) \rangle < \infty$ is only marginally non-optimal for the behavior described in [BK00], see Remark 3 after Theorem 1.1. As it turns out, the decay of $u(t, 0)$ is determined solely by upper *and* lower tails of $\mathrm{Prob}(\xi(0) \in \cdot)$. The reason why the intermediate part of the distribution does not play any role is that these tails give rise to two dominant and mutually competing mechanisms (field-shape optimization in the upper tail *versus* screening effect in the lower tail) whose balancing determines the decay rate. See Subsection 2.2 for more precise heuristic explanation.

1.2 Our assumptions

We proceed by stating precisely the needed assumptions, both on upper and lower tails of $\xi(0)$. First we restrict ourselves to dimension $d = 1$ for the sequel of this paper. In accord with [BK00], we consider the distributions with the upper tail of the form

$$\mathrm{Prob}\big(\xi(0) \geq -x\big) = \exp\big\{-x^{-\frac{\gamma}{1-\gamma}+o(1)}\big\}, \qquad x \downarrow 0, \tag{1.2}$$

for some $\gamma \in [0, 1)$. However, instead of the distribution function, it is more convenient to work with the cumulant generating function

$$H(\ell) = \log\langle e^{\ell\xi(0)} \rangle, \qquad \ell \geq 0. \tag{1.3}$$

The regime in (1.2) corresponds to the behavior $H(\ell) = -\ell^{\gamma+o(1)}$ as $\ell \to \infty$.

Assumption (H). *Let* $\operatorname{esssup}\xi(0) = 0$ *and suppose there are constants* $A > 0$ *and* $\gamma \in [0, 1)$*, and a positive increasing function* $t \mapsto \alpha_t$ *such that*

$$\lim_{t\to\infty} \frac{\alpha_t^3}{t} H\Big(\frac{t}{\alpha_t} y\Big) = -Ay^\gamma, \qquad y > 0. \tag{1.4}$$

The limit in (1.4) is necessarily uniform on compact sets in $(0, \infty)$, the pair (A, α_t) is unique up to a scaling transformation. Moreover, $t \mapsto \alpha_t$ is regularly varying and $\alpha_t = t^{\nu+o(1)}$ as $t \to \infty$ where $\nu = (1 - \gamma)/(3 - \gamma) \in (0, 1/3]$. In particular, $t/\alpha_t \to \infty$, i.e., Assumption (H) indeed controls the upper tails of $\xi(0)$. We say that H *is in the* γ*-class* if (1.4) holds.

Next we formulate our assumption on the lower tails of $\xi(0)$ at $\operatorname{essinf}\xi(0) = -\infty$. As the opposite case has already been handled in [BK00], we shall focus on the case where $\log(-\xi(0) \vee 1)$ is not integrable. Central to our attention are lower tails of the form

$$\mathrm{Prob}\big(\log(-\xi(0) \vee 1) > x\big) = x^{-\zeta+o(1)}, \qquad x \to \infty, \tag{1.5}$$

with some $\zeta \in [0, 1]$. In terms of the modified cumulant generating function

$$G(\ell) = -\log\big\langle(-\xi(0) \vee 1)^{-1/\ell}\big\rangle, \qquad \ell > 0, \tag{1.6}$$

the behavior (1.5) roughly corresponds to $G(\ell) = \ell^{-\zeta+o(1)}$ as $\ell \to \infty$. Note that G is positive and decreasing since $\operatorname{essinf}\xi(0) < -1$. The following is a weak regularity condition for G at infinity.

Assumption (G). *Let* $\langle \log(-\xi(0)\vee 1)\rangle = \infty$ *but* $\mathrm{Prob}(\xi(0)=-\infty)=0$. *Suppose that for each* $\eta\in(0,1)$ *there is a function* $\widetilde G_\eta\colon(0,\infty)\to(0,\infty)$ *with the following properties:*

(i) $\widetilde G_\eta(\ell)\le G(\ell)^{\eta+o(1)}$ *as* $\ell\to\infty$.
(ii) $\ell\mapsto 1/\widetilde G_\eta(\ell)$ *is increasing and concave for* ℓ *large enough.*
(iii) *The random variable* $1/\widetilde G_\eta(\log(-\xi(0)\vee 1))$ *has the first moment.*

Remark 1. As it turns out, Assumption (G) is needed only for the proof of the lower bound in our main result (see Theorem 1.1 below); the upper bound requires no assumptions at all. The role of Assumption (G) and particularly of its part (i) is the following: Abbreviate $Y=\log(-\xi(0)\vee 1)$ and note that, for any $\delta\in(0,1]$, $G(\ell)\le\langle Y^\delta\rangle\ell^{-\delta}$. Therefore, $G(\ell)\le\ell^{-\zeta_*+o(1)}$ where $\zeta_*=\sup\{\delta\ge 0\colon\langle Y^\delta\rangle<\infty\}$. However, a lower bound of the same (even asymptotic) form requires some regularity of $\ell\mapsto G(\ell)$ as $\ell\to\infty$, which is the essence of (i–iii).

Remark 2. In the view of Remark 1, it is immediate that Assumption (G) holds for $G(\ell)=\ell^{-\zeta+o(1)}$ with some $\zeta\in(0,1]$. The reason why we prefer the above (little cumbersome) setting as opposed to simple regularity of G is that many cases with $G(\ell)=\ell^{o(1)}$ are automatically included. Indeed, consider the following example: Let $\theta>0$ and $\mathrm{Prob}(\log(-\xi(0)\vee 1)\in dx)\sim C/[x\log^{1+\theta}(x)]dx$ as $x\to\infty$, where C is the normalizing constant. Then $G(\ell)\sim C'(\log\ell)^{-\theta}$ and Assumption (G) holds with $\widetilde G_\eta(\ell)=G(\ell)[\log\log(\ell\vee e)]^{1+\theta'}$ for any $\eta<1$ and any $\theta'>0$.

