Renormalization of Vector Fields and Diophantine Invariant Tori

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Abstract. We extend the renormalization group techniques that were developed originally for Hamiltonian flows to more general vector fields on $\mathbb{T}^d \times \mathbb{R}^\ell$. Each Diophantine vector $\omega \in \mathbb{R}^d$ determines an analytic manifold $\mathcal{W}$ of infinitely renormalizable vector fields, and each vector field on $\mathcal{W}$ is shown to have an elliptic invariant d-torus with frequencies $\omega_1, \omega_2, \ldots, \omega_d$. Analogous manifolds for particular classes of vector fields (Hamiltonian, divergence free, symmetric, reversible) are obtained simply by restricting $\mathcal{W}$ to the corresponding subspace. We also discuss nondegeneracy conditions, and the resulting reduction in the number of parameters needed in parametrized families to guarantee the existence of invariant tori.

1. Introduction

Classical KAM theory [1–3] shows that for every Diophantine vector $\omega \in \mathbb{R}^d$, there exist open sets of d-parameter families of Hamiltonian vector fields on $\mathbb{T}^d \times \mathbb{R}^d$, such that one of the family members has an invariant torus with frequency vector ω. (This applies to an individual Hamiltonian satisfying a non-degeneracy condition, if one considers the family of its translates.) Similar results have been obtained for different classes of near-linear vector fields, mostly by KAM type methods [3–6] or resummation of Lindstedt series [7–11]. Our goal is to obtain such results within the framework of renormalization transformations [12–26], and more importantly, to develop appropriate techniques for analyzing quasiperiodic motion in a large class of flows. The renormalization approach combines in a natural way the geometric and arithmetic aspects of the problem. It also applies to non-perturbative problems that are outside the reach of other known methods [20,24]. Our analysis is centered around three results that should be of independent interest: a normal form theorem for vector fields (Section 5), estimates on a multidimensional continued fractions expansion [25], and a stable manifold theorem for sequences of maps (Section 6).

A renormalization group analysis of Diophantine torus flows and/or Hamiltonian vector fields was carried out in [19,22] for $d = 2$, and more recently in [25] for $d \geq 2$. Earlier results covered a much smaller set of frequencies [16]. One of our goals is to extend the methods developed in these papers to a large class of vector field on $\mathcal{M} = \mathbb{T}^d \times \mathbb{R}^\ell$, and to do this in a way that allows for a unified treatment of Hamiltonian, divergence free, symmetric, reversible, and other types of vector fields. Some of our results are sufficiently general to be used e.g. in other problems involving renormalization. Despite the increase in scope, the analysis has in fact become simpler compared to previous work.

We note that the tori considered in this paper are elliptic, in the sense that they have zero Lyapunov exponents. It should be possible to adapt our method to hyperbolic situations, but we will not pursue this question here.

Denote by Φ_X the flow for a vector field X. In this paper, an invariant d-torus for X, with frequency vector $\omega \in \mathbb{R}^d$, is a continuous embedding Γ of $D_0 = \mathbb{T}^d \times \{0\}$ into the

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domain of X, with the property that $\Gamma\circ\Phi_K^t=\Phi_X^t\circ\Gamma$ for real times t, where $K=(\omega,0)$. Here, 0 denotes the zero vector in $\mathbb{R}^\ell$. We assume that ω satisfies a Diophantine condition

$$|\omega\cdot\nu|\geq\zeta\|\nu\|^{1-d-\beta}\,,\qquad \nu\in\mathbb{Z}\setminus\{0\}\,, \tag{1.1}$$

for some constants $\beta,\zeta>0$.

Our renormalization analysis (Sections 2,3,4) applies to vector fields that are close to K, after a change of variables, if necessary. We assume analyticity on a complex neighborhood D_ρ of D_0, characterized by the conditions $|\mathrm{Im}\,x_i|<\rho$ and $|y_j|<\rho$. We will also consider certain subclasses of vector fields, including Hamiltonian, divergence free, symmetric, and reversible vector fields. If G is a linear map on $\mathcal{M}$ that leaves D_ρ invariant, we call a vector field X on D_ρ *symmetric* with respect to G if $G^*X=X$, where $G^*X=G^{-1}X\circ G$ is the pullback of X under G. If $G\circ G$ is the identity, a vector field is called *reversible* with respect to G if $G^*X=-X$. Notice that $G^*X=\pm X$ implies that $G\circ\Phi_X^t=\Phi_X^{\pm t}\circ G$. In what follows, we will call a vector field symmetric if it is symmetric with respect to $G(x,y)=(x,-y)$, or reversible if it is reversible with respect to $G(x,y)=(-x,y)$.

