Photon number localization of ground states in nonrelativistic QED

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Abstract

One electron system minimally coupled to a quantized radiation field is considered. It is assumed that the quantized radiation field is *massless*, and *no* infrared cutoff is imposed. The Hamiltonian, H, of this system is defined as a self-adjoint operator acting on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$, where $\mathcal{F}$ is the Boson Fock space over $L^2(\mathbb{R}^3 \times \{1,2\})$. It is shown that the ground state, ψ_g, of H belongs to $\cap_{k=1}^\infty D(1 \otimes N^k)$, where N denotes the photon number operator of $\mathcal{F}$. Moreover it is shown that, for almost every electron position variable $x \in \mathbb{R}^3$ and for arbitrary $k \geq 0$, $\|(1 \otimes N^{k/2})\psi_g(x)\|_{\mathcal{F}} \leq D_k e^{-\delta|x|^{m+1}}$ with some constants m, D_k, and δ independent of k. In particular $\psi_g \in \cap_{k=1}^\infty D(e^{\beta|x|^{m+1}} \otimes N^k)$ for $0 < \beta < \delta/2$ is obtained.

1 Introduction

1.1 The Pauli-Fierz Hamiltonian

In this paper one spinless electron minimally coupled to a massless quantized radiation field is considered. It is the so-called Pauli-Fierz model of the nonrelativistic QED. The Hilbert space of state vectors of the system is given by

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F}$ denotes the Boson Fock space defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})\right],$$

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where $\otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\})$, $n \geq 1$, denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1,2\})$ and $\otimes_s^0 L^2(\mathbb{R}^3 \times \{1,2\}) = \mathbb{C}$. The Fock vacuum Ω is defined by $\Omega = \{1, 0, 0, ...\}$. Let

$$\mathcal{F}_0 = \{\oplus_{n=0}^\infty \Psi^{(n)} \in \mathcal{F} | \Psi^{(n)} = 0 \text{ for } n \geq m \text{ with some } m\}.$$

For each $\{k, j\} \in \mathbb{R}^3 \times \{1,2\}$, the annihilation operator $a(k,j)$ is defined by, for $\Psi = \oplus_{n=0}^\infty \Psi^{(n)} \in \mathcal{F}_0$,

$$\left(a(k,j)\Psi\right)^{(n)}(k_1, j_1..., k_n, j_n) = \sqrt{n+1}\Psi^{(n+1)}(k, j, k_1, j_1, ..., k_n, j_n).$$

The creation operator $a^*(k,j)$ is given by $a^*(k,j) = (a(k,j)\lceil_{\mathcal{F}_0})^*$. They satisfy the canonical commutation relations on $\mathcal{F}_0$

$$[a(k,j), a^*(k',j')] = \delta(k-k')\delta_{jj'},$$

$$[a(k,j), a(k',j')] = 0,$$
$$[a^*(k,j), a^*(k',j')] = 0.$$

The closed extensions of $a(k,j)$ and $a^*(k,j)$ are denoted by the same symbols respectively. The annihilation and creation operators smeared by $f \in L^2(\mathbb{R}^3)$ are formally written as

$$a^\sharp(f,j) = \int a^\sharp(k,j) f(k) dk, \quad a^\sharp = a \text{ or } a^*,$$

and act as

$$(a(f,j)\Psi)^{(n)} = \sqrt{n+1}\int f(k)\Psi^{(n+1)}(k, j, k_1, j_1..., k_n, j_n)dk,$$

$$(a^*(f,j)\Psi)^{(n)} = \frac{1}{\sqrt{n}}\sum_{j_l=j} f(k)\Psi^{(n-1)}(k_1, j_1, ..., \widehat{k_l, j_l}, ..., k_n, j_n),$$

where $\sum_{j_l=j}$ denotes to sum up j_l such that $j_l = j$, and $\widehat{X}$ means neglecting X. We work with the unit $\hbar = 1 = c$. The dispersion relation is given by

$$\omega(k) = |k|.$$

Then the free Hamiltonian H_f of $\mathcal{F}$ is formally written as

$$H_\mathrm{f} = \sum_{j=1,2}\int \omega(k) a^*(k,j) a(k,j) dk,$$

and acts as

$$(H_{\mathrm{f}}\Psi)^{(n)}(k_1, j_1, ..., k_n, j_n) = \sum_{j=1}^{n} \omega(k_j)\Psi^{(n)}(k_1, j_1, ..., k_n, j_n), \quad n \geq 1,$$

$$(H_{\mathrm{f}}\Psi)^{(0)} = 0$$

with the domain

$$D(H_{\mathrm{f}}) = \left\{ \Psi = \oplus_{n=0}^{\infty}\Psi^{(n)} \Big| \sum_{n=0}^{\infty} \|(H_{\mathrm{f}}\Psi)^{(n)}\|^2_{\otimes^n L^2(\mathbb{R}^3\times\{1,2\})} < \infty \right\}.$$

Since H_{f} is essentially self-adjoint and nonnegative, we denotes the self-adjoint extension of H_{f} by the same symbol H_{f}. Under the identification

$$\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx,$$

the quantized radiation field A with a form factor φ is given by the constant fiber direct integral

$$A = \int_{\mathbb{R}^3}^{\oplus} A(x) dx,$$

where $A(x)$ is the operator acting on $\mathcal{F}$ defined by

$$A(x) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int \frac{e(k,j)}{\sqrt{\omega(k)}} \left\{ a^*(k,j)e^{-ik\cdot x}\widehat{\varphi}(-k) + a(k,j)e^{ik\cdot x}\widehat{\varphi}(k) \right\} dk.$$

Here $\widehat{\varphi}$ denotes the Fourier transform of φ and $e(k,j)$, $j = 1, 2$, are polarization vectors such that $(e(k,1), e(k,2), k/|k|)$ forms a right-handed system, i.e., $k\cdot e(k,j) = 0$, $e(k,j)\cdot e(k,j') = \delta_{jj'}$, and $e(k,1) \times e(k,2) = k/|k|$ for almost every $k \in \mathbb{R}^3$. We fix polarization vectors through this paper.

The decoupled Hamiltonian is given by

$$H_0 = H_{\mathrm{p}} \otimes 1 + 1 \otimes H_{\mathrm{f}}.$$

Here

$$H_{\mathrm{p}} = \frac{1}{2}p^2 + V$$

denotes a particle Hamiltonian, where $p = (-i\nabla_{x_1}, -i\nabla_{x_2}, -i\nabla_{x_3})$ and $x = (x_1, x_2, x_3)$ are the momentum operator and its conjugate position operator in $L^2(\mathbb{R}^3)$, respectively, and $V : \mathbb{R}^3 \to \mathbb{R}$ an external potential. We are prepared to define the total Hamiltonian, H, of this system, which is give by the minimal coupling to H_0. I.e., we replace $p \otimes 1$ with $p \otimes 1 - eA$,

$$H = \frac{1}{2}(p \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_{\mathrm{f}},$$

where e denotes the charge of an electron.

1.2 Assumptions on V and fundamental facts

We give assumptions on external potentials. We say $V \in K_3$ (the three dimensional Kato class [22]) if and only if

$$\lim_{\epsilon \downarrow 0} \sup_{x \in \mathbb{R}^3} \int_{|x-y|<\epsilon} \frac{|V(y)|}{|x-y|} dy = 0,$$

and $V \in K_3^{\rm loc}$ if and only if $1_R V \in K_3$ for all $R \geq 0$, where $1_R(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| \geq R. \end{cases}$ Let us define classes K and $V_{\rm exp}$ as follows.

Definition 1.1 **(1)** *We say $V \in K$ if and only if $V = V_+ - V_-$ such that $V_\pm \geq 0$, $V_+ \in K_3^{\rm loc}$ and $V_- \in K_3$.*

(2) *We say $V \in V_{\rm exp}$ if and only if $V = Z + W$ such that $\inf Z > -\infty$, $Z \in L^1_{\rm loc}(\mathbb{R}^3)$, $W < 0$, and $W \in L^p(\mathbb{R}^3)$ for some $p > 3/2$.*

Definition 1.2 *Suppose that $V = Z + W \in V_{\rm exp} \cap K$, where the decomposition $Z + W$ is that of the definition of $V_{\rm exp}$.*

(1) *We say $V \in V(m)$, $m \geq 1$, if and only if $Z(x) \geq \gamma |x|^{2m}$ for $x \notin \mathcal{O}$ with a certain compact set $\mathcal{O}$ and with some $\gamma > 0$.*

(2) *We say $V \in V(0)$ if and only if $\liminf_{|x| \to \infty} W(x) > \inf \sigma(H)$, where $\sigma(H)$ denotes the spectrum of H.*

We introduce Hypothesis $\mathbb{H}_m$.

Hypothesis $\mathbb{H}_m$

(1) *$D(\Delta) \subset D(V)$ and there exists $0 \leq a < 1$ and $0 \leq b$ such that for $f \in D(\Delta)$,*

$$\|Vf\|_{L^2(\mathbb{R}^3)} \leq a \|\Delta f\|_{L^2(\mathbb{R}^3)} + b \|f\|_{L^2(\mathbb{R}^3)},$$

(2) *$\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$, and $\hat{\varphi}/\omega, \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$,*

(3) *$\inf \sigma_{\rm ess}(H_{\rm p}) - \inf \sigma(H_{\rm p}) > 0$, where $\sigma(H_{\rm p})$ (resp. $\sigma_{\rm ess}(H_{\rm p})$) denotes the spectrum (resp. essential spectrum) of $H_{\rm p}$,*

(4) *$V \in V(m)$.*

Proposition 1.3 *We assume (1) and (2) of $\mathbb{H}_m$. Then for arbitrary $e \in \mathbb{R}$, H is self-adjoint on $D(\Delta \otimes 1) \cap D(1 \otimes H_{\mathrm{f}})$ and bounded from below, moreover essentially self-adjoint on any core of $-\Delta \otimes 1 + 1 \otimes H_{\mathrm{f}}$.*

Proof: See [13, 14]. □

A typical example of V is the Coulomb potential $\dfrac{-eZ}{4\pi|x|}$, where $Z > 0$ denotes the charge of a nucleus.

Proposition 1.4 *Assume that*

$$\int_{\mathbb{R}^3} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk < \frac{Z^2}{2(4\pi)^2}.$$

Then $-eZ/(4\pi|x|) \in V(0)$ for all $e > 0$.

Proof: It is known that $-1/|x| \in K_3 \cap V_{\mathrm{exp}}$. Then we shall show $\inf\sigma(H) < 0$. Let $V = -eZ/(4\pi|x|)$ and f be the ground state of $H_{\mathrm{p}} = \frac{1}{2}p^2 + V$, $H_{\mathrm{p}} f = -E_0 f$, where $E_0 = e^2 Z^2/(2(4\pi)^2)$. Then we have

$$\inf\sigma(H) \leq (f \otimes \Omega, H f \otimes \Omega)_{\mathcal{H}} = (f, H_{\mathrm{p}} f)_{L^2(\mathbb{R}^3)} + \frac{e^2}{2}(f \otimes \Omega, A^2 f \otimes \Omega)_{\mathcal{H}}$$

$$= -E_0 + \frac{e^2}{2} \sum_{\mu=1,2,3} \int_{\mathbb{R}^3} \left(1 - \frac{k_\mu^2}{|k|^2}\right) \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk = -\frac{e^2}{2}\left(\frac{Z^2}{(4\pi)^2} - 2\int_{\mathbb{R}^3} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)} dk\right) < 0.$$

Thus the proposition follows. □

The number operator of $\mathcal{F}$ is defined by

$$N = \sum_{j=1,2} \int a^*(k,j) a(k,j) dk.$$

The operator N^k, $k \geq 0$, acts as, for $\Psi = \oplus_{n=0}^\infty \Psi^{(n)}$,

$$(N^k \Psi)^{(n)} = n^k \Psi^{(n)}$$

with the domain

$$D(N^k) = \left\{ \Psi = \oplus_{n=0}^\infty \Psi^{(n)} \Big| \sum_{n=0}^\infty n^{2k} \|\Psi^{(n)}\|^2_{\otimes^n L^2(\mathbb{R}^3 \times \{1,2\})} < \infty \right\}.$$

The following proposition is well known.

Proposition 1.5 *Suppose $\mathbb{H}_m$. Then there exists $e_0 \leq \infty$ such that for all $|e| \leq e_0$, (i) H has a ground state ψ_g, (ii) it is unique, (iii) $\|(1 \otimes N^{1/2})\psi_g\|_{\mathcal{H}} < \infty$, (iv) $\|\psi_g(x)\|_{\mathcal{F}} \leq De^{-\delta|x|^{m+1}}$ for almost every $x \in \mathbb{R}^3$ with some constants $D > 0$ and $\delta > 0$.*

Proof: See [5, 10] for (i) and (iii), [12] for (ii) and [15] for (iv). □

Remark 1.6 *It is not clear directly from Proposition 1.5 that $\psi_g \in D(e^{\delta|x|^{m+1}} \otimes N^{1/2})$.*

The condition

$$\mathcal{I} = \int_{\mathbb{R}^3} \frac{|\widehat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty \tag{1.1}$$

is called the infrared cutoff condition. (1.1) is *not* assumed in Proposition 1.5, which is due to [5]. For suitable external potentials, $e_0 = \infty$ is available in Proposition 1.5. This is established in [10]. In the case where $\inf_{\rm ess}(H_{\rm p}) - \inf \sigma(H_{\rm p}) = 0$, examples for H to have a ground state is investigated in [16, 18]. It is unknown, however, whether such a ground state decays in x exponentially or not. When electron includes spin, H has a twofold degenerate ground state for sufficiently small $|e|$, which is shown in [17].

