Dependence on the observational time intervals and domain of convergence of orbital determination methods

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Abstract. In the framework of the orbital determination methods, we study some
properties related to the algorithms developed by Gauss, Laplace and Mossotti.
In particular, we investigate the dependence of such methods upon the size of the
intervals between successive observations, encompassing also the case of two nearby
observations performed within the same night. Moreover we study the convergence
of Gauss algorithm by computing the maximal eigenvalue of the jacobian matrix
associated to the Gauss map. Applications to asteroids and Kuiper belt objects are
considered.

Keywords: Orbital determination, Gauss method, Laplace method, Mossotti method

1. Introduction

The determination of the orbital motion of a celestial body can be
obtained through the celebrated methods of Gauss or Laplace, once a
certain number (at least 3) of astronomical observations are available.
An alternative technique was developed by O.F. Mossotti in the XIX
century. The three methods (Gauss, Laplace and Mossotti) have been
extensively reviewed and compared in (Celletti and Pinzari, 2005). In
this work we want to explore the dependence of the three techniques
upon the observational time intervals. Let $t_1$, $t_2$, $t_3$ be the times of the
three observations; having fixed the intermediate time $t_2$, we vary the
time intervals $t_2 - t_1$ and $t_3 - t_2$, ranging from a few hours (whenever
two observations are performed on the same night) to several days. Two
sets of data are investigated: the first 10000 numbered asteroids and
615 Kuiper belt objects. While in the first case Gauss method provides
the best results, the orbital determination of Kuiper belt objects seems
to privilege Laplace method, being Mossotti’s technique intermediate
in all cases. Moreover the recovery of the orbits of the asteroidal belt
improves as the time intervals decrease, while it improves within the
Kuiper belt objects whenever the time intervals increase. A statistic
of the successful results in terms of the elliptic elements (semimajor
axis, eccentricity and inclination) is also performed. In the second part of the paper we concentrate on Gauss algorithm to investigate the stability domain of such method, by looking at the eigenvalues of the jacobian matrix associated to the Gauss map. We provide a numerical investigation performed on asteroids and Kuiper belt objects. We also develop an analytical estimate of the first order computation of the largest eigenvalue; we prove a proposition ensuring the convergence of Gauss method, which is related to the contractive character of the Gauss map, at least for small values of the observational times.

2. Implementation of Gauss, Laplace and Mossotti methods

2.1. Basics of the methods

With reference to a heliocentric frame let us denote the unknown elements of the asteroid as follows: $a$ is the semi–major axis, $e$ is the eccentricity, $i$ denotes the inclination, $\omega$ is the argument of perihelion, $\Omega$ is the longitude of the ascending node and $M$ is the mean anomaly at a fixed epoch $T$. We assume that the ecliptic geocentric longitudes and latitudes, say $\lambda_i$ and $\beta_i$, $i = 1, \cdots, N$, are given through $N$ observations at times $t_i$ referred to the epoch $T$. Moreover, let $t \to \vec{a}(t)$ denote the Sun–Earth vector, $t \to \vec{r}(t)$ is the Sun–asteroid vector, while $t \to \rho(t)$ is the geocentric distance and $t \to \vec{b}(t)$ with $|\vec{b}(t)| = 1$ denotes the Earth–object direction.

We assume to perform three observations at times $t_1$, $t_2$, $t_3$. The time intervals $t_{ij} = t_j - t_i$, $i, j = 1, 2, 3$, are regarded as small quantities of order $\varepsilon$; for some positive constants $\gamma_{12}$, $\gamma_{23}$, with $\gamma_{12} + \gamma_{23} = 1$, we set

$$
\varepsilon \equiv t_{13} \quad t_{12} = \gamma_{12} \varepsilon \quad t_{23} = \gamma_{23} \varepsilon .
$$

Let $\vec{h}$ be the unit vector perpendicular to the plane of the orbit; the coplanarity condition of the vectors $\vec{r}_i = \vec{a}_i + \rho_i \vec{b}_i$, $i = 1, 2, 3$, reads as

$$
n_{23} \vec{r}_1 - n_{13} \vec{r}_2 + n_{12} \vec{r}_3 = 0 ,
$$

where $n_{ij} = \vec{r}_i \wedge \vec{r}_j \cdot \vec{h}$ is twice the oriented area of the triangle spanned by $\vec{r}_i$ and $\vec{r}_j$. If $\vec{h}_1$, $\vec{h}_2$, $\vec{h}_3$ are linearly independent, one can express $\rho_i$ as linear functions (with coefficients of $O(\varepsilon^{-2})$) of the ratios $\frac{\Delta n_{ik}}{n_{ik}}$ with $i \neq l \neq k$. The first goal of Gauss method is to find a good approximation of $\rho_i$, say up to terms of $O(\varepsilon)$. To this end, let $S_{ij}$ be the areas of the elliptic sectors spanned between $t_i$ and $t_j$, and let $n_{ij} = \frac{n_{ij}}{S_{ij}}$, $f_{ij}$ be half the angle between $\vec{r}_i$, $\vec{r}_j$. Denote by $z = (P, Q)$ a
new set of quantities, called Gauss parameters, defined as

\[ P = \frac{n_{12}}{n_{23}} = \frac{\gamma_{12}}{\gamma_{23}} f(\eta_{12}, \eta_{23}), \]  

\[ Q = 2r_2^3 \left( \frac{n_{12} + n_{23}}{n_{13}} - 1 \right) = \gamma_{12} \gamma_{23} \varepsilon^2 g \left( \eta_{12}, \eta_{23}, \frac{r_1}{r_2}, \frac{r_2}{r_3}, f_{12}, f_{23} \right), \] 

where \( f \) and \( g \) are suitable functions differing from one up to \( O(\varepsilon^2) \), \( O(\varepsilon) \) respectively (see Celletti and Pinzari, 2005). The quantities \( \rho_i \) can be expressed in terms of \( P, Q \) as

\[ \rho_2 = G_2(P, Q, \rho_2), \quad \rho_1 = G_1(P, Q, \rho_2), \quad \rho_3 = G_3(P, Q, \rho_2) \]

for suitable functions \( G_i, i = 1, 2, 3 \) (see Appendix B for explicit expressions of the \( G_i \)). In particular \( \rho_2 = G_2(P, Q) \) is a solution of an implicit equation, from which we derive \( \rho_1 = \rho_1(P, Q), \rho_3 = \rho_3(P, Q) \). Finally, setting

\[ P_0 = \gamma_{12} / \gamma_{23}, \quad Q_0 = \gamma_{12} \gamma_{23} \varepsilon^2, \]

one finds that \( G_i(P, Q, \rho_2) = G_i(P_0, Q_0, \rho_2) + O(\varepsilon) \), namely \( \rho_i = \rho_{i,0} + O(\varepsilon) \), where \( \rho_{i,0} = \rho_i(P_0, Q_0) \).

Gauss Algorithm is inductively based on the following steps:

i) start from \( z_0 = (P_0, Q_0) \);

ii) given \( z_n = (P_n, Q_n) \), compute \( \rho_{2,n} = \rho_2(P_n, Q_n) \) trying to solve the implicit equation \( \rho_{2,n} = G_2(P_n, Q_n, \rho_{2,n}) \) and let, for \( i = 1, 3 \), \( \rho_{i,n} = \rho_i(P_n, Q_n) \). The three vectors \( \vec{r}_{i,n} = \vec{a}_i + \rho_{i,n} \vec{b}_i, i = 1, 2, 3 \) are shown to be coplanar;

iii) if the endpoints of \( \vec{r}_{1,n}, \vec{r}_{2,n}, \vec{r}_{3,n} \) are not on a straight line, there exists a unique conic \( C_n \) through \( \vec{r}_{1,n}, \vec{r}_{2,n}, \vec{r}_{3,n} \); compute the quantities \( \eta_{k,j,n}, \vec{f}_{k,j,n}, \vec{r}_{i,n} \) on \( C_n \);

iv) determine the new parameters \( z_{n+1} = (P_{n+1}, Q_{n+1}) \) through (2.2), where the r.h.s. are computed with \( \eta_{k,j,n}, \vec{f}_{k,j,n}, \vec{r}_{i,n} \) replacing \( \eta_{k,j}, \vec{f}_{k,j}, \vec{r}_i \). Such procedure defines the Gauss map \( \mathcal{F}(C, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) = \mathcal{F}_G \) as \( z_{n+1} = \mathcal{F}_G(z_n) \);

v) look for a fixed point of the Gauss map, motivated by the fact that a conic section \( C \) (on which a Keplerian motion takes place) is a solution of Gauss problem if and only if it corresponds to a fixed point of \( \mathcal{F}_G \).

We can finally summarize Gauss method (Gauss, 1963, see also Gallavotti, 1980) with the following

**THEOREM 2.1.** Let \( C, t_2, \gamma_{12}, \gamma_{23}, \varepsilon \) be such that \( \vec{b}_1, \vec{b}_2, \vec{b}_3 \) are linearly independent, and \( \partial \rho \partial G_2(P, Q, \rho) \big|_{\rho_2} \neq 1 \), where \( z = (P, Q) \) is the fixed point.
point of $T_G$, defined in (2.2). Let $D$ be the domain of definition of $T_G$, $U \subset D$ a neighborhood of $z$, $V$ a neighborhood of $\rho_2$, $\rho : z' = (P', Q') \in U \to \rho(P', Q') \in V$ be the smooth solution of $\diferential$ $\rho = G_2(P, Q, \rho)$ such that $\rho(P, Q) = \rho_2$. If $z_0 \in U$, the associated conic section $C_0$ verifies $C - C_0 = O(\varepsilon)$. Finally, if $z_n \in U$, the associated conic section $C_n$ verifies $C - C_n = O(\varepsilon^{n+1})$.

