The Bogoliubov model of Weakly Imperfect Bose-Gas*

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Abstract

We present a systematic account of known rigorous results about the Bogoliubov model of Weakly Imperfect Bose Gas (WIBG). This model is a basis of the celebrated Bogoliubov theory of superfluidity. The theory is based on two Bogoliubov's anstze: the first truncates the full Hamiltonian of the interacting bosons to produce the WIBG, whereas the second substitutes some operators by $c-$numbers (the Bogoliubov approximation). After some historical remarks, and physical and mathematical motivations of this Bogoliubov treatment of the WIBG, we turn to revision of the Bogoliubov's anstze from the point of view of rigorous quantum statistical mechanics. Since the exact calculation of the pressure and the behaviour of the Bose condensate in the WIBG are available, we review these results stressing the difference between them and the Bogliubov theory. One of the main feature of the mathematical analysis of the WIBG is that it takes into account quantum fluctuations ignored by the second Bogoliubov ansatz. It is these fluctuations which are responsible for indirect attraction between bosons in the fundamental mode. The latter is the origin of a nonconventional Bose condensation in this mode, which has a dynamical nature. A (generalized) conventional Bose-Einstein condensation appears in the WIBG only in the second stage as a result of the standard mechanism of the total particle density saturation. It coexists with the nonconventional condensation. We give also a review of some models related to the WIBG and to the Bogoliubov theory, where the similar two-stage Bose condensation may take place. They indicate possibilities to go beyond the Bogoliubov theory and the Hamiltonian for the WIBG.

Keywords : Bogoliubov theory of superfluidity, Bogoliubov Weakly Imperfect Gas, Bose-Einstein condensation, generalized condensations (conventional, nonconventional)
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*Dedicated to André VERBEURE on the occasion of his 60th birthday

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Introduction

"La seule vraie connaissance
est la connaissance des faits."
Buffon, *Histoire naturelle.*

The main object of the present review is the Bogoliubov model of Weakly Imperfect Bose Gas (WIBG). The corresponding Hamiltonian H_Λ^B is a *truncation* of the full Hamiltonian H_Λ of a Bose system with pair interactions between particles. For Bogoliubov the Hamiltonian H_Λ^B was a starting point for his celebrated *microscopic theory* of superfluidity [1–3]. This theory includes another famous Bogoliubov trick: approximation of some operators in H_Λ^B by $c-$numbers. Then the corresponding Hamiltonian $H_\Lambda^B(c)$ becomes bilinear. Its diagonalization by the Bogoliubov *u-v* transformation shows that for a *suitable choice* of $c-$numbers the system $H_\Lambda^B(c)$ has a Landau-type *excitation spectrum* [4, 5]. The latter considered as an explanation of *superfluidity* of the WIBG and, finally, as a microscopic theory of this phenomenon in liquid ^{4}He.

Notice that the both of Bogoliubov's ingenious inventions:

- *truncation* of the full Hamiltonian;
- *approximation* of operators by $c-$numbers,

have behind them a fundamental hypothesis about existence in the systems H_Λ and H_Λ^B of a *Bose-Einstein condensation* in *one* specific mode [6–8]. At this point we touch one of the most fascinating problem of contemporary mathematical physics - the proof of existence of *a condensation of bosons* in *interacting* systems.

The present paper is motivated by a recent progress in *rigorous results* concerning the Bogoliubov model of WIBG [9–16]. Since these results touch one of the fundamental part of the existing theory of Bose systems, we believe that this review gives more insight in understanding of interacting quantum models.

1.1 Historical remarks

In 1947 Bogoliubov published three revolutionary papers on the theory of interacting Bose-gas [1–3]. The famous paper [1] is in fact the English translation of the Russian version [2] presented to the Session of the Physical and Mathematical Departments of the Academy of Sciences of the USSR on October 21, 1946. The third paper [3] contains a lecture delivered on March 20, 1947, on a Seminar of the Physical Department of the Moscow State University. In this (almost unknown) paper Bogoliubov scrutinizes his theory and formulates it in a transparent, elegant form. Only in 1970 the paper has been reproduced in [17]. In this refined form the Bogoliubov theory was presented in 1949 in his famous "Lectures on Quantum Statistics", but in the Ukrainian language! The English edition appeared only in 1967 [18] and the Russian in [19].

Bogoliubov formulated his theory for temperature T=0 K. Throughout the papers [1–3, 17, 18], he stressed that his aim is to give a *microscopic* explanation of the *phenomenological* Landau's theory of superfluidity [4, 5]. It was Landau who, at the first time, understood that the properties of two substances ^{4}He and ^{3}He, which rest liquid (under normal pressure) even for T $\to$ 0 K, can be explained only by a new kind of quantum arguments. The Landau phenomenology was based on the following assumptions [4, 5]:

(*a*) at T=0 K, in quantum liquids, like ^{4}He (or ^{3}He), a *collective* (*coherent*) behaviour dominates the *individual* movement of atoms, i.e. the properties of these liquids can be entirely described by the *spectrum* of *collective excitation*;

(*b*) for liquid ^{4}He, this spectrum has *two branches* : "*phonons*" for long-wavelength excitations and "*rotons*" for a relatively short-wavelength collective excitations.

Therefore, according to Landau, the quantum liquid, as ^{4}He, at low temperatures, is nothing but a "gas of collective elementary excitations" or "quasi-particles". The second assumption ensures that this liquid is *superfluid*, i.e. the structure of the spectrum of excitations ensures that the liquid can move in a capillary without *friction* when the flow velocity is smaller than some critical value defined by the spectrum. Moreover, this structure of spectrum agrees with the low-temperature thermodynamic properties of liquid ^{4}He, such as specific heat and others [4, 5, 20].

Bogoliubov treated the Landau's assumptions in the framework of mathematical physics. From this point of view, the key-problem is to show that quantum mechanical behaviour of the *macroscopic* system of about 10^{23} atoms of ^{4}He is such that the *spectrum* of the corresponding *microscopic Hamiltonian* is close to the one proposed by Landau. It was a courageous programme, since calculations of the spectrum even for a *few-body* quantum system (e.g. atomic or molecular spectra beyond the hydrogen atom) pose enormous problems. Faced with this oppressive difficulty, Bogoliubov was guided by the Landau's bench-mark that (at least) the low energy part of the spectrum of about 10^{23} atoms of ^{4}He is defined by *coherent collective* movements of the system instead of *individual* ones. In the next (after [1–3]) paper [21] about kinetic equation in his theory of superfluidity, Bogoliubov alludes to low-energy excitations in crystals as a sound example of collective excitations. In fact, the *crystalline order* of atoms suppresses their individual displacements in favor of coherent collective movements which are nothing but the *sound waves* or *phonons*. For low-energy excitations, i.e. in harmonic approximation, spectrum of the corresponding macroscopic Hamiltonian for the microscopic system of atoms can be calculated *exactly*!

Since the system in question, i.e. ^{4}He, is liquid (and not solid !), the starting point of the Bogoliubov programme was to find a physical (or mathematical) mechanism which as in crystals, favors the collective motions of the "*helium jelly*", via some kind of ordering or coherence. Since the atoms of ^{4}He are *bosons* (in contrast to ^{3}He, which are *fermions*), a plausible conjecture relates to the *Bose-Einstein (BE) condensation* predicted for the *Perfect Bose Gas* (PBG) by Einstein [6] in 1925. It was Fritz London [22] who suggested, in 1938, that the transition to superfluidity in liquid ^{4}He might be an example of BE condensation. (Notice that superfluid phases in ^{3}He were discovered only in 1972 !). The basis of his arguments was that:

- the atoms of ^{4}He are *bosons*;
- the transition of the *normal* liquid ^{4}He (called He I) to *superfluid* phase He II (discovered by Kapitza [23] and Allen, Misener [24] in 1938) takes place at a temperature $\mathrm{T}_\lambda = 2,17$ K (along the vapor pressure curve); if the liquid ^{4}He is treated as a PBG, its temperature of BE condensation T_c would be very close to T_λ: $\mathrm{T}_c = 3,14$ K;
- the BE condensation in the PBG corresponds to a *macroscopic occupation* of the ground state which is related to collective *coherent properties* of the *condensate*.

Inspired by these observations, Tisza and London [25] tried to argue that the superfluidity is related to the motion of the BE condensate, moving as a whole. These attempts raised many objections. Since the spectrum of PBG Hamiltonian is known explicitly, the most obvious of objections is that this spectrum does not satisfy the *Landau criterion of superfluidity*!

In [1–3, 17, 18, 21, 26], Bogoliubov stressed that, in spite of a long-range coherence of condensate, elementary excitations in the PBG correspond to movements of individual atoms, i.e. "quasi-particles" simply coincide with particles. Therefore, accepting the idea that the BE condensation plays a crucial role in decoding a nature of superfluidity, he insisted that "an energy level scheme based on the solution of the quantum mechanical many-body problem with interactions, must be found" [18], Part 3.4. To summarize, the BE condensation *together* with interaction between bosons will *transform* individual excitations of the PBG into collective excitations of the "helium jelly" with a Landau-type spectrum.

1.2 Bose-Einstein condensation and superfluidity

Since the concept of the BE condensation is the *first* important ingredient of the Bogoliubov theory, let us discuss it in more details.

The phenomenon of macroscopic accumulation of boson particles in the ground state of the PBG was described first by Einstein [6] in 1925. He based his arguments on a combination of Bose Statistics and the classical expression for the density of states in phase-space. This combination plus difficulties in understanding of transition in finite volume provoked, in 1927, the Uhlenbeck criticism of the Einstein analysis [27]. It was only in 1937, when Kramers pointed out the importance of the thermodynamic limit for the sharp manifestation of transition into condensed phase, that Uhlenbeck withdrew his objection and pointed out that the Einstein prediction is correct in the thermodynamic limit [28]. In 1938, Fritz London [29] developed the concept of macroscopic occupation of the ground state and showed that it implies long-range coherence properties of the BE condensate.

After [29], the physical mechanism of this *conventional* BE condensation of the PBG became so transparent that, at present, it is an essential part of any standard textbook on statistical physics, see e.g. [30–33]. Let a PBG be enclosed in the three dimensional cubic box of volume V. Then the structure of the one-particle density of states (in the thermodynamic limit $V \to \infty$) is such that, for a given temperature $\theta = k_B\mathrm{T}$ (here k_B is the Boltzmann constant), there is a *critical* particle density $\rho_c^P(\theta)$, which corresponds to *maximal* total particle density under condition that the particle density in the ground state $\rho_0^P(\theta) = 0$. Notice that, in the limit $V \to \infty$, the one-particle ground state energy corresponds to the particle-momentum $p = 0$, i.e. to one-particle energy $\varepsilon(p) = p^2/2m = 0$, where m is the mass of bosons. Further increasing of total density beyond the *saturation* threshold $\rho_c^P(\theta)$ leads to macroscopic accumulation of particles in the ground state, i.e. $\rho_0^P(\theta) = \rho - \rho_c^P(\theta)$. This *saturation mechanism* of condensation does not change mean kinetic-energy density or pressure: they also attain their maximal values at the critical density $\rho_c^P(\theta)$. This picture tempted Tisza and London to interpret the condensate of frozen in momentum space bosons with $p = 0$ as a "superfluid component", and the rest of particles as a "normal component" which is the carrier of the total entropy of the system. Later, this picture was justified in the framework of the Landau phenomenological theory [4, 5, 20].

Now we would like to warn the reader accustomed to the conventional BE condensation that a rigorous analysis of this phenomenon on the level of limiting quantum Gibbs states is rather a nontrivial. It created a beautiful branch of mathematical physics, in particular related to representations of Canonical Commutation Relations. First results in this direction were due to Araki and Woods [34], Lewis and Pulè [35], Cannon [36], see also [37].

It is appropriate to mention here that the *conventional* BE condensation with macroscopic occupation of the ground state due to the saturation mechanism, is a rather subtle and fascinating matter, for an authoritative review about PBG, see Ziff, Uhlenbeck, Kac [7] and van den Berg, Lewis, Pulè [8], van den Berg, Lewis, Lunn [38]. In [8], the notion of *generalized* BE condensation is extensively discussed.

The concept of *generalized* (*conventional*) BE condensation was introduced by Girardeau [39], in 1960. He was motivated by the observation that, in one-dimensional model of impenetrable bosons, there is a sort of *generalized condensation* but no macroscopic occupation of the ground state. In fact, in 1963, Schultz [40] (and then Lenard [41] by another method) showed that this is not the case. In spite of that, the theoretical importance of this concept was realized by van den Berg, Lewis and Pulè. In [42] van den Berg and Lewis discussed rigorously and in detail the BE condensation in the PBG due to the standard saturation mechanism in a prism of volume V with sides V^{α_x}, V^{α_y}, V^{α_z}, where $\alpha_x \geq \alpha_y \geq \alpha_z > 0$ and $\alpha_x + \alpha_y + \alpha_z = 1$. A similar example was sketched by Casimir [43] in 1967 to point out the importance of distinguishing between thermodynamics and the coherence properties of BE condensation in the PBG. This example shows a condensation in "a narrow band" of levels. In [42] it was shown that:

(I) if $\alpha_x < 1/2$, then above the $\rho_c^P(\theta)$, the condensation starts *only* in the ground state;
(II) if $\alpha_x = 1/2$, then for $\rho > \rho_c^P(\theta)$, the condensation starts in the ground state and *infinitely many* levels close to the latter;
(III) if $\alpha_x > 1/2$, then the density of particles which does contribute in kinetic energy or entropy is $\rho_0^P(\theta) = \rho - \rho_c^P(\theta)$ (condensed particles), but *no levels* are macroscopically occupied.

The latter case is an example of *non-extensive* generalized BE condensation which satisfies the Girardeau's conditions [39]. More refined examples were given by van den Berg in [44] and [45]. In [46] and [47], van den Berg, Lewis and Pulè showed that the *generalized* BE condensation of *types II* or *III* can be provoked in the PBG by a *scaled external field.*

What is behind the *generalized* BE condensations in the PBG ? The van den Berg-Lewis-Pulè analysis shows that it is structure of the one-particle spectrum (spectrum of Laplacian) which distinguishes the *type I, type II or type III* condensations. On the other hand, thermodynamic properties (e.g. pressure in the thermodynamic limit or density of condensed particles) are independent of *type* of condensation [8]. This means that the *generalized* BE condensation is a kind of *"finite-size" effect* due to restricted geometry or (scaled) external field, which gets visible above the saturation density $\rho_c^P(\theta)$.

Since in the recent experiments with *trapped* Bose gases, we have the both: a restricted geometry and an external field, see e.g. [48] and [49], it does not clear which *type* of conventional BE condensation one gets in this case! Moreover, in spite of extreme dilution of atoms in the trap, it has become evident that one cannot explain experimental data (e.g. spatial distribution of condensed atoms) without taking into account the *interaction* between them [48–50]. But if one takes into account an *indirect* interaction between bosons, a *nonconventional* Bose condensation may come into existence [12].

To explain it, notice that by cooling a quantum gas or by increasing its density, one can make the thermal de Broglie wavelength comparable to the average separation distance between particles. In this case, the overlapping of wave-functions of individual atoms is so strong (*degenerate quantum gas*) that one cannot ignore anymore their statistics. In other words, even in the case of perfect gas, i.e. *no direct* interparticle interaction, the quantum particles are *not independent.* Calculation of the two-point correlation function [30] shows an *effective repulsion* between *fermions* but an *effective attraction* between *bosons.* This is the essential physics behind the conventional BE condensation : the only phase transition that occurs in the *absence* of (direct) interactions between particles! Therefore, an *effective attraction* does present in PBG. It makes this system *coherent* (BE condensation) at low temperatures, or high densities, and *unstable*: the grand partition function diverges for positive chemical potentials. Since bosons are very sensible with respect of attraction, any *direct attractive* interaction between particles leads to the *collapse* (fortunately, our world is essentially fermionic : the matter is stable ! [51]). But, if one counterbalances the direct attractive interaction by a repulsion stabilizing the system, the original attraction may be an origin of a new mechanism of condensation which we call a *nonconventional* (Bose) *condensation*. Thouless [52], in 1961, presents an instructive "back-of-the-envelope" calculations, which shows this new kind of condensation occurs in the Huang-Yang-Luttinger model [53]. In 1998, a *nonconventional* condensation was discovered in the Bogoliubov Weakly Imperfect Bose Gas (WIBG) by the authors of the present paper [12, 15, 16].

The difference between *conventional* and *nonconventional* condensations reflects the difference in the mechanism of their formation. The *conventional* BE condensation is a consequence of the balance between *entropy* and *kinetic energy*, whereas the *nonconventional* condensation results from the balance between *entropy* and *interaction energy*.

Rigorous analysis of the Huang-Yang-Luttinger (HYL) model and the proof of a prediction of Thouless [52] that the *nonconventional* (type I) condensation in the HYL model should appear with a *jump* is due to van den Berg, Dorlas, Lewis, Pulè [54–57]. A similar jump-behaviour of the *nonconventional* condensation was discovered in the WIBG [15]. Moreover, we proved that in this system condensation occurs in two stages : with increasing density (or chemical potential), the *nonconventional* (type I) condensation appears first with a jump, then, after a new critical density, the *conventional* (type I) BE condensation gradually springs up [16]. Therefore, conventional and nonconventional condensation may coexist. In [58, 59] we consider a model where a *nonconventional* (type I) condensation coexists with a *conventional* (type III, i.e. non-extensive) BE condensation.

As we see, the notion of the *condensate* of Bosons has to be reexamined when we consider interacting systems. Therefore, it is appropriate to clarify this notion, since the Bogoliubov theory *insists* on the necessity of interaction to give a microscopic explanation of superfluidity. Intuitively, the Bose condensate is a fraction of particles which does not move, "frozen in momentum space", with $p = 0$. This picture poses no problem for the PBG, where the operator of number of particles in a given mode is the integral of motion. Since single-particle quantum states are not proper for interacting system, Onsager and Penrose [60, 61], in 1956, tried to work out definition of *condensate* in this case. Their proposal was to identify condensation with an Off-Diagonal Long-Range Order (ODLRO), related to asymptotics of one-particle density matrix. This criterion shows that the mean-value of particle operator for the *mode* $p = 0$ can still be used as a characterization of a BE condensation. Moreover, they have given 8 % as an estimate of the fraction of particles in liquid ^{4}He having $p = 0$ at T=0 K.

Hohenberg and Platzman [62], in 1966, proposed to use deep-inelastic neutron scattering for experimental observation of the Bose condensate in ^{4}He. Cowley and Woods [63], in 1968, were the first, who announced that they observed about 17 % of condensate at T $= 1,1$ K. After that different groups repeated and improved experiments in this direction. The "Dubna-Obninsk group", at the first time, not only reexamined the fraction of the condensate: $(3,6 \pm 1,4)$ % at T $= 1,2$ K [64], but also measured its temperature dependence [65]. This data gives: $(2,2 \pm 0,2)$ % for the condensate at T=0 K and $\mathrm{T}_c = (2,24 \pm 0,04)$ K for the critical temperature. Recall that the λ-point for superfluidity corresponds to $\mathrm{T}_\lambda = 2,18$ K! Hence, the appearance of the condensate with a good accuracy coincides with appearance of superfluidity, while the latter at T=0 K attains 100 %.

Of course, they were only the first (far from to be definite!) experimental estimates of the

Bose condensate fraction in the liquid ^{4}He. After these pioneering attempts, now (as we have learned from referee) it is universally agreed by the "superfluid He community" that the best estimate of the condensate fraction is about 9% at T = 0 K, for details of the history and of the methods see [68] and the recent review by Sokol in [49]. Notice that this value of condensate is surprisingly close to the above Penrose-Onsager estimate [60, 61].

While the experimental evidence of the *condensate* in the liquid ^{4}He bolsters one of the main hypothesis of the Bogoliubov theory, we have to mention another experimental aspect directly related to *superfluidity*. Henshaw and Wood [66], in 1961 measured the spectrum of elementary excitations in the liquid ^{4}He by a weak-inelastic neutron scattering. Their data gave a sound evidence that a *main branch* of the excitations indeed has a Landau-type form with *connected* "*phonons*" and "*rotons*" parts, as it is predicted by the Bogoliubov theory! But there were at least two points that cause difficulties to fit these data with the theory of superfluidity:

- Application of the *Landau criterion* of superfluidity to the Henshaw-Woods spectrum gives for the *critical velocity* $v_c \approx 60\ m/s$ whereas superfluidity in capillaries disappears when velocity is of the other of *few* cm/s. Moreover, it depends sensitively on the diameter of the channel.

- The Henshaw-Woods spectrum does *not* change drastically when the temperature crosses T_λ: the Landau criterion still gives $v_c > 0$ for $T > T_\lambda$, whereas there is *no* superfluidity for these temperatures.

The attempts to explain these "misfittings" are concentrated around the idea that Landau-type spectrum discovered by Henshaw and Woods is only a *part* of a plethora of other type of "elementary" excitations *not covered* by the Bogoliubov theory, see e.g. [67] and the book [68].

Of course, it would be too naïve to imagine that the Bogoliubov theory is a theory of "real superfluidity" or an ultimate theory of ^{4}He. It is, first of all, a mathematical approach to microscopic theory of many-body interacting boson systems, which solves principal difficulties on the way leading to calculation of the ground-state energy and of the spectrum of such systems at T = 0 K, see Section 2.4. Moreover, the Bogoluibov phonon-maxon-roton dispersion branch is obviously only a part of the spectrum of the *full* (or even truncated) quantum-mechanical Hamiltonian of this system, for discussion see [84], [88] and Section 6.1. For example, being a "quadratic theory", the Bogoluibov approach does not take into account interaction between above excitations, which is known to lead to decay of excitations, to different kind of hybridizations, including appearance of pair-excitations branches, bound states of excitations etc. It is not the aim of the present paper to go into details of the hierarchy of excitations in the liquid ^{4}He, we address curious reader to e.g. a very detailed Griffin's book [68]. For "naïve" readers we mention only that the above limit on the critical velocity of superfluid ^{4}He in capillaries is determined essentially by excitations of *vertices,* that is obviously out of the scope of the standard Bogoluibov theory.

1.3 The Bogoliubov model of Weakly Imperfect Bose-Gas (WIBG)

With this brief history of Bose condensation in mind we return to Bogoliubov's determination to give a microscopic theory of Landau-type spectrum of elementary excitations in He II.

Since the BE condensation in the PBG does not produce the Landau-type spectrum (and by consequence, any superfluidity), Bogoliubov turned his attention to interacting Bose-systems. He insisted that a due regard must be extended to the boson interaction together with the BE condensation. Then, he immediately discovered that this problem can not be tackled by standard methods of perturbation theory.

It is the BE condensation that causes the difficulty. Since for the unperturbed system (i.e. for the PBG) the ground state corresponds to Bose condensate of particles in either in the mode with momentum $p = 0$ (BE condensation) or in a band around this mode (*generalized* BE condensation), *any* perturbation due to interaction is *larger* than kinetic energy in this state. Therefore, if we take only the kinetic energy operator of bosons as unperturbed part of the Hamiltonian, then we must face up to a theory of singular perturbations. Since these singularities are due to interactions with condensed particles, Bogoliubov formulated his *first ansatz* as a *truncation* of the full Hamiltonian which takes into account only interaction of particles *in* the condensate and interaction of condensed particles with particles *out* of the condensate. This *truncated* Hamiltonian, see [1] or [18], eq. (3.81), [17], eq. (28), he named as "nearly perfect" [1] or "weakly imperfect" [18] Bose-Einstein gas.

Below we call this truncated Hamiltonian the *Bogoliubov Model of Weakly Imperfect Bose Gas* (WIBG).

To proceed further, Bogoliubov formulated the *second ansatz*. Combining the *macroscopic* occupation of the ground state (i.e. the BE condensate in the mode $p = 0$) with *canonical commutation relations* of creation/annihilation operators in this mode, he proposed to treat the particle in the condensate *classically*. Due to this prescription the condensate acts as a *classical particle reservoir* (or external *classical complex field*) which noncondensate particles can enter and leave via scattering, see [1–3,18]. Notice that this famous Bogoliubov prescription amounts to *neglecting* the dynamical quantum behaviour of condensate, i.e. the *quantum fluctuations*. The corresponding, *approximating* to the WIBG, Hamiltonian with c-numbers instead of zero-mode creation/annihilation operators, has the following properties:

- it is a *bilinear* form in creation/annihilation operators of noncondensate particles;
- it *does not* conserve (noncondensate) particles.

The first allows us to diagonalize the *approximating* Hamiltonian of the WIBG by canonical Bogoliubov u-v transformation to obtain the spectrum of this system in the explicit form. After a judicious choice of c-numbers involved in this Hamiltonian, Bogoliubov got the Landau-type gapless spectrum of elementary excitations with the phonon-like branch for small wave vectors and with a "roton" minimum in domain close to the Henshaw-Woods experimental data. The u-v transformation shows that elementary excitations are a non-interacting gas of bosons. Bogoliubov stresses that all above is due to approximations (the *first* and the *second* ansätze) involved in his theory [1–3, 18]. Having taken into account the terms disregarded by the first ansatz, one goes beyond the Bogoliubov theory, which gives, e.g. an interaction between elementary excitations.

This programme was initiated by Beliaev in 1958 [69,70]. His approach forms the basis of the systematic application of the quantum field theory methods to boson systems *with condensate*, including *anomalous* propagators representing two particles going into or out of the condensate, Dyson-Beliaev equations etc. Since the *condensate* is treated in the frame-work of the *second* Bogoliubov *ansatz*, the Beliaev theory needs the formalism of the *grand-canonical ensemble* with the chemical potential defined by the density of particles and interaction. In particular, Beliaev bolstered up the Bogoliubov hypothesis about the elementary excitations interaction by demonstrating that in the long-wavelength limit $k \to 0$ the spectrum of the excitations contains an imaginary part (decay) which goes as $\sim k^5$ [70].

This approach was developed further in 1959 by Hugenholtz and Pines [71, 72], and was thoroughly reexamined in 1963 by Gavoret and Nozières [73,74]. They showed that the *singular* character of interaction in the Bose systems *with condensate* remarked by Bogoliubov manifests itself in divergences of perturbation theory at small momenta (*infrared divergences*). Gavoret and Nozières [74], scrutinized this question inspired by the higher-order calculations by Beliaev [70] and Hugenholtz-Pines [71]. They showed that:

- infrared divergences cancel out in all physical quantities;
- the energy gap in elementary excitation spectrum vanishes to all order of perturbation theory;

for further development see . [68, 75]
The last observation (known as the Hugenholtz-Pines-Gavoret-Nozières theorem) was discovered by Bogoliubov under the name $1/q^2$ theorem in his "Quasi-averages in Problems of Statistical Mechanics" published in 1963 [26] Part I, Section 2.8, and [76]. Instead of perturbation theory the Bogoliubov's proof of this theorem is based on local gauge transformations. This method clarifies why one is faced with appearance of a gap in the elementary excitations spectrum when one tries to "improve" the perturbation theory or the Bogoliubov theory "mismatching" the approximations, see [77–80]. Bogoliubov showed [26, 76] that the appearance of a gap in some models is a consequence of the lack of local gauge invariance of these models, see [81–84] for discussions.

The Bogoliubov theory [1–3, 18] as well as further development by Beliaev, Hugenholtz-Pines, Gavoret-Nozières were restricted to the zero temperature T=0. The first finite-temperature calculations were made in 1957 by Lee, Huang, and Yang [85] for a gas of hard spheres, see also [86]. For T $\neq$ 0 the thermally induced depletion of condensate is now important which may to spoil the validity of the both Bogoliubov's ansätze. To avoid the "mismatching" one has to treat the condensate in some consistent fashion. It was Popov (see e.g. [87,88]) who proposed in 1965 a generalization of the Bogoliubov theory for T $\neq$ 0 that gives the elementary excitation spectrum similar to that for T=0, but now with temperature-dependent condensate. For further details of the Beliaev-Popov theory we address readers to a recent review by Shi and Griffin [89].

1.4 Outline

In the present review we try to summarize recent *rigorous* results about the Bogoliubov model of the Weakly Imperfect Bose-Gas (WIBG).

The motivation comes from the following. Whilst the Bogoliubov *truncation* of the full Hamiltonian (the 1-st ansatz), which leads to the WIBG-model, reflects his vision of the rôle of different interactions in the non-ideal Bose-gas, his second ansatz is based on *two hypotheses*:

- The first one is that the Bose-Einstein (BE) condensation *does exist* in this non-ideal system and it occurs only in *one mode* (ground state) of the Perfect Bose-Gas (PBG).
- The second one is that we may to treat this conventional condensation "classically", i.e. *approximating* the creation/annihilation operators, corresponding to this mode, by *c-numbers.*

Neither the first, nor the second of these hypotheses have been ever justified rigorously for the WIBG. Moreover, Bogoliubov himself stressed many times in [3, 18, 21] that, since the perturbation theory around the PBG is highly singular, a Bose-gas with any interaction, however weak, may be qualitatively rather *different* from the non-interacting system. Therefore, the assumption that in the Bogoliubov WIBG we *do* have a Bose condensation must be proven. In the Bogoliubov theory, which starts with the *full* Hamiltonian, this problem is still open: the existence of the BE condensate in *one* mode (ground state) of the PBG was meant *ad hoc*, see an instructive discussion by Lieb [90] and Section 2.4. A challenge to mathematical physics was to *prove* this fact at least for the *truncated* WIBG-model related to the Bogoliubov theory. The second hypothesis is also too straightforward. First of all it sounds as "approximation" of *unbounded* creation/annihilation operators by some *bounded c-numbers.* Second, it excludes all *quantum fluctuations* related to the condensed mode, which may be a cause of indirect *effective interactions* between bosons in this mode as well as outside of it, not taken into account by direct interaction between zero- and nonzero-modes particles in the WIBG. It is well-known that fluctuations of the condensate in the PBG are "abnormal", see [7]. Therefore, effective interactions via mediation of the fluctuating condensate could be (and they are, see [13, 14]) important in the WIBG if condensation in this system *exists,* and if it is *similar* to the one in the PBG. At the first time, the problem of *exactness* of the *Bogoliubov approximation* for a non-ideal Bose-gas with *superstable* interaction [91] was considered rigorously in 1968 by Ginibre [92] and then by Buffet, de Smedt and Pulè [93]. A mathematical justification of this procedure for the Bogoliubov model of the WIBG is provided only recently in [15]. There we also proved that, the condensate in the WIBG starts as a *nonconventional* Bose condensate due to an effective attraction between bosons in the ground-state mode. This interaction results from the quantum fluctuations in this mode [13,14]. A conventional (generalized) Bose-Einstein condensation comes into existence *after* the nonconventional one [16].

Since there are very few rigorous results concerning the Bogoliubov model of the WIBG and the Bogoliubov theory, we believe that the present review will fill this gap, at least partially. We suppose that the important rôle of quantum fluctuations, which are behind the thermodynamic behaviour of the WIBG, is instructive for further attempts in theoretical understanding of interacting Bose-systems. This review is a status report on the WIBG from the point of view of *rigorous* results. We hope that it will discourage from performing sloppy manipulations with Bose condensations, quantum fluctuations and different kinds of ansätze. We believe that example of the exact solution of the WIBG-model (on the thermodynamic level) will provide a basis for further mathematical and theoretical studies of similar problems.

As any exact solution, this one should be considered as a *caricature* on physical reality, cf., for example, the well-known exact solution of the two-dimensional Ising model. The evident value of such models is in a confidence that one is able to control the properties of the model exactly. This allows to check rigorously fundamental principles related to a *modelized* phenomenon. Using again a magnetic analogy we recall the rigorous Peierls-Dobrushin argument proving spontaneous magnetization in the Ising model at low temperatures, which has nothing to do with magnetization in Fe, Ni or Co. Similarly, we do not claim that properties of the WIBG are directly related to recent experimental data on liquid helium, see [67], or on vapor of trapped alkali metals, see [48–50]. As far as physics is concerned the provenance of the WIBG-model hints at its probable(?) relevance to the limit of low temperatures and densities. But, since this model does not coincide with (another *caricature*) the PBG, the rigorously established mechanism of effective attraction between bosons due to quantum fluctuations, implying the nonconventional condensation, undoubtedly merits an attention. Is it relevant to *physics* ? The answer of the referee of the present paper is "it does not". The author's answer is "we don't know". But what is definitely interested is a breaking the "common wisdom" prove of existence of the nonconventional condensation, which appears not due to the standard *saturation* mechanism of the BE condensation, but due to the *interaction,* see Section 3.

As any review the present one has evidently many omissions. One of the major omission is any serious discussion of the rigorous results on dynamics of (collective) quantum fluctuations in the WIBG and on inhomogeneous Bose condensations (conventional and nonconventional). The latter might have interest for a justification of the Gross-Pitaevskii equation, which plays an important rôle in analysis of experimental data concerning the Bose condensate of trapped alkali metals [48–50, 94–99]. The only excuse for the authors is that there are yet very few rigorous results in these directions, see e.g. the paper [100] about mathematical status of collective quantum fluctuations in boson systems, or [101, 102] and the recent paper [103] giving the proof of exactness of the Gross-Pitaevskii energy functional for low density bosons in a trap, as well as discussion in Section 6.

Section 2

The Bogoliubov theory of superfluidity

"To construct a complete molecular theory of superfluidity it is necessary to consider the liquid helium as being a system of interacting atoms"
Bogoliubov, *Lectures on Quantum Statistics.*

"Faced with this failure, theorists retreated into the corner of low density gases with weak interaction"
Lieb, *The Bose Fluid.*

To fix notations and to define explicitly the Bogoliubov's two ansätze we briefly review in this section his microscopic approach to the superfluidity theory. We start by the Perfect Bose Gas (PBG) for the simplest case of the system enclosed in a cubic box with periodic boundary conditions to explain the phenomenon of the conventional Bose-Einstein condensation in the zero-mode single-particle state, see Section 2.1. In the next Section 2.2 we discuss the first ansatz of Bogoliubov, i.e. the truncation which leads to the Bogoliubov Hamiltonian of the WIBG. Then we discuss the Bogoliubov approximation, i.e. the second Bogoliubov's ansatz. In Section 2.3 we review the Landau criterion of superfluidity and the spectrum of elementary excitations results from the Bogoliubov theory. A brief survey of theories that go beyond the Bogoliubov theory is presented in Section 2.4.

2.1 Second quantization, Perfect Bose Gas and conventional Bose-Einstein condensation

Consider a system of bosons of mass m enclosed in a *cubic* box $\Lambda = L \times L \times L \subset \mathbb{R}^{d=3}$ of side length L. (In what follows we shall stick essentially the *physical* dimensionality $d = 3$

with comments (if necessary) concerning the cases $d \neq 3$.) Without loss of generality we may suppose *periodic* boundary conditions (b.c.) for wave functions on the surface $\partial\Lambda$. We denote by $V \equiv |\Lambda| = L^{d=3}$ the volume of the box Λ.

Then the *one-particle* Hilbert space $\mathcal{H}$ coincides with the space of (periodic) square-integrable functions $L^2(\Lambda)$ in Λ. The set of the one-particle quantum *states* $\{\psi_k(x) = V^{-1/2}e^{ikx}\}_{k\in\Lambda^*}$ (i.e. eigenfunctions of the one-particle self-adjoint Schrödinger operator $T_\Lambda^{(1)} = (-\hbar^2\Delta/2m)_{\text{periodic b.c.}}$) forms an orthonormal basis in $\mathcal{H} = L^2(\Lambda)$. Here the dual set of wave-vectors

$$\Lambda^* = \left\{k \in \mathbb{R}^{d=3} : \ k_\alpha = \frac{2\pi n_\alpha}{L},\ n_\alpha = 0, \pm 1, \pm 2, ...,\ \alpha = 1, 2, 3\right\}. \tag{2.1}$$

is defined by the periodic b.c. on $\partial\Lambda$, i.e. $k = 2\pi n/L$, $n \in \mathbb{Z}^3$.

The standard device used in the quantum many-body problems is to pass to the *second quantization*, for details see e.g. [104–106] and standard texts on quantum statistical mechanics like [18, 37, 107], [108] or [109]. We wish to emphasize that it is not essential for the theory of the PBG but it is very convenient device for our discussion of the Bogoliubov theory of the WIBG.

We recall that the *boson Fock space* $\mathcal{F}_B(\mathcal{H})$ over the one-particle space $\mathcal{H}$ is a Hilbert space constructed as the orthogonal direct sum

$$\mathcal{F}_B(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_B^{(n)} = \bigoplus_{n=0}^{\infty} \left(L^2(\Lambda^n)\right)_{\text{symm}}, \tag{2.2}$$

where

$$\mathcal{H}_B^{(n)} \equiv \left(L^2(\Lambda^n)\right)_{\text{symm}} \tag{2.3}$$

are *symmetrized* n-particle Hilbert spaces appropriate for bosons, and $\mathcal{H}_B^{(n=0)} = \mathbb{C}$ is the set of complex numbers. The space (2.2) is endowed by the scalar product

$$(\Psi, \Phi)_{\mathcal{F}_B} = \overline{\Psi}^{(0)}\Phi^{(0)} + \sum_{n=1}^{\infty} \left(\Psi^{(n)}, \Phi^{(n)}\right)_{\mathcal{H}_B^{(n)}}, \tag{2.4}$$

where

$$\left(\Psi^{(n)}, \Phi^{(n)}\right)_{\mathcal{H}_B^{(n)}} = \int_{\Lambda^n} dx_1...dx_n \overline{\Psi}^{(n)}(x_1, ..., x_n)\, \Phi^{(n)}(x_1, ..., x_n),$$

for any $\Psi^{(n)}, \Phi^{(n)} \in (L^2(\Lambda^n))_{\text{symm}}$. Recall that for any $f \in L^2(\Lambda)$ one can introduce *annihilation* and (conjugate) *creation* operators, $a(f)$ and $a^*(f)$, acting on $\mathcal{F}_B(\mathcal{H})$ as

$$\begin{aligned} (a(f)\Psi)^{(n)}(x_1, ..., x_n) &= \sqrt{n+1} \int_\Lambda dx \overline{f(x)}\, (\Psi)^{(n+1)}(x, x_1, ..., x_n), \\ (a^*(f)\Psi)^{(n)}(x_1, ..., x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\, (\Psi)^{(n-1)}(x_1, ..., \widehat{x}_j, ..., x_n), \end{aligned} \tag{2.5}$$

where $\widehat{x}_j$ means that the argument x_j is omitted (a "hole"). One can check that operators (2.5) verify on $\mathcal{F}_B(\mathcal{H})$ the Canonical Commutation Relations (CCR):

$$[a(f), a^*(f)]\,\Psi = (f,g)_{\mathcal{H}}\,\Psi. \tag{2.6}$$

Notice that $\Omega = (1,0,0,...)$ is a *vacuum vector* in the space $\mathcal{F}_B(\mathcal{H})$, since by definition (2.5):

$$a(f)\,\Omega = 0,\ \forall f \in \mathcal{H}, \tag{2.7}$$

whereas

$$\prod_{j=1}^{l} a^*(f_j)\,\Omega = \left(0,...,0,\Psi^{(l)},0,...\right), \tag{2.8}$$

where $\Psi^{(l)}$ is defined by the set $\{f_j\}_{j=1}^{l}$ according (2.5). Finally we denote by

$$a_k = a(\psi_k) \text{ and } a_k^* = a^*(\psi_k), \tag{2.9}$$

the usual annihilation and creation Bose-operators in modes $k \in \Lambda^*$, i.e. in one-particle quantum states $\{\psi_k(x)\}_{k\in\Lambda^*}$.

With these notations the second-quantized Hamiltonian $T_\Lambda \equiv \Gamma\left(T_\Lambda^{(1)}\right)$ of the PBG in the box Λ can be written (formally) as

$$T_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k. \tag{2.10}$$

Here $\varepsilon_k = \hbar^2 k^2/2m$ is the one-particle energy spectrum of free bosons in the modes $k \in \Lambda^*$. Similarly one defines the *total particle-number operator*

$$N_\Lambda = \sum_{k\in\Lambda^*} a_k^* a_k, \tag{2.11}$$

where $N_k \equiv a_k^* a_k$ is the *occupation-number operator* in the mode $k \in \Lambda^*$. Operators (2.10) and (2.11) are self-adjoint on some appropriate domains $dom(T_\Lambda)$ and $dom(N_\Lambda)$ dense in the space $\mathcal{F}_B(\mathcal{H})$. By the CCR (2.6) and (2.7), (2.8) one gets that the *spectrum* of the occupation-number operator is equal to $\sigma(N_k) = \{0,1,2,...\}$. Vectors

$$\Psi_{\{n_k\}} = \prod_{k\in\Lambda^*} \frac{(a_k^*)^{n_k}}{\sqrt{n_k!}}\Omega,\ n_k \in \sigma(N_k), \tag{2.12}$$

form a (canonical) *orthonormal basis* in $\mathcal{F}_B(\mathcal{H})$ and they are proper for the operators N_q, $q \in \Lambda^*$, and (2.10), (2.11):

$$\begin{aligned} N_q \Psi_{\{n_k\}} &= n_q \Psi_{\{n_k\}}, \\ N_\Lambda \Psi_{\{n_k\}} &= \left(\sum_{q\in\Lambda^*} n_q\right) \Psi_{\{n_k\}}, \\ T_\Lambda \Psi_{\{n_k\}} &= \left(\sum_{q\in\Lambda^*} \varepsilon_q n_q\right) \Psi_{\{n_k\}}. \end{aligned} \tag{2.13}$$

In particular, *restriction* of the set (2.12) on the *n-particle sector* $\mathcal{H}_B^{(n)}$ of the Fock space $\mathcal{F}_B(\mathcal{H})$:

$$\Psi_{\{n_k\}}^{(n)} = \delta_{n, \sum_{k\in\Lambda^*} n_k} \Psi_{\{n_k\}} \tag{2.14}$$

is an orthonormal basis in $\mathcal{H}_B^{(n)}$. By $T_\Lambda^{(n)} = T_\Lambda \lceil \mathcal{H}_B^{(n)}$ and $N_\Lambda^{(n)} = N_\Lambda \lceil \mathcal{H}_B^{(n)}$ we denote corresponding restrictions of operators (2.10) and (2.11) on the n-particle sector $\mathcal{H}_B^{(n)}$. Finally, formulae (2.5) get in the basis (2.12) a particular simple form:

$$\begin{aligned} a_q \Psi(..., n_q, ...) &= \sqrt{n_q}\Psi(..., n_q - 1, ...), \\ a_q^* \Psi(..., n_q, ...) &= \sqrt{n_q + 1}\Psi(..., n_q + 1, ...). \end{aligned} \tag{2.15}$$

Let the system, we are concerned with, consist of n bosons enclosed in Λ which are in thermal equilibrium with a reservoir of the *temperature* $\theta = k_B \mathrm{T} \equiv \beta^{-1}$. Then the *canonical* Gibbs state is defined by the *canonical density matrix* in $\mathcal{H}_B^{(n)}$:

$$D_\beta\left(T_\Lambda^{(n)}\right) = \frac{e^{-\beta T_\Lambda^{(n)}}}{Z_\Lambda^P(\beta, n)}, \tag{2.16}$$

where

$$Z_\Lambda^P(\beta, n) = Tr_{\mathcal{H}_B^{(n)}} e^{-\beta T_\Lambda^{(n)}} \tag{2.17}$$

is the PBG *canonical partition function* related to the density of the *free-energy*:

$$f_\Lambda\left[T_\Lambda^{(n)}\right] = -\frac{1}{\beta V}\ln Z_\Lambda^P(\beta, n). \tag{2.18}$$

One can verify that for $\beta > 0$ the operator $D_\beta\left(T_\Lambda^{(n)}\right)$ belongs to the Tr-class of bounded operators on $\mathcal{H}_B^{(n)}$, see e.g. [91]. Therefore, (2.17) and (2.18) are correctly defined, but explicit calculation of the free-energy density even in the *thermodynamic limit*: $V \to \infty$, $n/V \to \rho$, is a rather nontrivial problem, see [18].

That is why a common wisdom is to study the PBG in the *grand canonical ensemble*. The system now is in equilibrium with a thermostat of the temperature θ and *chemical potential* μ, which define a mean-value of the number of bosons enclosed in Λ. The corresponding *grand-canonical* Gibbs state is defined now by the *grand-canonical density matrix* on $\mathcal{F}_B(\mathcal{H})$:

$$D_\beta(T_\Lambda - \mu N_\Lambda) = \frac{e^{-\beta(T_\Lambda - \mu N_\Lambda)}}{\Xi_\Lambda^P(\beta, \mu)}, \tag{2.19}$$

which is a Tr-class operator on this space for $\beta > 0$ and $\mu < 0$, see below. Then the PBG *grand partition function*

$$\Xi_\Lambda^P(\beta, \mu) = Tr_{\mathcal{F}_B} e^{-\beta(T_\Lambda - \mu N_\Lambda)} \tag{2.20}$$

is related to the *grand-canonical pressure* by

$$p_\Lambda[T_\Lambda] = \frac{1}{\beta V}\ln \Xi_\Lambda^P(\beta, \mu). \tag{2.21}$$

To calculate the trace in (2.20) one can use the basis (2.12) and the properties (2.13):

$$\Xi_{\Lambda}^{P}(\beta,\mu) = \prod_{k\in\Lambda^*} \sum_{n_k=0}^{\infty} e^{-\beta(\varepsilon_k-\mu)n_k}. \tag{2.22}$$

Since for the periodic b.c. $\inf\limits_{k\in\Lambda^*} \varepsilon_k = 0$, the grand partition function of the PBG is bounded for $\mu < 0$ and

$$p_\Lambda[T_\Lambda] = \frac{1}{\beta V}\sum_{k\in\Lambda^*} \ln\left(1-e^{-\beta(\varepsilon_k-\mu)}\right)^{-1}. \tag{2.23}$$

Using again the basis (2.12) and (2.13) one readily gets for the grand-canonical *mean-value* of the occupation-number operator in the mode k:

$$\langle N_k\rangle_{T_\Lambda}(\beta,\mu) \equiv Tr_{\mathcal{F}_B}\left(D_\beta\left(T_\Lambda-\mu N_\Lambda\right)N_k\right) = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}. \tag{2.24}$$

Therefore, the average of the total particle density (cf. (2.13))

$$\rho_\Lambda^P(\beta,\mu) = \frac{1}{V}\sum_{k\in\Lambda^*}\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} = \partial_\mu p_\Lambda[T_\Lambda], \tag{2.25}$$

is again valid only for $\mu < 0$.

Now we are in position to come to the question of the Bose-Einstein condensation. First, notice that for allowed values of chemical potential, $\mu < 0$, there is *no macroscopic occupation* of any mode $k \in \Lambda^*$:

$$\lim_{\Lambda}\left\langle\frac{N_k}{V}\right\rangle_{T_\Lambda}(\beta,\mu) = 0,\ \ k\in\Lambda^*,\ \ \mu<0.$$

Here we denote the *thermodynamic limit* $V \to \infty$ by $\lim\limits_{\Lambda}$. Second, for $\mu < 0$ the both sums in (2.23) and (2.25) are regular (integral) Darboux sums which converge in the limit $V \to \infty$ to corresponding Riemann integrals:

$$\begin{aligned} p^P(\beta,\mu) &= \frac{1}{\beta(2\pi)^3}\int_{\mathbb{R}^3} d^3k \ln\left(1-e^{-\beta(\varepsilon_k-\mu)}\right)^{-1}, \\ \rho^P(\beta,\mu<0) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\left(e^{\beta(\varepsilon_k-\mu)}-1\right)^{-1}. \end{aligned} \tag{2.26}$$

If one introduces the standard thermal wavelength λ_β by

$$\lambda_\beta \equiv \left(\frac{2\pi\hbar^2\beta}{m}\right)^{1/2}, \tag{2.27}$$

and the function

$$g_\alpha(z) \equiv \sum_{s=1}^{\infty}\frac{z^s}{s^\alpha},\ \ 0\le z\le 1,\ \ \alpha>0, \tag{2.28}$$

then

$$\begin{aligned} p^P(\beta,\mu) &= \frac{1}{\beta\lambda_\beta^3} g_{5/2}\left(e^{\beta\mu}\right), \\ \rho^P(\beta,\mu<0) &= \frac{1}{\lambda_\beta^3} g_{3/2}\left(e^{\beta\mu}\right). \end{aligned} \tag{2.29}$$

The crucial observation now comes from the fact that, since

$$\begin{aligned} \lim_{\mu\to-\infty} \rho_\Lambda^P(\beta,\mu) &= 0, \\ \lim_{\mu\to 0^-} \rho_\Lambda^P(\beta,\mu) &= +\infty, \end{aligned} \tag{2.30}$$

and function $\rho_\Lambda^P(\beta,\mu)$, cf. (2.25), is strictly monotonous for $\mu<0$, the *finite-volume* equation:

$$\rho = \rho_\Lambda^P(\beta,\mu),\ \ \beta>0, \tag{2.31}$$

has *unique* solution $\mu_\Lambda^P(\theta,\rho)<0$ for *any* $\rho>0$. On the other hand, the monotonous function $g_{3/2}(z)$ attends the maximum at $z=1$. We denote by

$$\rho_c^P(\theta) \equiv \sup_{\mu<0} \rho^P(\beta,\mu) = \lim_{\mu\to 0^-} \rho^P(\beta,\mu) = \frac{1}{\lambda_\beta^3} g_{3/2}(1) \tag{2.32}$$

the *critical density* for the PBG. Then the *infinite-volume* equation, cf. (2.31),

$$\rho = \rho^P(\beta,\mu) \tag{2.33}$$

has *unique* solution $\mu^P(\theta,\rho)\le 0$ for $\rho\le\rho_c^P(\theta)$, and it has *no* solutions for $\rho>\rho_c^P(\theta)$! To solve this paradox (which leads to the BE condensation) one has to return to individual terms in the sum (2.25) corresponding to the particle density in each mode $k\in\Lambda^*$. Let us single out the one-particle ground-state term corresponding to the mode $k=0$. Then equation (2.31) reads as

$$\rho = \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} + \frac{1}{V}\sum_{k\in\Lambda^*\setminus\{0\}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}. \tag{2.34}$$

Let $\rho<\rho_c^P(\theta)$ in (2.34). Since for any $\mu<0$ the sum in (2.34) tends (for $V\to\infty$) to the integral $\rho^P(\beta,\mu)$, cf. (2.26), and the first term to zero, one gets:

$$\lim_\Lambda \mu_\Lambda^P(\theta,\rho) = \mu^P(\theta,\rho) < 0,\ \ \rho<\rho_c^P(\theta). \tag{2.35}$$

Let $\rho>\rho_c^P(\theta)$ in (2.34). Notice that by (2.1)

$$\inf_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k = \varepsilon_{\|k\|=\frac{2\pi}{L}} = \frac{\hbar^2}{2m}(2\pi)^2\frac{1}{V^{2/3}}. \tag{2.36}$$

Therefore, the Darboux sum in (2.34) converges for $V\to\infty$ to the integral $\rho^P(\beta,\mu)$ *even* for $\mu\to 0^-$. Since this limit is bounded by $\rho_c^P(\theta)$ the solution $\mu_\Lambda^P\left(\theta,\rho>\rho_c^P(\theta)\right)<0$ *must* tend to zero for $V \to \infty$ in such a way that

$$\begin{aligned}
\lim_{\Lambda} \left\langle \frac{N_{k=0}}{V} \right\rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P(\theta,\rho)\right) &= \lim_{\Lambda} \frac{1}{V} \frac{1}{e^{-\beta\mu_\Lambda^P(\theta,\rho)} - 1} \\
&= \lim_{\Lambda} \left(\rho - \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} - 1} \right) \\
&= \rho - \rho_c^P(\theta). \qquad (2.37)
\end{aligned}$$

¿From (2.37) we find for the asymptotics of the chemical potential:

$$\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right) \simeq \frac{1}{\beta V\left(\rho - \rho_c^P(\theta)\right)}, \ V \to \infty. \qquad (2.38)$$

Hence, the particles corresponding to the surplus density $\rho_0^P(\theta) = \rho - \rho_c^P(\theta)$ all reside in the mode $k = 0$:

$$\lim_{\Lambda} \left\langle \frac{N_{k=0}}{V} \right\rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)\right) = \rho_0^P(\theta) = \rho - \rho_c^P(\theta), \qquad (2.39)$$

which is density of the *conventional* BE condensation in *one mode*. Indeed, as it was mentioned above, cf. (2.36), for the *next* to the $k = 0$ mode and for $V \to \infty$, one gets:

$$\begin{aligned}
\frac{1}{V} \sup_{k \in \Lambda^* \setminus \{0\}} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)\right) &= \frac{1}{V} \frac{1}{e^{\beta\left(\varepsilon_{\|k\| = \frac{2\pi}{L}} - \mu_\Lambda^P(\theta,\rho)\right)} - 1} \\
&\simeq \frac{m}{2\pi^2 \beta \hbar^2} \frac{1}{L}. \qquad (2.40)
\end{aligned}$$

Finally, for $\rho = \rho_c^P(\theta)$ by virtue of the arguments (2.34)-(2.37) one gets instead of (2.38) the asymptotics:

$$\mu_\Lambda^P\left(\theta, \rho = \rho_c^P(\theta)\right) \simeq \frac{const}{V^\alpha}, \ 2/3 < \alpha < 1. \qquad (2.41)$$

This is a textbook presentation of the BE condensation, see [30–32]. The mathematical study of the infinite-volume quantum Gibbs states was initiated in 1963 by Araki, Woods [34]. The complete theory of the grand-canonical and canonical Gibbs states and calculation of the corresponding generating functional of the cyclic representation of the CCR is due to Lewis, Pulè [35] and Cannon [36], see also [37].

2.2 The Weakly Imperfect Bose-Gas and Bogoliubov approximation

As above let us consider a system of identical bosons of mass m in a cubic box $\Lambda \subset \mathbb{R}^3$ with periodic boundary conditions (torus Λ). If

$$\varphi(x) = \varphi(\|x\|) \in L^1\left(\mathbb{R}^3\right), \qquad \textit{hypothesis (A)}$$

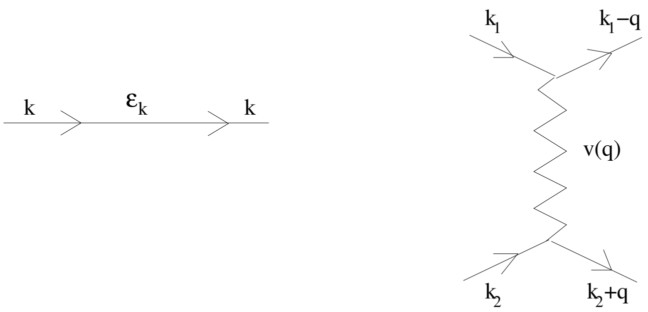

Figure 2.1: Kinetic-energy and interaction vertices of the full Hamiltonian.

denotes an absolutely integrable (real) *two-body* interaction potential. Then its (real) Fourier-transform

$$v(q) = \int_{\mathbb{R}^3} d^3x \varphi(x) e^{-iqx}, \;\; q \in \mathbb{R}^3, \tag{2.42}$$

exists and it is a continuous function vanishing at infinity. To ensure *stability*, i.e. convergence of the grand-canonical partition function, following Bogoliubov [1–3, 18] we suppose below that

$$v(0) \geq v(q) \geq 0 \text{ and } v(0) > 0, \qquad \textit{hypothesis (}\mathrm{B}),$$

see e.g. [91] and Appendix G. Notice that (2.42) for real $\varphi(x) = \varphi(\|x\|)$ implies $v(q) = v(\|q\|)$. The Hamiltonian for n of these *interacting* bosons is defined by a *self-adjoint* (s.-a.) extension

$$H_\Lambda^{(n)} = \left\{ \sum_{j=1}^{n} \left(-\frac{\hbar^2 \Delta_j}{2m} \right) + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \varphi_{\mathrm{per}}(x_i - x_j) \right\}_{\text{s-a}} \tag{2.43}$$

on the boson space $\mathcal{H}_B^{(n)}$ of symmetric functions on the *torus* Λ, see e.g. [37, 91]. Here $\varphi_{\mathrm{per}}(x) = \sum_{n\in\mathbb{Z}^3} \varphi(x + nL)$, $x \in \Lambda$, is the restriction of the potential $\varphi(x)$ on the torus Λ, i.e.

$$v(q) = \int_\Lambda d^3x \varphi_{\mathrm{per}}(x) e^{-iqx}.$$

Then the second-quantized Hamiltonian (2.43) acting on the boson Fock space $\mathcal{F}_B(\mathcal{H})$ has the form

$$H_\Lambda = T_\Lambda + U_\Lambda \equiv \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} v(q) a_{k_1-q}^* a_{k_2+q}^* a_{k_2} a_{k_1}. \tag{2.44}$$

The corresponding interacting vertex is illustrated by the Figure 2.1.

Notice that $\rho_c^P(\theta = 0) = 0$, see (2.32), then by virtue of (2.37) and (2.39) one has

$$\lim_{\theta\to 0^+} \rho_0^P(\theta) = \rho - \lim_{\theta\to 0^+} \rho_c^P(\theta) = \rho, \tag{2.45}$$

i.e. at *zero* temperature *all* particles are in condensate. This might be easily expected, since for the n-particle *ground state* ψ_0 of the PBG one gets:

$$(\psi_0, T_\Lambda \psi_0)_{\mathcal{F}_B} = \min_{\psi \in \mathcal{F}_B} (\psi, T_\Lambda \psi)_{\mathcal{F}_B} = 0, \tag{2.46}$$

i.e. by (2.12) we obtain that the normalized ground state has the form

$$\psi_0 = \psi_{\{n,0,0,\dots\}} = \frac{(a^*_{k=0})^n}{\sqrt{n!}}\Omega. \tag{2.47}$$

This means that *all* of n bosons are accumulated in the one-particle ground state which corresponds to the mode $k = 0$. Notice that the interaction energy in this state is equal to

$$(\psi_0, U_\Lambda \psi_0)_{\mathcal{F}_B} \equiv \left(\psi_0, \frac{1}{2V} \sum_{k_1,k_2,q \in \Lambda^*} v(q) a^*_{k_1-q} a^*_{k_2+q} a_{k_2} a_{k_1} \psi_0\right) \neq 0. \tag{2.48}$$

Let (in spite of the evident difficulty caused by (2.46) and (2.48)) one *suppose* that even for $v(q) \neq 0$ the *overwhelming majority of particles rests in the state* ψ_0, which is now *not* a ground state of (2.44). Then, according to Bogoliubov, the system (2.44) behaves as *two interacting subsystems*: the first one corresponds to the *condensate*, whereas the second one to a small amount of particles *outside the condensate*. Since

$$(a_k \psi_0, a_k \psi_0)_{\mathcal{F}_B} = n \delta_{k,0}, \tag{2.49}$$

Bogoliubov proposed (the *first ansatz*) to *truncate* the Hamiltonian (2.44) in such a way that one disregards in (2.44) the terms of the third and higher order in operators $a_{k\neq 0}$, $a^*_{k\neq 0}$. This truncation gives the Bogoliubov Hamiltonian for the WIBG, see eq. (3.81) in [18]:

$$H^B_\Lambda = T_\Lambda + U^B_\Lambda = T_\Lambda + U^{BD}_\Lambda + U^{BN}_\Lambda, \tag{2.50}$$

where

$$T_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a^*_k a_k, \tag{2.51}$$

$$\begin{aligned} U^{BD}_\Lambda &= \frac{v(0)}{2V} a_0^{*2} a_0^2 + \frac{v(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^* \setminus \{0\}} a^*_k a_k \\ &\quad + \frac{1}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} v(k)\, a_0^* a_0 \left(a^*_k a_k + a^*_{-k} a_{-k}\right), \end{aligned} \tag{2.52}$$

$$U^{BN}_\Lambda = \frac{1}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} v(k) \left(a^*_k a^*_{-k} a_0^2 + a_0^{*2} a_k a_{-k}\right). \tag{2.53}$$

Vertices corresponding to the interaction in the WIBG are illustrated by Figures 2.2 and 2.3. The vertex with three dotted lines is excluded by the momentum conservation.

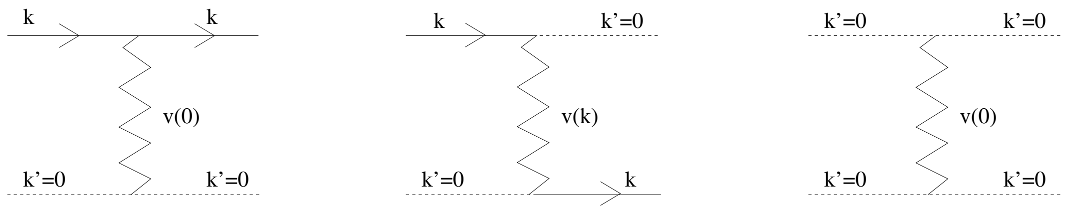

Figure 2.2: Diagonal-interaction vertices corresponding to U_Λ^{BD}.

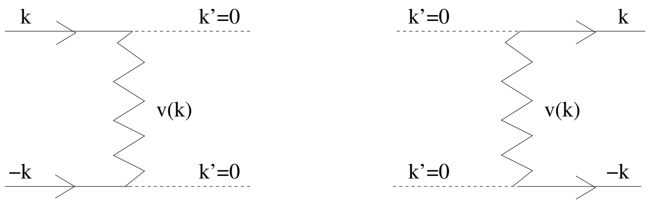

Figure 2.3: Nondiagonal-interaction vertices corresponding to U_Λ^{BN}.

Notice that even after this truncation the problem of the Hamiltonian (2.50) rests rather difficult. Still the perturbation U_Λ^B of the state ψ_0 is singular: one obviously has $\left(\psi_0, U_\Lambda^B \psi_0\right) \neq 0$ and $(\psi_0, T_\Lambda \psi_0) = 0$. Therefore, one poses the same question as above: why we have to believe that the *conventional* BE condensation in the mode $k = 0$ *persists* in the presence of the *truncated* interaction U_Λ^B? Bogoliubov does not answer this question. The response was found only recently [12, 15] and it will be our preoccupation in the next Sections 4, 5. Instead, he proposed a neat ingenious trick (the *second ansatz*) to treat the Hamiltonian H_Λ^B, *presupposing* that the conventional BE condensation in the mode $k = 0$ *does* exist!

His motivation comes from the observation that operators a_0, a_0^* enter H_Λ^B only in the form $a_0/\sqrt{V}$, $a_0^*/\sqrt{V}$. These operators for large V:
(i) almost commute,

$$\left[a_0/\sqrt{V}, a_0^*/\sqrt{V}\right] = 1/V, \tag{2.54}$$

(ii) are nontrivial in the state ψ_0,

$$\left(\psi_0, \frac{a_0^* a_0}{V}\psi_0\right)_{\mathcal{F}_B} = \frac{n}{V}, \tag{2.55}$$

if $n \sim V$, as it is in the case of BE condensation in the mode $k = 0$. Therefore, they could be *replaced* in a macroscopic system by *complex numbers*:

$$a_0/\sqrt{V} \to c,\ \ a_0^*/\sqrt{V} \to \bar{c}, \tag{2.56}$$

to be determined *self-consistently*. This *second ansatz* is called the *Bogoliubov approximation* (see Appendix B). Attempts of mathematical justification of this procedure and its intimate connection with representations of the CCR was a subject of several papers, see e.g. [37,110,111].

On the other hand, Ginibre [92] discussed this problem directly for the full Hamiltonian (2.44), see Appendix B.