1.3 Photon number localization and infrared singularities for a linear coupling model

The Nelson Hamiltonian [21] describes a linear coupling between a nonrelativistic particle and a scalar quantum field with a form factor φ. Let $\mathcal{H}_{\rm N} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\rm N}$, where $\mathcal{F}_{\rm N} = \bigoplus_{n=0}^{\infty} [\otimes_n^s L^2(\mathbb{R}^3)]$. The Nelson Hamiltonian is defined as a self-adjoint operator acting in the Hilbert space $\mathcal{H}_{\rm N}$, which is given by

$$H_{\rm N} = H_{\rm p} \otimes 1 + 1 \otimes H_{\rm f}^{\rm N} + g\phi,$$

where g denotes a coupling constant, $H_{\rm f}^{\rm N} = \int \omega(k) a^*(k) a(k) dk$ is the free Hamiltonian in $\mathcal{F}_{\rm N}$, and under identification $\mathcal{H}_{\rm N} \cong \int^{\oplus} \mathcal{F}_{\rm N} dx$, ϕ is defined by $\phi = \int^{\oplus} \phi(x) dx$ with

$$\phi(x) = \frac{1}{\sqrt{2}} \int \left\{ a^*(k) e^{-ikx} \frac{\widehat{\varphi}(k)}{\sqrt{\omega(k)}} + a(k) e^{ikx} \frac{\widehat{\varphi}(k)}{\sqrt{\omega(k)}} \right\} dk.$$

It has been established in [2, 4, 9, 24] that the Nelson Hamiltonian has the unique ground state, $\psi_g^{\rm N}$, under the condition $\mathcal{I} < \infty$. Let us denote the number operator in $\mathcal{H}_{\rm N}$ by the same symbol N as that of $\mathcal{H}$. In [6] it has been proven that $\psi_g^{\rm N}$ decays superexponentially, i.e.,

$$\|e^{+\beta(1 \otimes N)} \psi_g^{\rm N}\|_{\mathcal{H}_{\rm N}} < \infty \tag{1.2}$$

for arbitrary $\beta > 0$. This kind of results has been obtained in [11, Section 3] and [23] for relativistic polaron models, and [25, Section 8] for spin-boson models. Moreover in [6] we see that

$$\lim_{\mathcal{I}\to\infty} \|(1\otimes N^{1/2})\psi_g^{\mathrm{N}}\|_{\mathcal{H}_{\mathrm{N}}} = \infty. \tag{1.3}$$

Actually in the infrared divergence case,

$$\mathcal{I} = \infty, \tag{1.4}$$

it is shown in [19] that the Nelson Hamiltonian with some confining external potentials has no ground states in $\mathcal{H}_{\mathrm{N}}$. Then we have to take a non-Fock representation to investigate a ground state with (1.4). See [1, 3, 20] for details. That is to say, the number of bosons of ψ_g^{N} diverges and the ground state disappears as the infrared cutoff is removed. A method to show (1.2) and (1.3) is based on a path integral representation of $(\psi_g^{\mathrm{N}}, e^{+\beta(1\otimes N)}\psi_g^{\mathrm{N}})_{\mathcal{H}_{\mathrm{N}}}$. Precisely it can be shown that in the case $\mathcal{I} < \infty$ there exists a probability measure μ on $C(\mathbb{R};\mathbb{R}^3)$ such that for arbitrary $\beta > 0$,

$$(\psi_g^{\mathrm{N}}, e^{+\beta(1\otimes N)}\psi_g^{\mathrm{N}})_{\mathcal{H}_{\mathrm{N}}} = \int_{C(\mathbb{R};\mathbb{R}^3)} e^{-(g^2/2)(1-e^{+\beta})\int_{-\infty}^0 ds\int_0^\infty dtW(q_s-q_t,s-t)}\mu(dq), \tag{1.5}$$

where $(q_t)_{-\infty<t<\infty} \in C(\mathbb{R};\mathbb{R}^3)$, and

$$W(X,T) = \int_{\mathbb{R}^3} e^{-|T|\omega(k)}e^{ik\cdot X}\frac{|\widehat{\varphi}(k)|^2}{\omega(k)}dk. \tag{1.6}$$

Note that the double integral $\int_{-T}^0 ds\int_0^T dtW(q_s-q_t,s-t)$ is estimated uniformly in path and T as

$$\left|\int_{-T}^0 ds\int_0^T dtW(q_s-q_t,s-t)\right| \le \mathcal{I}. \tag{1.7}$$

This uniform bound is a core of the proof of identity (1.5).

1.4 The main theorems

In contrast to the Nelson Hamiltonian, for the Pauli-Fierz Hamiltonian, as is seen in Proposition 1.5, it is shown that the ground state, ψ_g, exists and $\|(1\otimes N^{1/2})\psi_g\|_{\mathcal{H}} < \infty$ even in the case $\mathcal{I} = \infty$. We may say that the infrared singularity for the Pauli-Fierz Hamiltonian is not so singular in comparison with the Nelson Hamiltonian, and one may expect that

$$\|e^{+\beta(1\otimes N)}\psi_g\|_{\mathcal{H}} < \infty \tag{1.8}$$

holds for some $\beta > 0$ under $\mathcal{I} = \infty$. Unfortunately, however, we can not show (1.8), since the similar path integral method as the Nelson Hamiltonian is not available on account of the appearance of the so-called *double stochastic integral* ([12]), instead of $\int_{-\infty}^0 ds \int_0^\infty dt W(q_s - q_t, s-t)$ in (1.5). The double stochastic integral is formally written as

$$\sum_{\mu,\nu=1,2,3} \int_{-\infty}^0 dq_{\mu,s} \int_0^\infty dq_{\nu,t} W_{\mu\nu}(q_s - q_t, s-t), \tag{1.9}$$

where

$$W_{\mu\nu}(X,T) = \int_{\mathbb{R}^3} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2} \right) e^{-|T|\omega(k)} e^{ik\cdot X} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk.$$

Actually we can not estimate (1.9) uniformly in path such as (1.7). Therefore we are not concerned here with (1.8). In place of this we will show the following theorems.

Theorem 1.7 *Assume $\mathbb{H}_m$. Then $\psi_g \in \bigcap_{k=1}^\infty D(1 \otimes N^{k/2})$.*

Theorem 1.8 *Assume $\mathbb{H}_m$. Then for a fixed $k \geq 0$ there exist positive constants D_k, and δ independent of k such that*

$$\|(1 \otimes N^{k/2})\psi_g(x)\|_{\mathcal{F}} \leq D_k e^{-\delta|x|^{m+1}} \tag{1.10}$$

for almost every $x \in \mathbb{R}^3$.

From Theorems 1.7 and 1.8 the following corollary is immediate.

Corollary 1.9 *Assume $\mathbb{H}_m$. Then $\psi_g \in \bigcap_{k=0}^\infty D(e^{\beta|x|^{m+1}} \otimes N^{k/2})$ for $\beta < \delta/2$.*

Proof: Since $\psi_g \in D(e^{2\beta|x|^{m+1}} \otimes 1) \cap D(1 \otimes N^{k/2})$ for all $k \geq 0$, the corollary follows from the fact that $D(e^{2\beta|x|^{m+1}} \otimes 1) \cap D(1 \otimes N^k) \subset D(e^{\beta|x|^{m+1}} \otimes N^{k/2})$. $\square$

Remark 1.10 *Theorem 1.7 automatically follows if one assumes that photons have artificial positive mass.*

1.5 Outline of proofs of the main theorems

In the following we mostly omit the tensor notation $\otimes$, e.g., we express as H_f for $1 \otimes H_\mathrm{f}$, $a^\sharp(k,j)$ for $1 \otimes a^\sharp(k,j)$, Δ for $\Delta \otimes 1$, $|x|$ for $|x| \otimes 1$, etc. The strategy of this paper is as follows. We check that in order to prove Theorem 1.7 it is enough to show that $\psi_g \in D(a(k_1,j_1)...a(k_l,j_l))$ and that

$$\sum_{j_1,...,j_l=1,2} \int \|a(k_1,j_1)...a(k_l,j_l)\psi_g\|_{\mathcal{H}}^2 dk_1...dk_l < \infty \tag{1.11}$$

for all $l \geq 0$. It is not easy, however, to show (1.11) because of no infrared cutoff condition. Nevertheless Bach-Fröhlich-Sigal [5] proved that (1.11) is finite for $l = 1$ even in the case $\mathcal{I} = \infty$. We extend their method for $l \geq 1$. To see it we first establish the following inequality:

$$\text{L.H.S. (1.11)} \leq C \left(\sum_{j_1,\ldots,j_l=1,2} \int \|b(k_1, j_1) \ldots b(k_l, j_l)\psi_g\|^2_{\mathcal{H}} dk_1 \ldots dk_l \right.$$

$$\left. + \sum_{m=0}^{l-1} \sum_{j_1,\ldots,j_m=1,2} \int \|a(k_1, j_1) \ldots a(k_m, j_m)(|x|+1)^l \psi_g\|^2_{\mathcal{H}} dk_1 \ldots dk_m \right) \tag{1.12}$$

with some positive constant C. Here $b(k,j)$ is a modified annihilation operator given by

$$b(k,j) = a(k,j) - i\frac{e}{\sqrt{2}}(x \cdot e(k,j))\frac{e^{-ik\cdot x}}{\sqrt{\omega(k)}}\widehat{\varphi}(k).$$

Moreover *if* $\psi_g \in D(N^{k/2})$ then we establish the identity

$$N^{k/2}\psi_g = e^{-tH}e^{tE}N^{k/2}\psi_g + e^{tE}[N^{k/2}, e^{-tH}]\psi_g,$$

and

$$\|N^{k/2}\psi_g(x)\|_{\mathcal{F}} \leq D_k e^{-\delta|x|^{m+1}} \tag{1.13}$$

for almost every $x \in \mathbb{R}^3$ with some positive constants D_k and δ, where

$$E = \inf\sigma(H).$$

Under these preparation we prove Theorem 1.7 by means of an induction. Let us assume that $\psi_g \in D(N^{k/2})$ for $k = 1, 2, \ldots, n-1$. Then we see that (1.11) holds for $l = 1, 2, \ldots, n-1$. We will prove two things. First by applying a variant of the pull-through formula we show that

$$\int \|b(k_1, j_1) \ldots b(k_l, j_l)\psi_g\|^2 dk_1 \ldots dk_l < \infty \tag{1.14}$$

and secondly by means of (1.13),

$$\int \|a(k_1, j_1) \ldots a(k_m, j_m)(|x|+1)^l \psi_g\|^2_{\mathcal{H}} dk_1 \ldots dk_m < \infty$$

for $m \leq l-1$. Thus we conclude that (1.11) follows for $l = n$. Since $\psi_g \in D(N^{1/2})$ is known, we obtain $\psi_g \in \bigcap_{k=1}^{\infty} D(1 \otimes N^{k/2})$.

New ingredients in this paper are to establish (1.12), (1.13) and (1.14). This paper is organized as follows. In Section 3 we establish (1.14) by means of pull-through formula. In Section 4 we give a proof of the main theorems. In Section 5 we show (1.13) by virtue of a functional integral representation.

2 Pull-through formula and exponential decay

2.1 Fundamental facts

We set

$$C^\infty(T) = \bigcap_{k=1}^{\infty} D(T^k).$$

Lemma 2.1 *We have $\psi_g \in C^\infty(|x|) \cap D(\Delta) \cap C^\infty(H_{\rm f})$.*

Proof: Since $\psi_g \in D(H) = D(\Delta) \cap D(H_{\rm f})$. Then $\psi_g \in D(\Delta)$ follows. By Proposition 1.5 (2) it holds that $\psi_g \in C^\infty(|x|)$. It is obtained in [8] that $H_{\rm f}^l (H-i)^{-l}$ is bounded for all $l \geq 0$. Then it follows that for arbitrary $l \geq 0$,

$$\|H_{\rm f}^l \psi_g\|_{\mathcal{H}} = \|H_{\rm f}^l (H-i)^{-l} (E-i)^l \psi_g\|_{\mathcal{H}} \leq |(E-i)^l| \|H_{\rm f}^l (H-i)^{-l}\| \|\psi_g\|_{\mathcal{H}}.$$

Then $\psi_g \in D(H_{\rm f}^l)$ for all $l \geq 0$. Thus the lemma follows. □

Let

$$\mathcal{F}_\omega = \mathcal{L}\{a^*(f_1, j_1)...a^*(f_n, j_n)\Omega, \Omega | f_j \in C_0^\infty(\mathbb{R}^3), j = 1, ..., n, n = 0, 1, ...\},$$

where $\mathcal{L}\{...\}$ denotes the set of the finite linear sum of $\{...\}$. We define

$$\mathcal{D} = C^\infty(|x|) \cap C^\infty(H_{\rm f}),$$

and

$$\mathcal{C} = C_0^\infty(\mathbb{R}^3) \hat{\otimes} \mathcal{F}_\omega.$$

Lemma 2.2 *Let $m \geq 0$ and $n \geq 0$. Then $(H_{\rm f} + 1)^n + |x|^m$ is self-adjoint on $D((H_{\rm f} + 1)^n) \cap D(|x|^m)$ and essentially self-adjoint on $\mathcal{C}$.*

Proof: The self-adjointness is trivial. Since $C_0^\infty(\mathbb{R}^3)$ and $\mathcal{F}_\omega$ are the set of analytic vectors of $|x|^m$ and $(H_{\rm f}+1)^n$ respectively, $C_0^\infty(\mathbb{R}^3)$ and $\mathcal{F}_\omega$ are cores of $|x|^m$ and $(H_{\rm f}+1)^n$ respectively. Hence $\mathcal{C} = C_0^\infty(\mathbb{R}^3) \hat{\otimes} \mathcal{F}_\omega$ is a core of $(H_{\rm f} + 1)^n + |x|^m$. □

Remark 2.3 *Let $p, q \geq 0$. From Lemma 2.2 it follows that for $\Psi \in \mathcal{D} \subset D((H_{\rm f}+1)^p + |x|^q)$ there exists a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \Psi$ and $((H_{\rm f} + 1)^p + |x|^q)\Psi_m \to ((H_{\rm f} + 1)^p + |x|^q)\Psi$ strongly as $m \to \infty$.*

For notational convenience we set $\mathbf{k} = (k, j) \in \mathbb{R}^3 \times \{1, 2\}$ and

$$\sum\!\!\!\!\!\!\int ...dk_1...dk_n = \sum_{j_1,...,j_n=1,2} \int ...dk_1...dk_n.$$

Let $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, ..., n$, and $\Psi \in \mathcal{C}$. Then it is well known and easily proven that

$$\sum\!\!\!\!\!\!\int |\prod_{j=1}^{n} f_j(k_j)| \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|_{\mathcal{H}} dk_1...dk_n \leq \epsilon(f_1, ..., f_n) \|(H_{\mathrm{f}} + 1)^{n/2}\Psi\|_{\mathcal{H}} \tag{2.1}$$

with some constant $\epsilon(f_1, ..., f_n)$ independent of Ψ.

Let A and B be operators. We say $f \in D(AB)$ if $f \in D(B)$ and $Bf \in D(A)$.

Lemma 2.4 *Let $\Psi \in \mathcal{D}$. Then there exists $\mathcal{M}_{\mathcal{D}}(\Psi) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that*

$$\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n)) \tag{2.2}$$

and

$$a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in \mathcal{D}. \tag{2.3}$$

for $(k_1, ..., k_n) \notin \mathcal{M}_{\mathcal{D}}(\Psi)$. Moreover assume that $\{\Psi_m\} \subset \mathcal{C}$ satisfies that $\Psi_m \to \Psi$ and $(H_{\mathrm{f}} + 1)^{n/2}\Psi_m \to (H_{\mathrm{f}} + 1)^{n/2}\Psi$ strongly as $m \to \infty$. Then there exists a subsequence $\{m'\} \subset \{m\}$ and $\mathcal{M}_{\mathcal{D}}(\Psi, \{\Psi_m\}, \{m'\}) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that for $(k_1, ..., k_n) \notin \mathcal{M}_{\mathcal{D}}(\Psi, \{\Psi_m\}, \{m'\})$, (2.2) and (2.3) are valid and

$$s - \lim_{m' \to \infty} a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m'} = a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi.$$

Proof: See Appendix A.