A different approach is provided by Laplace method, whose aim is to find an approximation of the position $\vec{r}$ and the velocity $\vec{v}$, so to determine the unknown orbit. Let $r = r(\rho) = |\vec{a} + \rho \vec{b}|$ be the heliocentric distance; using the equations of motion, one gets an implicit equation in the unknowns $\lambda, \beta, \hat{\lambda}, \hat{\beta}$:
\[
\rho = \frac{d_1}{d} \left( \frac{1}{r^3} - \frac{1}{a^3} \right) \equiv L(d_1/d, \rho) . \tag{2.4}
\]
Moreover, one finds that $\dot{\rho} = \frac{d_3}{d} \left( \frac{1}{r^3} - \frac{1}{a^3} \right)$, with $d = d(\lambda, \beta, \hat{\lambda}, \hat{\beta})$, $d_1 = d_1(\lambda, \beta, \hat{\lambda}, \hat{\beta})$, $d_2 = d_2(\lambda, \beta, \hat{\lambda}, \hat{\beta})$ (see Celletti and Pinzari, 2005, for the explicit expressions of $d, d_1, d_2$). Given the $N$ observations $(\lambda_1, \beta_1), (\lambda_2, \beta_2), \cdots, (\lambda_N, \beta_N)$, Laplace method (Laplace, 1780) consists in replacing $\lambda, \hat{\lambda}$ (equivalently $\beta, \hat{\beta}$) by the derivatives of some interpolating polynomials of degree $N - 1$ obtained through the observed data $(t_1, \lambda_1), (t_2, \lambda_2), \cdots, (t_N, \lambda_N)$ (equivalently $(t_1, \beta_1), (t_2, \beta_2), \cdots, (t_N, \beta_N)$).

An alternative technique was developed by Mossotti (Mossotti, 1942) and it is based on the following procedure. Writing the coplanarity condition among $\vec{r}(t), \vec{r}_2, \vec{v}_2$ as
\[
\vec{r}(t) = T(t)\vec{r}_2 + V(t)\vec{v}_2
\]
and developing the equation of motion $\ddot{\vec{r}} = -\frac{\vec{F}}{m}$ in Taylor series with initial data $\vec{r}(t_2) = \vec{r}_2, \vec{r}(t_2) = \vec{v}_2$, one obtains
\[
T(t) = 1 - \frac{(t - t_2)^2}{2r_2^3} h(t) , \quad V(t) = (t - t_2) k(t) ,
\]
where $h(t)$ and $k(t)$ are suitable functions; if $h_i$ and $k_i$ denote their values at times $t_i$, one can show that $h_i$ and $k_i$ differ from one up to $O(\varepsilon)$. Using (2.5) computed at $t_1$ and $t_3$, one can express $\rho_2$ and $\vec{v}_2$ as
\[
\rho_2 = M(h_1, h_3, k_1, k_3, \rho_2) = M(1, 1, 1, \rho_2) + O(\varepsilon)
\]
\[
\vec{v}_2 = \vec{N}(h_1, h_3, k_1, k_3, \rho_2) = \vec{N}(1, 1, 1, \rho_2) + O(\varepsilon) ,
\]
for suitable (vector) functions \( M, \bar{N} \). In conclusion, it turns out that \( \rho_2 \) is a solution of an implicit equation, which can be solved in analogy to Gauss method.

2.2. Iteration of the Methods

A major advantage of Gauss method with respect to the others is that it provides an iterative procedure to find better approximations of the solution. On the contrary, the methods of Laplace (implemented over 3 observations) and Mossotti were originally limited to the first order approximation. However, an iterative scheme can be implemented along the following lines.

Let us consider first the method of Laplace. Let \( R(t) \) denote the remainder function of order 3 of the series expansion of \( \lambda(t) \) around \( t_2 \), namely

\[
\lambda(t) = P(t) + R(t), \quad \text{with} \quad P(t) = \lambda_0(t_1, t_2, t_3) + \lambda_1(t_2, t_3)(t - t_1) + \lambda_2(t_3)(t - t_1)^2
\]

(\text{obviously} \( R(t_2) = 0 \)). In other words, \( \lambda_2 = \hat{\lambda}(t_2) \), \( \lambda_2 = \hat{\lambda}(t_2) \) are the derivatives of the interpolating polynomial \( t \to P(t) \) of degree 2 through \( \lambda_1 - R_1, \lambda_2, \lambda_3 - R_3 \) (here, \( R_i = R(t_i) \)), at times \( t_1, t_2, t_3 \). Similarly for \( \hat{\lambda}(t_2), \hat{\lambda}(t_2) \), where the remainder functions are denoted as \( S_1, S_3 \). When \( \lambda_2, \hat{\lambda}_2, \hat{\lambda}_2 \) are expressed as functions of \( R_1, R_3, S_1, S_3 \), equation (2.4), with \( t = t_2 \), takes the form (without changing the symbol for \( L \)) \( \rho_2 = L(R_1, R_3, S_1, S_3, \rho_2) \); the first approximation \( (N = 3) \) of Laplace corresponds to take \( R_i = S_i = 0 \) \( \forall i = 1, 3 \). We are therefore led to define a sequence of remainder functions \( \hat{R}_{i,n}, S_{i,n} \) as follows,

i) Start with \( R_{1,0} = R_{3,0} = 0 \) \( (S_{1,0} = S_{3,0} = 0) \).

ii) Given \( R_{1,n}, R_{3,n} \) \( (S_{1,n}, S_{3,n}) \), let \( \hat{\lambda}_n, \hat{\lambda}_n (\hat{\lambda}_n, \hat{\lambda}_n) \) be defined as the derivatives of the interpolating polynomial \( t \to P_n(t) \) \( (t \to Q_n(t)) \) of degree 2 through \( \lambda_1 - R_{1,n}, \lambda_2, \lambda_3 - R_{3,n} \) \( (\lambda_1 - S_{1,n}, \lambda_2, \lambda_3 - S_{3,n}) \) at times \( t_1, t_2, t_3 \), respectively. Let \( d_n = d(\lambda_2, \hat{\lambda}_n, \hat{\lambda}_n, \hat{\lambda}_n) \), \( d_{0,n} = d_1(\lambda_2, \hat{\lambda}_n, \hat{\lambda}_n) \), \( d_{2,n} = d_2(\lambda_2, \hat{\lambda}_n, \hat{\lambda}_n) \). If \( d_n \neq 0 \), compute the position \( \vec{r}_{2,n} \) and the velocity \( \vec{v}_{2,n} \). Let \( C_n \) be the conic describing a Keplerian motion with initial data \( \vec{r}_{2,n}, \vec{v}_{2,n} \) (whenever the latter vectors are not parallel), and let \( t \to \lambda_n(t), t \to \beta_n(t) \) be the motion of the angles.

iii) Define \( R_{4,n}, S_{4,n} \) as the remainders of order 3 of the Taylor expansion of \( t \to \lambda_n(t), t \to \beta_n(t) \) around \( t = t_2 \). Introduce the Laplace map \( \mathcal{F}_L \) as

\[
(R_{1,n+1}, R_{3,n+1}, S_{1,n+1}, S_{3,n+1}) = \mathcal{F}_L(R_{1,n}, R_{3,n}, S_{1,n}, S_{3,n})
\]

Like for Gauss, all fixed points of \( \mathcal{F}_L \) provide a solution of the problem, while the \( n^{th} \) iteration of \( \mathcal{F}_L \) gives an approximation of the unknown or-
bit up to terms of order $O(\varepsilon^n)$, provided that $d \neq 0$, $\partial_\rho L(R_1, R_3, S_1, S_3, \rho)|_{\rho_2} \neq 1$ and $(R_{1,n}, R_{3,n}, S_{1,n}, S_{3,n})$ belongs to a suitable neighborhood of $(R_1, R_3, S_1, S_3)$. Let us now present an iterative scheme for the method developed by Mossotti. Define the sequence $h_{i,n}, k_{i,n} (i = 1, 3)$ as follows.

i) Start with $h_{i,0} = k_{i,0} = 1$.

ii) Given $h_{i,n}, k_{i,n}$, let $\vec{r}_{2,n}, \vec{v}_{2,n}$ be the vectors obtained replacing $h_i, k_i$ with $h_{i,n}, k_{i,n}$. If $\vec{r}_{2,n}, \vec{v}_{2,n}$ are not parallel, let $\mathcal{C}_n$ be the corresponding conic. Finally, let $\vec{r}_{1,n}, \vec{r}_{3,n}$ denote the positions of the same body at times $t_1, t_3$, respectively.

iii) Define $h_{i,n+1}, k_{i,n+1}$ by means of the relations

$$T_{i,n+1} = 1 - \frac{(t_i - t_2)^2}{2r_{2,n}^3} h_{i,n+1}, \quad V_{i,n+1} = (t_i - t_2) k_{i,n+1}, \quad i = 1, 3,$$

where $T_{i,n+1}, V_{i,n+1}$ are the coefficients of the linear relations providing $\vec{r}_{1,n}, \vec{r}_{3,n}$ as a combination of $\vec{r}_{2,n}, \vec{v}_{2,n}$ in analogy to (2.5). Let the Mossotti map $\mathcal{F}_M$ be defined as

$$(h_{1,n+1}, h_{3,n+1}, k_{1,n+1}, k_{3,n+1}) = \mathcal{F}_M(h_{1,n}, h_{3,n}, k_{1,n}, k_{3,n}).$$

As for the previous methods, all fixed points of $\mathcal{F}_M$ define a solution of the problem, and the $n^{th}$ iteration of $\mathcal{F}_M$ provides an approximation of the unknown orbit up to terms of order $O(\varepsilon^n)$, whenever $\partial_\rho M(h_1, h_3, k_1, k_3, \rho)|_{\rho_2} \neq 1, \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3 \neq 0$ and for $(h_{1,n}, h_{3,n}, k_{1,n}, k_{3,n})$ in a suitable neighborhood of $(h_1, h_3, k_1, k_3)$.