To accept definitely the Bogoliubov Hamiltonian for the WIBG we have to gloss over a difficulty with *nonconservation* of particles number. Indeed, since Hamiltonians T_Λ, H_Λ or H_Λ^B *commute* with the total particle-number N_Λ, they conserve the number of particles. This means for example that one can consider a restriction of these Hamiltonians on n-particle sector $\mathcal{H}_B^{(n)}$ (*canonical ensemble* with n particles). On the other hand, invariance of these operators with respect to the *gauge transformations* $U_\varphi = e^{i\varphi N_\Lambda}$, $\varphi \in \mathbb{R}^1$, i.e.

$$U_\varphi T_\Lambda U_\varphi^* = T_\Lambda,\ U_\varphi H_\Lambda U_\varphi^* = H_\Lambda,\ U_\varphi H_\Lambda^B U_\varphi^* = H_\Lambda^B, \tag{2.57}$$

and (here $a_k^\# \equiv \{a_k \text{ or } a_k^*\}$)

$$U_\varphi a_k^\# U_\varphi^* = e^{\pm i\varphi} a_k^\# \tag{2.58}$$

imply (in contrast to (2.56) for $c \neq 0$) that

$$\left\langle a_k^\# \right\rangle_{T_\Lambda} (\beta,\mu) = \left\langle a_k^\# \right\rangle_{H_\Lambda} (\beta,\mu) = \left\langle a_k^\# \right\rangle_{H_\Lambda^B} (\beta,\mu) = 0,\ k \in \Lambda^*. \tag{2.59}$$

Let $H_\Lambda^B\left(c^\#\right)$ be the Hamiltonian of the WIBG *after* the Bogoliubov approximation (2.56):

$$\begin{aligned}
H_\Lambda^B\left(c^\#\right) &= T_\Lambda\left(c^\#\right) + U_\Lambda^B\left(c^\#\right) = T_\Lambda\left(c^\#\right) + U_\Lambda^{BD}\left(c^\#\right) + U_\Lambda^{BN}\left(c^\#\right),\\
T_\Lambda\left(c^\#\right) &\equiv \sum_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k a_k^* a_k;\\
U_\Lambda^{BD}\left(c^\#\right) &\equiv \frac{1}{2} v(0)\,|c|^4 V + v(0)\,|c|^2 \sum_{k\in\Lambda^*\setminus\{0\}} a_k^* a_k\\
&\quad + \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} v(k)\,|c|^2 \left[a_k^* a_k + a_{-k}^* a_{-k}\right];\\
U_\Lambda^{BN}\left(c^\#\right) &\equiv \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} v(k)\left[c^2 a_k^* a_{-k}^* + \overline{c}^2 a_k a_{-k}\right];
\end{aligned} \tag{2.60}$$

with $c^\# \equiv \{c \text{ or } \overline{c}\}$. Then the interaction $U_\Lambda^B\left(c^\#\right)$ does not commute with the particle-number operator N_Λ, i.e. it will no longer conserve particles because of creation and annihilation of pairs due to nondiagonal terms $c^2 a_k^* a_{-k}^*$ and $\overline{c}^2 a_k a_{-k}$ in $U_\Lambda^{BN}\left(c^\#\right)$ (2.60). Therefore, one cannot treat the *approximating* Hamiltonian (2.60) in canonical ensemble.

There are two ways around this problem. The first is the method proposed by Bogoliubov in the same year as his first paper, see [3] and [18, 21, 90, 112, 113]. Since $N_0 = a_0^* a_0$ is a non-negative self-adjoint operator, the operator $(N_0 + I)^{-1/2}$ is correctly defined and bounded. Let

$$\zeta_k = a_0^* \left(N_0 + \mathrm{I}\right)^{-1/2} a_k,\ \zeta_k^* = a_k^* \left(N_0 + \mathrm{I}\right)^{-1/2} a_0,\ k \in \Lambda^*. \tag{2.61}$$

Then for $k \neq 0$ these operators *satisfy* the CCR, and the Hamiltonian H_Λ^B (2.50) can be rewritten in the form:

$$H_\Lambda^B \ = \ \sum_{k\in\Lambda^*} \varepsilon_k \zeta_k^* \zeta_k + \frac{N_0}{V} v(0) \sum_{k\in\Lambda^*\setminus\{0\}} \zeta_k^* \zeta_k + \frac{v(0)}{2V} N_0 \left(N_0 - \mathrm{I}\right)$$

$$
\begin{aligned}
&+\frac{N_0}{2V}\sum_{k\in\Lambda^*\backslash\{0\}} v(k)\left(\zeta_k^*\zeta_k+\zeta_{-k}^*\zeta_{-k}\right)\\
&+\frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} v(k)\left(\zeta_k^*\zeta_{-k}^*\frac{N_0^{1/2}\left(N_0-\mathrm{I}\right)^{1/2}}{V}+\frac{N_0^{1/2}\left(N_0-\mathrm{I}\right)^{1/2}}{V}\zeta_k\zeta_{-k}\right). \qquad (2.62)
\end{aligned}
$$

Here we used that $\left(N_0+\mathrm{I}\right)^{-1/2}a_0 = a_0 N_0^{-1/2}$. Now the *Bogoliubov approximation* for the Hamiltonian (2.62) corresponds to

$$\frac{N_0}{V}\to|c|^2,\ \frac{N_0^{1/2}\left(N_0-\mathrm{I}\right)^{1/2}}{V}\to|c|^2. \qquad (2.63)$$

This gives the *same* result as in (2.60) after the canonical *gauge transformation* to boson operators

$$\widetilde{a}_k = a_k e^{-i\arg c},\ k\in\Lambda^*\backslash\{0\}. \qquad (2.64)$$

Notice that, since by (2.58) the operators (2.61) conserve the number of particles, the Hamiltonian (2.62) inherits this property even after the Bogoliubov approximation (2.63). Hence, now we may treat this approximating Hamiltonian in canonical ensemble with some parameter c. To exclude this uncertainty Bogoliubov proposed to eliminate the operator N_0 from (2.62) at the cost of further approximations, see [3,18] and discussion in [21]. Since

$$N'_\Lambda\equiv\sum_{k\in\Lambda^*\backslash\{0\}} a_k^*a_k=\sum_{k\in\Lambda^*\backslash\{0\}}\zeta_k^*\zeta_k,\ N_0=N_\Lambda-N'_\Lambda, \qquad (2.65)$$

he used approximation

$$\frac{N_0^2}{2V}+\frac{N_0}{V}\sum_{k\in\Lambda^*\backslash\{0\}}\zeta_k^*\zeta_k\simeq\frac{N_\Lambda^2}{2V} \qquad (2.66)$$

in the sum of the second and the third terms of (2.62) and

$$\frac{N_0}{V}\simeq\frac{N_\Lambda}{V}\to\rho \qquad (2.67)$$

in the third, the fourth and the fifth terms to arrive to an *approximating Hamiltonian* for the *canonical ensemble* with density ρ:

$$
\begin{aligned}
\widetilde{H}_\Lambda^B(\rho) &= \sum_{k\in\Lambda^*\backslash\{0\}}\left(\varepsilon_k+\rho v(k)\right)\zeta_k^*\zeta_k+\frac{\rho}{2}\sum_{k\in\Lambda^*\backslash\{0\}} v(k)\left(\zeta_k^*\zeta_{-k}^*+\zeta_k\zeta_{-k}\right)\\
&\quad +v(0)\frac{N_\Lambda^2}{2V}-\frac{1}{2}\rho v(0). \qquad (2.68)
\end{aligned}
$$

Since the last two terms are constants in the canonical ensemble we rest with a bilinear form (2.68) in Bose-operators $\{\zeta_k\}_{k\in\Lambda^*\backslash\{0\}}$.

The second method was essentially advocated by Beliaev [69, 70], Hugenholtz and Pines [71,81,114,115], and Tserkovnikov [116], Tolmachev [84]. Since the Bogoliubov approximation

(2.56) leads to nonconservation of particles, they proposed simply to use a *grand-canonical ensemble* from the very beginning. This means that instead of H_Λ^B one has to consider

$$H_\Lambda^B(\mu) = H_\Lambda^B - \mu N_\Lambda. \tag{2.69}$$

Then the Bogoliubov's *second ansatz* gives the approximating Hamiltonian in the form (cf. (2.60))

$$\begin{aligned} H_\Lambda^B\left(c^{\#},\mu\right) &= \sum_{k\in\Lambda^*\setminus\{0\}} \left[\varepsilon_k - \mu + v(0)|c|^2\right] a_k^* a_k \\ &+\frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} v(k)|c|^2\left[a_k^* a_k + a_{-k}^* a_{-k}\right] \\ &+\frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} v(k)\left[c^2 a_k^* a_{-k}^* + \overline{c}^2 a_k a_{-k}\right] \\ &-\mu|c|^2 V + \frac{1}{2}v(0)|c|^4 V. \end{aligned} \tag{2.70}$$

where $c^{\#} = \{c \text{ or } \overline{c}\}$. In fact the turn to the grand-canonical ensemble would not be very astonishing, since properties of the PBG are also more transparent in this ensemble. After the gauge transformation (2.64) it gets clear that, as above, the Hamiltonian (2.70) depends only on $|c|^2$. Then the procedure for determining $|c|^2$, proposed in [69, 71, 81, 84] gives the *variation principle*:

$$\frac{\partial}{\partial|c|^2}\left(\psi_0, H_\Lambda^B\left(c^{\#},\mu\right)\psi_0\right)_{\mathcal{F}_B} = \frac{\partial}{\partial|c|^2}\left(\frac{1}{2}v(0)|c|^4 V - \mu|c|^2 V\right) = 0, \tag{2.71}$$

where ψ_0 is the ground state (2.47) of the PBG. Hence one obtains:

$$\mu = v(0)|c|^2. \tag{2.72}$$

For a given total particle density ρ the chemical potential should be excluded from (2.72) by the subsidiary condition, which defines $|c|^2$ as a function of ρ. If one does this in the *first approximation* [69, 71, 81], then one gets $|c|^2 = \rho$, which returns us to (2.68), see discussion in Section 2.4.

Needless to say that calculations done with approximating Hamiltonian by this last method give the same results as those in canonical ensemble [3, 18, 21, 112, 113]. Below we shall consider the WIBG in the *grand-canonical ensemble* (β, μ).

2.3 Spectrum of elementary excitations and superfluidity

Since the ultimate aim of the Bogoliubov theory is a microscopic explanation of the superfluidity à la Landau, one has to calculate the spectrum of excitations either of the Hamiltonian $\widetilde{H}_\Lambda^B(\rho)$

(2.68) or the Hamiltonian $H_{\Lambda}^{B}\left(c^{\#},\mu\right)$ (2.70) with condition (2.72). The both give the same result, so we stuck with the grand-canonical ensemble.

Notice that Hamiltonian $H_{\Lambda}^{B}\left(c^{\#},\mu\right)$ is a bilinear form in boson operators $\left\{a_k^{\#}\right\}_{k\in\Lambda^*\backslash\{0\}}$. So, one can diagonalize it by the Bogoliubov canonical *u-v transformation* to a new set of *boson* operators $\left\{b_k^{\#}\right\}_{k\in\Lambda^*\backslash\{0\}}$. First, we perform the canonical gauge transformation (2.64) to $\left\{\widetilde{a}_k^{\#}\right\}_{k\in\Lambda^*\backslash\{0\}}$, then

$$\begin{aligned}\widetilde{a}_k &= \mathbf{u}_k b_k - \mathbf{v}_k b_{-k}^*,\\ \widetilde{a}_k^* &= \mathbf{u}_k b_k^* - \mathbf{v}_k b_{-k}.\end{aligned}\tag{2.73}$$

Here the real coefficients $\{\mathbf{u}_k = \mathbf{u}_{-k}\}_{k\in\Lambda^*\backslash\{0\}}$ and $\{\mathbf{v}_k = \mathbf{v}_{-k}\}_{k\in\Lambda^*\backslash\{0\}}$ satisfy the relations:

$$\mathbf{u}_k^2 - \mathbf{v}_k^2 = 1\tag{2.74}$$

to ensure the CCR for $\left\{b_k^{\#}\right\}_{k\in\Lambda^*\backslash\{0\}}$, and

$$2\mathbf{u}_k\mathbf{v}_k = \frac{h_k}{E_k^B},\ \mathbf{u}_k^2 + \mathbf{v}_k^2 = \frac{\varepsilon_k}{E_k^B},\tag{2.75}$$

where

$$\begin{aligned}&E_k^B = \sqrt{f_k^2 - h_k^2},\\ &f_k = \varepsilon_k - \mu + |c|^2\left[v\left(0\right) + v\left(k\right)\right],\\ &h_k = |c|^2\, v\left(k\right),\end{aligned}\tag{2.76}$$

to ensure diagonalization of the form (2.70). The equations (2.74) and (2.75) imply that

$$\mathbf{u}_k^2 = \frac{1}{2}\left(\frac{f_k}{E_k^B} + 1\right),\ \mathbf{v}_k^2 = \frac{1}{2}\left(\frac{f_k}{E_k^B} - 1\right).\tag{2.77}$$

Notice that the Bogoliubov transformation (2.73) is *trivial* if $v\left(k\right) = 0$, and it *makes sense* only if $f_k \geq h_k$. This constrains $|c|^2$ and μ to satisfy the inequality:

$$\mu \leq v\left(0\right)|c|^2 + \min_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k.\tag{2.78}$$

In terms of the new boson operators $\left\{b_k^{\#}\right\}_{k\in\Lambda^*\backslash\{0\}}$ the Hamiltonian (2.70) becomes:

$$\mathcal{H}_{\Lambda}^{B}\left(c^{\#},\mu\right) = \sum_{k\in\Lambda^*\backslash\{0\}} E_k^B b_k^* b_k + \frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}}\left(E_k^B - f_k\right) - \mu|c|^2 + \frac{v(0)}{2}|c|^4 V,\tag{2.79}$$

where

$$E_k^B = \sqrt{\left(\varepsilon_k - \mu + |c|^2 v(0)\right)\left(\varepsilon_k - \mu + |c|^2 v(0) + 2|c|^2 v(k)\right)}\tag{2.80}$$

is the spectrum of *elementary excitations* of (2.70). Now, if we take into account the first-order calculation (2.72) for the chemical potential (cf. condition (2.78)), we obtain the famous *Bogoliubov spectrum*:

$$E_k^B = \sqrt{\varepsilon_k\left(\varepsilon_k + 2v(k)|c|^2\right)}.\tag{2.81}$$

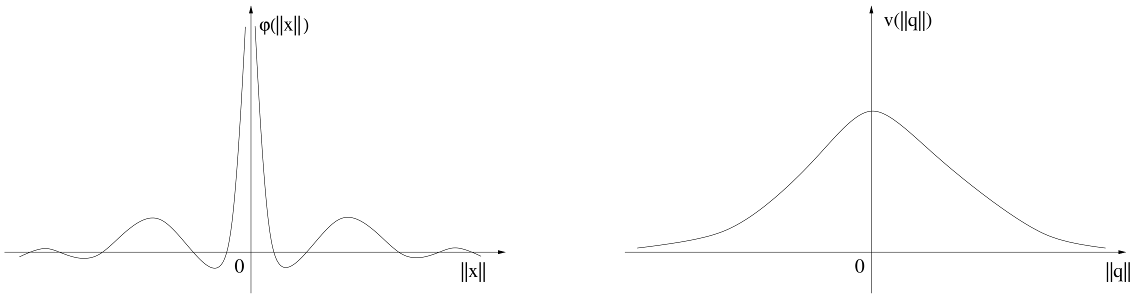

Figure 2.4: Interaction potential $\varphi(\|x\|)$ and its Fourier transform $v(\|q\|)$.

Recall that by the *hypotheses* (A) and (B)) (see Section 2.2) the two-body potential $\varphi(x)$ is spherically-symmetric, i.e. $\varphi(x) = \varphi(\|x\|)$. Then the Fourier transform

$$v(q) = 4\pi \int_0^\infty dr r \varphi(r) \frac{\sin(\|q\| r)}{\|q\|} \equiv v(\|q\|) \tag{2.82}$$

is also spherically-symmetric, (see Figure 2.4).

Suppose in addition the absolute integrability of $x^2\varphi(x) \in L^1(\mathbb{R}^3)$. Then one gets from (2.82) the expansion:

$$v(q) = v(0) + \frac{1}{2}\|q\|^2 v''(0) + o\left(\|q\|^2\right), \tag{2.83}$$

where $v''(0) \leq 0$, and $|v(q)| \leq const.\, \|q\|^{-1}$.

Now we return to the Bogoliubov spectrum (2.81). First, let $|c|^2 = 0$ (*no condensation*). Then the Bogoliubov theory is *trivial*:

$$H_\Lambda^B\left(c^\# = 0, \mu\right) = T'_\Lambda - \mu N'_\Lambda, \tag{2.84}$$

i.e. it coincides with a PBG with *excluded* mode $k = 0$. This kind of the PBG has in the thermodynamic limit the same properties as the standard PBG (see [117] and Section 4). In this case the Bogoliubov spectrum

$$E_k^B = \varepsilon_k \tag{2.85}$$

is identical to the spectrum of the PBG. Now let $|c|^2 \neq 0$. Then (see Figure 2.5) by (2.83) one gets:

$$E_k^B = \left\{ \begin{array}{ll} \left(\dfrac{\hbar^2}{m} v(0)\,|c|^2\right)^{1/2} \|k\| = \hbar \mathrm{w}\,\|k\|, & \text{for } \|k\| \to 0^+, \\ \varepsilon_k = \hbar^2 k^2/2m & , \text{ for } \|k\| \to +\infty. \end{array} \right\}. \tag{2.86}$$

So, the Bogoliubov spectrum is *gapless* and *phonon-like* for *small* $\|k\|$, whereas for *large* wave-vectors it behaves like *single-particle excitations* ε_k in the PBG. Since $v(q)$ attains its maximum at $q = 0$, one can choose the potential $\varphi(x)$ in such a way that

$$\left(\varepsilon_k\left(\varepsilon_k + 2|c|^2 v(\|k\|)\right)\right)' = 0 \text{ at } \|k\| = \|k_{\mathrm{rot}}\| \neq 0. \tag{2.87}$$

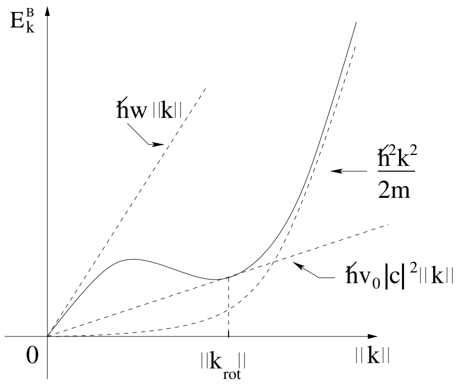

Figure 2.5: The Bogoliubov spectrum E_k^B.

In this case the Bogoliubov spectrum has local ("roton") minimum at $\|k_{\mathrm{rot}}\|$, see Figure 2.5.

On the other hand, by (2.81) one gets:

$$E_k^B \geq \|k\| \left(\frac{\hbar^2}{2m}\right)^{1/2} \left\{\min_k \left(\varepsilon_k + 2v\left(k\right)|c|^2\right)\right\}^{1/2} \equiv \hbar \mathrm{v}_0 \left(|c|^2\right) \|k\| \, , \tag{2.88}$$

see Figure 2.5.

These properties of the Bogoliubov spectrum give a *microscopic* interpretation of the Landau theory of superfluidity of quantum liquids [1,3,18,20,90,118,119]. Recall that the latter reposed on the following principle:

- *quantum* liquid rests *fluid* even for zero temperature;
- at low temperatures *collective* behaviour *dominates* the *individual* movement of atoms, i.e. apart *translations* (flow) the state of this liquid is entirely described by the *spectrum* of collective (elementary) *excitations*;
- by virtue of thermodynamic data [20, 120] (e.g. specific heat capacity) this spectrum for ^{4}He should be a *phonon-like* for the long-wave length collective excitations and should be *above* a straight line with *positive slope* with ("roton") *minimum* in the vicinity of $\|k_{\mathrm{rot}}\| \simeq 2\ \mathring{\mathrm{A}}^{-1}$.

The Bogoliubov theory shows that within some approximations the WIBG behaves as this quantum liquid with collective excitations described by the boson operators $\left\{b_k^{\#}\right\}_{k\in\Lambda^*\setminus\{0\}}$ and by the gapless Bogoliubov spectrum (2.81) with properties (2.86)-(2.88).

Suppose that a mass, M, of this Bogoliubov quantum liquid with velocity v in a channel (see Figure 2.6).

Let in the rest system of the liquid it be in its ground state $\mathcal{E}_0$. Then in the laboratory system this fluid will have energy $\mathcal{E}_{\mathrm{v}} = \mathcal{E}_0 + M\mathrm{v}^2/2$. Since any interaction of the quantum liquid with the "outside world" manifests as elementary excitations, the friction have to produce

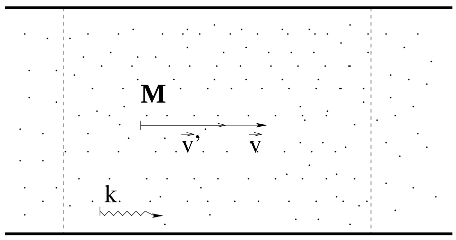

Figure 2.6: Friction due to excitation k by the walls of the channel.

them, decreasing the initial energy $\mathcal{E}_{\mathrm{v}}$. For the Bogoliubov quantum liquid (2.79) this means appearance of excitations by action of creation Bose-operators $\left\{b_k^{\#}\right\}_{k\in\Lambda^*\backslash\{0\}}$ on vacuum $\widetilde{\Omega}$ of the corresponding Fock space $\widetilde{\mathcal{F}}_B$. To make the effect of the friction more sound, suppose that, created by this process, excited state $b_k^*\widetilde{\Omega}$ (with one elementary excitation) corresponds to the wave-vector k parallel to the velocity v. Then the new energy $\mathcal{E}_{\mathrm{v}'}$ and the new momentum $\mathcal{P}' = M\mathrm{v}'$ of the flow are related with initial conditions $\mathcal{E}_{\mathrm{v}}$ and $\mathcal{P} = M\mathrm{v}$ by

$$\begin{aligned} &\mathcal{E}_{\mathrm{v}} = \mathcal{E}_{\mathrm{v}'} + E_k^B, \\ &M\mathrm{v} = M\mathrm{v}' + \hbar k. \end{aligned} \tag{2.89}$$

¿From (2.89) one readily gets:

$$\frac{1}{2}\hbar k\,(\mathrm{v} + \mathrm{v}') = E_k^B. \tag{2.90}$$

If excitations $b_k^*\widetilde{\Omega}$ results from *friction*, then $\|\mathrm{v}'\|$ must be *less* than the initial velocity $\|\mathrm{v}\|$. By virtue of (2.90) this means that the initial velocity *must* be bounded from below:

$$\|\mathrm{v}\| > \frac{E_k^B}{\hbar\,\|k\|} > \|\mathrm{v}'\|\,. \tag{2.91}$$

For the Bogoliubov quantum liquid one has (cf. (2.89)):

$$\inf_k \frac{E_k^B}{\hbar\,\|k\|} = \mathrm{v}_0\left(|c|^2\right) = \left\{ \begin{array}{l} > 0, \text{ if } |c|^2 > 0, \\ 0 \quad, \text{ if } |c|^2 = 0. \end{array} \right\} \tag{2.92}$$

Therefore, if $\mathrm{v}_0\left(|c|^2\right) > 0$ and if the initial velocity of the flow

$$\|\mathrm{v}\| < \mathrm{v}_0\left(|c|^2\right),$$

then the fluid will not be *able* to lose any momentum to the walls of the channel! Hence, it will display superfluid behaviour. This is the famous Landau's *criterion of superfluidity* of 1941 [4, 5] in application to the Bogoliubov theory.

Notice that for the PBG (*independent* of the presence or absence of the BE condensate) one has:

$$\inf_k \frac{\varepsilon_k}{\hbar\,\|k\|} = 0. \tag{2.93}$$

Consequently, by (2.91) the flow of the PBG in a channel is always able to lose energy via friction for any $\mathrm{v} \neq 0$. As it was claimed by Bogoliubov [1–3], the *coherency* due to the presence of the BE condensation is *not* enough to make the PBG superfluid. The spectrum of elementary excitations in this case is not *collective*: it corresponds to the *individual* movement of particles. The *same* we have for the Bogoliubov quantum fluid when the density of the BE condensate $|c|^2 = 0$, see (2.85).

The crucial rôle of *interaction* is that it transforms the *coherency* due to the BE condensation into *collective* behaviour of the system. In particular, it modifies spectrum of individual excitations ε_k into the Bogoliubov spectrum E_k^B. They are drastically different for $\|k\| \to 0$, i.e. where interaction dominates the kinetic energy, cf. discussion in Section 1.

Since in the first approximation one has (2.72) and $|c|^2 = \rho$ (see end of Section 2.2), the pressure equals:

$$p = -\partial_V \left(\psi_0, H_\Lambda^B \left(c^\#, \mu = v(0)|c|^2\right) \psi_0\right)_{\mathcal{F}_B} = \frac{1}{2}\rho^2 v(0), \tag{2.94}$$

and hence one has the velocity of sound defined by

$$s^2 = \partial_\rho p = \rho \frac{v(0)}{m}. \tag{2.95}$$

¿From (2.88), (2.92) and (2.95) one would expect that the critical velocity for superfluidity should be of *order* of magnitude of the *sound velocity*, which is supported independently by the Feynman theory of ^{4}He [118, 119].

2.4 Beyond the Bogoliubov theory

Since the Bogoliubov theory is founded on *two ansätze*:

- *truncation* of the full Hamiltonian,
- Bogoliubov's *approximation*, i.e. *substitution* of the zero-mode creation/annihilation operators by a *judiciously* chosen c-numbers,

one could imagine several directions to improve the Bogoliubov theory.

If one does not aspire to deal with the full Hamiltonian (2.44) but on the other hand does not like the idea of the Bogoliubov truncation either, then one could try the Bogoliubov *approximation* directly in the *full* Hamiltonian (2.44). With the substitution (2.56) the interaction U_Λ in (2.44) changes to $U_\Lambda\left(c^\#\right)$ with number of terms, depending how many *nonzero* modes are in the vertex (see Figure 2.7):

$$U_\Lambda\left(c^\#\right) \equiv U_{22} + U_{21} + U_{12} + U_{20} + U_{02} + U_{11} + U_{00}, \tag{2.96}$$

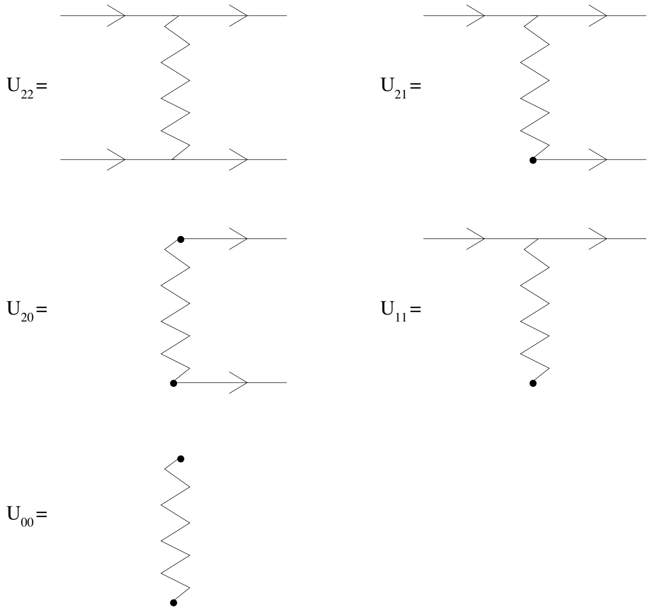

Figure 2.7: Full interaction with the classical reservoir of condensed particles.

where

$$\begin{aligned}
U_{22} &= \frac{1}{2V}\sum_{k_1,k_2,q\in\Lambda^*\backslash\{0\}} v(q)a^*_{k_1-q}a^*_{k_2+q}a_{k_2}a_{k_1},\\
U_{21} &= \frac{1}{2\sqrt{V}}\sum_{k_2,q\in\Lambda^*\backslash\{0\}} v(q)a^*_q a^*_{k_2-q}a_{k_2}c,\ U_{12}=U^*_{21},\\
U_{20} &= \frac{1}{2}\sum_{q\in\Lambda^*\backslash\{0\}} v(q)a^*_q a^*_{-q}c^2,\ U_{02}=U^*_{20},\\
U_{11} &= \frac{1}{2}\sum_{q\in\Lambda^*\backslash\{0\}} \left(v\left(0\right)+v(q)\right)a^*_q a_q\left|c\right|^2,\\
U_{00} &= \frac{1}{2}V\left|c\right|^4 v\left(0\right).
\end{aligned} \tag{2.97}$$

Notice that the Bogoliubov's part of interaction (2.50)-(2.53) corresponds in (2.97) to

$$\begin{aligned}
U^{BD}_{\Lambda}\left(c^{\#}\right) &= U_{11}+U_{00},\\
U^{BN}_{\Lambda}\left(c^{\#}\right) &= U_{02}+U_{20}.
\end{aligned} \tag{2.98}$$

The interactions $U_{21}+U_{12}$ and $U_{02}+U_{20}$, and hence $U_\Lambda\left(c^{\#}\right)$, do *not* conserve the number of particles, i.e. they are *not* gauge invariant:

$$e^{i\varphi N_\Lambda}U_\Lambda\left(c^{\#}\right)e^{-i\varphi N_\Lambda}\neq U_\Lambda\left(c^{\#}\right).$$

Therefore, following [26, 69–71, 76] we consider the Bogoliubov approximation in the grand-canonical ensemble, that gives (cf. (2.96), (2.97)):

$$H_{\Lambda}\left(c^{\#},\mu\right)=\sum_{k\in\Lambda^{*}\backslash\{0\}}\left(\varepsilon_{k}-\mu\right)N_{k}-V\mu\left|c\right|^{2}+U_{\Lambda}\left(c^{\#}\right). \tag{2.99}$$

After canonical transformation to boson operators $\widetilde{a}_k = e^{-i\arg c}a_k$, $\widetilde{a}_k^* = e^{i\arg c}a_k^*$, $k \in \Lambda^*\backslash\{0\}$, one finds that the Hamiltonian (2.99) depends only on $|c|$.

To avoid a discrepancy between prescription (2.56) and

$$\left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_\Lambda}(\beta,\mu)=0,$$

see (2.57)-(2.59), which persists even in the thermodynamic limit

$$\lim_{\Lambda}\left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_\Lambda}(\beta,\mu)=0, \tag{2.100}$$

Bogoliubov proposed ([26] Part I, [76]) to include in the Hamiltonian H_Λ a term *breaking* the gauge invariance in the *mode* $k=0$:

$$H_{\Lambda}\left(\nu^{\#}\right)=H_{\Lambda}-\sqrt{V}\left(\overline{\nu}a_0+\nu a_0^*\right),\ \nu=|\nu|\,e^{i\arg\nu}\in\mathbb{C}. \tag{2.101}$$

Then the Bogoliubov *quasi-average* (i.e. expectation in pure Gibbs phase defined by the gauge $\varphi=\arg\nu$ ([26] Part I, [76])) should to fit with (2.56):

$$\lim_{\{\nu\to0:\arg\nu=\varphi\}}\lim_{\Lambda}\left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_\Lambda\left(\nu^{\#}\right)}(\beta,\mu)=|c|\,e^{\pm i\varphi}, \tag{2.102}$$

in the condensed phase. Application of the Bogoliubov approximation to (2.101) change (2.99) on unimportant constant term.

The system described by the Hamiltonian (2.99) is a generalization of (2.70), known as a *model system with a (uniform) condensate*, see [26, 69, 70, 76].

It was Beliaev [69, 70], who considered this generalization of the Bogoliubov theory for the first time by systematic application of the methods of quantum field theory. The new feature associated with the existence of a BE condensate reservoir is that the Dyson equation should be replaced by the *pair* of *Dyson-Beliaev equations* for *normal* and *anomalous* propagator (Green's functions). The latter represents two particles going into or out of the condensate (cf. Figure 2.7). The model with a condensate contains two parameters μ and $|c|^2$, which are determined by the following procedure, cf. the case of $H_\Lambda^B\left(c^{\#},\mu\right)$. For given μ and $|c|^2$ one first calculates the ground state Φ_0 and eigenvalue $E_0\left(|c|^2,\mu\right)$ of (2.99) and the corresponding mean-value of particles outside the condensate:

$$\langle N_\Lambda'\rangle_{\Phi_0}\left(|c|^2,\mu\right)=\left(\Phi_0,N_\Lambda'\Phi_0\right)_{\mathcal{F}_B}. \tag{2.103}$$

Then one determine $|c|^2$ by the condition:

$$\partial_{|c|^2}\left(E_0\left(|c|^2,\mu\right)/V\right)=0, \tag{2.104}$$

and the chemical potential μ (for given total density ρ and $\theta = 0$) from the subsidiary equation:

$$\rho = |c|^2 + \left\langle \frac{N'_\Lambda}{V} \right\rangle_{\Phi_0} \left(|c|^2, \mu\right), \tag{2.105}$$

see [26, 69–76, 81, 84].

So the real problem now is to calculate, in some approximation, the ground state eigenvalue and the particle density. Beliaev verified [69] that in the *first-order* in interaction the only contribution in the ground-state energy comes from U_{00}, see (2.97), i.e. the trial ground-state wave-function $\Phi_0 = \psi_0$, see (2.47). Therefore,

$$E_0\left(|c|^2, \mu\right) = \frac{1}{2} V v(0) |c|^4 - \mu V |c|^2, \tag{2.106}$$

which gives by (2.104)

$$\mu = v(0) |c|^2, \tag{2.107}$$

compare with (2.72). Moreover, the particle density outside the condensate (depletion fraction) in the lowest order of the perturbation theory is

$$\left\langle \frac{N'_\Lambda}{V} \right\rangle_{\psi_0} \left(|c|^2, \mu\right) = \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \mathbf{v}_k^2. \tag{2.108}$$

By (2.77) and (2.107) one gets

$$\left\langle \frac{N'_\Lambda}{V} \right\rangle_{\psi_0} \left(|c|^2, \mu = |c|^2 v(0)\right) \sim o\left[(\rho v(0))^{3/2}\right]. \tag{2.109}$$

Hence, in the first-order we have by (2.105):

$$\rho = |c|^2. \tag{2.110}$$

With (2.107) and (2.110) Beliaev found that in the first-order the self-energy parts in Dyson-Beliaev equations gives exactly the Bogoliubov spectrum (2.81) with phonon velocity *be equal to* s, see (2.95).

In [70] Beliaev showed that in the next approximation the spectrum rests *gapless* and *phonon-like* in the long-wavelength limit $k \to 0$, while the interactions U_{21}, U_{12} (2.97) make the elementary excitations *unstable* with *damping* which goes, in this limit, as $\sim k^5$. These calculations demonstrate explicitly that the singularity of the perturbation theory indicated by Bogoliubov as a general feature of *models with a condensate* [3, 18, 21], manifests in *infrared divergences* (for small energies) of Feynman-Beliaev diagrams. This leads to a *non-analytic* dependence of the ground-state energy $E_0\left(|c|^2, \mu\right)$ and other observables on the parameter $(\rho v(k))$, cf. (2.109).

As we see the *model with condensate* (2.99) is a kind of *self-consistent* theory, since the condensate is supposed as an *a priori* datum (classical external field in the mode $k = 0$) which has to be determined from conditions that are "missing" in the frame of this model. They can be found from Green-function equations ([69] eqs (6.11)-(6.13)), variational principle ([71] eq. (3.5)), minimizing $E_0\left(|c|^2, \mu\right)$, or a dynamical equation for the zero mode ([26] Part I, Sect. 2.7, eq. (7.11)), ([116] eq. (3)), equivalent to the variational principle. Following [26, 81] we would like to warn against the "mismatching" of approximations in different parts of this self-consistent scheme. For example, if one tentatively improves the approximation of μ (2.107) by a more precise expression, then we lose the cancellation in (2.80) necessary to close the *gap* and to recover the Bogoliubov spectrum (2.81)! However, Hugenholtz and Pines in 1959 [71], showed that in a consistent calculation the spectral gap *vanished* to *all order* in perturbation theory, which is in fact an expansion in the potential $v(k)$ *and* in the density ρ, cf. (2.107), (2.109). In 1960 Bogoliubov showed ([26] Part I, Section 2.8, and [76]) that this *Hugenholtz-Pines theorem* follows from his $1/q^2$ theorem, which he proved in a *nonperturbative* way.

In the above context, this means that to improve the expression (2.80) for the spectrum E_k it is not sufficient *only* to substitute a more accurate than (2.107) approximation for μ. In this higher order of accuracy, besides $U_{20} + U_{02} + U_{11} + U_{00}$ one has to take into account the contribution $U_{21} + U_{12}$, see (2.97), as it was done by Beliaev [70].

Notice that the *Hugenholtz-Pines theorem* says nothing about the asymptotics of the spectrum $E_k \to 0$ for $k \to 0$. It was Gavoret and Nozières [73, 74], who extended the above theorem showing that in *all order* of the perturbation theory this asymptotics is *linear* with the slope which coincides with the *sound* velocity, see also [26] and [84] for applications of the $1/q^2$ theorem in this case.

Notice that the Bogoliubov theory and the model with a condensate are formulated on the level of approximating Hamiltonian, i.e. for *zero-temperature* $\theta = 0$. If one would try naively to substitute the variational principles (2.71), (2.104) for $\theta = 0$ by the corresponding one for thermodynamic potential, then relations (2.72) and (2.107) will be disbalanced (the BE condensate *depletion*), and we obtain a gap in the spectrum. The first consistent theory of the *model with condensate* for $\theta \neq 0$ was developed by Popov in 1965, see [87, 88] and also Tserkovnikov [116]. For a nice detailed discussion of the Popov generalization of the Bogoliubov theory on $\theta \neq 0$ we address the reader to a recent review article [89].

Notice that in the present paper we review the *exact* solution of the WIBG. Therefore, we need not self-consistent scheme or ansatz depending on the temperature.

Another way to relax the *first* Bogoliubov ansatz is a less severe truncation. This truncation was proposed by Zubarev and Tserkovnikov in 1958 [77] inspired by their studies of the Bardeen-Cooper-Schriffer (BCS) model where pair excitations play a central rôle. Later this idea was developed in several papers, see e.g. [78–80, 83, 111, 121]. The corresponding *truncated* Hamiltonian is known as the *Pair Hamiltonian* which has the form:

$$H_\Lambda^{Pair} = T_\Lambda + U_\Lambda^{Pair D} + U_\Lambda^{Pair N} \equiv T_\Lambda + U_\Lambda^{Pair}. \tag{2.111}$$

Here $U_\Lambda^{Pair D}$ is the *diagonal* part of the interaction in H_Λ^{Pair}:

$$\begin{aligned} U_\Lambda^{Pair D} &= \frac{1}{2V} \sum_{k_1, k_2 \in \Lambda^*} v(0) a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1} \\ &+ \frac{1}{2V} \sum_{k_1 \in \Lambda^*, k_2 \in \Lambda^* \setminus \{\pm k_1\}} v(k_1 - k_2) a_{k_2}^* a_{k_2} a_{k_1}^* a_{k_1}, \end{aligned} \tag{2.112}$$

and U_Λ^{PairN} is the corresponding *nondiagonal* part:

$$U_\Lambda^{PairN} = \frac{1}{2V} \sum_{k\in\Lambda^*, k'\in\Lambda^*\setminus\{k\}} v(k-k') a_k^* a_{-k}^* a_{-k'} a_{k'} . \tag{2.113}$$

¿From (2.112) and (2.113) it is clear that the full interaction U_Λ in (2.44) is truncated in the following way: first, put $q = 0$ or $q = k_1 - k_2$ and then $k_1 = k'$, $k_2 = -k'$, $q = k - k'$. Another evident remark is that interaction U_Λ^{Pair} contains the Bogoliubov interacting terms U_Λ^{BD} (2.52) and U_Λ^{BN} (2.53). To obtain (2.52) one has to truncate the both of the sums (2.112) by constrains: $k_1 = 0$, $k_2 \neq 0$ or $k_1 \neq 0$, $k_2 = 0$. Similarity, one gets (2.53) truncating (2.113) by conditions $k = 0$, $k' \neq 0$ or $k \neq 0$, $k' = 0$.

There are no rigorous mathematical results on the model H_Λ^{Pair} (2.111). Starting on [77] and [78] numerous articles on this model use different kinds of approximations or variational schemes.

Inspired by the success in the rigorous study of the BCS model, the early papers [77–80,111] used either a BCS-type variational principle or the Approximating Hamiltonian Method (AHM) which indeed gives the *exact* result for the BCS-model, see [122], [123] (translated in [26] Part II), and [124–128].

The main idea of the AHM grew from the Bogoliubov's approximation ansatz of 1947 applied to combinations of operators A_k and B_k, $k \in \Lambda^*$, by

$$A_k = a_k^* a_k - \alpha_k, \;\; B_k = a_{-k} a_k - \gamma_k, \tag{2.114}$$

where $\alpha = \{\alpha_k \in \mathbb{R}^1\}_{k\in\Lambda^*}$ and $\gamma = \left\{\gamma_k^\# \in \mathbb{C}\right\}_{k\in\Lambda^*}$ are set of *trial* parameters. In term of the operators (2.114) the Hamiltonian $H_\Lambda^{Pair}(\mu) = H_\Lambda^{Pair} - \mu N_\Lambda$ takes the form

$$H_\Lambda^{Pair}(\mu) = H_\Lambda^{Pair}(\alpha, \gamma; \mu) + W_\Lambda^{Pair}(\alpha, \gamma), \tag{2.115}$$

where

$$H_\Lambda^{Pair}(\alpha, \gamma; \mu) = E_\Lambda^{Pair}(\alpha, \gamma; \mu) + \sum_{k\in\Lambda^*} \left\{ f_k^{Pair} a_k^* a_k + \frac{1}{2}\left(h_k^{Pair} a_k^* a_{-k}^* + \overline{h_k^{Pair}} a_{-k} a_k \right) \right\}, \tag{2.116}$$

and

$$\begin{aligned} W_\Lambda^{Pair}(\alpha, \gamma) \;\; = \;\; & \frac{v(0)}{2V} \sum_{k_1, k_2\in\Lambda^*} A_{k_1}^* A_{k_2} + \frac{1}{2V} \sum_{k_1\in\Lambda^*, k_2\in\Lambda^*\setminus\{\pm k_1\}} v(k_1 - k_2) A_{k_1}^* A_{k_2} \\ & + \frac{1}{2V} \sum_{k\in\Lambda^*, k'\in\Lambda^*\setminus\{k\}} v(k-k') B_k^* B_{k'} . \end{aligned} \tag{2.117}$$

Here $E_\Lambda^{Pair}(\alpha, \gamma; \mu)$ corresponds to zero- and linear terms in development of the operator $H_\Lambda^{Pair}(\mu)$ around the point α_k and $\gamma_k^\# = \{\gamma_k \text{ or } \overline{\gamma_k}\}$:

$$\begin{aligned} E_\Lambda^{Pair}(\alpha, \gamma; \mu) \;\; = \;\; & - \left\{ \frac{1}{2V} \sum_{k_1, k_2\in\Lambda^*} v(0)\, \alpha_{k_1} \alpha_{k_2} + \frac{1}{2V} \sum_{k_1\in\Lambda^*, k_2\in\Lambda^*\setminus\{\pm k_1\}} v(k_1 - k_2) \alpha_{k_1} \alpha_{k_2} \right\} \\ & - \frac{1}{2V} \sum_{k\in\Lambda^*, k'\in\Lambda^*\setminus\{k\}} v(k-k') \overline{\gamma_k} \gamma_{k'} , \end{aligned} \tag{2.118}$$

and

$$\begin{aligned} f_k^{Pair} &= \varepsilon_k - \mu + \frac{v(0)}{V}\left\{\sum_{k'\in\Lambda}\alpha_{k'} - \tfrac{1}{2}\right\} + \frac{1}{V}\sum_{k'\in\Lambda^*\backslash\{\pm k\}} v(k-k')\,\alpha_{k'}, \\ h_k^{Pair} &= \frac{1}{2V}\sum_{k'\in\Lambda^*\backslash\{k\}} v(k-k')\,\gamma_{k'}. \end{aligned} \tag{2.119}$$

By virtue of the Bogoliubov inequality (Appendix D) one can estimate the difference of pressures corresponding to H_Λ^{Pair} (2.111) and to the approximating Hamiltonian $H_\Lambda^{Pair}(\alpha,\gamma)$:

$$\frac{1}{V}\left\langle W_\Lambda^{Pair}(\alpha,\gamma)\right\rangle_{H_\Lambda^{Pair}} \le p_\Lambda\left[H_\Lambda^{Pair}(\alpha,\gamma)\right] - p_\Lambda\left[H_\Lambda^{Pair}\right] \le \frac{1}{V}\left\langle W_\Lambda^{Pair}(\alpha,\gamma)\right\rangle_{H_\Lambda^{Pair}(\alpha,\gamma)}. \tag{2.120}$$

If the interaction satisfies $v(q)\ge 0$ (see *hypothesis* (B), Section 2.2), then

$$\frac{v(0)}{2V}\sum_{k_1,k_2\in\Lambda^*} A_{k_1}^* A_{k_2} \ge 0. \tag{2.121}$$

If, in addition, one supposes that $\varphi(x)\ge 0$ (*pure repulsion*), then $v(q)$ is a *positively-defined* function on Λ^* :

$$\frac{1}{2V}\sum_{k,k'\in\Lambda^*} v(k-k')\,\overline{z}_k z_{k'} \ge 0,\ \ z_k\in\mathbb{C}. \tag{2.122}$$

Therefore, the operator (cf. (2.117))

$$\frac{1}{2V}\sum_{k_1\in\Lambda^*,k_2\in\Lambda^*\backslash\{\pm k_1\}} v(k_1-k_2)A_{k_1}^*A_{k_2} + \frac{1}{2V}\sum_{k\in\Lambda^*,k'\in\Lambda^*\backslash\{k\}} v(k-k')B_k^*B_{k'} \ge 0 \tag{2.123}$$

is *non-negative*. By virtue of (2.121) the same would be true for the expectation in the left-hand side of the estimate (2.120), at least for $V\to\infty$, if we prove that

$$\lim_\Lambda \frac{1}{V^2}\left\langle \sum_{k\in\Lambda^*} v(k)\,A_k^*A_k + \sum_{k\in\Lambda^*} v(2k)\,A_k^*A_{-k}\right\rangle_{H_\Lambda^{Pair}} = 0 \tag{2.124}$$

and

$$\lim_\Lambda \frac{1}{V^2}\left\langle \sum_{k\in\Lambda^*} v(0)\,B_k^*B_k\right\rangle_{H_\Lambda^{Pair}} = 0. \tag{2.125}$$

The V^2 denominators in (2.124), (2.125) make these hypothesis plausible. If we admit them, then the estimate (2.120) gives a *variational problem* with the approximating Hamiltonian $H_\Lambda^{Pair}(\alpha,\gamma)$:

$$\partial_{\alpha_k} p_\Lambda\left[H_\Lambda^{Pair}(\alpha,\gamma)\right] = 0, \tag{2.126}$$

and

$$\partial_{\gamma_k} p_\Lambda\left[H_\Lambda^{Pair}(\alpha,\gamma)\right] = 0,\ \ \partial_{\overline{\gamma_k}} p_\Lambda\left[H_\Lambda^{Pair}(\alpha,\gamma)\right] = 0, \tag{2.127}$$

corresponding to a *minimum* of the trial pressure $p_\Lambda\left[H_\Lambda^{Pair}(\alpha,\gamma)\right]$. This leads to self-consistency equations:

$$\alpha_k = \langle a_k^* a_k\rangle_{H_\Lambda^{Pair}(\alpha,\gamma)}, \tag{2.128}$$

and

$$\gamma_k = \langle a_{-k} a_k\rangle_{H_\Lambda^{Pair}(\alpha,\gamma)}, \ \overline{\gamma_k} = \left\langle a_k^* a_{-k}^*\right\rangle_{H_\Lambda^{Pair}(\alpha,\gamma)}. \tag{2.129}$$

Since the approximating Hamiltonian, see (2.116), is bilinear, one can diagonalize it by the Bogoliubov u-v transformation (2.73) with

$$\mathbf{u}_k^2 = \frac{1}{2}\left(\frac{f_k^{Pair}}{E_k^{Pair}}+1\right), \ \mathbf{v}_k^2 = \frac{1}{2}\left(\frac{f_k^{Pair}}{E_k^{Pair}}-1\right), \tag{2.130}$$

where

$$E_k^{Pair} = \sqrt{\left(f_k^{Pair}\right)^2 - |h_k^{Pair}|^2}, \ k\in\Lambda^*. \tag{2.131}$$

Therefore, the equations (2.128) and (2.129) take the explicit form:

$$\alpha_k = \frac{1}{2}\left[\frac{f_k^{Pair}}{E_k^{Pair}} cth\left(\frac{1}{2}\beta E_k^{Pair}\right)-1\right], \tag{2.132}$$

and

$$\gamma_k = -\frac{1}{2}\frac{h_k^{Pair}}{E_k^{Pair}} cth\left(\frac{1}{2}\beta E_k^{Pair}\right), \ \overline{\gamma_k} = -\frac{1}{2}\frac{\overline{h_k^{Pair}}}{E_k^{Pair}} cth\left(\frac{1}{2}\beta E_k^{Pair}\right). \tag{2.133}$$

Now, to check the *exactness* of the approximating Hamiltonian (2.116) one has to prove that the right-hand side in the estimate is zero in the limit $V\to\infty$.

In spite of fairly convincing arguments, see e.g. [79,80,111], the above programme has never been realized rigorously. The difficulty is that no one has proved the *infinitely-parametric* variational principle for the model (2.111), i.e. the starting point of this programme, see discussion in [90]. If we *admit* the variational formula for approximating Hamiltonian (2.116), then one can *prove* that in the thermodynamic limit equations (2.129) or (2.133) have only trivial solution $\gamma_k = 0$ [83]. To make the solution nontrivial one may to consider (similar to the BCS model) an *attraction* in the interaction of pairs (2.113) [80,83].

Another remark concerns the spectrum E_k^{Pair}, for which by (2.119) we get

$$\lim_{k\to 0} E_k^{Pair} \neq 0, \tag{2.134}$$

known since the first variational calculations by [78]. If one separates the $k=0$ mode to treat the Bose-Einstein condensation, this gives for $\rho > \rho_c^{Pair}(\theta)$ that $f_{k=0}^{Pair} = \left|h_{k=0}^{Pair}\right|$ and

$$\lim_\Lambda \frac{\alpha_{k=0}}{V} = \lim_\Lambda \frac{|\gamma_{k=0}|}{V} = \rho_0^{Pair}(\theta) > 0, \tag{2.135}$$

which implies the *gap* in the spectrum:

$$E_{k=0}^{Pair} = 0 \text{ and } \lim_{k\to 0} E_k^{Pair} = \sqrt{4\rho_0^{Pair}(\theta)\, v(0)}, \tag{2.136}$$

see [77, 79, 80]. The result (2.136) does not satisfy the Hugenholtz-Pines-, or the Bogoliubov $1/q^2$-theorem about gapless excitations in the Bose-systems with condensate [71, 76], [26] Part I, Section 2.8.

In spite of nonrigorous status of (2.136), it should discourage us to attempt to go beyond the Bogoliubov Hamiltonian via the Pair Hamiltonian H_{Λ}^{Pair}. In fact, the proof of the $1/q^2$-theorem shows that the gap in the spectrum of elementary excitations of the Pair Hamiltonian is not an artifact of uncontrolled approximations, but a breaking of the local gauge symmetry in truncated theory might be regarded as a "fortuitous result".

Our discussion above shows that the gapless spectrum in the Bogoliubov theory is due to a delicate *balance* between two Bogoliubov's *ansätze* and the *tuning* of the value of the chemical potential, see (2.72), (2.80) and (2.107). The Bogoliubov's proof of the $1/q^2$-theorem (implying the Hugenholtz-Pines theorem) is nonperturbative and it is based on the *local* gauge transformations:

$$\begin{aligned} a(x) &\to e^{i\varphi(x)} a(x), \\ a^*(x) &\to e^{-i\varphi(x)} a^*(x). \end{aligned} \tag{2.137}$$

Here $x \in \Lambda$ (torus), $\varphi(x)$ is a continuous function on Λ and $\{a^{\#}(x)\}_{x\in\Lambda}$ are boson-field operators (cf. Section 2.1)

$$\begin{aligned} a(x) &= \sum_{k\in\Lambda^*} a_k \psi_k(x), \\ a^*(x) &= \sum_{k\in\Lambda^*} a_k \psi_k(x). \end{aligned} \tag{2.138}$$

Then the full grand-canonical Hamiltonian (2.44) can be rewritten in the form:

$$\begin{aligned} H_{\Lambda}(\mu) &= H_{\Lambda} - \mu N_{\Lambda} = \int\limits_{\Lambda} dx a^*(x) \left[-\frac{\hbar^2}{2m}\Delta - \mu\right] a(x) + \\ &+ \frac{1}{2}\int\limits_{\Lambda} dx \int\limits_{\Lambda} dy a^*(x) a^*(y) \varphi(x-y) a(y) a(x). \end{aligned} \tag{2.139}$$

Notice that this Hamiltonian is invariant with respect to the *local* gauge transformation (2.137), which leads to *local* particle number conservation, i.e. the *continuity* equation

$$\partial_t \rho(x) + \nabla j(x) = 0, \tag{2.140}$$

relating the *density* operator $\rho(x) = a^*(x) a(x)$ with the *current* density

$$j(x) = \frac{\hbar^2}{2mi}\{a^*(x)(\nabla a)(x) - (\nabla a^*)(x) a(x)\}.$$

Notice that, for example, the nondiagonal part (2.113) of H_{Λ}^{Pair}:

$$U_{\Lambda}^{PairN} = \frac{1}{2V}\int\limits_{\Lambda} dx \int\limits_{\Lambda} dx' \int\limits_{\Lambda} dz a^*(x+z) a^*(x'+z) \varphi(z) a(x') a(x)$$

is *not* locally gauge invariant, cf. (2.137). One sees that a particle absorbed at the point x can be recreated at the point $x+z$. The same concerns the interaction (2.112). Therefore, the appearance of the gap (2.136) is a consequence of the lack of local gauge invariance or of local conservation of particle density (2.140).

Similarly, the nondiagonal part U_Λ^{BN} (2.53) of the WIBG combined with the term $v(0)(a_0^*)^2(a_0)^2/2$ from (2.52) gives the interaction

$$\begin{aligned}\widetilde{U}_\Lambda^{BN} &= \frac{1}{2V}\int\limits_\Lambda dx_1 \int\limits_\Lambda dx_2 \varphi(x_1-x_2)\, a^*(x_1)\, a^*(x_2) \int\limits_\Lambda dx_3 a(x_3) \int\limits_\Lambda dx_4 a(x_4) + \\ &+\frac{1}{2V}\int\limits_\Lambda dx_4 a^*(x_4) \int\limits_\Lambda dx_3 a^*(x_3) \int\limits_\Lambda dx_2 \int\limits_\Lambda dx_1 \varphi(x_1-x_2)\, a(x_2)\, a(x_1)\,, \quad (2.141)\end{aligned}$$

which is *not* locally gauge invariant, see (2.137); and the same concerns the U_Λ^{BD} (2.52). Hence one has to anticipate a *gap* in the spectrum of elementary excitations for the *exact* solution of the Bogoliubov WIBG with Hamiltonian H_Λ^B (2.50). This conclusion is confirmed in [15,16], see Sections 4 and 5.

Another temptation to go beyond the Bogoliubov theory is related to the *full* Hamiltonian (2.44) or H_Λ in (2.139). Instead of consider the variational principle based on the Pair Hamiltonian part H_Λ^{Pair} (2.111) of the full problem, see [90], one can try to work out it for H_Λ. In 1965 Robinson [129] formulated such a principle for $T=0$, on a subset of the infinite-volume ground states, which coincide with translation-invariant *quasi-free* states. Then it has been generalized to $T\neq 0$ in [121].

Recall that the infinite-volume *quasi-free* states $\omega_{\rm qf}(-)$ in the set of translation-invariant (locally normal) states are completely determined by the following functions on the boson-field operators (2.138) in the limit (see Appendix F and [37]):

$$\omega_{\rm qf}\left(a^{\#}(x)\right) = c^{\#}, \quad (2.142)$$

$$\omega^T_{\rm qf}(a^*(x)\,a(y)) \equiv \omega_{\rm qf}(a^*(x)\,a(y)) - \omega_{\rm qf}(a^*(x))\,\omega_{\rm qf}(a(y)) \equiv \mathrm{d}(x-y)\,, \quad (2.143)$$

$$\omega^T_{\rm qf}(a(x)\,a(y)) \equiv \omega_{\rm qf}(a(x)\,a(y)) - \omega_{\rm qf}(a(x))\,\omega_{\rm qf}(a(y)) \equiv \mathrm{w}(x-y)\,. \quad (2.144)$$

Here $c\in\mathbb{C}$, d is a real-valued function, and w is a complex-valued function, both depending only on $\|x-y\|$. Then the thermodynamic functional, cf. (F.5) in Appendix F,

$$g_\mu(\omega) = e(\omega) - \beta^{-1}s(\omega) - \mu\,(n(\omega)-\rho)\,, \quad (2.145)$$

that we require for the variational principle, is the *free-energy* density $f(\omega)=e(\omega)-\beta^{-1}s(\omega)$ with the particle density constraint $n(\omega)=\rho$ taken via the Lagrange multiplier μ. For $\omega=\omega_{\rm qf}$ the functional can be calculated explicitly using the local normality and (F.2)-(F.4) (Appendix F):

$$n(\omega_{\rm qf}) = \omega_{\rm qf}(a^*(0)\,a(0)) = \mathrm{d}(0)+|c|^2 = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} dk\,\widehat{\mathrm{d}}(k) + |c|^2\,, \quad (2.146)$$

$$s(\omega_{\rm qf}) = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} dk\,\{[1+\widehat{\mathrm{r}}(k)]\ln[1+\widehat{\mathrm{r}}(k)] - \widehat{\mathrm{r}}(k)\ln[\widehat{\mathrm{r}}(k)]\}\,, \quad (2.147)$$

$$e(\omega_{\rm qf}) = \omega_{\rm qf}\left(a^*(0)\left(-\frac{\Delta}{2m}a\right)(0)\right) + \frac{1}{2}\int\limits_{\mathbb{R}^3} dx\varphi(x)\,\omega_{\rm qf}(a^*(x)\,a^*(0)\,a(0)\,a(x)) \quad (2.148)$$

where

$$\widehat{\mathrm{r}}(k)=\sqrt{\left(\widehat{\mathrm{d}}(k)+\frac{1}{2}\right)^{2}-|\widehat{\mathrm{w}}(k)|^{2}}-\frac{1}{2}. \tag{2.149}$$

Using (2.142)-(2.145) we obtain for the energy-density functional (2.148):

$$\begin{aligned} e\left(\omega_{\mathrm{qf}}\right) &= \frac{1}{(2\pi)^{3}}\int_{\mathbb{R}^{3}} dk\varepsilon_{k}\widehat{\mathrm{d}}(k)+\frac{1}{2}n^{2}\left(\omega_{\mathrm{qf}}\right)v(0)+\frac{|c|^{2}}{(2\pi)^{3}}\int_{\mathbb{R}^{3}} dk\widehat{\mathrm{d}}(k)v(k)+ \\ &+\frac{1}{2}\frac{c^{2}}{(2\pi)^{3}}\int_{\mathbb{R}^{3}} dk\overline{\widehat{\mathrm{w}}(k)}v(k)+\frac{1}{2}\frac{\overline{c}^{2}}{(2\pi)^{3}}\int_{\mathbb{R}^{3}} dk\widehat{\mathrm{w}}(k)v(k)+ \\ &+\frac{1}{2}\frac{1}{(2\pi)^{6}}\int_{\mathbb{R}^{3}} dk\int_{\mathbb{R}^{3}} dk'\widehat{\mathrm{d}}(k)v(k-k')\widehat{\mathrm{d}}(k')+\frac{1}{2}\frac{1}{(2\pi)^{6}}\int_{\mathbb{R}^{3}} dk\int_{\mathbb{R}^{3}} dk'\overline{\widehat{\mathrm{w}}(k)}v(k-k')\widehat{\mathrm{w}}(k'). \end{aligned} \tag{2.150}$$

To determine the equilibrium quasi-free state one has to minimize $g_{\mu}\left(\omega_{\mathrm{qf}}\right)$ with respect to the parameters $|c|$, $\varphi=\arg c$, $\widehat{\mathrm{d}}(k)$, $|\widehat{\mathrm{w}}(k)|$, $\psi(k)=\arg\widehat{\mathrm{w}}(k)$ and μ. The solution of the set of equations:

$$\partial_{|c|}g_{\mu}=0,\ \partial_{\varphi}g_{\mu}=0,\ \frac{\delta}{\delta\widehat{\mathrm{d}}(k)}g_{\mu}=0,\ \frac{\delta}{\delta|\widehat{\mathrm{w}}(k)|}g_{\mu}=0,\ \frac{\delta}{\delta\psi(k)}g_{\mu}=0,\ \partial_{\mu}g_{\mu}=0 \tag{2.151}$$

provided it exists and it is a global minimum, determines the *canonical equilibrium quasi-free state* $\omega_{\mathrm{qf}}(-)\equiv\omega_{\beta,\rho}(-)$. Notice that it is possible that (2.151) have *no* solution at all. For example, for the PBG, i.e. $v(q)\equiv 0$, the minimum of $g_{\mu}\left(\omega_{\mathrm{qf}}\right)$ occurs on the hyperplane $\widehat{\mathrm{r}}(k)=\widehat{\mathrm{d}}(k)$ which is *not* a solution of equations (2.151).

Retaining only those terms *linear* in $\widehat{\mathrm{d}}(k)$ and $\widehat{\mathrm{w}}(k)$ in the expression for the energy-density functional (2.150), one obtains the *simplified* form of the set of equations (2.151). As it is shown in [121] the solution of these equations gives the quasi-free $\omega_{\mathrm{qf}}^{B}(-)=\lim_{\Lambda}\omega_{\Lambda,c}^{B}(-)$, where the finite-volume state $\omega_{\Lambda,c}^{B}(-)=\langle-\rangle_{H_{\Lambda}^{B}(c^{\#})}$ is generated by $H_{\Lambda}^{B}\left(c^{\#}\right)$, see (2.60), with c satisfying (2.72). Therefore the equilibrium state corresponding to the Bogoliubov theory is a particular case *linear approximation* of the full variational problem on the set of quasi-free states.

Since the rigorous results for the full Hamiltonian (2.43), (2.44) (for $d=3$) are very rare, we would like to conclude this section by one of the very recent [130]. It concerns the proof of the (put forward many decades ago, see [90]) formula for the asymptotics of the *ground state* energy per particle ($n=\rho V$)

$$e_{0}(\rho)=\lim_{\Lambda}\left\{\frac{1}{n}\inf_{\|\psi\|=1,\psi\in\mathcal{H}_{B}^{(n)}}\left(\psi,H_{\Lambda}^{(n)}\psi\right)_{\mathcal{H}_{B}^{(n)}}\right\} \tag{2.152}$$

for *dilute* Bose gas. Let $\varphi(x)=\varphi(\|x\|=r)$ be non-negative spherically symmetric potential decreasing faster than r^{-3} at infinity. The *scattering length* a of the potential is defined by

$$a=\lim_{r\to+\infty}\left\{r-\frac{u_{0}(r)}{u_{0}'(r)}\right\}, \tag{2.153}$$

where $u_0(r)$ solves the zero-energy scattering equation

$$-\frac{\hbar^2}{m}u_0''(r)+\varphi(r)\,u_0(r)=0$$

for two-particle problem (i.e. for the reduced mass $m/2$) and for $u_0(0)=0$. Lieb and Yngvason [130] have recently proved that the low density asymptotic formula

$$\lim_{\rho a^3\to 0}\frac{m}{2\pi\hbar^2}\frac{e_0(\rho)}{\rho a}=1 \tag{2.154}$$

is exact. Since this formula is valid for the Lieb-Liniger exact solution of the one-dimensional Bose gas with a pair-wise repulsive δ-function potential [131], [132] (which presumably does not manifest a BE condensation), it seems that there is no BE condensation in the asymptotic limit (2.154) as well. On the other hand Bogoliubov [1–3] argued existence of the BE condensation and of the similar formula in this regime for the WIBG, see discussions in [90] and in [131], [130].

Notice that earlier , see e.g. [69], [78], [85], [114], [133], [134] , [135] several derivations of (2.154) were presented , even including higher order terms (cf. discussion at the beginning of this section) :

$$\frac{e_0(\rho)}{4\pi\mu\rho a}=1+\frac{128}{15\pi^{1/2}}(\rho a^3)^{1/2}+8(4\pi/3-3^{1/2})(\rho a^3)\ln(\rho a^3)+O(\rho a^3), \tag{2.155}$$

where we have denoted $\hbar^2/2m$ by μ for short. They all rely on different kind of assumptions about the ground-state or on specific summations of perturbation series that have never been proved to be exact , for review see [90]. At that time the only rigorous estimate of $e_0(\rho)/4\pi\mu\rho a$ from below and from above was due to Dyson [139] . After 40 years Lieb and Yngvason [130], [136] succeeded to refine the Dyson analysis for non-negative spherically symmetric potential (decreasing faster than r^{-3} at infinity) with $a>0$, to prove (2.154). Together with the study of the one-dimensional Bose gas with a pair-wise repulsive δ-function potential [131], [132] this in particular shows that the Bogoluibov *theory* (which *a priori* presupposed the existence of the BE condensation) may give a correct answer for the ground-state energy for the asymptotics (2.154) even without having to decide about the BE condensation. The analog of the formula (2.154) for a two-dimensional Bose gas gives $|\ln(\rho a^2)|^{-1}$ instead of a in the denominator, see [137]. Notice that in contrast to the three-dimensional case this ground-state energy density is not proportional to the density of particles! Recently, using the ideas of [134], Lieb and Solovej have extended this analysis to a "jellium" model of the one-component charged Bose gas [138]. For the case of the Coulomb interaction and high densities the first (mean-field) term in the above energy density representation (2.155) is zero by charge neutrality. Therefore, it is the next term that gives the first nontrivial input. In 1961 Foldy [140] used the Bogoliubov theory to calculate it in the high density limit, where the Bogoliubov method was claimed to be exact for this problem. Lieb and Solovej *proved* that this is true. This gives a rigorous example of verification of the Bogoliubov theory in a three-dimensional boson system.