Lemma 2.5 *The operator $|x|$ leaves $\mathcal{D}$ invariant.*

Proof: Let $\Psi \in \mathcal{D}$. It is clear that $|x|\Psi \in C^\infty(|x|)$. We choose a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \Psi$ and $((H_{\mathrm{f}} + 1)^{2n} + |x|^2)\Psi_m \to ((H_{\mathrm{f}} + 1)^{2n} + |x|^2)\Psi$ strongly as $m \to \infty$. In particular

$$|x|\Psi_m \to |x|\Psi \tag{2.4}$$

strongly as $m \to \infty$. $H_{\mathrm{f}}^n |x|\Psi_m$ is well defined and it is obtained that

$$\|H_{\mathrm{f}}^n |x|\Psi_m\|_{\mathcal{H}}^2 \leq \|H_{\mathrm{f}}^{2n}\Psi_m\|_{\mathcal{H}} \||x|^2\Psi_m\|_{\mathcal{H}} \leq \|((H_{\mathrm{f}} + 1)^{2n} + |x|^2)\Psi_m\|_{\mathcal{H}}^2.$$

Then $H_{\mathrm{f}}^n|x|\Psi_m$ converges strongly as $m \to \infty$. Since H_{f}^n is closed, by (2.4) we have $|x|\Psi \in D(H_{\mathrm{f}}^n)$. Here n is arbitrary, hence $|x|\Psi \in C^\infty(H_{\mathrm{f}})$. The proof is complete. $\square$

Let

$$\beta(\mathbf{k}) = \frac{e}{\sqrt{2}} \frac{e^{-ik\cdot x}}{\sqrt{\omega(k)}} e(\mathbf{k})\widehat{\varphi}(k)$$

and

$$b(\mathbf{k}) = e^{ix\cdot A} a(\mathbf{k}) e^{-ix\cdot A} = a(\mathbf{k}) - ix\cdot\beta(\mathbf{k}).$$

For simplicity we set $-ix\cdot\beta(\mathbf{k}_j) = \theta_j$. Then

$$b(\mathbf{k}_j) = a(\mathbf{k}_j) + \theta_j.$$

Lemma 2.6 *Let $\Psi \in \mathcal{C}$ and $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, ..., n$. Then there exists a constant $\epsilon'(f_1, ..., f_n)$ independent of Ψ such that*

$$\not\sum\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)| \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi\|_{\mathcal{H}} dk_1...dk_n \le \epsilon'(f_1, ..., f_n) \|((H_{\mathrm{f}}+1)^n + |x|^{2n})\Psi\|_{\mathcal{H}}. \quad (2.5)$$

Proof: Since $[\theta_j, a(\mathbf{k})] = 0$ on $\mathcal{C}$, we have

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi = (a(\mathbf{k}_1)+\theta_1)...(a(\mathbf{k}_n)+\theta_n)\Psi$$

$$= \sum_{l=0}^n \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \theta_{p_1}...\theta_{p_l} a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)\Psi,$$

where $\sum_{\{p_1,...,p_l\}\subset\{1,2,...,n\}}$ means summing up all the combinations to choose l numbers from $\{1, 2, ..., n\}$. Hence by (2.1),

$$\not\sum\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)| \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi\|_{\mathcal{H}} dk_1...dk_n$$

$$\le \sum_{l=0}^n \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \left(\prod_{i=1}^l \int \frac{|e\widehat{\varphi}(k)|}{\sqrt{2\omega(k)}} f_{p_i}(k)dk \right)$$
$$\times \epsilon(f_1, .., \widehat{f_{p_1}}, .., \widehat{f_{p_l}}, ..., f_n) \|(H_{\mathrm{f}}+1)^{(n-l)/2}|x|^l\Psi\|_{\mathcal{H}}.$$

Since $\|(H_{\mathrm{f}}+1)^{(n-l)/2}|x|^l\Psi\|_{\mathcal{H}} \le c_{nl}\|((H_{\mathrm{f}}+1)^n + |x|^{2n})\Psi\|_{\mathcal{H}}$ with some constant c_{nl}. Thus (2.5) follows. $\square$

Lemma 2.7 *Let $\Psi \in \mathcal{D}$. Then there exists $\mathcal{N}_{\mathcal{D}}(\Psi) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that*

$$\Psi \in D(b(\mathbf{k}_1)...b(\mathbf{k}_n)) \tag{2.6}$$

and

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi \in \mathcal{D}. \tag{2.7}$$

Moreover assume that $\{\Psi_m\} \subset \mathcal{C}$ satisfies that $\Psi_m \to \Psi$ and $((H_{\rm f}+1)^n + |x|^{2n})\Psi_m \to ((H_{\rm f}+1)^n + |x|^{2n})\Psi$ strongly as $m \to \infty$. Then there exists a subsequence $\{m'\} \subset \{m\}$ and $\mathcal{N}_{\mathcal{D}}(\Psi, \{\Psi_m\}, \{m'\}) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that for $(k_1, ..., k_n) \notin \mathcal{N}_{\mathcal{D}}(\Psi, \{\Psi_m\}, \{m'\})$, (2.6) and (2.7) are valid and

$$s - \lim_{m'\to\infty} b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi_{m'} = b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi.$$

Proof: See Appendix A.

2.2 Pull-through formula

Lemma 2.8 *We have*

$$\mathcal{C} \subset D(Hb(\mathbf{k}_1)...b(\mathbf{k}_n)) \cap D(b(\mathbf{k}_1)...b(\mathbf{k}_n)H) \tag{2.8}$$

for all $(k_1, ..., k_n) \in \mathbb{R}^{3n}$, and for $\Psi \in \mathcal{C}$,

$$[H, b(\mathbf{k}_1)...b(\mathbf{k}_n)]\Psi = \mathcal{R}_0\Psi + \mathcal{R}_1\Psi + \mathcal{R}_2\Psi.$$

Here

$$\mathcal{R}_0 = \mathcal{R}_0(\mathbf{k}_1, ..., \mathbf{k}_n) = -\sum_{p=1}^n \omega(k_p) b(\mathbf{k}_1)...b(\mathbf{k}_n),$$

$$\mathcal{R}_1 = \mathcal{R}_1(\mathbf{k}_1, ..., \mathbf{k}_n) = \sum_{p=1}^n \vartheta_1(\mathbf{k}_p) b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n),$$

$$\mathcal{R}_2 = \mathcal{R}_2(\mathbf{k}_1, ..., \mathbf{k}_n) = \sum_{p=1}^n \sum_{q<p} \vartheta_2(\mathbf{k}_p, \mathbf{k}_q) b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_q)}...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n),$$

and

$$\vartheta_1(\mathbf{k}) = \vartheta_1(\mathbf{k}, x, p) = \frac{i}{2}\{(x\cdot\beta(\mathbf{k}))k\cdot(p-eA) + k\cdot(p-eA)(x\cdot\beta(\mathbf{k}))\} - i\omega(k)(x\cdot\beta(\mathbf{k})),$$

$$\vartheta_2(\mathbf{k}, \mathbf{k}') = \vartheta_2(\mathbf{k}, \mathbf{k}', x) = (x\cdot\beta(\mathbf{k}))(x\cdot\beta(\mathbf{k}'))(k\cdot k').$$

Proof: (2.8) is trivial. On $\mathcal{C}$ we have

$$[H, b(\mathbf{k})] = -\omega(k)b(\mathbf{k}) + \vartheta_1(\mathbf{k}). \tag{2.9}$$

Moreover

$$[b(\mathbf{k}'), \vartheta_1(\mathbf{k})] = \vartheta_2(\mathbf{k}, \mathbf{k}'). \tag{2.10}$$

By (2.9) and (2.10) we have

$$[H, b(\mathbf{k}_1)...b(\mathbf{k}_n)]\Psi = \sum_{p=1}^n b(\mathbf{k}_1)...\left\{-\omega(k_p)b(\mathbf{k}_p) + \vartheta_1(\mathbf{k}_p)\right\}...b(\mathbf{k}_n)\Psi$$

$$= -\sum_{p=1}^n \omega(k_p)b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi + \sum_{p=1}^n \vartheta_1(k_p)b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n)\Psi$$

$$+\sum_{p=1}^n \sum_{q<p} \vartheta_2(\mathbf{k}_p, \mathbf{k}_q)b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_q})...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n)\Psi.$$

The lemma follows. □

$\overline{B}$ denotes the closure of B. We simply set $\overline{\mathcal{R}_1} = \overline{\mathcal{R}_1\lceil_{\mathcal{C}}}$.

Lemma 2.9 *Let $\Psi \in \mathcal{D} \cap D(\Delta)$. Then there exists $\mathcal{N}(\Psi) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that for $(k_1, ..., k_n) \notin \mathcal{N}(\Psi)$,*

$$\Psi \in D(\mathcal{R}_0(\mathbf{k}_1, ..., \mathbf{k}_n)) \cap D(\overline{\mathcal{R}_1}(\mathbf{k}_1, ..., \mathbf{k}_n)) \cap D(\mathcal{R}_2(\mathbf{k}_1, ..., \mathbf{k}_n)).$$

Proof: By Lemma 2.7, $\Psi \in D(b(\mathbf{k}_1)...b(\mathbf{k}_n))$ and $b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi \in \mathcal{D}$ for $(k_1, ..., k_n) \notin \mathcal{N}_{\mathcal{D}}(\Psi)$. Thus $b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi \in D(\sum_{p=1}^n \omega(k_p)) \cap D(\vartheta_2(\mathbf{k}_p, \mathbf{k}_q))$, which implies that $\Psi \in D(\mathcal{R}_0(\mathbf{k}_1, ..., \mathbf{k}_n)) \cap D(\mathcal{R}_2(\mathbf{k}_1, ..., \mathbf{k}_n))$. Next we shall prove $D(\overline{\mathcal{R}_1}(\mathbf{k}_1, ..., \mathbf{k}_n)) \ni \Psi$. Simply we set $K_n = ((H_\mathrm{f} + 1)^n + |x|^{2n})$. We have on $\mathcal{C}$

$$\mathcal{R}_1 = \sum_{p=1}^n ix \cdot \beta(\mathbf{k}_p)(k_p \cdot p)b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n) + \sum_{p=1}^n \mathcal{R}_x(\mathbf{k}_p)b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n),$$

where

$$\mathcal{R}_x(\mathbf{k}_p) = (-ie)(x \cdot \beta(\mathbf{k}_p))(k \cdot A) - \frac{i}{2}(i\beta(\mathbf{k}_p) \cdot k_p + x \cdot \beta(\mathbf{k}_p)|k_p|^2) - i\omega(k_p)(x \cdot \beta(\mathbf{k}_p)).$$

It follows that for $\Phi \in \mathcal{C}$,

$$\sum\!\!\!\!\!\!\!\int \| \prod_{j=1}^n f_j(k_j)\mathcal{R}_x(\mathbf{k}_p)b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n)\Phi\|_{\mathcal{H}} dk_1...dk_n \le c_1\|K_{m_1}\Psi\|_{\mathcal{H}}$$

with some constants c_1 and m_1, and

$$ix\cdot\beta(\mathbf{k}_p)(k_p\cdot p)b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)\Phi = ix\cdot\beta(\mathbf{k}_p)b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(k_p\cdot p)\Phi$$

$$+\sum_{q\neq p}\mathcal{R}_x(\mathbf{k}_p,\mathbf{k}_q)b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_q)}...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)\Phi, \tag{2.11}$$

where

$$\mathcal{R}_x(\mathbf{k}_p,\mathbf{k}_q) = ix\cdot\beta(\mathbf{k}_p)\left(-k_q\beta(\mathbf{k}_q)+ix\cdot\beta(\mathbf{k}_q)(k_p\cdot k_q)\right).$$

The second term of (2.11) is estimated as

$$\not{\sum}\|\prod_{j=1}^n f_j(k_j)\sum_{q\neq p}\mathcal{R}_x(\mathbf{k}_p,\mathbf{k}_q)b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_q)}...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)\Phi\|_{\mathcal{H}}dk_1...dk_n$$

$$\leq c_2\|K_{m_2}\Psi\|_{\mathcal{H}}$$

with some constants c_2 and m_2. By (2.5) the first term of (2.11) is estimated as

$$\not{\sum}\|\prod_{j=1}^n f_j(k_j)\left(x\cdot\beta(\mathbf{k}_p)\right)b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(k_p\cdot p)\Phi\|_{\mathcal{H}}dk_1...dk_n$$

$$\leq \epsilon'(f_1,...,\widehat{f_p},...,f_n)\int|f(k_p)|\frac{|e|}{\sqrt{2}}\frac{|\widehat{\varphi}(k_p)|}{\sqrt{\omega(k_p)}}\|K_{n-1}|x|(k_p\cdot p)\Phi\|_{\mathcal{H}}dk_p.$$

Let $Q=K_{n-1}$. Note that

$$\|Q|x|(k\cdot p)\Phi\|^2_{\mathcal{H}} = (|x|^2Q^2\Phi,(k_p\cdot p)^2\Phi)_{\mathcal{H}}+(\Phi,[(k_p\cdot p),Q^2|x|^2](k_p\cdot p)\Phi)_{\mathcal{H}}.$$

Since $[(k_p\cdot p),|x|]=-i(k_p\cdot x)/|x|$, we have $[(k_p\cdot p),Q^2|x|^2]=k_p\cdot P$, where

$$P=2\left\{(H_{\rm f}+1)^{n-1}(-i)\frac{x}{|x|}+(-i)x(|x|+1)^{2n-3}+(-i)\frac{x}{|x|}(|x|+1)^{2n-2}\right\}.$$

Then

$$\|Q|x|(k_p\cdot p)\Phi\|^2_{\mathcal{H}}\leq|k_p|^2\left(\||x|^2Q^2\Phi\|_{\mathcal{H}}\|\Delta\Phi\|_{\mathcal{H}}+\|P\Phi\|_{\mathcal{H}}\||p|\Phi\|_{\mathcal{H}}\right).$$