3. Dependence on the times of observations

In order to study the dependence on the intervals among the times of observations, we consider two samples given by the first 10000 numbered asteroids and by 615 Kuiper belt objects\(^1\). We apply Gauss, Mossotti and Laplace methods for different time intervals $t_{12}$ and $t_{23}$, where the central time $t_2$ is the real observational time as provided by the astronomical data (see footnote n. 1). Starting from the elements $(a, e, i, \omega, \Omega, M)$ at the epoch $t_2$, and given the time intervals $t_{12}$ and $t_{23}$, we compute the geocentric longitude and latitude at times $t_1, t_2, t_3$ by means of the coordinates of the object and that of the Earth (see Appendix A). Finally, we apply Gauss, Mossotti and Laplace methods,

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\(^1\) The astronomical data of the asteroids can be found on the web site “Astdys” at http://hamilton.dm.unipi.it/cgi-bin/astdys/astibo; the astronomical data of the Kuiper belt objects can be found at the ephemerides page by D. Jewitt at http://www.ifa.hawaii.edu/faculty/jewitt/kb.html.
iterating the procedure as described in the previous section until convergence is reached. In order to be sure that a given method converges in a significant range around the given time $t_2$ (and not only for the specific time $t_2$), we proceed as follows. Define $t_{ij}^n \equiv t_{ij} + n/2$, where $n = 0, \pm 1, \pm 2$; if the method converges for the above time lapses $t_{ij}^n$ and $t_{ij}^n$ ($n = 0, \pm 1, \pm 2$), then we say that the method is successful, otherwise we decide that the method fails.

We consider several choices of the time intervals $t_{ij}$ from 3 to 90 days. Moreover, to cover the case of two observations performed within the same night, we selected $t_{ij}$ of the order of some hours and $t_{ij}$ ranging from 5 to 30 days. The results are summarized in Table 1, where the first percentage refers to the asteroids, while the second number of each method refers to Kuiper belt objects. Concerning the main belt, one concludes that Gauss method provides the best result, while Mosso is more successful than Laplace; the opposite conclusion holds for the Kuiper belt objects. Moreover, the number of successful cases within the asteroidal belt increases as the time interval decreases, while (again) the opposite conclusion can be drawn for the Kuiper belt objects. As discussed in the following section, one might expect that whenever the time interval $\varepsilon$ among the observations is sufficiently small (say $\varepsilon < \varepsilon'$), Gauss method (as well as the other techniques) converges. Of course $\varepsilon$ depends on $C, \gamma_{12}, \gamma_{23}$ (and $t_2$), implying that smaller is $\varepsilon$, greater is the number of converging orbits for fixed values of $\gamma_{12}, \gamma_{23}$. On the other hand, the dependence of $\varepsilon$ on $\gamma_{12}, \gamma_{23}$ implies that $t_{ij}$ cannot be chosen too small, otherwise $C$ (as well as its approximants $C_n$) is badly determined. The latter effect is particularly relevant when the semimajor axis is large as it happens for the Kuiper’s belt (notice that the mean anomalies between two observations differ by $M_{ij} = t_{ij} a^{-3/2}$ and that the difference $\nu_{ij}$ between the true anomalies, and henceforth between the $t_{ij}$, goes to zero with $M_{ij}$).

In order to see the distribution of the previous results as functions of the semimajor axis, eccentricity and inclination, we compute the percentages of successful results of the first 10000 numbered asteroids by considering four different regions in $a, e, \iota$, each one being composed by 2500 objects. The results are provided in Table 2 for the time intervals $t_{12} = 1^d$ and $t_{23} = 5^d$ and in Table 3 for $t_{12} = t_{23} = 10^d$. We conclude that the success of all methods (slightly) grows if the semimajor axis increases. On the other hand, all methods seem to be independent on the value of the inclination, while only Laplace method is affected by the value of the eccentricity, performing better for lower eccentricities.
Table I. Percentage of successful results for Gauss, Mossoiti and Laplace methods; the first number refers to the asteroids (e.g. 99.86, first line of Gauss method), while the second to Kuiper's objects (e.g. 79.67, same line).

<table>
<thead>
<tr>
<th>$t_{12}$</th>
<th>$t_{23}$</th>
<th>Gauss</th>
<th>Mossoiti</th>
<th>Laplace</th>
</tr>
</thead>
<tbody>
<tr>
<td>3°</td>
<td>3°</td>
<td>99.86/79.67</td>
<td>99.55/92.03</td>
<td>99.00/93.33</td>
</tr>
<tr>
<td>5°</td>
<td>5°</td>
<td>99.87/93.33</td>
<td>99.45/93.98</td>
<td>98.90/93.98</td>
</tr>
<tr>
<td>10°</td>
<td>10°</td>
<td>99.78/93.98</td>
<td>99.23/94.30</td>
<td>98.73/94.63</td>
</tr>
<tr>
<td>15°</td>
<td>15°</td>
<td>99.56/94.47</td>
<td>99.27/94.47</td>
<td>98.54/94.63</td>
</tr>
<tr>
<td>30°</td>
<td>30°</td>
<td>99.45/94.63</td>
<td>99.36/94.47</td>
<td>98.17/94.63</td>
</tr>
<tr>
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<td>60°</td>
<td>98.77/94.63</td>
<td>98.41/94.63</td>
<td>96.00/94.63</td>
</tr>
<tr>
<td>90°</td>
<td>90°</td>
<td>96.80/94.63</td>
<td>96.73/94.63</td>
<td>94.32/94.63</td>
</tr>
<tr>
<td>10°</td>
<td>30°</td>
<td>99.60/94.63</td>
<td>99.45/94.63</td>
<td>98.01/94.63</td>
</tr>
<tr>
<td>5°</td>
<td>10°</td>
<td>99.82/94.47</td>
<td>99.56/94.63</td>
<td>98.63/94.63</td>
</tr>
<tr>
<td>1°</td>
<td>5°</td>
<td>99.77/73.32</td>
<td>99.72/54.79</td>
<td>98.82/93.17</td>
</tr>
<tr>
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<td>5°</td>
<td>99.87/17.40</td>
<td>99.77/78.53</td>
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<td>10°</td>
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<td>99.66/79.84</td>
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</tr>
<tr>
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<td>10°</td>
<td>99.81/53.17</td>
<td>99.67/88.62</td>
<td>98.55/94.30</td>
</tr>
<tr>
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<td>99.62/90.24</td>
<td>97.59/94.63</td>
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<tr>
<td>5°</td>
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<td>99.70/83.85</td>
<td>99.64/92.84</td>
<td>97.61/94.63</td>
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</tbody>
</table>

4. Convergence of Gauss algorithm: computation of the eigenvalues of the jacobian matrix

In the framework of theorem 2.1, we investigate whether $\mathcal{F}_G : z' = (\zeta_1, \zeta_2) \mapsto \mathcal{F}_G(z') \equiv (\mathcal{F}_{G}^{1}(z'), \mathcal{F}_{G}^{2}(z'))$ can be indefinitely iterated from the initial point $z_0$ and, eventually, if the $n$-th iterate $z_n = \mathcal{F}_G^n(z_0)$ tends to its fixed point $z = z(\zeta, t_{12}, \gamma_{12}, \gamma_{23, \epsilon})$. Let $W \subset U$ be a closed convex neighborhood of $z$; by Lagrange's theorem, if $z_1, z_2 \in W$, there exists $z_1^*, z_2^*$ belonging to the interval $(z_1, z_2)$, such that $\mathcal{F}_G(z_1) - \mathcal{F}_G(z_2) = \partial \mathcal{F}_G(z_1^*, z_2^*)(z_1 - z_2)$, where $\partial \mathcal{F}_G(z_1^*, z_2^*)$ has entries $\partial \mathcal{F}_G^1(z_1^*, z_2^*)$, for $z' = (\zeta_1, \zeta_2)$. Let us assume that the complex eigenvalues $\lambda_1(x, y), \lambda_2(x, y)$ of $\partial \mathcal{F}_G(x, y)$ verify, for $x, y \in W$

$$\lambda_1(x, y) \neq \lambda_2(x, y), \quad |\lambda_i(x, y)| \leq \theta < 1.$$  \hspace{1cm} (4.6)

For $z_1 \neq z_2 \in W$, let $\vec{v}_i \in \mathbb{C}^2$ denote the eigenvector corresponding to $\lambda_i \equiv \lambda_i(z_1^*, z_2^*)$; we define $d(z_1, z_2) \equiv |\alpha_1| + |\alpha_2|$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ are such that $z_1 - z_2 = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$. Otherwise, for $z_1 = z_2$ we set...
Table II. Percentage of successful results for Gauss, Mossotti and Laplace methods in terms of semimajor axis \(a\) (in AU), eccentricity \(e\), inclination \(i\) (in degrees). Each parameter region is composed by 2500 objects belonging to the first 10000 numbered asteroids. The time intervals are \(t_{12} = 1^h\) and \(t_{23} = 5^d\).