Section 3

The Bogoliubov theory, revisited

"Il y a deux choses que l'expérience doit apprendre: la première, c'est qu'il faut beaucoup corriger; la seconde, c'est qu'il ne faut pas trop corriger."
Delacroix

"But of equal importance is understanding, with a little hindsight, the failure of the theory, for its success led to regrettable tendency to take its predictions as gospel"
Lieb, *The Bose Fluid.*

In this section we take two *steps back* in the Bogoliubov theory to return on the level of the *first Bogoliubov ansatz*. This ansatz suggests to us the Bogoliubov *truncated* Hamiltonian (2.50) as a starting point of this theory. Since this ansatz is based on the hypothesis concerning the BE condensation in zero-mode and on its stability with respect to interactions, we present in Section 3.1 some rigorous results which cast doubt on this hypothesis. If nevertheless one accepts the Hamiltonian (2.50) for the WIBG, then one is faced with difficulties with the *second Bogoliubov ansatz*. In Sections 3.2, 3.3 we discuss the Bogoliubov prescription concerning *substitution* of the zero-mode annihilation and creation operators by c-numbers and concerning of their *choice*. In fact this substitution *eliminates quantum fluctuations*, which are mediators of supplementary *effective interactions* between bosons of the WIBG. This interaction in the zero-mode is *attractive*, whereas for bosons outside this mode it is *repulsive*. Then we show that the WIBG is *unstable* for *positive* chemical potentials: the grand-canonical partition function *diverges* for $\mu = |c|^2 v(0) > 0$, which is the choice of the Bogoliubov theory, see (2.72), (2.107). In the last Section 3.4 we revise the Bogoliubov theory in the high of these rigorous results.

3.1 Generalized Bose-Einstein condensation and the first Bogoliubov ansatz

Recall that the starting point of the Bogoliubov theory is based on two hypotheses (Section 2.2):

(a) there is a BE condensation in *one mode*;
(b) this condensation is *stable* with respect to weak interaction between particles.

These hypotheses inspired Bogoliubov on his *first* ansatz: the *truncation* of the full Hamiltonian (2.44) to the Hamiltonian (2.50) of the WIBG.

First we scrutinize the (most innocent) hypothesis (a), which concerns the PBG. As we discussed in Section 1.2 the conventional BE condensation is a *subtle* matter even for the Perfect Bose-Gas (PBG).

Let one take, instead of the *cubic box* $\Lambda = L\times L\times L$ with $|\Lambda| = V$, a *prism* $\Lambda = L_1\times L_2\times L_3$ of the *same* volume with sides of length $L_j = V^{\alpha_j}$, $j = 1, 2, 3$, such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. As in Section 2.1 we impose periodic boundary conditions on $\partial\Lambda$ and we consider a possibility of conventional BE condensation in this prism. This example is due to Casimir [43], van den Berg and Lewis [42]. Notice that up to (2.34) all arguments of Section 2.1 are valid for the prism. Rewrite equation (2.34) in the form:

$$\begin{aligned}\rho &= \frac{1}{V}\frac{1}{e^{-\beta\mu}-1}+\frac{1}{V}\sum_{k\in\{\Lambda^*:n_1\neq 0,n_2=n_3=0\}}\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}\\ &\quad+\frac{1}{V}\sum_{k\in\{\Lambda^*:n_j\neq 0,j=2\text{ or }3\}}\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}.\end{aligned}\tag{3.1}$$

For $\rho > \rho_c^P(\theta)$ equation (3.1) gives asymptotic behaviour of the solution $\mu_\Lambda^P\left(\theta,\rho>\rho_c^P(\theta)\right)$ and for distributions of bosons in the mode $k=0$ and in its vicinity.

$1°$. Let $\alpha_1 < 1/2$. Then (cf. (2.36)) for $V\to\infty$ one gets:

$$\inf_{k\in\Lambda^*\setminus\{0\}}\varepsilon_k = O\left(V^{-2\alpha_1}\right).\tag{3.2}$$

This means that

$$\lim_\Lambda\sup_{k\in\Lambda^*\setminus\{0\}}\frac{1}{V}\frac{1}{e^{\beta\varepsilon_k}-1}=0.$$

Therefore, the Darboux sum in (3.1) for $k\neq 0$ is well-defined and it converges to the integral

$$\rho^P(\beta,0)=\lim_\Lambda\rho_\Lambda^P\left(\beta,\mu_\Lambda^P\left(\theta,\rho>\rho_c^P(\theta)\right)\right),$$

see (2.26). Since $\rho > \rho_c^P(\theta) = \rho^P(\beta,0)$, the equation (3.1) implies that

$$\mu_\Lambda^P\left(\theta,\rho>\rho_c^P(\theta)\right)\simeq\frac{1}{\beta V\left(\rho-\rho_c^P(\theta)\right)},\ V\to\infty,\tag{3.3}$$

as it is in the case of the cube when $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, see (2.38). Consequently, one gets a *conventional* BE condensation in *one* mode $k = 0$:

$$\lim_{\Lambda} \frac{1}{V} \langle N_{k=0} \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P \left(\theta, \rho > \rho_c^P(\theta)\right)\right) = \rho_0^P(\theta) \equiv \rho - \rho_c^P(\theta) > 0,$$
$$\lim_{\Lambda} \frac{1}{V} \langle N_{k\neq 0} \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P \left(\theta, \rho > \rho_c^P(\theta)\right)\right) = 0.$$

2°. Now let $\alpha_1 = 1/2$. Then one gets:

$$\inf_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k = \frac{(2\pi\hbar)^2}{2m} \frac{1}{V}, \tag{3.4}$$

which corresponds to the mode with $(n_1 = 1, n_2 = 0, n_3 = 0)$, see (3.1). Since $\alpha_2 < 1/2$ and $\alpha_3 < 1/2$, we get

$$\lim_{\Lambda} \sup_{k \in \{\Lambda^* : n_j \neq 0, j = 2 \text{ or } 3\}} \frac{1}{V} \frac{1}{e^{\beta \varepsilon_k} - 1} = 0.$$

Then the Darboux sum corresponding to the last term in (3.1) converges to the integral $\rho^P(\beta, \mu)$ for all $\mu < 0$. Since this integral attains its $\sup_{\mu<0} \rho^P(\beta, \mu) = \rho_c^P(\theta)$ for $\mu \to 0^-$, from (3.1) one obtains for $\rho > \rho_c^P(\theta)$ that

$$\lim_{\Lambda} \left\{ \frac{1}{V} \frac{1}{e^{-\beta \mu_\Lambda^P(\theta,\rho)} - 1} + \frac{1}{V} \sum_{k \in \{\Lambda^* : n_1 \neq 0, n_2 = n_3 = 0\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} - 1} \right\} = \rho - \rho_c^P(\theta). \tag{3.5}$$

By virtue of $\alpha_1 = 1/2$ the left-hand side of (3.5) implies:

$$\mu_\Lambda^P \left(\theta, \rho > \rho_c^P(\theta)\right) = \frac{A}{V} + o\left(\frac{1}{V}\right), \ A > 0, \tag{3.6}$$

for $V \to \infty$. Since for the asymptotics (3.6) we have

$$\lim_{\Lambda} \left\{ \sum_{k \in \{\Lambda^* : (n_1, 0, 0)\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} - 1} \right\} = \frac{1}{\beta} \sum_{n_1 = 0, \pm 1, \pm 2, \ldots} \left(\frac{(2\pi\hbar)^2}{2m} n_1^2 + A \right)^{-1}, \tag{3.7}$$

the (3.5) implies

$$\frac{1}{\beta} \sum_{n_1 = 0, \pm 1, \pm 2, \ldots} \left(\frac{(2\pi\hbar)^2}{2m} n_1^2 + A \right)^{-1} = \rho - \rho_c^P(\theta), \tag{3.8}$$

where $A = A(\theta, \rho)$ is a *unique* root of this equation. Moreover, the limit (3.7) shows that for $\rho > \rho_c^P(\theta)$:

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P(\theta, \rho)\right) = \left\{ \begin{array}{ll} \beta^{-1} \left(\frac{(2\pi\hbar)^2}{2m} n_1^2 + A \right)^{-1}, & \text{for } k \in \{\Lambda^* : (n_1, 0, 0)\} \\ 0 & , \text{ for } k \in \{\Lambda^* : n_j \neq 0, j = 2 \text{ or } 3\} \end{array} \right\}. \tag{3.9}$$

This means that for the *long prism* with $\alpha_1 = 1/2$ there is *macroscopic* occupation of an *infinite* number of low-lying levels $k \in \{\Lambda^* : (n_1, 0, 0)\}$ including the zero-mode $k = 0$.

3°. Let $\alpha_1 > 1/2$, i.e. we deal with a *highly anisotropic* prism in direction $j = 1$.Then we have

$$\inf_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k = \frac{(2\pi\hbar)^2}{2m} \frac{1}{V^{2\alpha_1}}, \; 2\alpha_1 > 1, \tag{3.10}$$

which corresponds to the mode with $(n_1 = 1, n_2 = 0, n_3 = 0)$. Since for any $\mu < 0$ the right-hand side of (3.1) converges to the integral $\rho^P(\beta, \mu)$ which is monotonously increasing up to $\rho_c^P(\theta)$, when $\mu \to 0^-$, the solution $\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)$ of (3.1) has to have the asymptotics:

$$\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right) = \frac{B}{V^\delta} + o\left(\frac{1}{V^\delta}\right), \; B < 0, \; \delta > 0, \tag{3.11}$$

for $V \to \infty$. To calculate B and δ, we use that the first *two* terms in (3.1) may be represented as

$$\begin{aligned} \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1, 0, 0)\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} - 1} &= \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1, 0, 0)\}} \sum_{s=1}^\infty e^{-s\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} \\ &= \frac{1}{V} \sum_{s=1}^\infty e^{+s\beta\mu_\Lambda^P(\theta,\rho)} \sum_{n_1 = 0, \pm1, \pm2, \ldots} e^{-s\beta\left(\frac{(2\pi\hbar)^2}{2m} \frac{n_1^2}{V^{2\alpha_1}}\right)} . \end{aligned} \tag{3.12}$$

Notice that

$$\sum_{n_1 = 0, \pm1, \pm2, \ldots} e^{-\pi\lambda n_1^2} = \frac{1}{\sqrt{\lambda}} \sum_{\xi = 0, \pm1, \pm2, \ldots} e^{-\left(\pi\xi^2/\lambda\right)}, \tag{3.13}$$

where $\lambda = \beta \dfrac{2\pi\hbar^2}{m} s V^{-2\alpha_1}$. Taking into account (3.11)-(3.13) we find that only the term with $\xi = 0$ is important for (3.12) in the limit $V \to \infty$:

$$\rho - \rho_c^P(\theta) = \lim_\Lambda \left\{ \left(\frac{2\pi\hbar^2}{m} \beta \right)^{-1/2} \left\{ \frac{V^{\alpha_1 - 1}}{V^{\delta/2}} . V^\delta \right\} \frac{1}{V^\delta} \left\{ \sum_{s=1}^\infty e^{+\beta B\left(s/V^\delta\right)} \left(\frac{s}{V^\delta} \right)^{-1/2} \right\} \right\}. \tag{3.14}$$

The limit in (3.14) is *nontrivial* only for

$$\delta = 2\left(1 - \alpha_1\right). \tag{3.15}$$

Then one gets

$$0 < \rho - \rho_c^P(\theta) = \left(\frac{2\pi\hbar^2}{m} \right)^{-1/2} \beta^{-1/2} \int_0^\infty d\xi e^{\beta B \xi} \xi^{-1/2}, \tag{3.16}$$

where $B = B\left(\theta = \beta^{-1}, \rho\right) < 0$ is the *unique* root of equation (3.16), or

$$\rho - \rho_c^P(\theta) = \left(\frac{m}{2\hbar^2} \right)^{1/2} \frac{\theta}{\sqrt{|B(\theta, \rho)|}}, \tag{3.17}$$

since

$$I = \int_0^\infty dy e^{-y} y^{-1/2} = \sqrt{\pi}.$$

By virtue of (3.11), (3.15) and (3.17) we get that for $\rho > \rho_c^P(\theta)$ and $V \to \infty$:

$$\varepsilon_k - \mu_\Lambda^P(\theta,\rho) \simeq \left\{ \begin{array}{ll} \dfrac{\hbar^2}{2m}\left(\dfrac{2\pi n_1}{V^{\alpha_1}}\right)^2 + \dfrac{m/2\hbar^2}{\beta^2\left(\rho-\rho_c^P(\theta)\right)^2 V^{2(1-\alpha_1)}}, & k \in \{\Lambda^* : (n_1,0,0)\} \\ \dfrac{\hbar^2}{2m}\left[\left(\dfrac{2\pi n_2}{V^{\alpha_2}}\right)^2 + \left(\dfrac{2\pi n_3}{V^{\alpha_3}}\right)^2\right] + \dfrac{m/2\hbar^2}{\beta^2\left(\rho-\rho_c^P(\theta)\right)^2 V^{2(1-\alpha_1)}}, & k \in \{\Lambda^* : n_{j=2 \text{ or } 3} \neq 0\} \end{array} \right\}. \tag{3.18}$$

Since $\alpha_1 > 1/2$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the asymptotics (3.18) implies

$$\lim_\Lambda \frac{1}{V} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)\right) = 0, \ k \in \Lambda^*, \tag{3.19}$$

while

$$\lim_\Lambda \frac{1}{V^{2(1-\alpha_1)}} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)\right) = \frac{m/2\hbar^2}{\beta^2\left(\rho-\rho_c^P(\theta)\right)^2}, \ k \in \{\Lambda^* : (n_1,0,0)\}, \tag{3.20}$$

and (cf. (3.1), (3.12), (3.14))

$$\rho_0^P(\theta) = \rho - \rho_c^P(\theta) = \lim_\Lambda \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1,0,0)\}} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P(\theta,\rho)\right) > 0.$$

This means that for the *extremely long* prism with $\alpha_1 > 1/2$ for $\rho > \rho_c^P(\theta)$ there is *conventional* BE condensation with density $\rho_0^P(\theta) > 0$, whereas there is no *macroscopically* occupied levels in Λ^*.

Cases $\alpha_1 = 1/2$ and $\alpha_1 > 1/2$ are provided examples of *generalized condensation* in the Girardeau's sense [39]. This gives a motivation for the following van den Berg-Lewis-Pulè's *classification* of the generalized BE condensation [8, 42, 44] (see Appendix A):

- the condensation is called *type I* when a *finite* number of single-particle levels are macroscopically occupied;
- it is of *type II* when an *infinite* number of the levels are macroscopically occupied;
- it is called *type III*, or the *non-extensive* condensation, when *no* of the levels are macroscopically occupied whereas one has

$$\lim_{\delta \to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 \le \|k\| \le \delta\}} \langle N_k \rangle = \rho - \rho_c(\theta).$$

Remark 3.1 *Returning back to the Bogoliubov theory we have to recognize that the hypothesis (a) about one the one-mode BE condensation looses its grip if we pass to the PBG in a long prism. The latter is not an exotic situation if one considers superfluidity of extremely thin films of* ^{4}*He, see [20, 120].*

In [46, 47] it was shown that the *type III* condensation can be caused in the PBG by a weak *external field* or by a specific choice of the boundary conditions and geometry [44, 45]. But recently other examples of this *non-extensive* BE condensation were discovered for bosons in an *isotropic* box Λ with *repulsive interaction* which *spreads out* the conventional BE condensation of *type I* into *type III*, [58, 59, 141].

To avoid the influence of the shape on the BE condensation let us take the *cubic box* $\Lambda = L \times L \times L$. Imposing the periodic boundary conditions on $\partial\Lambda$, we consider an interacting boson system with the Hamiltonian:

$$H_\Lambda^I = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k\in\Lambda^*} v(0) a_k^* a_k^* a_k a_k, \ \ v(0) > 0. \tag{3.21}$$

Interaction in (3.21) results from a severe truncation of the full interaction (2.44). Below we shall demonstrate that even this small part of the total interaction can drastically change the distribution of condensed particles spreading out the BE condensation in the one mode $k = 0$ into non-extensive BE condensation of type III.

First we show that the interaction in (3.21) is so gentle that it even does not modify the pressure of the perfect system.

Theorem 3.2 *For the model (3.21) the thermodynamic limit of the pressure coincides with the pressure of the PBG:*

$$\lim_{\Lambda} p_\Lambda \left[H_\Lambda^I\right] = \lim_{\Lambda} p_\Lambda \left[T_\Lambda\right] = p^P(\beta,\mu). \tag{3.22}$$

Proof. Notice that $a_k^* a_k^* a_k a_k = N_k^2 - N_k$. Therefore, the grand partition function for the model (3.21) takes the form (cf. (2.22)):

$$\Xi_\Lambda^I(\beta,\mu) = Tr_{\mathcal{F}_B} e^{-\beta\left(H_\Lambda^I - \mu N_\Lambda\right)} = \prod_{k\in\Lambda^*} \sum_{n_k=0}^{\infty} e^{-\beta\left[(\varepsilon_k-\mu)n_k + \frac{v(0)}{2V}\left(n_k^2 - n_k\right)\right]}, \tag{3.23}$$

which converges for all $\mu \in \mathbb{R}^1$. Then for the pressure in the finite volume we have the estimates:

$$p_\Lambda[T_\Lambda] \geq p_\Lambda\left[H_\Lambda^I\right] \geq \frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln \sum_{n_k=0}^{[\ln V]} e^{-\beta\left[(\varepsilon_k-\mu)n_k + \frac{v(0)}{2V}\left(n_k^2 - n_k\right)\right]}. \tag{3.24}$$

By virtue of $n_k^2 - n_k \leq [\ln V]^2$ in (3.24) (here $[x]$ denotes the integer part of x) one gets:

$$\sum_{n_k=0}^{[\ln V]} e^{-\beta\left[(\varepsilon_k-\mu)n_k + \frac{v(0)}{2V}\left(n_k^2 - n_k\right)\right]} \geq \sum_{n_k=0}^{[\ln V]} e^{-\beta\left[(\varepsilon_k-\mu)n_k + \frac{v(0)}{2V}[\ln V]^2\right]} = e^{-\beta\frac{v(0)}{2V}[\ln V]^2} \left\{\frac{1 - e^{-\beta(\varepsilon_k-\mu)([\ln V]-1)}}{1 - e^{-\beta(\varepsilon_k-\mu)}}\right\}. \tag{3.25}$$

For $\mu < 0$ the left-hand side of (3.25) converges (uniformly in $k \in \Lambda^*$) to $\left(1 - e^{-\beta(\varepsilon_k - \mu)}\right)^{-1}$ when $V \to \infty$. Hence, in the thermodynamic limit, the lower estimate in (3.24) coincides with the upper estimate (cf. (2.23)):

$$\begin{aligned}\lim_{\Lambda} \frac{1}{\beta V} \sum_{k \in \Lambda^*} \ln \left\{ e^{-\beta \frac{v(0)}{2V}[\ln V]^2} \frac{1 - e^{-\beta(\varepsilon_k - \mu)([\ln V]-1)}}{1 - e^{-\beta(\varepsilon_k - \mu)}} \right\} &= \lim_{\Lambda} \frac{1}{\beta V} \sum_{k \in \Lambda^*} \ln \left(1 - e^{-\beta(\varepsilon_k - \mu)}\right)^{-1} \\ &= \lim_{\Lambda} p_\Lambda \left[T_\Lambda\right],\end{aligned}$$

which proves the equality (3.22). Since $\lim\limits_{\mu \to 0^-} \lim\limits_{\Lambda} p_\Lambda \left[T_\Lambda\right]$ is bounded, one extends $p^P(\beta, \mu)$ to $\mu = 0$ by continuity. ■

As usual, the total density of particles for the model (3.21) is defined in the grand-canonical ensemble for a finite volume V by (cf.(2.25))

$$\rho_\Lambda^I(\beta, \mu) = \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^I} (\beta, \mu) = \frac{1}{V} \sum_{k \in \Lambda^*} \langle N_k \rangle_{H_\Lambda^I} (\beta, \mu) = \partial_\mu p_\Lambda \left[H_\Lambda^I\right]. \tag{3.26}$$

Corollary 3.3 *Due to $v(0) > 0$ the pressure $p_\Lambda \left[H_\Lambda^I\right]$ is defined for all chemical potentials $\mu \in \mathbb{R}^1$, see (3.23). Moreover it is obviously a convex function of this parameter. The limit function $p^P(\beta, \mu)$ is defined only for $\mu \leq 0$ and is also a convex function, but only for $\mu < 0$. Since (3.22) is statement about convergence of convex functions of $\mu \in (-\infty, 0)$, by the Griffiths lemma (Appendix C) we get convergence of the corresponding derivatives (cf. (2.25), (3.26)):*

$$\lim_{\Lambda} \rho_\Lambda^I(\beta, \mu) = \lim_{\Lambda} \partial_\mu p_\Lambda \left[H_\Lambda^I\right] = \partial_\mu \lim_{\Lambda} p_\Lambda \left[H_\Lambda^I\right] = \partial_\mu p^P(\beta, \mu) = \rho^P(\beta, \mu),$$

on the very same interval $\mu \in (-\infty, 0)$. So, in the thermodynamic limit the total particle density $\rho^I(\beta, \mu)$ coincides with $\rho^P(\beta, \mu)$. Since $\lim\limits_{\mu \to 0^-} \rho^P(\beta, \mu)$ is bounded for $d > 2$, one defines $\rho^I(\beta, \mu = 0) = \rho^P(\beta, \mu = 0)$ by continuity.

¿From this corollary one deduces that, as in the case of the PBG, $\rho^I(\beta, \mu)$ attains its maximum $\rho_c^I(\theta) = \lim\limits_{\mu \to 0^-} \rho^I(\beta, \mu) = \rho_c^P(\theta)$ at $\mu = 0$, i.e. for densities $\rho > \rho_c^I(\theta) = \rho_c^P(\theta)$ one *must* have a BE condensation. On the other hand, for the expectation of the ground-state occupation number at $\mu = 0$ we get from (3.21):

$$\left\langle \frac{N_0}{\sqrt{V}} \right\rangle_{H_\Lambda^I} (\beta, \mu = 0) = \frac{\frac{1}{\sqrt{V}} \sum\limits_{n_0=0}^{\infty} \frac{n_0}{\sqrt{V}} e^{-\beta \frac{v(0)}{2} \left(\frac{n_0}{\sqrt{V}}\right)^2}}{\frac{1}{\sqrt{V}} \sum\limits_{n_0=0}^{\infty} e^{-\beta \frac{v(0)}{2} \left(\frac{n_0}{\sqrt{V}}\right)^2}}. \tag{3.27}$$

Since the numerator and the denominator in the right-hand side of (3.27) are nothing but Darboux sums for the corresponding integrals in the limit $V \to \infty$, we obtain that

$$\lim_{\Lambda} \frac{1}{\sqrt{V}} \langle N_0 \rangle_{H_\Lambda^I} (\beta, \mu = 0) = \frac{\int dx x e^{-\beta \frac{v(0)}{2} x^2}}{\int dx e^{-\beta \frac{v(0)}{2} x^2}} = \frac{1}{2} \sqrt{\frac{2\theta}{v(0)}}. \tag{3.28}$$

This means that there is no macroscopic occupation of the mode $k = 0$ because $\langle N_0 \rangle_{H^I_\Lambda} (\beta, \mu = 0)$ grows only as $V^{1/2}$ for $V \to \infty$. Moreover, this is true for any $k \in \Lambda^*$.

Theorem 3.4 *For any parameters $(\beta, \mu \leq 0)$ one has*

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{H^I_\Lambda} (\beta, \mu) = 0, \ \forall k \in \Lambda^*, \tag{3.29}$$

i.e. there is no macroscopic occupation of any mode $k \in \Lambda^*$.

Proof. By the Bogoliubov convexity inequality for pressure (see Appendix D) we get

$$\frac{v(0)}{2V^2} \left\langle \sum_{k \in \Lambda^*} \left(N_k^2 - N_k\right) \right\rangle_{H^I_\Lambda} \leq p_\Lambda [T_\Lambda] - p_\Lambda \left[H^I_\Lambda\right] \leq \frac{v(0)}{2V^2} \left\langle \sum_{k \in \Lambda^*} \left(N_k^2 - N_k\right) \right\rangle_{T_\Lambda} .$$

By virtue of $v(0) > 0$, $N_k^2 - N_k \geq (N_k - I)^2$ and Theorem 3.2, one has

$$\lim_{\Lambda} \frac{1}{V^2} \sum_{k \in \Lambda^*} \left\langle (N_k - I)^2 \right\rangle_{H^I_\Lambda} (\beta, \mu \leq 0) = 0. \tag{3.30}$$

Notice that by the Cauchy-Schwarz inequality for the Gibbs state we get

$$\langle N_k - I \rangle^2_{H^I_\Lambda} \leq \left\langle (N_k - I)^2 \right\rangle_{H^I_\Lambda} .$$

Then

$$0 \leq \left(\frac{1}{V} \langle N_k - I \rangle_{H^I_\Lambda} \right)^2 \leq \frac{1}{V^2} \left\langle (N_k - I)^2 \right\rangle_{H^I_\Lambda} \leq \frac{1}{V^2} \sum_{q \in \Lambda^*} \left\langle (N_q - I)^2 \right\rangle_{H^I_\Lambda} . \tag{3.31}$$

The estimates (3.30) and (3.31) give the assertion (3.29). ■

Resuming Corollary 3.3 and Theorem 3.4 we must admit that the model (3.21) gives an example of non-extensive (i.e. type III) BE condensation $\rho^I_0 (\theta) = \rho - \rho^I_c (\theta)$. Namely, for $\rho > \rho^I_c (\theta) = \rho^P_c (\theta)$ one has

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{H^I_\Lambda} \left(\beta, \mu^I_\Lambda \left(\theta, \rho > \rho^I_c (\theta)\right)\right) = 0, \ k \in \Lambda^*, \tag{3.32}$$

where $\mu^I_\Lambda (\theta, \rho)$ is defined by equation $\rho = \rho^I_\Lambda (\beta, \mu)$, and at the same time

$$\begin{aligned} \rho &= \lim_{\Lambda} \rho^I_\Lambda \left(\beta, \mu^I_\Lambda \left(\theta, \rho > \rho^I_c (\theta)\right)\right) \\ &= \rho^I_c (\theta) + \lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^* :\, 0 \leq \|k\| \leq \delta\}} \langle N_k \rangle_{H^I_\Lambda} \left(\beta, \mu^I_\Lambda \left(\theta, \rho > \rho^I_c (\theta)\right)\right) . \end{aligned} \tag{3.33}$$

Remark 3.5 *Returning back to the Bogoliubov theory we again recognize that there is a problem with the hypotheses (a) and (b) about the one-mode (i.e. type I) BE condensation. Even if it exists in PBG, a very gentle interaction (e.g. (3.21)), which does not change the pressure of the PBG, may drastically change the asymptotics of the occupation number expectations, cf. (3.28), (3.29). In the model (3.21) the interaction spreads out the type I condensation in one mode $k = 0$ into the type III BE condensation, which means that the Bogoliubov hypothesis (b) is a priori doubtful.*

The perturbation of the model (3.21) is a function of only occupation-number operators, i.e. this interaction commutes with the kinetic-energy operator. This class of interactions is called *diagonal*. In the series of papers [54–57, 142], van den Berg, Dorlas, Lewis and Pulè gave an exhausting analysis of a hole class of *diagonal models* (including the exact solution of the Huang-Yang-Luttinger model (4.8)) showing how subtle and various may be the BE condensation in presence of this type of interaction. Since the interaction is diagonal, their analysis is based on a refined Large Deviation Principle of the classical probability. If one accepts the first Bogoliubov ansatz, one faces with a more complicated *nondiagonal* interaction (2.53), which does not commute with T_Λ.

Therefore, the existence of the $k = 0$ mode BE condensation in the Bogoliubov WIBG (2.50) is screaming for a rigorous analysis.

3.2 Quantum fluctuations and the second Bogoliubov ansatz

Let us accept the *first* Bogoliubov ansatz, i.e. the *truncated* Hamiltonian (2.50). Then, anticipating that the BE condensation *persists* in the same mode $k = 0$ as for the PBG, Bogoliubov proposes to pass to the *approximating* Hamiltonian (2.60) with a judiciously chosen c-numbers instead of operators $a_0^\#/\sqrt{V}$. It is the *second* Bogoliubov ansatz.

As we mentioned in Section 2.4, this ansatz is a basis of theories with a *uniform* condensate, which is considered as a *classical particle reservoir* creating and absorbing bosons with $k \neq 0$, i.e. as a *classical external field.* Therefore, the Bogoliubov approximation $a_0^\#/\sqrt{V} \to c^\#$ *eliminates quantum fluctuations* in the mode $k = 0$, or quantum fluctuations of the eventual condensate. Mathematically this means the "approximation" of the unbounded operators $a_0^\#/\sqrt{V}$ by the following bounded operators $Ic^\#$.

Again, a rigorous analysis of the truncated Hamiltonian (2.50) requires a study of the rôle of these quantum fluctuations and the proof (or disproof) of this ansatz. It is well-known in Quantum Mechanics that nondiagonal (indirect) interactions are the source of effective diagonal interactions. A standard example in Quantum Many-Body theory is the effective (indirect) electron-electron interaction in the BCS theory which results from the electron-phonon (nondiagonal) interaction in the second order of perturbation theory, see e.g. [128, 143]. This is a typical exchange interaction due to quantum fluctuations of the background lattice (phonons).

Since the truncated Hamiltonian (2.50) contains a nondiagonal interaction U_Λ^{BN} of the $k = 0$ mode subsystem with bosons with $k \neq 0$ (see Figure 2.3):

$$U_\Lambda^{BN} = \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} v\left(k\right) \left(a_k^* a_{-k}^* a_0^2 + a_0^{*^2} a_k a_{-k}\right), \tag{3.34}$$

there is an *effective direct* interaction of bosons with $k \neq 0$ due to *transformations* of a couple of bosons with $k = 0$ into a couple with k, $-k$, and similarity an effective direct interaction of bosons in the mode $k = 0$, see [13].

The simplest way to calculate the corresponding two effective coupling constants $g_{\Lambda,pq}$ and $g_{\Lambda,00}$ is to use the Fröhlich transformation originally proposed to deduce the BCS model of

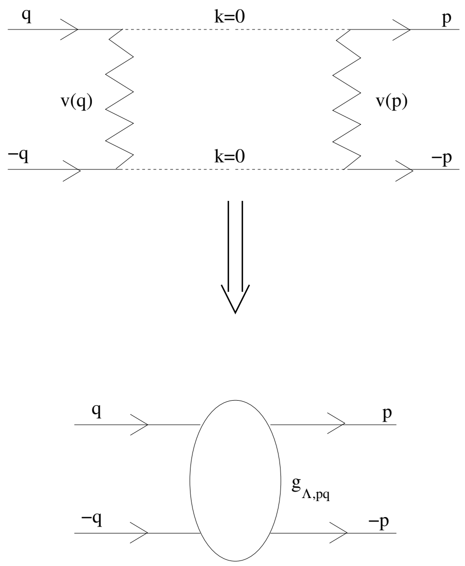

Figure 3.1: Effective interaction in modes $k \neq 0$.

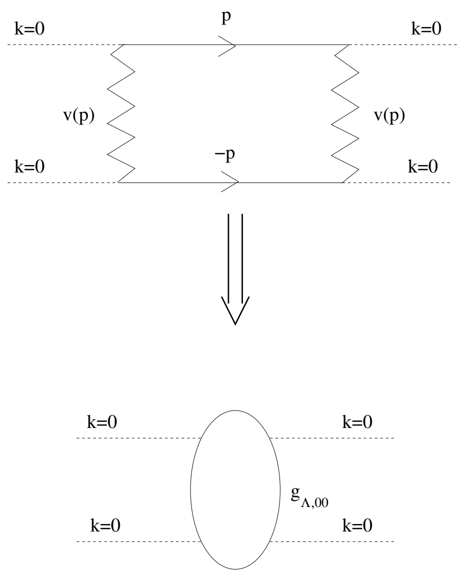

Figure 3.2: Effective interaction in the mode $k = 0$.

superconductivity, see e.g. [143]. This transformation consists of a unitary transformation of a Hamiltonian H

$$\widetilde{H} = e^{-iS} H e^{iS}, \tag{3.35}$$

with $S = S^*$ a self-adjoint operator. By developing e^{iS} and e^{-iS}, one obtains commutator series:

$$\widetilde{H} = H + i\,[H,S] - \frac{1}{2}\,[[H,S]\,,S] + ... \tag{3.36}$$

In the case of the Bogoliubov Hamiltonian $H^B_\Lambda(\mu)$, we have to search for such operator S that the nondiagonal part U^{BN}_Λ will be canceled, giving instead the terms of kind $(g_{\Lambda,00}/V)\, a_0^* a_0^* a_0 a_0$ and $(g_{\Lambda,pq}/V)\, a_p^* a_{-p}^* a_q a_{-q}$. To this end, we define the self-adjoint operator S by

$$S = \sum_{k\in\Lambda^*\setminus\{0\}} \left(\Phi(k) a_k^* a_{-k}^* a_0^2 + \overline{\Phi(k)} a_0^{*^2} a_k a_{-k} \right), \tag{3.37}$$

where $\Phi(k)$ is determined in order to cancel U^{BN}_Λ. Thus, by analogy with perturbation theory, to evaluate $g_{\Lambda,00}$ and $g_{\Lambda,kq}$, we have to calculate (3.36) to the second order in $v(k)$.
Therefore, we have

$$\begin{aligned} \widetilde{H}^B_\Lambda &= e^{-iS} H^B_\Lambda(\mu)\, e^{iS} = \widetilde{H}^B_{\Lambda,1} + \widetilde{H}^B_{\Lambda,2} + ... = \\ &= H^B_\Lambda(\mu) + i\left[H^B_\Lambda(\mu)\,,S\right] - \frac{1}{2}\left[\left[H^B_\Lambda(\mu)\,,S\right],S\right] + ..., \end{aligned} \tag{3.38}$$

and the first order term in $v(k)$ is equal to

$$\widetilde{H}^B_{\Lambda,1} = U^{BD}_\Lambda + U^{BN}_\Lambda + i\,[T_\Lambda(\mu)\,,S]\,. \tag{3.39}$$

Thus, to calculate $\Phi(k)$ we get the equation

$$U^{BN}_\Lambda + i\,[T_\Lambda(\mu)\,,S] = 0. \tag{3.40}$$

Straightforward calculation of commutator gives

$$i\,[T_\Lambda(\mu)\,,S] = 2i \sum_{k\in\Lambda^*\setminus\{0\}} \left\{ \varepsilon_k \Phi(k) a_{-k}^* a_k^* a_0^2 - \varepsilon_k \overline{\Phi(k)} a_0^{*^2} a_{-k} a_k \right\},$$

and so, we have to define $\Phi(k)$ by

$$\Phi(k) \equiv \frac{iv(k)}{4V\varepsilon_k}, \tag{3.41}$$

for every $k \in \Lambda^*\setminus\{0\}$. By (3.38) and (3.41), the second order term in $v(k)$ is equal (see (3.38)) to

$$\widetilde{H}^B_{\Lambda,2} = i\left[U^{BD}_\Lambda + U^{BN}_\Lambda + \frac{i}{2}\,[T_\Lambda(\mu)\,,S]\,,S \right]. \tag{3.42}$$

By virtue of (3.40) we find that

$$U^{BN}_\Lambda + \frac{i}{2}\,[T_\Lambda(\mu)\,,S] = \frac{1}{2} U^{BN}_\Lambda,$$

hence

$$\widetilde{H}^{B}_{\Lambda,2} = i\left[U^{BD}_{\Lambda}, S\right] + \frac{i}{2}\left[U^{BN}_{\Lambda}, S\right]. \tag{3.43}$$

After some straightforward calculations, one finds that $i\left[U^{BD}_{\Lambda}, S\right]$ does not give any term of kind $(g_{\Lambda,00}/V)\, a_0^* a_0^* a_0 a_0$ and $(g_{\Lambda,pq}/V)\, a_p^* a_{-p}^* a_q a_{-q}$ for $p,q \in \Lambda^*, p,q \neq 0$, which is not surprising. Indeed, one can remark that vertices corresponding to U^{BD}_{Λ} and S can not produce vertices of the kind $(g_{\Lambda,00}/V)\, a_0^* a_0^* a_0 a_0$ or $(g_{\Lambda,pq}/V)\, a_p^* a_{-p}^* a_q a_{-q}$. Moreover, a direct calculation of the second term of (3.43) gives:

$$\frac{i}{2}\left[U^{BN}_{\Lambda}, S\right] = \sum_{k,q\in\Lambda^*\backslash\{0\}} \frac{iv(q)}{4V}\left(\Phi(k)\left[a_0^{*^2} a_q a_{-q}, a_k^* a_{-k}^* a_0^2\right] + \overline{\Phi(k)}\left[a_{-q}^* a_q^* a_0^2, a_0^{*^2} a_{-k} a_k\right]\right) \tag{3.44}$$

with

$$\begin{aligned}\left[a_0^{*^2} a_{-q} a_q, a_k^* a_{-k}^* a_0^2\right] &= 2a_0^{*^2} a_0^2\left(1 + a_{-k}^* a_{-k} + a_k^* a_k\right)\delta_{q,k} - 2\left(1 + 2a_0^* a_0\right) a_k^* a_{-k}^* a_{-q} a_q \\ &= -\left[a_{-q}^* a_q^* a_0^2, a_0^{*^2} a_{-k} a_k\right]^*. \end{aligned}\tag{3.45}$$

Therefore, by (3.44) and (3.45) we get the effective interaction of bosons in the mode $k=0$ of the form:

$$\begin{aligned}\widetilde{H}^{B,I}_{\Lambda,2} &= -\sum_{k\in\Lambda^*\backslash\{0\}} \frac{[v(k)]^2}{8V^2\varepsilon_k}\left(a_0^{*^2} a_0^2 + a_0^2 a_0^{*^2}\right) \\ &= -\frac{a_0^{*^2} a_0^2}{V}\sum_{k\in\Lambda^*\backslash\{0\}} \frac{[v(k)]^2}{4V\varepsilon_k} - \frac{a_0^* a_0}{V}\sum_{k\in\Lambda^*\backslash\{0\}} \frac{[v(k)]^2}{2V\varepsilon_k} - \sum_{k\in\Lambda^*\backslash\{0\}} \frac{[v(k)]^2}{4V^2\varepsilon_k}. \end{aligned}\tag{3.46}$$

Thus, the coupling constant $g_{\Lambda,00}$ corresponding to the effective vertex of Figure 3.2 is equal to

$$g_{\Lambda,00} = -\frac{1}{V}\sum_{k\in\Lambda^*\backslash\{0\}} \frac{[v(k)]^2}{4\varepsilon_k}, \tag{3.47}$$

and

$$g_{00} \equiv \lim_{\Lambda} g_{\Lambda,00} = -\frac{1}{4(2\pi)^3}\int_{\mathbb{R}^3} d^3k\, \frac{[v(k)]^2}{\varepsilon_k} < 0. \tag{3.48}$$

On the other hand, the commutator (3.44) contains also the following terms

$$\widetilde{H}^{B,II}_{\Lambda,2} = \frac{1}{V}\sum_{k,q\in\Lambda^*\backslash\{0\}} \frac{v(k)v(q)}{8}\left(\frac{1}{\varepsilon_k} + \frac{1}{\varepsilon_q}\right)\frac{(1 + 2a_0^* a_0)}{V} a_k^* a_{-k}^* a_{-q} a_q. \tag{3.49}$$

It is known that in order to find the effective BCS-interaction between two electrons due to the phonons exchange, one has to project the result of the Fröhlich transformation on the state which is "empty" for phonons, i.e. on the fundamental phonon-state, see e.g. [143]. Since effective interaction between bosons with $k \neq 0$ is due to an exchange via condensate, we project $\widetilde{H}_{\Lambda,2}^{B,II}$ on the coherent state for the mode $k=0$, i.e. on the state $\psi = \psi_0(c) \otimes \psi'$ where

$$\psi_{0\Lambda}(c) \equiv e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V}c\right)^k (a_0^*)^k \Omega_0 \in \mathcal{F}_{B0},$$

see Appendix B, and $\psi' \in \mathcal{F}'_B$. Notice that $a_0 \psi_{0\Lambda}(c) = c\psi_{0\Lambda}(c)$.

We denote by $\mathcal{H}_0$ the one-dimensional Hilbert space spanned by $\psi_0(x) = 1/\sqrt{V}$ (the one particle constant function corresponding to the mode $k=0$, see Section 2.1) and by $\mathcal{H}_0^{\perp}$ the orthogonal complement ($\mathcal{H} = L^2(\Lambda) = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$), then $\mathcal{F}_{B0} \equiv \mathcal{F}_B(\mathcal{H}_0)$ and $\mathcal{F}'_B \equiv \mathcal{F}_B\left(\mathcal{H}_0^{\perp}\right)$ are the boson Fock space constructed on $\mathcal{H}_0$ and $\mathcal{H}_0^{\perp}$ respectively, see Section 2.1. Notice that the boson Fock space $\mathcal{F}_B(\mathcal{H})$ is naturally isomorphic to the tensor product $\mathcal{F}_B(\mathcal{H}_0) \otimes \mathcal{F}_B\left(\mathcal{H}_0^{\perp}\right)$, see Appendix B.
Then

$$\left(\psi_0(c), \frac{a_0^* a_0}{V}\psi_0(c)\right)_{\mathcal{F}_B} = |c|^2,$$

and the operator of the effective interaction due to the coherent condensate state is defined by the bilinear form :

$$\left(\psi_1, \widetilde{H}_{\Lambda,2}^{B,II}\psi_2\right)_{\mathcal{F}_B} = \frac{1}{V} \sum_{p,q \in \Lambda^* \backslash \{0\}} \frac{v(p)v(q)}{8} \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q}\right) \frac{\left(1+2|c|^2\right)}{V} \left(\psi'_1, a_p^* a_{-p}^* a_{-q} a_q \psi'_2\right)_{\mathcal{F}'_B}.$$

Hence, we obtain that the coupling constant $g_{\Lambda,pq}$, which corresponds to effective interaction presented on Figure 3.1 is equal to

$$g_{\Lambda,pq} = \frac{1}{8} v(p)v(q) \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q}\right) \frac{\left(1+2|c|^2\right)}{V}. \tag{3.50}$$

If we suppose that

$$\lim_{\Lambda} \frac{1}{V} \langle a_0^* a_0 \rangle_{H_\Lambda^B} \left(\beta = \theta^{-1}, \mu\right) = \rho_0^B(\theta, \mu), \tag{3.51}$$

is the density of the condensate in the mode $k=0$ for the Gibbs state defined by H_Λ^B. Then we get that the effective interaction between particles outside the mode $k=0$ is proportional to the density (3.51) :

$$g_{pq} \equiv \lim_{\Lambda} g_{\Lambda,pq} = \frac{v(p)v(q)}{4} \rho_0^B(\theta,\mu) \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q}\right) \geq 0. \tag{3.52}$$

Notice that if $\rho_0^B(\theta,\mu) \neq 0$, then repulsion $g_{pq} \to +\infty$ for $q \to 0$ or $p \to 0$.

Thus, the nondiagonal part U_Λ^{BN} of the Bogoliubov Hamiltonian implies an effective *attraction* between particles in the mode $k=0$ and effective *repulsion* of particles outside the condensate (mode $k \neq 0$). To estimate the interaction of particles in the mode $k=0$ for the

WIBG, one has to take into account besides (3.46) a *direct* repulsion corresponding to the first term in the diagonal part U_Λ^{BD} (2.52). Thus, we have to evaluate the sign of

$$\left(\frac{v(0)}{2}+g_{00}\right)=\frac{v(0)}{2}-\frac{1}{4\left(2\pi\right)^3}\int_{\mathbb{R}^3}d^3k\frac{\left[v(k)\right]^2}{\varepsilon_k}. \tag{3.53}$$

Therefore, the inequality

$$\left(\frac{v(0)}{2}+g_{00}\right)<0 \tag{3.54}$$

gives a sufficient condition that effective *attraction* prevails in zero-mode the bare repulsive term in U_Λ^{BD}.

Remark 3.6 *The calculations above show that even in the case, when the BE condensation persists the interaction $U_\Lambda^{BD}+U_\Lambda^{BN}$ in the WIBG (2.50), the quantum fluctuations of the condensate are not negligible. They are responsible for effective attraction of bosons in the zero-mode. Since the Bose statistics allows accumulation of particles, this attraction may be a source of instability of the WIBG or a new kind of condensation. In fact, this is proven in [15]. Therefore, the second Bogoliubov ansatz missed these phenomena caused by quantum fluctuations.*

3.3 Instability of the Weakly Imperfect Bose-Gas

We recall that a "judicious" choice of the constant $c^{\#}$ in the second Bogoliubov ansatz corresponds to the variational principal (2.71) or to the first-order perturbation calculation (2.107), which give the same relation $\mu=v\left(0\right)\left|c\right|^2\geq 0$. So, if the BE condensation in $k=0$ do *exist,* one must get $\mu>0$. The aim of this section is to show that this inequality is in a *conflict* with *stability* of the Bogoliubov Hamiltonian for the WIBG (2.50), see Angelescu, Verbeure and Zagrebnov [9]. First we prove the following assertion:

Theorem 3.7 *Let $H_\Lambda^{BD}=T_\Lambda+U_\Lambda^{BD}$ be diagonal part of the Bogoliubov Hamiltonian (2.50). Then*
(a) for any $\mu\leq 0$ one has

$$p^{BD}\left(\beta,\mu\right)=\lim_{\Lambda}p_\Lambda\left[H_\Lambda^{BD}\right]=p^P\left(\beta,\mu\right), \tag{3.55}$$

with $p^P\left(\beta,\mu\right)$ (2.26) corresponding to the PBG pressure,
(b) for $\mu>0$ one has

$$p^{BD}\left(\beta,\mu\right)=+\infty. \tag{3.56}$$

Proof. By definition of the Hamiltonian $H_\Lambda^{BD}=T_\Lambda+U_\Lambda^{BD}$ (2.50) and $v\left(k\right)\geq 0$ one gets

$$H_\Lambda^{BD}=T_\Lambda+\frac{v\left(0\right)}{V}N_0N_\Lambda-\frac{v\left(0\right)}{2V}\left(N_0^2+N_0\right)+\frac{1}{V}\sum_{k\in\Lambda^*\setminus\{0\}}v\left(k\right)N_0N_k\geq T_\Lambda. \tag{3.57}$$

Therefore, by direct calculation of the $Tr_{\mathcal{F}_B}$ we deduce that

$$p_\Lambda^{BD}(\beta,\mu) = p_\Lambda\left[H_\Lambda^{BD}\right] = \frac{1}{\beta V}\ln Tr_{\mathcal{F}_B} e^{-\beta\left(H_\Lambda^{BD}-\mu N_\Lambda\right)} = \frac{1}{\beta V}\ln \Xi_\Lambda^{BD}(\beta,\mu), \tag{3.58}$$

with

$$\Xi_\Lambda^{BD}(\beta,\mu) = \sum_{n_0=0}^{\infty}\left\{ e^{-\beta\left(\frac{v(0)}{2V}\left(n_0^2-n_0\right)-\mu n_0\right)} \prod_{k\in\Lambda^*\setminus\{0\}} \left(1-e^{-\beta\left(\varepsilon_k-\mu+\frac{[v(0)+v(k)]}{V}n_0\right)}\right)^{-1}\right\}. \tag{3.59}$$

Hence the two last equalities imply

$$p_\Lambda^{BD}(\beta,\mu) \geq \widetilde{p}_\Lambda^{P}(\beta,\mu) \equiv \frac{1}{\beta V}\sum_{k\in\Lambda^*\setminus\{0\}} \ln\left[\left(1-e^{[-\beta(\varepsilon_k-\mu)]}\right)^{-1}\right]. \tag{3.60}$$

Note that $\widetilde{p}_\Lambda^{P}(\beta,\mu)$ is the pressure of a Perfect Bose-Gas with *excluded* mode $k=0$ and $\widetilde{p}_\Lambda^{P}(\beta,\mu) < +\infty$ *only* for $\mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}$. Therefore by (3.59),

$$\begin{aligned} &p_\Lambda^{BD}(\beta,\mu) < +\infty \text{ for } \mu < \inf_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k = \varepsilon_{\|k\|=\frac{2\pi}{L}}, \\ &\lim_{\mu\to\varepsilon_{\|k\|=\frac{2\pi}{L}}} p_\Lambda^{BD}(\beta,\mu) = +\infty. \end{aligned} \tag{3.61}$$

Moreover the estimate (3.57) implies a second inequality

$$p_\Lambda\left[H_\Lambda^{BD}\right] \leq p_\Lambda\left[T_\Lambda\right] = \frac{1}{\beta V}\sum_{k\in\Lambda^*} \ln\left[\left(1-e^{[-\beta(\varepsilon_k-\mu)]}\right)^{-1}\right], \tag{3.62}$$

for $\mu < 0$. Since for $\mu < 0$ one has

$$\lim_\Lambda \widetilde{p}_\Lambda^{P}(\beta,\mu) = \lim_\Lambda p_\Lambda\left[T_\Lambda\right] = p^P(\beta,\mu), \tag{3.63}$$

in the thermodynamic limit, (3.60)-(3.63) imply (3.55) for $\mu < 0$. Then taking limit $\mu \to 0^-$ one gets (3.55) for $\mu = 0$. Moreover, inequality (3.60) implies in the thermodynamic limit (3.56). ■

Now we can prove that the WIBG exists only for $\mu \leq 0$.

Theorem 3.8 *For any $\mu \leq 0$ one has*

$$p^B(\beta,\mu) = \lim_\Lambda p_\Lambda\left[H_\Lambda^B\right] \geq p^P(\beta,\mu), \tag{3.64}$$

whereas for $\mu > 0$

$$p^B(\beta,\mu) = +\infty. \tag{3.65}$$

Proof. By the Bogoliubov inequality (Appendix D), we know that for any $\mu \leq 0$,

$$\frac{1}{V}\left\langle U_{\Lambda}^{BN}\right\rangle_{H_{\Lambda}^{B}} \leq p_{\Lambda}\left[H_{\Lambda}^{BD}\right] - p_{\Lambda}\left[H_{\Lambda}^{B}\right] \leq \frac{1}{V}\left\langle U_{\Lambda}^{BN}\right\rangle_{H_{\Lambda}^{BD}} = 0, \tag{3.66}$$

(the latter because U_{Λ}^{BN} is nondiagonal whereas H_{Λ}^{BD} is diagonal). Then by Theorem 3.7 and (3.66) we obtain (3.64) in the thermodynamic limit. The divergence (3.65) comes directly from (3.61) and (3.66). ■

Remark 3.9 *The last observation shows that only the trivial choice $c = 0$ in the second Bogoliubov ansatz is compatible with thermodynamic properties of the WIBG, which is unstable for $\mu > 0$ by (3.65).*

In fact one can say more about the WIBG for $\mu \leq 0$. The next statement is due to [9].

Theorem 3.10 *If $\mu \leq -1/2\varphi(0) \equiv \mu_*$, then the pressure of the WIBG coincides with the pressure of the PBG:*

$$p^{B}(\beta, \mu \leq \mu_*) = p^{P}(\beta, \mu \leq \mu_*). \tag{3.67}$$

Proof. Regrouping terms in (2.50) one gets

$$H_{\Lambda}^{B} = \widetilde{H}_{\Lambda} + \frac{1}{2V}\sum_{k \in \Lambda^* \backslash \{0\}} v(k)\left(a_0^* a_k + a_{-k}^* a_0\right)^*\left(a_0^* a_k + a_{-k}^* a_0\right) \geq \widetilde{H}_{\Lambda}, \tag{3.68}$$

where

$$\widetilde{H}_{\Lambda} = \sum_{k \in \Lambda^* \backslash \{0\}}\left(\varepsilon_k - \frac{v(k)}{2V} + \frac{v(0)}{V}N_0\right)N_k + \frac{v(0)}{2V}N_0^2 - \frac{1}{2}\varphi(0)N_0. \tag{3.69}$$

Hence, by the Bogoliubov inequality for (3.68) and for its diagonal part (3.69) (see Appendix D) we obtain in the thermodynamic limit for $\mu \leq 0$

$$p^{B}(\beta, \mu) \leq \lim_{\Lambda} p_{\Lambda}\left[\widetilde{H}_{\Lambda}\right] = \sup_{\rho_0 \geq 0} G(\beta, \mu; \rho_0). \tag{3.70}$$

Here

$$G(\beta, \mu; \rho_0) \equiv \left[-\frac{v(0)}{2}\rho_0^2 + \left(\mu + \frac{1}{2}\varphi(0)\right)\rho_0 + p^{P}(\beta, \mu - v(0)\rho_0)\right]. \tag{3.71}$$

If $\mu \leq -\frac{1}{2}\varphi(0)$, then $\sup_{\rho_0 \geq 0} G(\beta, \mu; \rho_0) = p^{P}(\beta, \mu)$. Therefore, by Theorem 3.8 and inequality (3.70) we get

$$p^{B}(\beta, \mu) = p^{P}(\beta, \mu), \text{ for } \left(\theta = \beta^{-1}, \mu\right) \in \Gamma_{\mu_*}^{B} = \left\{\theta \geq 0,\ \mu \leq -\frac{1}{2}\varphi(0) \equiv \mu_*\right\}. \tag{3.72}$$

■

So, the properties of the WIBG for very negative chemical potential ($\mu \leq -\mu_*$) are in some sense trivial. The interaction U_{Λ}^{B} is not able to change them with respect to the PBG, cf. the model (3.21) of Section 3.1. The question of behaviour of the WIBG in the interval $[\mu_*, 0]$ is less trivial. It got answer in our papers [12, 14–16]. This will be discussed in Sections 4 and 5.

3.4 From the Weakly Imperfect Bose-Gas to the Bogoliubov theory

Resuming the observations of the Sections 3.1-3.3, one has to admit that in various respects the Bogoliubov model of the WIBG is not very appropriate as the model of superfluidity. In the next Sections 4 and 5 we discuss an *exact solution* of the WIBG manifesting properties rather *far* from that follow from the Bogoliubov theory. This obviously means that the second Bogoliubov ansatz gives a solution of something *other* than the Bogoliubov WIBG.

The *first* main feature of the second Bogoliubov ansatz is that the Bogoliubov approximation $a_0^{\#}/\sqrt{V} \to c^{\#}$ *transfers* the system $H_\Lambda^B(\mu)$, see (2.69), which is *stable* only for $\mu \le 0$ (Theorem 3.8) into the system $H_\Lambda^B(c^{\#}, \mu)$, see (2.70), which is stable for $\mu \le |c|^2 v(0)$. The *second* feature is the choice $\mu = |c|^2 v(0)$ (2.72) coming from the variational principle only for the Landau-type terms in the variational energy, see (2.71), which is exactly the first order perturbation theory, cf. (2.106), (2.107). An attempt to go *beyond* this approximation, staying in the framework of the Hamiltonian (2.70), disbalances the relation (2.72) or (2.107), and leads to the *gap* in the spectrum of elementary excitations, see Section 2.4.

Notice that the *exact* solution of the WIBG also shows a *gap* in the spectrum of quasi-particles in a *condensed* phase. The nature of this gap hidden in the *nonlocal* interaction U_Λ^B (2.50) and, by consequence, in the violation of the local gauge invariance, which implies the violation of the local particle-number conservation, i.e. the continuity equation, sum rules etc, see [81] and discussions in Sections 2.4 and 6.1. Therefore, the difficulties with the *gap* in the Bogoliubov theory in higher approximations are "preprogrammed" by the *first* ansatz which leads to the Hamiltonian of the WIBG (2.50).

In a sense the Bogoliubov theory is a series of *recipes*, which, after the *first* ansatz about *truncated* Hamiltonian of the WIBG, give a formula (the *second* ansatz) saving the theory from *instability* for $\mu > 0$ and from the *gap* in the spectrum. If one accepts the *model with a condensate* (2.99) (i.e. the Bogoliubov approximation in the full Hamiltonian), then the consistent way out is the Beliaev-Hugenholtz-Pines-Gavoret-Nozières-Popov theory discussed in Section 2.4. On the other hand, since this is a kind of a self-consistent theory (without condensate fluctuations), it is alluring to try, first, to *stabilize* the WIBG and then to study the application of the Bogoliubov approximation and variational principle, i.e. of the second ansatz.

This approach was developed in papers [9–11] for a "minimal" stabilization of the Bogoliubov Hamiltonian (2.50) by adding the "forward scattering" repulsive interactions between particles above zero-mode:

$$H_\Lambda^{SB} \equiv H_\Lambda^B + \frac{v(0)}{2V} \sum_{k,k' \in \Lambda^* \backslash \{0\}} a_k^* a_{k'}^* a_{k'} a_k. \qquad (3.73)$$

The Hamiltonian (3.73) can be rewritten in the form

$$H_\Lambda^{SB} = T_\Lambda + U_\Lambda^{BN} + \frac{1}{V} \sum_{k \in \Lambda^* \backslash \{0\}} v(k)\, a_0^* a_0 a_k^* a_k + \frac{v(0)}{2V} N_\Lambda (N_\Lambda - I). \qquad (3.74)$$

Since

$$\begin{aligned}a_k^* a_{-k}^* a_0 a_0 + a_0^* a_0^* a_k a_{-k} + a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k}\right) &= \left(a_0^* a_k + a_{-k}^* a_0\right)^* \left(a_0^* a_k + a_{-k}^* a_0\right) \\ -a_k^* a_k - a_0^* a_0 &\geq -N_k - N_0,\end{aligned}$$

and $\varepsilon_k \geq 0$, $v(0) \geq v(k) \geq 0$, (see *hypothesis* (B), Section 2.2), one gets

$$\begin{aligned}H_\Lambda^{SB} &\geq T_\Lambda - \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} v(k) N_k - \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} v(k) N_0 + \frac{v(0)}{2V} N_\Lambda (N_\Lambda - I) \geq \\ &\geq T_\Lambda - \frac{1}{V} v(0) N_\Lambda - \frac{1}{2}\varphi(0) N_\Lambda + \frac{v(0)}{2V} N_\Lambda^2 \geq T_\Lambda - B N_\Lambda + \frac{A}{V} N_\Lambda^2 \qquad (3.75)\end{aligned}$$

for $B = v(0)/V + \frac{1}{2}\varphi(0)$ and $A = v(0)/2 > 0$. Then the Hamiltonian is *superstable* [91] (see also Appendix G), i.e. the grand-partition function converges for any $\mu \in \mathbb{R}^1$:

$$\Xi_\Lambda^{SB}(\beta,\mu) \equiv Tr_{\mathcal{F}_B}\left(e^{-\beta\left(\widetilde{H}_\Lambda^{SB} - \mu N_\Lambda\right)}\right) \leq \sum_{n=0}^{\infty} e^{-\beta\left[\frac{A}{V}n^2 - (B+\mu)n\right]} Tr_{\mathcal{H}_B^{(n)}}\left(e^{-\beta T_\Lambda^{(n)}}\right).$$

This implies that the pressure of the system (3.73):

$$\begin{aligned}p^{SB}(\beta,\mu) &= \lim_\Lambda p_\Lambda\left[H_\Lambda^{SB}\right] = \lim_\Lambda \frac{1}{\beta V} \ln \Xi_\Lambda^{SB}(\beta,\mu) \leq \\ &\leq \lim_\Lambda \frac{1}{\beta V} \ln \sum_{n=0}^{\infty} e^{-\beta\left[\frac{A}{V}n^2 - (B+\mu)n + f_\Lambda^P\left(\beta,\frac{n}{V}\right)\right]} = \\ &= \sup_{\rho\geq 0}\left[(B+\mu)\rho + f^P(\beta,\rho) - A\rho^2\right],\end{aligned}$$

exists for any $\mu \in \mathbb{R}^1$. Here $f_\Lambda^P(\beta,\rho) = f_\Lambda[T_\Lambda]$ is the free-energy density of the PBG ($\lim_\Lambda f_\Lambda^P(\beta,\rho) = f^P(\beta,\rho)$).

The next step is to introduce the Bogoliubov approximation of the $\widetilde{H}_\Lambda^{SB}$ in the following generalized form [11]:

$$\begin{aligned}H_\Lambda^{SB}(b) &= \sum_{k\in\Lambda^*\setminus\{0\}} \left[\varepsilon_k + |b|^2 v(k)\right] a_k^* a_k + \frac{1}{2} \sum_{k\in\Lambda^*\setminus\{0\}} v(k)\left[b^2 a_k^* a_{-k}^* + \overline{b}^2 a_{-k} a_k\right] \\ &\quad + \frac{v(0)}{2V} N_\Lambda (N_\Lambda - I), \qquad (3.76)\end{aligned}$$

where the parameter $|b|$ is related to the square root of the condensate density $|c|$ via a suitably chosen function:

$$|b| = g(|c|). \qquad (3.77)$$

Deviation of $g(x)$ from simple x is supposed to take into account the *quantum fluctuations* of the condensate. The variational principle (2.71) is substituted in [11] by a variational principle for the trial Hamiltonian $H_\Lambda^{SB}(b)$ on the quasi-free Gibbs states, see Appendix F for details.

The results can be formulated as follows. The model $H_{\Lambda}^{SB}(b)$ with self-consistency equation (3.77) for e.g.

$$g(|c|) = |c| \left(\frac{\lambda |c|}{1 + \lambda |c|} \right)^{1/2}, \ \lambda > 0, \tag{3.78}$$

is showing BE condensation in zero-mode via second-order phase transition and a linear asymptotics of the elementary excitation spectrum in condensed phase for $\|k\| \to 0$.

Quantum fluctuations in the model [11] have been taken into account on some phenomenological level via "form-factor" $g(|c|)$, which reflects possible effective interactions due to fluctuations of condensate. That is why it would be important to have a model where it is taken into account exactly and rigorously as it is done in Sections 4 and 5.

Section 4

Exact solution of the Weakly Imperfect Bose-Gas: pressure

"If we are to make progress with the problem of the existence of boson condensation in interacting systems, we must have a good conceptual understanding of existing results."
Lewis, *Why do bosons condense?*

The Section 3 highlights different problems of the standard Bogoliubov theory based on a truncation and the Bogoliubov approximation (see Section 2). One of the problem of this theory is a drastic difference between the WIBG and a caricature implied by the Bogoliubov approximation. An essential point is that in the Bogoliubov theory the gapless spectrum occurs for the *positive* chemical potential $\mu = v(0)|c|^2 > 0$, where the system corresponding to the WIBG is unstable (Theorem 3.8), i.e. where the corresponding grand-canonical pressure diverges in the thermodynamic limit.

The aim of this section is twofold: *first*, to show that the WIBG is less trivial than it is expressed by [9] and that its properties drastically depend on the interaction potential (2.42), which may imply a *nonconventional* Bose condensation in the zero-mode; *second*, to calculate exactly the corresponding WIBG pressure in domain where it does not coincide (cf. Theorem 3.10) with the PBG pressure. In the next Section 4.1, we study the thermodynamic behaviour of the diagonal part of the WIBG, which, in fact, manifests only one *conventional (generalized)* BE condensation. This means in particular that the thermodynamic *non-equivalence* between the WIBG and PBG is due to *nondiagonal* terms of interaction (2.53). In Section 4.2, we show a nontriviality of the WIBG in the sense that its pressure may not coincide with the PBG pressure (see condition (C) in Definition 4.5) in some domain of (θ, μ). In particular we shall see that this domain of nontriviality of the WIBG corresponds to existence of the Bose condensation in the zero-mode. Finally, Section 4.3 concerns the explicit calculation of the thermodynamic limit of the WIBG pressure whereas essential proofs of this section are collected in the last Section 4.4.