Denote by A^u the space of all vector fields $Y(x,y)=(u,My+v)$, with (u,v) a vector in $\mathbb{C}^d\times\mathbb{C}^\ell$ and M a complex $\ell\times\ell$ matrix. In Section 2, we will introduce Banach spaces $\mathcal{A}_\rho$ of vector fields that are analytic on D_ρ, and a projection operator $\mathbb{P}$ from $\mathcal{A}_\rho$ onto the subspace A^u. The subspace of functions in $\mathcal{A}_\rho$ that do not depend on the coordinate $y\in\mathbb{C}^\ell$ will be denoted by $\mathcal{A}_\rho^0$. A function will be called "real" if it takes real values for real arguments.

Theorem 1.1. *Let $K=(\omega,0)$ with $\omega\in\mathbb{R}^d$ Diophantine. Given $\rho>\delta>0$, there exists an open neighborhood B of K in $\mathcal{A}_\rho$, and a real analytic map $W:(\mathrm{I}-\mathbb{P})B\to\mathbb{P}B$, satisfying $W(0)=K$ and $DW(0)=0$, such that the following holds. Let $\mathcal{W}$ be the graph of W. Then every vector field $X\in\mathcal{W}$ has an elliptic invariant torus $\Gamma_X\in\mathcal{A}_\delta^0$ with frequency vector ω. The map $X\mapsto\Gamma_X$ is real analytic on $\mathcal{W}$. The restriction of W to symmetric vector fields takes values in the subspace of symmetric vector fields, and similar statement holds for reversible, Hamiltonian, and divergence free vector fields.*

This theorem, as well as the lemma below on parametrized families, will be proved in Section 4. The size of the neighborhood B is independent of ω, given the Diophantine constants and a lower bound on the norm of ω. We note that, for any fixed $\beta>0$, the measure of the set of vectors ω that violate (1.1) approaches zero as ζ tends to zero [27].

In what follows, $\mathcal{H}_\rho$ denotes either $\mathcal{A}_\rho$, or the subspace of $\mathcal{A}_\rho$ consisting of all vector fields in a given class (divergence free, Hamiltonian, symmetric, or reversible). The intersection of A^u with $\mathcal{H}_\rho$ will be denoted by H^u.

Theorem 1.1 has an obvious corollary concerning the existence of vector fields with invariant tori in N–parameter families, where N is the dimension of H^u. In particular, any analytic family $f:B\cap H^u\to\mathcal{H}_\rho$ sufficiently close to the family $f_0(s)=K+s$ intersects the manifold $\mathcal{W}\cap\mathcal{H}_\rho$ transversally, and Theorem 1.1 yields an invariant torus Γ_X for the vector field $X=f(s)$ in the intersection.

If we are just looking for families containing a vector field with frequency vector parallel (but not necessarily equal) to ω, then the number of necessary parameters is reduced by one. A further reduction is possible for vector fields that satisfy a nondegeneracy condition, so that some directions in A^u can be generated via translations. To be more precise, let V be some proper linear subspace of $\mathbb{C}^\ell$. Let $r > \rho > 0$, and let $Z = Z(x, y)$ be a real vector field in $\mathcal{H}_r$ that does not depend on the coordinate x, and that satisfies $\mathbb{P}Z = 0$.