<table>
<thead>
<tr>
<th></th>
<th>Gauss</th>
<th>Mossotti</th>
<th>Laplace</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 \leq a &lt; 2.341)</td>
<td>99.56</td>
<td>99.04</td>
<td>97.36</td>
</tr>
<tr>
<td>(2.341 \leq a &lt; 2.6144)</td>
<td>99.96</td>
<td>99.96</td>
<td>98.48</td>
</tr>
<tr>
<td>(2.6144 \leq a &lt; 3.0053)</td>
<td>99.80</td>
<td>99.96</td>
<td>99.52</td>
</tr>
<tr>
<td>(3.0053 \leq a &lt; 100)</td>
<td>99.76</td>
<td>99.92</td>
<td>99.92</td>
</tr>
<tr>
<td>(0 \leq e &lt; 0.094)</td>
<td>99.68</td>
<td>99.92</td>
<td>99.60</td>
</tr>
<tr>
<td>(0.094 \leq e &lt; 0.140244)</td>
<td>99.92</td>
<td>99.92</td>
<td>99.56</td>
</tr>
<tr>
<td>(0.140244 \leq e &lt; 0.187321)</td>
<td>99.84</td>
<td>99.64</td>
<td>98.52</td>
</tr>
<tr>
<td>(0.187321 \leq e &lt; 1)</td>
<td>99.64</td>
<td>99.40</td>
<td>97.60</td>
</tr>
<tr>
<td>(0 \leq i &lt; 3.2185)</td>
<td>99.72</td>
<td>99.76</td>
<td>98.68</td>
</tr>
<tr>
<td>(3.2185 \leq i &lt; 6.0218)</td>
<td>99.84</td>
<td>99.56</td>
<td>98.36</td>
</tr>
<tr>
<td>(6.0218 \leq i &lt; 10.918)</td>
<td>99.72</td>
<td>99.80</td>
<td>99.08</td>
</tr>
<tr>
<td>(10.918 \leq i &lt; 360)</td>
<td>99.80</td>
<td>99.76</td>
<td>99.16</td>
</tr>
</tbody>
</table>

\(d(z_1, z_2) = 0\). With this choice of the metric, \(F_G\) becomes a contraction on \(W\), being \(F_G(z_1) - F_G(z_2) = \lambda_1 \alpha_1 \vec{v}_1 + \lambda_2 \alpha_2 \vec{v}_2\). On the other hand, one can conclude by continuity that setting \(x = y = z(C, t_2, \gamma_{12}, \gamma_{23}, \varepsilon)\), if \(\lambda_i(z, z) \equiv \lambda_i(z)\) verify

\[
\lambda_1(z) \neq \lambda_2(z) \quad (4.7)
\]

and

\[
\mu(C, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) \equiv \text{Max } i=1,2 |\lambda_i(z)| < 1 , \quad (4.8)
\]

then, there exists a suitable closed convex set \(W\) containing \(z\) where (4.6) holds, namely, \(F_G\) is a contraction. As a consequence, its unique fixed point \(z\) in \(W\) can be obtained as the limit \(z = \lim_{n \to \infty} z_n\), starting from any \(z_0 \in W\). We will see (proposition 4.1 below) that, under slightly stronger assumptions than in theorem 2.1 (see (4.9), (4.10) below), condition (4.8) is always satisfied, provided \(\varepsilon\) is small enough.

The assumptions we make are the following:

i) the vectors \(\vec{b}_2 = b(t_2), \vec{b}_2 = \vec{b}(t_2), \vec{b}_2 = \vec{b}(t_2)\) are linearly independent:

\[
\vec{b}_2 \wedge \vec{b}_2 \cdot \vec{b}_2 \neq 0 ; \quad (4.9)
\]
Table III. Percentage of successful results for Gauss, Mosotti and Laplace methods in terms of semimajor axis $a$ (in AU), eccentricity $e$, inclination $i$ (in degrees). Each parameter region is composed by 2500 objects belonging to the first 10000 numbered asteroids. The time intervals are $t_{12} = 10^4$ and $t_{23} = 10^4$.

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<td>97.28</td>
<td>96.88</td>
</tr>
<tr>
<td>$2.341 \leq a &lt; 2.6144$</td>
<td>99.80</td>
<td>99.68</td>
<td>98.40</td>
</tr>
<tr>
<td>$2.6144 \leq a &lt; 3.0053$</td>
<td>99.88</td>
<td>99.96</td>
<td>99.72</td>
</tr>
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<td>100</td>
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</tr>
<tr>
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<td>99.80</td>
<td>95.56</td>
</tr>
<tr>
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<td>99.08</td>
<td>99.08</td>
</tr>
</tbody>
</table>

ii) setting $\tilde{a}_2 = \tilde{a}(t_2)$, one has

$$
\mathcal{D} \equiv 3 \frac{\tilde{b}_2 \wedge \tilde{\gamma}_2 \cdot \tilde{\gamma}_2 \cdot \rho_2 + \tilde{a}_2 \cdot \tilde{b}_2}{r_2^2} \neq 1 . \quad (4.10)
$$

**REMARK 4.1.** The independence of the $\tilde{b}_i$'s is required by Gauss algorithm. Indeed, for $\varepsilon < 1$ let us expand in Taylor series as

$$
\tilde{b}_1 = \tilde{b}_2 - \tilde{b}_2 \gamma_1 \varepsilon + \tilde{b}_2 \frac{\gamma_2^2}{2} \varepsilon^2 + o(\varepsilon^3)
$$

$$
\tilde{b}_2 = \tilde{b}_2 + \tilde{b}_2 \gamma_2 \varepsilon + \tilde{b}_2 \frac{\gamma_3^2}{2} \varepsilon^2 + o(\varepsilon^3) ;
$$

then, by (4.9) for $\varepsilon$ small one finds that $|\tilde{b}_1 \wedge \tilde{b}_2 \wedge \tilde{b}_3| = |\frac{1}{2} \tilde{b}_2 \wedge \tilde{b}_2 \cdot \tilde{b}_2 \gamma_1 \gamma_2 \varepsilon^3 + o(\varepsilon^4)| > 0$. With a similar argument, one finds that condition (4.10) implies that for $\varepsilon$ small $\partial_{\rho_2} G_2(P, Q, \rho_2) \neq 1$ allowing to solve Gauss equation.
PROPOSITION 4.1. For any $C$, $t_2$ such that conditions (4.9), (4.10) are satisfied, one has $\mu(C, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) \to 0$ as $\varepsilon \to 0$.

The proof is given in Appendix B.

In order to prove the contractive character of $F_G$ for $0 < \varepsilon < \varepsilon'$ for a suitable $\varepsilon'$ (and, consequently, the convergence of Gauss algorithm for $0 < \varepsilon < \varepsilon'$), at least if $\varepsilon$ is so small that the initial point $z_0 = (P_0, Q_0)$, defined in (2.3), belongs to $W$), we still need the assumption (4.7). In this context we provide in Appendix B a sufficient condition (corollary B.1), based on the computation of $\lambda_1(z)$, $\lambda_2(z)$ at the first order in $\varepsilon$.

4.1. Eigenvalues of $\partial F_G(z)$

Motivated by the previous discussion and by the fact that the explicit computation of $\mu$ is extremely long, we determine numerically the elements of the jacobian matrix $\partial F_G(z)$, which yield the eigenvalues $|\lambda_1(z)|$, $|\lambda_2(z)|$. We let $t_{12}$, $t_{23}$ vary, while $t_2$ is fixed equal to a given epoch (MJ 53450 for the asteroids, while it changes for Kuiper belt objects according to the astronomical data of footnote n. 1). More precisely, for each $C$ (with related set of elements $(a, e, i, \omega, \Omega, M)$ at time $t_2$) and for each choice of $t_{12}$, $t_{23}$, we compute the three vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$. Together with the three Sun–Earth vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, we obtain the Earth–object directions $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, which provide the Gauss map $F_G$ and its fixed point $z$.

The jacobian $\partial F_G(z)$ is computed through a polynomial interpolation. Let us consider, for example, the computation of the first element $\partial_p P'(P, Q)$ (for the other derivatives, the computation is quite similar), where $F_G = (P', Q')$. Having fixed $Q$, we choose an odd number (say, $2n + 1$) of points $P_i = P + ih$, $i = -n, \cdots, n$, equally spaced and symmetrically distributed around $P$ with constant step-size $h$, such that $2nh = 0.1$. Denoting by $F_{i}$ the value of $P'$ at $z_i = (P_i, Q)$, we approximate $\partial_p P'(P, Q)$ with the quantity $\sum_{|i| \leq n, i \neq 0} \frac{(-1)^{i+1}}{h^{n-i} |n-i| !} \frac{|n+1| !}{|n-i| ! |n+i| !} F_{i}$.

The overall number of nodes is such that the difference between the values of the derivatives is smaller than 0.001 as $n$ increases to $n + 1$. The computational details are provided in Appendix C.

4.2. Eigenvalues of asteroids and Kuiper belt objects

We compute the eigenvalues of the jacobian matrix of the Gauss map, following the algorithm outlined in the previous sections. Over a sample of 100 asteroids of the main belt we found 20 objects with at least one eigenvalue greater than one. Typically the graph of the maximum eigenvalue versus the time intervals $t_{12}$ or $t_{23}$ is provided in Figure 1
Figure 1. Maximum eigenvalue of the Jacobian matrix versus the time intervals \( t_{12} \) and \( t_{23} \). Left: asteroid number 8; Right: Kuiper belt object number 12.

(left panel), where \( t_{12} \) and \( t_{23} \) are taken between 0 and 90 days with a time-step equal to 5 days. This example refers to the asteroid nr. 8, whose elements are \( a = 0.2012 \ AU, e = 0.1563, i = 5.8869^\circ, \omega = 284.9649, \Omega = 111.0326, M = 81.1258 \) at epoch MJD 53450. A similar procedure was adopted for the 615 objects of the Kuiper belt; however, contrary to the main belt objects we have not found any sample showing an eigenvalue greater than one. A typical picture of the first eigenvalue of a Kuiper belt object is provided in Figure 1 (right panel), which corresponds to the Kuiper belt object nr. 12, whose elements are \( a = 42.3035 \ AU, e = 0.2174, i = 14.0299^\circ, \omega = 236.5808, \Omega = 56.2982, M = 336.4332 \) at epoch MJD 53400.5. We remark that in both cases the graph of \(|\lambda_1|\) versus \( t_{12}, t_{23} \) is roughly symmetric with respect to the line \( t_{12} = t_{23} \), where the eigenvalue approximately attains its minimum.