1.3 Main result

We begin by defining the scale function of the almost-sure asymptotics:

$$\frac{b_t}{\alpha_{b_t}^2}=-\log G(t),\qquad t>0. \tag{1.7}$$

In other words, $t\mapsto b_t$ is the inverse of the function $t\mapsto t\alpha_t^{-2}$ (which we may and shall assume to be strictly increasing), evaluated at $-\log G(t)$. Note that, since $\lim_{\ell\to\infty}G(\ell)=0$, we have $b_t\to\infty$ as $t\to\infty$. If $G(\ell)=\ell^{-\zeta+o(1)}$ as $\ell\to\infty$ for some $\zeta\in(0,1]$, then $\alpha_{b_t}^2=\zeta^\beta(\log t)^{\beta+o(1)}$, where $\beta=2\nu/(1-2\nu)=2(1-\gamma)/(1-3\gamma)\in(0,2]$. In the case $\zeta=0$, $\alpha_{b_t}=o(\log^\beta t)$ as $t\to\infty$.

Here is our main result. The constant $\widetilde\chi$ appearing in (1.8) depends only on A, γ and κ and will be defined in Subsection 2.1.

Theorem 1.1 *Let* $d=1$ *and suppose that Assumption (H) and Assumption (G) hold. Define* $t\mapsto b_t$ *as in* (1.7)*, and let* $\widetilde\chi$ *be the constant in Theorem 1.5 of* [BK00]. *Then*

$$\lim_{t\to\infty}\frac{\alpha_{b_t}^2}{t}\log u(t,0)=-\widetilde\chi,\qquad \mathrm{Prob}\text{-almost surely.} \tag{1.8}$$

Interestingly, if $\ell\mapsto G(\ell)$ has a power-law decay as $\ell\to\infty$, the lower-tail dependence of the rate can explicitly be computed. This allows for an easy comparison with the assertion in Theorem 1.5 of [BK00]. Let $t\mapsto b_t^*$ be the scale function introduced in [BK00]:

$$\frac{b_t^*}{\alpha_{b_t^*}^2}=\log t. \tag{1.9}$$

Recall $\zeta_* = \sup\{\delta \geq 0\colon \langle[\log(-\xi(0) \vee 1)]^\delta\rangle < \infty\}$. For G decaying with a power law, necessarily, $G(\ell) = \ell^{-\zeta_*+o(1)}$. The following is an immediate consequence of Theorem 1.1 and the regularity of $t \mapsto \alpha_t$:

Corollary 1.2 *Let $d = 1$, suppose $\mathrm{Prob}(\xi(0) = -\infty) = 0$ and suppose that Assumption (H) holds. Assume that either $\zeta_* = 1$ or $\zeta_* \in (0,1)$ and $G(\ell) = \ell^{-\zeta_*+o(1)}$ as $\ell \to \infty$. Then*

$$\lim_{t\to\infty} \frac{\alpha_{b_t^*}^2}{t} \log u(t,0) = -\zeta_*^{-\beta}\widetilde{\chi}, \qquad \mathrm{Prob}\text{-almost surely.} \tag{1.10}$$

where $\beta = 2(1-\gamma)/(1-3\gamma)$.

Remark 3. By comparison of Corollary 1.2 and Theorem 1.5 of [BK00], $\zeta_* = 1$ is necessary and sufficient for the assertion of the latter to hold, at least in the class of distribution with G decaying as a positive power. In particular, the condition that $\langle\log(-\xi(0) \vee 1)\rangle < \infty$ in [BK00] is only marginally non-optimal because Theorem 1.5 also literally holds if we just assume that $[\log(-\xi(0) \vee 1)]^\delta$ be integrable for any $\delta < 1$. This answers the first of the questions above.

Remark 4. The cases with $\zeta_* > 0$ have a different absolute size of the rate while the time dependence remains as for $\zeta_* = 1$. However, when $\zeta_* = 0$, also the time dependence changes. For instance, in the aforementioned example $\mathrm{Prob}(\log(-\xi(0) \vee 1) \in dx) \sim C/[x\log^{1+\theta}(x)]\,dx$ as $x \to \infty$ (see Remark 2), $\alpha_{b_t}^2 = [\log\log t]^{\beta+o(1)}$, which grows much slower than in the case $\zeta_* > 0$. For yet thicker lower tails, even slower growths are possible. We conclude that the result of Theorem 1.5 of [BK00] qualitatively changes only when $-\xi(0)$ lacks all positive logarithmic moments.

The remainder of this paper is organized as follows: In the next section we define some important objects and use them to give a heuristic outline of the proof. The actual proof comes in Section 3. Since many steps can almost literally be taken over from [BK00], we stay as terse as possible. The essentially novel part are Lemmas 3.2, 3.3, and 3.4.

2. Definitions and heuristics

2.1 Auxiliary objects

For the sake of both completeness and later reference, we will now introduce the objects needed to define the quantity $\widetilde{\chi}$ in Theorem 1.1. Then we proceed by recalling the Feynman-Kac representation and some formulas for Dirichlet eigenvalues.

2.1.1 Definition of $\widetilde{\chi}$. Let $\mathcal{F}_R$ be the set of continuous functions $f\colon \mathbb{R} \to [0,\infty)$ satisfying $\mathrm{supp}\, f \subset [-R,R]$ and having total integral equal to one. Let $C^+(R)$ (resp., $C^-(R)$) be the set of continuous functions $[-R,R] \to [0,\infty)$ (resp. $[-R,R] \to (-\infty,0]$). For H in the γ-class, let $\mathcal{H}_R\colon C^+(R) \to (-\infty,0]$ be the functional defined by

$$\mathcal{H}_R(f) = -A\int_{[-R,R]} f^\gamma\, 1_{\{f>0\}}\, dx, \tag{2.1}$$

where A is as in (1.4).