Given $\varepsilon > 0$, define

$$g_\varepsilon(z, v) = zK + \varepsilon\mathbb{P}J_v^* Z\,, \qquad z \in \mathbb{C}, \quad v \in V\,, \tag{1.2}$$

where $J_v(x, y) = (x, y + v)$. We assume that g_ε is non-degenerate, in the sense that the derivative $Dg_\varepsilon(0)$ is one-to-one. Let H_0^u be a linear subspace of H^u that is transversal to the range of $Dg_\varepsilon(0)$, and define

$$f_\varepsilon(s) = K + \varepsilon Z + s\,, \qquad s \in H_0^u\,. \tag{1.3}$$

We will see later that $f_\varepsilon(0)$ belongs to $\mathcal{W}$, for small $\varepsilon > 0$.

Given an open neighborhood b in some complex Banach space, denote by $\mathcal{F}(b)$ the space of all bounded analytic functions $f : b \to \mathcal{H}_r$, equipped with the sup-norm.

Lemma 1.2. *If $\varepsilon > 0$ is chosen sufficiently small, and if g_ε is non-degenerate, then, given an open neighborhood b_2 of the origin in H_0^u , there exists an open neighborhood B_2 of f_ε in $\mathcal{F}(b_2)$, such that the following holds. For every family $f \in B_2$ we can find a parameter value $s_f \in b_2$, and a nonzero complex number c_f , such that $X = c_f f(s_f)$ belongs to $\mathcal{W}$ and thus has an invariant torus $\Gamma_X \in \mathcal{A}_\delta^0$ with rotation vector ω. The maps $f \mapsto (c_f, s_f)$ and $f \mapsto \Gamma_X$ are real analytic on B_2 .*

This lemma includes cases where $Dg_\varepsilon(0)$ is onto and thus H_0^u trivial. In such a case, every vector field near $K + \varepsilon Z$ has an invariant torus with frequency vector ω. Consider e.g. the case of Hamiltonian vector fields, or symmetric vector fields with $\ell = d$. Then H^u is of dimension $\ell = d$. Taking for V some $(\ell - 1)$–dimensional subspace of $\mathbb{C}^\ell$ not containing ω, it is easy to write down examples (see e.g. below) of vector fields $Z \in \mathbb{P}\mathcal{H}_\rho$ for which g_ε is non-degenerate and thus $H_0^u = \{0\}$. Hamiltonian vector fields of this type are also called isoenergetically nondegenerate.

The following example covers several classes of vector fields.

Example. Consider a basis $\{w_1, w_2, \ldots, w_d\}$ for $\mathbb{R}^d$, with $w_d = \omega$. Let k be the minimum of $d - 1$ and ℓ. Define $X_j(x, y) = (y_j w_j, 0)$ for $1 \le j \le k$, and if $k < \ell$, define

$$X_j(x, y) = \bigl(0, (y_j - y_\ell)^2(e_j + e_\ell)\bigr)\,, \qquad X_\ell(x, y) = \bigl(0, (y_1^2 + \ldots + y_{\ell-1}^2)e_\ell\bigr)\,,$$

for $k < j < \ell$, where $\{e_1, e_2, \ldots, e_\ell\}$ denotes the standard basis for $\mathbb{R}^\ell$. Consider now a real vector field $Z = c_1X_1 + \ldots c_\ell X_\ell$, with $c_j \ne 0$ if and only if X_j belongs to $\mathcal{H}_\rho$.

In the case of general vector fields, $c_j \ne 0$ for all j, and we can choose $V = \mathbb{C}^\ell$. The resulting function g_ε is non-degenerate, and the parameter space H_0^u used in Lemma 1.2 is of dimension $d + \ell^2 - 1$. The same choice of Z and V can also be used for divergence free vector fields. In this case, H_0^u has dimension $d + \ell^2 - 2$. For reversible or Hamiltonian vector fields, $c_j = 0$ for $j > k$. Taking V to be the span of $\{e_1, \ldots, e_k\}$, we get again a non-degenerate function g_ε , and the parameter space H_0^u is of dimension $d - 1 - k$. In particular, if $k = d - 1$, then we are in the situation described above, where every vector field near $K + \varepsilon Z$ has an invariant torus with frequency vector ω.