Appendix

A. Computation of the longitude and latitude from the elliptic elements

We derive the ecliptic geocentric longitude and latitude from the elliptic elements, without taking into account topocentric corrections or aberrational effects. We restrict to consider \( e < 1 \). Let \( a, e, i, \omega, \Omega, M \) be the elliptic elements at a fixed reference epoch \( T = 0 \); let \( t_1, t_2, t_3 \) be the times of observations with \( t_{12} = t_2 - t_1, t_{23} = t_3 - t_2 \). The mean anomaly at time \( t_2 \) is given by \( M_2 = M(t_2) = M + nt_2 \), where \( n = ka^{-3/2} \) is the mean motion with \( k = 0.985608^\circ/\text{day} \). Similarly one has \( M_1 = M_2 - nt_{12}, M_3 = M_2 + nt_{23} \). The eccentric anomalies \( \xi_1, \xi_2, \ldots \) are...
ξ₃ at t₁, t₂, t₃ are obtained solving Kepler’s equation ξᵢ − e sin ξᵢ = Mᵢ (i = 1, 2, 3). Let \( \vec{s} = (x, y, z) \) be the coordinates of the asteroid in the orbital frame with the x axis coinciding with the perihelion line, i.e.,
\( x = a(\cos \xi − e), y = a(1 − e^2)^{1/2} \sin \xi, z = 0 \). Replacing ξ with ξ₁, ξ₂, ξ₃, one obtains the position vectors \( \vec{s}_1, \vec{s}_2, \vec{s}_3 \), which must be transformed in the ecliptic frame by means of the following three rotations:
a) a rotation of angle \( \omega \) around the z-axis;
b) a rotation of angle \( i \) around the x-axis;
c) a rotation of angle \( \Omega \) around the z-axis.

Let the resulting vectors in the ecliptic frame be denoted as \( \vec{s}^{(e)}_i \) (i = 1, 2, 3); with a similar procedure one obtains the Earth’s coordinates \( \vec{a}^{(e)}_i \) (i = 1, 2, 3). Defining the generic geocentric vectors as \( \vec{R} \equiv \vec{s}^{(e)} − \vec{a}^{(e)} \equiv (X, Y, Z) \), the longitude of \( \vec{R} \) is given by the expression \( \lambda = \tan^{-1}(Y/X) \) if \( X \geq 0 \) and \( \lambda = \tan^{-1}(Y/X) + \pi \) if \( X < 0 \), while the latitude is given by \( \beta = \sin^{-1}(Z/(X^2 + Y^2 + Z^2)^{1/2}) \).

B. Proof of proposition 4.1

In this appendix we give a proof of proposition 4.1 as a byproduct of proposition B.1 below. Moreover (see corollary B.1), we provide a sufficient condition to ensure that \( \mathcal{F}_G \) is a contraction for \( \varepsilon \) small. Let \( \mathcal{C} \) be a conic, and let \( \tilde{z} = (\tilde{P}, \tilde{Q}) \) be its Gauss parameters\(^2\). We recall that we keep \( t₂ \) fixed, while \( t₁, t₃ \) are varied; let \( \varepsilon \) be the time interval between the first and the third observation and, as in (2.1), let \( t₁ = t₂ − \gamma₁\varepsilon, \ t₃ = t₂ + \gamma₃\varepsilon \). Denote by \( \mathcal{F}_G : \ z = (P, Q) \rightarrow \ z' = (P', Q') \) the Gauss map, defined in a suitable neighborhood of \( \tilde{z} \). We want to compute the eigenvalues of the jacobian matrix of \( \mathcal{F}_G \), which we denote as \( \mathcal{J} = \mathcal{J}(\mathcal{C}, t₂, \gamma₁, \gamma₂, \gamma₃, \varepsilon) \equiv \{ \mathcal{J}_{ij} \}_{i,j=1,2} \).

PROPOSITION B.1. Fix \( t₂ \) and \( \mathcal{C} \) such that conditions (4.9), (4.10) are satisfied. Then, there exist \( \tilde{\mathcal{J}}_{11}, \tilde{\mathcal{J}}_{12}, \tilde{\mathcal{J}}_{21}, \tilde{\mathcal{J}}_{22} \) depending on \( \mathcal{C}, \ t₂, \ \gamma₁, \ \gamma₂, \ \gamma₃, \varepsilon \) such that
\[
\tilde{\mathcal{J}}_{11} = \partial_P P'(P, Q) = \tilde{\mathcal{J}}_{11}e + o(\varepsilon^2) \quad \tilde{\mathcal{J}}_{12} = \partial_Q P'(P, Q) = \tilde{\mathcal{J}}_{12} + o(\varepsilon) \\
\tilde{\mathcal{J}}_{21} = \partial_P Q'(P, Q) = \tilde{\mathcal{J}}_{21}e^2 + o(\varepsilon^2) \quad \tilde{\mathcal{J}}_{22} = \partial_Q Q'(P, Q) = \tilde{\mathcal{J}}_{22}e + o(\varepsilon^2).
\]

REMARK B.1. The eigenvalues \( \lambda₁, \lambda₂ \in \mathbb{C} \) of \( \mathcal{J} \) can be written as \( \lambda_j = i_j \varepsilon + o(\varepsilon^2) \) (j = 1, 2) with \( i_j = \tau \pm \sqrt{\tau^2 − \delta} \), where \( \tau = (\tilde{\mathcal{J}}_{11} + \tilde{\mathcal{J}}_{22})/2 \) and \( \delta = \tilde{\mathcal{J}}_{11}\tilde{\mathcal{J}}_{22} - \tilde{\mathcal{J}}_{21}\tilde{\mathcal{J}}_{12} \) are the semi-trace and determinant of \( \mathcal{J} = \{ \tilde{\mathcal{J}}_{ij} \}_{i,j=1,2} \). Moreover, if \( \mathcal{C}, \ t₂, \ \gamma₁, \ \gamma₃, \varepsilon \) are such that \( \Delta \equiv \tau^2 − \delta \neq 0 \), then, \( \lambda₁(\tilde{\varepsilon}) \neq \lambda₂(\tilde{\varepsilon}) \) for \( \varepsilon > 0 \) sufficiently small.

\(^2\) Barred quantities will refer to \( \mathcal{C} \).
COROLLARY B.1. Let \( \mathcal{C} \), \( t_2 \) verify (4.9), (4.10) and let \( \gamma_{12}, \gamma_{23} \) be chosen such that \( \Delta \neq 0 \). Then, there exists \( \varepsilon > 0 \) such that, if \( 0 < \varepsilon < \varepsilon \), the mapping \( \mathcal{F}_G : (W,d) \to \mathbb{R}^2 \) is a contraction.

Let us first recall the definition of the Gauss map, referring to (Celletti and Pinzari, 2005), for details. Let \( \rho_2(P,Q) \) be the solution of Gauss equation:

\[
\rho_2 = G_2(P,Q,\rho_2) \equiv -\bar{c}_2 \cdot \bar{a}_2 + \frac{\bar{c}_2 \cdot \bar{a}_1 + \bar{c}_2 \cdot \bar{a}_3}{P + 1} \left(1 + \frac{Q}{2r_2^3} \right),
\]

where \( \bar{c}_i = \frac{\bar{a}_i \cdot \bar{a}_k}{\bar{a}_i \cdot \bar{a}_k} \varepsilon_{jki} \), \( r_2 = |\bar{a}_2 + \rho_2 \bar{a}_2| \) and \( \varepsilon_{jki} = 1 \) if \( \{j, k, i\} \) is an even permutation of \( \{1, 2, 3\} \), \( \varepsilon_{jki} = -1 \) otherwise. Let \( \rho_1 \equiv \rho_1(P,Q) \), \( \rho_3 \equiv \rho_3(P,Q) \) be defined as

\[
\rho_1 = -\bar{c}_1 \cdot \bar{a}_1 + \frac{P + 1}{1 + \frac{Q}{2r_2^3} r_2^3} \bar{c}_1 \cdot \bar{a}_2 - P \bar{c}_1 \cdot \bar{a}_3 \equiv G_1(P,Q,\rho_2(P,Q)),
\]

\[
\rho_3 = \frac{1}{P} \bar{c}_3 \cdot \bar{a}_1 + \frac{P + 1}{P(1 + \frac{Q}{2r_2^3} r_2^3)} \bar{c}_3 \cdot \bar{a}_2 - \bar{c}_3 \cdot \bar{a}_3 \equiv G_3(P,Q,\rho_2(P,Q)).
\]

Let \( \bar{r}_i(P,Q), i = 1, 2, 3 \), be written as

\[
\bar{r}_i = \bar{r}_i(P,Q) = \bar{a}_i + \rho_i(P,Q) \bar{a}_i.
\]  

(B.12)

It can be shown (Celletti and Pinzari, 2005) that \( \bar{r}_1, \bar{r}_2, \bar{r}_3 \) are coplanar and define a unique conic \( \mathcal{C} = \mathcal{C}(P,Q) \) with a focus in their common origin. We also recall that the eccentricity \( e = e(P,Q) \) of \( \mathcal{C} \) and the argument of perihelion \( g = g(P,Q) \) are given by

\[
e = \frac{\sqrt{A^2 + B^2}}{|n_{12} + n_{23} - n_{31}|},
\]

\[
\cos g = \frac{B}{\sqrt{A^2 + B^2}} s, \quad \sin g = -\frac{A}{\sqrt{A^2 + B^2}} s,
\]  

