TOPOLOGICAL CONJUGACY BETWEEN APERIODIC TILING DYNAMICAL SYSTEMS

by

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Abstract

We extend to certain aperiodic tiling dynamical systems associated with an underlying substitution a geometric invariant for topological conjugacy.

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I. Introduction

We are concerned with the geometric symmetry of certain tilings of d-dimensional Euclidean space $\mathbb{E}^d$ by congruent copies, called "tiles", of "prototiles" drawn from some given finite collection $\mathcal{B}$. We assume each prototile is compact, connected, has dense interior, and boundary of zero volume. (The prototiles may incoroporate "colors" represented on the prototiles by embedded curves [HRS].) We are mainly interested here in the set $X^{ft}(\mathcal{B})$ of all possible tilings of $\mathbb{E}^d$ by such tiles. Together with the natural action of the Euclidean group $\mathcal{G}^d$ of $\mathbb{E}^d$ this is called a "finite type" tiling system, by analogy with the older subject of subshifts of finite type. There is a natural metrizable topology on $X^{ft}(\mathcal{B})$ in which $X^{ft}(\mathcal{B})$ is compact and such that the action of $\mathcal{G}^d$ is continuous. As a general reference to the terminology we refer to [HRS].

There are several known ways of producing interesting collections of prototiles. We are concerned here with a method based on substitution tilings. In this method one begins with a finite collection $\mathcal{A}$ of prototiles in $\mathbb{E}^d$ and a substitution function which generates a closed, Euclidean invariant subset $X^{sub}(\mathcal{A}) \subset X^{ft}(\mathcal{A})$ of "substitution tilings", each with a unique hierarchical structure. Under weak conditions the natural action of $\mathcal{G}^d$ on $X^{sub}(\mathcal{A})$ is uniquely ergodic.

Starting from some $X^{sub}(\mathcal{A})$ there is a rather general constructive theory which produces a set $\mathcal{B}(\mathcal{A})$ of prototiles such that: i) $X^{ft}(\mathcal{B}(\mathcal{A}))$ is uniquely ergodic; ii) there is a natural semi-conjugacy $\phi : X^{ft}(\mathcal{B}(\mathcal{A})) \to X^{sub}(\mathcal{A})$ for the actions of $\mathcal{G}^d$, which is bicontinuous off certain subsets of measure zero, the so-called "tilings with infinite level supertiles". The most general proof is by Goodman-Strauss [GS].

Our main result is that topological conjugacies between finite type tiling systems produced by a version of the Goodman-Strauss construction are sliding block codes. Thus a certain geometric feature of a substitution tiling system, the group of relative orientations, which was proven in [HRS] to be an invariant for topological conjugacy between substitution tiling systems, can be lifted to the finite type tiling systems associated to them by the construction of Goodman-Strauss. This gives an effective invariant for topological conjugacy between interesting finite type tiling systems. For finite type tiling systems, such as the Penrose kite & dart tilings of $\mathbb{E}^2$, in which the tiles in each tiling only appear in finitely many orientations, the relative orientation group shows up as a symmetry of the spectral measure, and is therefore an invariant not just of topological conjugacy but even of metric conjugacy. (Recall that we are dealing with uniquely ergodic systems.) The value of our main result is that it gives a (topological) conjugacy invariant among certain (aperiodic) finite type tiling systems such as the pinwheel systems, which are harder to classify than the tiling systems with finite orientations. For instance it shows that the finite type versions of the (1,2)-pinwheel and (3,4)-pinwheel are not topologically conjugate, an open problem in [HRS].

We begin with a lemma about tiling spaces with hierarchy. We omit the proof, which is similar to that in [P] or [RS].

Lemma. If a tiling system has the property that each tile in a tiling belongs to a unique supertile at every finite level then the set, of those tilings which contain an infinite level supertile boundary, has measure zero with respect to any finite invariant Borel measure.

Our main result is the following.

Theorem. The finite type tilings produced by the method of Goodman-Strauss [GS] support a substitution in the sense of [HRS]. If two such finite type tiling systems are topologically conjugate under the natural action of $\mathcal{G}^d$ then their relative orientation groups are inner isomorphic.

Proof. Let σ be a substitution rule on a set $\mathcal{A}$ of prototiles and let X_σ be the associated substitution tiling system. We assume that σ satisfies the hypotheses of both [HRS] and [GS].

We first show that a version of the Goodman-Strauss construction [GS] produces labeled tiles with a well-defined substitution rule. The construction labels each tile in a tiling with certain information about its position in the hierarchy, the supertile, edge and vertex packets. We take as our set $\mathcal{B}$ all labeled tiles which appear in tilings of X_σ. Let λ be the labeling function, taking tilings in X_σ to their labeled versions in $X^{ft}(\mathcal{B})$. The nature of the labeling is such that λ intertwines the actions of $\mathbb{E}^d$. It is easy to see that all the packet data for the first-level children of a tile can be inferred from that of the tile. We may thus define a substitution σ' on $\mathcal{B}$, geometrically the same as σ, but also passing packet data, such that $\lambda \circ \sigma = \sigma' \circ \lambda$.

The author remarks in [GS], Subsection 3.1, that the construction of matching rules works for this choice of $\mathcal{B}$. Let us call the system generated by matching rules $X^{mr}(\mathcal{B})$ (it is conjugate to a finite type system though presenting it as such may require modifying the set of prototiles). By adding restrictions we may assume $X^{mr}(\mathcal{B})$ has finite local complexity. According to Section 4, tilings in $X^{mr}(\mathcal{B})$ have hierarchy: Every tile in a tiling belongs to a unique supertile at every level, where the patch of labeled tiles in an nth-level supertile is the labeled patch of tiles for an nth-level supertile of a tiling in X_σ, or equivalently, the patch of labeled tiles for some nth-level supertile in $X_{\sigma'}$.

Intuitively, $X_\sigma, X_{\sigma'}$ and $X^{mr}(\mathcal{B})$ are the same space except for infinite level tile boundaries. Let $E_\sigma, E_{\sigma'}$ and $E^{mr}(\mathcal{B})$ be the sets of tilings in $X_\sigma, X_{\sigma'}$ and $X^{mr}(\mathcal{B})$, respectively, which contain an infinite level tile boundary. From the above we have

$$X^{mr}(\mathcal{B}) \setminus E^{mr}(\mathcal{B}) \subset X_{\sigma'}.$$

Let $\phi : X^{ft}(\mathcal{B}) \to X^{ft}(\mathcal{A})$ be the map which simply drops labels. Hierarchy for $X^{mr}(\mathcal{B})$ implies that the restriction of ϕ to $X^{mr}(\mathcal{B}) \setminus E^{mr}(\mathcal{B})$ is a homeomorphism with image contained in $X_\sigma \setminus E_\sigma$, and thus $\phi\left(X^{mr}(\mathcal{B})\right)$ contains X_σ. By the lemma, $X^{mr}(\mathcal{B})$ is uniquely ergodic. Consider the minimal component of $X^{mr}(\mathcal{B}) \cap X_{\sigma'}$. It is easily seen to be a substitution tiling system generated by the restriction of σ' to some subset of $\mathcal{B}$, satifies the hypotheses of [HRS], and has full measure.

A conjugacy between two such finite type systems thus induces a conjugacy between embedded full-measure substitution tiling systems. By a result in [HRS], the restriction of the conjugacy to these substitution tiling systems is a sliding block code. Hierarchy then implies that the conjugacy itself is a sliding block code. ∎

References

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