4.1 Diagonal part of the Bogoliubov model

4.1.1 Diagonal models and H_Λ^{BD}

Nowadays, the Bose-gas (2.44) with the full interaction

$$U_\Lambda \equiv \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} v(q) a^*_{k_1-q} a^*_{k_2+q} a_{k_2} a_{k_1}, \tag{4.1}$$

seems still beyond the reach of present methods of mathematical physics. Then to proceed (as we mentioned in Section 1) one must either use a very particular two-body potential $\varphi(x)$, or truncate the interaction (4.1).

There are in fact only few examples, when the full Hamiltonian (2.44) is solved on the *thermodynamic level* even for very special two-body interactions. Lieb and Liniger [131] diagonalized the full Hamiltonian (2.43), (2.44) for a *δ-function* pair potential in *one dimension* using the Bethe ansatz. Yang and Yang [144] used this result to calculate the pressure in this model, but the proof was given only in 1989 by Dorlas, Lewis, and Pulè [142]. Another example is to consider, instead of a *bona fide* two-body potential $\varphi(x)$, a scaled potential $\varphi_a(x) = a^d\varphi(x)$, $a > 0$, to calculate observables e.g. the pressure $p(\beta,\mu)$ in the *van der Waals limit*: $p^{VdW}(\beta,\mu) = \lim_{a\to 0^+}\lim_{\Lambda} p_\Lambda[T_\Lambda + U_\Lambda^a]$. This limit corresponds to a system with very long-range interaction of the mean-field type. This model was studied exactly by Lebowitz, Penrose [145] and Lieb [146]. In [147] the van der Waals limit of the interacting Bose-gas H_Λ^a is considered in a scaled external field. There it was shown that this model is not so simple as one could imagine on the first glance. For example the behaviour of the BE condensate in the model is *different* than in the Mean-Field (MF) Hamiltonian H_Λ^{MF} in the scaled external field. Here H_Λ^{MF} corresponds to a constant two-body potential in the box Λ:

$$\varphi^{MF}(x) = \frac{1}{V}\varphi_0\chi_\Lambda(x) = \left\{ \begin{array}{c} \varphi_0/V, \text{ for } x\in\Lambda \\ 0, \text{ for } x\notin\Lambda \end{array} \right\}, \tag{4.2}$$

i.e. $\varphi^{MF}_{\text{per}}(x) = \varphi^{MF}(x)$, $x\in\Lambda$. Therefore, by (2.42) one gets for $q\in\Lambda^*$:

$$v(q) = \varphi_0\delta_{q,0} = \left\{ \begin{array}{c} \varphi_0, \text{ for } q=0 \\ 0, \text{ for } q\neq 0 \end{array} \right\}. \tag{4.3}$$

Hence, $\varphi_0 = v(0)$ and (4.1) takes the form

$$U_\Lambda^{MF} \equiv \frac{v(0)}{2V} \sum_{k_1,k_2\in\Lambda^*} a^*_{k_1} a^*_{k_2} a_{k_2} a_{k_1} = \frac{v(0)}{2V}\left(N_\Lambda^2 - N_\Lambda\right). \tag{4.4}$$

Since the last term in (4.4) has order $o(V)$, the thermodynamic (bulk) properties of the model $H_\Lambda^{MF} = T_\Lambda + U_\Lambda^{MF}$ coincide with those described by the Hamiltonian

$$H_\Lambda^{IBG} = T_\Lambda + \frac{v(0)}{2V} N_\Lambda^2, \tag{4.5}$$

known as the *Imperfect Bose-Gas* (IBG) model [30]. The model H_Λ^{MF} (or H_Λ^{IBG}) is the *first* of the series of *diagonal models*, which one gets from the full Hamiltonian by *truncation* of the interaction (4.1). The interaction (4.4) corresponds to cut-off in (4.1) the terms with $q \neq 0$, i.e. to the "forward scattering" interaction.

The class of *diagonal models* is singled out by the property that the *interaction* $U_\Lambda^{(\cdot)}$ is a function of *only* the free-gas occupation-number operators $\{N_k\}_{k\in\Lambda^*}$. This implies $\left[N_k, U_\Lambda^{(\cdot)}\right] = 0$, $k \in \Lambda^*$ (the local *gauge invariance*) and, in particular, $\left[T_\Lambda, U_\Lambda^{(\cdot)}\right] = 0$. The *first* means that Hamiltonian $H_\Lambda^{(\cdot)} = T_\Lambda + U_\Lambda^{(\cdot)}$ conserves the number of particles in each mode:

$$e^{i\varphi N_k} H_\Lambda^{(\cdot)} e^{-i\varphi N_k} = H_\Lambda^{(\cdot)}, \ k \in \Lambda^*. \tag{4.6}$$

Whereas the *second* allows to consider $H_\Lambda^{(\cdot)}$ in the proper basis of the occupation-number operators $\left\{\Psi_{\{n_k\}}\right\}_{\{n_k\}}$, see (2.12) and (2.13), which *maps* the *quantum* problem in the Fock space $\mathcal{F}_B(\mathcal{H})$ on the *classical probability* problem for occupation numbers $\{n_k = 0, 1, 2, ...\}_{k\in\Lambda^*}$ as *random variable*, see e.g. (2.14).

This was an approach developed by the "Dublin school" in 1988-1993. It gives a beautiful application of the Varadhan large deviation principle to the occupation measure to solve a class of *diagonal* boson models. *Truncating* the full interaction (4.1) they distinguish four of them:

$$\begin{aligned}
H_\Lambda^{MF} &= T_\Lambda + U_\Lambda^{MF}, && (4.7)\\
H_\Lambda^{HYL} &= H_\Lambda^{MF} + \frac{v(0)}{2V}\left(N_\Lambda^2 - \sum_{k\in\Lambda^*} N_k^2\right), && (4.8)\\
H_\Lambda^{PMF} &= T_\Lambda + \frac{1}{2V}\sum_{k,k'\in\Lambda^*} v(k'-k) N_k N_{k'}, && (4.9)\\
H_\Lambda^{FD} &= H_\Lambda^{PMF} + \frac{v(0)}{2V}\left(N_\Lambda^2 - \sum_{k\in\Lambda^*} N_k^2\right), && (4.10)
\end{aligned}$$

here $v(0) > 0$.

The first of this models has been studied exhaustively in [148–153]. It is of interest because the pathological aspects of the PBG are removed by the interaction (4.4) or (4.5), without destroying the BE condensation and creating the gap in the spectrum (see discussion in Sections 2.4 and 6.1). Since interactions (4.4), (4.5) are superstable [91] (see Appendix G):

$$H_\Lambda^{MF} \geq -BN_\Lambda + A\frac{N_\Lambda^2}{V},$$

the grand-canonical partition function converges for all chemical potentials, i.e. in contrast to the PBG the pressure

$$p_\Lambda\left[H_\Lambda^{MF}\right]=\frac{1}{\beta V}\ln\sum_{n=0}^{\infty}e^{\beta\left(\left(\mu-\frac{v(0)}{2V}\right)n+\frac{v(0)}{V}n^2\right)}Z_\Lambda^P\left(\beta,n\right), \tag{4.11}$$

(where $Z_\Lambda^P\left(\beta,n\right)$ is the canonical partition function of the PBG) exists for *all* $\mu\in\mathbb{R}^1$ (*domain of stability*) even in the thermodynamic limit. To solve the model (4.7) on the *thermodynamic level*, i.e. to calculate pressure or free-energy density in the *thermodynamic* (bulk) *limit*, one has to remark that the interaction U_Λ^{MF} is simply a *constant* in the *canonical ensemble* for a fixed total number of particles. Therefore, the corresponding *free-energy density* is simply

$$f^{MF}\left(\beta,\rho\right)=\lim_{\Lambda}f_\Lambda\left[H_\Lambda^{MF}\right]=f^P\left(\beta,\rho\right)+\frac{v\left(0\right)}{2}\rho^2. \tag{4.12}$$

Since $v\left(0\right)>0$, the function (4.12) is strictly convex for $\rho\geq 0$. Then by the Laplace large deviation principle one gets from (4.11) and (4.12) that

$$p^{MF}\left(\beta,\rho\right)=\lim_{\Lambda}p_\Lambda\left[H_\Lambda^{MF}\right]=\sup_{\rho\geq 0}\left\{\mu\rho-f^P\left(\beta,\rho\right)-\frac{v\left(0\right)}{2}\rho^2\right\}, \tag{4.13}$$

for any $\mu\in\mathbb{R}^1$. Notice that by (4.12) the canonical chemical potential is equal to:

$$\mu^{MF}\left(\theta,\rho\right)=\partial_\rho f^{MF}\left(\beta,\rho\right)=\left\{\begin{array}{l}\partial_\rho f^P\left(\beta,\rho\right)+v\left(0\right)\rho,\text{ for }\rho\leq\rho_c^P\left(\theta\right)\\ v\left(0\right)\rho,\text{ for }\rho>\rho_c^P\left(\theta\right)\end{array}\right\}, \tag{4.14}$$

where $\rho_c^P\left(\theta\right)=\rho^P\left(\beta,\mu=0\right)$ is the critical density for the PBG (2.32). Hence there is a critical chemical potential:

$$\mu_c^{MF}\left(\theta\right)\equiv\mu^{MF}\left(\theta,\rho_c^P\left(\theta\right)\right)=v\left(0\right)\rho_c^P\left(\theta\right)>0. \tag{4.15}$$

It is less trivial to prove that the BE condensation *persists* in the mean-field model (4.7), see [149, 150, 153]. It is not destroyed by the interaction (4.4) and it is described by the equation:

$$\rho_0^{MF}\left(\theta,\mu^{MF}\left(\theta,\rho\right)\right)=\left\{\begin{array}{l}0,\text{ for }\rho\leq\rho_c^P\left(\theta\right)\\ \rho-\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}d^3k\left(e^{\beta\left(\varepsilon_k-\mu^{MF}(\theta,\rho)+v(0)\rho\right)}-1\right)^2,\text{ for }\rho>\rho_c^P\left(\theta\right)\end{array}\right\}, \tag{4.16}$$

where chemical potential for given density is *uniquely* defined by (4.14). Notice that the BE condensation in (4.16) results from the double limit, see [150, 153]:

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0\leq\|k\|\leq\delta\}}\left\langle N_k\right\rangle_{H_\Lambda^{MF}}=\rho_0^{MF}\left(\theta,\mu\right), \tag{4.17}$$

cf. Section 3.1. Our toy model (3.21) shows that repulsion between bosons occupying the same levels may spread out the BE condensation of type I into type III (see Appendix A). The interaction U_Λ^{MF} (4.4) includes the both: repulsions of bosons on the same levels as well as repulsions between bosons on different levels. Therefore, the question of type of the BE condensation (4.17) needs a complementary analysis, cf. [149,150]. Remark that relation (4.15) for the critical chemical potential $\mu_c^{MF}(\theta)$ coincides with (2.107), (2.110) in the second Bogoliubov ansatz, although $\rho_0^{MF}\left(\theta, \mu_c^{MF}(\theta)\right) = 0$.

The next model (4.8) is motivated by the hard-sphere boson gas. It was introduced by Huang, Yang, and Luttinger in [53], then discussed in [30] and [52]. To separate (4.8) as a truncation of the full interaction (4.1), notice that U_Λ can be write as a sum $U_\Lambda^{FD} + U_\Lambda^{OD}$, where

$$U_\Lambda^{FD} \equiv \frac{v(0)}{2V}\left(N_\Lambda^2 - N_\Lambda\right) + \frac{1}{2V}\sum_{\{k_1,k_2\in\Lambda^*:k_1\neq k_2\}} v(k_2-k_1)N_{k_1}N_{k_2}, \tag{4.18}$$

is the *full diagonal part* of (4.1), and

$$U_\Lambda^{OD} \equiv \frac{1}{2V}\sum_{k_1,k_2\in\Lambda^*}\sum_{\{q\in\Lambda^*:q\neq 0,q\neq k_1-k_2\}} v(q)a_{k_1-q}^*a_{k_2+q}^*a_{k_2}a_{k_1}, \tag{4.19}$$

is the *off-diagonal part*.

Now, the two body interaction $\varphi(x)$ is of a very short-range, e.g. $\varphi(x) \to a\delta(x)$, then the Fourier transform (2.42) gives:

$$v(q) = a = v(0) > 0. \tag{4.20}$$

Therefore, the full diagonal interaction (4.18) takes the form

$$U_\Lambda^{HYL} = U_\Lambda^{MF} + \frac{v(0)}{2V}\left(N_\Lambda^2 - \sum_{k\in\Lambda^*} N_k^2\right), \tag{4.21}$$

which leads to (4.8): $H_\Lambda^{HYL} = T_\Lambda + U_\Lambda^{HYL}$.

According to Bogoliubov and Zubarev [1,154,155], the interaction U_Λ^{OD} (4.19) which is not local gauge invariant, i.e. does not conserve the number of particles in modes, should deplete the BE condensation manifested by the PBG in the mode $k=0$. Their estimates in the first approximation for the interaction U_Λ^{BN} (2.50) showed that it is of the order $\sim (\rho v(0))^{3/2}$. See Section 2.4 and [84,156] for further development of ideas of [154,155]. This asymptotics means that the Bogoliubov theory is non-analytic in parameter $\rho v(0)$, but it is still consistent for $\rho v(0) << 1$, see discussion in Section 2. On the other hand, in [53], and especially in [52], it was argued that the negative sign in the interaction (4.21) (in contrast to our toy model (3.21) where this sign is positive) may to enhance the BE condensate in the mode $k=0$, and even to give rise to a jump discontinuity in the amount of condensate as the chemical potential increases. The first rigorous study of the H_Λ^{HYL} model was given in [54,56]. The proof uses a large deviation result for the occupation measure associated with the pressure functional. It verifies the prediction of Thouless [52] about the *jump* in the density of condensate at critical value of the chemical potential (or density). Another peculiarity of the Huang-Yang-Luttinger (HYL) model is that (in contrast to the PBG or the MF model where there is no condensation for $d \leq 2$) it manifests condensation for *any* $d > 0$! These properties show that the condensate in the HYL model has a different nature than the BE condensation in the PBG or in the MF model. In latter cases (as well as in the toy model (3.21)) the condensation is due to the *saturation* of density of bosons outside any band $\|k\| > \delta$, see Section 2 and 3. There is no such a saturation for $d \leq 2$, that is why one has no conventional BE condensation in these models for $d \leq 2$. Whereas in HYL model it is competition between MF *repulsion* and *attraction* (see the last term in (4.21)) which leads to condensation independent of dimensionality. We call this *kind* of condensation, induced by *attraction* mechanism, a *nonconventional* Bose condensation, see Appendix A. The HYL model is the first model where this new kind of condensation was discovered. Another one is the (nondiagonal!) Bogoliubov model of the WIBG which also manifests the *nonconventional* condensation, see Section 5.

The *Perturbed Mean-Field* (PMF) model (4.9) was studied in [55,157], where the pressure and the BE condensate behaviour were calculated explicitly for various kernels $v(k' - k)$.

The importance of the *Full Diagonal* (FD) model (4.10) with interaction U_Λ^{FD} (4.18) is that for a large class of interaction kernels $v(k' - k) \geq 0$, and for $v\,(0)$ substituted by $a > v\,(0)$, the *Thouless effect* (i.e. the *nonconventional* condensation) persists; for details we address the reader to the paper [57].

In our recent papers [58,59] we study new *diagonal* models which manifests the two kinds of condensations. They occur in *two* stages: for intermediate densities one has a *nonconventional* Bose condensation in the lowest mode $k = 0$, whereas for large densities this condensation coexists with conventional (*generalized*) type III BE condensation. The nature of the *nonconventional* condensation in [58, 59] is related to a *gap* in the spectrum, i.e. to the interaction

$$W_\Lambda^0 = \varepsilon_0 N_0 + \frac{v\,(0)}{2V} a_0^* a_0^* a_0 a_0, \;\; \varepsilon_0 < 0, \;\; v\,(0) > 0. \tag{4.22}$$

At the first time this rôle of the artificial gap was discovered in [93]. The interaction ensuring the non-extensive (i.e. the type III) BE condensation on the second stage, coincides with our toy model interaction (3.21), for $k \neq 0$. So, the Hamiltonian has the form

$$H_\Lambda^{II} = T_\Lambda + \left\{ \varepsilon_0 N_0 + \frac{v\,(0)}{2V} a_0^* a_0^* a_0 a_0 \right\} + \frac{v\,(0)}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} a_k^* a_k^* a_k a_k. \tag{4.23}$$

Therefore, in spite of extreme simplicity of the diagonal Hamiltonian H_Λ^{II}, the exact solution shows a rather nontrivial behaviour of the Bose-system (4.23), see [58, 59].

This motivates an importance to study the *diagonal* part $H_\Lambda^{BD} = \left(T_\Lambda + U_\Lambda^{BD}\right)$ (2.51), (2.52) of the Bogoliubov WIBG *before* we pass to the full Bogoliubov Hamiltonian H_Λ^B (2.50). The pressure associated with

$$\begin{aligned} H_\Lambda^{BD} \;\; &= \;\; T_\Lambda + U_\Lambda^{BD} = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{v\,(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^* \setminus \{0\}} a_k^* a_k \\ &+ \frac{1}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} v\,(k)\, a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k}\right) + \frac{v\,(0)}{2V} a_0^{*^2} a_0^2, \end{aligned} \tag{4.24}$$

is calculated in Section 3.3, Theorem 3.7. We recall that in the thermodynamic limit it coincides with that for the PBG:

$$p^{BD}(\beta,\mu) = \lim_{\Lambda} p_{\Lambda}\left[H_{\Lambda}^{BD}\right] = \lim_{\Lambda} p_{\Lambda}^{BD}(\beta,\mu) = p^{P}(\beta,\mu). \tag{4.25}$$

Therefore, there is no difference between the models H_{Λ}^{BD} and T_{Λ} on this thermodynamic level.

4.1.2 Conventional (generalized) Bose-Einstein condensation

Let us consider the total particle density

$$\rho_{\Lambda}^{BD}(\beta,\mu) \equiv \left\langle \frac{N_{\Lambda}}{V} \right\rangle_{H_{\Lambda}^{BD}} (\beta,\mu) = \partial_{\mu} p_{\Lambda}^{BD}(\beta,\mu). \tag{4.26}$$

We recall that $\langle - \rangle_{H_{\Lambda}^{(\cdot)}}(\beta,\mu)$ denotes the Gibbs state associated with a Hamiltonian $H_{\Lambda}^{(\cdot)}$ in the grand canonical ensemble.

Theorem 4.1 *Let $v(k)$ satisfy hypotheses (A) and (B) (Section 2.2) Then the total particle density (4.26) in the thermodynamic limit:*

$$\rho^{BD}(\beta,\mu) \equiv \lim_{\Lambda} \rho_{\Lambda}^{BD}(\beta,\mu)$$

coincides with that for the PBG (cf. (2.26)):

$$\rho^{BD}(\beta,\mu \leq 0) = \rho^{P}(\beta,\mu \leq 0). \tag{4.27}$$

Proof. Since functions $\left\{p_{\Lambda}^{BD}(\beta,\mu)\right\}_{\Lambda\subset\mathbb{R}^3}$ (3.58) are convex for $\mu \leq 0$ and the limit $p^{P}(\beta,\mu)$ (2.26) is differentiable for $\mu < 0$, by the Griffiths lemma (Appendix C) and by (4.26) we have

$$\lim_{\Lambda} \rho_{\Lambda}^{BD}(\beta,\mu) = \lim_{\Lambda} \partial_{\mu} p_{\Lambda}^{BD}(\beta,\mu) = \partial_{\mu} p^{P}(\beta,\mu) = \rho^{P}(\beta,\mu),$$

i.e. (4.27) for $\mu < 0$. Taking in (4.27) the limit $\mu \to 0^{-}$ we extend this equality to $\mu \in (-\infty, 0]$. ■

Remark 4.2 *The convexity of the pressure $p_{\Lambda}^{BD}(\beta,\mu)$ (3.58) for $0 < \mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}$ (3.61) implies*

$$\frac{p_{\Lambda}^{BD}(\beta,\mu) - p_{\Lambda}^{BD}(\beta,0)}{\mu} \leq \partial_{\mu} p_{\Lambda}^{BD}(\beta,\mu) = \rho_{\Lambda}^{BD}(\beta,\mu).$$

Therefore from (3.61) we have

$$p_{\Lambda}^{BD}\left(\beta,\mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}\right) < +\infty, \;\; \rho_{\Lambda}^{BD}\left(\beta,\mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}\right) < +\infty,$$

whereas

$$\lim_{\mu\to\varepsilon_{\|k\|=\frac{2\pi}{L}}} p_{\Lambda}^{BD}(\beta,\mu) = +\infty, \quad \lim_{\mu\to\varepsilon_{\|k\|=\frac{2\pi}{L}}} \rho_{\Lambda}^{BD}(\beta,\mu) = +\infty. \tag{4.28}$$

¿From (4.25) and (4.27), one gets that the diagonal part of the WIBG H_{Λ}^{BD} is *thermodynamically* equivalent to the PBG in the sense that their pressure and particle densities coincide in the thermodynamic limit. In particular the diagonal interaction U_{Λ}^{BD} (2.52) is negligible in this limit, with respect to the kinetic-energy part T_{Λ}.
By (4.27), we have (for $d = 3$) a critical density $\rho_c^{BD}(\theta)$ which is equal to $\rho_c^{P}(\theta)$ (2.32):

$$\rho_c^{BD}(\theta) = \rho_c^{P}(\theta) \equiv \rho^{P}\left(\theta^{-1}, \mu = 0\right) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \left(e^{\varepsilon_k/\theta} - 1\right)^{-1} d^3k < \infty. \tag{4.29}$$

By virtue of Remark 4.2, for any particle density ρ there is a chemical potential

$$\mu_{\Lambda}^{BD}(\theta, \rho) < \varepsilon_{\|k\|=\frac{2\pi}{L}}, \tag{4.30}$$

which is a solution of the equation:

$$\rho_{\Lambda}^{BD}(\beta, \mu) = \rho. \tag{4.31}$$

Since $\varepsilon_{\|k\|=\frac{2\pi}{L}} > 0$, then $\mu_{\Lambda}^{BD}(\theta, \rho) \gtrless 0$, but in the thermodynamic limit by (4.30) one gets

$$\lim_{\Lambda} \mu_{\Lambda}^{BD}(\theta, \rho) = 0.$$

Therefore, similar to the PBG (Section 2.1), for $\rho > \rho_c^{BD}(\theta) = \rho_c^{P}(\theta)$ one has to have a *conventional* BE condensation due to a saturation mechanism indicated by the existence of $\rho_c^{BD}(\theta)$ (4.29) (cf. Appendix A). To prove this result we establish first the following estimate for the mean-value of occupation numbers:

Lemma 4.3 *Let interaction $v(k)$ satisfy hypotheses (A) and (B) (Section 2.2). Then for $\|k\| > 2\pi/L$ we have*

$$\langle N_k \rangle_{H_{\Lambda}^{BD}} \left(\beta, \mu_{\Lambda}^{BD}(\theta, \rho)\right) \leq \frac{1}{e^{\beta\left(\varepsilon_k - \mu_{\Lambda}^{BD}(\theta,\rho)\right)} - 1}. \tag{4.32}$$

Proof. Using the correlation inequalities for the quantum Gibbs state $\omega_{\Lambda}^{BD}(-) \equiv \langle - \rangle_{H_{\Lambda}^{BD}}(\beta, \mu)$ (see Appendix E):

$$\beta\omega_{\Lambda}^{BD}\left(X^* \left[H_{\Lambda}^{BD}(\mu), X\right]\right) \geq \omega_{\Lambda}^{BD}(X^*X) \ln \frac{\omega_{\Lambda}^{BD}(X^*X)}{\omega_{\Lambda}^{BD}(XX^*)}, \tag{4.33}$$

where X is an observable from domain of the commutator $\left[H_{\Lambda}^{BD}(\mu), .\right]$ and $H_{\Lambda}^{BD}(\mu) = H_{\Lambda}^{BD} - \mu N_{\Lambda}$, we obtain

$$\beta\omega_{\Lambda}^{BD}\left(a_k^* \left[H_{\Lambda}^{BD}(\mu), a_k\right]\right) \geq \omega_{\Lambda}^{BD}(N_k) \ln \frac{\omega_{\Lambda}^{BD}(N_k)}{\omega_{\Lambda}^{BD}(N_k) + 1}, \tag{4.34}$$

for $X = a_k$ and any $\mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}$ (see Remark 4.2). Since for $\|k\| > 2\pi/L$

$$\left[H_{\Lambda}^{BD}(\mu), a_k\right] = -\left(\varepsilon_k - \mu + [v(0) + v(k)] \frac{a_0^* a_0}{V}\right) a_k,$$

by $v(k) \geq 0$ one gets that

$$\omega_\Lambda^{BD}\left(a_k^*\left[H_\Lambda^{BD}(\mu), a_k\right]\right) \leq -(\varepsilon_k - \mu)\,\omega_\Lambda^{BD}(N_k). \tag{4.35}$$

Therefore, the inequalities (4.34) and (4.35) imply

$$\beta(\varepsilon_k - \mu)\,\omega_\Lambda^{BD}(N_k) \leq \omega_\Lambda^{BD}(N_k)\ln\frac{\omega_\Lambda^{BD}(N_k)+1}{\omega_\Lambda^{BD}(N_k)}, \tag{4.36}$$

Since $\mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}$ and $\|k\| > 2\pi/L$, one has $\varepsilon_k > \mu$. Then, to estimate $x \equiv \omega_\Lambda^{BD}(N_k) = \langle N_k\rangle_{H_\Lambda^{BD}}(\beta,\mu)$, we have to solve the inequality

$$\beta(\varepsilon_k - \mu)\,x \leq x\ln\frac{x+1}{x}, \tag{4.37}$$

for $x \geq 0$. A nontrivial solution of the equation

$$\beta(\varepsilon_k - \mu)\,x = x\ln\frac{x+1}{x}$$

is

$$x_1 = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}, \tag{4.38}$$

for any $\mu < \varepsilon_{\|k\|=\frac{2\pi}{L}}$. Therefore $x \equiv \omega_\Lambda^{BD}(N_k) \leq x_1$ solved (4.37) that implies (4.32) for $\|k\| > 2\pi/L$ and $\mu = \mu_\Lambda^{BD}(\theta,\rho) < \varepsilon_{\|k\|=\frac{2\pi}{L}}$ (cf. (4.30)). ■

Theorem 4.4 *Let $v(k)$ satisfy hypotheses (A) and (B) (Section 2.2). Then for $\rho > \rho_c^{BD}(\theta) = \rho_c^P(\theta)$ (4.29):*
(i) we have:

$$\lim_\Lambda \left\langle \frac{N_0}{V}\right\rangle_{H_\Lambda^{BD}}\left(\beta, \mu_\Lambda^{BD}(\theta,\rho)\right) = 0, \tag{4.39}$$

and the same for any fixed k with $\|k\| > \frac{2\pi}{L}$, i.e.

$$\lim_\Lambda \left\langle \frac{N_k}{V}\right\rangle_{H_\Lambda^{BD}}\left(\beta, \mu_\Lambda^{BD}(\theta,\rho)\right) = 0; \tag{4.40}$$

(ii) whereas one has:

$$\lim_\Lambda \frac{1}{V}\sum_{\left\{k\in\Lambda^*, \|k\|=\frac{2\pi}{L}\right\}} \langle N_k\rangle_{H_\Lambda^{BD}}\left(\beta, \mu_\Lambda^{BD}(\theta,\rho)\right) = \rho - \rho_c^{BD}(\theta) = \rho - \rho_c^P(\theta) > 0, \tag{4.41}$$

i.e. a (generalized) conventional BE condensation of type I (cf. Appendix A).

Proof. (i) In order to control the expectation in the mode $k=0$ we introduce an auxiliary Hamiltonian

$$H^{BD}_{\Lambda,\gamma_0} \equiv H^{BD}_{\Lambda} - \gamma_0 a^*_0 a_0,$$

and we set

$$p^{BD}_{\Lambda}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho), \gamma_0\right) \equiv p_{\Lambda}\left[H^{BD}_{\Lambda,\gamma_0}\right] \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}_B} e^{-\beta\left(H^{BD}_{\Lambda,\gamma_0} - \mu^{BD}_{\Lambda}(\theta,\rho)N_{\Lambda}\right)},$$

where $\mu^{BD}_{\Lambda}(\theta,\rho)$ is defined by equation (4.31). Similar to (3.59) we calculate the explicit form of $p^{BD}_{\Lambda}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho), \gamma_0\right)$. By $v(0) > 0$ it exists for every $\gamma_0 \in \mathbb{R}^1$. From this explicit form we obtain in the thermodynamic limit that

$$\lim_{\Lambda} p^{BD}_{\Lambda,\gamma_0}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho), \gamma_0\right) = p^P(\beta, 0), \tag{4.42}$$

for any γ_0. Since

$$\left\langle \frac{N_0}{V} \right\rangle_{H^{BD}_{\Lambda,\gamma_0}} \left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right) = -\partial_{\gamma_0} p^{BD}_{\Lambda}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho), \gamma_0\right),$$

and $\left\{p^{BD}_{\Lambda}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho), \gamma_0\right)\right\}_{\Lambda}$ is a set of convex functions of $\gamma_0 \in \mathbb{R}^1$, then combining (4.42) with the Griffiths lemma (Appendix C) we deduce:

$$\lim_{\Lambda} \left\langle \frac{N_0}{V} \right\rangle_{H^{BD}_{\Lambda,\gamma_0}} \left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right) = \partial_{\gamma_0} p^P(\beta, 0) = 0,\ \ \gamma_0 \in \mathbb{R}^1, \tag{4.43}$$

which proves (4.39) when we put $\gamma_0 = 0$.
Since $\|k\| > \frac{2\pi}{L}$, the thermodynamic limit (4.40) is a direct consequence of Lemma 4.3.
(ii) For a fixed total density ρ we have the identity:

$$\begin{aligned}\frac{1}{V}\sum_{\left\{k\in\Lambda^*, \|k\| = \frac{2\pi}{L}\right\}} \langle N_k\rangle_{H^{BD}_{\Lambda}}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right) \;&=\; \rho - \frac{1}{V}\sum_{\left\{k\in\Lambda^* : \|k\| > \frac{2\pi}{L}\right\}} \langle N_k\rangle_{H^{BD}_{\Lambda}}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right) \\ &\quad - \left\langle \frac{N_0}{V}\right\rangle_{H^{BD}_{\Lambda}}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right).\end{aligned} \tag{4.44}$$

Now, by Lemma 4.3 and (4.39), (4.44) we find for (4.41) the estimate from below:

$$\lim_{\Lambda} \frac{1}{V}\sum_{\left\{k\in\Lambda^*, \|k\| = \frac{2\pi}{L}\right\}} \langle N_k\rangle_{H^{BD}_{\Lambda}}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right) \geq \rho - \rho^{BD}_c(\theta), \tag{4.45}$$

where $\rho^{BD}_c(\theta) = \rho^P_c(\theta)$ is defined by (4.29). To obtain an estimate from above we use that

$$\lim_{\Lambda} \frac{1}{V}\sum_{\left\{k\in\Lambda^*, \|k\| = \frac{2\pi}{L}\right\}} \langle N_k\rangle_{H^{BD}_{\Lambda}}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right) \leq \lim_{\Lambda} \frac{1}{V}\sum_{\{k\in\Lambda^*, 0<\|k\|\leq\delta\}} \langle N_k\rangle_{H^{BD}_{\Lambda}}\left(\beta, \mu^{BD}_{\Lambda}(\theta,\rho)\right), \tag{4.46}$$

and

$$\begin{aligned}\frac{1}{V}\sum_{\{k\in\Lambda^*,0<\|k\|\leq\delta\}}\langle N_k\rangle_{H_\Lambda^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right) &= \rho-\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|>\delta\}}\langle N_k\rangle_{H_\Lambda^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right)\\ &\quad-\left\langle\frac{N_0}{V}\right\rangle_{H_\Lambda^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right).\end{aligned}\tag{4.47}$$

Now we introduce the auxiliary Hamiltonian

$$H_{\Lambda,\gamma}^{BD}=H_\Lambda^{BD}-\gamma\sum_{\{k\in\Lambda^*:\|k\|>\delta>0\}}a_k^*a_k,$$

where $\gamma<\varepsilon_\delta=\hbar^2\delta^2/2m$. Let

$$p_\Lambda^{BD}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho),\gamma\right)\equiv p_\Lambda\left[H_{\Lambda,\gamma}^{BD}\right]$$

be the pressure corresponding to $H_{\Lambda,\gamma}^{BD}$ for $\mu=\mu_\Lambda^{BD}(\theta,\rho)$. We can calculate the explicit form of $p_\Lambda^{BD}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho),\gamma\right)$ and find that

$$\lim_\Lambda p_\Lambda^{BD}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho),\gamma\right)=p^P(\beta,0,\gamma)\tag{4.48}$$

where $p^P(\beta,0,\gamma)$ is the pressure of the PBG with a new free-particle spectrum: $\varepsilon_k\to\varepsilon_k-\gamma\chi_{(\delta,+\infty)}(\|k\|)$ for $\gamma\in(-\infty,\varepsilon_\delta)$. Here $\chi_{(a,b)}(\|k\|)$ is the characteristic function of interval $(a,b)\subset\mathbb{R}^1$. Notice that $\left\{p_\Lambda^{BD}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho),\gamma\right)\right\}_\Lambda$ is a set of convex functions of $\gamma\in(-\infty,\varepsilon_\delta)$ and

$$\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|>\delta\}}\langle N_k\rangle_{H_{\Lambda,\gamma}^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right)=\partial_\gamma p_\Lambda^{BD}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho),\gamma\right).$$

Therefore, using the Griffiths lemma (Appendix C) from (4.48) we deduce for $\gamma=0$ that

$$\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|>\delta\}}\langle N_k\rangle_{H_\Lambda^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right)=\frac{1}{(2\pi)^3}\int_{\|k\|>\delta}\frac{d^3k}{(e^{\beta\varepsilon_k}-1)}.\tag{4.49}$$

By (4.43) and (4.49), the identity (4.47) implies

$$\lim_{\delta\to0^+}\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*,0<\|k\|\leq\delta\}}\langle N_k\rangle_{H_\Lambda^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right)=\rho-\rho_c^{BD}(\theta),$$

(cf. (4.29)). Therefore, by (4.46) we deduce

$$\lim_\Lambda\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\langle N_k\rangle_{H_\Lambda^{BD}}\left(\beta,\mu_\Lambda^{BD}(\theta,\rho)\right)\leq\rho-\rho_c^{BD}(\theta),$$

which combining with (4.45) implies (4.41). ■

In fact, by (4.25) and by Theorem 4.4 the Bose-system (4.24) coincides with a PBG with *excluded* mode $k = 0$. Indeed for any fixed density ρ, the ground state configuration is defined by the energy

$$\mathcal{E}_{\inf}^{BD} \equiv \inf_{\{n_k\}_{k\in\Lambda^*}} \left(\Psi^{(n)}_{\{n_k\}}, H_\Lambda^{BD}\Psi^{(n)}_{\{n_k\}}\right)_{\mathcal{H}_B^{(n)}}$$

directly deduced from (3.57). Then for a system of $n = \rho V = \rho L^3$ particles, let us consider the two following cases:

(i) All particles are accumulated in the mode $k = 0$ (2.47). Then $n = \rho V = n_0$ and the corresponding energy of the system is equal to (see (3.57)):

$$\mathcal{E}_0^{BD} = \frac{L^3 v(0)}{2}\rho^2 - \frac{v(0)\rho}{2}. \tag{4.50}$$

(ii) All particles are on the first $2d = 6$ modes k with kinetic energy $\varepsilon_{\|k\|=\frac{2\pi}{L}}$, i.e. the first modes next to the zero-mode are occupied. So $n = \rho V = n_{\|k\|=\frac{2\pi}{L}}$ and the corresponding energy of the system equals (see (3.57))

$$\mathcal{E}_{\|k\|=\frac{2\pi}{L}}^{BD} = \varepsilon_{\|k\|=\frac{2\pi}{L}}\rho V = \frac{L\left(2\pi\hbar\right)^2\rho}{2m}. \tag{4.51}$$

Since the monotonicity of ε_k implies

$$\mathcal{E}_{\inf}^{BD} = \min\left\{\mathcal{E}_0^{BD}, \mathcal{E}_{\|k\|=\frac{2\pi}{L}}^{BD}\right\},$$

for a sufficiently large volume $V = L^3$ one gets by (4.50) and (4.51):

$$\frac{L\left(2\pi\hbar\right)^2\rho}{2m} < \frac{L^3 v(0)\rho^2}{2} - \frac{v(0)\rho}{2}. \tag{4.52}$$

Hence we obtain that

$$\mathcal{E}_{\inf}^{BD} = \mathcal{E}_{\|k\|=\frac{2\pi}{L}}^{BD},$$

i.e. in the thermodynamic limit a *conventional* BE condensation should appear in modes $\{k \in \Lambda^*, \|k\| = 2\pi/L\}$. The reasoning above confirms what is proven in Theorem 3.7 and 4.4: the *direct repulsion*

$$\frac{v\left(0\right)}{2V} a_0^{*2} a_0^2 \tag{4.53}$$

(cf. (4.24) and Figure 2.2) suppresses the macroscopic occupation of the mode $k = 0$. Of course, in the thermodynamic limit, the diagonal part of the WIBG can not be really distinguished from the PBG with included mode $k = 0$. In the next section we show that it is the *nondiagonal* interaction (2.53) that makes the essential difference between WIBG and PBG. In particular this *nondiagonal* term (2.53) competes with the diagonal interaction (4.53) and may imply a *nonconventional* Bose condensation in mode $k = 0$.

4.2 Stability domain: (non) triviality of the Weakly Imperfect Bose-Gas and condensation

By Theorem 3.8 the stability domain of the WIBG in the grand-canonical ensemble:

$$Q^B = \left\{ (\theta, \mu) : p^B (\beta, \mu) = \lim_{\Lambda} p_\Lambda \left[H^B_\Lambda (\mu) \right] = \lim_{\Lambda} p^B_\Lambda (\beta, \mu) < +\infty \right\}, \tag{4.54}$$

is equal to

$$Q^B = Q \equiv \{ (\theta, \mu) : \theta \geq 0, \ \mu \leq 0 \} ; \tag{4.55}$$

We recall that here

$$p^B_\Lambda (\beta, \mu) \equiv p_\Lambda \left[H^B_\Lambda \right] \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_B} e^{-\beta \left(H^B_\Lambda - \mu N_\Lambda \right)} \tag{4.56}$$

is the grand-canonical pressure associated with the WIBG. Moreover the paper [9] conjectured, on the basis of the equality

$$p^B (\beta, \mu \leq \mu^*) = p^P (\beta, \mu \leq \mu^*)$$

for $\mu^* = -\varphi (0) / 2$ (see Theorem 3.10), that the WIBG is thermodynamically equivalent to the PBG for the whole domain (4.55).

The aim of the present section is to show that the thermodynamic behaviour of the WIBG less trivial and that it drastically depends on the interaction potential $v (q)$ (2.42). As a consequence, we show that the nontriviality of the WIBG is related to existence of Bose condensation in mode $k = 0$.

First we recall the two assumptions (A) and (B) on the spherically-symmetric interaction potential $\varphi (x)$ (*hypotheses* (A) and (B) of Section 2.2):
(A) the real $\varphi \in L^1 (\mathbb{R}^3)$ (absolute integrability of $\varphi (x)$ for $x \in \mathbb{R}^3$);
(B) the (real) Fourier transform of $\varphi (x)$ (2.42) satisfies: $0 \leq v (q) \leq v (0)$ for $q \in \mathbb{R}^3$ and $v (0) > 0$.
Recall that the properties of $\varphi (x)$ imply $v (q) = v (\|q\|)$. Now we formulate a supplementary condition, which will distinguish the WIBG from the PBG.

Definition 4.5 *The interaction potential $\varphi (x)$ satisfies the condition (C), if*

$$v(0) < \frac{1}{2 (2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{v(k)^2}{\varepsilon_k}, \qquad \textit{hypothesis (C).}$$

Remark 4.6 *Notice that for low dimensions $d = 1, 2$ divergence of the integral ensures the hypothesis (C) for any interaction potential $\varphi (x)$ satisfying (A) and (B). On the other hand, it is obvious that, if $\varphi (x)$ satisfies hypotheses (A) and (B), then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ (strong repulsion) the potential $\varphi_\lambda (x) = \lambda \varphi (x)$ satisfies the hypothesis (C).*

Theorem 3.8 states that $p^B(\beta,\mu) \geq p^P(\beta,\mu)$ in the stability domain Q (4.55). Motivated by this inequality we define the domain

$$D^B \equiv \left\{(\theta,\mu) \in Q : p^B(\beta,\mu) > p^P(\beta,\mu)\right\}. \tag{4.57}$$

By $\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right)$ we define the pressure associated with the approximating Hamiltonian $H_\Lambda^B\left(c^\#,\mu\right)$ (2.70):

$$\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}'_B} e^{-\beta H_\Lambda^B\left(c^\#,\mu\right)}, \tag{4.58}$$

where $\mathcal{F}'_B \equiv \mathcal{F}_B\left(\mathcal{H}_0^\perp\right)$ is the Fock space constructed on the one-particle Hilbert space $\mathcal{H}_0^\perp$ orthogonal to $\mathcal{H}_0$, spanned by the $k=0$ mode wave-functions proportional to $\psi_0(x) = 1/\sqrt{V}$ (see Appendix B).
Since the Hamiltonian $H_\Lambda^B\left(c^\#,\mu\right)$ can be diagonalized (2.79), one can calculate the pressure $\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right)$ explicitly:

$$\begin{aligned}
&\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) = \xi_\Lambda(\beta,\mu;x) + \eta_\Lambda(\mu;x),\\
&\xi_\Lambda(\beta,\mu;x) = \frac{1}{\beta V}\sum_{k\in\Lambda^*\backslash\{0\}} \ln\left(1-e^{-\beta E_k^B}\right)^{-1},\\
&\eta_\Lambda(\mu;x) = -\frac{1}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}\left(E_k^B - f_k\right) + \mu x - \tfrac{1}{2}v(0)\,x^2.
\end{aligned} \tag{4.59}$$

Here $x = |c|^2 \geq 0$ and

$$\begin{aligned}
f_k &= \varepsilon_k - \mu + x\left[v(0) + v(k)\right],\\
h_k &= x v(k),\\
E_k^B &= \sqrt{f_k^2 - h_k^2},
\end{aligned} \tag{4.60}$$

see (2.76). By Proposition B.2 of the Appendix B we have:

$$p_\Lambda^B(\beta,\mu) \geq \sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right), \text{ for } (\theta,\mu)\in Q. \tag{4.61}$$

¿From Corollary B.7 (Appendix B) we know that, if the interaction potential $v(k)$ satisfies hypotheses (A), (B) and (C) (Definition 4.5) then there are (θ,μ) such that

$$\lim_\Lambda \sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) > p^P(\beta,\mu).$$

Therefore, by (4.57) and (4.61), the domain $D^B \neq \{\varnothing\}$.
At this stage we can already extract the existence of a $k=0$ mode Bose condensation in the domain D^B, where $p^B(\beta,\mu) \neq p^P(\beta,\mu)$. This condensation is explicated in Section 5.1.

Theorem 4.7 *Let*

$$\rho_0^B(\theta,\mu) \equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B}(\beta,\mu) \tag{4.62}$$

be the particle density in the mode $k = 0$ of the WIBG (2.50) in the thermodynamic limit. Then the domain D^B coincides with:

$$D^B = \left\{ (\theta, \mu) \in Q : \ \rho_0^B(\theta, \mu) \equiv \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} (\beta, \mu) > 0 \right\}, \tag{4.63}$$

i.e. with domain of the Bose condensation existence in the mode $k = 0$.

Proof. Put

$$\widehat{H}_\Lambda^B \equiv H_\Lambda^B + \frac{1}{2}\varphi(0)\, a_0^* a_0. \tag{4.64}$$

Then by (3.70) and (3.71) we obtain

$$\lim_{\Lambda} p_\Lambda \left[\widehat{H}_\Lambda^B \right] \leq \sup_{\rho_0 \geq 0} \left\{ G(\beta, \mu; \rho_0) - \frac{1}{2}\varphi(0)\, \rho_0 \right\} = p^P(\beta, \mu). \tag{4.65}$$

By the Bogoliubov inequality for H_Λ^B and $\widehat{H}_\Lambda^B$ (see Appendix D) we find

$$p_\Lambda \left[H_\Lambda^B \right] - \frac{\varphi(0)}{2} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} (\beta, \mu) \leq p_\Lambda \left[\widehat{H}_\Lambda^B \right]. \tag{4.66}$$

By Theorem 3.8, one has

$$p^P(\beta, \mu) \leq p^B(\beta, \mu). \tag{4.67}$$

Consequently by (4.65), (4.66) and (4.67), in the thermodynamic limit one gets the estimate:

$$p^P(\beta, \mu) - \frac{\varphi(0)}{2}\rho_0^B(\theta, \mu) \leq p^B(\beta, \mu) - \frac{\varphi(0)}{2}\rho_0^B(\theta, \mu) \leq p^P(\beta, \mu).$$

Therefore $p^B(\beta, \mu) > p^P(\beta, \mu)$ (4.57) if and only if $\rho_0^B(\theta, \mu) > 0$ (4.63). ■

4.3 Theorem about exactness of the Bogoliubov approximation

Since the pressure $p^B(\beta, \mu)$ is not equal to $p^P(\beta, \mu)$ only in domain D^B (4.57), where $\rho_0^B(\theta, \mu) > 0$ (cf. Theorem 4.7), the aim of this section is to identify $p^B(\beta, \mu)$ in this domain.

Theorem 4.8 *In the thermodynamic limit the pressure of the WIBG is equal to:*

$$p^B(\beta, \mu) = \lim_{\Lambda} p_\Lambda^B(\beta, \mu) = \sup_{c \in \mathbb{C}} \widetilde{p}^B\left(\beta, \mu; c^{\#}\right) \equiv \lim_{\Lambda} \left\{ \sup_{c \in \mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta, \mu; c^{\#}\right) \right\}, \tag{4.68}$$

in the whole stability domain $Q^B = Q \equiv \{(\theta, \mu) : \theta \geq 0, \ \mu \leq 0\}$, see (4.58)-(4.60).

Proof. Notice that the equality (4.68) is true for $(\theta,\mu)\in Q\backslash D^{B}$. Indeed, since by the definition of D^{B} (4.57) and by estimate (4.61),

$$D_{0}^{B}\equiv\left\{(\theta,\mu)\in Q:\ \lim_{\Lambda}\ \sup_{c\in\mathbb{C}}\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#}\right)>p^{P}\left(\beta,\mu\right)\right\}\subseteq D^{B},$$

(cf. (B.14) and Corollary B.7 in Appendix B), for $(\theta,\mu)\in Q\backslash D^{B}$, we have

$$p^{B}\left(\beta,\mu\right)=p^{P}\left(\beta,\mu\right),$$

and, consequently

$$\sup_{c\in\mathbb{C}}\widetilde{p}^{B}\left(\beta,\mu;c^{\#}\right)=\widetilde{p}^{B}\left(\beta,\mu;0\right)=p^{P}\left(\beta,\mu\right),$$

by virtue of (4.59), (4.60).

The proof of the theorem for $(\theta,\mu)\in D^{B}$ follows the idea of paper [92]. There Ginibre proves exactness of the Bogoliubov approximation for a non-ideal Bose-gas (2.44) with superstable interaction [91] (see Appendix G). It is the case of $v\left(k\right)$ satisfying the hypothesis (B), see Appendix G. But after truncation of (2.44) the Hamiltonian H_{Λ}^{B} (2.50) for WIBG is *not* superstable: by Theorem 3.8, the system (2.50) is unstable for $\mu>0$! Hence, the essential difficulty comes from the relaxation of the superstability to the stability for $\mu\leq 0$.

Up to this difficulty (see Section 4.4 for details) the proof in domain D^{B} runs as follows: Since in the Hamiltonian $H_{\Lambda}^{B}\left(c^{\#},\mu\right)$ (2.70) the (operator) gauge symmetry is broken, we introduce the Hamiltonian $H_{\Lambda}^{B}\left(\nu^{\#}\right)$ defined by

$$\begin{aligned}H_{\Lambda}^{B}\left(\nu^{\#}\right)&=H_{\Lambda}^{B}-\sqrt{V}\left(\overline{\nu}a_{0}+\nu a_{0}^{*}\right),\\ H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)&=H_{\Lambda}^{B}\left(\nu^{\#}\right)-\mu N_{\Lambda},\end{aligned}$$

with sources $\nu\in\mathbb{C}$ breaking the symmetry of H_{Λ}^{B}, here $\nu^{\#}=\{\nu\text{ or }\overline{\nu}\}$.
Similar to (4.61) (see Proposition B.2), we have the following estimate

$$\sup_{c\in\mathbb{C}}\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\leq p_{\Lambda}\left[H_{\Lambda}^{B}\left(\nu^{\#}\right)\right],\ \text{for}\ (\theta,\mu)\in Q,\tag{4.69}$$

with

$$\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\equiv\frac{1}{\beta V}\ln Tr_{\mathcal{F}_{B}^{\prime}}e^{-\beta H_{\Lambda}^{B}\left(c^{\#},\nu^{\#},\mu\right)},$$

see (4.58)-(4.60). Combining (4.69) with the Bogoliubov inequality for $H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)$ and $H_{\Lambda}^{B}\left(c^{\#},\mu,\nu^{\#}\right)$ (Appendix D) we obtain:

$$\begin{aligned}0&\leq\Delta_{\Lambda}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\equiv p_{\Lambda}\left[H_{\Lambda}^{B}\left(\nu^{\#}\right)\right]-\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\leq\\ &\leq\frac{1}{V}\left\langle H_{\Lambda}^{B}\left(c^{\#},\mu,\nu^{\#}\right)-H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}.\end{aligned}\tag{4.70}$$

Let $A\equiv a_{0}-\sqrt{V}c$, $A^{*}\equiv a_{0}^{*}-\sqrt{V}\overline{c}$. Then by a Taylor expansion around $a_{0}^{\#}$ one gets:

$$H_{\Lambda}^{B}\left(c^{\#},\mu,\nu^{\#}\right)-H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\ =\ -A^{*}\left[a_{0},H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right]+h.c.$$

$$
\begin{aligned}
&+\frac{1}{2}A^{*^2}\left[a_0,\left[a_0,H_\Lambda^B\left(\mu,\nu^{\#}\right)\right]\right]+h.c.\\
&+A^*\left[a_0,\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),a_0^*\right]\right]A\\
&-\frac{1}{2}A^{*^2}\left[a_0,\left[a_0,\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),a_0^*\right]\right]\right]A+h.c.\\
&+\frac{1}{4}A^{*^2}\left[a_0,\left[a_0,\left[\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),a_0^*\right],a_0^*\right]\right]\right]A^2. \qquad (4.71)
\end{aligned}
$$

To finish the proof one has to estimate with help of (4.71) the upper bound (4.70) in the thermodynamic limit in the vicinity of $\nu = 0$, see Section 4.4 for details. Overcoming the problem of stability we found the following asymptotics [15]:

$$0 \leq \inf_{c\in\mathbb{C}} \Delta_\Lambda\left(\beta,\mu;c^{\#},\nu^{\#}\right) \equiv p_\Lambda\left[H_\Lambda^B\left(\nu^{\#}\right)\right] - \sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right) \leq \delta_\Lambda \equiv const\frac{1}{\sqrt{V}}, \qquad (4.72)$$

for each compact $C_0 \subset D^B$ and $|\nu| \leq r_0$ (cf. (4.144)). Put

$$p^B\left(\beta,\mu;\nu^{\#}\right) \equiv \lim_\Lambda p_\Lambda\left[H_\Lambda^B\left(\nu^{\#}\right)\right],$$

then (4.72) implies that locally uniformly in D^B, one gets

$$p^B\left(\beta,\mu;\nu^{\#}\right) = \sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu;c^{\#},\nu^{\#}\right), \text{ for } |\nu| \leq r_0,\ r_0 > 0, \qquad (4.73)$$

where by (4.59), (4.60) we have:

$$
\begin{aligned}
\widetilde{p}^B\left(\beta,\mu;c^{\#},\nu^{\#}\right) &= \frac{1}{\beta\left(2\pi\right)^3}\int_{\mathbb{R}^3} d^3k \ln\left(1-e^{-\beta E_k^B\left(|c|^2\right)}\right)^{-1}\\
&-\frac{1}{\beta\left(2\pi\right)^3}\int_{\mathbb{R}^3} d^3k\left[E_k^B\left(|c|^2\right)-f_k\left(|c|^2\right)\right]\\
&+\mu\left|c\right|^2-\frac{1}{2}v\left(0\right)\left|c\right|^4+\left(\nu\overline{c}+\overline{\nu}c\right), \qquad (4.74)
\end{aligned}
$$

(see Theorem 4.23 and Corollary 4.25). ■

Theorem 4.8 gives an *exact solution* of WIBG (2.50) on the level of thermodynamics with explicit formulae (4.73), (4.74) for the pressure. Theorems B.4 and B.6 (Appendix B) provide the following corollaries:

Corollary 4.9 *Let $v(k)$ satisfy hypotheses (A), (B) and*

$$v(0) \geq \frac{1}{2\left(2\pi\right)^3}\int_{\mathbb{R}^3} d^3k\frac{v(k)^2}{\varepsilon_k}. \qquad (4.75)$$

Then

$$p^B\left(\beta,\mu\right) = \sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu;c^{\#}\right) = \widetilde{p}^B\left(\beta,\mu;0\right) = p^P\left(\beta,\mu\right), \qquad (4.76)$$

for any $(\theta,\mu) \in Q^B = Q$, i.e. $D^B = \{\varnothing\}$, see (4.57), (4.63).

Corollary 4.10 *Let $v(k)$ verify the hypotheses (A), (B) and (C) (see Definition 4.5). Then there are $\mu_0 < 0$ and $\theta_0(\mu) > 0$ for $\mu \in [\mu_0, 0)$, such that*

$$p^B(\beta,\mu) = \sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu;c^{\#}\right) = \widetilde{p}^B\left(\beta,\mu;\widehat{c}^{\#}(\beta,\mu)\neq 0\right) > p^P(\beta,\mu), \tag{4.77}$$

for $(\theta,\mu)\in D^B$ characterized by

$$D^B = \{(\theta,\mu)\in Q : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\}, \tag{4.78}$$

(cf. (4.57), (4.63)). Moreover for $(\theta,\mu)\in Q\backslash D^B$ one has

$$p^B(\beta,\mu) = \sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu;c^{\#}\right) = p^P(\beta,\mu). \tag{4.79}$$

4.4 Proofs

The aim of this section is to give the promised details of the proof of the Theorem 4.8. For that we use the Approximation Hamiltonian Method originally invented for quantum systems with attractive interactions (see e.g. [126]). This method has been adapted by Ginibre [92] to prove the exactness of the Bogoliubov approximation for a non-ideal Bose-gas (2.44) with superstable (repulsive) interaction ([91], Appendix G). Below we improve the line of reasoning à la Ginibre to make it applicable to the WIBG [15].

Since the (operator) gauge symmetry is broken in $H_\Lambda^B\left(c^{\#},\mu\right)$ (2.70), we introduce the following Hamiltonians:

$$\begin{aligned} H_\Lambda^B\left(\nu^{\#}\right) &= H_\Lambda^B - \sqrt{V}\left(\overline{\nu}a_0 + \nu a_0^*\right), \\ H_\Lambda^B\left(\mu,\nu^{\#}\right) &= H_\Lambda^B\left(\nu^{\#}\right) - \mu N_\Lambda, \end{aligned} \tag{4.80}$$

with external sources defined by $\nu^{\#} = \{\nu \text{ or } \overline{\nu}\}$. Then by Proposition B.2 and by the Bogoliubov inequality for $H_\Lambda^B\left(\mu,\nu^{\#}\right)$ and $H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right)$ (see Appendix D) one gets:

$$\begin{aligned} 0 \ &\leq \ \Delta_\Lambda\left(\beta,\mu;c^{\#},\nu^{\#}\right) \equiv p_\Lambda\left[H_\Lambda^B\left(\nu^{\#}\right)\right] - \widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right) \leq \\ &\leq \ \frac{1}{V}\left\langle H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right) - H_\Lambda^B\left(\mu,\nu^{\#}\right)\right\rangle_{H_\Lambda^B\left(\nu^{\#}\right)}. \end{aligned} \tag{4.81}$$

The pressure $\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right)$ corresponds to the Hamiltonian $H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right)$. A Taylor expansion around $a_0^{\#}$ gives:

$$\begin{aligned} H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right) - H_\Lambda^B\left(\mu,\nu^{\#}\right) \ = \ & -A^*\left[a_0, H_\Lambda^B\left(\mu,\nu^{\#}\right)\right] + h.c. \\ & + \frac{1}{2}A^{*^2}\left[a_0,\left[a_0,H_\Lambda^B\left(\mu,\nu^{\#}\right)\right]\right] + h.c. \\ & + A^*\left[a_0,\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),a_0^*\right]\right]A \\ & - \frac{1}{2}A^{*^2}\left[a_0,\left[a_0,\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),a_0^*\right]\right]\right]A + h.c. \\ & + \frac{1}{4}A^{*^2}\left[a_0,\left[a_0,\left[\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),a_0^*\right],a_0^*\right]\right]\right]A^2, \end{aligned} \tag{4.82}$$

with $A \equiv a_0 - \sqrt{V}c$ and $A^* \equiv a_0^* - \sqrt{V}\overline{c}$.

Remark 4.11 *Explicit calculations show that the third and the fourth order terms (in $A^{\#}$) of (4.82) are bounded from above:*

$$-\frac{v(0)}{\sqrt{V}}\left(\overline{c}A^{*}AA+cA^{*}A^{*}A\right)-\frac{v(0)}{2V}A^{*^{2}}A^{2}=2v(0)\left|c\right|^{2}A^{*}A-$$
$$-\frac{v(0)}{2V}\left(A^{2}+2\sqrt{V}cA\right)^{*}\left(A^{2}+2\sqrt{V}cA\right)\leq 2v(0)\left|c\right|^{2}A^{*}A. \tag{4.83}$$

Remark 4.12 *After some algebra, the terms of the first and the second order in (4.82) can be combined in*

$$-\frac{1}{2}\left[A^{*}A,\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A^{*}A\right]\right]+2A^{*}\left[A,\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A^{*}\right]\right]A$$
$$-\frac{3}{2}A^{*}\left[A,H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right]-\frac{3}{2}\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A^{*}\right]A. \tag{4.84}$$

Lemma 4.13 *One has the following inequality:*

$$\left\langle\left[A^{*}A,\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A^{*}A\right]\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}\geq 0. \tag{4.85}$$

Proof. Denote by $(.,.)_{H_{\Lambda}}$ the positive semi-definite Bogoliubov-Duhamel scalar product with respect to a Hamiltonian H_{Λ} (see, e.g. [37, 92]):

$$(X,Y)_{H_{\Lambda}}\equiv\frac{1}{\beta\Xi_{\Lambda}(\beta,\mu)}Tr_{\mathcal{F}_{B}}\left(\int_{0}^{\beta}d\tau e^{-(\beta-\tau)H_{\Lambda}(\mu)}X^{*}e^{-\tau H_{\Lambda}(\mu)}Y\right). \tag{4.86}$$

Then one has $(\mathbf{1},Y)_{H_{\Lambda}}=\langle Y\rangle_{H_{\Lambda}}$ and

$$\beta\left(\left[X,H_{\Lambda}(\mu)\right],\left[X,H_{\Lambda}(\mu)\right]\right)_{H_{\Lambda}}=\left\langle\left[X,\left[H_{\Lambda}(\mu),X^{*}\right]\right]\right\rangle_{H_{\Lambda}}\geq 0. \tag{4.87}$$

Applying (4.87) to $H_{\Lambda}(\mu)=H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)$ and $X=A^{*}A$, one gets (4.85). ■

Lemma 4.14 *The following estimate holds:*

$$-2\left\langle A^{*}\left[A,H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}\leq\left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}$$
$$+\left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right]^{*}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}$$
$$+2\beta^{-1}\left\langle\left\{A,A^{*}\right\}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}, \tag{4.88}$$

with $\{X,Y\}\equiv XY+YX$.

Proof. By the spectral decomposition of the Hamiltonian $H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)$:

$$H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\psi_{n}=E_{n}\psi_{n},$$

one gets

$$\left\langle\left\{A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right\}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} = \frac{1}{\Xi_{\Lambda}^{B}\left(\beta,\mu,\nu^{\#}\right)}\sum_{m,n}\left|\left(\psi_{m},A\psi_{n}\right)_{\mathcal{F}_{B}}\right|^{2} \times\left(E_{m}-E_{n}\right)\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right), \tag{4.89}$$

where

$$\Xi_{\Lambda}^{B}\left(\beta,\mu,\nu^{\#}\right)=Tr_{\mathcal{F}_{B}}e^{-\beta H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)} \tag{4.90}$$

is the grand canonical partition function associated with the Hamiltonian $H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)$. Since

$$\frac{1}{2}\left(e^{x}+e^{y}\right)-\frac{1}{2}\left|e^{x}-e^{y}\right|\leq\frac{e^{x}-e^{y}}{x-y}\leq\frac{1}{2}\left(e^{x}+e^{y}\right),\ \ (x,y)\in\mathbb{R}^{2}, \tag{4.91}$$

we find:

$$\begin{aligned}\beta\left(E_{m}-E_{n}\right)\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right) &\leq 2\left(e^{-\beta E_{n}}-e^{-\beta E_{m}}\right)\\ &\quad+\beta\left(E_{m}-E_{n}\right)\left|e^{-\beta E_{n}}-e^{-\beta E_{m}}\right|\\ &\leq 2\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right)\\ &\quad+\beta\left(E_{m}-E_{n}\right)\left(e^{-\beta E_{n}}-e^{-\beta E_{m}}\right),\end{aligned} \tag{4.92}$$

which by (4.89) implies

$$\begin{aligned}\left\langle\left\{A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right\}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} &\leq 2\beta^{-1}\left\langle AA^{*}+A^{*}A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}\\ &\quad+\left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}.\end{aligned} \tag{4.93}$$

Notice that

$$\begin{aligned}-2\left\langle A^{*}\left[A,H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} &= \left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}\\ &\quad+\left\langle\left\{A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right\}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}.\end{aligned} \tag{4.94}$$

Therefore combining (4.93) with (4.94), we obtain (4.88). ■

Corollary 4.15 *Since*

$$\left\langle A^{*}\left[A,H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}=\left\langle\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A^{*}\right]A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)},$$

by the estimate (4.88) the mean value of the last two terms of (4.84) is bounded from above:

$$\begin{aligned}-3\left\langle A^{*}\left[A,H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} &\leq \frac{3}{2}\left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A\right]\right]+h.c.\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}+\\ &\quad+3\beta^{-1}\left\langle AA^{*}+A^{*}A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}.\end{aligned} \tag{4.95}$$

Since we are looking for the estimate of (4.82) (and consequently of (4.84)) from *above*, the inequalities (4.83), (4.85) and (4.95) show that one has to estimate only the mean value of the second term in (4.84). Before to proceed further, we establish some important lemmata and corollaries.

Lemma 4.16 *For $(\theta,\mu)\in Q$ and $\nu\in\mathbb{C}$, we have*

$$p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right]\le \widetilde{p}_\Lambda^P\left(\beta,\mu\right)+\left\{\frac{1}{\beta V}\sum_{n_0=0}^{\infty}e^{\frac{\beta}{2}\left[\left(\varphi(0)+2\right)n_0-v(0)n_0^2/V\right]}\right\}+\left|\nu\right|^2, \tag{4.96}$$

where $\widetilde{p}_\Lambda^P\left(\beta,\mu\right)$ is the pressure of the PBG with excluded mode $k=0$.

Proof. Since

$$\left(a_0^*-\overline{\nu}\sqrt{V}\right)\left(a_0-\nu\sqrt{V}\right)\ge 0,$$

we obtain

$$-\sqrt{V}\left(\overline{\nu}a_0+\nu a_0^*\right)\ge -a_0^*a_0-\left|\nu\right|^2V.$$

Then inequality (4.96) follows immediately from the estimate (cf. (3.68), (3.69)):

$$\begin{aligned}H_\Lambda^B\left(\nu^\#\right)-\mu N_\Lambda &\ge \sum_{k\in\Lambda^*\setminus\{0\}}\left(\varepsilon_k-\mu-\frac{v(k)}{2V}\right)N_k+\frac{v(0)}{2V}N_0^2\\ &\quad-\left(\mu+\frac{1}{2}\varphi\left(0\right)+1\right)N_0-\left|\nu\right|^2V.\end{aligned}$$

■

Corollary 4.17 *¿From (4.96) in the thermodynamic limit we obtain:*

$$p^B\left(\beta,\mu;\nu^\#\right)\le p^P\left(\beta,\mu\right)+\frac{1}{2}\sup_{\rho\ge 0}\left\{\left(\varphi\left(0\right)+2\right)\rho-v\left(0\right)\rho^2\right\}+\left|\nu\right|^2 \tag{4.97}$$

for $(\theta,\mu)\in Q$ and $\nu\in\mathbb{C}$.