Our proof of Theorem 1.1 and Lemma 1.2 is based on renormalization group techniques. The general idea in this approach is to take a continued fractions algorithm, acting on frequency vectors, and to "lift" it to a space of vector fields in some appropriate way. We choose a multidimensional continued fractions expansion [25] which, starting from a Diophantine vector $\omega_0 \in \mathbb{R}^d$, produces a sequence of vectors $\omega_n = \eta_n^{-1} T_n^{-1} \omega_{n-1}$, where T_n is a matrix in $\mathrm{SL}(d, \mathbb{Z})$ and η_n an appropriate normalization constant. The matrices T_n can be used e.g. to construct successive rational approximants to ω_0. Our renormalization group (RG) transformation $\mathcal{R}_n$ that corresponds to the matrix T_n^{-1} has the property that it maps $K_{n-1} = (\omega_{n-1}, 0)$ to $K_n = (\omega_n, 0)$. Other properties will be given below.

We start by describing a single RG step. It involves a "scaling" of the torus variable x by a matrix in $\mathrm{SL}(d, \mathbb{Z})$, whose transpose is strongly contracting on the orthogonal complement of some unit vector $\omega \in \mathbb{R}^d$. Given such a matrix T, and a nonzero real number μ, define

$$\mathcal{S}_\mu(x, y) = (x, \mu y), \qquad \mathcal{T}(x, y) = \bigl(Tx, \widehat{T} y\bigr), \qquad x \in \mathbb{T}^d, \quad y \in \mathbb{R}^\ell. \tag{1.4}$$

Here, $\widehat{T}$ is either the $\ell \times \ell$ identity matrix, or if desired for the renormalization of Hamiltonian vector fields (where $\ell = d$), the inverse of the transpose of T. The scaling of a vector field X on $\mathcal{M}$ is then given by $(\mathcal{S}_\mu \mathcal{T})^* X$, the pullback of X under $\mathcal{S}_\mu \mathcal{T}$. Recall that the pullback of a vector field X, defined on the range of a differentiable map U, is given by $U^* X = (DU)^{-1}(X \circ U)$.

Notice that scaling by $\mathcal{T}^*$ is a singular operation on spaces of analytic vector fields, since it shrinks the domain of analyticity in the expanding direction of T. Although the domain loss is of order one (not small), it is possible to associate with $X \in \mathcal{A}_\varrho$ a change of variables $\mathcal{U}_X$, which is close to the identity for X close to $K = (\omega, 0)$, such that the renormalized vector field

$$\mathcal{R}(X) = \eta^{-1} \mathcal{T}^* \mathcal{S}_\mu^* \mathcal{U}_X^* X \tag{1.5}$$

belongs again to $\mathcal{A}_\varrho$.

To be more specific, we will identify (in Section 2) a subspace of "resonant" vector fields, containing K, such that the restriction of $\mathcal{T}^* \mathcal{S}_\mu^*$ to this subspace is compact, and in fact analyticity improving, for small $\mu > 0$. Then, using a general result from Section 5, we show that there exists an analytic map $X \mapsto \mathcal{U}_X$, defined near K, which makes $\mathcal{U}_X^* X$ resonant. In other words, the resonant vector fields, which behave well under scaling, can be regarded as a local normal form for vector fields. We note that $\mathcal{U}_K$ is the identity, so the transformation $\mathcal{R}$ maps K to $\widetilde{K} = (\widetilde{\omega}, 0)$, where $\widetilde{\omega} = \eta^{-1} T^{-1} \omega$.

Theorem 1.3. *Let $\varrho > 0$. Given a Diophantine unit vector $\omega_0 \in \mathbb{R}^d$, there exists a sequence of matrices $T_n \in \mathrm{SL}(d, \mathbb{Z})$, and a corresponding sequence of transformations $\mathcal{R}_n$ of the form (1.5), such that the following holds. For $n = 1, 2, \ldots$ define $\omega_n = \eta_n^{-1} T_n \omega_{n-1}$, with $\eta_n > 0$ chosen in such a way that ω_n is a unit vector. Then $\mathcal{R}_n$ is well defined and analytic in some open neighborhood $\mathcal{D}_{n-1}$ of $K_{n-1} = (\omega_{n-1}, 0)$ in $\mathcal{A}_\varrho$. The set $\mathcal{W}$ of infinitely renormalizable vector fields X_0 in $\mathcal{D}_0$, characterized by the property that*

$X_n = \mathcal{R}_n(X_{n-1})$ belongs to $\mathcal{D}_n$ for $n = 1, 2, \ldots$, is the graph of an analytic function W with the properties described in Theorem 1.1 (if $\varrho + \delta < \rho$), where $\omega = \omega_0$ and $B = \mathcal{D}_0$.