Let $\mathcal{L}_R\colon C^-(R) \to [0,\infty]$ be the Legendre transform of $\mathcal{H}_R$:

$$\mathcal{L}_R(\psi) = \sup\bigl\{(f,\psi) - \mathcal{H}_R(f)\colon f \in C^+(R),\, \operatorname{supp} f \subset \operatorname{supp}\psi\bigr\}, \tag{2.2}$$

where $(f,\psi) = \int f(x)\psi(x)dx$. Conventionally, $\mathcal{L}_R(0) = \infty$. If H is in the γ-class with a $\gamma \in [0,1)$, $\mathcal{L}_R(\psi)$ can explicitly be computed: for any $\psi \in C^-(R)$, $\psi \not\equiv 0$,

$$\mathcal{L}_R(\psi) = \begin{cases} (1-\gamma^{-1})(A\gamma)^{\frac{1}{1-\gamma}} \int |\psi(x)|^{-\frac{\gamma}{1-\gamma}}\, dx, & \text{if } \gamma \in (0,1), \\ -A|\operatorname{supp}\psi|, & \text{if } \gamma = 0, \end{cases} \tag{2.3}$$

where $|\operatorname{supp}\psi|$ is the Lebesgue measure of $\operatorname{supp}\psi$. (Here $\mathcal{L}_R(\psi) = \infty$ whenever $\gamma \in (0,1)$ and the integral diverges.)

The last object we need is the principal eigenvalue of the operator $\kappa\Delta + \psi$ on $L^2([-R,R])$ with Dirichlet boundary conditions:

$$\lambda_R(\psi) = \sup\bigl\{(\psi, g^2) - \kappa\|\nabla g\|_2^2\colon g \in C_{\mathrm{c}}^\infty(\operatorname{supp}\psi, \mathbb{R}), \|g\|_2 = 1\bigr\}, \tag{2.4}$$

with the interpretation $\lambda_R(0) = -\infty$. Then

$$\widetilde{\chi} = -\sup_{R>0} \sup\bigl\{\lambda_R(\psi)\colon \psi \in C^-(R),\, \mathcal{L}_R(\psi) \le 1\bigr\}. \tag{2.5}$$

As was proved in [BK00], $\widetilde{\chi} \in (0,\infty)$.

Remark 5. In $d = 1$, the minimizer of an associated variational problem (namely, that for the annealed or moment asymptotics) can explicitly be computed, see [BK98]. Proposition 1.4 of [BK00] then allows $\widetilde{\chi}$ to be evaluated in a closed form. Except for $\gamma = 0$, no such expression is known in higher dimensions.

2.1.2 Feynman-Kac formula, Dirichlet eigenvalues. Let $(X(s))_{s\in[0,\infty)}$ be the continuous-time simple random walk on $\mathbb{Z}$ with generator $\kappa\Delta^{\mathrm{d}}$. We use $\mathbb{E}_x$ to denote the expectation with respect to the walk starting at x. The Feynman-Kac representation for $u(t,\cdot)$ then reads

$$u(t,x) = \mathbb{E}_x\Bigl[\exp\Bigl\{\int_0^t \xi\bigl(X(s)\bigr)\,ds\Bigr\}\Bigr]. \tag{2.6}$$

Given $R > 0$, let $Q_R = [-R,R] \cap \mathbb{Z}$ and let $u_R(t,x)$ be the solution to the system (1.1) in Q_R and Dirichlet boundary condition $u_R(\cdot,x) = 0$ for $x \notin Q_R$. Let τ_R be the first exit time from Q_R, i.e., $\tau_R = \inf\{s > 0\colon X(s) \notin Q_R\}$. Then

$$u_R(t,x) = \mathbb{E}_x\Bigl[\exp\Bigl\{\int_0^t \xi\bigl(X(s)\bigr)\,ds\Bigr\} 1\{\tau_R > t\}\Bigr] \tag{2.7}$$

Note that $R \mapsto u_R(t,x)$ is increasing.

In the forthcoming developments we will also need the principal Dirichlet eigenvalue of the operator $\kappa\Delta^{\mathrm{d}} + \xi$ in the box $z + Q_R$ centered at z:

$$\lambda^{\mathrm{d}}_{z;R}(\xi) = \sup\Bigl\{\sum_{x\in Q_R} \xi(x+z)g(x)^2 + \kappa\sum_{x\in Q_R} g(x)[\Delta^{\mathrm{d}} g](x)\colon g \in \ell^2(Q_R),\, \|g\|_2 = 1\Bigr\}. \tag{2.8}$$

Note that, by the standard eigenvalue expansion (see [BK00]),

$$\mathrm{e}_R(z)^2 e^{t\lambda^{\mathrm{d}}_{z;R}(\xi)} \le u_R(t,z) \le \#Q_R\, e^{t\lambda^{\mathrm{d}}_{z;R}(\xi)}, \tag{2.9}$$

where $\mathrm{e}_R(\cdot)$ is the ℓ^2-normalized principal eigenvector in Q_R. In particular, the logarithmic asymptotics of $u_R(t,z)$ and the asymptotics of $t\lambda^{\mathrm{d}}_{z;R}(\xi)$ coincide provided $R = R(t)$ does not grow too fast with t (which ensures that $t \mapsto \mathrm{e}_{R(t)}(z)^2$ does not decay too fast).

2.2 Heuristic explanation

As alluded to in the introduction, (1.8) results from the competition of two mechanisms: (1) searching for optimal shapes of the potential by the walk in (2.6) and (2) screening off far away sites by regions of strongly negative potential. Let us describe this interplay in detail. To avoid cluttering of indices we often use $\alpha(b_t)$ in the place of α_{b_t}.

Consider a "macrobox" $Q_{r(t)} = [-r(t), r(t)] \cap \mathbb{Z}$ with $r(t) \approx \exp[b_t\alpha(b_t)^{-2}]$, where we think of b_t as of a yet undetermined scale function. Fix $R > 0$ and a shape function $\psi \in C^-(R)$ satisfying $\mathcal{L}_R(\psi) < 1$. A Borel-Cantelli argument shows that there exists a randomly located microbox in $Q_{r(t)}$, with diameter $2R\alpha(b_t)$, where ξ is shaped like $\psi_t(\cdot) \approx \psi(\cdot/\alpha(b_t))/\alpha(b_t)^2$. Let us assume that R and ψ approximately maximize (2.5), i.e., $\lambda_R(\psi) \approx -\widetilde{\chi}$. Then the dominating strategy for the walk is to move in a short time to that favorable microbox and spend the rest of the time until t in it. The contribution coming from the long stay in the microbox is roughly $\exp[t\lambda_{R\alpha(b_t)}(\psi_t)]$, which can be approximated by $\exp[t\alpha(b_t)^{-2}\lambda_R(\psi)] \approx \exp[-t\alpha(b_t)^{-2}\widetilde{\chi}]$, using the scaling properties of the Laplace operator.