Hence

$$\|Q|x|(k_p\cdot p)\Phi\|_{\mathcal{H}}\leq|k_p|\left(c_3\|K_{m_3}\Psi\|_{\mathcal{H}}+c'\|\Delta\Phi\|_{\mathcal{H}}\right)$$

follows with some constants c_3, c' and m_3. Thus for $\Phi\in\mathcal{C}$

$$\not{\sum}\|\prod_{j=1}^n f_j(k_j)\mathcal{R}_1(\mathbf{k}_1,...,\mathbf{k}_n)\Phi\|_{\mathcal{H}}dk_1...dk_n\leq c\|K_m\Phi\|_{\mathcal{H}}+c'\|\Delta\Phi\|_{\mathcal{H}} \tag{2.12}$$

follows with some constants c and m. Set $K = -\Delta + K_m = -\Delta + |x|^{2m} + (H_\mathrm{f}+1)^m$. Then K is self-adjoint on $D(-\Delta + |x|^{2m}) \cap D((H_\mathrm{f}+1)^m)$ and essentially self-adjoint on $\mathcal{C}$. Then for $\Psi \in \mathcal{D} \cap D(\Delta)$ there exists a sequence $\{\Psi_l\} \subset \mathcal{C}$ such that $\Psi_l \to \Psi$ and $K\Psi_l \to K\Psi$ strongly as $l \to \infty$. By (2.12) it follows that

$$\not{\sum} \| \prod_{j=1}^n f_j(k_j) \mathcal{R}_1(\mathbf{k}_1, ..., \mathbf{k}_n) \Psi_l \|_\mathcal{H} dk_1 ... dk_n \leq c \|K_m \Psi_l\|_\mathcal{H} + c' \|\Delta \Psi_l\|_\mathcal{H}.$$

Then there exist $\mathcal{N}_\mathcal{D}(\Psi)' \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero and a subsequence $\{l'\} \subset \{l\}$ such that $\mathcal{R}_1(\mathbf{k}_1, ..., \mathbf{k}_n)\Psi_{l'}$ strongly converges as $l' \to \infty$ for $(k_1, ..., k_n) \notin \mathcal{N}_\mathcal{D}(\Psi)'$. Then $\Psi \in D(\overline{\mathcal{R}_1}(\mathbf{k}_1, ..., \mathbf{k}_n))$ for $(k_1, ..., k_n) \notin \mathcal{N}_\mathcal{D}(\Psi)'$. Set

$$\mathcal{N}(\Psi) = \mathcal{N}_\mathcal{D}(\Psi) \cup \mathcal{N}_\mathcal{D}(\Psi)'.$$

We get the desired results. □

The following lemma is a variant of the pull-through formula. Recall that $E = \inf\sigma(H)$.

Lemma 2.10 *For $(k_1, ..., k_n) \notin \mathcal{N}(\psi_g)$, the following (1), (2) and (3) hold;*

(1) $\psi_g \in D(b(\mathbf{k}_1)...b(\mathbf{k}_n)) \cap D(\mathcal{R}_0) \cap D(\overline{\mathcal{R}_1}) \cap D(\mathcal{R}_2)$,

(2) $b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g \in D(H)$,

(3)

$$\left(H - E + \sum_{p=1}^n \omega(k_p) \right) b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g = \overline{\mathcal{R}_1}\psi_g + \mathcal{R}_2\psi_g. \tag{2.13}$$

In particular it follows that for $(k_1, ..., k_n) \notin \mathcal{N}(\psi_g)$ and $(k_1, ..., k_n) \neq (0, ..., 0)$,

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g = \left(H - E + \sum_{p=1}^n \omega(k_p) \right)^{-1} \left(\overline{\mathcal{R}_1}\psi_g + \mathcal{R}_2\psi_g \right). \tag{2.14}$$

Proof: Note that $\psi_g \in \mathcal{D} \cap D(\Delta) = C^\infty(|x|) \cap C^\infty(H_\mathrm{f}) \cap D(\Delta)$. Then (1) follows from Lemma 2.9. Since $\mathcal{C}$ is a core of H, we have $\phi_m \in \mathcal{D}$ such that $\phi_m \to \psi_g$ and $H\phi_m \to H\psi_g = E\psi_g$ strongly as $m \to \infty$. Then we have for $\phi \in \mathcal{C}$

$$(H\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)\phi_m)_\mathcal{H} = \sum_{j=0,1,2} (\phi, \mathcal{R}_j\phi_m)_\mathcal{H} + (\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)H\phi_m)_\mathcal{H}.$$

It follows that

$$\lim_{m\to\infty}(H\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)\phi_m)_{\mathcal{H}} = \lim_{n\to\infty}(b^*(\mathbf{k}_n)...b^*(\mathbf{k}_1)H\phi, \phi_m)_{\mathcal{H}}$$

$$= (b^*(\mathbf{k}_n)...b^*(\mathbf{k}_1)H\phi, \psi_g)_{\mathcal{H}} = (H\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g)_{\mathcal{H}},$$

$$\lim_{m\to\infty}(\phi, \mathcal{R}_j\phi_m)_{\mathcal{H}} = \lim_{m\to\infty}(\mathcal{R}_j^*\phi, \phi_m)_{\mathcal{H}} = (\mathcal{R}_j^*\phi, \psi_g)_{\mathcal{H}} = (\phi, \overline{\mathcal{R}_j}\psi_g)_{\mathcal{H}},$$

and

$$\lim_{m\to\infty}(\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)H\phi_m)_{\mathcal{H}} = \lim_{n\to\infty}(b^*(\mathbf{k}_n)...b^*(\mathbf{k}_1)\phi, H\phi_m)_{\mathcal{H}}$$

$$= (b^*(\mathbf{k}_n)...b^*(\mathbf{k}_1)\phi, E\psi_g)_{\mathcal{H}} = E(\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g)_{\mathcal{H}}.$$

Hence

$$(H\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g)_{\mathcal{H}} = \sum_{j=0,1,2}(\phi, \overline{\mathcal{R}_j}\psi_g)_{\mathcal{H}} + E(\phi, b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g)_{\mathcal{H}}.$$

Then $b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g \in D(H)$ and we have

$$Hb(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g = \sum_{j=0,1,2}\overline{\mathcal{R}_j}\psi_g + Eb(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g.$$

Note that $\overline{\mathcal{R}_0}\psi_g = \mathcal{R}_0\psi_g$ and $\overline{\mathcal{R}_2}\psi_g = \mathcal{R}_2\psi_g$. Then (2.13) follows. □

2.3 Exponential decay of $N^{k/2}\psi_g$

Lemma 2.11 *Suppose that $\psi_g \in D(N^{k/2})$. Then there exist positive constants D_k, and δ independent of k such that*

$$\|N^{k/2}\psi_g(x)\|_{\mathcal{F}} \le D_k e^{-\delta|x|^{m+1}}$$

for almost every $x \in \mathbb{R}^3$. In particular $N^{k/2}\psi_g \in D(e^{\delta|x|^{m+1}})$.

The proof of Lemma 2.11 is based on a functional integral representation of e^{-tH}. Essential ingredients of the proof have been obtained in [13]. The proof is, however, long and complicated. Then we move it to Appendix B.

Lemma 2.12 *Suppose that $\psi_g \in D(N^{k/2})$. Then $|x|^l\psi_g \in D(N^{k/2})$ for all $l \ge 0$.*

Proof: This lemma is immediately follows from Lemma 2.11 and the following fundamental lemma. □

Lemma 2.13 *Let $\mathcal{K}$ be a Hilbert space, and A and B self-adjoint operators such that $[e^{-itsA}, e^{-isB}] = 0$ for $s, t \in \mathbb{R}$. Suppose that $\phi \in D(A) \cap D(B)$ and $A\phi \in D(B)$. Then $B\phi \in D(A)$ with $AB\phi = BA\phi$.*

Proof: It follows that $t^{-1}(e^{-itA} - 1)e^{-isB}\phi = t^{-1}e^{-isB}(e^{-itA} - 1)\phi$. Take $t \to 0$ on the both sides. Then it follows that $e^{-isB}\phi \in D(A)$ with $Ae^{-isB}\phi = e^{-isB}A\phi$. From this identity we have $s^{-1}A(e^{-isB} - 1)\phi = s^{-1}(e^{-isB} - 1)A\phi$. Take $s \to 0$ on the both sides. Since A is closed and assumption $A\phi \in D(B)$, we see that $B\phi \in D(A)$ and $AB\phi = BA\phi$. $\square$

Proof of Lemma 2.12

In Lemma 2.13 we set $\mathcal{K} = \mathcal{H}$, $A = N^{k/2}$ and $B = |x|^l$. Since $\psi_g \in D(N^{k/2}) \cap D(|x|^l)$ and $N^{k/2}\psi_g \in D(|x|^l)$ by Lemma 2.11, the lemma follows. $\square$

3 Proof of the main theorems

Lemma 3.1 *Assume that $\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n))$ for almost every $(k_1, ..., k_n) \in \mathbb{R}^{3n}$ and that*

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|^2_{\mathcal{H}} dk_1...dk_n < \infty. \tag{3.1}$$

Then $\Psi \in D(\prod_{j=1}^n (N - j + 1)^{1/2})$ and

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|^2_{\mathcal{H}} dk_1...dk_n = \|\prod_{j=1}^n (N - j + 1)^{1/2}\Psi\|^2_{\mathcal{H}}.$$

Proof: We identify $\mathcal{H}$ as

$$\mathcal{H} \cong \bigoplus_{n=0}^\infty L^2(\mathbb{R}^3 \times (\mathbb{R}^3 \times \{1, 2\})^{\times n}_{\rm sym}). \tag{3.2}$$

We note that

$$\left(\prod_{j=1}^n (N - j + 1)^{1/2}\Psi\right)^{(k)} = \begin{cases} 0, & k = 0, 1, ..., n - 1, \\ \sqrt{k(k-1)...(k-n+1)}\Psi^{(k)}, & k \geq n. \end{cases}$$

By the definition of $a(\mathbf{k})$ we have

$$(a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi)^{(l)}(x, \mathbf{k}'_1, ..., \mathbf{k}'_l) = \sqrt{l+1}\sqrt{l+2}...\sqrt{l+n}\Psi^{(l+n)}(x, \mathbf{k}_1, ..., \mathbf{k}_n, \mathbf{k}'_1, ..., \mathbf{k}'_l).$$

Then

$$\|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|^2_{\mathcal{H}}$$

$$= \sum_{l=0}^{\infty}(l+1)(l+2)...(l+n)\sum\!\!\!\!\!\!\int \|\Psi^{(l+n)}(\cdot, \mathbf{k}_1, ..., \mathbf{k}_n, \mathbf{k}'_1, ..., \mathbf{k}'_l)\|^2_{L^2(\mathbb{R}^3)} dk'_1...dk'_l.$$

Hence

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|^2_{\mathcal{H}} dk_1...dk_n$$

$$= \sum_{l=0}^{\infty}(l+1)(l+2)...(l+n)\sum\!\!\!\!\!\!\int \|\Psi^{(l+n)}(\cdot, \mathbf{k}_1, ..., \mathbf{k}_n, \mathbf{k}'_1, ..., \mathbf{k}'_l)\|^2_{L^2(\mathbb{R}^3)} dk'_1...dk'_l dk_1...dk_n$$

$$= \sum_{k \geq n} k(k-1)(k-2)...(k-n+1)\sum\!\!\!\!\!\!\int \|\Psi^{(k)}(\cdot, \mathbf{k}_1, ..., \mathbf{k}_k)\|^2_{L^2(\mathbb{R}^3)} dk_1...dk_k$$

$$= \| \prod_{j=1}^{n}(N-j+1)^{1/2}\Psi\|^2_{\mathcal{H}}.$$

Here on the third line we can interchange $\sum\!\!\!\!\!\!\int$ and $\sum_{l=0}^{\infty}$ by (3.1) and the monotone convergence theorem. Thus the lemma follows. $\square$

Lemma 3.2 *Assume that $\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n))$ for almost every $(k_1, ..., k_n) \in \mathbb{R}^{3n}$ and, that for $n = 1, 2, ..., k$,*

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|^2_{\mathcal{H}} dk_1...dk_n < \infty.$$

Then $\Psi \in D(N^{k/2})$.

Proof: By Lemma 3.1,

$$\Psi \in \bigcap_{k=1}^{n} D(\prod_{j=1}^{k}(N^n - j + 1)^{1/2}). \tag{3.3}$$

Define $\Psi_n = \oplus_{m=0}^{\infty}\Psi_n^{(m)} \in \mathcal{H}$ under the identification (3.2) by

$$\Psi_n^{(m)} = \begin{cases} \Psi^{(m)}, & m \leq n, \\ 0, & m > n. \end{cases}$$

Let $W_n = \prod_{j=1}^{n}(N-j+1)$. For example $N = W_1, N^2 = W_2 + W_1, N^3 = W_3 + 3W_2 + W_1, N^4 = W_4 + 6W_3 + 7W_2 + W_1$, etc. One can inductively see that there exist constants $a_j, j = 1, ..., k$, such that on $\mathcal{F}_0$,

$$N^k = \sum_{j=1}^{n} a_j W_j.$$

Then it follows that

$$\|N^{k/2}\Psi_n\|^2_{\mathcal{H}} = a_1\|W_1^{1/2}\Psi_n\|^2_{\mathcal{H}} + a_2\|W_2^{1/2}\Psi_n\|^2_{\mathcal{H}} + ... + a_k\|W_k^{1/2}\Psi_n\|^2_{\mathcal{H}}. \tag{3.4}$$

As $n \to \infty$, from (3.3) it follows that the right hand side of (3.4) converges to

$$a_1 \|W_1^{1/2}\Psi\|^2_{\mathcal{H}} + a_2 \|W_2^{1/2}\Psi\|^2_{\mathcal{H}} + ... + a_k \|W_k^{1/2}\Psi\|^2_{\mathcal{H}}.$$

Since $N^{k/2}$ is closed, $\Psi \in D(N^{k/2})$ follows. $\square$

We set

$$\mathcal{R}_\omega = \mathcal{R}_\omega(k_1, ..., k_n) = \left(H - E + \sum_{p=1}^{n} \omega(k_p)\right)^{-1}.$$

Lemma 3.3 *There exist $\delta_1(\cdot) \in L^2(\mathbb{R}^3)$ and $\delta_2(\cdot,\cdot) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ such that for $\Psi \in \mathcal{D}$,*

$$\|\overline{\mathcal{R}_\omega \vartheta_1(\mathbf{k}_q)}\Psi\|_{\mathcal{H}} \leq \delta_1(k_q)\|(|x|+1)\Psi\|_{\mathcal{H}}, \tag{3.5}$$

and

$$\|\overline{\mathcal{R}_\omega \vartheta_2(\mathbf{k}_q, \mathbf{k}_p)}\Psi\|_{\mathcal{H}} \leq \delta_2(k_q, k_p)\||x|^2\Psi\|_{\mathcal{H}}. \tag{3.6}$$