The set $\mathcal{W}$ can be regarded as the (local) stable manifold for the transformations $\mathcal{R}_1, \mathcal{R}_2, \ldots$. A stable manifold theorem that applies to such sequences of maps will be proved in Section 6. Section 2 deals with a single renormalization group transformation $\mathcal{R}$, using a normal form theorem proved in Section 5. The composition of such transformations $\mathcal{R}_n$, according to a multidimensional continued fractions expansion [25], will be described in Section 3. Section 4 is devoted to the construction of invariant tori.

2. A single renormalization step

In this section we give a precise definition RG transformation $\mathcal{R}$ and describe some of its properties. A matrix $T \in \mathrm{SL}(d, \mathbb{Z})$ is assumed to be given, subject to certain conditions that will be specified below.

2.1. Spaces and basic estimates

Unless specified otherwise, our norm on $\mathbb{C}^n$ is $\|v\| = \sup_j |v_j|$. Another norm that will be used is $|v| = \sum_j |v_j|$. For linear operators between normed linear spaces, including matrices, we will always use the operator norm, unless stated otherwise. Let C be some finite dimensional complex Banach space. Denote by D_ρ the set of all vectors (x, y) in $\mathbb{C}^d \times \mathbb{C}^\ell$, characterized by $\|\mathrm{Im}\, x\| < \rho$ and $\|y\| < \rho$. We consider functions on $\mathbb{T}^d \times \mathbb{C}^\ell$ with values in C, that extend analytically to D_ρ and continuously to the boundary of D_ρ. Our norm on the space $\mathcal{A}_\rho(C)$ of such functions f is given in terms of the Fourier-Taylor series of f as follows:

$$\|f\|_\rho = \sum_{\nu,\alpha} \|f_{\nu,\alpha}\| e^{\rho|\nu|} \rho^{|\alpha|}, \qquad f(x, y) = \sum_{\nu,\alpha} f_{\nu,\alpha} e^{i\nu\cdot x} y^\alpha, \tag{2.1}$$

where $\nu \cdot x = \sum_j \nu_j x_j$ and $y^\alpha = \prod_j y_j^{\alpha_j}$. The sum in this equation ranges over all $\nu \in \mathbb{Z}^d$ and $\alpha \in \mathbb{N}^\ell$. If it is clear what space C is being considered, or irrelevant, we will simply write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(C)$. The operator norm of a continuous linear map L on $\mathcal{A}_\rho$ will be denoted by $\|L\|_\rho$.

Later on, for the construction of invariant tori, we will also use non-analytic functions, with real domain $D_0 = \mathbb{T}^d \times \{0\}$. Denote by $\mathcal{A}_0$ the Banach space of continuous functions $f : D_0 \to \mathbb{C}^d$, for which the norm $\|f\|_0 = \sum_\nu \|f_\nu\|$ is finite, where $\{f_\nu\}$ are the Fourier coefficients of f. This space can be viewed as a $\rho \to 0$ limit of the spaces $\mathcal{A}_\rho$ defined above.

Proposition 2.1. *Let $X \in \mathcal{A}_\rho(C)$ and $Z \in \mathcal{A}_{\rho'}(\mathbb{C}^{d+\ell})$, with $0 \le \rho' \le \rho$. Then*
(a) $\|X(x, y)\| \le \|X\|_\rho$ for all $(x, y) \in D_\rho$.
(b) $(DX)Z \in \mathcal{A}_{\rho'}(C)$ and $\|(DX)Z\|_{\rho'} \le (\rho - \rho')^{-1} \|X\|_\rho \|Z\|_{\rho'}$, if $\rho' < \rho$.
(c) $X \circ (\mathrm{I} + Z) \in \mathcal{A}_{\rho'}(C)$ and $\|X \circ (\mathrm{I} + Z)\|_{\rho'} \le \|X\|_\rho$, if $\rho' + \|Z\|_{\rho'} \le \rho$.