The size of the macrobox is determined by the amount of mass the walk loses on the way from the origin to the favorable microbox, while traveling through long stretches of large negative potential. A calculation shows that the penalty it pays is roughly of order $\exp[-\sum_{x=1}^{r(t)} \log(-\xi(x) \vee 1)]$. (An optimal strategy is not to spend more than $(-\xi(x) \vee 1)^{-1}$ time units at each site x on the way.) Under our assumptions on the lower tails of $\xi(0)$, a Borel-Cantelli argument shows that this penalty is roughly $\exp[-G^{-1}(1/r(t))]$ where G^{-1} denotes the inverse function of G.

As it turns out, the two mechanisms run at optimal "speed" when the two exponents are roughly of the same order, i.e., $G^{-1}(1/r(t)) \approx t\alpha(b_t)^{-2} \approx t$, because $\alpha_{b_t} \ll t$. Recalling that $r(t) \approx \exp[b_t\alpha(b_t)^{-2}]$, this reasoning leads to (1.7). A fine tuning of $r(t)$ makes the contribution from the travel to the microbox negligible compared to the contribution from the stay in it, i.e., we shall in fact have $G^{-1}(1/r(t)) = o(t\alpha(b_t)^{-2})$. Hence, we obtain (1.8) with $\widetilde{\chi}$ as in (2.5).

3. Proof of Theorem 1.1

As in [BK00], the main result will be proved by separately proving upper and lower bounds in (1.8). The proof of Corollary 1.2 comes at the very end of this section.

3.1 The upper bound

Recall the notation of Subsection 2.1, in particular that $Q_R = [-R, R] \cap \mathbb{Z}$. Let

$$r(t) = -\frac{3}{G(t)} \log G(t). \tag{3.1}$$

Note that $r(t) = t^{\zeta+o(1)}$ as $t \to \infty$ if $\widetilde{G}(\ell) = \ell^{-\zeta+o(1)}$ as $\ell \to \infty$. Abbreviate $B_R(t) = Q_{r(t)+2\lfloor R \rfloor}$.

The crux of the proof of the upper bound in Theorem 1.1 is the following generalization of Proposition 4.4 of [BK00] adapted to the new definition of $r(t)$.

Proposition 3.1 *There exists a constant $C = C(\kappa) > 0$ and a random variable $C_\xi \in (0,\infty)$ such that,* Prob*-almost surely, for all $R, t > C$,*

$$u(t,0) \le C_\xi e^{-t} + e^{Ct/R^2} 3r(t) \exp\Big\{t \max_{z\in B_R(t)} \lambda^{\mathrm{d}}_{z;2R}(\xi)\Big\}. \tag{3.2}$$

Proof of Theorem 1.1, upper bound. With Proposition 3.1 in the hand, the proof goes along very much the same lines as in [BK00]. Indeed, let $r(t)$ be as in (3.1) and set R in (3.2) to be $R\alpha(Kb_t)$, where $K > 0$ will be chosen later and R will tend to ∞. Let H be in the γ-class and recall that $\alpha_t = t^{\nu+o(1)}$ with $\nu = (1-\gamma)/(3-\gamma)$.

Abbreviate $B(t) = B_{R\alpha(Kb_t)}(t)$ and $\lambda(z) = \lambda^{\mathrm{d}}_{z;2R\alpha(Kb_t)}(\xi)$, and note that $r(t) \le e^{o(t\alpha(b_t)^{-2})}$. Then, using also that $\lim_{t\to\infty} \alpha(Kb_t)/\alpha(b_t) = K^\nu$, we have from (3.2) that

$$\limsup_{t\to\infty} \frac{\alpha_{b_t}^2}{t} \log u(t,0) \le \frac{C}{K^{2\nu}R^2} + \limsup_{t\to\infty} \Big[\alpha_{b_t}^2 \max_{z\in B(t)} \lambda(z)\Big], \tag{3.3}$$

Prob-almost surely. Abbreviating $M(t) = \max_{z\in B(t)} \lambda(z)$, we have to prove that, for any $\varepsilon > 0$,

$$\limsup_{t\to\infty} \alpha_{b_t}^2 M(t) \le -\widetilde{\chi} + \varepsilon, \qquad \text{Prob-almost surely,} \tag{3.4}$$

for some appropriate $K \in (0,\infty)$ and sufficiently large R.

Note that the eigenvalues $\lambda(z)$ have identical distribution. Furthermore, their exponential moments can be estimated by

$$\limsup_{R\to\infty} \limsup_{t\to\infty} \frac{\alpha_{b_t}^2}{b_t} \log\big\langle e^{Kb_t\lambda(z)}\big\rangle \le -K^{1-2\nu}\chi, \tag{3.5}$$

where $\chi \in (0,\infty)$ is a constant related to $\widetilde{\chi}$, see [BK00]. Since $t \mapsto M(t)$ is increasing and $t \mapsto \alpha_{b_t}$ slowly varying, it suffices to prove (3.4) for t taking only a discrete set of values; the main difference compared to [BK00] is that now we take

$$\frac{1}{G(t)} \in \{e^n : n \in \mathbb{N}\}. \tag{3.6}$$

Let $G(t) = e^{-n}$ and note that (1.7) implies that $b_t \alpha_{b_t}^{-2} = n$. The proof now proceeds exactly as in [BK00]: We let $p_n(\varepsilon) = \mathrm{Prob}(M(t)\alpha_{b_t}^2 \ge -\widetilde{\chi} + \varepsilon)$ and use the Chebyshev inequality and (3.5) to derive that $p_n(\varepsilon)$ is summable on n for all $\varepsilon > 0$, provided K is chosen appropriately and R is sufficiently large. The claim is finished using the Borel-Cantelli lemma. ☐

It remains to prove Proposition 3.1. In [BK00], the choice $t \log t$ for $r(t)$ allowed us to use a simple probability estimate for the simple random walk; in particular, the corresponding bound (3.2) held true uniformly in all non-positive potentials. In our present cases, $r(t)$ is typically much smaller than $t \log t$ and the potential has to cooperate to get the bound (3.2). Unlike in [BK00], the role of the potential is actually dominant in the cases of our present interest.