Lemma 4.18 *For any $\mu<0$ and $\nu\in\mathbb{C}$ one has the estimate*

$$\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^B\left(\nu^\#\right)}\le g_\Lambda\left(\beta,\mu;\nu^\#\right)<\infty. \tag{4.98}$$

Proof. For any $\mu<0$ there is $\delta>0$ such that $\mu+\delta<0$. Then by the Bogoliubov inequality (see Appendix D) we obtain

$$\delta\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^B\left(\nu^\#\right)}\le p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)-\delta N_\Lambda\right]-p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right]. \tag{4.99}$$

Therefore, by Lemma 4.16, we get (4.98) for

$$g_\Lambda\left(\beta,\mu;\nu^\#\right)\equiv\frac{1}{\delta}\left(p_\Lambda^B\left(\beta,\mu+\delta;\nu^\#\right)-p_\Lambda^B\left(\beta,\mu;\nu^\#\right)\right). \tag{4.100}$$

■

Corollary 4.19 *In the thermodynamic limit the estimate (4.98), with (4.100), gives:*

$$\rho^B\left(\beta,\mu;\nu^{\#}\right) \equiv \lim_{\Lambda}\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^B(\nu^{\#})} \leq g\left(\beta,\mu;\nu^{\#}\right), \tag{4.101}$$

where

$$g\left(\beta,\mu;\nu^{\#}\right) \equiv \frac{1}{\delta}\left(p^B\left(\beta,\mu+\delta;\nu^{\#}\right) - p^B\left(\beta,\mu;\nu^{\#}\right)\right).$$

In fact for $\mu < 0$, $\nu \in \mathbb{C}$ the Griffiths lemma (see Appendix C) implies

$$\rho^B\left(\beta,\mu;\nu^{\#}\right) = \partial_\mu p^B\left(\beta,\mu;\nu^{\#}\right). \tag{4.102}$$

Corollary 4.20 *By (4.98) we obtain:*

$$\begin{aligned}&\left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B(\nu^{\#})} \leq g_\Lambda\left(\beta,\mu;\nu^{\#}\right),\\ &\left|\left\langle\frac{a_0^*}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^{\#})}\right| = \left|\left\langle\frac{a_0^*}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^{\#})}\right| \leq \sqrt{g_\Lambda\left(\beta,\mu;\nu^{\#}\right)}.\end{aligned} \tag{4.103}$$

Remark 4.21 *To optimize the estimate (4.81), we have to look for* $\sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right)$. *Since by Definition B.1 and (4.80) one has*

$$H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right) = H_\Lambda^B\left(c^{\#},\mu\right) - V\left(\nu\overline{c} + \overline{\nu}c\right) \geq H_\Lambda^B\left(c^{\#},\mu\right) - V\left(|\nu|^2|c|^2 + 1\right), \tag{4.104}$$

Then by (B.10) for $(\theta,\mu)\in Q$ *and fixed* $\nu^{\#}$ *there is* $A \geq 0$ *such that*

$$\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right) \leq A - \frac{1}{2}v(0)|c|^4. \tag{4.105}$$

Thus for any compact $K \subset Q \times \{\nu \in \mathbb{C}\}$, *the optimal value of* $|c|$ *is bounded by a positive constant* $M_K < \infty$.

Lemma 4.22 *For* $(\theta,\mu) \in D^B$ *(cf. (4.57), (4.63)),there are two non-negative, locally bounded in* D^B, *functions*

$$\begin{aligned}a &= a\left(\theta,\mu,\nu^{\#}\right),\\ b &= b\left(\theta,\mu,\nu^{\#}\right),\end{aligned} \tag{4.106}$$

and $r_0 > 0$ *such that for* $|\nu| \leq r_0$ *we have:*

$$\left\langle A^*\left[A,\left[H_\Lambda^B\left(\mu,\nu^{\#}\right),A^*\right]\right]A\right\rangle_{H_\Lambda^B(\nu^{\#})} \leq a\left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^{\#})} + b. \tag{4.107}$$

Proof. (1) The pressure $p_\Lambda\left[H_\Lambda^B\left(\nu^{\#}\right)\right]$ is bounded from below and from above uniformly on any compact $K \subset Q\times\{\nu : |\nu| < r_0\}$ (see (B.5) and (4.96)). Since the family

$$\left\{p_\Lambda\left[H_\Lambda^B\left(\nu^{\#}\right)\right] = p_\Lambda^B\left(\beta,\mu;\nu^{\#}\right)\right\}_{\Lambda\subset\mathbb{R}^3}$$

consists of convex functions of the chemical potential $\mu < 0$, by compactness argument (see [158], Ch.II, §10), there is a subsequence $\left\{p^B_{\Lambda_j}\left(\beta,\mu;\nu^{\#}\right)\right\}_{j=1}^{\infty}$ which converges uniformly in μ on any compact $C_\mu \subset \mathbb{R}^1_-$ and for fixed β,ν to the convex function $p^B\left(\beta,\mu;\nu^{\#}\right)$, i.e. converges locally uniformly in $\mathbb{R}^1_-$.

(2) The grand-canonical pressure has the form (see (4.90)):

$$p^B_\Lambda\left(\beta,\mu;\nu^{\#}\right) = \frac{1}{\beta V}\ln\left\{\sum_{n=0}^{\infty} e^{\beta V\left(\mu\frac{n}{V} - f^B_\Lambda\left(\beta,\frac{n}{V};\nu^{\#}\right)\right)}\right\} = \frac{1}{\beta V}\ln \Xi^B_\Lambda\left(\beta,\mu;\nu^{\#}\right), \tag{4.108}$$

where

$$f^B_\Lambda\left(\beta,\rho;\nu^{\#}\right) \equiv -\frac{1}{\beta V}\ln Tr_{\mathcal{H}^{(n)}_B} e^{-\beta H^B_\Lambda\left(\nu^{\#}\right)},\ \rho \geq 0, \tag{4.109}$$

is the corresponding free-energy density with $\mathcal{H}^{(n)}_B$ a symmetrized $n(=[V\rho])$-particle Hilbert space (cf. (2.3)). For $(\theta,\mu)\in D^B$ (cf. (4.63)), we have $\rho^B_0(\theta,\mu) > 0$. Consequently

$$\left\{\partial_\mu p^B_\Lambda\left(\beta,\mu;\nu^{\#}\right) = \left\langle\frac{N_\Lambda}{V}\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)} > 0\right\}_\Lambda$$

are continuous convex functions of $\mu \in [\mu_0(\theta)+\varepsilon, 0)$, $\varepsilon > 0$, even in the thermodynamic limit. By the Griffiths lemma (see Appendix C) one gets $\partial_\mu p^B = \lim_\Lambda \partial_\mu p^B_\Lambda$, which is a continuous functions of $\mu \in [\mu_0(\theta)+\varepsilon, 0)$, $\varepsilon > 0$. Notice that $\partial_\mu p^B$ can be extended to $\mu = 0$ by continuity. Then by a Tauberian theorem proved in [159], the existence of the limit $p^B\left(\beta,\mu;\nu^{\#}\right)$ entails the existence of the limit for the free-energy densities:

$$f^B\left(\beta,\rho;\nu^{\#}\right) = \lim_\Lambda f^B_\Lambda\left(\beta,\rho;\nu^{\#}\right), \tag{4.110}$$

which is uniform on the interval

$$\rho \in I_{\varepsilon_{1,2}} = \left[\partial_\mu p^B\left(\beta,\mu_0(\theta)+\varepsilon_1;\nu^{\#}\right), \partial_\mu p^B\left(\beta,-\varepsilon_2;\nu^{\#}\right)\right],\ \varepsilon_{1,2} > 0. \tag{4.111}$$

In fact on this interval the limit (4.110) coincides with its convex envelope (the Legendre transformation):

$$f^B\left(\beta,\rho;\nu^{\#}\right) = E.C.\left\{f^B\left(\beta,\rho;\nu^{\#}\right)\right\} = \sup_{\mu\leq 0}\left\{\mu\rho - p^B\left(\beta,\mu;\nu^{\#}\right)\right\}. \tag{4.112}$$

(3) By virtue of (4.110) and (4.112), for $V_j = |\Lambda_j|$ large enough, functions $\left\{f^B_{\Lambda_j}\left(\beta,\rho;\nu^{\#}\right)\right\}_{j=1}^{\infty}$ are strictly convex on $I_{\varepsilon_{1,2}}$ and for $\mu \in [\mu_0(\theta)+\varepsilon_1, 0]$,

$$\sup_{\frac{n}{V}}\left(\mu\frac{n}{V} - f^B_\Lambda\left(\beta,\frac{n}{V};\nu^{\#}\right)\right) = \mu\overline{\rho}_\Lambda - f^B_\Lambda\left(\beta,\overline{\rho}_\Lambda;\nu^{\#}\right) \equiv -F_\Lambda\left(\beta,\mu;\overline{\rho}_\Lambda,\nu^{\#}\right), \tag{4.113}$$

$\overline{\rho}_\Lambda(\mu) \in I_{\varepsilon_{1,2}}$. Then for $\left|\frac{n}{V} - \overline{\rho}_\Lambda\right| > \xi > 0$, one gets

$$F_\Lambda\left(\beta,\mu;\frac{n}{V},\nu^{\#}\right) > F_\Lambda\left(\beta,\mu;\overline{\rho}_\Lambda,\nu^{\#}\right) + \gamma \equiv \overline{F}_\Lambda + \gamma,\ \gamma > 0, \tag{4.114}$$

and for $\left|\frac{n}{V}-\overline{\rho}_\Lambda\right|<\xi'<\xi$, we obtain

$$\overline{F}_\Lambda \le F_\Lambda\left(\beta,\mu;\frac{n}{V},\nu^{\#}\right)\le \overline{F}_\Lambda+\frac{\gamma}{2}. \tag{4.115}$$

By (4.108) one gets that for $\frac{n}{V}-\overline{\rho}_\Lambda>\xi$ there are two constants $a_{1,2}>0$ such that

$$F_\Lambda\left(\beta,\mu;\frac{n}{V},\nu^{\#}\right)>a_1+a_2\left(\frac{n}{V}-\overline{\rho}_\Lambda-\xi\right). \tag{4.116}$$

(4) (*Large Deviation Principle for the particle density*) By a standard reasoning (see e.g. [148, 160, 161]), one gets from the grand-canonical distribution of particles and (4.114)-(4.116) that probabilities:

$$\begin{aligned}
q_{\Lambda,I} &= \mathbb{P}_\Lambda\left\{0\le\frac{n}{V}<\overline{\rho}_\Lambda-\xi\right\}=\left(\Xi^B_\Lambda\right)^{-1}\sum_{0\le n<V(\overline{\rho}_\Lambda-\xi)} e^{-\beta V F_\Lambda\left(\beta,\mu;\frac{n}{V},\nu^{\#}\right)}\le \\
&\le V\left(\overline{\rho}_\Lambda-\xi\right)e^{-\beta V\left(\overline{F}_\Lambda+\gamma\right)}, &(4.117)\\
q_{\Lambda,II} &= \mathbb{P}_\Lambda\left\{\overline{\rho}_\Lambda-\xi\le\frac{n}{V}<\overline{\rho}_\Lambda+\xi\right\}\ge \\
&\ge \left(\Xi^B_\Lambda\right)^{-1}\sum_{V(\overline{\rho}_\Lambda-\xi')\le n<V(\overline{\rho}_\Lambda+\xi')} e^{-\beta V F_\Lambda\left(\beta,\mu;\frac{n}{V},\nu^{\#}\right)} \\
&\ge 2\left(\Xi^B_\Lambda\right)^{-1}\xi' V e^{-\beta V\left(\overline{F}_\Lambda+\frac{\gamma}{2}\right)}, &(4.118)\\
q_{\Lambda,III} &= \mathbb{P}_\Lambda\left\{\frac{n}{V}\ge\overline{\rho}_\Lambda+\xi\right\}\le\left(\Xi^B_\Lambda\right)^{-1}\sum_{n\ge V(\overline{\rho}_\Lambda+\xi)} e^{-\beta V\left[a_1+a_2\left(\frac{n}{V}-\overline{\rho}_\Lambda-\xi\right)\right]}. &(4.119)
\end{aligned}$$

Since $q_{\Lambda,I}+q_{\Lambda,II}+q_{\Lambda,III}=1$, these estimates imply that

$$\lim_{\Lambda} q_{\Lambda,II}=1, \tag{4.120}$$

for any $\xi>0$.
(5) Now we can apply the Large Deviation Principle (4.120) for the particle density in domain D^B to obtain (4.107). By explicit calculations we have

$$\left[A,\left[H^B_\Lambda\left(\mu,\nu^{\#}\right),A^*\right]\right]=\frac{v(0)}{V}N_\Lambda-\mu+\frac{1}{V}\sum_{k\in\Lambda^*}v(k)\,a^*_k a_k. \tag{4.121}$$

Consequently by the condition (B) on the interaction potential, i.e. $v(0)\ge v(k)\ge 0$ and $v(0)>0$, we get

$$\left\langle A^*\left[A,\left[H^B_\Lambda\left(\mu,\nu^{\#}\right),A^*\right]\right]A\right\rangle_{H^B_\Lambda(\nu^{\#})}\le 2v(0)\left\langle A^*\frac{N_\Lambda}{V}A\right\rangle_{H^B_\Lambda(\nu^{\#})}-\mu\left\langle A^*A\right\rangle_{H^B_\Lambda(\nu^{\#})}. \tag{4.122}$$

Since

$$\left\langle A^*\frac{N_\Lambda}{V}A\right\rangle_{H^B_\Lambda(\nu^{\#})}=\left\langle\left(\frac{N_\Lambda}{V}-\overline{\rho}_\Lambda\right)A^*A\right\rangle_{H^B_\Lambda(\nu^{\#})}+\overline{\rho}_\Lambda\left\langle A^*A\right\rangle_{H^B_\Lambda(\nu^{\#})}, \tag{4.123}$$

we have to estimate from above the first term in the left-hand side of (4.123). To this end we follow (4.117)-(4.120):
(I)

$$\left(\Xi_\Lambda^B\right)^{-1}\sum_{0\leq n<V(\overline{\rho}_\Lambda-\xi)}\left(\frac{n}{V}-\overline{\rho}_\Lambda\right)e^{-\beta VF_\Lambda}\left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^\#)}\left(\beta,n;\nu^\#\right)\leq 0. \tag{4.124}$$

(II)

$$\begin{aligned}&\left(\Xi_\Lambda^B\right)^{-1}\sum_{V(\overline{\rho}_\Lambda-\xi)\leq n<V(\overline{\rho}_\Lambda+\xi)}e^{\beta\mu n}\left(\tfrac{n}{V}-\overline{\rho}_\Lambda\right)Tr_{\mathcal{H}_B^{(n)}}\left(e^{-\beta H_\Lambda^B(\nu^\#)}A^*A\right)\leq\\&\leq 2\xi\left(\Xi_\Lambda^B\right)^{-1}\sum_{n=0}^{\infty}e^{\beta\mu n}Tr_{\mathcal{H}_B^{(n)}}\left(e^{-\beta H_\Lambda^B(\nu^\#)}A^*A\right)=\xi\left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^\#)}\left(\beta,\mu;\nu^\#\right).\end{aligned} \tag{4.125}$$

(III)

$$\begin{aligned}&\left(\Xi_\Lambda^B\right)^{-1}\sum_{n\geq V(\overline{\rho}_\Lambda+\xi)}\left(\tfrac{n}{V}-\overline{\rho}_\Lambda\right)e^{-\beta VF_\Lambda}\left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^\#)}\left(\beta,n;\nu^\#\right)\leq\\&\leq\left(\Xi_\Lambda^B\right)^{-1}\sum_{n\geq V(\overline{\rho}_\Lambda+\xi)}\left(\tfrac{n}{V}-\overline{\rho}_\Lambda\right)e^{-\beta V\left[a_1+a_2\left(\frac{n}{V}-\overline{\rho}_\Lambda-\xi\right)\right]}2\left(n+|c|^2V\right)=\\&=2\left(\Xi_\Lambda^B\right)^{-1}\sum_{n\geq V(\overline{\rho}_\Lambda+\xi)}\left(\tfrac{n}{V}-\overline{\rho}_\Lambda\right)\left(n-V\overline{\rho}_\Lambda\right)e^{-\beta Va_1}e^{-\beta a_2(n-V(\overline{\rho}_\Lambda+\xi))}+\\&+2\left(\Xi_\Lambda^B\right)^{-1}V\left(\overline{\rho}_\Lambda+|c|^2\right)e^{-\beta Va_1}\sum_{n\geq V(\overline{\rho}_\Lambda+\xi)}e^{-\beta a_2(n-V(\overline{\rho}_\Lambda+\xi))}=\\&=q_{\Lambda,II}\left(\xi'V^2\right)^{-1}e^{-\beta V^{\frac{\gamma}{2}}}\sum_{n\geq V(\overline{\rho}_\Lambda+\xi)}\left(n-V\overline{\rho}_\Lambda\right)^2e^{-\beta a_2(n-V(\overline{\rho}_\Lambda+\xi))}+\\&+q_{\Lambda,II}V^{-1}\left(\overline{\rho}_\Lambda+|c|^2\right)e^{-\beta V^{\frac{\gamma}{2}}}\sum_{n\geq V(\overline{\rho}_\Lambda+\xi)}e^{-\beta a_2(n-V(\overline{\rho}_\Lambda+\xi))}\leq const\times e^{-\beta V^{\frac{\gamma}{2}}}.\end{aligned} \tag{4.126}$$

Combining (4.124)-(4.126) with (4.122) and (4.123), we find that for any compact $C_\mu\subset(\mu_0(\theta),0)$, $|\nu|\leq r_0$ and compact $C_\beta\subset\mathbb{R}_+^1$, one has

$$\left\langle A^*\left[A,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A^*\right]\right]A\right\rangle_{H_\Lambda^B(\nu^\#)}\leq a\left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^\#)}+b, \tag{4.127}$$

for positive bounded $a>0$, $b>0$, which depend on C_μ, C_β, and r_0, i.e. for a, b locally bounded in D^B. ■
Now we are in position to prove the main statement of this section about *exactness* of the Bogoliubov approximation for the WIBG.

Theorem 4.23 *Let $(\theta,\mu)\in D^B$. Then*

$$\lim_{\Lambda}\left\{p_\Lambda^B\left(\beta,\mu,\nu^\#\right)-\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right)\right\}=0 \tag{4.128}$$

locally uniformly in D^B for $|\nu|\leq r_0$, $r_0>0$.

Proof. From the main inequality (4.81) one obtains

$$\begin{aligned}0&\leq\inf_{c\in\mathbb{C}}\Delta_\Lambda\left(\beta,\mu;c^\#,\nu^\#\right)\equiv\Delta_\Lambda\left(\beta,\mu;\widehat{c}_\Lambda^\#\left(\beta,\mu,\nu^\#\right),\nu^\#\right)\\&\leq\frac{1}{V}\left\langle H_\Lambda^B\left(c^\#,\mu,\nu^\#\right)-H_\Lambda^B\left(\mu,\nu^\#\right)\right\rangle_{H_\Lambda^B(\nu^\#)}.\end{aligned} \tag{4.129}$$

By virtue of (4.82)-(4.84), estimates (4.83) (4.85), (4.88), (4.95), (4.107) and Remark 4.21 there are positive constants u and w independent of the volume V, such that

$$\begin{aligned}&\frac{1}{V}\left\langle H_{\Lambda}^{B}\left(c^{\#},\mu,\nu^{\#}\right)-H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})}\leq u+\frac{w}{2}\times\\&\times\left\langle\left\{\left(a_0^{*}-\sqrt{V}c^{*}\right),\left(a_0-\sqrt{V}c\right)\right\}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})},\end{aligned}\tag{4.130}$$

locally uniformly in D^B. Put $c\equiv\left\langle a_0/\sqrt{V}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})}$ which is bounded (see (4.103)). Then

$$\Delta_{\Lambda}\left(\beta,\mu;\widehat{c}_{\Lambda}^{\#},\nu^{\#}\right)\leq\Delta_{\Lambda}\left(\beta,\mu;\left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})},\nu^{\#}\right),$$

and estimates (4.129), (4.130) give

$$0\leq\inf_{c\in\mathbb{C}}\Delta_{\Lambda}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\leq\frac{u}{V}+\frac{w}{2V}\left\langle\left\{\left(a_0^{*}-\langle a_0^{*}\rangle\right),\left(a_0-\langle a_0\rangle\right)\right\}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})},\tag{4.131}$$

where we use for short $\left\langle a_0^{\#}\right\rangle\equiv\left\langle a_0^{\#}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})}$. Let

$$\delta a_0^{\#}\equiv a_0^{\#}-\left\langle a_0^{\#}\right\rangle.$$

By the spectral decomposition of the Hamiltonian $H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)$:

$$H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)\psi_n=E_n\psi_n,$$

one obtains

$$\left\langle\left\{\delta a_0^{*},\delta a_0\right\}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})}=\frac{1}{\Xi_{\Lambda}^{B}\left(\beta,\mu;\nu^{\#}\right)}\sum_{m,n}\left|A_{mn}\right|^2\left(e^{-\beta E_n}+e^{-\beta E_m}\right),\tag{4.132}$$

with $A_{mn}=\left(\psi_m,\delta a_0\psi_n\right)$. Notice that

$$\begin{aligned}\left(e^{-\beta E_n}+e^{-\beta E_m}\right)&=2\frac{\left(e^{-\beta E_n}-e^{-\beta E_m}\right)}{\beta\left(E_m-E_n\right)}\frac{\beta\left(E_m-E_n\right)}{2}\frac{\left(e^{-\beta E_n}+e^{-\beta E_m}\right)}{\left(e^{-\beta E_n}-e^{-\beta E_m}\right)}\\&=2\frac{\left(e^{-\beta E_n}-e^{-\beta E_m}\right)}{\beta\left(E_m-E_n\right)}\frac{\beta\left(E_m-E_n\right)}{2}cth\left(\frac{\beta\left(E_m-E_n\right)}{2}\right).\end{aligned}$$

Since for $x\in\mathbb{R}^1$,

$$chx-\frac{shx}{x}=\sum_{n=1}^{\infty}\frac{2n}{(2n+1)!}x^{2n}\leq\frac{xshx}{3},$$

then $xcthx\leq 1+x^2/3$. Hence for (4.132) we obtain the estimate

$$\begin{aligned}\left\langle\left\{\delta a_0^{*},\delta a_0\right\}\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})}&\leq\frac{2}{\Xi_{\Lambda}^{B}\left(\beta,\mu;\nu^{\#}\right)}\sum_{m,n}\left|A_{mn}\right|^2\frac{\left(e^{-\beta E_n}-e^{-\beta E_m}\right)}{\beta\left(E_m-E_n\right)}\\&+\frac{1}{6\ \Xi_{\Lambda}^{B}\left(\beta,\mu,\nu^{\#}\right)}\sum_{m,n}\left|A_{mn}\right|^2\left(e^{-\beta E_n}-e^{-\beta E_m}\right)\beta\left(E_m-E_n\right)\\&\leq 2\left(\delta a_0^{*},\delta a_0\right)_{H_{\Lambda}^{B}(\nu^{\#})}+\frac{\beta}{6}\left\langle\left[\delta a_0^{*},\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),\delta a_0\right]\right]\right\rangle_{H_{\Lambda}^{B}(\nu^{\#})},\end{aligned}$$

which implies

$$\frac{1}{2}\left\langle\left\{\delta a_0^*, \delta a_0\right\}\right\rangle_{H_\Lambda^B(\nu^\#)} \leq \left(\delta a_0^*, \delta a_0\right)_{H_\Lambda^B(\nu^\#)} + \frac{\beta}{12}\left\langle\left[\delta a_0^*, \left[H_\Lambda^B\left(\mu, \nu^\#\right), \delta a_0\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)}. \tag{4.133}$$

The inequality (4.133) is proven in [162], see also [163]. By the condition (B) on the interaction (see Section 2.2 or Section 4.2) and Lemma 4.18 we have:

$$\left\langle\left[\delta a_0^*, \left[H_\Lambda^B\left(\mu, \nu^\#\right), \delta a_0\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)} \leq 2v(0)\, g_\Lambda\left(\beta, \mu; \nu^\#\right) - \mu, \tag{4.134}$$

see (4.121). By (3.60), (3.66) and (4.101), we have a uniform bound:

$$g_\Lambda\left(\beta, \mu; \nu^\#\right) < g_0, \tag{4.135}$$

for each compact $C_0(\mu < 0) \subset D^B$ and $|\nu| \leq r_0$. Therefore, the estimate (4.131) on this compact $C_0(\mu < 0)$ set takes the form:

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_\Lambda\left(\beta, \mu; c^\#, \nu^\#\right) \leq \frac{1}{V}\left[\widetilde{u} + w\left(\delta a_0^*, \delta a_0\right)_{H_\Lambda^B(\nu^\#)}\right]. \tag{4.136}$$

Now we can proceed along the standard reasoning of the Approximation Hamiltonian Method (see [126, 163]). First we note that

$$\left(\delta a_0^*, \delta a_0\right)_{H_\Lambda^B(\nu^\#)} = \frac{1}{\beta}\partial_\nu \partial_{\overline{\nu}} p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right]. \tag{4.137}$$

By the (canonical) gauge transformation $a_0 \to a_0 e^{i\varphi}$, $\varphi = \arg\nu$, one finds that in fact

$$p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right] = p_\Lambda^B\left(\beta, \mu; |\nu| \equiv r\right).$$

Then passing in (4.137) to polar coordinates (r, φ) we obtain:

$$\left(\delta a_0^*, \delta a_0\right)_{H_\Lambda^B(\nu^\#)} = \frac{1}{4\beta r}\partial_r\left(r\partial_r p_\Lambda^B\right). \tag{4.138}$$

Let $c = |c|\, e^{i\psi}$, $\psi = \arg c$. Then from (4.80) and (4.81), we obtain

$$\begin{aligned}\inf_{c \in \mathbb{C}} \Delta_\Lambda\left(\beta, \mu; c^\#, \nu^\#\right) &= \inf_{|c|, \psi} \Delta_\Lambda\left(\beta, \mu; |c|\, e^{\pm i\psi}, re^{\pm i\varphi}\right) \\ &= \inf_{|c|} \widehat{\Delta}_\Lambda\left(\beta, \mu; |c|\, e^{\pm i\varphi}, r\right) \equiv \inf_{|c|} \widetilde{\Delta}_\Lambda(r).\end{aligned} \tag{4.139}$$

Consequently, by (4.136) we find that

$$\int_R^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\, dr \leq \frac{1}{V}\left\{\widetilde{u}\frac{(R+\varepsilon)^2 - R^2}{2} + \frac{w}{4\beta}\left(r\partial_r p_\Lambda^B\right)\Big|_R^{R+\varepsilon}\right\}, \tag{4.140}$$

for $[R, R+\varepsilon] \subset [0, r_0]$. Note that by (4.103) et (4.135), we have

$$\partial_r p_\Lambda^B = 2\left|\left\langle a_0/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu\#)}\right| \leq 2g_0^{\frac{1}{2}}, \ \ (\theta,\mu) \in C_0 \subset D^B, |\nu| \leq r_0. \tag{4.141}$$

Therefore (4.140) takes the form

$$\int_R^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\, dr \leq \frac{1}{V}\left\{\widetilde{u}\frac{(R+\varepsilon)^2 - R^2}{2} + \frac{w}{2\beta} g_0^{\frac{1}{2}}(2R+\varepsilon)\right\}. \tag{4.142}$$

Since by Corollary 4.20 and Remark 4.21, one has

$$\left|\partial_r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\right| \leq 2g_\Lambda^{\frac{1}{2}} + 2\left|\widehat{c}_\Lambda\right| \leq 2\left(g_0^{\frac{1}{2}} + M\right), \ \ r \in [R, R+\varepsilon].$$

we obtain

$$\inf_{|c|} \widetilde{\Delta}_\Lambda(R) \leq \inf_{|c|} \widetilde{\Delta}_\Lambda(r) + 2(r-R)\left(g_0^{\frac{1}{2}} + M\right).$$

Hence,

$$\inf_{|c|} \widetilde{\Delta}_\Lambda(R)\frac{(R+\varepsilon)^2 - R^2}{2} \leq \int_R^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\, dr + 2\left(g_0^{\frac{1}{2}} + M\right)\left(\frac{r^3}{3} - R\frac{r^2}{2}\right)\Big|_R^{R+\varepsilon},$$

and by (4.142), we find

$$\inf_{|c|} \widetilde{\Delta}_\Lambda(R) \leq \frac{1}{V}\left\{\widetilde{u} + \frac{w}{\beta} g_0^{\frac{1}{2}} \varepsilon^{-1}\right\} + \left(g_0^{\frac{1}{2}} + M\right)\varepsilon\frac{R + \frac{2}{3}\varepsilon}{R + \frac{1}{2}\varepsilon}. \tag{4.143}$$

Notice that $\varepsilon > 0$ is still arbitrary. Minimizing the right-hand side of (4.143) with respect to ε, one obtains for large V the optimal value of $\varepsilon \sim 1/\sqrt{V}$. Hence, for $V \to \infty$ one gets from (4.143) the asymptotic estimate

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_\Lambda\left(\beta, \mu; c^\#, \nu^\#\right) \leq \delta_\Lambda \equiv const\frac{1}{\sqrt{V}} \tag{4.144}$$

which is valid for each compact $C_0 \subset D^B$ and $|\nu| \leq r_0$. One gets (4.128) for any $(\theta, \mu) \in D^B$ by extension of (4.143) to $\mu = 0$ by continuity. ■

Corollary 4.24 *Let $(\theta,\mu) \in D^B$. Then, if one considers the Bogoliubov approximation for the statistical operator W_Λ*

$$W_\Lambda = e^{-\beta H_\Lambda^B(\mu,\nu^\#)},$$

then one gets

$$\lim_\Lambda \left\{p_\Lambda^B\left(\beta,\mu,\nu^\#\right) - \sup_{c\in\mathbb{C}} p_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right)\right\} = 0 \tag{4.145}$$

locally uniformly in D^B for $|\nu| \leq r_0$, $r_0 > 0$. Here

$$p_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}'_B} W_\Lambda\left(c^\#\right),$$

(cf. Definition B.1) and $|\nu| \leq r_0$, $r_0 > 0$.

Proof. Let denote by $\mathcal{F}_{B0} = \mathcal{F}_B\left(\mathcal{H}_0\right)$ a boson Fock space constructed on the one-dimensional Hilbert space $\mathcal{H}_0$ (see Appendix B). Using for calculation of $Tr_{\mathcal{F}_B}\left(-\right) = Tr_{\mathcal{F}_{B0}\otimes\mathcal{F}'_B}\left(-\right)$ a product-basis in $\mathcal{F}_B$ (B.1), (B.7), one gets (cf. Definition B.1):

$$Tr_{\mathcal{F}_B}\left(W_\Lambda\right) \geq \sup_{\{\psi'_n\}_{n=1}^\infty}\sum_{n=1}^\infty\left(\psi_{0\Lambda}\left(c\right)\otimes\psi'_n, e^{-\beta H_\Lambda^B\left(\mu,\nu^\#\right)}\psi_{0\Lambda}\left(c\right)\otimes\psi'_n\right)_{\mathcal{F}_B} \equiv Tr_{\mathcal{F}'_B}W_\Lambda\left(c^\#\right),$$

where $\{\psi'_n\}_{n=1}^\infty$ is an arbitrary orthonormal basis in $\mathcal{F}'_B$. Then by the Peierls-Bogoliubov inequality (see Appendix D) and by Definition B.1 one gets:

$$\left(\psi_{0\Lambda}\left(c\right)\otimes\psi'_n, e^{-\beta H_\Lambda^B\left(\mu,\nu^\#\right)}\psi_{0\Lambda}\left(c\right)\otimes\psi'_n\right)_{\mathcal{F}_B} \geq e^{-\beta\left(\psi'_n, H_\Lambda^B\left(c^\#,\mu,\nu^\#\right)\psi'_n\right)_{\mathcal{F}'_B}}.$$

Hence

$$\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right) \leq p_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right) \leq p_\Lambda^B\left(\beta,\mu,\nu^\#\right), \tag{4.146}$$

which by Theorem 4.23 implies the thermodynamic limit (4.145). ■

Corollary 4.25 *Since the thermodynamic limits exist:*

$$\begin{aligned}\widetilde{p}^B\left(\beta,\mu;c^\#,\nu^\#\right) &= \lim_\Lambda \widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right) = \lim_\Lambda\left(\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) + \left(\nu\overline{c} + \overline{\nu}c\right)\right),\\ \widetilde{p}^B\left(\beta,\mu;\widetilde{c}^\#\left(\beta,\mu;\nu^\#\right),\nu^\#\right) &= \lim_\Lambda\left[\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#,\nu^\#\right)\right] = \sup_{c\in\mathbb{C}}\widetilde{p}^B\left(\beta,\mu;c^\#,\nu^\#\right),\end{aligned} \tag{4.147}$$

by virtue of the locally uniform estimate (4.144) and of extension by continuity to $\mu = 0$ we get

$$p^B\left(\beta,\mu;\nu^\#\right) = \lim_\Lambda p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right] = \sup_{c\in\mathbb{C}}\widetilde{p}^B\left(\beta,\mu;c^\#,\nu^\#\right), \tag{4.148}$$

for $(\theta,\mu) \in D^B$, $|\nu| \leq r_0$. Then by the limit $|\nu| \to 0$ one gets

$$p^B\left(\beta,\mu\right) = \sup_{c\in\mathbb{C}}\widetilde{p}^B\left(\beta,\mu;c^\#\right), \tag{4.149}$$

i.e. the assertion of the Theorem 4.8.

Section 5

Nonconventional and conventional condensations in the Weakly Imperfect Bose-Gas

"Le travail éloigne de nous trois grands maux: l'ennui, le vice et le besoin."
Voltaire

In the last section, we have found the *exact* solution of the Bogoliubov WIBG in the sense that in the thermodynamic limit the corresponding WIBG pressure is calculated explicitly in the grand-canonical ensemble (Theorem 4.8, Corollaries 4.9 and 4.10). We have also shown that this pressure is different from the PBG pressure only in the domain D^B, where there is a macroscopic accumulation of particles in the mode $k = 0$. This specific kind of the Bose condensation exists exclusively due to interaction under condition (C): for $v(q) \equiv 0$ the domain $D^B = \{\varnothing\}$. On the other hand, we know that the diagonal part of the WIBG manifests, for high particle densities, a *conventional* BE condensation induced by the mechanism of saturation (Theorem 4.4).

Therefore the aim of this section is to show that, in fact, the WIBG has a complex thermodynamic behaviour. Namely, it may manifest *two kinds* of condensation: at the *first* stage it shows a *nonconventional* Bose condensation induced by interaction and at the *second* stage one has a *conventional* BE condensation induced by the standard mechanism of saturation. In Section 5.1, we make explicit the first (*nonconventional*) Bose condensation and we highlight the effect of interaction responsible of this phenomenon. Whereas in Section 5.2, we prove the existence of the *conventional* BE condensation for high particle densities, which may coexist with the *nonconventional* one. In Section 5.3 we discuss the thermodynamic behaviour and the two kinds of Bose condensations in the WIBG at low dimensions $d = 1, 2$.

5.1 Nonconventional Bose condensation

¿From Corollaries 4.9 and 4.10 the pressure $p^B(\beta,\mu)$ for the WIBG is known explicitly (see (4.74), (4.79)). By Theorem 4.7 we know that the domain D^B (4.57) corresponds to existence of the Bose condensation in the mode $k=0$:

$$D^B = \{(\theta,\mu)\in Q: \ |\widehat{c}(\beta,\mu)|>0\} = \left\{(\theta,\mu)\in Q: \ \rho_0^B(\theta,\mu)>0\right\}, \tag{5.1}$$

see (4.63) and (4.77). Here $\widehat{c}^{\#}(\beta,\mu) = \left\{\widehat{c}(\beta,\mu) \text{ or } \overline{\widehat{c}(\beta,\mu)}\right\} > 0$ are defined by

$$\sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu;c^{\#}\right) = \widetilde{p}^B\left(\beta,\mu;\widehat{c}^{\#}(\beta,\mu)\neq 0\right), \ \ (\theta,\mu)\in D^B, \tag{5.2}$$

see (4.77). In this section we study this nonconventional condensation in more details. In particular we show a discontinuity of the particle density $\rho_0^B(\theta,\mu)$ in the mode $k=0$ on the line of transitions ∂D^B. The first statement concerns the gauge symmetry breaking in domain D^B; there we use the Bogoliubov concept of quasi-averages ([26] Part I, Section 1.2 or [76]).

5.1.1 The gauge symmetry breaking

Theorem 5.1 *Let $D^B \neq \{\varnothing\}$, i.e. the interaction potential $v(q)$ satisfy the hypothesis (C) (Definition 4.5). Then for the quasi-averages below one gets:*

$$\lim_{\{\nu\to 0:\arg\nu=\varphi\}} \lim_{\Lambda} \left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^{\#})} = e^{\pm i\varphi}\,|\widehat{c}(\beta,\mu)| = \left\{ \begin{array}{c} \neq 0, (\theta,\mu)\in D^B \\ = 0, (\theta,\mu)\in Q\backslash\overline{D^B} \end{array} \right\}. \tag{5.3}$$

Here for short we denote by $\langle - \rangle_{H_\Lambda} \equiv \langle - \rangle_{H_\Lambda}(\beta,\mu)$ the grand-canonical Gibbs state corresponding to a Hamiltonian H_Λ.

Proof. By the global gauge transformation $U_\varphi = e^{i\varphi N_\Lambda}$, $\varphi = \arg\nu$,

$$\widetilde{a}_0^{\#} \equiv U_\varphi a_0^{\#} U_\varphi^* = e^{\mp i\varphi} a_0^{\#}, \ a_0^{\#} = \{a_0 \text{ or } a_0^*\},$$

(cf. (2.57), (2.58)) we get

$$\begin{array}{l} \widetilde{H}_\Lambda^B(\mu,r) = U_\varphi H_\Lambda^B\left(\mu,\nu^{\#}\right)U_\varphi^* = \widetilde{H}_\Lambda^B - \mu\widetilde{N}_\Lambda - \sqrt{V}r\left(\widetilde{a}_0+\widetilde{a}_0^*\right), \\ p_\Lambda\left[H_\Lambda^B\left(\nu^{\#}\right)\right] = p_\Lambda\left[U_\varphi H_\Lambda^B\left(\nu^{\#}\right)U_\varphi^*\right] = p_\Lambda^B\left(\beta,\mu;r=|\nu|\right). \end{array} \tag{5.4}$$

By virtue of

$$0 = \left\langle \left[\widetilde{H}_\Lambda^B(\mu,r), \widetilde{N}_\Lambda\right]\right\rangle_{\widetilde{H}_\Lambda^B(r)} = r\sqrt{V}\left\langle \widetilde{a}_0 - \widetilde{a}_0^*\right\rangle_{\widetilde{H}_\Lambda^B(r)},$$

and of

$$0 \leq \left\langle \left[\widetilde{N}_\Lambda, \left[\widetilde{H}_\Lambda^B(\mu,r), \widetilde{N}_\Lambda\right]\right]\right\rangle_{\widetilde{H}_\Lambda^B(r)} = r\sqrt{V}\left\langle \widetilde{a}_0 + \widetilde{a}_0^*\right\rangle_{\widetilde{H}_\Lambda^B(r)},$$

(cf. (4.87)), we obtain

$$\langle\widetilde{a}_0\rangle_{\widetilde{H}_\Lambda^B(r)} = \langle\widetilde{a}_0^*\rangle_{\widetilde{H}_\Lambda^B(r)} \geq 0. \tag{5.5}$$

Since

$$\begin{aligned}\partial_r^2 p_\Lambda^B(\beta,\mu;r) &= \beta\left(\left\{(\widetilde{a}_0+\widetilde{a}_0^*)-\langle\widetilde{a}_0+\widetilde{a}_0^*\rangle_{\widetilde{H}_\Lambda^B(r)}\right\},\left\{(\widetilde{a}_0+\widetilde{a}_0^*)-\langle\widetilde{a}_0+\widetilde{a}_0^*\rangle_{\widetilde{H}_\Lambda^B(r)}\right\}_{\widetilde{H}_\Lambda^B(r)}\right)\\ &\geq 0,\end{aligned}\tag{5.6}$$

(see (4.86) and (4.137)), by (4.73) the sequence of the convex (for $r \geq 0$) functions $\left\{p_\Lambda^B(\beta,\mu;r)\right\}_\Lambda$ converges to the (convex) function

$$\begin{aligned}\widehat{p}^B(\beta,\mu;r) &\equiv \sup_{c\in\mathbb{C}}\widetilde{p}^B\left(\beta,\mu;c^\#,\nu^\#\right)=\sup_{\substack{|c|\geq 0\\ \psi=\arg c}}\widetilde{p}^B\left(\beta,\mu;|c|\,e^{\pm i\psi},|\nu|\,e^{\pm i\varphi}\right)\\ &= \widetilde{p}^B\left(\beta,\mu;|\widehat{c}(\beta,\mu;r)|\,e^{\pm i\varphi},|\nu|\,e^{\pm i\varphi}\right),\end{aligned}\tag{5.7}$$

locally uniformly in $D^B\times[0,r_0]$, see (4.74) and (4.139). By explicit calculations one finds from (4.74) and (5.7) that the derivatives

$$\begin{aligned}&0\leq\partial_r\widehat{p}^B(\beta,\mu;r)=2\,|\widehat{c}(\beta,\mu;r)|\leq C_1,\\ &0\leq\partial_r^2\widehat{p}^B(\beta,\mu;r)=2\partial_r\,|\widehat{c}(\beta,\mu;r)|\leq C_2,\end{aligned}\tag{5.8}$$

are continuous and bounded in $D^B\times[0,r_0]$. Therefore, the Griffiths lemma (Appendix C) implies

$$\lim_\Lambda\partial_r p_\Lambda\left[\widetilde{H}_\Lambda^B(r)\right]=\lim_\Lambda\left\langle\frac{\widetilde{a}_0+\widetilde{a}_0^*}{\sqrt{V}}\right\rangle_{\widetilde{H}_\Lambda^B(r)}=2\,|\widehat{c}(\beta,\mu;r)|\,,$$

or by (5.5),

$$\begin{aligned}\lim_\Lambda\left\langle\widetilde{a}_0/\sqrt{V}\right\rangle_{\widetilde{H}_\Lambda^B(r)}&=|\widehat{c}(\beta,\mu;r)|\,,\\ \lim_\Lambda\left\langle\widetilde{a}_0^*/\sqrt{V}\right\rangle_{\widetilde{H}_\Lambda^B(r)}&=|\widehat{c}(\beta,\mu;r)|\,.\end{aligned}\tag{5.9}$$

Returning back to original creation/annihilation operators, one obtains from (5.9)

$$\begin{aligned}\lim_\Lambda\left\langle a_0/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)}&=e^{+i\varphi}\,|\widehat{c}(\beta,\mu;r)|\,,\\ \lim_\Lambda\left\langle a_0^*/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)}&=e^{-i\varphi}\,|\widehat{c}(\beta,\mu;r)|\,.\end{aligned}\tag{5.10}$$

Then the first part of the statement (5.3) follows from (5.10) and the continuity of the solution $\widehat{c}(\beta,\mu;r)$ at $r=0$, whereas the second part follows from (5.1). ■

Corollary 5.2 *Notice that by the gauge invariance we have*

$$\left\langle a_0^\#/\sqrt{V}\right\rangle_{H_\Lambda^B}=\left\langle a_0^\#/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#=0)}=0.\tag{5.11}$$

Therefore, we have a gauge-symmetry breaking phase transition across the line ∂D^B:

$$\lim_{\nu\to 0}\lim_\Lambda\left\langle a_0^\#/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)}\neq\lim_\Lambda\lim_{\nu\to 0}\left\langle a_0^\#/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)},\tag{5.12}$$

as soon as the condensate $\rho_0^B(\theta,\mu)\neq 0$.

The very same observation is obviously valid for the PBG, i.e. for $T_\Lambda\left(\nu^{\#}\right)$, see [26] Part I, Section 1.2. Therefore, the gauge symmetry breaking indicates in general the presence of the condensate order parameter in the system.

5.1.2 Bose condensation in the fundamental mode $k = 0$

Since the domain D^B (4.57) of nontriviality of the WIBG corresponds to the existence of a Bose condensation in the zero-mode (5.1) (Theorem 4.7), by (5.10) it is plausible that $\rho_0^B(\theta,\mu) = |\widehat{c}(\beta,\mu)|^2$. We would like to warn reader that Theorem 5.1 does not imply this.
Notice that the Cauchy-Schwarz inequality for states gives

$$\langle a_0^*\rangle_{H_\Lambda^B(\nu^{\#})}\langle a_0\rangle_{H_\Lambda^B(\nu^{\#})} \leq \langle a_0^* a_0\rangle_{H_\Lambda^B(\nu^{\#})}.$$

Hence, by (4.62) and (5.3) one gets

$$|\widehat{c}(\beta,\mu)|^2 \leq \lim_{\nu\to 0}\lim_{\Lambda}\left\langle \frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B(\nu^{\#})} = \rho_0^B(\theta,\mu). \tag{5.13}$$

To prove equality in (5.13) we proceed as follows. Let us introduce an auxiliary Hamiltonian:

$$H_{\Lambda,\gamma_0}^B = H_\Lambda^B + \gamma_0 a_0^* a_0,\ \gamma_0 \in \mathbb{R}^1. \tag{5.14}$$

We denote by

$$p_\Lambda^B(\beta,\mu,\gamma_0) = p_\Lambda\left[H_{\Lambda,\gamma_0}^B\right] \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}_B} e^{-\beta H_{\Lambda,\gamma_0}^B(\mu)} \tag{5.15}$$

the pressure associated with H_{Λ,γ_0}^B in the grand canonical ensemble, and by $p^B(\beta,\mu,\gamma_0)$ the corresponding thermodynamic limit. To calculate the pressure $p^B(\beta,\mu,\gamma_0)$, we use the line of reasoning of Sections 4.3 and 4.4. Consequently we start with the approximated pressure

$$\widetilde{p}_\Lambda^B\left(\beta,\mu,\gamma_0;c^{\#}\right) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}'_B} e^{-\beta H_{\Lambda,\gamma_0}^B\left(c^{\#},\mu\right)},\ c\in\mathbb{C},$$

associated with the trial Hamiltonian $H_{\Lambda,\gamma_0}^B\left(c^{\#},\mu\right)$ (see Definition B.1 in Appendix B). We denote by $\widetilde{p}^B\left(\beta,\mu,\gamma_0;c^{\#}\right)$ the corresponding thermodynamic limit.

Remark 5.3 *By (3.68), (3.70) and (3.71) we find*

$$p^B(\beta,\mu,\gamma_0) \leq \lim_{\Lambda} p_\Lambda\left[\widetilde{H}_\Lambda + \gamma_0 a_0^* a_0\right] = \sup_{\rho_0\geq 0} G(\beta,\mu,\gamma_0;\rho_0) < +\infty,\ \gamma_0\in\mathbb{R}^1,$$

with

$$G(\beta,\mu,\gamma_0;\rho_0) \equiv \left[-\frac{v(0)}{2}\rho_0^2 + \left(\mu-\gamma_0+\frac{1}{2}\varphi(0)\right)\rho_0 + p^P(\beta,\mu-v(0)\rho_0)\right].$$

Therefore we have:

Theorem 5.4 *Let interaction potential $v(q)$ satisfy hypotheses (A) and (B). Then for $\gamma_0 \in \mathbb{R}^1$, one gets*

$$p^B(\beta,\mu,\gamma_0) = \sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu,\gamma_0;c^{\#}\right) \equiv \lim_{\Lambda}\left\{\sup_{c\in\mathbb{C}} \widetilde{p}^B_\Lambda\left(\beta,\mu,\gamma_0;c^{\#}\right)\right\}, \tag{5.16}$$

in the stability domain $Q^B = Q \equiv \{(\theta,\mu) : \theta \geq 0,\ \mu \leq 0\}$ (cf. (4.54), (4.55)).

Proof. The proof is an almost literal replica of the one of the Theorem 4.8:
(a) Let us define $H^B_{\Lambda,\gamma_0}\left(\nu^{\#}\right)$ by

$$H^B_{\Lambda,\gamma_0}\left(\nu^{\#}\right) = H^B_{\Lambda,\gamma_0} - \sqrt{V}\left(\nu a_0^* + \overline{\nu} a_0\right), \tag{5.17}$$

and $D^B_{\gamma_0}$ by

$$D^B_{\gamma_0} \equiv \left\{(\theta,\mu)\in Q:\ p^B\left(\beta,\mu,\gamma_0;\nu^{\#}=0\right) = p_\Lambda\left[H^B_{\Lambda,\gamma_0}\left(\nu^{\#}=0\right)\right] > p^P(\beta,\mu)\right\}. \tag{5.18}$$

If $v(q)$ satisfies also the hypothesis (C), then by Remark 5.3 the domain $D^B_{\gamma_0} \neq \{\varnothing\}$ for $\gamma_0 < -\mu_0$, see Corollary 4.10. For $\gamma_0 \geq 1/2\varphi(0)$ one gets $D^B_{\gamma_0} = \{\varnothing\}$.
(b) Since the Bogoliubov approximation (cf. Definition B.1) gives the estimate of the pressure $p_\Lambda\left[H^B_{\Lambda,\gamma_0}\left(\nu^{\#}\right)\right]$ from below (see Proposition B.2):

$$\sup_{c\in\mathbb{C}} \widetilde{p}^B_\Lambda\left(\beta,\mu,\gamma_0;c^{\#},\nu^{\#}\right) \leq p_\Lambda\left[H^B_{\Lambda,\gamma_0}\left(\nu^{\#}\right)\right],$$

by the Bogoliubov inequality (Appendix D) we get:

$$\begin{aligned} 0 &\leq \Delta_{\Lambda,\gamma_0}\left(\beta,\mu;c^{\#},\nu^{\#}\right) \equiv p_\Lambda\left[H^B_{\Lambda,\gamma_0}\left(\nu^{\#}\right)\right] - \widetilde{p}^B_\Lambda\left(\beta,\mu,\gamma_0;c^{\#},\nu^{\#}\right) \\ &\leq \frac{1}{V}\left\langle H^B_{\Lambda,\gamma_0}\left(\widehat{c}^{\#},\mu,\nu^{\#}\right) - H^B_{\Lambda,\gamma_0}\left(\mu,\nu^{\#}\right)\right\rangle_{H^B_{\Lambda,\gamma_0}(\nu^{\#})}, \end{aligned} \tag{5.19}$$

cf. (4.81).
(c) For operators $A^{\#} \equiv a_0^{\#} - \sqrt{V}c^{\#}$ and for a Taylor expansion of $H^B_{\Lambda,\gamma_0}\left(\widehat{c}^{\#},\mu,\nu^{\#}\right)$ around $a_0^{\#}$ one obtains the estimate:

$$\begin{aligned} 0 &\leq \inf_{c\in\mathbb{C}} \Delta_{\Lambda,\gamma_0}\left(\beta,\mu;c^{\#},\nu^{\#}\right) = \Delta_{\Lambda,\gamma_0}\left(\beta,\mu;\widehat{c}^{\#}_{\Lambda,\gamma_0}\left(\beta,\mu;\nu^{\#}\right),\nu^{\#}\right) \leq \\ &\leq u_{\gamma_0} + \frac{w_{\gamma_0}}{2}\left\langle\left\{\left(a_0^* - \sqrt{V}\overline{c}\right),\left(a_0 - \sqrt{V}c\right)\right\}\right\rangle_{H^B_\Lambda(\nu^{\#})}. \end{aligned} \tag{5.20}$$

by repeating the line of reasoning developed in Remarks 4.11-4.21. The only difference with the case $\gamma_0 = 0$ comes from

$$\left[A,\left[H^B_{\Lambda,\gamma_0}\left(\mu,\nu^{\#}\right),A^*\right]\right] = \left[A,\left[H^B_\Lambda\left(\mu,\nu^{\#}\right),A^*\right]\right] + \gamma_0,$$

cf. (4.134), and the note that $\lim_{\gamma_0\to 0} u_{\gamma_0} = u$ et $\lim_{\gamma_0\to 0} w_{\gamma_0} = w$.
(d) Put $c^{\#} \equiv \left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H^B_{\Lambda,\gamma_0}(\nu^{\#})}$ in the left-hand side of (5.20). The same arguments as in Theorem 4.23 give the asymptotic estimate

$$0 \leq \inf_{c\in\mathbb{C}} \Delta_{\Lambda,\gamma_0}\left(\beta,\mu;c^{\#},\nu^{\#}\right) \leq \delta_{\Lambda,\gamma_0} \equiv \mathit{const}\,(\gamma_0)\,\frac{1}{\sqrt{V}} \tag{5.21}$$

for $(\theta,\mu)\in Q\backslash\partial D^B_{\gamma_0}$ and $|\nu|\le r_0$ for some fixed r_0.
(e) Therefore, in the case of empty or non-empty domain $D^B_{\gamma_0}$ (5.18) we have

$$p^B\left(\beta,\mu,\gamma_0;\nu^{\#}\right)=\lim_{\Lambda} p_\Lambda\left[H^B_{\Lambda,\gamma_0}\left(\nu^{\#}\right)\right]=\lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}^B_\Lambda\left(\beta,\mu,\gamma_0;c^{\#},\nu^{\#}\right)\right], \tag{5.22}$$

for $|\nu|\le r_0, r_0>0$ and $(\theta,\mu)\in Q\backslash\partial D^B_{\gamma_0}$. Hence for $\nu^{\#}=0$ the thermodynamic limit (5.22) implies the Theorem 5.4. ■

Corollary 5.5 *We have*

$$\rho^B_0(\theta,\mu)=\lim_{\Lambda}\left\langle\frac{a_0^*a_0}{V}\right\rangle_{H^B_\Lambda}=\lim_{\Lambda}\left\langle\frac{a_0^*}{\sqrt{V}}\right\rangle_{H^B_\Lambda}\lim_{\Lambda}\left\langle\frac{a_0}{\sqrt{V}}\right\rangle_{H^B_\Lambda}=\left|\widehat{c}\left(\beta,\mu\right)\right|^2. \tag{5.23}$$

Proof. Since

$$\partial^2_{\gamma_0}p^B_\Lambda(\beta,\mu,\gamma_0)=\frac{\beta}{V}\left(\left(a_0^*a_0-\langle a_0^*a_0\rangle_{H^B_{\Lambda,\gamma_0}}\right),\left(a_0^*a_0-\langle a_0^*a_0\rangle_{H^B_{\Lambda,\gamma_0}}\right)\right)_{H^B_{\Lambda,\gamma_0}}\ge 0,$$

functions $\left\{p^B_\Lambda(\beta,\mu,\gamma_0)\right\}_\Lambda$ (5.15) are convex for $\gamma_0\in\mathbb{R}^1$ and derivable for $|\gamma_0|\le-\mu_0$ (Remark 5.3). By explicit calculations we find

$$\begin{aligned}\sup_{c\in\mathbb{C}}\widetilde{p}^B\left(\beta,\mu,\gamma_0;c^{\#}\right)&=\widetilde{p}^B\left(\beta,\mu,\gamma_0;\widehat{c}^{\#}_{\gamma_0}(\beta,\mu)\right)\\&=\widetilde{p}^B\left(\beta,\mu;\widehat{c}^{\#}_{\gamma_0}(\beta,\mu)\right)-\gamma_0\left|\widehat{c}_{\gamma_0}(\beta,\mu)\right|^2.\end{aligned} \tag{5.24}$$

Hence

$$\partial_{\gamma_0}\widetilde{p}^B\left(\beta,\mu,\gamma_0;\widehat{c}^{\#}_{\gamma_0}(\beta,\mu)\right)=-\left|\widehat{c}_{\gamma_0}(\beta,\mu)\right|^2<const, \tag{5.25}$$

for $(\theta,\mu)\in Q$ and $|\gamma_0|\le-\mu_0$. Therefore by the Griffiths lemma (Appendix C) and by Theorem 5.4, we obtain

$$\lim_{\Lambda}\partial_{\gamma_0}p^B_\Lambda(\beta,\mu,\gamma_0)=\lim_{\Lambda}\left(-\left\langle\frac{a_0^*a_0}{V}\right\rangle_{H^B_{\Lambda,\gamma_0}}\right)=-\left|\widehat{c}_{\gamma_0}(\beta,\mu)\right|^2. \tag{5.26}$$

By continuity of (5.26) for $\gamma_0=0$ we get (5.23). ■
We conclude this section by analysis of the Bose condensate density $\rho^B_0(\theta,\mu)$ behaviour.

Theorem 5.6 *If the interaction potential $v(k)$ satisfies hypotheses (A)-(C) (see Section 4.2), then the domain $D^B\neq\{\varnothing\}$ and the Bose condensate undergo a jump on the boundary ∂D^B:*

$$\rho^B_0(\theta,\mu)\equiv\lim_{\Lambda}\left\langle\frac{a_0^*a_0}{V}\right\rangle_{H^B_\Lambda}(\beta,\mu)=\left\{\begin{array}{c}\left|\widehat{c}(\beta,\mu)\right|^2>0,\ (\theta,\mu)\in D^B\\ 0,\ (\theta,\mu)\in Q\backslash\overline{D^B}\end{array}\right\}. \tag{5.27}$$

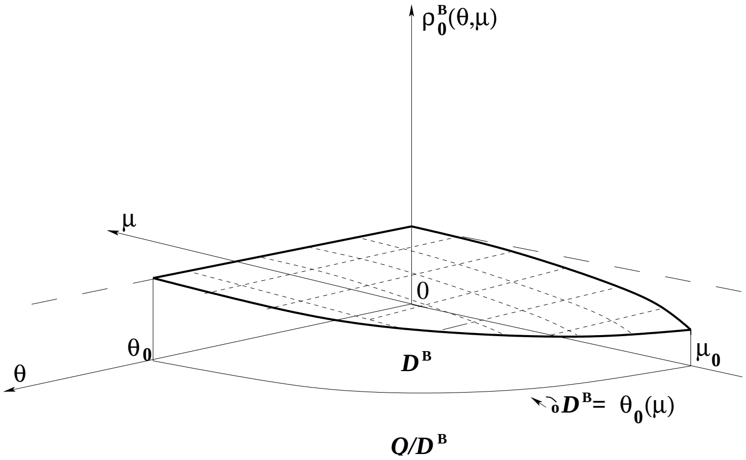

Figure 5.1: Illustration of the nonconventional Bose condensate density $\rho_0^B(\theta,\mu)$ as a function of μ and θ.

Proof. By (5.23), the analysis of $\rho_0^B(\theta,\mu)$ corresponds to the one of $\widehat{x}(\theta,\mu) \equiv |\widehat{c}(\beta,\mu)|^2$ (5.2) which maximizes the pressure (4.74), $\nu^{\#} = 0$. Therefore the limit (5.27) is a direct consequence of Theorem B.9 from Appendix B. ■

Behaviour of the nonconventional Bose condensate density (5.27) is illustrated by Figure 5.1.

Theorem 5.6 shows the existence of the phase transition on the line ∂D^B with appearance of Bose condensation in the mode $k = 0$ in domain D^B. Below we shall see (Section 5.2) that this condensation saturates by the critical value $\rho_0^B(\theta,0)$. This means that for densities $\rho > \rho_0^B(\theta,0)$ the *second* kind of condensation, due to the saturation mechanism, appears and *coexists* with the *first* kind of condensation $\rho_0^B(\theta,0)$. In fact in the next Section 5.1.3 we show that the *nonconventional* condensation is induced by attraction between bosons in the mode $k = 0$. Whereas there exists $\rho_c^B(\theta)$ such that the conventional BE condensation starts for $\rho > \rho_c^B(\theta) > \rho_0^B(\theta,0)$ via a saturation mechanism, see Section 5.2.

5.1.3 Quantum interpretation

In Section 3.2 we have already seen that the nondiagonal part of the WIBG U_Λ^{BN} (2.53) implies an *effective attraction* between particles in the zero-mode (see (3.47), (3.48) and Figure 3.2). Moreover (3.50)-(3.52) indicate that in the presence of Bose condensate in mode $k = 0$, the nondiagonal interaction U_Λ^{BN} induces an *effective repulsion* between particles outside the zero-mode. Hence in the lowest order of the perturbation theory the nondiagonal interaction U_Λ^{BN} favors an accumulation of particle in mode $k = 0$. Since there is a direct repulsive interaction between particles in mode $k = 0$

$$\frac{v(0)}{2V} a_0^{*^2} a_0^2$$

(see the diagonal interaction U_{Λ}^{BD} (2.52)), to evaluate the balance of interaction in the zero-mode for the WIBG one should estimate the sign of

$$\left(\frac{v(0)}{2}+g_{00}\right)=\frac{v(0)}{2}-\frac{1}{4\left(2\pi\right)^{3}}\int_{\mathbb{R}^{3}}d^{3}k\frac{\left[v(k)\right]^{2}}{\varepsilon_{k}}, \tag{5.28}$$

see (3.53) and (3.54). In fact by Theorem 5.6 we have rigorously shown that the condition (C) (see Definition 4.5) interpreted in (5.28) as a competition between U_{Λ}^{BD} and U_{Λ}^{BN} in the framework of perturbation theory:

$$v(0)-\frac{1}{2\left(2\pi\right)^{3}}\int_{\mathbb{R}^{3}}d^{3}k\frac{\left[v(k)\right]^{2}}{\varepsilon_{k}}=\frac{1}{2}\left(\frac{v(0)}{2}+g_{00}\right)<0, \tag{5.29}$$

gives a *sufficient* and *necessary* condition for non-equivalence between the WIBG and the PBG (cf. Corollary 4.10). Moreover, under (5.29), a Bose condensation $\rho_{0}^{B}\left(\theta,\mu\right)\neq 0$ occurs in the WIBG for $k=0$ in domain

$$D^{B}=\left\{(\theta,\mu)\in Q:\mu_{0}<\mu\leq 0, 0\leq\theta<\theta_{0}\left(\mu\right)\right\}$$

see (4.57), (4.63), Figure 5.1 and Theorem 5.6.

We call this condensation due to interaction between bosons as *nonconventional*, since its mechanism of formation is drastically different from those for the *conventional* BE condensation, see Appendix A.

Notice that if

$$\rho_{0}^{B}\left(\theta,\mu\right)=0 \tag{5.30}$$

then $g_{pq}=0$ (3.52). For this case the nondiagonal interaction U_{Λ}^{BN} has no influence on the thermodynamic behaviour of the WIBG in the sense that the WIBG pressure and the one corresponding to its diagonal part H_{Λ}^{BD} (4.24) coincide in the thermodynamic limit (see Theorem 3.7 and Corollary 4.10). In the next section we show that similar to the diagonal model H_{Λ}^{BD} (4.24) (see Theorem 4.4), the WIBG manifests for $\rho>\rho_{c}^{B}\left(\theta\right)$ a *conventional* BE condensation in modes next to the zero-mode due to the particle-density saturation.

5.2 Conventional Bose-Einstein condensation

5.2.1 Preliminary study: stability domain in finite volume

We have seen that for a *finite* isotropic box $\Lambda\subset\mathbb{R}^{3}$ (or $\mathbb{R}^{d>2}$), the PBG total particle density $\rho_{\Lambda}^{P}\left(\beta,\mu\right)$ is *finite* for $\mu<0$ but becomes *infinite* for $\mu\rightarrow 0^{-}$ (2.30). In particular, the corresponding stability domain is

$$Q_{\Lambda}^{P}\equiv\left\{(\theta,\mu):p_{\Lambda}\left[T_{\Lambda}\right]<+\infty\right\}=\left\{(\theta,\mu):\theta\geq 0,\ \mu<0\right\}.$$

although in the thermodynamic limit we have

$$\lim_{\mu\rightarrow 0^{-}}p^{P}\left(\beta,\mu\right)\equiv p^{P}\left(\beta,0\right)<+\infty,$$

see (2.26), i.e. its domain of stability, in the thermodynamic limit, equals

$$Q^P \equiv \left\{(\theta,\mu): \ \lim_{\Lambda} p_\Lambda\left[T_\Lambda\right] < +\infty\right\} = Q \equiv \left\{(\theta,\mu): \theta \geq 0, \ \mu \leq 0\right\}.$$

It is divergence $\lim_{\mu\to 0^-} \rho_\Lambda^P(\beta,\mu) = +\infty$ before the thermodynamic limit and convergence (saturation) after it: $\lim_{\mu\to 0^-}\lim_{\Lambda} \rho_\Lambda^P(\beta,\mu) = \rho_c^P(\theta) < +\infty$ (for $d > 2$) which is the origin of the *conventional* BE condensation in the fundamental mode $k = 0$ predicted by Einstein [6] in 1925, see Sections 1, 2. A lack of the total-particle saturation for $d \leq 2$ implies the well-known absence of the BE condensation in the PBG for low dimensions $d = 1, 2$.