The proof of these estimates is straightforward and will be omitted. In what follows, we always assume that $\rho > 0$, unless specified otherwise.

2.2. Resonant and nonresonant modes

Let $0 < \rho' < \varrho$ be given. These domain parameters are now considered fixed for the entire RG analysis. Choose $\gamma \geq 1$ and $\chi \geq \|\widehat{T}\|$. Let μ and τ be positive real numbers, satisfying

$$e^{\rho'/2}\mu < \hat{\mu} \equiv \frac{\rho'}{\chi\varrho}\,, \qquad \tau \leq \frac{\rho'}{2\varrho}\,, \qquad \tau\ln\bigl(\hat{\mu}/\mu\bigr) \leq \frac{\rho'}{2(\gamma+1)}\,. \tag{2.2}$$

Consider the matrix norm $|M| = \sup_{|v|=1}|Mv|$.

Definition 2.2. *Denote by S the generator of the one-parameter group of scalings $\mu \mapsto \mathcal{S}_\mu^*$. Given any subset J of $I = \mathbb{Z}^d \times \{-1,0,1,2,\ldots\}$, define $P(J)$ to be the joint spectral projection in $\mathcal{A}_\rho$ for the operators $(-i\nabla_x, S)$, associated with the eigenvalues (ν,k) in J. Let now I^+ be the set of all pairs $(\nu,k) \in I$ satisfying $|T^*\nu| \leq \tau|\nu|$ or $|T^*\nu| \leq \gamma^{-1}\tau k$, and let I^- be its complement in I. The projection onto the "resonant" and "nonresonant" subspace of $\mathcal{A}_\rho$ is defined as $\mathbb{I}^+ = P\bigl(I^+\bigr)$ and $\mathbb{I}^+ = P\bigl(I^-\bigr)$, respectively. In addition, we define $\mathbb{E}_k = P\bigl(\{(0,k)\}\bigr)$, for every integer $k \geq -1$, and $\mathbb{E} = \sum_k \mathbb{E}_k$. Notice that $\mathbb{E}X$ is the torus average of X.*

The following proposition shows that, unlike in KAM theory [1–3], resonant modes are easy to deal with in the RG approach.

Lemma 2.3. *Consider the two linear transformations $\mathcal{S}_\mu$ and $\mathcal{T}$ defined in (1.4). If the condition (2.2) holds, then $\mathcal{T}^*\mathcal{S}_\mu^*$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho'}$ to $\mathcal{A}_\varrho$, satisfying*

$$\begin{aligned} \bigl\|\mathcal{T}^*\mathcal{S}_\mu^*\mathbb{E}_k X\bigr\|_\varrho &\leq N(T)\bigl(\mu/\hat{\mu}\bigr)^k\|\mathbb{E}_k X\|_{\rho'} \\ \bigl\|\mathcal{T}^*\mathcal{S}_\mu^*\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\bigr\|_\varrho &\leq N(T)\bigl(b\mu/\hat{\mu}\bigr)^\gamma\|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_{\rho'}\,, \end{aligned} \tag{2.3}$$

where $N(T) = \bigl\|T^{-1}\bigr\| + \bigl|\widehat{T}^{-1}\bigr|\bigl|\widehat{T}\bigr|$.

Proof. By our choice of norm (2.1), it suffices to verify the given bounds for vector fields $X = P(J)Y$, with J containing a single point. Let

$$J = \{(\nu,k)\}\,, \qquad A = \varrho|T^*\nu| - \rho'|\nu| + k\ln\bigl(\mu/\hat{\mu}\bigr)\,. \tag{2.4}$$

Then it follows essentially from the definitions that

$$\|\mathcal{T}^*\mathcal{S}_\mu^* P(J)Y\|_\varrho \leq N(T)e^A\|P(J)Y\|_{\rho'}\,. \tag{2.5}$$

Setting $\nu = 0$ yields the first bound in (2.3).