Proof: By the closed graph theorem there exists a constant C such that

$$\|(-\Delta + H_{\rm f})\Psi\|_{\mathcal{H}} \leq C\|(H+1)\Psi\|_{\mathcal{H}}.$$

First we shall prove that $\overline{\mathcal{R}_\omega(p \cdot k_q)}$ and $\overline{\mathcal{R}_\omega(A \cdot k_q)}$ are bounded with

$$\|\overline{\mathcal{R}_\omega(p \cdot k_q)}\| \leq c_1(k_q) \tag{3.7}$$

and

$$\|\overline{\mathcal{R}_\omega(A \cdot k_q)}\| \leq c_2(k_q), \tag{3.8}$$

where $c_1(k_q) = \sqrt{(|k_q| + |1+E|)C}$ and $c_2(k_q) = \sqrt{2}\,(c_1(k_q)+1)\,(2\|\widehat{\varphi}/\omega\| + \|\widehat{\varphi}/\sqrt{\omega}\|)$. Let $\Psi \in \mathcal{C}$. Since

$$\|(p \cdot k_q)\Psi\|^2_{\mathcal{H}} \leq |k_q|^2(\Psi, C(H+1)\Psi) \leq |k_q|^2 C\left\{\|(H-E)^{1/2}\Psi\|^2_{\mathcal{H}} + |1+E|\|\Psi\|^2_{\mathcal{H}}\right\},$$

we see that

$$\|(p \cdot k_q)\mathcal{R}_\omega\Psi\|^2_{\mathcal{H}} \leq C|k_q|\|\Psi\|^2_{\mathcal{H}} + C|1+E|\|\Psi\|^2_{\mathcal{H}}.$$

Thus (3.7) follows. Note that

$$\|a(f,j)\Psi\|_{\mathcal{H}} \leq \|f/\sqrt{\omega}\|\|H_{\rm f}^{1/2}\Psi\|_{\mathcal{F}},$$

and

$$\|a^*(f,j)\Psi\|_{\mathcal{F}} \leq \|f/\sqrt{\omega}\|\|H_{\rm f}^{1/2}\Psi\|_{\mathcal{F}} + \|f\|\|\Psi\|_{\mathcal{F}}.$$

Since

$$\|(A\cdot k_q)\Psi\|_{\mathcal{H}} \leq \sqrt{2}|k_q|\left(2\|\widehat{\varphi}/\omega\|+\|\widehat{\varphi}/\sqrt{\omega}\|\right)\left(\|H_{\mathrm{f}}^{1/2}\Psi\|_{\mathcal{H}}+\|\Psi\|_{\mathcal{H}}\right)$$

and

$$\|H_{\mathrm{f}}^{1/2}\Psi\|^2_{\mathcal{H}} \leq C(\Psi,(H+1)\Psi)_{\mathcal{H}} \leq C\|(H-E)^{1/2}\Psi\|^2_{\mathcal{H}}+C|1+E|\|\Psi\|^2_{\mathcal{H}},$$

we have

$$|k_q|^2\|H_{\mathrm{f}}^{1/2}\mathcal{R}_\omega\Psi\|^2_{\mathcal{H}} \leq C|k_q|\|\Psi\|^2_{\mathcal{H}}+C|1+E|\|\Psi\|^2_{\mathcal{H}}.$$

Hence

$$\begin{aligned}\|(A\cdot k_q)\mathcal{R}_\omega\Psi\|_{\mathcal{H}} &\leq \sqrt{2}\left(2\|\widehat{\varphi}/\omega\|+\|\widehat{\varphi}/\sqrt{\omega}\|\right)\left(|k_q|\|H_{\mathrm{f}}^{1/2}\mathcal{R}_\omega\Psi\|_{\mathcal{H}}+|k_q|\|\mathcal{R}_\omega\Psi\|_{\mathcal{H}}\right)\\ &\leq \sqrt{2}\left\{\sqrt{(|k_q|+|1+E|)C}+1\right\}(2\|\widehat{\varphi}/\omega\|+\|\widehat{\varphi}/\sqrt{\omega}\|)\|\Psi\|_{\mathcal{H}}.\end{aligned}$$

Thus (3.8) follows. We have on $\mathcal{C}$

$$\vartheta_1(\mathbf{k}) = i(p-eA)\cdot k(x\cdot\beta(k))+\frac{i}{2}\left(i\beta(\mathbf{k})\cdot k+x\cdot\beta(\mathbf{k})|k|^2\right)-i\omega(k)(x\cdot\beta(\mathbf{k})).$$

Then by (3.7) and (3.8) we have for $\Psi\in\mathcal{C}$,

$$\|\mathcal{R}_\omega i(p-eA)\cdot k_p,(x\cdot\beta(k_p))\Psi\|_{\mathcal{H}} \leq (c_1(k_p)+|e|c_2(k_p))\frac{|e|}{\sqrt{2}}\frac{|\widehat{\varphi}(k_p)|}{\omega(k_p)}\||x|\Psi\|_{\mathcal{H}},$$

$$\|\mathcal{R}_\omega(-i\omega(k_p)(x\cdot\beta(\mathbf{k}_p)))\Psi\|_{\mathcal{H}} \leq \frac{|e|}{\sqrt{2}}\sqrt{\omega(k_p)}|\widehat{\varphi}(k_p)|\||x|\Psi\|_{\mathcal{H}},$$

and

$$\|\mathcal{R}_\omega\frac{i}{2}\left(i\beta(\mathbf{k}_p)\cdot k_p+x\cdot\beta(\mathbf{k}_p)|k_p|^2\right)\Psi\|_{\mathcal{H}} \leq \frac{1}{2}\frac{|e|}{\sqrt{2}}\left(\frac{|\widehat{\varphi}(k_p)|}{\omega(k_p)}\|\Psi\|_{\mathcal{H}}+|\widehat{\varphi}(k_p)|\||x|\Psi\|_{\mathcal{H}}\right).$$

Since $\|\widehat{\varphi}\|<\infty$, $\|\sqrt{\omega}\widehat{\varphi}\|<\infty$ and $\|\widehat{\varphi}/\omega\|<\infty$, (3.5) follows for $\Psi\in\mathcal{C}$. By a limiting argument it can be extended for $\Psi\in\mathcal{D}$. (3.6) is rather easier than (3.5). We have for $\Psi\in\mathcal{C}$,

$$\|\mathcal{R}_\omega\vartheta_2(\mathbf{k}_p,\mathbf{k}_q)\Psi\|_{\mathcal{H}} \leq \frac{e^2}{2}\sqrt{\omega(k_p)\omega(k_q)}|\widehat{\varphi}(k_q)\widehat{\varphi}(k_p)|\||x|^2\Psi\|_{\mathcal{H}}.$$

Thus the lemma follows from a limiting argument and $\|\sqrt{\omega}\widehat{\varphi}\|<\infty$. □

From Lemma 3.3 the next lemma immediately follows.

Lemma 3.4 *For almost every $(k_1, ..., k_n)$ it follows that*

$$\psi_g \in D(b(\mathbf{k}_1)...b(\mathbf{k}_n)) \bigcap [\cap_p D(b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1))]$$

$$\bigcap [\cap_{q<p}(b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_q)}...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2)] \tag{3.9}$$

and

$$\|b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g\|_{\mathcal{H}} \leq \sum_{p=1}^{n} \delta_1(k_p)\|b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\psi_g\|_{\mathcal{H}}$$

$$+\sum_{p=1}^{n}\sum_{q<p}\delta_2(k_p,k_q)\|b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_q)}...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\psi_g\|_{\mathcal{H}}.$$

Proof: Note that for $(k_1, ..., k_n) \notin \mathcal{N}(\psi_g)$ and $(k_1, ..., k_n) \neq (0, ..., 0)$,

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g = \mathcal{R}_\omega(k_1, ..., k_n)\overline{\mathcal{R}_1}(\mathbf{k}_1, ..., \mathbf{k}_n)\psi_g + \mathcal{R}_\omega(k_1, ..., k_n)\mathcal{R}_2(\mathbf{k}_1, ..., \mathbf{k}_n)\psi_g.$$

Let $\Psi \in \mathcal{C}$ and $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, ..., n$. It is obtained that

$$\not\!\!\sum |\prod_{j=1}^{n} f_j(k_j)|\|\mathcal{R}_\omega(k_1, ..., k_n)\overline{\mathcal{R}_1}(\mathbf{k}_1, ..., \mathbf{k}_n)\Psi\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq \sum_{p=1}^{n}\not\!\!\sum |\prod_{j=1}^{n} f_j(k_j)|\delta_1(k_p)\|b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\Psi\|_{\mathcal{H}} dk_1...dk_n. \tag{3.10}$$

Similarly we obtain that

$$\not\!\!\sum |\prod_{j=1}^{n} f_j(k_j)|\|\mathcal{R}_\omega(k_1, ..., k_n)\mathcal{R}_2(k_1, ..., k_n)\Psi\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq \sum_{p=1}^{n}\not\!\!\sum |\prod_{j=1}^{n} f_j(k_j)|\sum_{q<p}\delta_2(k_p,k_q)\|b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_q)}...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\Psi\|_{\mathcal{H}} dk_1...dk_n. \tag{3.11}$$

We choose a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \psi_g$ and $((H_\mathrm{f}+1)^K + |x|^{2K})\Psi_m \to ((H_\mathrm{f}+1)^K + |x|^{2K})\psi_g$ strongly as $m \to \infty$ for sufficiently large K. Note that $|x|^j\Psi_m \to |x|^j\Psi$ and $((H_\mathrm{f}+1)^n + |x|^{2n})|x|^j\Psi_m \to ((H_\mathrm{f}+1)^n + |x|^{2n})|x|^j\Psi$ strongly as $m \to \infty$ for $j = 1, 2$, since K is sufficiently large. By Lemma 2.7 there exists a subsequence $\{m'\} \subset \{m\}$ such that for almost every $(k_1, ..., k_n) \in \mathbb{R}^{3n}$, (3.9) follows and

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi_{m'} \to b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g, \tag{3.12}$$

$$b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\Psi_{m'} \to b(\mathbf{k}_1)...\widehat{b(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\psi_g, \tag{3.13}$$

and

$$b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_q)}...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\Psi_{m'} \to b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_q)}...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\psi_g \tag{3.14}$$

strongly as $m' \to \infty$. Moreover

$$\sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)\delta_1(k_p)| \|b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\Psi_{m'}\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq \left(\int \delta_1(k_p)|f_p(k_p)|dk_p\right) \epsilon'(f_1,...,\widehat{f}_p,...,f_n)\|((H_{\mathrm{f}}+1)^{n-1}+|x|^{2n-2})(|x|+1)\Psi_{m'}\|_{\mathcal{H}}$$

and

$$\sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)\delta_2(k_p,k_q)| \|b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_q)}...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\Psi_{m'}\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq \left(\int \delta_2(k_p,k_q))|f_p(k_p)f_q(k_q)|dk_p dk_q\right)$$

$$\times \epsilon'(f_1,...,\widehat{f}_p,...,\widehat{f}_q,...,f_n)\|((H_{\mathrm{f}}+1)^{n-2}+|x|^{2n-4})|x|^2\Psi_{m'}\|_{\mathcal{H}}.$$

Then we have by (3.10) and (3.11)

$$\sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)| \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi_{m'}\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq \sum_{p=1}^n \sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)\delta_1(k_p)| \|b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\Psi_{m'}\|_{\mathcal{H}} dk_1...dk_n$$

$$+\sum_{p=1}^n \sum_{q<p} \sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)\delta_2(k_p,k_q)| \|b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_q)}...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\Psi_{m'}\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq C\|((H_{\mathrm{f}}+1)^K+|x|^{2K})\Psi_{m'}\|_{\mathcal{H}} \leq C'\|((H_{\mathrm{f}}+1)^K+|x|^{2K})\psi_g\|_{\mathcal{H}}$$

with some constant C and C'. Thus by the Lebesgue dominated convergence theorem and (3.12), (3.13) and (3.14), we have

$$\sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)| \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g\|_{\mathcal{H}} dk_1...dk_n$$

$$\leq \sum_{p=1}^n \sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)\delta_1(k_p)| \|b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)(|x|+1)\psi_g\|_{\mathcal{H}} dk_1...dk_n$$

$$+\sum_{p=1}^n \sum_{q<p} \sum\!\!\!\!\!\!\!\int |\prod_{j=1}^n f_j(k_j)\delta_2(k_p,k_q)| \|b(\mathbf{k}_1)...b\widehat{(\mathbf{k}_q)}...b\widehat{(\mathbf{k}_p)}...b(\mathbf{k}_n)|x|^2\psi_g\|_{\mathcal{H}} dk_1...dk_n$$

Since $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j=1,...,n$, are arbitrary, the lemma follows. $\square$

Lemma 3.5 *Let $\Psi \in \mathcal{D}$. Then for almost every $(k_1, ..., k_n) \in \mathbb{R}^{3n}$ it follows that*

$$\Psi \in D(b(\mathbf{k}_1)...b(\mathbf{k}_n)) \bigcap [\cap_{l=0}^n \cap_{\{p_1,..,p_l\}\subset\{1,..,n\}} D(a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)|x|^l)]$$

and

$$\|b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi\|_{\mathcal{H}}$$
$$\leq \sum_{l=0}^n \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \prod_{j=1}^l \frac{|e\widehat{\varphi}(k_{p_j})|}{\sqrt{2\omega(k_{p_j})}} \|a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)|x|^l\Psi\|_{\mathcal{H}}. \tag{3.15}$$

Proof: Take a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \Psi$ and $(H_{\rm f}^K + |x|^{2K} + 1)\Psi_m \to (H_{\rm f}^K + |x|^{2K} + 1)\Psi$ strongly as $m \to \infty$ for sufficiently large K. (3.15) is valid for Ψ replaced by Ψ_m, since

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi_m = (a(\mathbf{k}_1) + \theta_1)...(a(\mathbf{k}_n) + \theta_n)\Psi_m$$
$$= \sum_{l=0}^n \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \theta_{p_1}...\theta_{p_l} a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)\Psi_m.$$

Note that $|x|^l\Psi_m \to |x|^l\Psi$ and $((H_{\rm f} + 1)^n + |x|^{2n})|x|^l\Psi_m \to ((H_{\rm f} + 1)^n + |x|^{2n})|x|^l\Psi$ strongly as $m \to \infty$, since K is sufficiently large. By Lemmas 2.4 and 2.7 there exists a subsequence $\{m'\} \subset \{m\}$ such that for almost every $(k_1, ..., k_n) \in \mathbb{R}^{3n}$,

$$b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi_{m'} \to b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi$$

and

$$a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)|x|^l\Psi_{m'} \to a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)|x|^l\Psi.$$