(B.13)

where \( n_{ij} = n_{ij}(P,Q) \) is the oriented area of the triangle formed by \( \bar{r}_i, \bar{r}_j, s \equiv \text{sgn}(n_{12} + n_{23} - n_{31}) \) and, denoting by \( \nu_{ij} \equiv 2 \bar{r}_i \) the angle formed by \( \bar{r}_i, \bar{r}_j \), one has

\[
A \equiv r_2(r_3 - r_1) + r_1(r_2 - r_3) \cos \nu_{12} + r_3(r_1 - r_2) \cos \nu_{23},
\]

\[
B \equiv -r_1(r_2 - r_3) \sin \nu_{12} + r_3(r_1 - r_2) \sin \nu_{23}.
\]  

(B.14)

Moreover, let \( P, Q \) be expressed by

\[
P = \frac{n_{12}}{n_{23}}, \quad Q = 2r_2^3 \left( \frac{n_{12} + n_{23}}{n_{31}} - 1 \right)
\]  

(B.15)
and let \( \eta_{ij} = \eta_{ij}(P, Q) \) denote the ratio of the area of the triangle formed by \( r_i \) and \( r_j \) with the corresponding conic sector. The Gauss map \( \mathcal{F}_G \) is finally defined by \( z' = \mathcal{F}_G(z) \), where \( z' = (P', Q') \) takes the form

\[
P' = \frac{\gamma_{12} \eta_{12}}{\gamma_{23} \eta_{23}}, \quad Q' = \varepsilon^2 \frac{\gamma_{12} \gamma_{23} r_2^3}{r_1 r_3 \cos \gamma_{12} \cos \gamma_{23} \cos \gamma_{13}} \cdot (B.16)
\]

The proof of proposition B.1 is obtained through some technical lemmas which provide estimates of the derivatives of \( \eta_{ij}, r_i/r_j, f_{ij} \) appearing in (B.16) (assumptions (4.9), (4.10) are assumed throughout all this appendix).

**Lemma B.1.** There exist two constants \( R_P, R_Q \) depending on \( \tilde{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23} \), such that for \( i = 1, 2, 3 \) one has

\[
\partial_p \varepsilon \rho_i(\bar{P}, \bar{Q}) = R_P + o(\varepsilon), \quad \partial_q \varepsilon^2 \rho_i(\bar{P}, \bar{Q}) = R_Q + o(\varepsilon) \cdot (B.17)
\]

**Proof:** Using (4.11), (4.9) and \( \bar{a}_3 - \bar{a}_1 = \bar{a}_2 \varepsilon + o(\varepsilon^2) \), denoting for short \( \bar{B} = -2 \frac{\bar{a}_3 - \bar{a}_1}{\bar{a}_1 \bar{a}_2 \bar{a}_3 \varepsilon^2} \) and recalling that \( \bar{a}_2 = \frac{\bar{a}_3 - \bar{a}_1}{\bar{a}_1 \bar{a}_2 \bar{a}_3} \), one has

\[
\bar{a}_2 = \bar{B} + o(\varepsilon^{-1}), \quad \bar{a}_2 \cdot (\bar{a}_3 - \bar{a}_1) = \bar{B} \cdot \bar{a}_2 \varepsilon + o(1) \cdot (B.18)
\]

The implicit function theorem shows that \( \rho_2(P, Q) \) is a smooth function of \( (P, Q) \), such that

\[
\partial_p \rho_2(P, Q) = \frac{\partial_p G_2(\bar{P}, \bar{Q}, \bar{p}_2)}{1 - \partial_p G_2(\bar{P}, \bar{Q}, \bar{p}_2)} = \frac{\frac{\bar{a}_3 \cdot (\bar{a}_3 - \bar{a}_1)}{\gamma_{12} \gamma_{23} \gamma_{13} \varepsilon^2}}{1 + \frac{3\bar{a}_2 \cdot \bar{a}_3}{2\bar{a}_2} \frac{\bar{Q}}{\gamma_{12} \gamma_{23} \gamma_{13}} (\bar{p}_2 + \bar{a}_2 \cdot \bar{b}_2)} + o(\varepsilon) \cdot (B.19)
\]

An explicit expression up to \( o(1) \) is obtained using (B.18), (B.19) and the estimates for \( \bar{P}, \bar{Q} \) given by \( \bar{P} = \frac{2\gamma_{12}}{\gamma_{23}} + o(\varepsilon^2), \quad \bar{Q} = \frac{\gamma_{12} \gamma_{23} \varepsilon^2}{\gamma_{12} \varepsilon} + o(1) \):

\[
\partial_p \rho_2(P, Q) = -\frac{2}{1 - \bar{D}} \frac{\bar{b}_2 \wedge \bar{b}_2 \cdot \bar{a}_2}{\bar{b}_2 \wedge \bar{b}_2 \cdot \bar{b}_2} \gamma_{23} + o(1) \equiv R_P \frac{\varepsilon}{\gamma_{12} \varepsilon} + o(1) \cdot
gn
\]

Similar computations allow to conclude that \( \partial_p \rho_1(\bar{P}, \bar{Q}) \equiv R_P \frac{\varepsilon}{\gamma_{12} \varepsilon} + o(1) \) and that \( \partial_q \rho_2(P, Q) \equiv \frac{R_Q}{\varepsilon} + o(1) \). Concerning the derivative with respect to \( Q \), one finds that

\[
\partial_q \rho_2(P, Q) = \frac{\partial_q G_2(P, Q, \bar{p}_2)}{1 - \partial_p G_2(P, Q, \bar{p}_2)}
\]
\[
\frac{1}{2} \left( \frac{1}{2} \left[ \bar{\mathcal{R}}_2 \cdot \bar{a}_3 - \frac{\bar{\mathcal{R}}_3 \cdot (\vec{r}_3 - \vec{r}_1)}{r_{p+1}} \right] - \frac{\bar{\mathcal{R}}_4 \cdot (\vec{r}_2 + \vec{b}_2)}{2r_{p+1}^2} \right) \frac{1}{2r_{p+1}^2}
\]

and one easily finds that \( \mathcal{R}_Q \) in (B.17) takes the expression
\[
\mathcal{R}_Q \equiv \frac{1}{2} \frac{\partial \bar{\mathcal{R}}_4}{r_{p+1}^2}.
\]

As a corollary of the previous lemma we have the following result.

**Lemma B.2.** For any \( i \neq j \), there exist constants \( \mathcal{R}_P^{\bar{\mathcal{R}}_4} \), \( \mathcal{R}_Q^{\bar{\mathcal{R}}_4} \) depending on \( \bar{\mathcal{R}}_4 \), \( t_2 \), \( \gamma_{12}, \gamma_{23} \), such that \( \partial_p (r_i \cdot r_j) = \mathcal{R}_P^{\bar{\mathcal{R}}_4} + o(\varepsilon) \), \( \partial_q e (r_i \cdot r_j) = \mathcal{R}_Q^{\bar{\mathcal{R}}_4} + o(\varepsilon) \) (a similar expression is valid also for \( r_i / r_j \)).

Next we have the following

**Lemma B.3.** There exist two constants \( N_P, N_Q \) depending on \( \bar{\mathcal{R}}_4 \), \( t_2 \), \( \gamma_{12}, \gamma_{23} \), such that
\[
\partial_p n_{13}(P, Q) = N_P + o(\varepsilon), \quad \partial_q e_{13}(P, Q) = N_Q + o(\varepsilon). \quad (B.20)
\]

**Proof:** Let \( \vec{k}(P, Q) = \frac{\vec{r}_1(P, Q) \wedge \vec{r}_3(P, Q)}{\| \vec{r}_1(P, Q) \wedge \vec{r}_3(P, Q) \|} \) be a unit vector normal to the plane formed by \( \vec{r}_1, \vec{r}_2, \vec{r}_3 \). Then, \( n_{13}(P, Q) = \vec{r}_1 \wedge \vec{r}_3 \cdot \vec{k} \) and
\[
\partial_p n_{13}(P, Q) = \left( \partial_p \vec{r}_1 \wedge \vec{r}_3 \cdot \vec{k} + \vec{r}_1 \wedge \partial_p \vec{r}_3 \cdot \vec{k} \right) \bigg|_{(P, Q)}.
\]

Last term is zero, since \( \partial_p \vec{k}(P, Q) \) is perpendicular to \( \vec{k} \) and therefore it is linearly dependent with \( \vec{r}_1, \vec{r}_3 \). For the remaining terms, using (B.12) we have
\[
\left( \partial_p \vec{r}_1 \wedge \vec{r}_3 \cdot \vec{k} \right) \bigg|_{(P, Q)} = (\vec{a}_1 \wedge \vec{a}_3 \cdot \vec{k} + \vec{b}_3 \vec{b}_1 \wedge \vec{b}_3 \cdot \vec{k}) \partial_p \rho_1(P, Q),
\]
\[
\left( \vec{r}_1 \wedge \partial_p \vec{r}_3 \cdot \vec{k} \right) \bigg|_{(P, Q)} = (\vec{a}_1 \wedge \vec{b}_3 \cdot \vec{k} + \vec{b}_1 \vec{b}_3 \wedge \vec{b}_3 \cdot \vec{k}) \partial_p \rho_3(P, Q).
\]

By (4.11) the two terms in parenthesis are both equal to \( \vec{b}_2 \wedge \vec{b}_2 \cdot \vec{k} \varepsilon \) up to \( o(\varepsilon^2) \), while for the first term we remark that \( \vec{b}_2 \wedge \vec{a}_3 + \vec{a}_1 \wedge \vec{b}_3 = (\vec{a}_2 \wedge \vec{b}_2 - \vec{a}_2 \wedge \vec{b}_2) \varepsilon + o(\varepsilon^2) \). Casting together the previous formulae and using lemma B.1, we conclude that
\[
\partial_p n_{13}(P, Q) = \left( \vec{a}_2 \wedge \vec{b}_2 \cdot \vec{k} - \vec{a}_2 \wedge \vec{b}_2 \cdot \vec{k} + 2 \vec{b}_2 \vec{b}_2 \wedge \vec{b}_2 \cdot \vec{k} \right) \mathcal{R}_P + o(\varepsilon),
\]

which can be written as \( \partial_p n_{13}(P, Q) \equiv N_P + o(\varepsilon) \) for a suitable constant \( N_P \). In a similar way one obtains the second of (B.20).
REMARK B.2. Similar results hold for $\tilde{\partial} n_{12}(P, Q)$, $\tilde{\partial} n_{23}(P, Q)$. More precisely, for $(i, j) = (1, 2), (2, 3)$, one has $\partial_P n_{ij} = N_P \gamma_{ij} + o(\varepsilon)$, $\partial_Q n_{ij} = N_Q \gamma_{ij} + o(\varepsilon)$. As a consequence of lemmas B.1, B.3 and of the previous remark, a similar estimate holds for $\nu_{ij}$, being $\sin \nu_{ij} = n_{ij}/(r_i r_j)$.