Lemma 3.2 *For any $b \in (2\kappa, \infty)$ there is a random variable $C(\xi) \in (0, \infty)$ such that,* Prob-*almost surely,*

$$u(t,0) - u_R(t,0) \le C(\xi)\Bigg(\prod_{x=0}^{R} \frac{b}{-\xi(x) \vee b} + \prod_{x=-R}^{0} \frac{b}{-\xi(x) \vee b}\Bigg), \qquad R \in \mathbb{N}, t \ge 0. \tag{3.7}$$

Proof. Let $(X_k)_{k\in\mathbb{N}_0}$ be the embedded discrete-time simple random walk on $\mathbb{Z}$ and let $\ell_n(x)$ be its local times defined by $\ell_n(x) = \sum_{k=1}^n 1\{X_k = x\}$. Let $\mathbb{E}_y^{\mathrm{d}}$ denote the expectation with respect to the discrete-time walk, starting at $y \in \mathbb{Z}$. Abbreviate $\xi_k = \xi(X_k)$ and $\widehat{u}_R(t,0) = u(t,0) - u_R(t,0)$. Then, by (2.6) and (2.7),

$$\widehat{u}_R(t,0) = e^{-2\kappa t} \sum_{n \ge R} (2\kappa)^n \, \mathbb{E}_0^{\mathrm{d}} \bigg[\int_{\triangle_n(t)} dt_1 \dots dt_n \, \exp\Big\{\sum_{k=0}^{n} \xi_k t_k\Big\} 1\{\operatorname{supp} \ell_n \not\subset Q_R\}\bigg], \tag{3.8}$$

where $\triangle_n(t) = \{(t_1, \dots, t_n) \in (0,\infty)^n \colon t_1 + \dots + t_n \le t\}$, and t_0 is a shorthand for $t - (t_1 + \dots + t_n)$.

Fix $b > 2\kappa$ and define

$$\mathcal{A}_n = \big\{x \in \operatorname{supp} \ell_n \colon \xi(x) \le -b\big\}. \tag{3.9}$$

Let

$$\mathcal{I}_n = \big\{k \in \{1, \dots, n\} \colon X_k \notin \mathcal{A}_n\big\} \tag{3.10}$$

be the set of all the times at which the walk visits a point x with $\xi(x) > -b$.

By relaxing the constraint $t_1 + \dots + t_n \le t$ in $\triangle_n(t)$ to $t_k \le t$ for every $k \in \mathcal{I}_n$, neglecting the terms with $k \in \mathcal{I}_n \cup \{0\}$ in the exponential, and integrating out $t_1, \dots, t_n$, we get

$$\widehat{u}_R(t,0) \le e^{-2\kappa t} \sum_{n \ge R} \sum_{m=0}^{n} \frac{(2\kappa t)^m}{m!} \mathbb{E}_0^{\mathrm{d}} \bigg[1\big\{\#\mathcal{I}_n = m\big\}\, 1\{\operatorname{supp} \ell_n \not\subset Q_R\} \prod_{0<k\le n \colon k \notin \mathcal{I}_n} \frac{2\kappa}{-\xi_k}\bigg]. \tag{3.11}$$

Neglecting the first indicator and the restriction to $m \le n$, we can carry out the sum over m in (3.11) and find that

$$\widehat{u}_R(t,0) \le \sum_{n \ge R} \mathbb{E}_0^{\mathrm{d}} \bigg[1\{\operatorname{supp} \ell_n \not\subset Q_R\} \prod_{x \in \mathcal{A}_n} \Big(\frac{2\kappa}{-\xi(x)}\Big)^{\ell_n(x)}\bigg]. \tag{3.12}$$

On $\{\operatorname{supp} \ell_n \not\subset Q_R\}$, the walk visits either all sites in $\{0, \dots, R\}$ or all sites in $\{-R, \dots, 0\}$. Hence, we can estimate

$$1\{\operatorname{supp} \ell_n \not\subset Q_R\} \prod_{x \in \mathcal{A}_n} \frac{2\kappa}{-\xi(x)} \le \prod_{x=1}^{R} \frac{b}{-\xi(x) \vee b} + \prod_{x=-R}^{1} \frac{b}{-\xi(x) \vee b}. \tag{3.13}$$

The claim (3.7) then follows from the assertion

$$\sum_{n=1}^{\infty} \mathbb{E}_0^{\mathrm{d}} \bigg[\prod_{x \in \mathcal{A}_n} \Big(\frac{2\kappa}{b}\Big)^{\ell_n(x)-1}\bigg] < \infty \qquad \text{Prob-almost surely}, \tag{3.14}$$

where we used that $\xi(x) \le -b$ whenever $x \in \mathcal{A}_n$. (The term with $x = 0$ in (3.7) can be added or removed at the cost of changing $C(\xi)$ by a finite amount.)

Let us prove that (3.14) holds. First we note that $\mathcal{A}_n$ contains in every sufficiently large interval in $\mathbb{Z}$ a positive fraction of sites. Indeed, put $p = \operatorname{Prob}(\xi(0) > -b) \in (0,1]$ and note that by Cramér's theorem we have $\mathrm{Prob}(\#(\mathcal{A}_n \cap I) \le \frac{p}{2}\#I) \le e^{-c\#I}$ for every bounded interval $I \subset \mathbb{Z}$ and some $c > 0$ independent of I. A routine application of the Borel-Cantelli lemma implies that

$$\forall\,\text{interval } I \subset [-n,n]\cap\mathbb{Z}: \qquad \#I \ge n^{1/4} \implies \#(\mathcal{A}_n \cap I) > \frac{p}{2}\#I, \tag{3.15}$$

for n large enough, Prob-almost surely.