For the diagonal part of the WIBG the mechanism of saturation is very close to the one in the PBG (cf. Section 4.1.2). The only difference with respect to the PBG concerns the divergence of the particle density, which in a *finite* volume occurs not for $\mu \to 0^-$ but for $\mu \to \varepsilon_{\|k\|=\frac{2\pi}{L}} > 0$. This implies a *conventional* BE condensation not directly in the mode $k = 0$ but in the $(2d = 6)$ modes $\|k\| = 2\pi/L$ (cf. Theorem 4.4) next to $k = 0$.

Consequently, since in the thermodynamic limit the WIBG stability domain $Q^B = Q$ (cf. Theorem 3.8, (4.54) and (4.55)), to proceed further in evaluation of the particle-density saturation we are interested to study the WIBG stability domain in finite volume:

$$Q_\Lambda^B \equiv \left\{(\theta,\mu): p_\Lambda^B(\beta,\mu) < +\infty\right\}. \tag{5.31}$$

If the saturation takes place for $\mu \to 0^-$, then one would guess that the WIBG manifests a *conventional* BE condensation.

Theorem 5.7 *Let $v(k)$ satisfy hypotheses (A) and (B) (see Section 2.2 or Section 4.2). Then there exists $\varepsilon_{\Lambda,1}$:*

$$\varepsilon_{\Lambda,1} \in \left[\inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right], \inf_{k\neq 0}\varepsilon_k\right],$$

such that for $\mu < \varepsilon_{\Lambda,1}$

$$p_\Lambda^B(\beta,\mu) < +\infty, \tag{5.32}$$

with

$$\lim_{\mu\to\varepsilon_{\Lambda,1}} p_\Lambda^B(\beta,\mu) = +\infty. \tag{5.33}$$

Proof. By inequality (3.68) and $v(k) \geq 0$ one has

$$p_\Lambda\left[H_\Lambda^B\right] \leq p_\Lambda\left[\widetilde{H}_\Lambda - \frac{v(0)}{V}\sum_{k\in\Lambda^*\setminus\{0\}} N_0 N_k\right] < +\infty,$$

for $\mu < \inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right]$, where

$$\begin{aligned} p_\Lambda\left[\widetilde{H}_\Lambda - \frac{v(0)}{V}\sum_{k\in\Lambda^*\setminus\{0\}} N_0 N_k\right] &= \frac{1}{\beta V}\sum_{k\in\Lambda^*\setminus\{0\}} \ln\left\{1 - e^{-\beta\left[\varepsilon_k - \left(\mu + \frac{v(k)}{2V}\right)\right]}\right\}^{-1} \\ &\quad + \frac{1}{\beta V}\ln\sum_{n_0=0}^{\infty} e^{\beta V\left[\left(\mu+\frac{1}{2}\varphi(0)\right)\left(\frac{n_0}{V}\right) - \frac{v(0)}{2}\left(\frac{n_0}{V}\right)^2\right]}. \end{aligned}$$

Hence for $\mu < \inf\limits_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right]$ we have

$$p_\Lambda^B(\beta,\mu) < +\infty. \tag{5.34}$$

Moreover the estimates (3.66) and (4.28) imply that

$$\lim_{\mu\to\inf\limits_{k\neq 0}\varepsilon_k} p_\Lambda^B(\beta,\mu) = +\infty. \tag{5.35}$$

By (5.34) and (5.35), we deduce that there exists

$$\varepsilon_{\Lambda,1} \in \left[\inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right], \inf_{k\neq 0}\varepsilon_k\right]$$

such that the pressure $p_\Lambda^B(\beta,\mu)$ is bounded for $\mu < \varepsilon_{\Lambda,1}$, whereas

$$\lim_{\mu\to\varepsilon_{\Lambda,1}} p_\Lambda^B(\beta,\mu) = +\infty. \tag{5.36}$$

■

By Theorem 5.7, the stability domain Q_Λ^B (5.31) of $p_\Lambda^B(\beta,\mu)$ is equal to

$$Q_\Lambda^B = \{(\theta,\mu) : \theta \geq 0,\ \mu < \varepsilon_{\Lambda,1}\}.$$

Of course, in the thermodynamic limit one has

$$\lim_\Lambda Q_\Lambda^B = Q^B = Q,$$

see (4.54), (4.55). Since for a sufficiently large volume $V = L^3$ we have

$$\varepsilon_{\Lambda,1} \in \left[\inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right], \inf_{k\neq 0}\varepsilon_k\right], \tag{5.37}$$

with

$$\inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right] = \varepsilon_{\|k\|=\frac{2\pi}{L}} - \frac{v\left(\|k\| = \frac{2\pi}{L}\right)}{2V} > 0, \tag{5.38}$$

the pressure $p_\Lambda^B(\beta,\mu)$ is defined for a small interval of *positive* chemical potentials μ: $0 \leq \mu < \varepsilon_{\Lambda,1}$. Now one could proceed with convexity arguments, similar to those for the H_Λ^{BD} (see Section 4.1.2), to argue the existence of conventional BE condensation in the vicinity of $\varepsilon_{\Lambda,1}$, or in modes $\|k\| = 2\pi/L$, see (5.38), for large total density ρ.

This allows us to guess that the *conventional* BE condensation should be very close to the one manifested by the diagonal part of the WIBG (4.24). In Section 5.2.2 we shall see that this assumption is correct.

5.2.2 Two stages of condensation in the Weakly Imperfect Bose-Gas

Thermodynamic study of the WIBG in Section 5.1 is realized by fixing the chemical potential μ, i.e. in the grand canonical ensemble (θ, μ). On the other hand, we know that the *conventional* BE condensation in the PBG (2.39) starts, when the grand-canonical particle density $\rho^P(\beta,\mu)$ is saturated: $\rho_c^P(\theta) = \sup_{\mu\leq 0} \rho^P(\beta,\mu) < +\infty$ (2.32), i.e. when the total density $\rho > \rho_c^P(\theta)$, see Section 2.1. Notice that by Remark 4.21 and Theorem 5.6 the nonconventional condensation density $\rho_0^B(\theta,\mu)$ is bounded. Therefore, our next step is to verify whether it is true for the total particle density $\rho^B(\beta,\mu)$ in the grand canonical ensemble for the WIBG in the thermodynamic limit:

$$\rho^B(\beta,\mu) \equiv \lim_{\Lambda} \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B}.$$

By the Griffiths lemma (Appendix C) and by Theorem 4.8 (4.68), the particle density for the WIBG in the grand canonical ensemble is equal to

$$\begin{aligned}\rho^B(\beta,\mu) &= \lim_{\Lambda} \frac{1}{V}\sum_{k\in\Lambda^*} \langle N_k\rangle_{H_\Lambda^B} = \lim_{\Lambda} \partial_\mu p_\Lambda^B(\beta,\mu) \\ &= \partial_\mu \widetilde{p}^B(\beta,\mu;0) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \left(e^{\beta(\varepsilon_k-\mu)}-1\right)^{-1} d^3k = \rho^P(\beta,\mu),\end{aligned} \tag{5.39}$$

for $(\theta, \mu<0) \in Q\backslash\overline{D^B}$ (cf. (2.26)), and to

$$\begin{aligned}\rho^B(\beta,\mu) &= \partial_\mu \widetilde{p}^B\left(\beta,\mu;\widehat{c}^{\#}(\beta,\mu)\neq 0\right) \\ &= |\widehat{c}(\beta,\mu)|^2 + \frac{1}{(2\pi)^3}\left[\int_{\mathbb{R}^3} \frac{f_k}{E_k^B}\left(e^{\beta E_k^B}-1\right)^{-1} + \frac{h_k^2}{2E_k^B(f_k+E_k^B)} d^3k\right]_{c=\widehat{c}(\beta,\mu)}\end{aligned} \tag{5.40}$$

for $(\theta,\mu<0)\in D^B$, see (4.74) with $\nu^{\#}=0$.

Then from (5.39) and (5.40), we see that the total particle density $\rho^B(\beta,\mu)$ attains its maximal (critical) value $\rho_c^B(\theta)\equiv\rho^B(\beta,\mu=0)$ at $\mu=0$:
(i) for $\theta>\theta_0\,(\mu=0)$ one gets

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \left(e^{\beta\varepsilon_k}-1\right)^{-1} d^3k = \rho_c^P(\theta) < +\infty, \tag{5.41}$$

see (2.32),
(ii) for $\theta<\theta_0\,(\mu=0)$ one has

$$\begin{aligned}\rho_c^B(\theta) &= \frac{1}{(2\pi)^3}\left[\int_{\mathbb{R}^3} \frac{f_k}{E_k^B}\left(e^{\beta E_k^B}-1\right)^{-1} + \frac{h_k^2}{2E_k^B(f_k+E_k^B)} d^3k\right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}} \\ &\quad + |\widehat{c}(\beta,\mu=0)|^2 < +\infty,\end{aligned} \tag{5.42}$$

by virtue of monotonicity of $|\widehat{c}(\beta,\mu)|^2$ and the estimate $|\widehat{c}(\beta,\mu=0)|^2 \leq const$, see (4.74) and Remark 4.21.

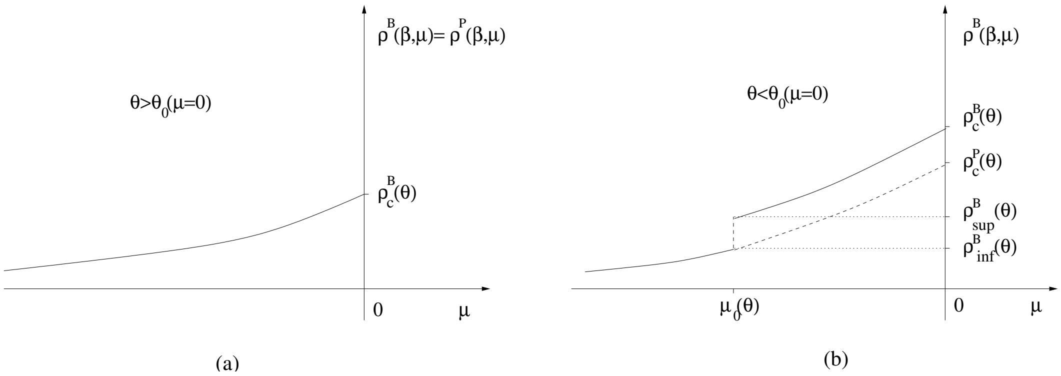

Figure 5.2: Illustration of the grand-canonical total particle density for the model H_{Λ}^{B} as a function of the chemical potential μ at fixed temperature $\theta = \beta^{-1}$: (a) if $\theta > \theta_0(0)$: the graph of $\rho^B(\beta,\mu) = \rho^P(\beta,\mu)$, where $\rho_c^P(\theta)$ is the critical density for the PBG (notice that $\rho_c^B(\theta) = \rho_c^P(\theta) \equiv \rho^P(\theta,0)$); (b) if $\theta < \theta_0(0)$: the graph of $\rho^B(\beta,\mu) \geq \rho^P(\beta,\mu)$, notice that in this case $\rho_c^B(\theta) > \rho_c^P(\theta)$.

Remark 5.8 *By convexity of $p^B(\beta,\mu)$ with respect to μ one gets that the function $\rho^B(\beta,\mu)$ (5.39), (5.40) is monotonous and*

$$\lim_{\mu\to\mu_0(\theta)^-} \rho^B(\beta,\mu) \equiv \rho^B_{\inf}(\theta) < \lim_{\mu\to\mu_0(\theta)^+} \rho^B(\beta,\mu) \equiv \rho^B_{\sup}(\theta), \tag{5.43}$$

where $\mu_0 = \mu_0(\theta)$ is the function inverse to the function $\theta_0(\mu)$, see (4.78), and correspondingly:

$$\lim_{\theta\to\theta_0(0)^+} \rho_c^B(\theta) < \lim_{\theta\to\theta_0(0)^-} \rho_c^B(\theta). \tag{5.44}$$

The total density $\rho^B(\beta,\mu)$ is illustrated by Figure 5.2.

The aim of the present section is to analyze the thermodynamic behaviour of the WIBG as a function of the total particle density ρ.

Since for $\rho < \rho_c^B(\theta)$ (cf. (5.41), (5.42)) the thermodynamic study of the WIBG corresponds to the one in the grand canonical ensemble (θ,μ) (see below Remark 5.10), the essential problem concerns its thermodynamic behaviour for densities $\rho \geq \rho_c^B(\theta)$. Below we show, for these densities, the existence of a *conventional (generalized)* BE condensation in the WIBG.

First we note that by Theorem 5.7 there is $\varepsilon_{\Lambda,1}$:

$$\varepsilon_{\Lambda,1} \in \left[\inf_{k\neq 0}\left[\varepsilon_k - \frac{v(k)}{2V}\right], \inf_{k\neq 0} \varepsilon_k = \varepsilon_{\|k\|=\frac{2\pi}{L}}\right],$$

such that

$$p_{\Lambda}^B(\beta,\mu<\varepsilon_{\Lambda,1}) < +\infty, \tag{5.45}$$

and

$$\lim_{\mu\to\varepsilon_{\Lambda,1}} p_\Lambda^B(\beta,\mu) = +\infty. \tag{5.46}$$

Since $\varepsilon_{\Lambda,1} > 0$, by convexity of $p_\Lambda^B(\beta,\mu)$ as a function of μ, we have

$$\begin{aligned} \frac{p_\Lambda^B(\beta,\mu) - p_\Lambda^B(\beta,0)}{\mu} &\geq \partial_\mu p_\Lambda^B(\beta,\mu) = \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu),\ \ \mu < 0; \\ \frac{p_\Lambda^B(\beta,\mu) - p_\Lambda^B(\beta,0)}{\mu} &\leq \partial_\mu p_\Lambda^B(\beta,\mu) = \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu),\ \ \mu > 0. \end{aligned}$$

Consequently by (5.45) and (5.46) we get

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu) < +\infty, \text{ for } \mu < 0, \tag{5.47}$$

although

$$\lim_{\mu\to\varepsilon_{\Lambda,1}} \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu) = +\infty. \tag{5.48}$$

Hence by the monotonicity of $\langle \frac{N_\Lambda}{V} \rangle_{H_\Lambda^B}(\beta,\mu)$ as a function of $\mu < \varepsilon_{\Lambda,1}$ (i.e. by strict convexity of $p_\Lambda^B(\beta,\mu)$) one gets the following corollary:

Corollary 5.9 *For any ρ, there is a unique root $\mu_\Lambda^B(\theta,\rho) < \varepsilon_{\Lambda,1} < \varepsilon_{\|k\|=\frac{2\pi}{L}}$ (notice that, in general, $\mu_\Lambda^B(\theta,\rho)$ could be negative or positive) of the equation*

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu) = \rho, \tag{5.49}$$

and

$$\lim_\Lambda \mu_\Lambda^B\left(\theta, \rho \geq \rho_c^B(\theta)\right) = 0. \tag{5.50}$$

Remark 5.10 *Notice that for $\rho < \rho_c^B(\theta)$ the monotonicity of $\rho_\Lambda^B(\theta,\mu)$ for $\mu \leq \varepsilon_{\Lambda,1}$ implies that (5.49) has a unique solution such that*

$$\mu^B(\theta,\rho) = \lim_\Lambda \mu_\Lambda^B(\theta,\rho) < 0$$

independent of the presence of the nonconventional condensation, see (5.39) and (5.40).

From Section 5.1.2 and monotonicity of $|\widehat{c}(\beta,\mu)|$ the WIBG condensation in the mode $k=0$ is saturated for $\mu \to 0^-$ either by $|\widehat{c}(\beta,0)|^2 > 0$ (for $\theta < \theta_0(0)$), or by $|\widehat{c}(\beta,0)|^2 = 0$ (for $\theta > \theta_0(0)$), see (5.27) and Figure 5.1. Therefore, by (5.39)-(5.42) and by (5.47), (5.48), the saturation of the total particle density $\rho^B(\beta,\mu)$ for $\mu \to 0^-$ should imply a *conventional* Bose condensation in the modes *next* to $k=0$, since the $k=0$ mode is already occupied by the saturated nonconventional condensate. For discussion of coexistence of these *two kinds* of condensations in the framework of a simplified models, see e.g. recent papers [58, 59].

To control the condensation for $k \neq 0$ we introduce an auxiliary Hamiltonian

$$H^B_{\Lambda,\gamma} = H^B_\Lambda - \gamma \sum_{k\in\Lambda^*, a<\|k\|<b} a^*_k a_k, \tag{5.51}$$

for $0 < a < b$. We define the corresponding pressure by

$$p^B_\Lambda(\beta,\mu,\gamma) = p_\Lambda\left[H^B_{\Lambda,\gamma}\right] \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}_B} e^{-\beta H^B_{\Lambda,\gamma}(\mu)}. \tag{5.52}$$

Recall that by (4.78) $\mu_0(\theta)$ is the function (inverse to $\theta_0(\mu)$) which defines a borderline of domain D^B, see Figure 5.1.

Theorem 5.11 *Let $\gamma \in [-\delta,\delta]$ where $0 \le \delta \le \varepsilon_a/2$ and $\varepsilon_a = \inf_{\|k\|\ge a} \varepsilon_k$. Then there exists a domain $D^B_\delta \subset D^B$:*

$$D^B_\delta \equiv \{(\theta,\mu) \in Q : \mu_0 < \mu_0(\delta) \le \mu \le 0,\ 0 \le \theta \le \theta_0(\mu,\delta) < \theta_0(\mu)\} \tag{5.53}$$

(see Figure 5.3) such that uniformly in $\gamma \in [-\delta,\delta]$ one has

$$\left| p^B_\Lambda(\beta,\mu,\gamma) - \sup_{c\in\mathbb{C}} \widetilde{p}^B_\Lambda\left(\beta,\mu,\gamma;c^\#\right)\right| \le \frac{K(\delta)}{\sqrt{V}} \tag{5.54}$$

for V sufficiently large and for:

$$\begin{aligned}&(i)\ \ (\theta,\mu) \in D^B_\delta,\ if\ \mu^B_\Lambda\left(\theta,\rho \ge \rho^B_c(\theta)\right) \le 0;\\ &(ii)\ \ (\theta,\mu) \in D^B_\delta \cup \left\{(\theta,\mu) : 0 \le \mu \le \mu^B_\Lambda\left(\theta,\rho \ge \rho^B_c(\theta)\right),\ 0 \le \theta \le \theta_0(\mu = 0,\delta)\right\},\\ &if\ \mu^B_\Lambda\left(\theta,\rho \ge \rho^B_c(\theta)\right) \ge 0.\end{aligned} \tag{5.55}$$

Proof. The existence of the domain D^B_δ follows from the proof of Theorem 4.8. This means that the estimate (5.54) is stable with respect to local perturbations of the free-particle spectrum: $\varepsilon_k \to \varepsilon_k - \gamma\chi_{(a,b)}(\|k\|)$ for $|\gamma| \le \delta \le \varepsilon_a/2$ in a reduced domain $D^B_\delta \subset D^B$. Here $\chi_{(a,b)}(\|k\|)$ is the characteristic function of interval $(a,b) \subset \mathbb{R}^1$. Extension in (5.55) is due to continuity of the pressure $p^B_\Lambda(\beta,\mu,\gamma)$ and the trial pressure $\widetilde{p}^B_\Lambda\left(\beta,\mu,\gamma;c^\#\right)$ in parameters $\gamma \in [-\delta,\delta]$ and $\mu \le \mu^B_\Lambda\left(\theta,\rho \ge \rho^B_c(\theta)\right)$, see Corollary 5.9. ∎

Corollary 5.12 *Let $\rho \ge \rho^B_c(\theta)$, see (5.41), (5.42). Then for $\theta < \theta_0(0)$ one has*

$$\begin{aligned}\lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*, a<\|k\|<b} \langle N_k\rangle_{H^B_\Lambda}\left(\beta,\mu^B_\Lambda(\theta,\rho)\right) &= \frac{1}{(2\pi)^3}\left[\int_{a<\|k\|<b} \frac{f_k}{E^B_k}\left(e^{\beta E^B_k}-1\right)^{-1} d^3k\right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}}\\ &+\frac{1}{(2\pi)^3}\left[\int_{a<\|k\|<b} \frac{h^2_k}{2E^B_k\left(f_k+E^B_k\right)} d^3k\right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}}\end{aligned} \tag{5.56}$$

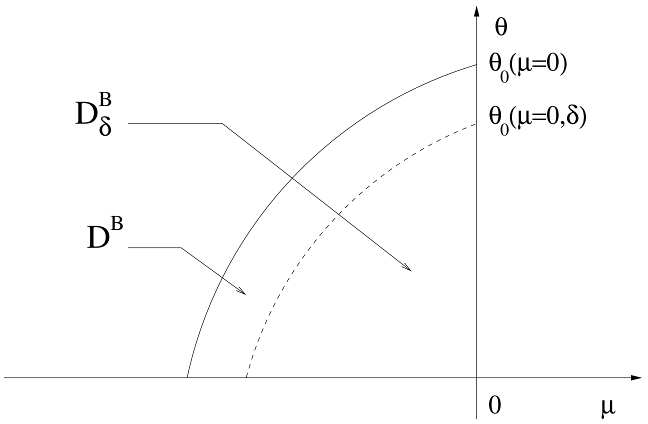

Figure 5.3: Illustration of the domain $D^B_\delta \subset D^B$.

whereas for $\theta > \theta_0(0)$ one gets

$$\lim_\Lambda \frac{1}{V} \sum_{k\in\Lambda^*, a<\|k\|<b} \langle N_k \rangle_{H^B_\Lambda} \left(\beta, \mu^B_\Lambda(\theta,\rho)\right) = \frac{1}{(2\pi)^3} \int_{a<\|k\|<b} \left(e^{\beta\varepsilon_k} - 1\right)^{-1} d^3k. \tag{5.57}$$

Proof. Consider the sequence of functions (5.52)

$$\left\{p^B_\Lambda\left(\beta, \mu^B_\Lambda(\theta,\rho), \gamma\right)\right\}_\Lambda,$$

where chemical potential $\mu^B_\Lambda(\theta,\rho)$ is a solution of (5.49). By (5.51), (5.52), we have

$$\partial_\gamma p^B_\Lambda\left(\beta, \mu^B_\Lambda(\theta,\rho), \gamma\right) = \frac{1}{V} \sum_{k\in\Lambda^*, a<\|k\|<b} \langle N_k \rangle_{H^B_{\Lambda,\gamma}} \left(\beta, \mu^B_\Lambda(\theta,\rho)\right) \tag{5.58}$$

and $\left\{p^B_\Lambda\left(\beta, \mu^B_\Lambda(\theta,\rho), \gamma\right)\right\}_\Lambda$ are convex functions of $\gamma \in [-\delta, \delta]$. Then Theorem 5.11 combining with the Griffiths lemma (Appendix C) implies

$$\begin{aligned} \lim_\Lambda \partial_\gamma p^B_\Lambda\left(\beta, \mu^B_\Lambda(\theta,\rho), \gamma\right) &= \lim_\Lambda \frac{1}{V} \sum_{k\in\Lambda^*, a<\|k\|<b} \langle N_k \rangle_{H^B_{\Lambda,\gamma}} \left(\beta, \mu^B_\Lambda(\theta,\rho)\right) \\ &= \partial_\gamma \lim_\Lambda \sup_{c\in\mathbb{C}} \widetilde{p}^B_\Lambda\left(\beta, \mu^B_\Lambda(\theta,\rho), \gamma; c^\#\right), \end{aligned} \tag{5.59}$$

for $\gamma \in [-\delta, \delta]$. By explicit calculations in the right-hand side of (5.59) (see (4.74) with $\nu^\# = 0$), one obtains the limits (5.56) and (5.57) for $\gamma = 0$. ■

Definition 5.13 *The mean particle values*

$$\omega^B_\Lambda(N_k) = \langle N_k \rangle_{H^B_\Lambda}(\beta, \mu)$$

and

$$\omega^B_{\Lambda,\rho}(N_k) = \langle N_k \rangle_{H^B_\Lambda}\left(\beta, \mu^B_\Lambda(\theta,\rho)\right)$$

are defined on the discrete set Λ^* (2.1). Below we denote by $\left\{\omega^B_\Lambda(N_k)\right\}_{k\in\mathbb{R}^3}$ (or $\left\{\omega^B_{\Lambda,\rho}(N_k)\right\}_{k\in\mathbb{R}^3}$) a continuous interpolation of these values from the set Λ^* to $\mathbb{R}^3$.

To evaluate the *type* of the second (conventional) condensation for $\rho > \rho_c^B(\theta)$, we need the following lemma:

Lemma 5.14 *Let interaction $v(k)$ verify hypotheses (A) and (B). Then for $\|k\| > 2\pi/L$ one has:*

$$\langle N_k \rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta,\rho)\right) \leq \frac{1}{e^{B_k\left(\mu_\Lambda^B(\theta,\rho)\right)} - 1} + \frac{\beta v(k)}{2V\left[1 - e^{-B_k\left(\mu_\Lambda^B(\theta,\rho)\right)}\right]} \langle N_0 \rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta,\rho)\right), \quad (5.60)$$

with

$$B_k\left(\mu = \mu_\Lambda^B(\theta,\rho)\right) \equiv \beta\left[\varepsilon_k - \mu_\Lambda^B(\theta,\rho) - \frac{v(k)}{2V}\right]. \quad (5.61)$$

Proof. By the correlation inequalities for the quantum Gibbs state $\omega_\Lambda^B(-) = \langle - \rangle_{H_\Lambda^B}(\beta,\mu)$ (see Appendix E):

$$\beta\omega_\Lambda^B\left(X^*\left[H_\Lambda^B(\mu), X\right]\right) \geq \omega_\Lambda^B(X^*X)\ln\frac{\omega_\Lambda^B(X^*X)}{\omega_\Lambda^B(XX^*)}, \quad (5.62)$$

where X is an observable from domain of the commutator $\left[H_\Lambda^B(\mu), .\right]$, we obtain

$$\beta\omega_\Lambda^B\left(a_k^*\left[H_\Lambda^B(\mu), a_k\right]\right) \geq \omega_\Lambda^B(N_k)\ln\frac{\omega_\Lambda^B(N_k)}{\omega_\Lambda^B(N_k)+1}, \quad (5.63)$$

for $X = a_k$, with $k \in \Lambda^*$ verifying $\|k\| > 2\pi/L$. Since

$$\left[H_\Lambda^B(\mu), a_k\right] = -\left(\varepsilon_k - \mu + [v(0) + v(k)]\frac{a_0^* a_0}{V}\right)a_k - \frac{v(k)}{V}a_0^2 a_{-k}^*,$$

one gets for $\mu = \mu_\Lambda^B(\theta,\rho)$ that

$$\begin{aligned}
\omega_{\Lambda,\rho}^B\left(a_k^*\left[H_\Lambda^B\left(\mu_\Lambda^B(\theta,\rho)\right), a_k\right]\right) &= -\left[\varepsilon_k - \mu_\Lambda^B(\theta,\rho)\right]\omega_{\Lambda,\rho}^B(N_k) \\
&\quad - [v(0) + v(k)]\frac{\omega_{\Lambda,\rho}^B(a_0^* a_0 N_k)}{V} \\
&\quad - v(k)\frac{\omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right)}{V}.
\end{aligned} \quad (5.64)$$

We use for short $\omega_{\Lambda,\rho}^B(-) \equiv \langle - \rangle_{H_\Lambda^B}\left(\beta, \mu_\Lambda^B(\theta,\rho)\right)$. Notice that $\omega_{\Lambda,\rho}^B\left(a_k^*\left[H_\Lambda^B\left(\mu_\Lambda^B(\theta,\rho)\right), a_k\right]\right) \in \mathbb{R}^1$, then by (5.64) $\omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right) \in \mathbb{R}^1$. Therefore,

$$2\omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right) = \omega_{\Lambda,\rho}^B\left(a_0^2 a_k^* a_{-k}^*\right) + \omega_{\Lambda,\rho}^B\left(a_k a_{-k} a_0^{*2}\right). \quad (5.65)$$

Moreover, since the functions ε_k and $v(k)$ are even for $(k \to -k)$, we have

$$\omega_{\Lambda,\rho}^B(a_0^* a_0 N_k) = \omega_{\Lambda,\rho}^B(a_0^* a_0 N_{-k}). \quad (5.66)$$

Thus (5.64)-(5.66) imply

$$\begin{aligned}\omega_{\Lambda,\rho}^{B}\left(a_{k}^{*}\left[H_{\Lambda}^{B}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right),a_{k}\right]\right) &= -\left[\varepsilon_{k}-\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right]\omega_{\Lambda,\rho}^{B}\left(a_{k}^{*}a_{k}\right)\\ &\quad -\frac{v\left(k\right)}{2V}\omega_{\Lambda,\rho}^{B}\left(a_{0}^{2}a_{k}^{*}a_{-k}^{*}+a_{0}^{*2}a_{k}a_{-k}\right)\\ &\quad -\frac{\left[v\left(0\right)+v\left(k\right)\right]}{2V}\omega_{\Lambda,\rho}^{B}\left(a_{0}^{*}a_{0}\left(N_{k}+N_{-k}\right)\right).\end{aligned} \tag{5.67}$$

Now by identity

$$\begin{aligned}a_{0}^{2}a_{k}^{*}a_{-k}^{*}+a_{0}^{*2}a_{k}a_{-k}+a_{0}^{*}a_{0}a_{k}^{*}a_{k}+a_{0}^{*}a_{0}a_{-k}^{*}a_{-k} &= \left(a_{0}^{*}a_{k}+a_{-k}^{*}a_{0}\right)^{*}\left(a_{0}^{*}a_{k}+a_{-k}^{*}a_{0}\right)\\ &\quad -a_{k}^{*}a_{k}-a_{0}^{*}a_{0},\end{aligned} \tag{5.68}$$

we deduce from (5.67) the estimate:

$$\begin{aligned}\omega_{\Lambda,\rho}^{B}\left(a_{k}^{*}\left[H_{\Lambda}^{B}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right),a_{k}\right]\right) &\leq -\left[\varepsilon_{k}-\mu_{\Lambda}^{B}\left(\theta,\rho\right)-\frac{v\left(k\right)}{2V}\right]\omega_{\Lambda,\rho}^{B}\left(N_{k}\right)\\ &\quad +\frac{v\left(k\right)}{2V}\omega_{\Lambda,\rho}^{B}\left(a_{0}^{*}a_{0}\right).\end{aligned} \tag{5.69}$$

Therefore, combining (5.63) with (5.69) we find:

$$B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)\omega_{\Lambda,\rho}^{B}\left(N_{k}\right)-\beta\frac{v\left(k\right)}{2V}\omega_{\Lambda,\rho}^{B}\left(a_{0}^{*}a_{0}\right)\leq\omega_{\Lambda,\rho}^{B}\left(N_{k}\right)\ln\frac{\omega_{\Lambda,\rho}^{B}\left(N_{k}\right)+1}{\omega_{\Lambda,\rho}^{B}\left(N_{k}\right)}, \tag{5.70}$$

with $B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)$ defined by (5.61). Since by Corollary 5.9

$$\mu_{\Lambda}^{B}\left(\theta,\rho\right)<\varepsilon_{\Lambda,1}<\varepsilon_{\|k\|=\frac{2\pi}{L}}=\inf_{k\neq0}\varepsilon_{k}$$

and $\|k\|>2\pi/L$, one has $B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)>0$. Hence, to estimate $x\equiv\omega_{\Lambda,\rho}^{B}\left(N_{k}\right)\geq0$, we have to solve the inequality

$$B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)x-\beta\frac{v\left(k\right)}{2V}\omega_{\Lambda,\rho}^{B}\left(a_{0}^{*}a_{0}\right)\leq x\ln\frac{x+1}{x}. \tag{5.71}$$

Notice that the solution of (5.71) is the set $\{0\leq x\leq x_{1}\}$, where x_1 is a solution of the equation

$$B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)x_{1}-\beta\frac{v\left(k\right)}{2V}\omega_{\Lambda,\rho}^{B}\left(a_{0}^{*}a_{0}\right)=x_{1}\ln\frac{x_{1}+1}{x_{1}}.$$

Let

$$x_{2}=\frac{1}{e^{B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)}-1}, \tag{5.72}$$

be a nontrivial solution of the equation

$$B_{k}\left(\mu_{\Lambda}^{B}\left(\theta,\rho\right)\right)x=x\ln\frac{x+1}{x}.$$

Then the inequality $x \leq x_1$ can be rewritten as

$$x \leq x_2 + (x_1 - x_2). \tag{5.73}$$

Since the function $f(x) \equiv x \ln \frac{x+1}{x}$ defined for $x \geq 0$ (we put $f(x=0)=0$) is concave, we have

$$\frac{f(x_1) - f(x_2)}{f'(x_2)} \leq x_1 - x_2,$$

from which by (5.72), (5.73) one gets (5.60) when $\|k\| > 2\pi/L$. ■

Now we are in position to prove the main statement of this section about the *conventional* BE condensation occurring in the WIBG for densities $\rho > \rho_c^B(\theta)$.

Theorem 5.15 *Let $\rho > \rho_c^B(\theta)$. Then we have that:*
(i)

$$\lim_{\Lambda} \left\langle \frac{N_0}{V} \right\rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta, \rho)\right) = \left\{ \begin{array}{c} |\widehat{c}(\beta, 0)|^2, \ \theta < \theta_0(0) \\ 0, \ \theta > \theta_0(0) \end{array} \right\}; \tag{5.74}$$

(ii) for any $k \in \Lambda^*$, such that $\|k\| > \frac{2\pi}{L}$,

$$\lim_{\Lambda} \left\langle \frac{N_k}{V} \right\rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta, \rho)\right) = 0; \tag{5.75}$$

(iii) for $\theta < \theta_0(0)$ and for all $k \in \Lambda^*$, such that $\|k\| > \delta > 0$,

$$\lim_{\Lambda} \langle N_k \rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta, \rho)\right) = \left[\frac{f_k}{E_k^B} \left(e^{\beta E_k^B} - 1\right)^{-1} + \frac{h_k^2}{2E_k^B\left(f_k + E_k^B\right)} \right]_{\substack{c=\widehat{c}(\beta,0) \\ \mu=0}}, \tag{5.76}$$

whereas for $\theta > \theta_0(0)$

$$\lim_{\Lambda} \langle N_k \rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta, \rho)\right) = \frac{1}{e^{\beta \varepsilon_k} - 1}; \tag{5.77}$$

(iv) the double limit

$$\widetilde{\rho}_0^B(\theta) \equiv \lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \leq \delta\}} \langle N_k \rangle_{H_\Lambda^B} \left(\beta, \mu_\Lambda^B(\theta, \rho)\right) = \rho - \rho_c^B(\theta), \tag{5.78}$$

which means that the WIBG manifests a conventional (generalized) BE condensation $\widetilde{\rho}_0^B(\theta) > 0$ in modes next to the zero-mode due to the particle density saturation.

Proof. (i) Since by (5.50) we have

$$\lim_{\Lambda} \mu_\Lambda^B\left(\theta, \rho > \rho_c^B(\theta)\right) = 0, \tag{5.79}$$

the thermodynamic limit in (5.74) results from Theorem 5.4 and Corollary 5.5 for $\mu = 0$ (see (5.27)).
(ii) Since $\|k\| > 2\pi/L$ and $\Lambda = L \times L \times L$ is cubic, which excludes generalized Bose-Einstein

condensation due to anisotropy (see [42] or Section 3.1), the thermodynamic limit in (5.75) follows from $\mu_\Lambda^B(\theta,\rho) < \varepsilon_{\|k\|=\frac{2\pi}{L}}$ (see Corollary 5.9) and Lemma 5.14.
(iii) Let us consider $g_\theta(k)$ defined for $k \in \mathbb{R}^3$, $\|k\| > \delta > 0$ by

$$g_\theta(k) \equiv \lim_\Lambda \omega_{\Lambda,\rho}^B(N_k), \tag{5.80}$$

where $\omega_{\Lambda,\rho}^B(-)$ stands for the state $\omega_\Lambda^B(-)$ with $\mu = \mu_\Lambda^B(\theta,\rho)$ (cf. Definition 5.13). Notice that by Lemma 5.14 and the fact that by Corollary 5.9 we have

$$\mu_\Lambda^B(\theta,\rho) < \varepsilon_{\Lambda,1} < \inf_{k\neq 0} \varepsilon_k = \varepsilon_{\|k\|=\frac{2\pi}{L}},$$

the thermodynamic limit (5.80) exists and it is uniformly bounded for $\|k\| > \delta > 0$. Moreover, for any interval $(a>\delta, b)$ we have convergence of Darboux sums to the integral:

$$\lim_\Lambda \frac{1}{V} \sum_{k\in\Lambda^*, \|k\|\in(a,b)} \omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3} \int\limits_{\|k\|>\delta} g_\theta(k)\,\chi_{(a,b)}(\|k\|)\, d^3k,$$

where again $\chi_{(a,b)}(\|k\|)$ is the characteristic function of (a,b). Then Corollary 5.12 implies that

$$\frac{1}{(2\pi)^3} \int\limits_{\|k\|>\delta} g_\theta(k)\,\chi_{]a,b[}(\|k\|)\, d^3k = \frac{1}{(2\pi)^3} \int\limits_{\|k\|>\delta} \xi_\theta(k)\,\chi_{(a,b)}(\|k\|)\, d^3k, \tag{5.81}$$

where $\xi_\theta(k)$ is a continuous function on $k \in \mathbb{R}^3$ defined by (5.56), (5.57), i.e.

$$\xi_\theta(k) \equiv \left[\frac{f_k}{E_k^B} \left(e^{\beta E_k^B} - 1 \right)^{-1} + \frac{h_k^2}{2E_k^B\left(f_k + E_k^B\right)} \right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}}, \tag{5.82}$$

for $\theta < \theta_0(0)$ and

$$\xi_\theta(k) \equiv \frac{1}{e^{\beta\varepsilon_k} - 1}, \tag{5.83}$$

for $\theta > \theta_0(0)$. Since the relation (5.81) is valid for any interval $(a>\delta, b) \subset \mathbb{R}^1$ one gets

$$g_\theta(k) = \xi_\theta(k),\ \ k \in \mathbb{R}^3,\ \ \|k\| > \delta > 0,$$

which by (5.80), (5.82) and (5.83), implies (5.76) and (5.77).
(iv) For a fixed total density ρ, we have the identity

$$\frac{1}{V} \sum_{\{k\in\Lambda^*, 0<\|k\|\leq\delta\}} \omega_{\Lambda,\rho}^B(N_k) = \rho - \omega_{\Lambda,\rho}^B\left(\frac{a_0^* a_0}{V}\right) - \frac{1}{V} \sum_{\{k\in\Lambda^*: \|k\|>\delta\}} \omega_{\Lambda,\rho}^B(N_k). \tag{5.84}$$

By Corollary 5.12 for $a=\delta$ and $b\to+\infty$ we obtain for $\theta < \theta_0(0)$ and $\rho > \rho_c^B(\theta)$:

$$\begin{aligned} \lim_\Lambda \frac{1}{V} \sum_{\{k\in\Lambda^*: \|k\|>\delta\}} \omega_{\Lambda,\rho}^B(N_k) &= \frac{1}{(2\pi)^3} \left[\int\limits_{\|k\|>\delta} \frac{f_k}{E_k^B} \left(e^{\beta E_k^B} - 1 \right)^{-1} d^3k \right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}} \\ &+ \frac{1}{(2\pi)^3} \left[\int\limits_{\|k\|>\delta} \frac{h_k^2}{2E_k^B\left(f_k+E_k^B\right)} d^3k \right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}}, \end{aligned} \tag{5.85}$$

and for $\theta > \theta_0(0)$

$$\lim_{\Lambda} \frac{1}{V} \sum_{k\in\Lambda^*, \|k\|>\delta} \omega^B_{\Lambda,\rho}(N_k) = \frac{1}{(2\pi)^3} \int_{\|k\|>\delta} \left(e^{\beta\varepsilon_k} - 1\right)^{-1} d^3k. \tag{5.86}$$

Now, from (5.41), (5.42), (5.74), (5.84)-(5.86), we deduce (5.78) by taking the limit $\delta \to 0^+$. ■

Therefore, according to (5.78) (and in a close similarity to the analysis of the diagonal part of the WIBG, see Section 4.1.2, or to [58]) for $\theta > \theta_0(0)$ and $\rho > \rho_c^B(\theta) = \rho_c^P(\theta)$ (5.41) the WIBG manifests only *one kind* of condensation, namely the *conventional* BE condensation which occurs in modes $k \neq 0$, whereas for $\theta < \theta_0(0)$ it manifests for $\rho > \rho_c^B(\theta) > \rho_c^P(\theta)$ (5.42) this kind of condensation at the *second* stage after the *nonconventional* Bose condensation $|\widehat{c}(\beta,0)|^2$, see (5.74). For classification of different condensations, see Section 3.1 and Appendix A.

Remark 5.16 *In domain: $\theta < \theta_0(0)$, $\rho > \rho_c^B(\theta)$, we have coexistence of these two kinds of condensations, namely:*
- the nonconventional one which starts when ρ becomes larger than $\rho^B_{\text{sup}}(\theta)$, see (5.43) and Figure 5.2 (b), and which attains its maximal value $\rho_0^B(\theta,0)$ for $\rho = \rho_c^B(\theta) > \rho_0^B(\theta,0)$, see (5.42);
- and the conventional Bose condensation $\widetilde{\rho}_0^B(\theta)$ which appears when $\rho > \rho_c^B(\theta)$, see (5.78).

Since the Bose-Einstein condensation (5.78) occurs in modes $k \neq 0$, it should be classified as a *generalized* condensation. According to the van den Berg-Lewis-Pulè's classification [8,42,44] (see Section 3.1 or Appendix A), by (5.75) and (5.78) we can deduce only that the *generalized conventional* condensation in the WIBG can be either a condensation of type I in modes $\|k\| = 2\pi/L$, or a condensation of type III if modes $\|k\| = 2\pi/L$ are not macroscopically occupied, or finally it can be a combination of the two. The next statement rules out these last two possibilities.

Corollary 5.17 *For $\rho > \rho_c^B(\theta)$ and for periodic boundary conditions, the (generalized) conventional condensation (5.78) is of type I in the first $2d(=6)$ modes next to the zero-mode $k = 0$, i.e.*

$$\widetilde{\rho}_0^B(\theta) = \lim_{\Lambda} \frac{1}{V} \sum_{\left\{k\in\Lambda^*, \|k\|=\frac{2\pi}{L}\right\}} \omega^B_{\Lambda,\rho}(a_k^* a_k) = \rho - \rho_c^B(\theta). \tag{5.87}$$

Proof. Since for any $\delta > 0$ we have the identity

$$\begin{aligned} \frac{1}{V} \sum_{\left\{k\in\Lambda^*, \|k\|=\frac{2\pi}{L}\right\}} \omega^B_{\Lambda,\rho}(N_k) &= \rho - \omega^B_{\Lambda,\rho}\left(\frac{a_0^* a_0}{V}\right) - \frac{1}{V} \sum_{\left\{k\in\Lambda^*, \frac{2\pi}{L}<\|k\|<\delta\right\}} \omega^B_{\Lambda,\rho}(N_k) \\ &\quad - \frac{1}{V} \sum_{\{k\in\Lambda^*: \|k\|\geq\delta\}} \omega^B_{\Lambda,\rho}(N_k), \end{aligned}$$

by Lemma 5.14 we get the estimate

$$\begin{aligned}\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega_{\Lambda,\rho}^{B}\left(N_k\right) \;\geq\; & \rho-\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\frac{2\pi}{L}<\|k\|<\delta\right\}}\frac{1}{e^{B_k\left(\mu_\Lambda^B(\theta,\rho)\right)}-1}\\ & -\frac{\beta\omega_{\Lambda,\rho}^{B}\left(\frac{a_0^*a_0}{V}\right)}{2V}\sum_{\left\{k\in\Lambda^*,\frac{2\pi}{L}<\|k\|<\delta\right\}}\frac{v\left(k\right)}{1-e^{-B_k\left(\mu_\Lambda^B(\theta,\rho)\right)}}\\ & -\frac{1}{V}\sum_{\left\{k\in\Lambda^*:\|k\|\geq\delta\right\}}\omega_{\Lambda,\rho}^{B}\left(N_k\right)-\omega_{\Lambda,\rho}^{B}\left(\frac{a_0^*a_0}{V}\right), \qquad (5.88)\end{aligned}$$

with $B_k\left(\mu_\Lambda^B\left(\theta,\rho\right)\right)$ defined by (5.61). Since by Corollary 5.9 one has

$$\mu_\Lambda^B\left(\theta,\rho\right)<\varepsilon_{\Lambda,1}<\inf_{k\neq 0}\varepsilon_k=\varepsilon_{\|k\|=\frac{2\pi}{L}},$$

from (5.41)-(5.42), and (5.85)-(5.86), we deduce that

$$\lim_\Lambda\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega_{\Lambda,\rho}^{B}\left(a_k^*a_k\right)\geq\rho-\rho_c^B\left(\theta\right) \qquad (5.89)$$

by taking the limit $\delta\rightarrow 0^+$ in the right-hand side of (5.88) *after* the thermodynamic limit. Hence, combining the inequality

$$\lim_\Lambda\frac{1}{V}\sum_{\left\{k\in\Lambda^*,\|k\|=\frac{2\pi}{L}\right\}}\omega_{\Lambda,\rho}^{B}\left(N_k\right)\leq\lim_\Lambda\frac{1}{V}\sum_{\left\{k\in\Lambda^*,0<\|k\|<\delta\right\}}\omega_{\Lambda,\rho}^{B}\left(N_k\right)$$

with Theorem 5.15 (see (5.78)) and (5.89)), we obtain (5.87). ■

Therefore, for temperature θ and total particle density ρ as parameters, we obtain *three* regimes in thermodynamic behaviour of the condensate in WIBG when $\theta<\theta_0\left(0\right)$ (see Figures 5.1 and 5.2):
(i) for $\rho\leq\rho_{\text{inf}}^B\left(\theta\right)$, there is no condensation;
(ii) for $\rho_{\text{sup}}^B\left(\theta\right)\leq\rho\leq\rho_c^B\left(\theta\right)$, there is a *nonconventional* Bose condensation (5.27) in the mode $k=0$ due to nondiagonal interaction in the Bogoliubov Hamiltonian (see Section 5.1.3);
(iii) for $\rho_c^B\left(\theta\right)\leq\rho$, there is a *second* kind of condensation: the *conventional* type I BE condensation which occurs *after* the *nonconventional* one; it appears due to the standard mechanism of the total particle density saturation (Corollary 5.17).

When $\theta>\theta_0\left(0\right)$, there are only *two* regimes in thermodynamic behaviour of condensate: they correspond to $\rho\leq\rho_c^B\left(\theta\right)$ with no condensation and to $\rho_c^B\left(\theta\right)<\rho$ with a *conventional* condensation as in (iii). Hence, for $\theta>\theta_0\left(0\right)$ the condensation in the WIBG coincides with the type I generalized BE condensation in the diagonal part of the WIBG (or in the PBG with *excluded* mode $k=0$), see Theorem 4.4 and 5.17.

5.3 Complementary remarks

5.3.1 Résumé

In Section 5.1 we proved the existence of *nonconventional* condensation of bosons for $k = 0$ for negative μ and $\theta < \theta_0(0)$ in the WIBG, see Figure 5.1. The physical reason for this phenomenon is an *effective attraction* between bosons in the mode $k = 0$ (see Sections 3.2 and 5.1.3):

$$-\left\{\frac{1}{V^2}\sum_{k\in\Lambda^*\backslash\{0\}}\frac{[v(k)]^2}{4\varepsilon_k}\right\}a_0^{*^2}a_0^2$$

which has to dominate the direct repulsion in (2.52):

$$\frac{v(0)}{2V}a_0^{*^2}a_0^2.$$

This is formalized as the condition (C) on the interaction potential $v(k)$ (Definition 4.5) .

In Section 5.2 we have shown that conventional BE condensation is also possible in the WIBG. It is a *generalized* BE condensation of *type I* in modes $\|k\| = 2\pi/L$. This second kind of condensation appears when $\rho \geq \rho_c^B(\theta)$ in accordance with standard mechanism of the total particle density saturation, see Corollary 5.17, well-know for the PBG.

Therefore, combining Section 5.1.2 with Section 5.2.2 for $\theta < \theta_0(0)$ (see Figures 5.1 and 5.2) we obtain the following scenario of behaviour of condensation in the WIBG for $d = 3$ and hypotheses (A)-(C) on the interaction:

(*i*) for $\rho \leq \rho_{\inf}^B(\theta)$ (5.43), there is no condensation;
(*ii*) for $\rho_{\sup}^B(\theta) \leq \rho \leq \rho_c^B(\theta)$ (5.42), (5.43), a *nonconventional* condensation (5.27) appears in the mode $k = 0$ jumping from 0 to $\rho_0^B(\theta,\mu)$, i.e. via the *first* order phase transition, see Figure 5.1;
(*iii*) for $\rho_c^B(\theta) \leq \rho$, the WIBG manifests a *conventional* BE condensation of type I: $\widetilde{\rho}_0^B(\theta) = \rho - \rho_c^B(\theta)$ (via continuous phase transition, Corollary 5.17), simultaneously with *nonconventional* one. Therefore, two kinds of condensations coexist.

Notice that for $\theta > \theta_0(0)$ the thermodynamic behaviour of the WIBG coincides with that of its diagonal part H_Λ^{BD} (4.24), i.e. with that of the PBG with *excluded* mode $k = 0$. Therefore, it manifests only the BE condensation of type I, see Corollary 4.10 and Corollary 5.17.

For $\theta < \theta_0(0)$, the thermodynamic behaviour of the WIBG is related to the two recent models [58] defined respectively by Hamiltonians

$$H_\Lambda^0 \equiv T_\Lambda + U_\Lambda^0, \tag{5.90}$$

and

$$H_\Lambda^1 = H_\Lambda^0 + U_\Lambda^1, \tag{5.91}$$

where

$$T_\Lambda = \sum_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k a_k^* a_k, \ \varepsilon_{k\neq 0} = \hbar^2 k^2/2m,$$

$$U_\Lambda^0 = \varepsilon_0 a_0^* a_0 + \frac{g_0}{V} a_0^* a_0^* a_0 a_0, \ \varepsilon_0 < 0, \ g_0 > 0, \tag{5.92}$$

$$U_\Lambda^1 = \frac{1}{V} \sum_{k\in\Lambda^*\backslash\{0\}} g_k(V)\, a_k^* a_k^* a_k a_k,$$

with $0 < g_k(V) \leq \gamma_k V^{\alpha_k}$ for $k \in \Lambda^*\backslash\{0\}$, $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$, which is a generalization of (4.22), (4.23) where $g_0 = g_k(V) = v(0)/2$. Notice that in these models, $\varepsilon_0 < 0$ is *not* equal to $\varepsilon_{\|k\|=0} = 0$. In the paper [58] we show possibility of coexistence of two kinds of Bose condensations in the both models (5.90) and (5.91). The behaviour of the WIBG for $\theta < \theta_0(0)$ is closer to the model (5.90) than (5.91) in the sense that the Bose gas (5.90) manifests the same three-stages scenario $(i)-(iii)$ of thermodynamic behaviour as above. The difference is in the absence of the limiting temperature $\theta_0(0)$ and of discontinuity of the nonconventional condensate and the total particle density as function of μ. The peculiarity of the model (5.91) is that under conditions $g_{k\neq 0}(V) \geq g_- > 0$ or $\inf_{\|k\|<\delta_0, V} g_k(V) > 0$ in a band $\delta_0 > 0$, the repulsion U_Λ^1 (5.92) spreads out the *conventional* BE condensation (originally of type I in modes $\|k\| = 2\pi/L$) into BE condensation of type III (cf. [58, 141], see also our toy model (3.21)). Notice that the *conventional* BE condensation persists in the model (5.91) even if for $k \in \Lambda^*\backslash\{0\}$ $g_k(V) = \gamma_k V^{\alpha_k} \underset{V\to+\infty}{\longrightarrow} +\infty$ $(\alpha_k \leq \alpha_+ < 1)$ which has a similarity with the WIBG. There the effective two-bosons *repulsion* for p verifying $\|p\| = 2\pi/L = 2\pi/V^{1/3}$, and $q \neq 0$

$$g_{\Lambda,pq} a_p^* a_{-p}^* a_{-q} a_q,$$

is such that the "form-factor" $g_{\Lambda,pq} > 0$ diverges with volume as $V^{2/3}$, see Section 3.2, eq. (3.50). However, an important difference is that this *effective* interaction (which is due to non-diagonal term (2.53)) is *not* able to spread out the BE condensation into the type III as in the model (5.91).

Remark 5.18 *The jump of the $\rho_0^B(\theta,\mu)$ and $\rho^B(\beta,\mu)$ on the line ∂D^B indicates that the nonconventional Bose condensate appears in the WIBG via the first order transition. Concerning the thermodynamic behaviour of the WIBG for densities*

$$\rho_{\inf}^B(\theta) < \rho < \rho_{\inf}^B(\theta),$$

in our paper [16] we conjectured the possibility of a mixture of two pure states $\omega^B(-)$ corresponding, respectively, to the state without and with nonconventional condensation in mode $k = 0$. This would be similar to what is known for some mean-field Bose-gas models [148, 152, 153].

5.3.2 The Weakly Imperfect Bose-Gas in dimensions $d < 3$

The preceding analysis in Sections 4 and 5 concerns the thermodynamic behaviour of the WIBG in dimension $d = 3$. Therefore in this section we study dependence of the *conventional* and *nonconventional* condensates on dimension d.

First let us recall that, for the PBG in low dimensions $d \leq 2$, the critical density $\rho_c^P(\theta)$ (2.32) becomes *infinite*:

$$\rho_c^P(\theta) = \sup_{\mu<0} \rho^P(\beta,\mu) = \lim_{\mu\to 0^-} \rho^P(\beta,\mu) = \lim_{\mu\to 0^-} \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{d^d k}{(e^{(\varepsilon_k-\mu)/\theta}-1)} = +\infty.$$

The essential consequence of this absence of a saturation of $\rho^P(\beta,\mu)$ is the uniqueness of the solution

$$\mu^P(\theta,\rho) = \lim_{\Lambda} \mu_\Lambda^P(\theta,\rho) < 0, \tag{5.93}$$

of the equation

$$\rho = \rho^P(\beta,\mu), \ \beta > 0,$$

see (2.31) for any total density ρ. Hence, by (5.93) one gets

$$\lim_{\Lambda} \frac{1}{V} \sup_{k\in\Lambda^*} \langle N_k\rangle_{T_\Lambda}\left(\beta, \mu_\Lambda^P(\theta,\rho)\right) = \lim_{\Lambda} \frac{1}{V} \frac{1}{e^{-\beta\mu_\Lambda^P(\theta,\rho)}-1} = 0$$

for *any* $\rho > 0$, i.e. no macroscopic occupation of any mode $k \in \Lambda^*$. To exclude the non-extensive (type III) BE condensation (see Appendix A) one has to remark that for any ρ and $d \leq 2$ the equation

$$\rho = \lim_{\delta\to 0^+} \frac{1}{(2\pi)^d} \int\limits_{\|k\|>\delta} \frac{d^d k}{(e^{\beta(\varepsilon_k-\mu)}-1)}$$

has the same solution (5.93) as above. Therefore

$$\lim_{\delta\to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k\in\Lambda^*, \|k\|<\delta\}} \frac{1}{\left(e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)}-1\right)} = 0,$$

which rules out the condensation of type III. In fact the existence of the (saturated) critical density $\rho_c^P(\theta)$ (2.32) is a *necessary* condition for existence of *conventional* BE condensation.

Concerning the WIBG we have to distinguish two regimes: the first, when the *nonconventional* Bose condensation $\rho_0^B(\theta,\mu) = 0$ and the second, when $\rho_0^B(\theta,\mu) \neq 0$.

1°. Let $\rho_0^B(\theta,\mu) = 0$, i.e. either $D^B = \{\varnothing\}$ or $\theta > \theta_0(0)$, see (5.27). Then similar to (5.41) one gets a critical density

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \left(e^{\beta\varepsilon_k}-1\right)^{-1} d^d k = \rho_c^P(\theta), \tag{5.94}$$

i.e. for dimensions $d > 2$ the critical density (5.94) is bounded, whereas for low dimensions $d \leq 2$, we have

$$\rho_c^B(\theta) = \sup_{\mu<0} \rho^B(\beta,\mu) = +\infty.$$

Consequently for *any* $\rho > 0$,

$$\mu^B(\theta,\rho) = \lim_{\Lambda} \mu^B_\Lambda(\theta,\rho) < 0,$$

is a unique solution of the equation

$$\rho = \rho^B(\beta,\mu), \ \beta > 0,$$

(cf. Remark 5.10). Therefore by Lemma 5.14, we obtain

$$\lim_{\Lambda} \frac{1}{V} \sup_{k\in\Lambda^*\setminus\{0\}} \langle N_k\rangle_{T_\Lambda}\left(\beta,\mu^B_\Lambda(\theta,\rho)\right) \leq \lim_{\Lambda} \frac{1}{V}\frac{1}{e^{-\beta\mu^B_\Lambda(\theta,\rho)}-1} = 0,$$

for *any* $\rho > 0$, i.e. there is no *conventional* BE condensation in any mode $k \neq 0$ and the non-extensive BE condensation is ruled out as above for the PBG. In fact in this regime, for low dimensions $d = 1, 2$, the thermodynamic behaviour of the WIBG is similar to its diagonal part or to the PBG with excluded mode $k = 0$.

$2°$. Now let $\rho^B_0(\theta,\mu) \neq 0$, i.e. $D^B \neq \{\varnothing\}$ and $(\theta,\mu) \in D^B$, see (5.27). First, notice that the *nonconventional* condensation $\rho^B_0(\theta,\mu) = |\widehat{c}(\beta,\mu)|^2$ (5.27) is uniformly bounded in D^B by $|\widehat{c}(\beta,0)|^2$ (5.74). Second the condition (C) (Definition 4.5)

$$v(0) < \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d} d^dk\frac{v(k)^2}{\varepsilon_k}, \tag{5.95}$$

which is necessary for $D^B \neq \{\varnothing\}$ (Theorem 5.6), is trivially verified for low dimensions $d = 1, 2$ since the integral in (5.95) diverges (cf. Remark 4.6). Hence for $d \leq 2$, this kind of condensation appears when interaction satisfies only hypotheses (A) and (B) (see Section 2.2 or Section 4.2). If $\rho^B_0(\theta,\mu) = |\widehat{c}(\beta,\mu)|^2 > 0$ (5.27), i.e. $\theta < \theta_0(0)$ (see Figure 5.1), then by (5.42) extended to dimension d one gets the critical density

$$\begin{aligned}\rho^B_c(\theta) \quad &= \quad \frac{1}{(2\pi)^d}\left[\int\limits_{\mathbb{R}^d} \frac{f_k}{E^B_k}\left(e^{\beta E^B_k}-1\right)^{-1} + \frac{h_k^2}{2E^B_k\left(f_k+E^B_k\right)}d^dk\right]_{\substack{c=\widehat{c}(\beta,0)\\ \mu=0}} \\ &\quad + |\widehat{c}(\beta,\mu=0)|^2 < +\infty,\end{aligned} \tag{5.96}$$

even for $d \leq 2$. Indeed, since $\rho^B_0(\theta,\mu) = |\widehat{c}(\beta,\mu=0)|^2 > 0$ for any $\mu \leq 0$ one gets $E^B_k > 0$ (4.60) that assures the convergence of the integral in (5.96) for all d. Consequently, under condition $\rho^B_0(\theta,\mu) > 0$ the arguments of Section 5.2 remain valid for *any* dimension d. Hence, for low dimensions $d \leq 2$ and high densities $\rho > \rho^B_c(\theta < \theta_0(0))$ we find the *conventional* BE condensation coexisting with the *nonconventional* one as it is described by Theorem 5.11 and Corollary 5.17.

In the WIBG for any d, the *nonconventional* condensate $\rho^B_0(\theta,\mu)$ always *saturates* and *implies* a saturation of the total particle density in the grand-canonical ensemble (θ,μ).

It makes a difference with models (5.90) and (5.91) [58]: there for $d \leq 2$ one has no conventional BE condensation for all $\theta \geq 0$, while the nonconventional one may exist for sufficiently low temperatures. This difference comes from the fact that in the WIBG the nonconventional condensation $\rho^B_0(\theta,\mu) = |\widehat{c}(\beta,\mu)|^2$ ensures the convergence of the integral (5.96) for $d \leq 2$, whereas in the models (5.90) and (5.91) it does not.

Section 6

Concluding remarks

Wagner: "Allein die Welt! des Menshen Herz und Geist! Möcht jeglicher doch was davon erkennen."
Faust: "Ja, was man so erkennen heißt! Wer darf das kind beim rechten Namen nennen?"
Goethe, *Faust.*

6.1 The Weakly Imperfect Bose-Gas versus the Bogoliubov theory

In Sections 4 and 5 we presented an exact solution of the Bogoliubov WIBG model H_{Λ}^{B} (2.50) originally conceived as a starting point for explanation of superfluity [1–5].

(i) We have shown that thermodynamic properties of the model drastically depend on the interaction potential. If it satisfies the *hypothesis* (C), see Section 4.2, then it is the *nondiagonal* part of interaction U_{Λ}^{BN} that makes the model nontrivial, i.e. non-equivalent to the PBG, see Theorem 4.8 and Corollary 4.10.
Therefore, we have answered the question formulated in [9] by showing that the solution depends on the potential. In particular we established that the hypothesis (C) (Definition 4.5) is necessary and sufficient for the WIBG to be *non-equivalent* to the PBG in a subdomain D^B of the domain of stability

$$Q^B \equiv \left\{(\theta,\mu) : p^B(\beta,\mu) = \lim_{\Lambda} p_{\Lambda}^{B}(\beta,\mu) < +\infty\right\} = Q \equiv \{(\theta,\mu) : \theta \geq 0,\ \mu \leq 0\},$$

(cf. (4.54), (4.55)). If interaction does not verify the hypothesis (C), one has $D^B = \{\varnothing\}$. In this case the thermodynamic properties of the WIBG coincide with those of the PBG. By virtue of Remark 4.6 if interaction potential $\varphi(x)$ satisfies (C), then there exists $\lambda_0 > 0$ such that potential $\varphi_\lambda(x) = \lambda\varphi(x)$ does not satisfy (C) for all $\lambda < \lambda_0$ (*weak-potential* limit).

(*ii*) We have shown that the Bogoliubov approximation (Appendix B) for the WIBG is *exact* in the sense that in the thermodynamic limit the pressure of the Hamiltonian H_Λ^B and of the corresponding approximating Hamiltonian coincide, see Theorem 4.8. This enables us to calculate explicitly the pressure $p^B(\beta,\mu)$, see (4.68). On the other hand, this exact solution is rather different from results of Bogoliubov's treatment [1–3, 18, 19] of the Hamiltonian H_Λ^B (2.50). This is because his treatment involves *additional* hypotheses, which are equivalent to modifications of the original Bogoliubov Hamiltonian (2.50), see discussions in Sections 2 and 3.

(*iii*) We found that for any interaction potential satisfying conditions (A)-(C) there is a domain

$$D^B = \{(\theta,\mu) \in Q : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\} \subset Q^B$$

where the pressure $p^B(\beta,\mu) \neq p^P(\beta,\mu)$. We showed (Theorem 4.7) that in fact the domain D^B coincides with the domain of existence of a *nonconventional* condensation in the mode $k = 0$:

$$D^B = \left\{(\theta,\mu) \in Q : \rho_0^B(\theta,\mu) \equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu) > 0 \right\}, \tag{6.1}$$

cf. (4.63). This kind of condensate is exclusively due to the interaction satisfying the hypothesis (C). In this case the nondiagonal part U_Λ^{BN} induces an effective attraction between bosons in the mode $k = 0$ that leads to a macroscopic occupation of this level when $\theta < \theta_0(\mu)$.