LEMMA B.4. There exist two constants $S_P, S_Q$ depending on $\tilde{C}, t_2$, $\gamma_{12}, \gamma_{23}$, such that

$$\partial_P (\sin \nu_{12} + \sin \nu_{23} - \sin \nu_{13})(P, Q) = S_P \varepsilon^2 + o(\varepsilon^4),$$

$$\partial_Q (\sin \nu_{12} + \sin \nu_{23} - \sin \nu_{13})(\tilde{P}, \tilde{Q}) = S_Q \varepsilon + o(\varepsilon^3).$$

Next step is to evaluate the derivatives of the eccentricity $e(P, Q)$ of $\tilde{C}(P, Q)$.

LEMMA B.5. There exist two constants $E_P, E_Q$ depending on $\tilde{C}, t_2$, $\gamma_{12}, \gamma_{23}$, such that $\partial_P e(P, Q) = E_P + o(\varepsilon)$, $\partial_Q e(P, Q) = E_Q + o(\varepsilon)$.

Proof: From (B.13), we obtain $(\tilde{\partial} \equiv (\partial_P, \partial_Q))$: 

$$\tilde{\partial} e(P, Q) = \frac{\tilde{\partial} \sqrt{A^2 + B^2}(P, Q)}{|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}|} - \varepsilon \frac{\tilde{\partial} (n_{12} + n_{23} - n_{13})(P, Q)}{|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}|}.$$

therefore, we can take $E_P = E_P^1 - \varepsilon E_P^2$, $E_Q = E_Q^1 - \varepsilon E_Q^2$, where $E_P^i, E_Q^i$ are such that

$$\varepsilon \frac{\partial_P \sqrt{A^2 + B^2}(\tilde{P}, \tilde{Q})}{|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}|} = E_P^1 + o(\varepsilon),$$

$$\varepsilon^2 \frac{\partial_Q \sqrt{A^2 + B^2}(P, Q)}{|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}|} = E_Q^1 + o(\varepsilon) \quad (B.21)$$

and

$$\varepsilon \frac{\partial_P (n_{12} + n_{23} - n_{13})(P, Q)}{|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}|} = E_P^2 + o(\varepsilon),$$

$$\varepsilon^2 \frac{\partial_Q (n_{12} + n_{23} - n_{13})(\tilde{P}, \tilde{Q})}{|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}|} = E_Q^2 + o(\varepsilon). \quad (B.22)$$

To prove (B.21) we proceed as follows. From the second of (B.15) with $(P, Q) = (P, Q)$, one has

$$|\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}| = n_{13}\frac{\bar{Q}}{2r_2^2} = \sqrt{\bar{P}\gamma_{12}\gamma_{23}} \frac{\varepsilon^3}{2r_2^2} + o(\varepsilon^4), \quad (B.23)$$
where $\tilde{p}$ is the parameter of $\tilde{C}$ and $\tilde{n}_{13} = \sqrt{\tilde{p}} \varepsilon + O(\varepsilon^2)$. Using (B.13) one has

$$
\tilde{\theta} \sqrt{A^2 + B^2(P, Q)} = \tilde{g} \cdot \tilde{R}^\perp,
$$

(B.24)

where $R^\perp = (s B, -s A)$. Therefore we need to evaluate $\tilde{\theta} A(P, \tilde{Q})$, $\tilde{\theta} B(\tilde{P}, \tilde{Q})$. To this end, rewrite (B.14) as

$$
A = -r_1(r_2 - r_3)(1 - \cos \nu_{12}) - r_3(r_1 - r_2)(1 - \cos \nu_{23})
$$

$$
B = r_1 r_3 (\sin \nu_{12} + \sin \nu_{23} - \sin \nu_{13}) - (n_{12} + n_{23} - n_{13}),
$$

(B.25)

where we used $n_{ij} = r_i r_j \sin \nu_{ij}$. From (B.25) one has

$$
\tilde{\theta} A(\tilde{P}, \tilde{Q}) = \sum_{i=1}^{3} \tilde{A}_i,
$$

$$
\tilde{\theta} B(\tilde{P}, \tilde{Q}) = \sum_{i=1}^{3} \tilde{B}_i,
$$

where

$$
\tilde{A}_1 = -\tilde{\theta} r_1(\tilde{P}, \tilde{Q}) (\tilde{r}_2 - \tilde{r}_3)(1 - \cos \nu_{12}) - \tilde{\theta} r_3(\tilde{P}, \tilde{Q}) (\tilde{r}_1 - \tilde{r}_2)(1 - \cos \nu_{23}),
$$

$$
\tilde{A}_2 = -\tilde{r}_1 \tilde{\theta} (r_2 - r_3)(\tilde{P}, \tilde{Q}) (1 - \cos \nu_{12}) - \tilde{r}_3 \tilde{\theta} (r_1 - r_2)(\tilde{P}, \tilde{Q}) (1 - \cos \nu_{23}),
$$

$$
\tilde{A}_3 = -\tilde{\theta} (r_2 - \tilde{r}_3) \tilde{\theta} (1 - \cos \nu_{12})(\tilde{P}, \tilde{Q}) - \tilde{r}_3 (\tilde{r}_1 - \tilde{r}_2) \tilde{\theta} (1 - \cos \nu_{23})(\tilde{P}, \tilde{Q}),
$$

$$
\tilde{B}_1 = \tilde{\theta} r_1 \tilde{r}_3 (\sin \nu_{12} + \sin \nu_{23} - \sin \nu_{13}) + \tilde{r}_1 \tilde{\theta} r_3 (\sin \nu_{12} + \sin \nu_{23} - \sin \nu_{13}),
$$

$$
\tilde{B}_2 = \tilde{\theta} r_3 (\sin \nu_{12} + \sin \nu_{23} - \sin \nu_{13}),
$$

$$
\tilde{B}_3 = -\tilde{\theta} (n_{12} + n_{23} - n_{13}).
$$

Using Taylor formula for $\tilde{r}_1$, $\tilde{r}_3$, $\nu_1$, $\tilde{r}_3$ and recalling lemmas B.1, B.2, B.4, we find that for suitable constants $\tilde{A}_P$, $\tilde{A}_Q$, $\tilde{B}_P$, $\tilde{B}_Q$, one has

$$
\partial_P A(\tilde{P}, \tilde{Q}) = \tilde{A}_P \varepsilon^2 + O(\varepsilon^3),
$$

$$
\partial_Q A(\tilde{P}, \tilde{Q}) = \tilde{A}_Q \varepsilon + O(\varepsilon^2),
$$

$$
\partial_P B(\tilde{P}, \tilde{Q}) = \tilde{B}_P \varepsilon^2 + O(\varepsilon^3),
$$

$$
\partial_Q B(\tilde{P}, \tilde{Q}) = \tilde{B}_Q \varepsilon + O(\varepsilon^2)
$$

(B.26)

The proof of (B.21) is obtained casting together (B.26), (B.24) and (B.23). The proof of (B.22) is quite similar: using (B.15) we have

$$
\frac{\tilde{\theta} [n_{12} + n_{23} - n_{13}] (\tilde{P}, \tilde{Q})}{\tilde{n}_{12} + \tilde{n}_{23} - \tilde{n}_{13}} = \frac{\tilde{\theta} n_{13}(\tilde{P}, \tilde{Q})}{\tilde{n}_{13}} + \frac{\tilde{\theta} Q(\tilde{P}, \tilde{Q})}{Q} - 3 \frac{\tilde{\theta} r_2(\tilde{P}, \tilde{Q})}{\tilde{r}_2}.
$$

Therefore, by lemmas B.1, B.3, we obtain (B.22).