Now we prove that with high probability, there are sufficiently large intervals which are traversed from one end to the other at least twice by the random walk $(X_k)_{k=0,\dots,n}$. Fix $K_n = \lfloor 3\log n\rfloor$ and abbreviate $k_n = \lfloor n/K_n\rfloor$. We divide the walk into K_n pieces $(X_k^{(i)})_{k=0,\dots,k_n}$ (neglecting a small overshoot) with $X_k^{(i)} = X_{(i-1)k_n+k} - X_{(i-1)k_n}$ for $i = 1,\dots,K_n$. Note that these K_n walks are independent copies of each other. Let us introduce the events

$$B_n = \bigcap_{i=1}^{K_n-1}\bigl\{\mathrm{sgn}X_{k_n}^{(i)} = \mathrm{sgn}X_{k_n}^{(i+1)}\bigr\} \qquad\text{and}\qquad C_n = \bigcup_{i=1}^{K_n}\bigl\{\max_{1\le k\le k_n}|X_k^{(i)}| \le L_n\bigr\}, \tag{3.16}$$

where $L_n = \sqrt{k_n/(\eta\log n)}$. It is elementary that $\mathbb{P}_0^{\mathrm{d}}(B_n) \le 2^{-K_n+1} \le n^{-2+o(1)}$ as $n\to\infty$. Furthermore, with the help of a concatenation argument and convergence of simple random walk to Brownian motion we derive that $\mathbb{P}_0^{\mathrm{d}}(C_n) \le n^{-2+o(1)}$, whenever $\eta > 0$ is large enough. Now we estimate

$$\mathbb{E}_0^{\mathrm{d}}\Bigl[\prod_{x\in\mathcal{A}_n}\Bigl(\frac{2\kappa}{b}\Bigr)^{\ell_n(x)-1}\Bigr] \le \mathbb{P}_0^{\mathrm{d}}(B_n) + \mathbb{P}_0^{\mathrm{d}}(C_n) + \mathbb{E}_0^{\mathrm{d}}\Bigl[1_{B_n^{\mathrm{c}}\cap C_n^{\mathrm{c}}}\prod_{x\in\mathcal{A}_n}\Bigl(\frac{2\kappa}{b}\Bigr)^{\ell_n(x)-1}\Bigr]. \tag{3.17}$$

Note that, on $B_n^{\mathrm{c}} \cap C_n^{\mathrm{c}}$, there is an interval $I \subset [-n,n]\cap\mathbb{Z}$ with $\#I \ge L_n$ such that every point of I is visited by at least two of the subwalks, i.e., we have $\ell_n(x) \ge 2$ for any $x \in I$. If n is sufficiently large, we deduce from (3.15) that there are at least $pL_n/2$ points x with $\ell_n(x) \ge 2$. By using this in (3.17), we have

$$\mathbb{E}_0^{\mathrm{d}}\Bigl[\prod_{x\in\mathcal{A}_n}\Bigl(\frac{2\kappa}{b}\Bigr)^{\ell_n(x)-1}\Bigr] \le n^{-2+o(1)} + \Bigl(\frac{2\kappa}{b}\Bigr)^{L_np/2}, \qquad n\to\infty. \tag{3.18}$$

The right hand side is clearly summable on $n\in\mathbb{N}$ since $2\kappa/b < 1$. This finishes the proof. □

Our next task is to get a good estimate on the size of the products in (3.7).

Lemma 3.3 *Suppose that* $\langle\log(-\xi(0)\vee 1)\rangle = \infty$. *Then, for all* $b \ge 1$,

$$\lim_{n\to\infty}\frac{1}{G^{-1}(1/n)}\sum_{x=1}^{\lfloor 2n\log n\rfloor}\log\Bigl(\frac{-\xi(x)\vee b}{b}\Bigr) = \infty \qquad \mathrm{Prob}\text{-almost surely.} \tag{3.19}$$

Proof. Abbreviate $N_n = \lfloor 2n\log n\rfloor$ and let $b \ge 1$. Then

$$\log\Bigl(\frac{-\xi(x)\vee b}{b}\Bigr) \ge \log(-\xi(x)\vee 1) - \log b. \tag{3.20}$$

Using this estimate and the Chebyshev inequality, we have for any $\theta > 0$ that

$$\mathrm{Prob}\Bigl(\sum_{x=1}^{N_n}\log\bigl(\tfrac{-\xi(x)\vee b}{b}\bigr) \le \theta G^{-1}(1/n)\Bigr) \le \exp\bigl\{-N_nG(1/\lambda) + N_n\lambda\log b + \lambda\theta G^{-1}(1/n)\bigr\}, \tag{3.21}$$

for any $\lambda > 0$. Set $\lambda = 1/G^{-1}(1/n)$ and note that we have $G(1/\lambda)/\lambda \to \infty$ as $\lambda \downarrow 0$, due to $\langle \log(-\xi(0) \vee 1) \rangle = \infty$. Consequently, the term with $\log b$ is negligible and the right-hand side of (3.21) is bounded by $n^{-2+o(1)}$. The claim is finished by the Borel-Cantelli lemma. $\square$

Proof of Proposition 3.1. Pick any $b \in (2\kappa, \infty)$. Let t_0 be so large such that the sum in (3.19) with $n = \lceil 1/G(t) \rceil$ for all $t \geq t_0$ exceeds $G^{-1}(1/n)$. Note that $r(t) \geq \lfloor 2n \log n \rfloor$. Combining the results of Lemma 3.2 for $R = r(t)$ and Lemma 3.3, we derive, for sufficiently large n resp. t, the bound

$$u(t,0) - u_{r(t)}(t,0) \leq 2C(\xi) \exp\bigl(-G^{-1}(1/n)\bigr), \tag{3.22}$$

where $C(\xi)$ is the constant from (3.7). But $G^{-1}(1/n) \geq t$ by our choice of n, which means that $u(t,0) - u_{r(t)}(t,0) \leq C_\xi e^{-t}$, where $C_\xi = 2C(\xi) \vee e^{t_0}$. The rest of the argument does not involve the particular form of $r(t)$ and can directly be taken over from [BK00]. $\square$

3.2 The lower bound

Unlike the upper bound, the lower bound was basically proved already in [BK00], up to a change of the spatial scale and Lemma 3.4 below. For this reason, we shall only indicate the necessary changes.