(*iv*) It was shown (Theorem 5.1) that the gauge symmetry of the Gibbs state for the WIBG is broken in D^B and that $\rho_0^B(\theta,\mu)$ changes its value on boundary ∂D^B from $\rho_0^B(\theta,\mu) = 0$ to $\rho_0^B(\theta,\mu) \neq 0$ discontinuously (Theorem 5.6). Therefore, when interaction verifies the hypothesis (C), the WIBG manifests a first-order phase transition on the line ∂D^B.

(*v*) Moreover, the approximating Hamiltonian $H_\Lambda^B\left(\widehat{c}_\Lambda^{\#},\mu\right)$, which is thermodynamically equivalent to $H_\Lambda^B(\mu)$ (Theorem 4.8), has a *gap* in the quasi-particles spectrum in the presence of the *nonconventional* Bose condensate, i.e. $\lim_{k\to 0} E_k^B \neq 0$ in the domain D^B. This indicates again that the Bogoliubov's ansätze imply modifications of the original Bogoliubov Hamiltonian (2.50) to produce a microscopic description of the superfluidity with a gapless approximating Hamiltonian, see Sections 2, 3 and discussion below.

(*vi*) Finally we have shown, that the total particle density $\rho^B(\beta,\mu)$ is bounded in Q^B and that it attains a critical value

$$\rho_c^B(\theta) = \sup_{\mu\leq 0} \rho^B(\beta,\mu)$$

at $\mu = 0$, see Section 5.2.2. This implies (Theorem 5.15) that, independent of the hypothesis (C), the WIBG manifests for $\rho > \rho_c^B(\theta)$ a conventional (generalized, type I) BE condensation which starts in the first ($2d = 6$) modes next to $k = 0$, see Corollary 5.17. So, if $\theta < \theta_0(0)$, then the condensation in the WIBG occurs in two stages: the first stage corresponds to the nonconventional one, when $\mu < \mu_0(\theta)$ ($\rho^B(\beta, \mu) > \rho^B_{\text{sup}}(\theta)$), then the conventional BE condensation starts for $\mu = 0$ as soon as $\rho > \rho^B_c(\theta)$.

The physical reason for the difference between the *exact* solution of the model H^B_Λ and the Bogoliubov theory [1–3, 18] is in a different treatment of *quantum fluctuations*. Since this difference is decisive, we open discussion by fluctuations.

It is the quantum fluctuations of the operators $a^\#_0/\sqrt{V}$ that imply an *effective attraction* between bosons with $k = 0$ in WIBG (see Section 3.2). This attraction is the cause of two phenomena: an instability of the WIBG for $\mu > 0$ (Theorem 3.8) known since [9], and a *nonconventional* condensation of bosons in the $k = 0$ mode for negative μ, when the *effective attraction* between them *dominates* a direct repulsion in (2.52), see hypothesis (C), Definition 4.5 and inequality (3.54) of Section 3.2 (Theorem 4.7).

In contrast to that, the Bogoliubov treatment of his model H^B_Λ (2.50) is based on the approximation:

$$a^\#_0/\sqrt{V} \to c^\#,$$

(Appendix B), i.e. on the *elimination* of the *quantum fluctuations* that makes the Hamiltonian $H^B_\Lambda\left(c^\#, \mu\right)$ (2.70) *stable* for a larger chemical potential domain: $\mu \leq v(0)|c|^2$ (i.e. even for $0 < \mu \leq v(0)|c|^2$ where the model (2.50) does *not* exist!), see Remark B.3. To make this treatment *self-consistent* and to gain the well-known *gapless* spectrum of quasi-particles excitations, the Bogoliubov's judicious choice of the parameter $|c|^2$ comes from the maximization of only the *nonfluctuating* "Landau's part" of the trial pressure (4.58)-(4.60), i.e. of $\mu x - \frac{1}{2}v(0)x^2$ (for discussions see [81, 115, 164]). This choice bolsters the assertion of elimination of the quantum fluctuations in the Bogoliubov theory of superfluidity, but at the same time creates great debates about the rôle of the quantum fluctuations in the full Hamiltonian (2.44) in the presence of condensate, see Section 3 and [84, 87–89, 111, 116, 151, 164]. Even in the PBG the quantum fluctuations merit a special study [7, 152]. Recently they have become an object of rigorous analysis, see [165–167].

In [12–16] we have found the exact solution of the simplest *nondiagonal* Hamiltonian H^B_Λ (2.50) invented by Bogoliubov (WIBG). Instead of the Bogoliubov treatment of this Hamiltonian, we studied the WIBG rigorously, without any *a priori* ansätze or approximations. Our results $(i) - (vi)$ show that *quantum fluctuations* of operators $a^\#_0/\sqrt{V}$ make the properties of WIBG drastically different from the Bogoliubov treatment. This evidently means that the Bogoliubov theory of WIBG is something else than a straightforward study of the model (2.50), see discussion in Section 2 and 3.

At the first glance it may seem a *paradox* (cf. the above remark about fluctuations) that by (ii) the Bogoliubov approximation is *exact* for calculations of the *thermodynamic* properties of the WIBG, Theorem 4.8. In fact the quantum fluctuations are not forgotten. The value c is not fixed by $\mu = v(0)|c|^2$ as in the Bogoliubov theory, but it is chosen to *maximize* the pressure $\widetilde{p}^B\left(\beta, \mu; c^\#\right)$ ((4.74) with $\nu^\# = 0$), which in addition to the "Landau part" $\mu|c|^2 - \frac{1}{2}v(0)|c|^4$ contains also a *nontrivial* "fluctuating part", see [12–16] and [164] for discussion.

Our rigorous study of WIBG shows that, in contrast to the *aim* of the Bogoliubov theory, the nonconventional Bose condensate implies a gap in the quasi-particles excitations spectrum of the approximating Hamiltonian, see (v). In fact, the nature of this gap is well-known, see e.g. [81] and discussion in Section 2.4. The interaction in the *truncated* Hamiltonian H_Λ^B (2.50) (in contrast to (2.44) or (2.139)) is *nonlocal*. For example, the nondiagonal interaction U_Λ^{BN} (2.53) combined with the repulsive diagonal term in the mode $k = 0$

$$\frac{v(0)}{2V} a_0^{*^2} a_0^2,$$

(cf. (2.52)), gets the form (2.141):

$$\begin{aligned}
&\frac{1}{2}\int_\Lambda dx_1 \int_\Lambda dx_2 \varphi(x_1 - x_2)\, a^*(x_1)\, a^*(x_2)\, \frac{1}{V}\int_\Lambda dx_3 a(x_3) \int_\Lambda dx_4 a(x_4) + \\
&\frac{1}{2}\frac{1}{V}\int_\Lambda dx_4 a^*(x_4) \int_\Lambda dx_3 a^*(x_3) \int_\Lambda dx_2 \int_\Lambda dx_1 \varphi(x_1 - x_2)\, a(x_2)\, a(x_1)\,,
\end{aligned} \tag{6.2}$$

Therefore, in contrast to the full interaction (2.139), the truncated interaction U_Λ^B (2.50) violates the local gauge invariance (2.137): $a(x) \to e^{i\varphi(x)} a(x)\,,\ a^*(x) \to e^{-i\varphi(x)} a^*(x)\,.$ As a consequence it violates the Bogoliubov $1/q^2$-theorem ([26] Part I) and the Hugenholtz - Pines - Gavoret - Nozières analysis which guarantee a gapless spectrum of quasi-particles excitations *even* if one treats the $k = 0$ mode operators using the Bogoliubov approximation, see an instructive discussion in [81] , [75] and the literature quoted there.

Bogoliubov [76] and Hohenberg [168] in 1962, then Bell in 1963 [169] noticed that non-local interactions similar to (6.2) break some exact relations known as the sum-rules for the particle-number fluctuations [68]. In fact, these rules are related to the local particle-number conservation law, or to the equation of continuity (2.140). Then these authors suggested that this breaking of the sum-rules should imply a “gap” in the spectrum of quasi-particles. Our analysis indicates that for the WIBG this observation is exact on the level of the quasi-particle spectrum of the approximating Hamiltonian.

Notice that it is not the second, but the first Bogoliubov ansatz (truncation of the full interaction), which is responsible for the gap in the spectrum. It does not mean that any truncation of the full Hamiltonian leads to this effect. For example, the truncated Hamiltonian H_Λ^{MF} (4.7) does not violate the local gauge invariance, since the interaction U_Λ^{MF} (4.4) has the form:

$$U_\Lambda^{MF} = \frac{v(0)}{2V}\left(\int_\Lambda dx a^*(x)\, a(x)\right)^2. \tag{6.3}$$

The model with $H_\Lambda^{MF}(\mu) = H_\Lambda^{MF} - \mu N_\Lambda$ is exactly soluble [148–150, 153]. It is known that the BE condensation exists in this model for $\rho > \rho_c^P(\theta)$, but the spectrum of quasi-particles excitations of the corresponding approximating Hamiltonian is gapless: it coincides with the free-gas spectrum ε_k. Of course, the latter is a simple consequence of the Bogoliubov $1/q^2$-theorem ([26] Part I).

We would like to stress that it is the spectrum of $H_{\Lambda}^{MF}(\mu)$, which is gapless for $\rho > \rho_c^P(\theta)$. In contrast to that the spectrum E_k^{MF} of the Hamiltonian approximating to H_{Λ}^{MF} has *always* a gap [153]:

$$E_k^{MF} = \varepsilon_k + v(0)\rho. \tag{6.4}$$

The corresponding chemical potential behaves as (4.14):

$$\mu^{MF}(\theta,\rho) = \left\{ \begin{array}{l} \partial_\rho f^P(\beta,\rho) + v(0)\rho, \text{ for } \rho \le \rho_c^P(\theta) \\ v(0)\rho, \text{ for } \rho > \rho_c^P(\theta) \end{array} \right\}, \tag{6.5}$$

where $f^P(\beta,\rho)$ is the PBG free-energy density [149, 150, 153]. Therefore, the spectrum of the Hamiltonian approximating to $H_{\Lambda}^{MF}\left(\mu^{MF}(\theta,\rho)\right)$ has a gap for $\rho \le \rho_c^P(\theta)$ whereas it has no gap, when $\rho > \rho_c^P(\theta)$. Recall that in the latter case the gauge symmetry is broken because of the BE condensation, see (4.15), (4.16):

$$\lim_{\nu\to 0, \arg\nu=\varphi} \lim_{\Lambda} \left\langle \frac{a_0^{\#}}{\sqrt{V}} \right\rangle_{H_{\Lambda}^{MF}(\nu^{\#})} (\beta,\mu) = \left\{ \begin{array}{l} 0,\ \mu \le \mu_c^{MF}(\theta) = v(0)\rho_c^P(\theta) \\ \sqrt{\rho_0^{MF}(\theta,\mu)} e^{\pm i\varphi},\ \mu > \mu_c^{MF}(\theta) \end{array} \right\}, \tag{6.6}$$

where the left-hand side of (6.6) is the quasi-average with the Hamiltonian

$$H_{\Lambda}^{MF}\left(\nu^{\#}\right) = H_{\Lambda}^{MF} - \sqrt{V}\left(\nu a_0^* + \overline{\nu} a_0\right). \tag{6.7}$$

These remarks show that the breaking of the continuous gauge symmetry by the BE condensation does not imply a gapless spectrum of quasi-particles in the system with the Hamiltonian H_{Λ}^{MF} (4.7). We can not refer in this case to the Goldstone theorem [170] because of the *long-range* interaction (6.3) in this model, which gives a gap in the spectrum (6.4).

[A similar example is given by the Curie-Weiss-Heisenberg (CWH) model with the continuous $O(n)$-symmetry [163]. The spectrum of this system is gapless above the Curie temperature, but (because of the long-range interaction) it has a gap proportional to the spontaneous magnetization below this temperature. Therefore, in this model one has no a gapless spin-wave picture suggested by the Goldstone theorem in the phase with the broken symmetry [82]]

Since the interaction in the WIBG (2.52) and (2.53) is also of a long-range, there is no reason to anticipate a gapless spectrum of H_{Λ}^{B} à la Goldstone, when there is the gauge symmetry breaking due to the condensation, see Section 5.1.1. In this sense the WIBG behaves similar to the H_{Λ}^{MF} model. Since for $\theta = 0$ and $\mu = 0$ the Hamiltonian $H_{\Lambda}^{B}(\mu = 0) = H_{\Lambda}^{B}$ is equivalent to $H_{\Lambda}^{B}\left(\widehat{c}_{\Lambda}^{\#}, \mu = 0\right)$, see (2.70) and Section 4.3, where $\widehat{c}_{\Lambda}^{\#}(\beta = \infty, 0) \neq 0$ (5.3), one gets for the quasi-particles spectrum (4.60) of the approximating Hamiltonian in the thermodynamic limit:

$$E_k^B(\mu = 0) = \left\{ \left(\varepsilon_k + \left|\widehat{c}^{\#}(\beta = \infty, 0)\right|^2 v(0)\right) \left(\varepsilon_k + \left|\widehat{c}^{\#}(\beta = \infty, 0)\right|^2 (2v(0) + v(k))\right) \right\}^{1/2}, \tag{6.8}$$

with a gap due to the nonconventional Bose condensation. The appearance of the conventional BE condensation does not change (6.8).

If we return to the Hamiltonian $H_{\Lambda}^{B}(\mu)$, then the gap (6.8) becomes even more sound ($\mu \le 0$), see (4.60). So, in contrast to the model $H_{\Lambda}^{MF}(\mu)$ (which is locally gauge invariant and has no gap in the presence of the BE condensate), the Bogoliubov WIBG $H_{\Lambda}^{B}(\mu)$ has no gap only in the domain $Q\backslash\overline{D^{B}}$, where the WIBG is equivalent to the PBG, see Section 4.

The above discussion of the spectrum of excitations and the Goldstone theorem is not completely consistent, since it is based on the Hamiltonian $H_{\Lambda}^{B}(\mu)$ and on the thermodynamically equivalent Hamiltonian $H_{\Lambda}^{B}\left(\widehat{c}_{\Lambda}^{\#},\mu\right)$ (2.70). The latter is sufficient to calculate exactly the pressure of the WIBG but *not* dynamical properties such as the spectrum of the collective quantum fluctuations related to the broken symmetry and to the Goldstone mode [82]. This was recently a subject of the paper by Michoel and Verbeure [100], where they put ahead the idea of the quantum fluctuations operator algebra (Goldstone boson normal coordinates) to study the collective excitations in a modified WIBG-model, which is stabilized by the MF interaction, see discussion in Section 3.4. Their analysis shows a gap in the spectrum of the collective excitations, which bolsters the above discussion on the quasi-particles spectrum in the WIBG. Notice that in the framework of the Bogoliubov theory the spectra of the both types of excitations coincide, see e.g. [68], [115], [168]. The Gavoret-Nozières Theorem [73], [74] says that it is true for the *model with condensate* (see Section 2.4) in the low-energy limit.

We would like to conclude by a remark inspired by the example [132], which shows that the structure of the spectrum of excitations of a boson system could be rather complex even in the regime, where the Bogoluibov theory gives a correct result for the ground state energy. This Lieb's exact solution of the one-dimensional gas of Bose particles interacting via repulsive delta-function potential manifests *two* branches of boson excitation spectra with sound velocity which agrees with macroscopic compressibility. In spite of restrictions to the particular interaction and dimensionality, this example appeals for attention with classification of the hierarchy of different "excitations" (elementary, quasi-particle, collective), since all of them are components of a *unique* dynamical spectrum of the system Hamiltonian, see discussion in [132].

6.2 The nonconventional condensation versus conventional BE condensation

We saw that the WIBG may manifest two kinds of Bose condensations (see Section 5.3.1). By its mechanism of formation, the first kind of condensation differs from the standard (*conventional*) BE condensations known since the PBG. We call it *nonconventional* condensation. The indication on this new kind of condensation in the WIBG appeared in 1998 [12, 13]. Detailed study of the nonconventional Bose condensation in the WIBG is published by the authors in [14–16], and in [58, 59] for models, which mimic the WIBG. In fact this kind of condensation was discovered in 1983 by Buffet, de Smedt and Pulè for a boson gas with an energy gap [93]. This phenomenon was then found in the Huang-Yang-Luttinger (HYL) (4.8) (see [54, 56]) and in the Full Diagonal models (4.10) [57] (since they contain some attractive interactions) under the name the "Thouless phenomenon", see discussion in Section 4.1.

In papers [58, 59] we present exactly soluble models with *diagonal* interactions (cf. (5.90) and (5.91)) which display a two-stage condensation close to that in the WIBG:

- the model exists (the grand-canonical partition function converges) only for $\mu \leq 0$;

- there is $\widetilde{\mu}_0 < 0$ such that there is *no condensation* for $\mu < \widetilde{\mu}_0$;

- a macroscopic occupation $\rho_0(\theta, \mu)$ of the mode $k = 0$ starts at $\mu = \widetilde{\mu}_0$ due to an instability implied by the interaction (a *nonconventional* Bose condensation);

- since $\rho_0(\theta, \mu)$ and the total density of particles $\rho(\theta, \mu)$ attain their maxima at $\mu = 0$, a *conventional* BE condensation occurs for densities $\rho > \rho(\theta, \mu = 0)$, but in a generalized sense, i.e. in modes close to $k = 0$;

- this *generalized* Bose condensation persists, if there is a *repulsive interaction* between bosons in modes $k \neq 0$ similar to the effective two-boson repulsion in the WIBG for these modes (see discussions in Section 5.3.1); it was Michoel and Verbeure [141] who proved that such repulsion may spread out the BE condensation of type I into the type III.

The studies of the WIBG [12–16] (see Section 5) and of these models [54,56–59,93] highlight different thermodynamic properties of the *nonconventional* condensation compared to the *conventional* one:

1°. The *nonconventional* Bose condensation precedes the *conventional* one.

2°. In contrast to the *conventional* BE condensation the *nonconventional* one is almost *independent* of dimension d : e.g., it exists even for $d \leq 2$ (see Section 5.3.2).

3°. The *nonconventional* Bose condensation may *saturate* for large densities $\rho > \rho_c$. In this case, a second, *conventional,* BE condensation occurs in the second stage. Notice that the model presented in paper [93] gives an example, where the *nonconventional* condensation has no saturation. Consequently, for this system there is only one kind of condensation.

4°. The *nonconventional* Bose condensation is rather robust with respect to "small" perturbations, see [58, 59] for details.

5°. The phase transitions corresponding to the *nonconventional* Bose condensation may be of the *first order*. In particular this is the case of the WIBG: the density of the *nonconventional* condensation $\rho_0^B(\theta, \mu)$ is discontinue on ∂D^B (see Section 5.1, Theorem 5.6; for reader's convenience, we recall below the Figure 5.1 under the name Figure 6.1). On the other hand the transition corresponding to the nonconventional Bose condensation in models [58, 59] is of the second order.

In papers [12–16] and [58,59,93], the *nonconventional* Bose condensation is of *type I* in the fundamental mode, although the *conventional* BE condensation may be of types I, II, or III. It seems that there are no *a priori* arguments preventing *nonconventional* condensations of type II or III.

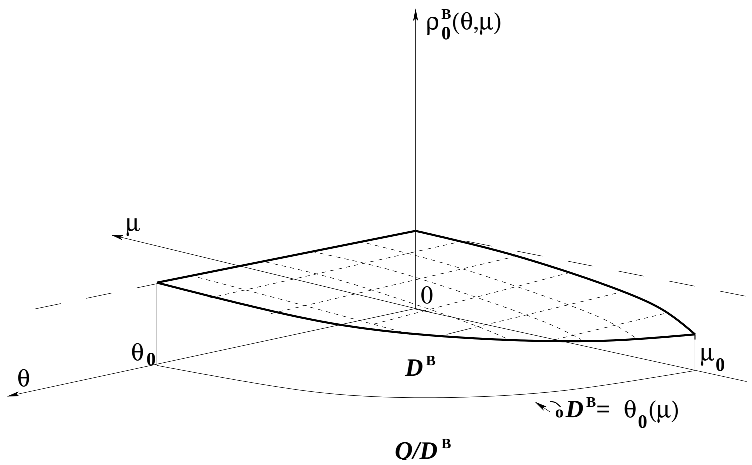

Figure 6.1: Illustration of the nonconventional Bose condensate density $\rho_0^B(\theta, \mu)$ as a function of μ and θ in the WIBG.

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Appendix A

Classification of Bose condensations

A.1 The van den Berg-Lewis-Pulè's classification of conventional generalized condensations

For reader's convenience we recall here a nomenclature of (generalized*)* Bose-Einstein (BE) condensations according to [8, 42, 44]:

- the condensation is called *type I* when a *finite* number of single-particle levels are macroscopically occupied (e.g. the standard BE condensation in the one mode for the Perfect Bose Gas (PBG) is of type I);
- it is of *type II* when an *infinite* number of the levels are macroscopically occupied;
- it is called *type III*, or the *non-extensive* condensation, when *no* of the levels are macroscopically occupied whereas one has

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0\leq\|k\|\leq\delta\}}\langle N_k\rangle=\rho-\rho_c\left(\theta\right).$$

An example of these different condensations is given in [42]. This paper demonstrates that three types of BE condensation can be realized in the case of the PBG in an *anisotropic* rectangular box $\Lambda\subset\mathbb{R}^3$ of volume $V=|\Lambda|=L_1L_2L_3$ and with the *Dirichlet boundary conditions.* Let $L_1=V^{\alpha_1}$, $L_2=V^{\alpha_2}$, $L_3=V^{\alpha_3}$ for $\alpha_1+\alpha_2+\alpha_3=1$ and $\alpha_1\geq\alpha_2\geq\alpha_3>0$. If $\alpha_1<1/2$, then for sufficient large density ρ, we have the Bose-Einstein condensation of *type I* in the fundamental mode $k=\left(\frac{2\pi}{L_1},\frac{2\pi}{L_2},\frac{2\pi}{L_3}\right)$. For $\alpha_1=1/2$ one gets a condensation of *type II* characterized by a macroscopic occupation of infinite package of modes $k=\left(\frac{2\pi n}{L_1},\frac{2\pi}{L_2},\frac{2\pi}{L_3}\right)$, $n\in\mathbb{N}$, whereas for $\alpha_1>1/2$ we obtain a condensation of *type III.* For further details see Section 3.1 where the anisotropic PBG is studied for the case of the *periodic boundary conditions.* In [46, 47] it was shown that type III condensation can be caused in the PBG by a weak external potential or (see [44,45]) by a specific choice of boundary conditions and geometry. Another example of the *non-extensive* condensation is given in an *isotropic* box Λ by our toy model (3.21) for imperfect bosons with *repulsive interaction* which spreads out the *conventional* BE condensation of type I into the type III BE condensation (see also [58, 141]).

A.2 Nonconventional and conventional Bose condensations

Here we classify Bose condensations by their mechanisms of formation. In the overwhelming majority of papers (cf. [42,44–47,141]), the condensation is due to *saturation* of the total particle density, originally discovered by Einstein [6] in the Bose gas without interaction (PBG). We call it *conventional* Bose-Einstein condensation [7].

The existence of condensations, which is induced by *interaction*, is pointed out in recent papers [13, 15, 16, 58, 59]. It is also the case of the Huang-Yang-Luttinger (HYL) (4.8) or the Full Diagonal models (4.10), since the Bose condensation manifested by the two models exists in presence of *attractive* interactions [54, 56, 57]. In particular notice that in the HYL model (4.8) the condensation appears via a first order phase transition [54, 56], see also Section 4.1.1. This is also the case of the Bogoliubov WIBG (2.50) [15, 16] (see Section 5). We call this kinds of condensation induced by attraction mechanism a *nonconventional* Bose condensation. Since in recent experiments the Bose-systems manifesting condensations are non-ideal [94–96], the concept of condensation induced by interaction is rather natural.

Remark A.1 *As it is shown in the present review (see Section 5.2), the nonconventional condensation in the WIBG does not exclude the appearance of the conventional BE condensation when total density of particles grows and exceeds some saturation limit $\rho_c^B(\theta)$ (5.41), (5.42).*

Remark A.2 *To appreciate the notion of nonconventional condensation let us remark that in the WIBG for $d = 1, 2$, this kind of condensation exists and implies the saturation of the total particle density, inducing for larger densities $\rho > \rho_c^B(\theta)$ a conventional (generalized) BE condensation (see Section 5.3.2).*

Remark A.3 *A nonconventional Bose condensation can always be characterized by its type. Therefore, formally one obtains six kinds of condensations: a nonconventional versus conventional of types I, II, or III.*

Appendix B

The Bogoliubov approximation

The aim of this appendix is to remind a definition of the *Bogoliubov approximation*:

$$a_0^{\#}/\sqrt{V} \to c^{\#} \in \mathbb{C},$$

à la Ginibre [92] (cf. Section 2.2 or Section 4.2). Then we apply this approximation to the WIBG (2.50). Finally we analyze the thermodynamic properties of the corresponding approximating pressure.

B.1 Definition of the Bogoliubov approximation

Let a system of identical bosons be enclosed in a cubic box Λ of the volume $V = |\Lambda|$ with periodic boundary conditions on $\partial\Lambda$. Let $\psi_0(x) = 1/\sqrt{V}$ be the one particle constant function corresponding to the mode $k = 0$ (see Section 2.1). We denote by $\mathcal{F}_{B0} \equiv \mathcal{F}_B(\mathcal{H}_0)$ the boson Fock space constructed on the one-dimensional Hilbert space $\mathcal{H}_0$ spanned by ψ_0. If $\mathcal{F}'_B \equiv \mathcal{F}_B\left(\mathcal{H}_0^{\perp}\right)$ is the Fock space constructed on the orthogonal complement $\mathcal{H}_0^{\perp}$, then the Fock space $\mathcal{F}_B(\mathcal{H})$ (2.2) constructed on the Hilbert space $\mathcal{H} = L^2(\Lambda) = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$ is naturally isomorphic to the tensor product:

$$\mathcal{F}_B(\mathcal{H}) \approx \mathcal{F}_B(\mathcal{H}_0) \otimes \mathcal{F}_B\left(\mathcal{H}_0^{\perp}\right), \tag{B.1}$$

see [104], Ch II.4. For any complex number $c \in \mathbb{C}$, we consider in $\mathcal{F}_B(\mathcal{H}_0)$ a coherent vector

$$\psi_{0\Lambda}(c) \equiv e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V}c\right)^k (a_0^*)^k \, \Omega_0, \tag{B.2}$$

where Ω_0 is the vacuum of $\mathcal{F}_B(\mathcal{H})$. Then $a_0\psi_{0\Lambda}(c) = c\sqrt{V}\psi_{0\Lambda}(c)$. Using this concept, Ginibre [92] defines the *Bogoliubov approximation* (cf. Section 2.2) for a Hamiltonian H_Λ in $\mathcal{F}_B(\mathcal{H})$ as follows:

Definition B.1 *[92] The Bogoliubov approximation $H_\Lambda\left(c^{\#}, \mu\right)$ for a Hamiltonian $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$ on $\mathcal{F}_B(\mathcal{H}) \approx \mathcal{F}_B(\mathcal{H}_0) \otimes \mathcal{F}_B\left(\mathcal{H}_0^{\perp}\right)$ is the operator defined on $\mathcal{F}'_B = \mathcal{F}_B\left(\mathcal{H}_0^{\perp}\right)$ by its quadratic form*

$$\left(\psi'_1, H_\Lambda\left(c^{\#}, \mu\right) \psi'_2\right)_{\mathcal{F}'_B} \equiv \left(\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda(\mu)\, \psi_{0\Lambda}(c) \otimes \psi'_2\right)_{\mathcal{F}_B} \tag{B.3}$$

for $\psi_{0\Lambda}(c) \otimes \psi'_{1,2}$ in the form-domain of $H_\Lambda(\mu)$. Here $c^\# = \{c \text{ or } \overline{c}\}$. Remind that

$$N_\Lambda = \sum_{k\in\Lambda^*} N_k = \sum_{k\in\Lambda^*} a_k^* a.$$

B.2 Application to the Weakly Imperfect Bose-Gas

Notice that the self-adjoint operator H_Λ^B (2.50) is defined on a dense domain in the boson Fock space $\mathcal{F}_B(\mathcal{H})$ ($\mathcal{H} = L^2(\Lambda)$). Then by Definition B.1, the *Bogoliubov approximation* for the Bogoliubov Hamiltonian (2.50) gets the form:

$$\begin{aligned} H_\Lambda^B\left(c^\#,\mu\right) &= \sum_{k\in\Lambda^*\backslash\{0\}} \left[\varepsilon_k - \mu + v(0)|c|^2\right] a_k^* a_k \\ &+ \frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} v(k)|c|^2\left[a_k^* a_k + a_{-k}^* a_{-k}\right] \\ &+ \frac{1}{2}\sum_{k\in\Lambda^*\backslash\{0\}} v(k)\left[c^2 a_k^* a_{-k}^* + \overline{c}^2 a_k a_{-k}\right] \\ &-\mu|c|^2 V + \frac{1}{2}v(0)|c|^4 V \end{aligned} \tag{B.4}$$

(see Section 2.3, (2.70)). This formulation of the *Bogoliubov approximation* [1, 18] allows to connect the thermodynamics of the WIBG H_Λ^B with thermodynamics of the system, described by the approximating Hamiltonian $H_\Lambda^B\left(c^\#,\mu\right)$:

Proposition B.2 *[9] For $(\theta,\mu) \in Q$, we have the following estimate*

$$\sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) \le p_\Lambda^B(\beta,\mu) \equiv p_\Lambda\left[H_\Lambda^B\right], \tag{B.5}$$

with

$$\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}'_B} e^{-\beta H_\Lambda^B\left(c^\#,\mu\right)} = \frac{1}{\beta V}\ln\left(\sup_{\{\psi'_n\}_{n=1}^\infty}\sum_{n=1}^\infty e^{-\beta\left(\psi'_n, H_\Lambda^B\left(c^\#,\mu\right)\psi'_n\right)_{\mathcal{F}'_B}}\right), \tag{B.6}$$

where the sup *is over all orthonormal bases $\{\psi'_n\}_{n=1}^\infty$ in $\mathcal{F}'_B = \mathcal{F}_B\left(\mathcal{H}_0^\perp\right)$ contained in the form-domain of $H_\Lambda^B\left(c^\#,\mu\right)$.*

Proof. Since

$$Tr_{\mathcal{F}_B(\mathcal{H})}(-) = Tr_{\mathcal{F}_B(\mathcal{H}_0)\otimes\mathcal{F}_B\left(\mathcal{H}_0^\perp\right)}(-), \tag{B.7}$$

for any Hamiltonian $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$ satisfying

$$Tr_{\mathcal{F}_B(\mathcal{H})}\left(e^{-\beta(H_\Lambda - \mu N_\Lambda)}\right) < +\infty, \tag{B.8}$$

one gets:

$$Tr_{\mathcal{F}_B(\mathcal{H})}\left(e^{-\beta H_\Lambda(\mu)}\right) \geq \sum_{n=1}^{\infty}\left(\psi_0(c)\otimes\psi_n', e^{-\beta H_\Lambda(\mu)}\psi_0(c)\otimes\psi_n'\right)_{\mathcal{F}_B},$$

with $\{\psi_n'\}_{n=1}^{\infty}$ an arbitrary orthonormal base of $\mathcal{F}_B'$. Then, if $\psi_0(c)\otimes\psi_n'$ is in the form-domain of $H_\Lambda(\mu)$, by Definition B.1 and by the Peierls-Bogoliubov inequality (Appendix D) we get

$$\left(\psi_0(c)\otimes\psi_n', e^{-\beta H_\Lambda(\mu)}\psi_0(c)\otimes\psi_n'\right)_{\mathcal{F}_B} \geq e^{-\beta\left(\psi_n', H_\Lambda\left(c^{\#},\mu\right)\psi_n'\right)_{\mathcal{F}_B'}},$$

which implies the estimate:

$$Tr_{\mathcal{F}_B(\mathcal{H})}\left(e^{-\beta H_\Lambda(\mu)}\right) \geq \sup_{\{\psi_n'\}_{n=1}^{\infty}} \sum_{n=1}^{\infty} e^{-\beta\left(\psi_n', H_\Lambda\left(c^{\#},\mu\right)\psi_n'\right)_{\mathcal{F}_B'}}. \tag{B.9}$$

By (B.8)

$$\sup_{\{\psi_n'\}_{n=1}^{\infty}} \sum_{n=1}^{\infty} e^{-\beta\left(\psi_n', H_\Lambda\left(c^{\#},\mu\right)\psi_n'\right)_{\mathcal{F}_B'}}$$

is finite and corresponds to the *trace* on $\mathcal{F}_B'$ which is equal to

$$Tr_{\mathcal{F}_B'}\left(e^{-\beta H_\Lambda\left(c^{\#},\mu\right)}\right) = \sup_{\{\psi_n'\}_{n=1}^{\infty}} \sum_{n=1}^{\infty} e^{-\beta\left(\psi_n', H_\Lambda\left(c^{\#},\mu\right)\psi_n'\right)_{\mathcal{F}_B'}},$$

see e.g. [104, 171]. In the case $H_\Lambda(\mu) = H_\Lambda^B(\mu)$ and $H_\Lambda\left(c^{\#},\mu\right) = H_\Lambda^B\left(c^{\#},\mu\right)$ (B.4), by (B.6) and (B.9) we deduce (B.5) for $(\theta,\mu) \in Q^B = Q \equiv \{(\theta,\mu) : \theta \geq 0,\ \mu \leq 0\}$ (cf. (4.54) and (4.55)). ■

B.2.1 Thermodynamics of the Bogoliubov approximating Hamiltonian

Since the Hamiltonian (B.4) can be diagonalized by the Bogoliubov u-v transformation (see Section 2.3, (2.73), (2.76) and (2.79)), the pressure $\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#}\right)$ associated with $H_\Lambda^B\left(c^{\#},\mu\right)$ has the following explicit form:

$$\begin{aligned}
\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#}\right) &= \xi_\Lambda(\beta,\mu;x) + \eta_\Lambda(\mu;x),\\
\xi_\Lambda(\beta,\mu;x) &= \frac{1}{\beta V}\sum_{k\in\Lambda^*\backslash\{0\}} \ln\left(1-e^{-\beta E_k^B}\right)^{-1},\\
\eta_\Lambda(\mu;x) &= -\frac{1}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}\left(E_k^B - f_k\right) + \mu x - \frac{1}{2}v(0)\,x^2,
\end{aligned} \tag{B.10}$$

with $x = |c|^2 \geq 0$ and

$$\begin{aligned}
f_k &= \varepsilon_k - \mu + x\left[v(0) + v(k)\right],\\
h_k &= xv(k),\\
E_k^B &= \sqrt{f_k^2 - h_k^2}.
\end{aligned} \tag{B.11}$$

Remark B.3 *By Theorem 3.8 the stability domain of the WIBG in the thermodynamic limit is*

$$(\theta, \mu) \in Q^B = Q \equiv \{(\theta, \mu) : \theta \geq 0,\ \mu \leq 0\}. \tag{B.12}$$

But in contrast to H_Λ^B the pressure $\widetilde{p}_\Lambda^B\left(\beta, \mu; c^\#\right)$ calculated with the approximating Hamiltonian $H_\Lambda^B\left(c^\#, \mu\right)$ is well-defined instead of $\mu \leq 0$ for $\mu \leq v(0)|c|^2$.

The aim of this Section is to analyze the trial pressure $\sup\limits_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta, \mu; c^\#\right)$ in the thermodynamic limit. In particular, similar to the domain

$$D^B \equiv \left\{(\theta, \mu) \in Q : p^B(\beta, \mu) > p^P(\beta, \mu)\right\}, \tag{B.13}$$

(cf. 4.57), we shall study the domain

$$D_0^B \equiv \left\{(\theta, \mu) \in Q :\ \lim_\Lambda\ \sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta, \mu; c^\#\right) > p^P(\beta, \mu)\right\}, \tag{B.14}$$

in order to extract conditions implying $D_0^B \neq \{\varnothing\}$, since by (B.5) this gives $D_0^B \subset D^B \neq \{\varnothing\}$.

Theorem B.4 *[9] Let interaction $v(k)$ satisfy hypotheses (A), (B) (Section 2.2 or 4.2) and*

$$v(0) \geq \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}. \tag{B.15}$$

Then

$$\sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta, \mu; c^\#\right) = \widetilde{p}_\Lambda^B(\beta, \mu; 0) = \widetilde{p}_\Lambda^P(\beta, \mu),$$

where

$$\widetilde{p}_\Lambda^P(\beta, \mu) \equiv \frac{1}{\beta V}\sum_{k\in\Lambda^*\setminus\{0\}} \ln\left[\left(1 - e^{[-\beta(\varepsilon_k - \mu)]}\right)^{-1}\right],$$

cf. (3.60). Consequently, in the thermodynamic limit (see (3.63)), one gets

$$\lim_\Lambda\left[\sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta, \mu; c^\#\right)\right] = p^P(\beta, \mu). \tag{B.16}$$

Lemma B.5 *[12, 15] Let interaction $v(k)$ verify the hypotheses (A), (B) (Section 2.2 or 4.2) and (C) (Definition 4.5):*

$$v(0) < \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}, \tag{B.17}$$

Then there is a chemical potential $\mu_0 < 0$ such that (see (B.10))

$$\lim_\Lambda\left(\sup_{x\geq 0} \eta_\Lambda(\mu; x)\right) = \eta(\mu; \overline{x}(\mu) > 0) > 0 \text{ for } \mu \in (\mu_0, 0], \tag{B.18}$$

where $\overline{x}(\mu)$ is maximizer of $\eta(\mu; x)$: $\sup\limits_{x\geq 0} \eta(\mu; x) = \eta(\mu; \overline{x}(\mu))$.

Proof. By the explicit formulae (B.10) and (B.11), we readily get that for $\mu \leq 0$:
(a) $\eta_\Lambda(\mu; x = 0) = 0$ and $\eta_\Lambda(\mu; x) \leq const - \frac{1}{2}v(0)\, x^2$;
(b) $\partial_x \eta_\Lambda(\mu; x = 0) = \mu$ and

$$\partial_x^2 \eta_\Lambda(\mu; x = 0) = \frac{1}{2V} \sum_{k \in \Lambda^* \backslash \{0\}} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} - v(0)\,.$$

Since

$$\lim_\Lambda \frac{1}{2V} \sum_{k \in \Lambda^* \backslash \{0\}} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} = \frac{1}{2\,(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{(\varepsilon_k - \mu)},$$

the hypothesis (C) (B.17) implies the existence of $\widetilde{\mu} < 0$ such that

$$\lim_\Lambda \partial_x^2 \eta_\Lambda(\mu > \widetilde{\mu}; x = 0) > 0.$$

By virtue of (a), (b), and $\lim_\Lambda \partial_x \eta_\Lambda(\mu = 0; x = 0) = 0$, this means that

$$\lim_\Lambda \left(\sup_{x \geq 0} \eta_\Lambda(\mu = 0; x) \right) = \eta\left(\mu = 0; \overline{x}(\mu = 0) > 0\right) > 0\ . \tag{B.19}$$

Therefore, by continuity of (B.19) on the interval $(\widetilde{\mu}, 0]$ we get the existence of $\mu_0 : \widetilde{\mu} \leq \mu_0 < 0$, such that one has (B.18). ■

Theorem B.6 *[12, 15] Let interaction $v(k)$ satisfy the hypotheses (A), (B) (Section 2.2 or 4.2) and (C) (Definition 4.5 or (B.17)). Then, for any $\mu \in (\mu_0, 0]$, there is $\theta_0(\mu) > 0$ such that one has*

$$D_0^B = \{(\theta, \mu) \in Q : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\} \neq \{\varnothing\}\,, \tag{B.20}$$

with μ_0 defined by Lemma B.5 and D_0^B by (B.14).

Proof. First we note that by (B.10) et (B.11) one has $\xi_\Lambda(\beta, \mu; x = 0) = \widetilde{p}_\Lambda^P(\beta, \mu)$ and in addition:

$$\begin{aligned} &(i)\ \ \partial_x \xi_\Lambda(\beta, \mu; x) \leq 0 \text{ and } \lim_{x \to +\infty} \xi_\Lambda(\beta, \mu; x) = 0, \text{ for any } \Lambda; \\ &(ii)\ \ \partial_\theta \xi_\Lambda(\beta, \mu; x) \geq 0 \text{ and } \lim_{\theta \to 0} \xi_\Lambda(\beta, \mu; x) = 0, \text{ for any } \Lambda. \end{aligned} \tag{B.21}$$

Next, by lemma B.5 for $\mu = \mu_0 < 0$ we have

$$\lim_\Lambda \left(\sup_{x \geq 0} \eta_\Lambda(\mu_0; x) \right) = \eta(\mu_0; 0) = \eta\left(\mu_0; \overline{x}(\mu_0) > 0\right) = 0\ . \tag{B.22}$$

Hence, according to (B.21) (see (i)) and (B.22) one obtains:

$$\begin{aligned} (iii)\ \text{if } \theta > 0 :\ \lim_\Lambda \left[\sup_{c \in \mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta, \mu_0; c^\#\right) \right] &= \sup_{x \geq 0} \left[\xi(\beta, \mu_0; x) + \eta(\mu_0; x)\right] \\ = \widetilde{p}^B\left(\beta, \mu_0; c^\# = 0\right) &= p^P(\beta, \mu)\,. \end{aligned} \tag{B.23}$$

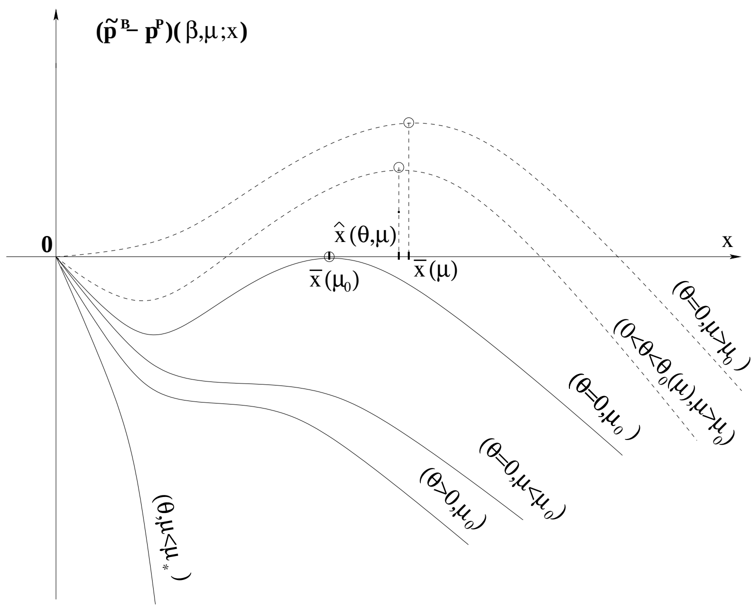

Figure B.1: Illustration of the Bogoliubov approximation variational problem: behaviour of the difference between the trial pressure $\widetilde{p}^B\left(\beta,\mu;c^{\#}\right)$ for the WIBG and the PBG pressure $p^P\left(\beta,\mu\right)$ as a function of the variational parameter $x=\left|c\right|^2$ for different values of (θ,μ). Nontrivial suprema are indicated by empty circles.

and by (B.21) (see (ii)) and (B.22), we find:

$$\begin{aligned}(i\nu)\ \text{ if } \theta &= 0: \lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta=\infty,\mu_0;c^{\#}\right)\right]=\widetilde{p}^B\left(\beta=\infty,\mu_0;c^{\#}=0\right)\\ &= \widetilde{p}^B\left(\beta=\infty,\mu_0;c^{\#}\right)|_{\left|c\right|^2=\overline{x}(\mu_0)>0}=0,\end{aligned}$$

see Figure B.1.

Now by (B.10), (B.21) and Lemma B.5 one obtains that for $\mu_0<\mu\le 0$

$$\lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta,\mu>\mu_0;c^{\#}\right)\right]\ge\eta\left(\mu>\mu_0;\overline{x}\left(\mu\right)>0\right)>0. \tag{B.24}$$

Since by (B.21) (see (ii)), the pressure $p^P\left(\beta,\mu\le 0\right)$ is monotonously decreasing for $\theta\searrow 0$, there is a temperature $\theta_0\left(\mu\right)\ \left(=\beta_0^{-1}\left(\mu\right)\right)$ such that for $\theta<\theta_0\left(\mu>\mu_0\right)$ we get from (B.24)

$$\begin{aligned}p^P\left(\beta>\beta_0\left(\mu\right),\mu>\mu_0\right) &< \eta\left(\mu>\mu_0;\overline{x}\left(\mu\right)>0\right)\\ &< \lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta>\beta_0\left(\mu\right),\mu>\mu_0;c^{\#}\right)\right].\end{aligned} \tag{B.25}$$

Therefore, (B.5) and (B.25) imply (B.20) for $(\theta,\mu)\in D_0^B$. ■

Corollary B.7 *[12, 15] From (B.5) and (B.20) we deduce that if the condition (C) (B.17) is verified, one has*

$$D^B \supseteq D_0^B = \{(\theta,\mu) \in Q : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\},$$

(cf. (B.13), (B.14)). Here $\mu_0 < 0$ is defined in Lemma B.5 and $\theta_0(\mu)$ in Theorem B.6.

The hypothesis (C) defined by (B.17) (or Definition 4.5) is sufficient to guarantee that $\mu_0 < 0$, i.e. $D^B \supseteq D_0^B \neq \{\varnothing\}$. On the other hand, the contrary condition (B.15) implies only the triviality (B.16) of the lower bound (B.5) for $p^B(\beta,\mu)$ but not $D^B = \{\varnothing\}$. Therefore, for the moment we do not know whether the condition (C) is *necessary* for $D^B \neq \{\varnothing\}$. For the answer to this question, see Sections 4.3 and 4.4 where we show that, in fact, $D^B = D_0^B$.

Remark B.8 *Let interaction $v(k)$ satisfy the hypotheses (A), (B) and (C).There is $\widetilde{\mu} < 0$ such that for $\mu \leq \widetilde{\mu}$ one has*

$$v(0) \geq \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} d^3k, \tag{B.26}$$

and consequently $\partial_x^2 \eta(\mu \leq \widetilde{\mu}; x = 0) \leq 0$ (cf. the proof of lemma B.5). One can represent the inequality (B.26) as

$$\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} v(k) \left\{ \frac{v(k)}{2(\varepsilon_k - \mu)} - \frac{v(0)}{\varphi(0)} \right\} \leq 0. \tag{B.27}$$

Since by hypothesis (B) and by $\mu \leq 0$ we have

$$\frac{v(k)}{2(\varepsilon_k - \mu)} \leq \frac{v(0)}{(-2\mu)},$$

the condition $\mu < -\frac{1}{2}\varphi(0) \equiv \mu_*$ (see Section 3, Theorem 3.10) implies (B.27), i.e. $\mu_* \leq \widetilde{\mu}$ (see Figure B.1). Therefore, a local convexity condition (B.26) for $\eta_\Lambda(\mu; x)$ is intimately related to the condition ensuring $p^B(\beta,\mu) = p^P(\beta,\mu)$. In particular, notice that for the condition (B.17) (or (B.15)) the inequality (B.26) is valid for any $\mu \leq \mu_*$.

B.2.2 Maximizer of the trial pressure $\widetilde{p}^B\left(\beta,\mu;c^\#\right)$

We conclude the study of the approximating Hamiltonian $H_\Lambda^B\left(c^\#,\mu\right)$ by analysis of the function $\widehat{x}(\theta,\mu) \equiv |\widehat{c}(\beta,\mu)|^2$, which is a solution of the equation

$$\sup_{c^\# \in \mathbb{R}} \widetilde{p}^B\left(\beta,\mu;c^\#\right) = \widetilde{p}^B\left(\beta,\mu;\widehat{c}^\#(\beta,\mu)\right).$$

which by Theorem 5.6 is related to the WIBG Bose condensate density in mode $k = 0$. Here the pressure $\widetilde{p}^B\left(\beta,\mu;c^\#\right)$ corresponds to the thermodynamic limit of $\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right)$ and has the form:

$$\widetilde{p}^B\left(\beta,\mu;c^\#\right) = \xi\left(\beta,\mu;x \equiv |c|^2\right) + \eta\left(\mu;x \equiv |c|^2\right), \tag{B.28}$$

where (cf. (B.10), (B.11) in the thermodynamic limit)

$$
\begin{aligned}
\xi(\beta,\mu;x) &= \frac{1}{(2\pi)^3\beta}\int_{\mathbb{R}^3} \ln\left(1-e^{-\beta E_k^B}\right)^{-1} d^3k \\
\eta(\mu;x) &= \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \left(f_k - E_k^B\right) d^3k + \mu x - \frac{1}{2}v(0)\,x^2, \\
f_k &= \varepsilon_k - \mu + x\left[v(0)+v(k)\right],\ h_k = xv(k),\ E_k^B = \sqrt{f_k^2-h_k^2}.
\end{aligned}
\tag{B.29}
$$

Theorem B.9 *Let the interaction potential $v(q)$ satisfy the hypotheses (A)-(C). Then for the maximizer of the trial pressure $\widetilde{p}^B\left(\beta,\mu;c^{\#}\right)$ we have*

$$
\widehat{x}(\theta,\mu) = \left\{ \begin{array}{l} 0 \ for\ (\theta,\mu)\in Q\backslash\overline{D}_0^B \\ >0\ for\ (\theta,\mu)\in D_0^B \end{array} \right\}
$$

with Q and D_0^B defined respectively by (B.12) and (B.14). Notice that $\widehat{x}(\theta,\mu)$ is discontinuous on the boundary ∂D_0^B.

Proof. (1) For $\mu\le 0$ the function (B.28) is differentiable with respect to $x=|c|^2\ge 0$ and

$$
\lim_{|c|^2\to\infty}\widetilde{p}^B\left(\beta,\mu;c^{\#}\right) = -\infty. \tag{B.30}
$$

Hence, $\sup\limits_{x\ge 0}(\xi+\eta)(\beta,\mu;x)$ is attained either at $x=0$, or at a positive solution of the equation

$$
\begin{aligned}
0 &= \partial_x(\xi+\eta)(\beta,\mu;x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(1-e^{\beta E_k^B}\right)^{-1}\partial_x E_k^B d^3k \\
&\quad -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(\partial_x E_k^B - \partial_x f_k\right)d^3k + \mu - xv(0),
\end{aligned}
\tag{B.31}
$$

see Figure B.1.

(2) By definitions (B.29) and by the hypotheses (A), (B) (see Section 2.2 or 4.2) on the potential $v(k)$ one obtains

$$
\partial_x f_k = v(0)+v(k),\partial_x E_k^B = \left(E_k^B\right)^{-1}\left(f_k v(0) + (f_k-h_k)\,v(k)\right)\ge 0,
$$

for $\mu\le 0$, $x\ge 0$ and any $k\in\mathbb{R}^3$. Therefore, by (B.31) we have

$$
\partial_x\widetilde{p}^B\left(\beta,\mu;c^{\#}=0\right)\le\partial_x\eta(\mu;x=0)\equiv\partial_x\widetilde{p}^B\left(\beta=\infty,\mu;c^{\#}=0\right)=\mu. \tag{B.32}
$$

(3) By explicit calculation one finds that $\partial_\mu\partial_x\eta(\mu;x)\ge 0$ for $\mu\le 0$ and $x\ge 0$. Hence

$$
\partial_x\eta(\mu;x)\le\partial_x\eta(\mu=0;x), \tag{B.33}
$$

and $\partial_x \eta(\mu = 0; x)$ is a concave function of $(0, \infty)$.
(4) Since the potential $v(k)$ satisfies the condition (C), we get

$$\partial_x^2 \eta(\mu = 0; x) = -v(0) + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{[v(k)]^2}{\varepsilon_k} d^3k \geq 0. \tag{B.34}$$

Since $\eta(\mu = 0; x = 0) = 0$, (B.34) means that the trial pressure

$$\widetilde{p}^B\left(\beta = \infty, \mu; c^{\#}\right) = \eta(\mu = 0; x)$$

attains its $\sup_{x \geq 0}$ at the point $\widehat{x}(\theta = 0, \mu = 0) > 0$. Then by continuity for $(\theta \geq 0, \mu \leq 0)$ we have

$$D_0^B = \{(\theta, \mu) \in Q : \ \widehat{x}(\theta, \mu) > 0\} \neq \{\varnothing\},$$

see Lemma B.5, Theorem B.6 and Figure B.1.
(5) Fix $\mu \in D_0^B$ and $\theta = 0$. Then, according to (B.32),

$$\partial_x \widetilde{p}^B\left(\beta = \infty, \mu; c^{\#} = 0\right) = \mu \leq 0.$$

But explicit calculation (see (B.29)) gives $\partial_x^2 \widetilde{p}^B\left(\beta = \infty, \mu; c^{\#}\right) > 0$ that ensures

$$|\widehat{c}(\beta = \infty, \mu)|^2 = \widehat{x}(\theta = 0, \mu) \equiv \overline{x}(\mu) > 0,$$

(see Figure B.1), i.e.

$$\widetilde{p}^B\left(\beta = \infty, \mu; c^{\#} = 0\right) < \widetilde{p}^B\left(\beta = \infty, \mu; |\widehat{c}(\beta = \infty, \mu)|^2\right). \tag{B.35}$$

(6) Since $\partial_x \xi(\beta, \mu; x) < 0$ (see (B.31)) and

$$\partial_\theta \partial_x \xi(\beta, \mu; x) = \frac{(-1)}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{\beta^2 E_k^B e^{\beta E_k^B}}{\left(1 - e^{\beta E_k^B}\right)^2} \partial_x E_k^B < 0, \tag{B.36}$$

there is a critical temperature $\theta_0(\mu)$ (cf. Theorem B.6) such that for $\mu \in D_0^B$ and $\theta = \theta_0(\mu)$, we have:

$$\begin{aligned} \sup_{x \geq 0} \left[\xi(\beta_0(\mu), \mu; x) + \eta(\mu; x)\right] &= \xi(\beta_0(\mu), \mu; 0) + \eta(\mu; 0) \\ &= \xi(\beta_0(\mu), \mu; \widehat{x}(\theta_0(\mu), \mu) > 0) \\ &\quad + \eta(\mu; \widehat{x}(\theta_0(\mu), \mu) > 0), \end{aligned} \tag{B.37}$$

whereas for $\theta < \theta_0(\mu)$ the supremum is attained at $x = \widehat{x}(\theta, \mu) > 0$ and for $\theta > \theta_0(\mu)$, it "*jumps*" to $\widehat{x}(\theta, \mu) = 0$ (see Figure B.1). ■

Appendix C

The Griffiths lemma

Lemma C.1 (Griffiths) *[172, 173] Let $\{f_n\}_{n\geq 1}$ be a sequence of convex functions on $(a,b) = I \subset \mathbb{R}^1$. If there exists a pointwise limit*

$$\lim_{n\to\infty} f_n(x) = f(x), \;\; x \in I,$$

then

$$\begin{array}{c} \lim\limits_{n\to\infty} \inf \partial_x f_n(x-0) \geq \partial_x f(x-0), \\ \lim\limits_{n\to\infty} \sup \partial_x f_n(x+0) \leq \partial_x f(x+0). \end{array}$$

In particular, if $f_n(x)$ and $f(x)$ are differentiable at some point $x_0 \in I$, then

$$\lim_{n\to\infty} \partial_x f_n(x_0) = \partial_x f(x_0).$$

Proof. Let $\{f_n(x)\}_{n\geq 1}$ be the sequence defined above, and

$$\forall x \in I, \lim_{n\to\infty} f_n(x) = f(x)$$

Then, for every $l \in I$ and for every $x \in [a+l, b-l]$

$$\begin{aligned} f_n(x+l) &\geq f_n(x) + l\partial_x f_n(x) \\ f_n(x-l) &\geq f_n(x) - l\partial_x f_n(x) \end{aligned}$$

and thus

$$\begin{aligned} \partial_x f_n(x) &\leq \frac{f_n(x+l) - f_n(x)}{l} \\ \partial_x f_n(x) &\geq \frac{f_n(x) - f_n(x-l)}{l} \end{aligned}$$

Consequently in the limit $n \to \infty$ we find

$$\begin{aligned} \lim_{n\to\infty} \partial_x f_n(x) &\leq \partial_x f(x+0) \equiv \lim_{l\to 0^+} \frac{f(x+l) - f(x)}{l}, \\ \lim_{n\to\infty} \partial_x f_n(x) &\geq \partial_x f(x-0) \equiv \lim_{l\to 0^+} \frac{f(x) - f(x-l)}{l}. \end{aligned}$$

■

Appendix D

The Bogoliubov (convexity) inequality

The aim of this appendix is to prove the Bogoliubov convexity inequality. In the case of the classical mechanics it is a simple consequence of convexity of the Gibbs exponent as function of the interaction potential. It is less trivial for the non-abelian case of the Quantum Statistical Mechanics, see [91, 125, 163, 171, 174, 175].

First we recall that in Quantum Statistical Mechanics the *density matrix* operator ρ_β for a finite system with the Hamiltonian H and temperature β^{-1} belongs to Tr-class $\mathcal{C}_1(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$ of the vector quantum states of the system. This means that

$$\rho_\beta(H) = \frac{1}{Z_\beta} e^{-\beta H} \in \mathcal{C}_1(\mathcal{H}), \tag{D.1}$$

with the normalizing factor

$$Z_\beta = Tr_{\mathcal{H}} e^{-\beta H} < +\infty, \ \beta > 0. \tag{D.2}$$

Notice that $\mathcal{H}$ varies from finite-dimensional spaces for quantum spin lattice systems up to Fock spaces for continuous systems. We denote by $(.,.)_{\mathcal{H}}$ the scalar product on $\mathcal{H}$.

Since the Hamiltonian H is *self-adjoint* and $e^{-\beta H} \in \mathcal{C}_1(\mathcal{H})$ there is an *orthonormal* basis $\{\varphi_n\}_{n=1}^{\infty}$: the eigenvectors of H with real eigenvalues E_n, i.e. $H\varphi_n = E_n \varphi_n$ and $e^{-\beta H}\varphi_n = e^{-\beta E_n}$, $n \geq 1$. The family $\left\{e^{-\beta H}\right\}_{\beta>0}$ got the name *Gibbs semigroup* generated by H, see [174].

Lemma D.1 *(Jensen inequality) Let ξ be a real random variable with expectation $\mathbb{E}(|\xi|) < \infty$. For any real convex function g on $\mathbb{R}^1$ one has*

$$\mathbb{E}(g(\xi)) \geq g(\mathbb{E}(\xi)). \tag{D.3}$$

Proof. By convexity of g there are two numbers $x_0 \in \mathbb{R}^1$ and $\lambda(x_0)$ such that

$$g(x) \geq g(x_0) + \lambda(x_0)(x - x_0). \tag{D.4}$$

Let $x = \xi$ and $x_0 = \mathbb{E}(\xi)$. Then by (D.4)

$$\mathbb{E}(g(\xi)) \geq g(\mathbb{E}(\xi)),$$

which proves (D.3). ■

Lemma D.2 *(Peierls-Bogoliubov inequality) Let* $\left\{e^{-\beta H}\right\}_{\beta>0}$ *be a Gibbs semigroup generated by the self-adjoint Hamiltonian* H. *Then for any orthonormal basis* $\{\eta_n\}_{n=1}^{\infty}$ *one gets:*

$$Tr_{\mathcal{H}}e^{-\beta H} \geq \sum_{n=1}^{\infty} \exp\left\{-\beta\left(\eta_n, H\eta_n\right)_{\mathcal{H}}\right\}. \tag{D.5}$$

Proof. Let $u \in \mathcal{H}$ be a unit vector, i.e. $\|u\| = (u,u)_{\mathcal{H}}^{1/2} = 1$. Then $u = \sum_{n=1}^{\infty} u_n\varphi_n$ and

$$\left(u, e^{-\beta H}u\right)_{\mathcal{H}} = \sum_{n=1}^{\infty} |u_n|^2 e^{-\beta E_n}. \tag{D.6}$$

Since $\|u\|^2 = \sum_{n=1}^{\infty} |u_n|^2 = 1$, we can consider (D.6) as an expectation of the convex function of the random variable $\{E_n\}_{n=1}^{\infty}$ with respect to the probability distribution defined by $\left\{|u_n|^2\right\}_{n=1}^{\infty}$. Then by the Jensen inequality (D.3) one gets

$$\left(u, e^{-\beta H}u\right)_{\mathcal{H}} \geq e^{-\beta \sum\limits_{n=1}^{\infty} |u_n|^2 E_n} = e^{-\beta (u,Hu)_{\mathcal{H}}}. \tag{D.7}$$

In particular, for any orthonormal basis $\{\eta_n\}_{n=1}^{\infty}$ we obtain the inequality

$$Tr_{\mathcal{H}}e^{-\beta H} = \sum_{n=1}^{\infty} \left(\eta_n, e^{-\beta H}\eta_n\right)_{\mathcal{H}} \geq \sum_{n=1}^{\infty} e^{-\beta(\eta_n, H\eta_n)_{\mathcal{H}}},$$

which proves (D.5). The rest is a consequence of invariance of the $Tr_{\mathcal{H}}(.)$ with respect to the choice of an orthonormal basis in $\mathcal{H}$. ■

Theorem D.3 *(Bogoliubov convexity inequality) Let* H_1 *and* H_2 *be two self-adjoint generators of Gibbs semigroups. Suppose that* $Tr_{\mathcal{H}}\left[e^{-\beta H_1}(H_2 - H_1)\right]$ *and* $Tr_{\mathcal{H}}\left[e^{-\beta H_2}(H_2 - H_1)\right]$ *are bounded. Then*

$$\beta\left\langle H_2 - H_1\right\rangle_{H_2} \leq \ln Tr_{\mathcal{H}}e^{-\beta H_1} - \ln Tr_{\mathcal{H}}e^{-\beta H_2} \leq \beta\left\langle H_2 - H_1\right\rangle_{H_1} \tag{D.8}$$

where

$$\left\langle H_2 - H_1\right\rangle_{H_2} = Tr_{\mathcal{H}}\left[\rho_\beta(H_2)(H_2 - H_1)\right], \tag{D.9}$$

$$\left\langle H_2 - H_1\right\rangle_{H_1} = Tr_{\mathcal{H}}\left[\rho_\beta(H_1)(H_2 - H_1)\right]. \tag{D.10}$$

Proof. Let $\left\{\varphi_n^{(2)}\right\}_{n=1}^{\infty}$ be an orthonormal basis of eigenvectors of H_2. Now by (D.5) and then by (D.3) one gets

$$\frac{Tr_{\mathcal{H}}e^{-\beta H_1}}{Tr_{\mathcal{H}}e^{-\beta H_2}} = \frac{\sum\limits_{n=1}^{\infty}\left(\varphi_n^{(2)}, e^{-\beta(H_2+H_1-H_2)}\varphi_n^{(2)}\right)_{\mathcal{H}}}{Tr_{\mathcal{H}}e^{-\beta H_2}} \geq \frac{\sum\limits_{n=1}^{\infty} e^{-\beta E_n^{(2)}} e^{-\beta\left(\varphi_n^{(2)},(H_1-H_2)\varphi_n^{(2)}\right)_{\mathcal{H}}}}{\sum\limits_{n=1}^{\infty} e^{-\beta E_n^{(2)}}} \geq$$

$$
\begin{aligned}
&\geq \exp\left\{-\beta\left(\frac{\sum_{n=1}^{\infty} e^{-\beta E_n^{(2)}}\left(\varphi_n^{(2)}, (H_1 - H_2)\,\varphi_n^{(2)}\right)_{\mathcal{H}}}{\sum_{n=1}^{\infty} e^{-\beta E_n^{(2)}}}\right)\right\} = \\
&= \exp\left\{-\beta\left\langle H_1 - H_2\right\rangle_{H_2}\right\}. \qquad \text{(D.11)}
\end{aligned}
$$

If we rename H_1 and H_2, then (D.11) reads as

$$
\frac{Tr_{\mathcal{H}} e^{-\beta H_2}}{Tr_{\mathcal{H}} e^{-\beta H_1}} \geq \exp\left\{-\beta\left\langle H_2 - H_1\right\rangle_{H_1}\right\}. \qquad \text{(D.12)}
$$

The inequalities (D.11) and (D.12) imply (D.8). ■

Corollary D.4 *For two grand-canonical pressures*

$$
p_\Lambda\left[H_{\Lambda,i}\right] = \frac{1}{\beta V} \ln Tr_{\mathcal{F}_B} e^{-\beta\left(H_{\Lambda,i} - \mu N_\Lambda\right)},\ i = 1, 2,
$$

one gets the Bogoliubov inequality:

$$
\frac{1}{V}\left\langle H_{\Lambda,2} - H_{\Lambda,1}\right\rangle_{H_{\Lambda,2}} \leq p_\Lambda\left[H_{\Lambda,1}\right] - p_\Lambda\left[H_{\Lambda,2}\right] \leq \frac{1}{V}\left\langle H_{\Lambda,2} - H_{\Lambda,1}\right\rangle_{H_{\Lambda,1}}, \qquad \text{(D.13)}
$$

which we use throughout the present review. Similar inequality we get for the corresponding free-energy densities :

$$
\frac{1}{V}\left\langle H_{\Lambda,2} - H_{\Lambda,1}\right\rangle_{H_{\Lambda,2}} \leq f_\Lambda\left[H_{\Lambda,2}\right] - f_\Lambda\left[H_{\Lambda,1}\right] \leq \frac{1}{V}\left\langle H_{\Lambda,2} - H_{\Lambda,1}\right\rangle_{H_{\Lambda,1}}, \qquad \text{(D.14)}
$$

with

$$
f_\Lambda\left[H_{\Lambda,i}\right] = -\frac{1}{\beta V} \ln Tr_{\mathcal{H}_B} e^{-\beta H_{\Lambda,i}},\ i = 1, 2.
$$

Appendix E

The correlation inequalities for quantum states

E.1 States for finite systems

We recall [37] that in the standard formulation of quantum statistical mechanics of an assembly of particles confined to a bounded region $\Lambda \subset \mathbb{R}^d$ the *observables* correspond to the self-adjoint operators A, B, ...$\in \mathcal{A}$, and the *states* $\omega_\Lambda(.)$ correspond to a *density matrices* $\rho_\omega^{(\Lambda)}$ in a Hilbert space $\mathcal{H}$. Thus the *expectation value* of an observable A for the state ω_Λ is

$$\omega_\Lambda(A) = Tr_{\mathcal{H}}\left(\rho_\omega^{(\Lambda)} A\right) \equiv \langle A \rangle_\omega^{(\Lambda)}. \tag{E.1}$$

According to the Gibbs *ansatz* the density matrix for the system in *equilibrium* with temperature $\theta = \beta^{-1}$ is defined by the Hamiltonian of the system H_Λ:

$$\rho_\omega^{(\Lambda)} = \frac{e^{-\beta H_\Lambda}}{Tr_{\mathcal{H}} e^{-\beta H_\Lambda}}, \ \ \langle A \rangle_\omega^{(\Lambda)} = \langle A \rangle_{H_\Lambda}, \tag{E.2}$$

see Section 2.1 for states in the grand-canonical ensemble $\mathcal{H} = \mathcal{F}_B\left(= \mathcal{F}_B\left(L^2(\Lambda)\right)\right)$.