Thus the proof is complete. $\square$

Lemma 3.6 *Let $\Psi \in \mathcal{D}$. Then for almost every $(k_1, ..., k_n) \in \mathbb{R}^{3n}$,*

$$\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n)) \bigcap [\cap_{l=0}^n \cap_{\{p_1,..,p_l\}\subset\{1,..,n\}} D(b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_{p_1}})...b(\widehat{\mathbf{k}_{p_l}})...b(\mathbf{k}_n)|x|^l)]$$

and

$$\|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|$$
$$\leq \sum_{l=0}^n \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \prod_{j=1}^l \frac{|e\widehat{\varphi}(k_{p_j})|}{\sqrt{2\omega(k_{p_j})}} \|b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_{p_1}})...b(\widehat{\mathbf{k}_{p_l}})...b(\mathbf{k}_n)|x|^l\Psi\|_{\mathcal{H}}.$$

Proof: Note $b(\mathbf{k}_1)...b(\mathbf{k}_n) = (a(\mathbf{k}_1)-\theta_1)...(a(\mathbf{k}_n)-\theta_n)$. The lemma is proven in the similar way as Lemma 3.5. $\square$

Lemma 3.7 *Suppose that* $|x|^z\Psi \in D(N^{n/2})\cap\mathcal{D}$ *for* $z = m, m+1, ..., m+n$. *Then*

$$\sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\mathbf{k}_n)|x|^m\Psi\|_{\mathcal{H}}^2 dk_1...dk_n$$

$$\leq 2^n\sum_{l=0}^{n}\sum_{\{p_1,..,p_l\}\subset\{1,..,n\}}\left(\left\|\frac{e\widehat{\varphi}}{\sqrt{2\omega}}\right\|^2\right)^l \|\prod_{j=1}^{n-l}(N-j+1)^{1/2}|x|^{m+l}\Psi\|_{\mathcal{H}}^2 < \infty. \tag{3.16}$$

Proof: We have by Lemma 3.5 and $|\sum_{j=1}^N x_j|^2 \leq N\sum_{j=1}^N x_j^2$,

$$\sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\mathbf{k}_n)|x|^m\Psi\|_{\mathcal{H}}^2 dk_1...dk_n \leq 2^n\sum_{l=0}^{n}\sum_{\{p_1,..,p_l\}\subset\{1,..,n\}}\left(\left\|\frac{e\widehat{\varphi}}{\sqrt{2\omega}}\right\|^2\right)^l \times$$

$$\times\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)|x|^{m+l}\Psi\|_{\mathcal{H}}^2 dk_1..\widehat{dk}_{p_1}..\widehat{dk}_{p_l}..dk_n.$$

By the assumption it follows that $|x|^{m+l}\Psi \in D(N^{n/2})$. Thus we see that

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\widehat{\mathbf{k}_{p_1}})...a(\widehat{\mathbf{k}_{p_l}})...a(\mathbf{k}_n)|x|^{m+l}\Psi\|_{\mathcal{H}}^2 dk_1..\widehat{dk}_{p_1}..\widehat{dk}_{p_l}..dk_n$$

$$= \|\prod_{j=1}^{n-l}(N-j+1)^{1/2}|x|^{m+l}\Psi\|_{\mathcal{H}}^2 < \infty.$$

Hence we conclude (3.16). $\square$

We set the right hand side of (3.16) by $\mathcal{R}_{n,m}(\Psi)$, i.e.,

$$\mathcal{R}_{n,m}(\Psi) = 2^n\sum_{l=0}^{n}\sum_{\{p_1,..,p_l\}\subset\{1,..,n\}}\left(\left\|\frac{e\widehat{\varphi}}{\sqrt{2\omega}}\right\|^2\right)^l \|\prod_{j=1}^{n-l}(N-j+1)^{1/2}|x|^{m+l}\Psi\|_{\mathcal{H}}^2.$$

Lemma 3.8 *Let* $\Psi \in \mathcal{D}$. *Then we have*

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|_{\mathcal{H}}^2 dk_1...dk_n$$

$$\leq 2^n\left\{\sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi\|_{\mathcal{H}}^2 dk + \sum_{l=1}^{n}\sum_{\{p_1,..,p_l\}\subset\{1,..,n\}}\left(\left\|\frac{e\widehat{\varphi}}{\sqrt{2\omega}}\right\|^2\right)^l \mathcal{R}_{n-l,l}(\Psi)\right\}.$$

Proof: We have by Lemma 3.6,

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi\|_{\mathcal{H}}^2 dk_1...dk_n \leq 2^n \sum_{l=0}^{n} \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \left(\left\| \frac{e\widehat{\varphi}}{\sqrt{2\omega}} \right\|^2 \right)^l \times$$

$$\times \int \|b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_{p_1}})...b(\widehat{\mathbf{k}_{p_l}})...b(\mathbf{k}_n)|x|^l\Psi\|_{\mathcal{H}}^2 dk_1..\widehat{dk}_{p_1}..\widehat{dk}_{p_l}..dk_n. \tag{3.17}$$

The term with $l = 0$ in (3.17) is just $\sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\Psi\|_{\mathcal{H}}^2 dk_1...dk_n$. The lemma follows from Lemma 3.7. □

Proof of Theorem 1.7

We prove the theorem by means of an induction. $\psi_g \in D(N^{1/2})$ is known. Suppose that $\psi_g \in D(N^{k/2})$ for $k = 1, ..., n-1$. Then

$$\|N^{l/2}|x|^m\psi_g\|_{\mathcal{H}} < \infty \tag{3.18}$$

follows for all $m \geq 0$ and $l \leq n-1$ by Lemma 2.12. By Lemma 3.8

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\psi_g\|_{\mathcal{H}}^2 dk_1...dk_n$$

$$\leq 2^n \left\{ \sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g\|_{\mathcal{H}}^2 dk_1...dk_n + \sum_{l=1}^{n} \sum_{\{p_1,..,p_l\}\subset\{1,..,n\}} \left(\left\| \frac{e\widehat{\varphi}}{\sqrt{2\omega}} \right\|^2 \right)^l \mathcal{R}_{n-l,l}(\Psi) \right\}.$$

By (3.18) we see that $\mathcal{R}_{n-l,l}(\Psi) < \infty$. From Lemma 3.4 it follows that

$$\sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\mathbf{k}_n)\psi_g\|_{\mathcal{H}}^2 dk_1...dk_n$$

$$\leq \delta_1 \sum_{p=1}^{n} \sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n)(|x|+1)\psi_g\|_{\mathcal{H}}^2 dk_1..\widehat{dk_p}..dk_n$$

$$+\delta_2 \sum_{p=1}^{n} \sum_{q<p} \sum\!\!\!\!\!\!\int \|b(\mathbf{k}_1)...b(\widehat{\mathbf{k}_q})...b(\widehat{\mathbf{k}_p})...b(\mathbf{k}_n)|x|^2\psi_g\|_{\mathcal{H}}^2 dk_1..\widehat{dk_q}..\widehat{dk_p}..dk_n, \tag{3.19}$$

where $\delta_1 = \int \delta_1(k)^2 dk$ and $\delta_2 = \int \delta_2(k,k')^2 dk dk'$. Then the right hand side of (3.19) is finite by Lemma 3.7. Hence

$$\sum\!\!\!\!\!\!\int \|a(\mathbf{k}_1)...a(\mathbf{k}_n)\psi_g\|_{\mathcal{H}}^2 dk_1...dk_n < \infty$$

follows, which implies that $\psi_g \in D(N^{n/2})$ by Lemma 3.2. Thus the theorem follows. □

Proof of Theorem 1.8

This follows from Theorem 1.7 and Lemma 2.11. □

4 Appendix

4.1 Appendix A

Lemma 4.1 *Let $\Psi \in D(H_{\rm f}^{n/2})$. Then there exists $\mathcal{M}(\Psi) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that*

$$\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n)) \tag{4.1}$$

for $(k_1, ..., k_n) \notin \mathcal{M}(\Psi)$. Moreover assume that $\{\Psi_m\} \subset \mathcal{C}$ satisfies that $\Psi_m \to \Psi$ and $(H_{\rm f}+1)^{n/2}\Psi_m \to (H_{\rm f}+1)^{n/2}\Psi$ strongly as $m \to \infty$. Then there exists a subsequence $\{m'\} \subset \{m\}$ and $\mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that (4.1) follows and

$$\lim_{m'\to\infty} a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m'} = a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi$$

for $(k_1, ..., k_n) \notin \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\})$.

Proof: We fix a sequence $\{\Psi_m\}$. The lemma is proven inductively. Note that

$$\|(H_{\rm f}+1)^p\Psi\| \le \|(H_{\rm f}+1)^q\Psi\| \tag{4.2}$$

for $p \le q$. By (2.1) we see that

$$\int\!\!\!\!\!\!\sum |f_1(k_1)|\|a(\mathbf{k}_1)\Psi_m\|_{\mathcal{H}} dk_1 \le \epsilon(f_1)\|(H_{\rm f}+1)^{1/2}\Psi_m\|_{\mathcal{H}} \tag{4.3}$$

for arbitrary $f_1 \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. The right hand side of (4.3) converges as $m \to \infty$ by (4.2). Then the left hand side of (4.3) is a Cauchy sequence. Then there exist $\mathbf{N}_1(\Psi) \subset \mathbb{R}^3$ with the Lebesgue measure zero and a subsequence $\{m_1\} \subset \{m\}$ such that $a(\mathbf{k}_1)\Psi_{m_1}$ converges strongly as $m_1 \to \infty$ for $k_1 \notin \mathbf{N}_1(\Psi)$. Since $a(\mathbf{k}_1)$ is closed, it follows that for $k_1 \notin \mathbf{N}_1(\Psi)$, $\Psi \in D(a(\mathbf{k}_1))$ and

$$s-\lim_{m_1\to\infty} a(\mathbf{k}_1)\Psi_{m_1} = a(\mathbf{k}_1)\Psi.$$

For Ψ_{m_1} we have by (2.1)

$$\int\!\!\!\!\!\!\sum |f_1(k_1)f_2(k_2)|\|a(\mathbf{k}_1)a(\mathbf{k}_2)\Psi_{m_1}\|_{\mathcal{H}} dk_1dk_2 \le \epsilon(f_1,f_2)\|(H_{\rm f}+1)\Psi_{m_1}\|_{\mathcal{H}}$$

for arbitrary $f_1, f_2 \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Then we also see that there exist $\mathbf{N}_2(\Psi) \subset \mathbb{R}^3 \times \mathbb{R}^3$ with the Lebesgue measure zero and a subsequence $\{m_2\} \subset \{m_1\}$ such that $a(\mathbf{k}_1)a(\mathbf{k}_2)\Psi_{m_2}$ converges strongly as $m_2 \to \infty$ for $(k_1, k_2) \notin \mathbf{N}_2(\Psi)$. Since $a(\mathbf{k}_2)\Psi_{m_2} \to$

$a(\mathbf{k}_2)\Psi$ strongly as $m_2 \to \infty$ for $k_2 \notin \mathbf{N}_1(\Psi)$ and $a(\mathbf{k}_1)$ is closed, we see that for $(k_1, k_2) \notin \mathbf{N}_2(\Psi) \cup [\mathbb{R}^3 \times \mathbf{N}_1(\Psi)]$, $a(\mathbf{k}_2)\Psi \in D(a(\mathbf{k}_1))$ and

$$s - \lim_{m_2 \to \infty} a(\mathbf{k}_1)a(\mathbf{k}_2)\Psi_{m_2} = a(\mathbf{k}_1)a(\mathbf{k}_2)\Psi.$$

Repeating this procedure we see that there exist subsets $\mathbf{N}_j(\Psi) \subset \mathbb{R}^{3j}$, $j = 1, ..., n$, with the Lebesgue measure zero and subsequences $\{m_n\} \subset \{m_{n-1}\} \subset ... \subset \{m\}$ such that for $(k_1, ..., k_n) \notin \mathbf{N}_n(\Psi)$, $a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m_n}$ converges strongly as $m_n \to \infty$ and $a(\mathbf{k}_2)\ldots a(\mathbf{k}_n)\Psi_{m_n} \to a(\mathbf{k}_2)\ldots a(\mathbf{k}_n)\Psi$ strongly as $m_n \to \infty$ for $(k_2, ..., k_n) \notin \mathbf{N}_{n-1}(\Psi) \cup [\mathbb{R}^3 \times \mathbf{N}_{n-2}(\Psi)] \cup \ldots \cup [\mathbb{R}^{3(n-2)} \times \mathbf{N}_1(\Psi)]$. Let

$$\mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) = \mathbf{N}_n(\Psi) \cup [\mathbb{R}^3 \times \mathbf{N}_{n-1}(\Psi)] \cup \ldots \cup [\mathbb{R}^{3(n-1)} \times \mathbf{N}_1(\Psi)].$$

Since $a(\mathbf{k}_1)$ is closed, we see that for $(k_1, ..., k_n) \notin \mathcal{M}_p(\Psi, \{\Psi_m\}, \{m'\})$,

$$a(\mathbf{k}_2)\ldots a(\mathbf{k}_n)\Psi \in D(a(\mathbf{k}_1))$$

and

$$s - \lim_{m_n \to \infty} a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m_n} = a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi.$$

Thus the proof is complete. $\square$

We define $\mathrm{ad}^l_A(B)$ by $\mathrm{ad}^0_A(B) = B$ and $\mathrm{ad}^l_A(B) = [A, \mathrm{ad}^{l-1}_A(B)]$. Note that on $\mathcal{F}_\omega$

$$[H_{\mathrm{f}}^p, a(\mathbf{k}_1)...a(\mathbf{k}_n)] = \sum_{l=1}^{p} \binom{p}{l} \mathrm{ad}^l_{H_{\mathrm{f}}}(a(\mathbf{k}_1)...a(\mathbf{k}_n))H_{\mathrm{f}}^{n-l},$$

$$\mathrm{ad}^l_{H_{\mathrm{f}}}(a(\mathbf{k}_1)...a(\mathbf{k}_n)) = \sum_{p_1=0}^{l} \sum_{p_2=0}^{l-p_1} \cdots \sum_{p_n=0}^{l-\sum_{i=1}^{n-1} p_i} \binom{l}{p_1} \binom{l-p_1}{p_2} \cdots \binom{l-\sum_{i=1}^{n-1} p_i}{p_n}$$
$$\times \mathrm{ad}^{p_1}_{H_{\mathrm{f}}}(a(\mathbf{k}_1))\mathrm{ad}^{p_2}_{H_{\mathrm{f}}}(a(\mathbf{k}_2)) \ldots \mathrm{ad}^{p_n}_{H_{\mathrm{f}}}(a(\mathbf{k}_n)),$$

and

$$\mathrm{ad}^p_{H_{\mathrm{f}}}(a(\mathbf{k})) = (-1)^p \omega(k)^p a(\mathbf{k}).$$