First we prove the following converse of Lemma 3.3:

Lemma 3.4 *Fix $\eta \in (0,1)$ and let $\widetilde{G}_\eta$ satisfy (ii) and (iii) in Assumption (G). Then there exists a $\varrho \in (0,\infty)$ such that*

$$\limsup_{n\to\infty} \frac{1}{\widetilde{G}_\eta^{-1}(\varrho/n)} \sum_{x=1}^{n} \log\bigl(-\xi(x) \vee 1\bigr) \leq 1 \qquad \text{Prob-almost surely.} \tag{3.23}$$

Proof. The argument is based on the asymptotic sublinearity of $1/\widetilde{G}_\eta$ at infinity. However, in order to have sublinearity on the whole interval $(0,\infty)$, we first construct an auxiliary modification of $\widetilde{G}_\eta$.

Let $x_0 > 0$ be such that $1/\widetilde{G}_\eta$ is positive, increasing, and concave on $[x_0,\infty)$. Let D_0 to be the right derivative of $1/\widetilde{G}_\eta$ at x_0. Define $\widehat{G}_\eta \colon (0,\infty) \to (0,\infty)$ by the formula

$$1/\widehat{G}_\eta(x) = \begin{cases} D_0 x & \text{if } x \leq x_0, \\ 1/\widetilde{G}_\eta(x) + D_0 x_0 - 1/\widetilde{G}_\eta(x_0) & \text{if } x > x_0. \end{cases} \tag{3.24}$$

Note that $1/\widehat{G}_\eta$ is positive, increasing, concave and hence sublinear on $(0,\infty)$. Moreover, Assumption (G)(iii) holds true for $\widetilde{G}_\eta$ replaced by $\widehat{G}_\eta$.

For $a \geq 1$, abbreviate $Y_a(x) = \log(-\xi(x) \vee a)$. Choose $a = e^{x_0}$ and estimate, for $n \to \infty$,

$$\frac{1}{\widetilde{G}_\eta\Bigl(\sum_{x=1}^n Y_a(x)\Bigr)} \leq \frac{1+o(1)}{\widehat{G}_\eta\Bigl(\sum_{x=1}^n Y_a(x)\Bigr)} \leq (1+o(1)) \sum_{x=1}^{n} \frac{1}{\widehat{G}_\eta(Y_a(x))}, \tag{3.25}$$

where we used the fact that $\sum_{x=1}^n Y_a(x) \to \infty$ almost surely, and sublinearity of $1/\widehat{G}_\eta$. Since we have that $\langle 1/\widehat{G}_\eta(Y_a(x)) \rangle < \infty$, the Strong Law of Large Numbers tells us that the right-hand side of (3.25) is almost surely no more than ϱn, where for ϱ we can take, for instance,

$$\varrho = 2\bigl\langle 1/\widehat{G}_\eta(Y_a(0)) \bigr\rangle. \tag{3.26}$$

Hence, we derive

$$\sum_{x=1}^{n} Y_1(x) \le \sum_{x=1}^{n} Y_a(x) \le \widetilde{G}_\eta^{-1}(\varrho/n), \tag{3.27}$$

which directly yields the desired claim. □

Another important ingredient is the following adaptation of the crucial Proposition 5.1 of [BK00] to the present situation. For $\eta \in (0,1)$, choose ϱ as in Lemma 3.4 and let this time

$$\gamma_t = \frac{\varrho}{\widetilde{G}_\eta(t\alpha_{b_t}^{-3})} \tag{3.28}$$

be the size of the macrobox Q_{γ_t} (see Subsection 2.2). Note that $t^{\eta\zeta+o(1)} \le \gamma_t \le t^{\zeta+o(1)}$ as $t \to \infty$ if $G(\ell) = \ell^{-\zeta+o(1)}$ as $\ell \to \infty$. Suppose without loss of generality that $t \mapsto \gamma_t$ is increasing.

Define for each $\psi \in C^-(R)$ a "microbox"

$$Q^{(t)} = \begin{cases} Q_{R\alpha(b_t)} & \text{if } \gamma \ne 0, \\ Q_{R\alpha(b_t)} \cap \operatorname{supp}\psi_t & \text{if } \gamma = 0, \end{cases} \tag{3.29}$$

where $\psi_t\colon \mathbb{Z} \to (-\infty, 0]$ is the function $\psi_t(\cdot) = \psi(\cdot/\alpha_{b_t})/\alpha_{b_t}^2$. The crucial input for the lower bound is the following claim, which says that, with probability one provided $\mathcal{L}_R(\psi) < 1$ and t is large, there is at least one microbox $Q^{(t)}$ in Q_{γ_t}, where ξ is no less than (the accordingly shifted) ψ_t.

Proposition 3.5 *Let $R > 0$ and fix $\psi \in C^-(R)$ satisfying $\mathcal{L}_R(\psi) < 1$. Let $\varepsilon > 0$ and suppose Assumptions (G) and (H) hold. Then the following holds almost surely: For each $\eta \in (\mathcal{L}_R(\psi), 1)$, there is a $t_0 = t_0(\xi, \psi, \varepsilon, R, \eta) < \infty$ such that for each $t \ge t_0$, there is a $y_t \in Q_{\gamma_t}$ with*

$$\xi(z + y_t) \ge \psi_t(z) - \varepsilon\alpha_{b_t}^{-2} \qquad \forall z \in Q^{(t)}. \tag{3.30}$$

Proof. We begin by formalizing the event in (3.30); in order to later approximate continuous t by a discrete variable, we write $\varepsilon/2$ instead of ε:

$$A_y^{(t)} = \bigcap_{z \in Q^{(t)}} \bigl\{\xi(y+z) \ge \psi_t(z) - \tfrac{\varepsilon}{2\alpha(b_t)^2}\bigr\}. \tag{3.31}$$

Note that the probability of $A_y^{(t)}$ does not depend on y and note that different $A_y^{(t)}$'s are independent if the y's have distance larger than $3R\alpha(b_t)$ from each other. The proof of Lemma 5.5 in [BK00] shows that $\operatorname{Prob}(A_0^{(t)}) \ge G(t)^{\mathcal{L}_R(\psi)+o(1)}$ as $t \to \infty$ (the only modification required is to replace every occurrence of t in the meaning $\exp\{b_t\alpha(b_t)^{-2}\}$ by $1/G(t)$).