We remark that (B.26) allows to evaluate the derivatives of the true anomaly $\nu_2 = -\tilde{g}$; indeed, taking the gradient of $\tan \nu_2 = A/B$ (see
\[ \delta \nu_2 (\bar{P}, \bar{Q}) = \cos^2 \bar{v}_2 \left[ \frac{\beta A(P, Q)}{B} - \frac{A}{B^2} \beta B(\bar{P}, \bar{Q}) \right] = \bar{s} \frac{\cos \bar{\gamma}}{\sqrt{A^2 + B^2}} \left[ \delta A(P, Q) + \tan \bar{\gamma} \delta B(\bar{P}, \bar{Q}) \right], \]

where \( \sqrt{A^2 + B^2} = \bar{c} |\bar{n}_{12} + \bar{n}_{23} - \bar{n}_{13}| = \frac{\bar{c} \sqrt{\gamma_{31} \gamma_{32}}}{2\bar{r}_2} \epsilon^3 + o(\epsilon^4) \) (see (B.13), (B.23)). Therefore we obtain the following

**Lemma B.6.** There exist two constants \( N_P^2, N_Q^2 \) depending on \( \bar{c} \), \( t_2 \), \( \gamma_{12}, \gamma_{23} \), such that

\[ \epsilon \partial_P \nu_2 (\bar{P}, \bar{Q}) = N_P^2 + o(\epsilon), \quad \epsilon^2 \partial_Q \nu_2 (\bar{P}, \bar{Q}) = N_Q^2 + o(\epsilon). \]

Finally we are able to compute the lowest orders of the quantities \( \eta_{ij} = \eta_{ij}(P, Q) = n_{ij} / S_{ij} \) appearing in the definition of \( \mathcal{F}_G \) (see (B.16)). For simplicity we assume to deal with an elliptic trajectory, i.e. \( \bar{c} < 1 \), though the results can be extended to any value of the eccentricity. Let \( z = (P, Q) \) vary in a small neighborhood of \( \bar{z} = (\bar{P}, \bar{Q}) \). If \( \xi_i = \xi_i(P, Q) \) denotes the eccentric anomaly and if \( M_i = M_i(P, Q) = \xi_i - \epsilon \sin \xi_i \) is the mean anomaly, the quantity \( \eta_{23} \) can be expressed as

\[ \eta_{23} = \frac{\sin(\xi_3 - \xi_2) - \epsilon (\sin \xi_3 - \sin \xi_2)}{M_{23}} = 1 - \frac{\xi_{23} - \sin \xi_{23}}{M_{23}}, \]

where \( \xi_{ij} = \xi_j - \xi_i, M_{ij} = M_j - M_i \). Therefore we have

\[ \bar{\delta} \eta_{23}(\bar{P}, \bar{Q}) = - \bar{\delta} (\xi_{23} - \sin \xi_{23}) + \frac{(\xi_{23} - \sin \xi_{23})}{M_{23}^2} \bar{\delta} M_{23} \]

\[ = - \left( \frac{\xi_{23}^2}{2 M_{23}} + o \left( \frac{\xi_{23}^4}{M_{23}^2} \right) \right) \bar{\delta} \xi_{23}(\bar{P}, \bar{Q}) \]

\[ + \left( \frac{\xi_{23}^2}{6 M_{23}^2} + o \left( \frac{\xi_{23}^5}{M_{23}^2} \right) \right) \bar{\delta} M_{23}(P, Q) \]

\[ = [\epsilon \mathcal{E}_1 + o(\epsilon^3)] \bar{\delta} \xi_{23}(P, Q) + [\epsilon \mathcal{E}_2 + o(\epsilon^3)] \bar{\delta} M_{23}(P, Q) \]  

(B.27)

where we used \( M_{ij} = \gamma_{ij} \bar{c}^{-3/2} \epsilon, \xi_{ij} = M_{ij} / (1 - \epsilon \cos \xi_i) + o(\epsilon^2) \), with \( \mathcal{E}_1, \mathcal{E}_2 \) being two suitable constants. We proceed to compute \( \bar{\delta} \xi_{23}(\bar{P}, \bar{Q}), \bar{\delta} M_{23}(P, Q) \). Using the classical relations

\[ \xi_i = 2 \tan^{-1} \left( f(\epsilon) \tan \frac{\nu_i}{2} \right), \quad f(\epsilon) \equiv \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \]
and recalling lemmas B.5 and B.6, one finds that
\[
\begin{align*}
\partial_P \xi_{23}(\bar{P}, \bar{Q}) = X^{23}_P &+ o(\varepsilon), \quad \partial_Q \varepsilon \xi_{23}(\bar{P}, \bar{Q}) = X^{23}_Q + o(\varepsilon), \\
\partial_P M_{23}(\bar{P}, \bar{Q}) = M^{23}_P &+ o(\varepsilon), \quad \partial_Q \varepsilon M_{23}(\bar{P}, \bar{Q}) = M^{23}_Q + o(\varepsilon),
\end{align*}
\]
(B.28)

for some quantities \(X^{23}_Q, X^{23}_Q, M^{23}_P, M^{23}_Q\) depending only on \(\bar{c}, \gamma_{12}, \gamma_{23}\).

Inserting (B.28) in (B.27), we obtain the following

**Lemma B.7.** Let \(i \neq j \in \{1, 2, 3\}\). There exist two constants \(\mathcal{E}^{ij}_P, \mathcal{E}^{ij}_Q\) depending on \(\bar{c}, t_2, \gamma_{12}, \gamma_{23}\), such that
\[
\partial_P \eta_{ij}(P, Q) = \mathcal{E}^{ij}_P \varepsilon + o(\varepsilon^2), \quad \partial_Q \eta_{ij}(P, Q) = \mathcal{E}^{ij}_Q + o(\varepsilon).
\]

We are finally ready to complete the

**Proof of proposition B.1.** From the definition of the Gauss map (B.16), one has
\[
\begin{align*}
\bar{d} P'(\bar{P}, \bar{Q}) &= \bar{\eta}_{12} \left[ \bar{\eta}_{12}(P, Q) - \bar{\eta}_{23}(P, Q) \right] \\
\bar{d} Q'(P, Q) &= \gamma_{12} \gamma_{23} \bar{q} \left[ \varepsilon^2 \bar{d} r_2/r_3(\bar{P}, \bar{Q}) + \varepsilon^2 \bar{d} r_2/r_3(\bar{P}, \bar{Q}) \right] \\
+ \gamma_{12} \gamma_{23} \bar{q} \left[ \varepsilon^2 \frac{\bar{d} \eta_{12}(\bar{P}, \bar{Q})}{\bar{\eta}_{12}} + \varepsilon^2 \frac{\bar{d} \eta_{23}(\bar{P}, \bar{Q})}{\bar{\eta}_{23}} \right] - \gamma_{12} \gamma_{23} \bar{q} \cdot \\
\left[ \varepsilon^2 \tan f_{12} \bar{d} f_{12}(\bar{P}, \bar{Q}) + \varepsilon^2 \tan f_{23} \bar{d} f_{23}(\bar{P}, \bar{Q}) + \varepsilon^2 \tan f_{13} \bar{d} f_{13}(\bar{P}, \bar{Q}) \right],
\end{align*}
\]

where \(\bar{q} \equiv \frac{\bar{q}}{\gamma_{12} \gamma_{23} \varepsilon^2} = \frac{r_{ij}}{r_j} \bar{\eta}_{12} \bar{\eta}_{23} \frac{1}{\cos f_{12} \cos f_{23} \cos f_{13}}.\) For \(i \neq j\), let
\[
\bar{r}_{ij} = 1 + o(\varepsilon), \quad \bar{r}_j = 1 + o(\varepsilon), \quad \cos f_{ij} = 1 + o(\varepsilon^2);\) therefore \(\bar{q} = 1 + o(\varepsilon)\) and using lemma B.7 to evaluate \(\bar{d} \eta_{ij}(P, Q)\), lemma B.2 to evaluate \(\bar{d} [r_i/r_j](P, Q)\) and the remark B.2 to evaluate \(\bar{d} f_{ij}(P, Q) = \bar{d} \nu_{ij}(P, Q)/2\), we find the result of proposition B.1.

**C. Computation of the derivatives by polynomial interpolation**

Suppose we want to compute the derivative at some point \(\bar{x}\) of the function \(x \to f(x)\), using a polynomial interpolation. Let \(x_i = \bar{x} + ih,\)
\( i = -n, \ldots, n \) be the nodes around \( \bar{x} \) and let \( y_i = f(x_i) \); we define the interpolating Laplace polynomial \( P_n \) of degree \( 2n \) as

\[ P_n(x) = \sum_{i=-n}^{n} \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} y_i. \]

After the change of variable \( s = (x - \bar{x})/h \), one obtains

\[ P_n(\bar{x} + sh) = \sum_{i=-n}^{n} \frac{\prod_{j \neq i}(s - j)}{\prod_{j \neq i}(i - j)} y_i \equiv Q_n(s). \]

The derivative \( df(\bar{x})/dx \) is approximated by \( dP_n(\bar{x})/dx = h^{-1} dQ_n(0)/ds \).

Let us consider first the term with \( i = 0 \):

\[ \prod_{j \neq 0}(s - j) \prod_{j \neq 0}(-j)^{y_0} = \frac{(s - n)(s - n + 1) \cdots (s - 1)(s + 1) \cdots (s + n - 1)(s + n)}{(-1)^n(j!)^2} y_0. \]

This term is an even function of \( s \), so that its derivative at \( s = 0 \) is zero. On the other hand, deriving (through Leibnitz rule) with respect to \( s \) the remaining terms of the sum, for any \( i \neq 0 \) one has:

\[ \frac{(s - n) \cdots [s - (i + 1)][s - (i - 1)] \cdots (s - 1)(s + 1) \cdots (s + n)}{(i - n) \cdots (-1)(1) \cdots (i + n)} y_i \]

\[ = \frac{(s - n) \cdots [s - (i + 1)][s - (i - 1)] \cdots (s - 1)(s + 1) \cdots (s + n)}{(-1)^n-i(n-i)! (n+i)!} y_i; \]

computing these terms at \( s = 0 \), the only one which survives is given by

\[ \frac{(-1)^{i+1}}{i} \frac{(n!)^2}{(n-i)! (n+i)!} y_i \quad (i \neq 0). \]

Finally, one concludes that

\[ \frac{dP_n}{dx}(\bar{x}) = \frac{1}{h} \frac{dQ_n}{ds}(0) = \sum_{|i| \leq n, \ i \neq 0} \frac{(-1)^{i+1}}{h} \frac{(n!)^2}{(n-i)! (n+i)!} y_i. \]

References


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