Hence we have

$$[H_{\mathrm{f}}^p, a(\mathbf{k}_1)...a(\mathbf{k}_n)] = \sum_{l=1}^{p} \binom{p}{l} \sum_{p_1=0}^{l} \sum_{p_2=0}^{l-p_1} \cdots \sum_{p_n=0}^{l-\sum_{i=1}^{n-1} p_i} \binom{l}{p_1} \binom{l-p_1}{p_2} \cdots \binom{l-\sum_{i=1}^{n-1} p_i}{p_n}$$
$$\times (-1)^l \omega(k_1)^{p_1} \omega(k_2)^{p_2} ... \omega(k_n)^{p_n} a(\mathbf{k}_1)...a(\mathbf{k}_n). \tag{4.4}$$

Lemma 4.2 *Let $\Psi \in C^\infty(H_{\rm f})$. Then there exists $\mathcal{M}_\infty(\Psi) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that for $(k_1, ..., k_n) \notin \mathcal{M}_\infty(\Psi)$,*

$$\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n))$$

and

$$a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in C^\infty(H_{\rm f}).$$

Proof: Let $\{\Psi_m\} \subset \mathcal{C}$ be such that $\Psi_m \to \Psi$ and $(H_{\rm f}+1)^q\Psi_m \to (H_{\rm f}+1)^q\Psi$ strongly as $m \to \infty$ for $q = (n/2) + p$. In particular $(H_{\rm f}+1)^{n/2}\Psi_m \to (H_{\rm f}+1)^{n/2}\Psi$ strongly as $m \to \infty$. By Lemma 4.1, there exists a subsequence $\{m'\} \subset \{m\}$ such that for $(k_1, ..., k_n) \notin \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\})$,

$$\Psi \in D(a(\mathbf{k}_1)...a(\mathbf{k}_n))$$

and

$$\lim_{m' \to \infty} a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m'} = a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi. \tag{4.5}$$

We reset m' as m. By (4.4), for $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, ..., n$,

$$\begin{aligned}
&\not\!\!\sum |\prod_{j=1}^n f_j(k_j)| \|H_{\rm f}^p a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_m\|_{\mathcal{H}} dk_1...dk_n \\
&\leq \not\!\!\sum |\prod_{j=1}^n f_j(k_j)| \|a(\mathbf{k}_1)...a(\mathbf{k}_n)H_{\rm f}^p\Psi_m\|_{\mathcal{H}} dk \\
&+ \sum_{l=1}^p \binom{p}{l} \sum_{p_1=0}^{l} \sum_{p_2=0}^{l-p_1} \cdots \sum_{p_n=0}^{l-\sum_{i=1}^{n-1} p_i} \binom{l}{p_1} \binom{l-p_1}{p_2} \cdots \binom{l-\sum_{i=1}^{n-1} p_i}{p_n} \\
&\times \not\!\!\sum |\prod_{j=1}^n f_j(k_j)\omega(k_j)^{p_j}| \|a(\mathbf{k}_1)...a(\mathbf{k}_n)H_{\rm f}^{n-l}\Psi_m\|_{\mathcal{H}} dk_1...dk_n \\
&\leq \epsilon(f_1, ..., f_n) \|(H_{\rm f}+1)^{n/2} H_{\rm f}^p \Psi_m\|_{\mathcal{H}} \\
&+ \sum_{l=1}^p \binom{p}{l} \sum_{p_1=0}^{l} \sum_{p_2=0}^{l-p_1} \cdots \sum_{p_n=0}^{l-\sum_{i=1}^{n-1} p_i} \binom{l}{p_1} \binom{l-p_1}{p_2} \cdots \binom{l-\sum_{i=1}^{n-1} p_i}{p_n} \\
&\times \epsilon(\omega^{p_1} f_1, ..., \omega^{p_n} f_n) \|(H_{\rm f}+1)^{n/2} H_{\rm f}^{(p-l)} \Psi_m\|_{\mathcal{H}} \\
&\leq C \|(H_{\rm f}+1)^{(n/2)+p}\Psi_m\|_{\mathcal{H}}
\end{aligned} \tag{4.6}$$

with some constant C. The right hand side of (4.6) converges strongly as $m \to \infty$. Since $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, ..., n$, are arbitrary, there exist $\mathbf{N}_p(\Psi) \subset \mathbb{R}^{3n}$ with the

Lebesgue measure zero and a subsequence $\{m'\} \subset \{m\}$ such that $H_{\rm f}^p a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m'}$ strongly converges as $m' \to \infty$ for $(k_1,...,k_n) \notin \mathbf{N}_p(\Psi)$. Since $H_{\rm f}^p$ is closed, we obtain by (4.5) that

$$a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in D(H_{\rm f}^p)$$

for $(k_1,...,k_n) \notin \Omega_p = \mathbf{N}_p(\Psi) \bigcup \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\})$. Define

$$\mathcal{M}_\infty(\Psi) = \bigcup_p \Omega_p.$$

Then it follows that $a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in C^\infty(H_{\rm f})$ for $(k_1,...,k_n) \notin \mathcal{M}_\infty$. $\square$

Proof of Lemma 2.4

Let $\{\Psi_m\} \subset \mathcal{C}$ be such that $\Psi_m \to \Psi$ and $((H_{\rm f}+1)^{n/2} + |x|^{2p})\Psi_m \to ((H_{\rm f}+1)^{n/2} + |x|^{2p})\Psi$ strongly as $m \to \infty$. From Lemma 4.2 it follows that for $(k_1,...,k_n) \notin \mathcal{M}_\infty(\Psi)$,

$$a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in C^\infty(H_{\rm f})$$

and from Lemma 4.1

$$s - \lim_{m' \to \infty} a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m'} = a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \tag{4.7}$$

with some subsequence $\{m'\}$ for $(k_1,...,k_n) \notin \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\})$. We reset m' as m. Let $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1,...,n$. Since $[|x|^p, a(\mathbf{k}_1)...a(\mathbf{k}_n)]\Psi_m = 0$, we have

$$\left(\not{\sum} | \prod_{j=1}^n f_j(k_j)| \| |x|^p a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_m\|_{\mathcal{H}} dk_1...dk_n \right)^2$$

$$\leq \epsilon(f_1,...,f_n)^2 \| |x|^{2p}\Psi\|_{\mathcal{H}} \|(H_{\rm f}+1)\Psi_m\|_{\mathcal{H}} \leq \epsilon(f_1,...,f_n)^2 \|((H_{\rm f}+1) + |x|^{2p})\Psi_m\|_{\mathcal{H}}^2.$$

Since the right hand side converges as $m \to \infty$, there exist $\mathbf{N}_p(\Psi)' \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero and a subsequence $\{m'\}$ such that $|x|^p a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi_{m'}$ strongly converge as $m' \to \infty$ for $(k_1,...,k_n) \in \mathbf{N}_p(\Psi)'$. Since $|x|^p$ is closed and by (4.7),

$$a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in D(|x|^p)$$

follows for $(k_1,...,k_n) \notin \Omega_p' = \mathbf{N}_p(\Psi)' \bigcup \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\})$. Then for $(k_1,...,k_n) \notin \cup_p \Omega_p'$,

$$a(\mathbf{k}_1)...a(\mathbf{k}_n)\Psi \in C^\infty(|x|).$$

Let

$$\mathcal{M}_{\mathcal{D}}(\Psi, \{\Psi_m\}, \{m'\}) = \mathcal{M}_\infty(\Psi) \bigcup [\cup_p \Omega_p'].$$

Then the lemma follows. $\square$

Proof of Lemma 2.6

Applying (2.5) instead of (2.1), we can show the lemma in the similar way as Lemmas 4.1, 4.2 and 2.4. $\square$

4.2 Appendix B

In this section we prove Lemma 2.11. In [13] we proved that e^{-tH} maps $D(N^{k/2})$ into itself for the case when $V = 0$. We extend this result for some nonzero potential V. We see that if $\psi_g \in D(N^{k/2})$ then the identity

$$N^{k/2}\psi_g = e^{-tH}e^{tE}N^{k/2}\psi_g + e^{tE}[N^{k/2}, e^{-tH}]\psi_g \tag{4.8}$$

is well defined. Using (4.8) we shall prove that $\|N^{k/2}\psi_g(x)\|_{\mathcal{F}}$ decays exponentially. To see it we prepare some probabilistic notations.

It is known that there exist a probability space (Q, μ) and Gaussian random variables $(\phi(f), f \in \oplus^3 L^2_{\text{real}}(\mathbb{R}^3))$ such that

$$\int_Q \phi(f)\phi(g)\mu(d\phi) = \frac{1}{2}\sum_{\mu,\nu=1,2,3}\int_{\mathbb{R}^3}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right)\overline{\hat{f}_\mu(k)}\hat{g}_\nu(k)dk.$$

For a general $f \in \oplus^3 L^2(\mathbb{R}^3)$, we set $\phi(f) = \phi(\Re f) + i\phi(\Im f)$. It is also known that there exists a unitary operator implementing $1 \cong \Omega$, $L^2(Q) \cong \mathcal{F}$ and $\phi(\oplus^3_{\nu=1}\delta_{\mu\nu}\lambda(\cdot - x)) \cong A_\mu(x)$, where λ is the inverse Fourier transform of

$$\hat{\lambda} = \widehat{\varphi}/\sqrt{\omega}.$$

The free Hamiltonian in $L^2(Q)$ corresponding to $H_{\rm f}$ in $\mathcal{F}$ is denoted by $\widetilde{H}_{\rm f}$. To have a functional integral representation of $e^{-t\widetilde{H}_{\rm f}}$ we go through another probability space (Q_0, ν_0) and Gaussian random variables $(\phi_0(f), f \in \oplus^3 L^2_{\text{real}}(\mathbb{R}^4))$ such that

$$\int_{Q_0}\phi_0(f)\phi_0(g)\nu_0(d\phi_0) = \frac{1}{2}\sum_{\mu,\nu=1,2,3}\int_{\mathbb{R}^4}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right)\overline{\hat{f}_\mu(k, k_0)}\hat{g}_\nu(k, k_0)dkdk_0.$$

Here $\phi_0(f)$ is also extended to $f \in \oplus^3 L^2(\mathbb{R}^4)$ such as $\phi(f)$. Let $j_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)$ be the isometry defined by

$$\widehat{j_t f}(k, k_0) = \frac{e^{-itk_0}}{\sqrt{\pi}}\sqrt{\omega(k)/(\omega(k)^2 + |k_0|^2)}\hat{f}(k)$$

and $J_t : L^2(Q) \to L^2(Q_0)$ by

$$J_t :\phi(f_1)\ldots\phi(f_n):=:\phi_0([\oplus^3 j_t]f_1)\ldots\phi_0([\oplus^3 j_t]f_n):,$$

$$J_t 1 = 1.$$

Here $:X:$ denotes the Wick power of X inductively defined by

$$:\phi_*(f):= \phi_*(f),$$

$$:\phi_*(f)\phi_*(f_1)\ldots\phi_*(f_n):= \phi_*(f)\ :\phi_*(f_1)\ldots\phi_*(f_n):$$
$$-\sum_{j=1}^n (\phi_*(f_j), \phi_*(f))_{L^2(Q_*)}\ :\phi_*(f_1)\ldots\widehat{\phi_*(f_j)}\ldots\phi_*(f_n):,$$

where $Q_* = Q, Q_0$ and $\phi_* = \phi, \phi_0$. Then J_t can be extended to an isometry and $J_t^* J_s = e^{-|t-s|\widetilde{H}_{\mathrm{f}}}$ follows for $t, s \in \mathbb{R}$. We identify $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$ with $L^2(\mathbb{R}^3; L^2(Q))$. Under this identification $\Psi \in \mathcal{H}$ can be regarded as $L^2(Q)$-valued L^2-function on $\mathbb{R}^3$, i.e., $\Psi(x) \in L^2(Q)$ for almost every $x \in \mathbb{R}^3$. In [13, Lemma 4.9] we established that

$$\left(e^{-tH}\Psi\right)(x) = \mathbb{E}_x^Q \left(e^{-\int_0^t V(X_s)ds} \mathcal{J}_t \Psi(X_t)\right)$$

for almost every $x \in \mathbb{R}^3$. Here $(X_t)_{t\geq 0} = (X_{1,t}, X_{2,t}, X_{3,t})_{t\geq 0} \in C([0,\infty); \mathbb{R}^3)$ denotes an $\mathbb{R}^3$-valued continuous path, $\mathbb{E}_x^Q$ an $L^2(Q)$-valued expectation value with respect to the wiener measure P_x on $C([0,\infty); \mathbb{R}^3)$ with $P_x(X_0 = x) = 1$, and

$$\mathcal{J}_t = \mathcal{J}_t(x, X.) : L^2(Q) \to L^2(Q)$$

is given by

$$\mathcal{J}_t = J_0^* e^{-ie\phi_0(K(x,X.))} J_t,$$

where $K(x, X.)$ is a $\oplus^3 L^2(\mathbb{R}^4)$-valued stochastic integral defined by

$$K = K(x, X.) = \oplus_{\mu=1,2,3} \int_0^t j_s \lambda(\cdot - X_s) dX_{\mu,s}.$$

Let N and N_0 be the number operators in $L^2(Q)$ and $L^2(Q_0)$, respectively. Note that

$$J_t N = N_0 J_t$$

on a dense domain. The expectation value with respect to P_x is denoted by $\mathbb{E}_x$. We show a fundamental inequality.

Lemma 4.3 *Let $\xi = \xi(x, X.) = \|K(x, X.)\|_{\oplus^3 L^2(\mathbb{R}^4)}$. Then, for all $m \geq 0$,*

$$\mathbb{E}_x\left(\xi^{2m}\right) \leq \frac{3(2m)!}{2^m} t^{m-1} \mathbb{E}_x\left(\int_0^t \|j_s\lambda(\cdot - X_s)\|^{2m}_{L^2(\mathbb{R}^4)} ds\right) = \frac{3(2m)!}{2^m} t^m \|\widehat{\varphi}/\sqrt{\omega}\|^{2m}. \quad (4.9)$$

In particular $\sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left(\xi^{2m}\right) < \infty$.

Proof: See [13, Theorem 4.6]. □

Lemma 4.4 *For each $(x, X.) \in \mathbb{R}^3 \times C([0,\infty);\mathbb{R}^3)$ and $\Psi \in D(N^{k/2})$,*

$$\|[N^{k/2}, \mathcal{J}_t(x, X.)]\Psi\|_{L^2(Q)} \leq P_k(\xi)\|(N+1)^{k/2}\Psi\|_{L^2(Q)},$$

with some polynomial $P_k(\cdot)$.