In order to prove our claim, it is sufficient to show the summability of

$$p_t = \operatorname{Prob}\Bigl(\bigcap_{y \in Q_{\gamma_t}} \bigl(A_y^{(t)}\bigr)^{\mathrm{c}}\Bigr) \tag{3.32}$$

over all $t > 0$ such that $1/G(t) \in \{e^n \colon n \in \mathbb{N}\}$. (The sufficiency follows from the facts that $\alpha(b_t)/\alpha(b_{et}) \to 1$ as $t \to \infty$ and that $t \mapsto b_t$ and $t \mapsto \gamma_t$ are increasing. The error terms are absorbed into an extra $\varepsilon/2$ in (3.30) compared to (3.31), see [BK00].)

Using the independence of $A_y^{(t)}$ for $y \in B(t) = Q_{\gamma_t} \cap \lfloor 3R\alpha(b_t) \rfloor \mathbb{Z}$ and the bound $\mathrm{Prob}(A_0^{(t)}) \geq G(t)^{\mathcal{L}_R(\psi)+o(1)}$, we easily derive

$$p_t \leq \big(1 - G(t)^{\mathcal{L}_R(\psi)+o(1)}\big)^{\#B(t)} \leq \exp\Big\{-\frac{G(t)^{\mathcal{L}_R(\psi)+o(1)}}{\alpha(b_t)\widetilde{G}_\eta(t\alpha(b_t)^{-3})}\Big\}, \tag{3.33}$$

where we used that $\#B(t) \geq 2\gamma_t/(3R\alpha(b_t))$ and then applied the definition of γ_t.

Use concavity of $1/\widetilde{G}_\eta$ to estimate $1/\widetilde{G}_\eta(t\alpha(b_t)^{-3}) \geq \alpha(b_t)^{-3}/\widetilde{G}_\eta(t)$ and use Assumption (G)(i) to bound $\widetilde{G}_\eta(t)$ by $G(t)^{\eta+o(1)}$. Furthermore, since $\alpha(b_t)$ is bounded from above by a positive power of $b_t\alpha(b_t)^{-2}$, we see from (1.7) that $\alpha(b_t) = G(t)^{o(1)}$. Applying all this reasoning on the right-hand side of (3.33), we see that $p_t \leq \exp(-G(t)^{\mathcal{L}_R(\psi)-\eta+o(1)})$ as $t \to \infty$, which is summable on the sequence of t such that $1/G(t) \in \{e^n : n \in \mathbb{N}\}$. This finishes the proof. □

Now we finish the proof of our main result.

Proof of Theorem 1.1, lower bound. Let $\varepsilon > 0$ and fix $R > 0$ and $\psi \in C^-(R)$ such that $\mathcal{L}_R(\psi) < 1$. Let $\eta \in (\mathcal{L}_R(\psi), 1)$, define γ_t as in (3.28) and let y_t be as in Proposition 3.5; suppose $y_t \geq 0$ without loss of generality. Let $r_x = [-\xi(x) \vee 1]^{-1}$. As in [BK00], the lower bound will be obtained by restricting the walk in (2.6) to perform the following: The walk keeps jumping toward y_t, spending at most time r_x at each site x such that it reaches y_t before time γ_t. Then it stays at y_t until time γ_t and then within $y_t + Q^{(t)}$ for the remaining time $t - \gamma_t$.

Inserting this event into (2.6) and invoking Markov property at time γ_t we get

$$u(t, 0) \geq \mathrm{II} \times \mathrm{III}, \tag{3.34}$$

where the same argument as in [BK00] shows that $\mathrm{III} \geq e^{t\alpha(b_t)^{-2}[\lambda_R(\psi)-\varepsilon]}$ for large t, while for II we have

$$\begin{aligned} \mathrm{II} &= \int_{\Delta_{y_t}(\gamma_t)} dt_0 \dots dt_{y_t-1}\, e^{-2\kappa\gamma_t} \exp\Big\{\sum_{k=0}^{n} \xi_k t_k\Big\} \prod_{x=0}^{y_t-1} 1\{t_x \leq r_{x-1}\} \\ &\geq e^{-2\kappa\gamma_t} \prod_{x=0}^{y_t-1} \Big[r_x e^{r_x\xi(x)}\Big] \geq e^{-(2\kappa+1)\gamma_t} \exp\Big\{-\sum_{x=0}^{y_t-1} \log\big(-\xi(x) \vee 1\big)\Big\}, \end{aligned} \tag{3.35}$$

where we recalled the notation of (3.8). Now $y_t \leq \gamma_t$, so using Lemma 3.4 we have that

$$\mathrm{II} \geq e^{-(2\kappa+1)\gamma_t} \exp\big\{-\widetilde{G}_\eta^{-1}(\varrho/\gamma_t)(1 + o(1))\big\} = e^{-(2\kappa+1)\gamma_t - t\alpha(b_t)^{-3}(1+o(1))}, \tag{3.36}$$

where we used the definition of γ_t. Since $1/\widetilde{G}_\eta$ is asymptotically concave, $\gamma_t = \varrho/\widetilde{G}_\eta(t\alpha_{b_t}^{-3}) \leq O(t\alpha_{b_t}^{-3})$ and the exponent is $o(t\alpha_{b_t}^{-2})$. Consequently,

$$u(t, 0) \geq e^{t\alpha(b_t)^{-2}[\lambda_R(\psi)-\varepsilon+o(1)]}, \tag{3.37}$$

where $o(1)$ still depends on η. The proof is finished by letting $t \to \infty$ (which eliminates the dependence on η), optimizing over ψ and R with $\mathcal{L}_R(\psi) < 1$ and letting $\varepsilon \downarrow 0$. □

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