Dynamical of the system is also defined (formally) by their Hamiltonian according to the formula

$$\tau_t(A) = e^{iH_\Lambda t} A e^{-iH_\Lambda t}. \tag{E.3}$$

Then, by cyclicity of the $Tr_{\mathcal{H}}$ one gets for the Gibbs state (E.2) the identity:

$$\omega_\Lambda(\tau_t(A) B) = \omega_\Lambda(A \tau_{t+i\beta}(B)), \ A, B \in \mathcal{A}, \tag{E.4}$$

known as *Kubo-Martin-Schwinger* (KMS) conditions. On the other hand this condition implies that ω_Λ is *stationary* in time: $\omega_\Lambda(\tau_t(A)) = \omega_\Lambda(A)$ (take $B = I$ in (E.4)). For the system with Hamiltonian H_Λ the KMS conditions *uniquely* specify the Gibbs state given by the density matrix (E.2).

Thus the KMS states for finite systems are *identical* to quantum Gibbs states defined by density matrices (E.2).

E.2 KMS-condition and correlation inequalities

It has been proved by Fannes and Verbeure [176], that there is a *hierarchy* of correlation inequalities related to KMS-states [177] or [26] (Part I Section 1.5) and [178–181]. Some of these inequalities are *equivalent* to the KMS conditions (E.4). In Sections 4.1.2 and 5.2.2 we use the following

Proposition E.1 *Let $\omega_\Lambda(-)$ be a finite-volume KMS-state. Then for any X from domain of the commutator $[H_\Lambda, .]$ one has the correlation inequality:*

$$\beta\omega_\Lambda(X^*[H_\Lambda, X]) \geq \omega_\Lambda(X^*X)\ln\frac{\omega_\Lambda(X^*X)}{\omega_\Lambda(XX^*)}, \tag{E.5}$$

Here we suppose that

$$S(u,v) \equiv u\ln\left(\frac{u}{v}\right) = \left\{\begin{array}{l} u\ln u - u\ln v,\ \textit{for } u,v \geq 0,\ u+v>0 \\ 0,\ \textit{for } u=v=0 \end{array}\right\}. \tag{E.6}$$

Remark E.2 *In fact (E.5) is equivalent to (E.4), see [37,176,181], but for purposes of Sections 4.1.2 and 5.2.2 only (E.4) $\Rightarrow$ (E.5) is necessary.*

Proof. (1) Notice that the function (E.6) is jointly convex in (u,v) and homogeneous of degree 1:

$$S(\lambda u, \lambda v) = \lambda S(u,v),\ u,v,\lambda \in \mathbb{R}^1_+. \tag{E.7}$$

Let $\sum_{i=1}^{n}\xi_i = 1$, $\xi_i \in \mathbb{R}^1_+$. Then by convexity one gets

$$S\left(\sum_{i=1}^{n}\xi_i u_i, \sum_{i=1}^{n}\xi_i v_i\right) \leq \sum_{i=1}^{n}\xi_i S(u_i, v_i) \tag{E.8}$$

for all finite sequences $\{u_i\}_{i=1}^n$, $\{v_i\}_{i=1}^n \subset \mathbb{R}^1_+$. Therefore, (E.8) can be extended for any problem *probability* measure ν on $\mathbb{R}^1_+$:

$$S(\nu(f_1), \nu(f_2)) \leq \nu(S(f_1, f_2)), \tag{E.9}$$

where f_1, f_2 are continuous bounded non-negative functions on $\mathbb{R}^1_+$ and $\nu(f) = \int_{\mathbb{R}^1_+} d\nu(x) f(x)$.

By (E.7) the inequality (E.8) can be extended to all finite (non-normalized) sequences $\{\lambda_i \geq 0\}_{i=1}^n$ instead of $\{\xi_i\}_{i=1}^n$. Therefore, (E.9) is valid for any finite measure μ on $\mathbb{R}^1_+$:

$$S(\mu(f_1), \mu(f_2)) \leq \mu(S(f_1, f_2)). \tag{E.10}$$

(2) Using the state $\omega_\Lambda(-)$ and dynamics $\tau_t(-)$, see (E.3), we introduce two functionals (finite measures):

$$u_A\left(\widehat{f}\right) = \int_{\mathbb{R}^1} dt f(t)\,\omega_\Lambda(A^*\tau_t(A)), \tag{E.11}$$

$$v_A\left(\widehat{f}\right) = \int_{\mathbb{R}^1} dt f(t)\,\omega_\Lambda(\tau_t(A)A^*), \tag{E.12}$$

defined on the smooth functions $\widehat{f}$ with compact support supp$f = \left\{k : \widehat{f}(k) \neq 0\right\} \subset \mathbb{R}^1$, where

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}^1} dt f(t) e^{-ikt} \tag{E.13}$$

is the Fourier transform of f. Notice that by (E.11)-(E.13) one has

$$u_A\left(\widehat{f} = 1\right) = \omega_\Lambda(A^*A), \; v_A\left(\widehat{f} = 1\right) = \omega_\Lambda(AA^*), \tag{E.14}$$

and by (E.4) we obtain

$$u_A\left(\widehat{f}\right) = v_A\left(L_\beta \widehat{f}\right), \tag{E.15}$$

where $L_\beta(k) = e^{-\beta k}$.
(3) From inequality (E.10) and (E.14), (E.15), one gets:

$$\begin{aligned} S\left(\omega_\Lambda(A^*A), \omega_\Lambda(AA^*)\right) &= S\left(u_A(1), v_A(1)\right) = S\left(u_A(1), u_A\left(L_\beta^{-1}\right)\right) \\ &\leq u_A\left(S\left(1, L_\beta^{-1}\right)\right) = u_A(\ln L_\beta), \end{aligned} \tag{E.16}$$

see (E.6).
(4) Let $\widehat{f}(k) \to k\widehat{f}(k) \equiv \widehat{f}_1(k)$. Then

$$f_1(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}^1} dt f(t) e^{ikt} = \frac{1}{i} f'(t).$$

Therefore, by (E.3) we obtain:

$$\begin{aligned} u_A\left(\widehat{f}_1\right) &= \frac{1}{i} \int\limits_{\mathbb{R}^1} dt f'(t) \omega_\Lambda(A^*\tau_t(A)) = i \int\limits_{\mathbb{R}^1} dt f(t) \omega_\Lambda(A^*\partial_t \tau_t(A)) = \\ &= \int\limits_{\mathbb{R}^1} dt f(t) \omega_\Lambda(A^*[\tau_t(A), H_\Lambda]). \end{aligned} \tag{E.17}$$

Now put $\widehat{f}(k) = -\beta$, i.e. $f(t) = -\beta\delta(t)$. Then (E.17) implies

$$u_A(\ln L_\beta) = \beta\omega_\Lambda(A^*[H_\Lambda, A]). \tag{E.18}$$

Hence, (E.16) and (E.18) give (E.5) for $A = X$. ■

Appendix F

Quasi-free states

A general (infinite volume) Gibbs state ω of a Bose system is determined if all its correlation functions: $\omega\left(a^*\left(f_1\right)...a^*\left(f_n\right)a\left(g_1\right)...a\left(g_m\right)\right)$, for all $n,\ m \geq 0$ and all test functions $f_i,\ g_i$ are known.

A particular class of states, called quasi-free states [182], is defined by the property that all its correlation functions can be expressed as sums of products of one- and two-point correlations:

$$\text{if } Q_{1...n} = \prod_{i=1}^{n}\left(a^{\#}\left(f_i\right) - \omega\left(a^{\#}\left(f_i\right)\right)\right), \tag{F.1}$$

then

$$\omega\left(Q_{1...n}\right) = \sum_{P\in\mathcal{P}} \omega\left(Q_{P_1}\right)...\omega\left(Q_{P_r}\right)$$

where the sum runs over the set $\mathcal{P}$ of all partitions $P = \{P_1,...,P_r\}$ of $\{1,...,n\}$ into two-point sets P_i, and the operators within each P_i are in the same order as in $Q_{1...n}$.

Let us now consider a model defined by the finite-volume Hamiltonians H_Λ. For each translation-invariant (general) state ω, one defines particle-, energy-, and entropy-densities by:

$$n\left(\omega\right) = \lim_{\Lambda} \frac{1}{V}\omega_\Lambda\left(N_\Lambda\right) \tag{F.2}$$

$$e\left(\omega\right) = \lim_{\Lambda} \frac{1}{V}\omega_\Lambda\left(H_\Lambda\right) \tag{F.3}$$

$$s\left(\omega\right) = \lim_{\Lambda} \frac{1}{V}S\left(\omega_\Lambda\right) \tag{F.4}$$

respectively, where N_Λ is the particle-number operator in Λ, ω_Λ denotes the restriction of the state ω to local observables in the box Λ and $S\left(\omega_\Lambda\right)$ is its entropy.

The variational principle of statistical mechanics is as follows. The equilibrium states of the model H_Λ at inverse temperature $\beta = \left(kT\right)^{-1}$ and particle density ρ are the minimizers of the free energy: $f\left(\omega\right) = e\left(\omega\right) - \beta^{-1}s\left(\omega\right)$, over all translation-invariant states ω, such that $n\left(\omega\right) = \rho$.

It is convenient to take into account the constraint by the Lagrange multiplier method. One defines:

$$g_\mu\left(\omega\right) = f\left(\omega\right) - \mu\left(n\left(\omega\right) - \rho\right) \tag{F.5}$$

and solves the unconstrained minimum problem for g_μ. Let $\omega_{\beta,\mu}$ denote the minimizer of g_μ, $\mu \in \mathbb{R}^1$; one determines $\mu(\rho)$ from the subsidiary condition:

$$n(\omega_{\beta,\mu}) = \rho. \tag{F.6}$$

Then $\omega_{\beta,\mu(\rho)}$ is an equilibrium state and $g_{\mu(\rho)}\left(\omega_{\beta,\mu(\rho)}\right)$ is the equilibrium pressure functional corresponding to β and ρ. Note that the physical pressure equals $-g_\mu(\omega)$.

A model H_Λ is quasi-free if the associated energy density (F.3) is expressed solely in terms of the one- and two-point functions of ω, i.e. $\omega\left(a^{\#}(x)\right)$ and $\omega\left(a^{\#}(x)\, a^{\#}(y)\right)$. In such cases, the variational principle can be simplified considerably, in that the minimum of g_μ (F.5) can be taken only within the class of quasi-free states. Indeed, every state ω determines a quasi-free state ω_{qf}: it has the same one- and two-point correlations as ω, and, of course, all its higher correlations are put equal to zero. Then, $n(\omega_{qf}) = n(\omega)$ and, by assumption, $e(\omega_{qf}) = e(\omega)$. Moreover, it is always true (cf. for example [183]) that $s(\omega_{qf}) \geq s(\omega)$. Therefore, $g_\mu(\omega_{qf}) \leq g_\mu(\omega)$ for all states ω, the assertion follows.

In order to apply the variational principle, we need a convenient parametrization of the set of quasi-free states. First, we consider the one-point function:

$$c \equiv \omega(a(x)) \tag{F.7}$$

as a parameter characterizing the $k = 0$ mode, for example we have for finite volume V, $\omega(a_0) = V^{1/2} c$ and $\omega(a_0^* a_0) = V\,|c|^2$.

The two-point correlations are more conveniently described in Fourier representation, i.e. we take the function $\omega(a_k^* a_k)$ and $\omega\left(a_k^* a_{-k}^*\right) = \overline{\omega(a_k a_{-k})}$ for $k \in \mathbb{R}^3 \setminus \{0\}$ as non-trivial parameters. Remark that we consider quasi-free states which are not necessarily gauge invariant, which enables us to take into account spontaneous symmetry breaking. Indeed, models (2.50) and (3.73) have *global gauge symmetry* (expressing the conservation of their particle number); model (3.76) is *not invariant* under global gauge transformations, but is instead invariant for gauge transformations in the $k = 0$ mode (expressing conservation of the number of particles with $k = 0$). The thermodynamic limit of finite-volume Gibbs states shares the symmetries of the Hamiltonian, but, like any equilibrium state, it has a decomposition into ergodic states, and the latter are not necessarily preserving the symmetry of the Hamiltonian. The quasi-free minimizers we are looking for are the ergodic equilibrium states [182]. In both cases (3.73) and (3.76), it can be checked that, in a minimizing state ω, the phases of the complex numbers $\omega\left(a_k^* a_{-k}^*\right)$ are independent of k : for model (3.73) they equal the phase of $\overline{c}^2$, while for (3.76) they equal the phase of $\overline{b}^2$, which, by the self-consistency equations, should be put equal to that of $\overline{c}^2$. Thereby, the energy density $e(\omega)$ is independent of this phase. Therefore, if $c \neq 0$ in a minimizing state ω, then all states obtained from ω by simultaneously changing the phase of $\omega\left(a_k^* a_{-k}^*\right)$ and of $\overline{c}^2$ will also be minimizers.

So, for simplicity of the presentation, we shall assume henceforth, without any loss, that c and $\omega\left(a_k^* a_{-k}^*\right)$ are real (and, of course, $b \in \mathbb{R}^1$ in (3.76)). Also, because $\varepsilon_k = \varepsilon_{-k}$ and $v(k) = v(-k)$, one can assume $\omega(a_k^* a_k) = \omega\left(a_{-k}^* a_{-k}\right)$.

In order to compute (and express in a simple form) the entropy functional $s(\omega)$, it is convenient to use another parametrization of the set of quasi-free states. The new parameters will be a triple (c, m_k, α_k) $(k \in \mathbb{R}^3 \setminus \{0\})$, where $c \in \mathbb{C}$ is again the one-point function, and m_k, α_k are real-valued, even functions, subject to the constraint:

$$m_k \geq 1,\ \forall k \in \mathbb{R}^3 \setminus \{0\}\,. \tag{F.8}$$

The old parameters are expressed in terms of (m_k, α_k) by:

$$\omega\left(a_k^* a_k\right) = \frac{1}{2}\left(m_k \cosh 2\alpha_k - 1\right),\ \omega\left(a_k^* a_{-k}^*\right) = -\frac{1}{2} m_k \sinh 2\alpha_k. \tag{F.9}$$

It is clear that constraint (F.8) expresses the positivity of ω.

The new parameters appear naturally by going, via a Bogoliubov transformation, to a gauge-invariant formulation of ω, i.e. in terms of new boson operators $\eta_k^{\#}$:

$$a_k = \eta_k \cosh \alpha_k + \eta_{-k}^* \sinh \alpha_k,\ \ a_{-k} = \eta_{-k} \cosh \alpha_k + \eta_k^* \sinh \alpha_k, \tag{F.10}$$

such that $\omega\left(\eta_k^* \eta_{-k}^*\right) = 0$. The latter equation defines $\alpha_k = \alpha_{-k}$, and $m_k = m_{-k}$ is taken as

$$m_k = 2\omega\left(\eta_k^* \eta_k\right) + 1. \tag{F.11}$$

In the gauge-invariant formulation, and using equation (F.11), the entropy density functional is calculated without difficulty and it has the following expression [184]:

$$s(\omega) = \frac{1}{(2\pi)^3} \int dk \left(\frac{m_k + 1}{2} \log \frac{m_k + 1}{2} - \frac{m_k - 1}{2} \log \frac{m_k - 1}{2}\right). \tag{F.12}$$

Taking into account (F.9), it is now an easy matter to compute the functional $g_\mu(\omega)$ entering the variational principle.

Let us consider now the model (3.76). One has to minimize with respect to (c, m_k, α_k), the pressure

$$\begin{aligned} g_\mu^{SB}(\omega; b) \ = \ & \frac{1}{2(2\pi)^3} \int dk \left[\left(\varepsilon_k + b^2 v(k)\right)\left(m_k \cosh 2\alpha_k - 1\right) - b^2 v(k)\, m_k \sinh \alpha_k\right] \\ & + \frac{1}{2} v(0)\, n(\omega)^2 - \mu\left(n(\omega) - \rho\right) \\ & - \frac{1}{\beta(2\pi)^3} \int dk \left(\frac{m_k + 1}{2} \log \frac{m_k + 1}{2} \log \frac{m_k - 1}{2}\right), \end{aligned} \tag{F.13}$$

where $n(\omega)$ is given by equation:

$$n(\omega) = c^2 + \frac{1}{2(2\pi)^3} \int dk \left(m_k \cosh 2\alpha_k - 1\right),$$

see [11]. Among the minimizing states at a given μ, which are, of course, labeled by b, one has to select those for which b is related to the one-point function c according to the self-consistency

equations of the model (3.76).
The Euler equation for $g_{\mu}^{SB}(\omega;b)$ are:

$$\begin{aligned}
&cx(\omega)=0\\
&\tanh 2\alpha_k=\frac{c^2v(k)}{\varepsilon_k+x(\omega)+b^2v(k)}\\
&\frac{1}{\beta}\log\frac{m_k+1}{m_k-1}=\left(\varepsilon_k+x(\omega)+b^2v(k)\right)\cosh 2\alpha_k-b^2\sinh\alpha_k
\end{aligned}\tag{F.14}$$

where $x(\omega)$ is defined by equation:

$$x(\omega)=v(0)\,n(\omega)-\mu.$$

The self-consistency equations should be added to (F.14) as a separate equation.

Appendix G

Stable and Superstable interactions

To fix notations we start by recalling some facts about the full Hamiltonian (2.43), (2.44). We consider a system of n bosons of mass m enclosed in a cubic box $\Lambda = L\times L\times L \subset \mathbb{R}^{d=3}$ of volume $V \equiv |\Lambda| = L^3$. The interaction two-body potential is defined by $\varphi(x)$. The corresponding Hamiltonian is defined by a *self-adjoint* (s-a) extension

$$H_\Lambda^{(n)} = \left\{ \sum_{j=1}^{n} \left(-\frac{\hbar^2 \Delta_j}{2m} \right) + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \varphi_{\mathrm{per}}(x_i - x_j) \right\}_{\text{s-a}}$$

in the boson Hilbert space $\mathcal{H}_B^{(n)} = (L^2(\Lambda^n))_{\mathrm{symm}}$ of symmetric functions on the *torus* Λ [37,91]. Here we consider the periodic boundary conditions on $\partial\Lambda$, and then $\varphi_{\mathrm{per}}(x) = \sum_{n\in\mathbb{Z}^3} \varphi(x + nL)$, $x \in \Lambda$, is the restriction of the potential $\varphi(x)$ on the torus Λ. Therefore, after the second quantization (see Section 2.1), the corresponding Hamiltonian acting in the boson Fock space $\mathcal{F}_B(\mathcal{H})$ over $\mathcal{H} = L^2(\Lambda)$ equals

$$\begin{aligned} H_\Lambda &= T_\Lambda + U_\Lambda = \int_\Lambda a^*(x) \left(-\frac{\hbar^2\Delta}{2m} \right) a(x)\, d^3x + \\ &+ \frac{1}{2} \int_\Lambda\int_\Lambda a^*(x)\, a^*(y)\, \varphi_{\mathrm{per}}(x-y)\, a(x)\, a(y)\, d^3x d^3y, \end{aligned} \tag{G.1}$$

where $a^{\#}(x)$ are basic boson-field operators in the boson Fock space $\mathcal{F}_B(\mathcal{H})$.

Remark G.1 *Under assumption that the particle interaction is defined by a translation-invariant absolutely integrable two-body potential $\varphi(x)$ (hypothesis (A), (2.42)), the full Hamiltonian (G.1) can be written as (2.44).*

Remark G.2 *The operator*

$$N_\Lambda = \int_\Lambda a^*(x)\, a(x)\, dx$$

is the total particle-number operator and $[N_\Lambda, H_\Lambda] = 0$.

Of course the interaction between particles cannot be arbitrary: it must become sufficiently weak with the increase of the inter-particle distance to ensure the thermodynamic limit, and it becomes repulsive on short distances to prevent the collapse of an infinite number of particles [91].

A *necessary* condition for existence of thermodynamics, i.e. for existence of a domain $(\theta = \beta^{-1}, \mu)$, where the corresponding *grand partition function*

$$\Xi_\Lambda(\beta, \mu) = Tr_{\mathcal{F}_B} e^{-\beta(H_\Lambda - \mu N_\Lambda)} \tag{G.2}$$

exists, is the following condition of *stability*:

Definition G.3 *The interaction U_Λ is stable if there is $B \geq 0$ such that*

$$U_\Lambda \geq -BN_\Lambda, \tag{G.3}$$

i.e. in the canonical ensemble the corresponding n-particles interaction $U_\Lambda^{(n)}$ acting on $\mathcal{H}_B^{(n)}$ satisfies

$$U_\Lambda^{(n)} \geq -Bn, \tag{G.4}$$

for any $n \geq 0$. We say also that the corresponding Hamiltonian $H_\Lambda = T_\Lambda + U_\Lambda$ is stable.

Since for the translation-invariant two-body potential φ in Λ with periodic boundary conditions one has:

$$U_\Lambda^{(n)} = \frac{1}{2} \sum_{1 \leq i < j \leq n}^{n} \varphi_{\text{per}}(x_i - x_j), \tag{G.5}$$

the stability condition depends only on the interaction potential $\varphi(x)$.

Proposition G.4 *[91] Let $\varphi(x)$ be a real semicontinuous function on $\mathbb{R}^3$ satisfying $\varphi(x) = \varphi(-x)$ then the following properties are equivalent:*
(i) the corresponding n-particles interaction $U_\Lambda^{(n)}$ (G.5) is stable,
(ii) the interaction potential $\varphi(x)$ verifies the inequality:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(x_i - x_j) \geq 0, \tag{G.6}$$

for any $n \geq 0$ and all $x_1, ..., x_n \in \mathbb{R}^3$.

The important condition of the stability is given by:

Proposition G.5 *[91] Let (real) $\varphi(x) \in L^1(\mathbb{R}^3)$ (hypothesis (A)) and its (real) Fourier transform $v(q) \geq 0$. Then $\varphi(x) = \varphi(-x)$ and the interaction $U_\Lambda^{(n)}$ is stable.*

Proof. The first assertion follows from reality of $\varphi(x)$ and from the Fourier representation

$$\varphi(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dq v(q) e^{iqx}.$$

For the second assertion note that:

$$\begin{aligned} \sum_{l=1}^{n} \sum_{m=n}^{n} z_l \overline{z_m} \varphi(x_l - x_m) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dq v(q) \left(\sum_{l=1}^{n} z_l e^{iqx_l} \right) \left(\sum_{m=1}^{n} \overline{z_m} e^{-iqx_m} \right) \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dq v(q) \left| \sum_{j=1}^{n} z_j e^{iqx_j} \right|^2 \geq 0, \end{aligned} \tag{G.7}$$

for all $x_1, ..., x_n \in \mathbb{R}^{d=3}$ and for all $z_1, ..., z_n \in \mathbb{C}$. Taking $z_1 = z_2 = ... = z_n = 1$ one gets (G.6). ■

Remark G.6 *A simple example of catastrophic interaction potential for (G.1) is a pair potential verifying:*

$$v(0) = \int_{\mathbb{R}^3} d^3x \varphi(x) < 0, \tag{G.8}$$

for which the Hamiltonian (G.1) is not stable for any $\mu \in \mathbb{R}^1$ *[185]. The conditions of Proposition G.5 and (G.8) have to be compared with hypotheses (A) and (B) in Section 2.2 (see also Section 4.2).*

For bosons, the condition of stability (G.3) (or (G.4)) implies an upper bound of the grand-partition function

$$\Xi_\Lambda(\beta, \mu) = Tr_{\mathcal{F}_B} e^{-\beta(H_\Lambda - \mu N_\Lambda)} \leq \sum_{n=0}^{\infty} e^{\beta(B+\mu)n} Tr_{\mathcal{H}_B^{(n)}} \left(e^{-\beta T_\Lambda^{(n)}} \right). \tag{G.9}$$

Since

$$\Xi_\Lambda^P(\beta, \widetilde{\mu}) = Tr_{\mathcal{F}_B} e^{-\beta(T_\Lambda - \widetilde{\mu} N_\Lambda)} = \sum_{n=0}^{\infty} e^{\beta \widetilde{\mu} n} Tr_{\mathcal{H}_B^{(n)}} \left(e^{-\beta T_\Lambda^{(n)}} \right)$$

is the grand-partition function of the PBG (2.20), one gets

$$\Xi_\Lambda(\beta, \mu) \leq \Xi_\Lambda^P(\beta, \widetilde{\mu} = \mu + B). \tag{G.10}$$

Therefore, since

$$\Xi_\Lambda^P(\beta, \widetilde{\mu}) = \prod_{k \in \Lambda^*} \left(1 - e^{-\beta(\varepsilon_k - \widetilde{\mu})}\right)^{-1} < +\infty,$$

only for $\widetilde{\mu} < 0$ (see Section 2.1), by (G.10), for an *interacting Bose-Gas* the immediate application of the stability condition (G.3) implies convergence of the corresponding grand-partition function $\Xi_\Lambda(\beta, \mu) < +\infty$ *only* for $\mu < -B$.

Notice that in contrast to that, in the case of the *Perfect Fermi-Gas,* in the Fermion Fock space $\mathcal{F}_F(\mathcal{H})$ the corresponding grand-partition function equals

$$\Xi^P_{\Lambda,\text{fermi}}(\beta,\widetilde{\mu}) \equiv Tr_{\mathcal{F}_F} e^{-\beta(T_\Lambda-\widetilde{\mu}N_\Lambda)} = \prod_{k\in\Lambda^*}\left(1+e^{-\beta(\varepsilon_k-\widetilde{\mu})}\right),$$

and exists for *any* $\widetilde{\mu}\in\mathbb{R}^1$. Therefore, by (G.10) extended to an *interacting Fermi-Gas*, the stability condition (G.3) implies the convergence of the corresponding grand-partition function:

$$\Xi_{\Lambda,\text{fermi}}(\beta,\mu) = Tr_{\mathcal{F}_F} e^{-\beta(T_\Lambda-\mu N_\Lambda)} < \Xi^P_{\Lambda,\text{fermi}}(\beta,\widetilde{\mu}=\mu+B) < +\infty,$$

for *any* $\mu\in\mathbb{R}^1$.

In the case of classical particles (Boltzmann statistics) the corresponding grand-partition function of the *Perfect Classical Gas* is also defined for *any* $\mu\in\mathbb{R}^1$:

$$\Xi^P_{\Lambda,\text{classical}}(\beta,\widetilde{\mu}) = \sum_{n=0}^{\infty}\frac{e^{\beta\widetilde{\mu}n}}{n!}(2\pi\theta m)^{3n/2} = \exp\left\{Ve^{\beta\widetilde{\mu}}(2\pi\theta m)^{3/2}\right\} < +\infty.$$

Consequently, by (G.10) extended to an *interacting Classical Gas*,

$$\Xi_{\Lambda,\text{classical}}(\beta,\mu) = \sum_{n=0}^{\infty}\frac{e^{\beta\mu n}}{n!}Tr_{\mathcal{H}_B^{(n)}}\left(e^{-\beta H_\Lambda^{(n)}}\right) < \Xi^P_{\Lambda,\text{classical}}(\beta,\widetilde{\mu}=\mu+B)$$

exists also for *any* $\mu\in\mathbb{R}^1$.

For *bosons*, a sufficient condition to obtain the convergence of the grand-partition function (G.2) for all $\mu\in\mathbb{R}^1$ is the condition of *superstability* [91]:

Definition G.7 *The interaction term U_Λ (or the corresponding Hamiltonian $H_\Lambda = T_\Lambda + U_\Lambda$) is superstable if there are two constants $B\geq 0$ and $A>0$ such that*

$$U_\Lambda \geq -BN_\Lambda + \frac{A}{V}N_\Lambda^2, \tag{G.11}$$

i.e. in the canonical ensemble the corresponding n-particles interaction $U_\Lambda^{(n)}$ acting on $\mathcal{H}_B^{(n)}$ satisfies

$$U_\Lambda^{(n)} \geq -Bn + \frac{A}{V}n^2, \tag{G.12}$$

for all $n\geq 0$.

For a Hamiltonian H_Λ verifying (G.11) the estimate of the grand-partition function from above is

$$\Xi_\Lambda(\beta,\mu) \equiv Tr_{\mathcal{F}_B}\left(e^{-\beta(H_\Lambda-\mu N_\Lambda)}\right) \leq \sum_{n=0}^{\infty} e^{-\beta\left[\frac{A}{V}n^2-(B+\mu)n\right]}Tr_{\mathcal{H}_B^{(n)}}\left(e^{-\beta T_\Lambda^{(n)}}\right). \tag{G.13}$$

Notice that for any $n\geq 0$

$$Z_\Lambda^P(\beta,n) = Tr_{\mathcal{H}_B^{(n)}} e^{-\beta T_\Lambda^{(n)}} < +\infty$$

is the *canonical* partition function of the PBG related to the *free-energy* density:

$$f_\Lambda\left[T_\Lambda^{(n)}\right] = -\frac{1}{\beta V}\ln Z_\Lambda^P(\beta, n),$$

see (2.17) and (2.18). Then, since, for any $n \geq 0$, this density is bounded from below: $f_\Lambda\left[T_\Lambda^{(n)}\right] > -a$, the inequality (G.13) implies the convergence of the grand-partition function $\Xi_\Lambda(\beta,\mu)$ for any $\mu \in \mathbb{R}^1$:

$$\Xi_\Lambda(\beta,\mu) \leq \sum_{n=0}^{\infty} e^{-\beta\left(\frac{A}{V}n^2-(B+\mu)n+Vf_\Lambda\left[T_\Lambda^{(n)}\right]\right)} \leq e^{\beta V a}\sum_{n=0}^{\infty} e^{-\beta\left(\frac{A}{V}n^2-(B+\mu)n\right)} < +\infty,$$

and the upper bound for the pressure:

$$\lim_\Lambda p_\Lambda[H_\Lambda] \leq a + \lim_\Lambda \frac{1}{\beta V}\ln\sum_{n=0}^{\infty} e^{-\beta V\left(A\left(\frac{n}{V}\right)^2-(B+\mu)\frac{n}{V}\right)} = a + \sup_{x>0}\left\{(B+\mu)x - Ax^2\right\}.$$

Remark G.8 *The superstability (G.11), (G.12) is ensured by the following properties of the pair potential [91, 185–187]: let $\varphi_1(x)$ create a stable interaction and $\varphi_2(x)$ be non-negative continuous function with $\varphi_2(0) > 0$. Then the interaction U_Λ corresponding to $\varphi(x) = \varphi_1(x) + \varphi_2(x)$ is superstable.*

Return to our hypotheses on the interaction potential:
(A) a real $\varphi(x) \in L^1(\mathbb{R}^3)$;
(B) its (real) Fourier transform $v(q)$ (2.42) satisfies: $v(0) > 0$ and $0 \leq v(q) \leq v(0)$ for $q \in \mathbb{R}^3$.
Since $v(q)$ is continuous and $v(0) > 0$, we have a representation: $v(q) = v_1(q) + v_2(q)$, where the both terms are chosen to be continuous, non-negative and such that

$$\varphi_2(0) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} dq v_2(q) > 0.$$

Then by Proposition G.5 and Remark G.8 the interaction U_Λ corresponding to $v(q)$ is *superstable*. Therefore the full Hamiltonian (G.1) is *superstable,* i.e. the grand-partition function (G.2) converges for all $\mu \in \mathbb{R}^1$.

Remark G.9 *The integrability of $\varphi(x) \in L^1(\mathbb{R}^3)$ (A) implies the existence of its Fourier transform $v(q)$ and allows to consider the full Hamiltonian (G.1) in the orthonormal basis $\Psi_{\{n_k\}}$ (2.12) of proper vectors for the kinetic-energy part T_Λ (2.13), i.e. (G.1) can be written as*

$$H_\Lambda = T_\Lambda + U_\Lambda \equiv \sum_{k\in\Lambda^*}\varepsilon_k a_k^* a_k + \frac{1}{2V}\sum_{k_1,k_2,q\in\Lambda^*} v(q) a_{k_1-q}^* a_{k_2+q}^* a_{k_2} a_{k_1},$$

see (2.44).

Remark G.10 *Of course if $\varphi(x) = 0$, i.e. if we consider the PBG, the stability condition (G.3) is verified for $B = 0$, whereas the superstability condition (G.11) is obviously failed. Hence the grand-partition function converges only for $\mu < 0$, see (G.10).*

Bibliography

[1] N.N. Bogoliubov, On the theory of superfluidity, J. Phys. (USSR) 11 (1947) 23–32.

[2] N.N. Bogoliubov, About the theory of superfluidity, Izv. Akad. Nauk USSR 11 (1947) 77–90.

[3] N.N. Bogoliubov, Energy levels of the imperfect Bose-Einstein gas, Bull. Moscow State Univ. 7 (1947) 43–56.

[4] L.D. Landau, The theory of superfluidity of Helium II, J. Phys. (USSR) 5 (1941) 71.

[5] L.D. Landau, On the theory of superfluidity of Helium II, J. Phys. (USSR) 11 (1947) 91.

[6] A. Einstein, Quantentheorie des einatomigen idealen Gases, Sitzungsberichte der Preussischen Akademie der Wissenschaften I (1925) 3–14.

[7] R.M. Ziff, G.E. Uhlenbeck and M. Kac, The Ideal Bose-Einstein Gas, Revisited, Phys. Rep. 32(C) (1977) 169–248.

[8] M. van den Berg, J.T. Lewis and J.V. Pulè, A general theory of Bose-Einstein condensation, Helv. Phys. Acta 59 (1986) 1271–1288.

[9] N. Angelescu, A. Verbeure and V.A. Zagrebnov, On Bogoliubov's model of superfluidity, J. Phys. A: Math.Gen. 25 (1992) 3473–3491.

[10] N. Angelescu and A. Verbeure, Variational solution of a superfluidity model, Physica A 216 (1995) 386–396.

[11] N. Angelescu, A. Verbeure and V.A. Zagrebnov, Superfluidity III, J. Phys. A: Math.Gen. 30 (1997) 4895–4913.

[12] J.-B. Bru and V.A. Zagrebnov, Exact phase diagram of the Bogoliubov Weakly Imperfect Bose gas, Phys. Lett. A 244 (1998) 371–376.

[13] J.-B. Bru and V.A. Zagrebnov, Quantum interpretation of thermodynamic behaviour of the Bogoliubov Weakly Imperfect Bose gas, Phys. Lett. A 247 (1998) 37–41.

[14] J.-B. Bru and V.A. Zagrebnov, Thermodynamic Behavior of the Bogoliubov Weakly Imperfect Bose Gas, in: Mathematical Results in Statistical Mechanics, eds S. Miracle-Sole and al. (World Scientific, Singapore, 1999) p. 313-321.

[15] J.-B. Bru and V.A. Zagrebnov, Exact solution of the Bogoliubov Hamiltonian for Weakly Imperfect Bose gas, J. Phys. A: Math.Gen. 31 (1998) 9377–9404.

[16] J.-B. Bru and V.A. Zagrebnov, On condensations in the Bogoliubov Weakly Imperfect Bose-Gas, J. Stat. Phys. 99 (2000) 1297-1338.

[17] N.N. Bogoliubov, Energy levels of the imperfect Bose-Einstein gas, in: Collection of papers, Vol. 2 (Naukova Dumka, Kiev, 1970) p. 242-257.

[18] N.N. Bogoliubov, Lectures on Quantum Statistics, Vol. 1 : Quantum Statistics (Gordon and Breach Science Publishers, New York-London-Paris, 1970).

[19] N.N. Bogoliubov, Lectures on Quantum Statistics, in: Collection of papers Vol. 2 (Naukova Dumka, Kiev, 1970) p. 217-493.

[20] I.M. Khalatnikov, An Introduction to the Theory of Superfluidity (Benjamin, New York, 1965).

[21] N.N. Bogoliubov, Kinetic equations in the theory of superfluidity, JETP 18 (1948) 622–630.

[22] F. London, The λ-Phenomenon of Liquid Helium and the Bose-Einstein Degeneracy, Nature 141 (1938) 643.

[23] P. Kapitza, Nature 141 (1938) 74.

[24] J.F. Allen and A.D. Misener, Nature 141 (1938) 75.

[25] F. London, Superfluids, Vol 2: Macroscopic Theory of Superfluid Helium, (John Wiley

[26] N.N. Bogoliubov, Lectures on Quantum Statistics: Quasi-Averages, Vol. 2 (Gordon and Breach-Science Publishers, New York-London-Paris, 1970).

[27] G.E. Uhlenbeck, Thesis, Leyden, 1927.

[28] B. Kahn and G.E. Uhlenbeck, On the theory of condensation, Physica 5 (1938) 399–415.

[29] F. London, On the Bose-Einstein condensation, Phys. Rev. 54 (1938) 947–954.

[30] K. Huang, Statistical Mechanics (Wiley, New York, 1963).

[31] I.A. Kvasnikov, Thermodynamics and Statistical Physics: Theory of Equilibrium Systems (Izdatelstvo MGU, Moscow, 1991).

[32] T.C. Dorlas, Statistical Mechanics, Fundamentals and Model Solutions (Inst. Phys. Publishing, Bristol, 1999).

[33] D.Ya, Petrina. Mathematical Foundations of Quantum Statistical Mechanics (Kluwer Academic Publishers, 1995).

[34] H. Araki and E.J. Woods, Representations of the Canonical Commutation Relations Describing a Nonrelativistic Infinite Free Bose Gas, J. Math. Phys. 4 (1963) 637–662.

[35] J.T. Lewis and J.V. Pulè, The Equilibrium States of the Free Boson Gas, Commun. Math. Phys. 36 (1974) 1–18.

[36] J.T. Cannon, Infinite Volume Limits of the Canonical Free Bose Gas States on the Weyl Algebra, Commun. Math. Phys. 29 (1973) 89–104.

[37] O. Brattelli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, Vol. 2. , 2nd ed : Equilibrium States, models in Quantum Statistical Mechanics (Springer, New York-London-Paris, 1996).

[38] M. van den Berg, J.T. Lewis and M. Lunn, On the general theory of Bose-Einstein condensation and the state of the free boson gas, Helv. Phys. Acta 59 (1986) 1289–1310.

[39] M. Girardeau, Relationship between systems of impenetrable bosons and fermions in one dimension, J. Math. Phys. 1 (1960) 516–523.

[40] T. Schultz, Note on the one-dimensional gas of impenetrable point particle bosons, J. Math. Phys. 4 (1963) 666–671.

[41] A. Lenard, Momentum Distribution in the Ground State of the One-Dimensional System of Impenetrable Bosons, J. Math. Phys. 5 (1964) 930-943.

[42] M. van den Berg and J.T. Lewis, On generalized condensation in the free boson gas, Physica A 110 (1982) 550–564.

[43] H.B.G. Casimir, On Bose-Einstein condensation, in: Fundamental Problems in Statistical Mechanics III, ed. E.G.D. Cohen (North-Holland Publishing Company, Amsterdam, 1968) p. 188-196.

[44] M. van den Berg, On boson condensation into an infinite number of low-lying levels, J. Math. Phys. 23 (1982) 1159–1161.

[45] M. van den Berg, On condensation in the free-bosons gas and the spectrum of the Laplacian, J. Stat. Phys. 31 (1983) 623–637.

[46] M. van den Berg and J.T. Lewis, On the free boson gas in a weak external potential, Commun. Math. Phys. 81 (1981) 475–494.

[47] J.V. Pulè, The free boson gas in a weak external potential, J. Math. Phys. 24 (1983) 138–142.

[48] A.S. Parkins and D.F. Walls, The Physics of Trapped Dilute-Gas Bose-Einstein Condensates, Phys. Rep. 303 (1998) 1–80.

[49] Bose-Einstein condensation, ed. A. Griffin, D.W. Snoke and S. Stringari (Univ. Press, Cambridge, 1996).

[50] F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Mod. Phys. 71 (1999) 463–512.

[51] E.H. Lieb, The stability of matter, Rev. Mod. Phys. 48 (1976) 553–569.

[52] D.J. Thouless, The quantum mechanics of many body systems (Academic Press, New York, 1961).

[53] K. Huang, C.N. Yang and J.M. Luttinger, Imperfect Bose gas with hard-sphere interactions, Phys. Rev. 105 (1957) 776–784.

[54] M. van den Berg, J.T. Lewis and J.V. Pulè, The large deviation principle and some models of an interacting boson gas, Commun. Math. Phys. 118 (1988) 61–85.

[55] M. van den Berg, T.C. Dorlas, J.T. Lewis and J.V. Pulè, A perturbed meanfield model of an interacting boson gas and the large deviation principle, Commun. Math. Phys. 127 (1990) 41–69.

[56] M. van den Berg, T.C. Dorlas, J.T. Lewis and J.V. Pulè, The pressure in the Huang-Yang-Luttinger model of an interacting boson gas, Commun. Math. Phys. 128 (1990) 231–245.

[57] T.C. Dorlas, J.T. Lewis and J.V. Pulè, The Full Diagonal Model of a Bose Gas, Commun. Math. Phys. 156 (1993) 37–65.

[58] J.-B. Bru and V.A. Zagrebnov, Exactly soluble model with two kinds of Bose-Einstein condensations, Physica A 268 (1999) 309–325.

[59] J.-B. Bru and V.A. Zagrebnov, A model with coexistence of two kinds of Bose condensations, J. Phys. A: Math.Gen. 33 (2000) 449–464.

[60] O. Penrose, On the Quantum Mechanics of Helium II, Phil. Mag. 42 (1951) 1373.

[61] O. Penrose and L. Onsager, Bose-Einstein condensation and Liquid Helium, Phys. Rev. 104 (1956) 576–584.

[62] P.C. Hohenberg and P.M. Platzman, High-Energy Neutron scattering from liquid ^{4}He, Phys. Rev. 152 (1966) 198–200.

[63] R.A. Cowley and A.D.B. Woods, Neutron Scattering from Liquid Helium at High Energies, Phys. Rev. Lett. 21 (1968) 787–789.

[64] L. Aleksandrov, V.A. Zagrebnov, Zh.A. Kozlov, V.A. Parfenov and V.B. Priezzhev, High energy neutron scattering and the Bose condensate in He II, Sov. Phys.-JETP 41 (1975) 915–918.

[65] E.V. Dokukin, Zh.K. Kozlov, V.A. Parfenov and A.V. Puchkev, Investigation of the temperature dependence of the density of the Bose condensate in helium-4 in connection with the superfluidity phenomenon, Sov. Phys.-JETP 48 (1978) 1146–1149.

[66] D.G. Henshaw and A.D.B. Woods, Modes of Atomic Motions in Liquid Helium by Inelastic Scattering of Neutrons, Phys. Rev. 121 (1961) 1266–1274.

[67] N.M. Blagoveshchenskii, I.V. Bogoyavlenskii, L.V. Karnatsevich, V.G. Kolobrodov, Zh.A. Kozlov, V.B. Priezzhev, A.V. Puchkov, A.N. Skomorokhov and V.S. Yarunin, Structure of the excitation spectrum of liquid ^{4}He, Phys. Rev. B 50 (1994) 16550–16565.

[68] A. Griffin, Excitations in a Bose-Condensated Liquid (Cambridge Univ. Press, Cambridge, 1993).

[69] S.T. Beliaev, Application of the methods of quantum field theory to a system of bosons, Sov. Phys.-JETP 7 (1958) 289–299.

[70] S.T. Beliaev, Energy spectrum of a non-ideal Bose gas, Sov. Phys.-JETP 7 (1958) 299–307.

[71] N.M. Hugenholtz and D. Pines, Ground State Energy and Excitation spectrum of system of Interacting Bosons, Phys. Rev. 116 (1959) 489–506.

[72] N.M. Hugenholtz, Ground State of a System of Interacting Bosons, Physica 26 (1960) 170-173.

[73] J. Gavoret, Thèse, Univ. de Paris, juin 1963.

J. Gavoret, Application de la théorie des perturbations à l'étude d'un liquide de Bose au zéro absolu, Ann. de Phys. (Paris) 8 (1963) 441-491.

[74] J. Gavoret and Ph. Nozières, Structure of the Perturbation Expansion for the Bose Liquid of Zero temperature, Ann. of Phys. (N. Y.). 28 (1964) 349–399.

[75] E.Talbot and A.Griffin, High- and Low-Frequency Behaviour of Response Functions in a Bose Liquid: One-Loop Aproximation, Ann. of Phys. (N. Y.) 151 (1983) 71-98.

[76] N.N. Bogoliubov, Quasi-averages in Problems of Statistical Mechanics, in: Collection of papers, Vol. 3 (Naukova Dumka, Kiev, 1971) p. 174-243.

[77] D.N. Zubarev and Yu.A. Tserkovnikov, On the theory of phase transition in non-ideal Bose-gas, Dokl. Akad. Nauk USSR. 120 (1958) 991–994.

[78] M. Girardeau and R. Arnowitt, Theory of Many-Boson Systems. Pair Theory, Phys. Rev. 113 (1959) 755–761.

[79] G. Wentzel, Thermodynamically Equivalent Hamiltonian for Some Many-Body Problems, Phys. Rev. 120 (1960) 1572–1575.

[80] M. Luban, Statistical Mechanics of a Nonideal Boson Gas: Pair Hamiltonian Model, Phys. Rev. 128 (1962) 965–987.

[81] N.M. Hugenholtz, Quantum Theory of Many-Body Systems, Reports on Progress in Physics 28 (1965) 201–247.

[82] H. Wagner, Long-Wavelength Excitations and the Goldstone Theorem in Many-Particle System with "Broken Symmetries", Z. für Physik 195 (1966) 273–299.

[83] J.V. Pulè and V.A. Zagrebnov, A pair Hamiltonian model of a non-ideal Boson gas, Ann. Inst. Henri Poincaré 59 (1993) 421–444.

[84] V.V. Tolmachev, Theory of Bose-Gas (Izd. MGU, Moscow, 1969).

[85] T.D. Lee, K. Huang and C.N. Yang, Eigenvalues and Eigenfunctions of a Bose System of Hard Spheres and its Low-Temperature Properties, Phys. Rev. 106 (1957) 1135–1145.

[86] K. Huang, Imperfect Bose Gas, in: Studies in Statistical Mechanics, vol II, ed. J.De Boer and G.E. Uhlenbeck (North-Holland Publ. Co., Amsterdam. 1964) p. 3-110.

[87] V.N. Popov, Functional Integrals in Quantum Field Theory and Statistical Physics (Dordrecht, Reidel, 1983).

[88] V.N. Popov, Functional Integrals and Collective Excitations (Univ. Press, Cambridge, 1987).

[89] H. Shi and A. Griffin, Finite-temperature excitations in a dilute Bose-condensated gas, Phys. Rep. 304 (1998) 1–87.

[90] E.H. Lieb, The Bose fluid, in: Lectures in Theoretical Physics, Vol. VII C. ed. W.E. Briffin (University of Colorado, Boulder, 1965).

[91] D. Ruelle, Statistical Mechanics: Rigorous Results (Benjamin, New York, 1969).

[92] J. Ginibre, On the Asymptotic Exactness of the Bogoliubov Approximation for many Bosons Systems, Commun. Math. Phys. 8 (1968) 26–51.

[93] E. Buffet, Ph. de Smedt and J.V. Pulè, The condensate equation for some Bose systems, J. Phys. A: Math.Gen. 16 (1983) 4307–4324.

[94] K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn and W. Ketterle, Bose-Einstein Condensation in a Gas of Sodium Atoms, Phys. Rev. Lett. 75(22) (1995) 3969–3973.

[95] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman and E.A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, Science 269 (1995) 198–201.

[96] C.C. Bradley, C.A. Sackett, J.J. Tollet and R.G. Hulet, Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions, Phys. Rev. Lett. 75 (1995) 1687–1690.

[97] E.P. Gross, Structure of a Quantized Vortex in Boson Systems, Nuovo Cimento. 20 (1961) 454–466.

[98] L.P. Pitaevskii, Vortex lines in an imperfect Bose gas, Sov. Phys.-JETP 13 (1961) 451–454.

[99] E.P. Gross, Hydrodynamics of a superfluid condensate, J. Math. Phys. 4 (1963) 195–207.

[100] T. Michoel and A. Verbeure, Goldstone Boson Normal Coordinates in Interacting Bose Gases, J. Stat. Phys. 96 (1999) 1125–1161.

[101] E.H. Lieb, R. Seiringer and J. Yngvason, Bosons in a Trap: A Rigorous Derivation of the Gross-Pitaevskii Energy Functional, Phys.Rev.A 61 (2000) 043602-1–043602-13.

[102] E.H. Lieb, R. Seiringer and J. Yngvason, The Ground State Energy and Density of Interacting Bosons in a Trap in : Quantum Theory and Symmetries, Goslar, 1999, eds H.-D.Doebner et al (World Scientific, 2000) 101–110.

[103] E.H. Lieb, R. Seiringer and J. Yngvason, A Rigorous Derivation of the Gross-Pitaevskii Energy Functional for Two-Dimensional Bose Gas, to appear in Commun.Math.Phys. (2000).

[104] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. I: Functional Analysis (Academic Press, New York-London, 1972).

[105] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. II: Fourier analysis, Self-adjointness (Academic Press, New York-London, 1972).

[106] J. Glimm and A. Jaffe, Quantum Physics. A functional integral point of view (Springer, Berlin, 1981).

[107] W. Thirring, A course in Mathematical Physics, Vol. 4: Quantum Mechanics of Large systems (Springer, Wien, 1983).

[108] A.A. Abrikosov, L.P. Gorkov and I.E. Dzyaloshinskii, Methods of Quantum Field Theory in Statistical Physics (Prentice-Hall, Englewood Cliffs N.J., 1963).

[109] A.L. Fetter and J.D. Walecka, Quantum Theory of Many Particle Systems (McGraw-Hill, New York, 1971).

[110] H. Ezawa, Quantum Mechanics of a Many-Boson System and the Representation of Canonical Variables, J. Math. Phys. 6 (1965) 380–404.

[111] H. Ezawa and M. Luban, Criterion for Bose-Einstein condensation and Representation of Canonical Commutation Relations, J. Math. Phys. 8 (1967) 1285.

[112] M. Girardeau, Variational Method for the Quantum Statistics of Interacting Particles, J. Math. Phys. 3 (1962) 131–139.

[113] A.J. Kromminga and M. Bolsterli, Perturbation Theory of Many-Boson Systems, Phys. Rev. 128 (1962) 2887–2897.

[114] D. Pines, The Many-body Problems (W.A. Benjamin Inc., New York, 1962).

[115] D. Pines and Ph. Nozières, The Theory of Quantum Liquids Vol. 2: Superfluid Bose Liquids (Addison-Wesley Publishing Company Inc., Redwood City, 1989).

[116] Yu.A. Tserkovnikov, Theory of the imperfect Bose-Gas for non-zero temperature, Doklady Acad. Nauk USSR 143 (1962) 832–835.

[117] Vl.V. Papoyan and V.A. Zagrebnov, On generalized Bose-Einstein condensation in the almost-ideal boson gas, Helv. Phys. Acta 63 (1990) 557–564.

[118] R.P. Feynman, Application of Quantum Mechanics to Liquid Helium, in: Progress in Low Temperature Physics, Vol. 1, Ch. 2. ed. J. Gorter (North-Holland, Amsterdam, 1955) p. 199.

[119] R.P. Feynman and M. Cohen, Energy spectrum of the Excitations in Liquid Helium, Phys. Rev. 102 (1956) 1189.

[120] J. Wilks, An introduction to liquid helium (Claredon, Oxford, 1970).

[121] R.H. Critchley and A.I. Solomon, A Variational Approach to Superfluidity, J. Stat. Phys. 14 (1976) 381–393.

[122] N.N. Bogoliubov, D.N. Zubarev and Yu.A. Tserkovnikov, Asymptotically exact solution for the model Hamiltonian of superconductivity, Sov. Phys.-JETP 39 (1960) 120–129.

[123] N.N. Bogoliubov, On the model Hamiltonian of superconductivity, in: Collection of papers, Vol. 3 (Naukova Dumka, Kiev, 1971) p. 110-173.

[124] N.N. Bogoliubov, On some problems of the theory of superconductivity, Physica 26 (1960) S1–S16.

[125] N.N. Bogoliubov (Jr), A method of studying model Hamiltonians (Pergamon, Oxford, 1977).

[126] N.N. Bogoliubov (Jr), J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, Some classes of exactly soluble models of problems in Quantum Statistical Mechanics : the method of the approximating Hamiltonian, Russian Math. Surveys 39 (1984) 1–50.

[127] R. Haag, The mathematical structure of the Bardeen-Cooper-Schriffer model, Nuovo Cimento 25 (1962) 287–299.

[128] N.N. Bogoliubov, V.V. Tolmachev and D.V. Shirkov, A New Method in the Theory of Superconductivity (Consultants Bureau Inc., New York, 1959).

[129] D. W. Robinson, The ground state of the Bose gas, Commun. Math. Phys. 1 (1965) 159–174.

[130] E.H. Lieb and J. Yngvason, The Ground State Energy of a Dilute Bose Gas, in : Differential Equations and Mathematical Physics , University of Alabama, Birmingham 1999, eds R.Weikard and G.Weinstein (Internat Press , 2000) 271-282.

[131] E.H. Lieb and W. Liniger, Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State , Phys. Rev. 130 (1963) 1605–1616.

[132] E.H. Lieb , Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum , Phys. Rev. 130 (1963) 1616–1624.

[133] E.H. Lieb, Simplified Approach to the Ground State Energy of an Imperfect Bose Gas , Phys. Rev. 130 (1963) 2518–2528.

[134] E.H. Lieb and A.Sakakura, Simplified Approach to the Ground State Energy of an Imperfect Bose Gas. II. The Charged Bose Gas at High Density, Phys. Rev. 133 (1964) A899–A906.

[135] E.H. Lieb and W. Liniger, Simplified Approach to the Ground State Energy of an Imperfect Bose Gas. III. Applications to the One-Dimensional Model , Phys. Rev. 134 (1964) A312–A315.

[136] E.H. Lieb and J. Yngvason, Ground State Energy of the Low Density Bose Gas, Phys. Rev. Lett. 80 (1998) 2504-2507.

[137] E.H. Lieb and J. Yngvason, The Ground State Energy of a Dilute Two-Dimensional Bose Gas, to be published in J.Stat.Phys. (2000).

[138] E.H. Lieb and J.P.Solovej, Ground State Energy of the One-Component Charged Bose Gas, Preprint, mp_arc 00-303.

[139] F.J.Dyson, Ground-State Energy of a Hard-Sphere Gas, Phys. Rev. 106 (1957) 20–24.

[140] L.L.Foldy, Charged Bose Gas, Phys. Rev. 124 (1961) 649–651; Errata ibid. 125 (1962) 2208.

[141] T. Michoel and A. Verbeure, Nonextensive Bose-Einstein condensation model, J. Math. Phys. 40 (1999) 1268–1279.

[142] T.C. Dorlas, J.T. Lewis and J.V. Pulè, The Yang-Yang thermodynamic formalism and large deviations, Commun. Math. Phys. 124 (1989) 365–402.

[143] C. Kittel, Quantum Theory of Solids (John Wiley and Sons Inc., New York, 1963).

[144] C.N. Yang and C.P. Yang, Thermodynamics of one dimensional system of bosons with repulsive delta-function interaction, J. Math. Phys. 10 (1969) 1115–1122.

[145] J. Lebowitz and O. Penrose, Rigorous Treatment of the Van der Waals-Maxwell Theory of the Liquid-Vapor Transition, J. Math. Phys. 7 (1966) 98–113.

[146] E.H. Lieb, Quantum-Mechanical Extension of the Lebowitz-Penrose Theorem on the Van der Waals Theory, J. Math. Phys. 7 (1966) 1016–1024.

[147] Ph. de Smedt and V.A. Zagrebnov, Van der Waals Limit of an Interacting Bose Gas in a Weak External Field, Phys. Rev. A 35 (1987) 4763–4769.

[148] E.B. Davies, The thermodynamic limit for an imperfect boson gas, Commun. Math. Phys. 28 (1972) 69–86.

[149] M. Fannes and A. Verbeure, The condensed phase of the imperfect Bose gas, J. Math. Phys. 21 (1980) 1809–1818.

[150] M. van den Berg, J.T. Lewis and Ph. de Smedt, Condensation in the Imperfect Boson Gas, J. Stat. Phys. 37 (1984) 697–707.

[151] E. Buffet and J.V. Pulè, Fluctuations Properties of the Imperfect Boson Gas, J. Math. Phys. 24 (1983) 1608–1616.

[152] J.T. Lewis, J.V. Pulè and V.A. Zagrebnov, The Large Deviation Principle for the Kac Distribution, Helv. Phys. Acta 61 (1988) 1063–1078.

[153] Vl.V. Papoyan and V.A. Zagrebnov, The ensemble equivalence problem for Bose systems (non-ideal Bose gas), Theor. Math. Phys. 69 (1986) 1240–1253.

[154] N.N. Bogoliubov and D.N. Zubarev, Wave function of the ground-state of interacting Bose-particles, JETP 28 (1955) 129–139.

[155] D.N. Zubarev, Distribution function of non-ideal Bose-gas for zero temperature, JETP 29 (1955) 881–882.

[156] I.A. Vakarchuk, Density matrices of a many-boson system at low temperatures, Theor. Math. Phys. 23 (1975) 496–504.

[157] T.C. Dorlas, J.T. Lewis and J.V. Pulè, Condensation in some perturbed meanfield models of a Bose gas, Helv. Phys. Acta 64 (1991) 1200–1224.

[158] R.T. Rockafellar, Convex Analysis (Princeton University Press, Princeton-New Jersey, 1972).

[159] R.A. Minlos and A.Ja. Povzner, Thermodynamic limit for entropy, Trans. Moscow Math. Soc. 17 (1967) 269–300.

[160] J.T. Lewis, Mark Kac seminar on probability and physics: The Large Deviation Principle in Statistical Mechanics, syllabus 17 (Centrum voor Wiskunde en Informatica CWI (1985-1987), Amsterdam, 1988).

[161] J.T. Lewis and C.E. Pfister, Thermodynamic probability theory: some aspects of large deviations, Russian Math. Surveys 50 (1995) 279–317.

[162] B. Harris, Bounds for certain thermodynamic averages, J. Math. Phys. 8 (1967) 1044–1045.

[163] N.N. Bogoliubov (Jr), J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, The Approximating Hamiltonian Method in Statistical Physics (Publ. Bulgarian Akad. Sciences, Sofia, 1981).

[164] P.B. Weichman, Crossover scaling in a dilute Bose superfluid near zero temperature, Phys. Rev. B 38 (1988) 8739–8759.

[165] N. Angelescu, J.G. Brankov and A. Verbeure, General one-particle fluctuations of the ideal Bose gas, J. Phys. A: Math.Gen. 29 (1996) 3341–3356.

[166] M. Broidioi and A. Verbeure, Scaling Behaviour in the Bose Gas, Commun. Math. Phys. 174 (1996) 635–660.

[167] J.L. Lebowitz, M. Lenci and H. Spohn, Large Deviations for Ideal Quantum Systems, Preprint (1999).

[168] P.C. Hohenberg, Thesis, Harvard University, 1962.

P.C. Hohenberg and P.C.Martin, Microscopic Theory of Superfluid Helium, Ann. of Phys. (N. Y.) 34 (1965) 291-359.

[169] J.S. Bell, Fluctuation Compressibility Theorem and Its Application to the Pairing Model, Phys. Rev. 129 (1963) 1896–1900.

[170] M. Fannes, J.V. Pulè and A. Verbeure, Goldstone Theorem for Bose systems, Lett. Math. Phys. 6 (1982) 385–389.

[171] B. Simon, The Statistical Mechanics of Lattice Gases, Vol. I (Princeton Univ. Press, Princeton, 1993).

[172] R. Griffiths, A Proof that the Free Energy of a Spin System is extensive, J. Math. Phys. 5 (1964) 1215–1222.

[173] K. Hepp and E.H. Lieb, Equilibrium Statistical Mechanics of Matter Interacting with the Quantized Radiation Field, Phys. Rev. A 8 (1973) 2517–2525.

[174] V.A. Zagrebnov, Perturbations of Gibbs Semigroups, Commun. Math. Phys. 120 (1989) 653–664.

[175] H.D. Maison, Analyticity of the partition function for finite quantum systems, Commun. Math. Phys. 22 (1971) 166–172.

[176] M. Fannes and A. Verbeure, Correlation Inequalities and Equilibrium States II, Commun. Math. Phys. 57 (1977) 165–171.

[177] N.N. Bogoliubov, Physik Abhandl Sowjet Union, 6 (1962) 113–229.

[178] G. Roepstorff, Correlation inequalities in quantum statistical mechanics and their application in the Kondo problem, Commun. Math. Phys. 46 (1976) 253–262.

[179] H. Araki, On KMS states of a C^*-dynamical system, in: Proc. of the Second Japan-U.S. Seminar on C^*-algebra and Application to Physics, Los Angeles 1977, eds. H. Araki and R. Kadison (Springer-Verlag, Berlin-Heidelberg-New York, 1978).

[180] L.G. Sewell, KMS conditions and local thermodynamic stability of quantum lattice systems II, Commun. Math. Phys. 55 (1977) 53–61.

[181] M. Fannes and A. Verbeure, Correlation Inequalities and Equilibrium States I, Commun. Math. Phys. 55 (1977) 125–131.

[182] J. Manuceau and A. Verbeure, Quasi-Free States of the C.C.R. - Algebra and Bogoliubov Transformations, Commun. Math. Phys. 9 (1968) 293–302.

[183] M. Fannes, P.N.M. Sisson, A. Verbeure and J.C. Wolf, Equilibrium State and Free Energy of an Infinite Mode Dicke Maser Model, Ann. Phys. 98 (1976) 38–49.

[184] M. Fannes, The entropy density of quasi-free states for a continuous boson system, Ann. Inst. Henri Poincaré. 28 (1978) 187–196.

[185] R.L. Dobrushin, Investigation of the Conditions of the Asymptotic Existence of the Configurational Integral of the Gibbs Distribution, Teorija Verojatn. Prim. 9 (1964) 626–643. (in russian)

[186] D. Ruelle, Statistical Mechanics of Quantum Systems of Particles, Helv. Phys. Acta 36 (1963) 789–799.

[187] M. Fisher and D. Ruelle, The Stability of Many-Particle Systems, J. Math. Phys. 7 (1966) 260–270.