Proof: Note that for each $(x, X.)$, $\mathcal{J}_t = \mathcal{J}_t(x, X.)$ maps $D(N^{k/2})$ into itself. We have

$$\begin{aligned}
[N^{k/2}, \mathcal{J}_t]\Psi &= J_0^* e^{-ie\phi_0(K)}[e^{ie\phi_0(K)} N_0^{k/2} e^{-ie\phi_0(K)} - N_0^{k/2}] J_t\Psi \\
&= J_0^* e^{-ie\phi_0(K)} \left\{\left(N_0 - e\phi_0'(K) + \frac{e^2}{2}\xi\right)^{k/2} - N_0^{k/2}\right\} J_t\Psi \\
&= -\mathcal{J}_t N^{k/2}\Psi + J_0^* e^{-ie\phi_0(K)} \left\{\left(N_0 - e\phi_0'(K) + \frac{e^2}{2}\xi\right)^{k/2}\right\} J_t\Psi,
\end{aligned}$$

where $\phi_0'(K) = i[N_0, \phi_0(K)]$. We see that

$$\|\mathcal{J}_t N^{k/2}\Psi\|_{L^2(Q)} \leq \|N^{k/2}\Psi\|_{L^2(Q)}.$$

Note that

$$\|\phi_0(K)\Psi\| \leq \sqrt{2}\xi\|(N_0+1)^{1/2}\Psi\|.$$

Then it is obtained that

$$\|\left(N_0 - e\phi_0'(K) + \frac{e^2}{2}\xi\right)^k J_t\Psi\|_{L^2(Q)} \leq R_k(\xi)\|(N+1)^k\Psi\|_{L^2(Q)}$$

with some polynomial $R_k(\cdot)$. Then

$$\begin{aligned}
\|[N^{k/2}, \mathcal{J}_t]\Psi\|_{L^2(Q)} &\leq R_k(\xi)\|(N+1)^{k/2}\Psi\|_{L^2(Q)} + \|N^{k/2}\Psi\|_{L^2(Q)} \\
&\leq (R_k(\xi)+1)\|(N+1)^{k/2}\Psi\|_{L^2(Q)}.
\end{aligned}$$

Thus the proof is complete. □

Proposition 4.5 *Let $1 \le p \le \infty$ and $a \ge 0$. Then there exists a constant $c_p = c_p(a)$ such that*

$$\sup_{x\in\mathbb{R}^3} \left| \mathbb{E}_x \left(e^{-a\int_0^t V(X_s)ds} f(X_t) \right) \right| \le c_p \|f\|_{L^p(\mathbb{R}^3)}. \tag{4.10}$$

Proof: See [22, Theorem B.1.1]. □

Lemma 4.6 *We see that e^{-tH} maps $D(N^{k/2})$ into itself.*

Proof: Let $\Phi, \Psi \in D(N^{k/2})$. We have

$$(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}} = \int \left((N^{k/2}\Phi)(x), \mathbb{E}_x^Q \left(e^{-\int_0^t V(X_s)ds} \mathcal{J}_t \Psi(X_t) \right) \right)_{L^2(Q)} dx$$

$$= \int \mathbb{E}_x \left\{ \left(N^{k/2}\Phi(x), \mathcal{J}_t\Psi(X_t) \right)_{L^2(Q)} e^{-\int_0^t V(X_s)ds} \right\} dx.$$

Then

$$(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}} = \int \mathbb{E}_x \left\{ \left(\Phi(x), \mathcal{J}_t N^{k/2}\Psi(X_t) \right)_{L^2(Q)} e^{-\int_0^t V(X_s)ds} \right\} dx$$

$$+ \int \mathbb{E}_x \left\{ \left(\Phi(x), [N^{k/2}, \mathcal{J}_t]\Psi(X_t) \right)_{L^2(Q)} e^{-\int_0^t V(X_s)ds} \right\} dx.$$

Hence we have by Lemma 4.4

$$|(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}}| \le \int \mathbb{E}_x \left(e^{-\int_0^t V(X_s)ds} \|\Phi(x)\|_{L^2(Q)} \|N^{k/2}\Psi(X_t)\|_{L^2(Q)} \right) dx \tag{4.11}$$

$$+ \int \mathbb{E}_x \left(P_k(\xi) e^{-\int_0^t V(X_s)ds} \|\Phi(x)\|_{L^2(Q)} \|(N+1)^{k/2}\Psi(X_t)\|_{L^2(Q)} \right) dx. \tag{4.12}$$

The first term (4.11) is estimated as

$$(4.11) = \left(\|\Phi(\cdot)\|_{L^2(Q)}, e^{-tH_{\mathrm{p}}} \|N^{k/2}\Psi(\cdot)\|_{L^2(Q)} \right)_{L^2(\mathbb{R}^3)} \le e^{-tE_{\mathrm{p}}} \|\Phi\|_{\mathcal{H}} \|N^{k/2}\Psi\|_{\mathcal{H}},$$

where $E_{\mathrm{p}} = \inf \sigma(H_{\mathrm{p}})$. The second term (4.12) is estimated as

$$(4.12) \le \int \|\Phi(x)\|_{L^2(Q)} \times$$

$$\times \left(\mathbb{E}_x P_k(\xi)^2 e^{-2\int_0^t V(X_s)ds} \right)^{1/2} \left(\mathbb{E}_x \|(N+1)^{k/2}\Psi(X_t)\|^2_{L^2(Q)} \right)^{1/2} dx$$

$$\le \int \|\Phi(x)\|_{L^2(Q)} \left(\mathbb{E}_x P_k(\xi)^4 \right)^{1/4} \times$$

$$\times \left(\mathbb{E}_x e^{-4\int_0^t V(X_s)ds} \right)^{1/4} \left(\mathbb{E}_x \|(N+1)^{k/2}\Psi(X_t)\|^2_{L^2(Q)} \right)^{1/2} dx.$$

By Lemma 4.3 we have

$$\theta = \sup_{x\in\mathbb{R}^3} \left(\mathbb{E}_x P_k(\xi)^4\right)^{1/4} < \infty,$$

and by (4.10),

$$\eta = \sup_{x\in\mathbb{R}^3} \left(\mathbb{E}_x e^{-4\int_0^t V(X_s)ds}\right)^{1/4} < \infty.$$

Then we have

$$(4.12) \le \theta\eta \int \|\Phi(x)\|_{L^2(Q)} \left(\mathbb{E}_x \|(N+1)^{k/2}\Psi(X_t)\|^2_{L^2(Q)}\right)^{1/2} dx$$

$$\le \theta\eta \left(\int \|\Phi(x)\|^2_{L^2(Q)} dx\right)^{1/2} \left(\int \mathbb{E}_x \|(N+1)^{k/2}\Psi(X_t)\|^2_{L^2(Q)} dx\right)^{1/2}$$

$$= \theta\eta \|\Phi\|_{\mathcal{H}} \|(N+1)^{k/2}\Psi\|_{\mathcal{H}}.$$

Thus we conclude that

$$|(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}}| \le \|\Phi\|_{\mathcal{H}} \left(e^{-tE_{\mathrm{p}}} \|N^{k/2}\Psi\|_{\mathcal{H}} + \theta\eta \|(N+1)^{k/2}\Phi\|_{\mathcal{H}}\right).$$

This implies that $e^{-tH}\Psi \in D(N^{k/2})$. □

Lemma 4.7 *Assume that $\psi_g \in D(N^{k/2})$. Then $\sup_{x\in\mathbb{R}^3} \|N^{k/2}\psi_g(x)\|_{L^2(Q)} < \infty$.*

Proof: By Lemma 4.6 the identity $N^{k/2}\psi_g = e^{tE} e^{-tH} N^{k/2}\psi_g + e^{tE}[N^{k/2}, e^{-tH}]\psi_g$ is well defined, and we obtained that

$$N^{k/2}\psi_g(x) = e^{tE}\mathbb{E}^Q_x \left(e^{-\int_0^t V(X_s)ds} \mathcal{J}_t N^{k/2}\psi_g(X_t)\right) + e^{tE}\mathbb{E}^Q_x \left(e^{-\int_0^t V(X_s)ds} [N^{k/2}, \mathcal{J}_t]\psi_g(X_t)\right)$$

for almost every $x \in \mathbb{R}^3$. We see that by Lemma 4.4

$$\|N^{k/2}\psi_g(x)\|_{L^2(Q)} \le e^{tE}\mathbb{E}_x \left(e^{-\int_0^t V(X_s)ds} \|N^{k/2}\psi_g(X_t)\|_{L^2(Q)}\right) \tag{4.13}$$

$$+ e^{tE}\mathbb{E}_x \left(e^{-\int_0^t V(X_s)ds} P_k(\xi) \|(N+1)^{k/2}\psi_g(X_t)\|_{L^2(Q)}\right). \tag{4.14}$$

By (4.10) it is obtained that

$$\sup_{x\in\mathbb{R}^3} (4.13) < \infty. \tag{4.15}$$

(4.14) is estimated as

$$(4.14) \le \left(\mathbb{E}_x P_k(\xi)^2\right)^{1/2} \left(\mathbb{E}_x e^{-2\int_0^t V(X_s)ds} \|(N+1)^{k/2}\psi_g(X_t)\|^2_{L^2(Q)}\right)^{1/2}.$$

By (4.10) we yield that

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}_x \left(e^{-2\int_0^t V(X_s)ds} \|(N+1)^{k/2}\psi_g(X_t)\|^2_{L^2(Q)} \right) < \infty,$$

and by Lemma 4.3, $\sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left(P_k(\xi)^2\right) < \infty$. Hence

$$\sup_{x\in\mathbb{R}^3} (4.14) < \infty. \tag{4.16}$$

Thus the lemma follows from (4.15) and (4.16). $\square$

Proof of Lemma 2.11

It is enough to prove the lemma for sufficiently large $|x|$ by Lemma 4.7. Set $\theta = \sup_{x\in\mathbb{R}^3} \|(N+1)^{k/2}\psi_g(x)\|_{L^2(Q)} < \infty$. We have by (4.13) and (4.14) for almost every $x \in \mathbb{R}^3$

$$\begin{aligned} \|N^{k/2}\psi_g(x)\|_{L^2(Q)} &\leq \mathbb{E}_x \left(e^{-\int_0^t V(X_s)ds}(1+P_k(\xi)) \right) e^{tE}\theta \\ &\leq \left\{ \mathbb{E}_x \left((1+P_k(\xi))^2 \right) \right\}^{1/2} \left(\mathbb{E}_x e^{-2\int_0^t V(X_s)ds} \right)^{1/2} e^{tE}\theta. \end{aligned}$$

By (4.9) we have

$$\mathbb{E}_x \left((1+P_k(\xi))^2 \right) \leq Q_k(t),$$

where Q_k is some polynomial of the same degree as P_k. Then we have

$$\|N^{k/2}\psi_g(x)\|_{L^2(Q)} \leq \theta Q_k(t) e^{tE} \mathbb{E}_x \left(e^{-2\int_0^t V(X_s)ds} \right).$$

Here t is arbitrary. Take $t = t(x) = |x|^{1-m}$. Then by [7] we see that there exist positive constants D and δ such that for sufficiently large $|x|$,

$$e^{t(x)E} \mathbb{E}_x \left(e^{-2\int_0^{t(x)} V(X_s)ds} \right) \leq De^{-\delta|x|^{m+1}}.$$

In the case of $m \geq 1$ it is trivial that $Q_k(t(x)) \leq \theta'$ with some constant θ' independent of x. Hence

$$\|N^{k/2}\psi_g(x)\|_{L^2(Q)} \leq \theta\theta' De^{-\delta|x|^{m+1}}$$

follows for sufficiently large $|x|$. Thus the lemma follows for $m \geq 1$. In the case of $m = 0$, we see that $\|N^{k/2}\psi_g(x)\|_{L^2(Q)} \leq \theta Q_k(|x|)De^{-\delta|x|}$, and hence

$$\|N^{k/2}\psi_g(x)\|_{L^2(Q)} \leq \theta D' e^{-\delta'|x|}$$

follows for $\delta' < \delta$ with some constant D' for sufficiently large $|x|$. The lemma is complete. $\square$

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References

[1] A. Arai, Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, *Rev. Math. Phys.* **13** (2001), 1075–1094.

[2] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151** (1997), 455–503.

[3] A. Arai, M. Hirokawa, and F. Hiroshima, On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoff, *J. Funct. Anal.* **168** (1999), 470–497.

[4] V. Bach, J. Fröhlich, I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles, *Adv. Math.* **137** (1998), 299–395.

[5] V. Bach, J. Fröhlich, I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* **207** (1999), 249–290.

[6] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos and H. Spohn, Gibbs measure associated with particle-field system, *Rev. Math. Phys.*, **14** (2002), 173–198.

[7] R. Carmona, Pointwise bounds for Schrödinger operators, *Commun. Math. Phys.* **62** (1978), 97–106.

[8] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic electromagnetic fields in a mode of quantum-mechanical matter interacting with the quantum radiation field, *Adv. in Math.* **164** (2001), 349–398.

[9] C. Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, *Ann. Henri Poincaré* **1** (2000), 443–459.

[10] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* **145** (2001), 557–595.

[11] L. Gross, The relativistic Polaron without cutoffs, *Commun. Math. Phys.* **31** (1973), 25–73.

[12] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics I, *J. Math. Phys.* **40** (1999), 6209–6222, II, *J. Math. Phys.* **41** (2000), 661–674.

[13] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585–613.

[14] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré*, **3** (2002), 171–201.

[15] F. Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation fields, to be published in *Int. J. Mod .Phys. B.*

[16] F. Hiroshima and H. Spohn, Enhanced binding through coupling to a quantum field, *Ann. Henri Poincaré* **2** (2001), 1159–1187.

[17] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, *Adv. Theor. Math. Phys.* **5** (2001), 1091–1104.

[18] C. Hainzl, V. Vougalter and S. A. Vugalter, Enhanced binding in non-relativistic QED, mp-arc 01-455, preprint, 2001.

[19] J. Lőrinczi, R. A. Minlos and H. Spohn, The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3** (2001), 1–28.

[20] J. Lőrinczi, R. A. Minlos and H. Spohn, Infrared regular representation of the three dimensional massless Nelson model, *Lett. Math. Phys.* **59** (2002), 189–198.

[21] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[22] B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526. *J. Funct. Anal.* **32** (1979), 97–101.

[23] A. Sloan, The polaron without cutoffs in two space dimensions, *J. Math. Phys.* **15** (1974), 190–201.

[24] H. Spohn, Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* **44** (1998), 9–16.

[25] H. Spohn, Ground state(s) of the spin-boson Hamiltonian, *Commun. Math. Phys.* **123** (1989), 277–304.