In order to prove the second bound, assume that (ν,k) belongs to I^+, and that $\nu \neq 0$. Consider first the case $|T^*\nu| \leq \tau|\nu|$. Then $|\nu| \geq \tau^{-1}$, and we obtain

$$\begin{aligned} A &\leq (\varrho\tau - \rho')|\nu| + k\ln\bigl(\mu/\hat{\mu}\bigr) \leq (\varrho - \rho'/\tau) - \ln\bigl(\mu/\hat{\mu}\bigr) \\ &\leq -\rho'/(2\tau) - \ln\bigl(\mu/\hat{\mu}\bigr) \leq \gamma\ln\bigl(\mu/\hat{\mu}\bigr)\,. \end{aligned} \tag{2.6}$$

In the last inequality we have used condition (2.2).

Now consider the case $\tau|\nu| < |T^*\nu| \le \frac{\tau}{\gamma}k$. By using that $\varrho\frac{\tau}{\gamma} \le \frac{\varrho'}{2\gamma} = \ln(c)$, and $k > \gamma$, we find that

$$A \le \varrho\tau\gamma^{-1}k + k\ln\bigl(\mu/\hat\mu\bigr) \le k\ln\bigl(b\mu/\hat\mu\bigr) \le \gamma\ln\bigl(b\mu/\hat\mu\bigr)\,. \tag{2.7}$$

The second bound in (2.3) now follows from (2.6) and (2.7). **QED**

This lemma shows that the scaling $\mathcal{T}^*\mathcal{S}_\mu^*$ is a contraction (for small μ) on the resonant subspace of $\mathcal{A}_\varrho$, except for a small number of non-contracting directions. In order to exploit this property, given a vector field X that is not necessarily resonant, we first perform a change of variables $\mathcal{U}_X$, such that

$$\mathbb{I}^-\mathcal{U}_X^* X = 0\,. \tag{2.8}$$

The corresponding linearized equation, which needs to be solved in the process, is of the form $\mathbb{I}^-(X - [X,Z]) = 0$, where $[X,Z] = (DZ)X - (DX)Z$. In order to guarantee the existence of a solution, we need to make the following assumptions.

Assume that there exists a $d-1$ dimensional subspace of $\mathbb{R}^d$ where the transpose T^* of T contracts distances by a factor of at least $\tau/2\sqrt{d}$. Let ω be a unit vector in $\mathbb{R}^d$ that is perpendicular to this subspace, and set $K = (\omega, 0)$.

Proposition 2.4. *Choose $\sigma > 0$ such that $2\sqrt{d}\,\sigma\|T\| \le \tau$. If Z belongs to $\mathbb{I}^-\mathcal{A}'_r$ then*

$$\|[K,Z]\|_r \ge \sigma\|Z\|_r\,, \qquad \|[K,Z]\|_r \ge \frac{\sigma r}{r+\gamma+1}\|DZ\|_r\,. \tag{2.9}$$

Proof. Assume that (ν,k) belongs to I^-. In other words, $|T^*\nu| > \tau|\nu|$ and $|T^*\nu| > \frac{\tau}{\gamma}k$. Consider the decomposition $\nu = \nu_\| + \nu_\perp$ into a vector $\nu_\|$ parallel to ω and a vector $\nu_\perp$ perpendicular to ω. By using that $|\nu_\perp| \le \sqrt{d}|\nu|$, we obtain

$$\sqrt{d}\,\sigma|\nu| < \|T\|^{-1}\frac{\tau}{2}|\nu| \le \|T\|^{-1}\bigl(|T^*\nu| - |T^*\nu_\perp|\bigr) \le \|T\|^{-1}|T^*\nu_\|| \le |\nu_\|| \le \sqrt{d}\,|\omega\cdot\nu|\,,$$

and in particular, $\sigma < |\omega\cdot\nu|$. Similarly, we have

$$\sqrt{d}\,\frac{\sigma}{\gamma}k \le \|T\|^{-1}\frac{\tau}{2\gamma}k \le \frac{1}{2}\|T\|^{-1}|T^*\nu| \le \|T\|^{-1}\bigl(|T^*\nu| - |T^*\nu_\perp|\bigr) \le \sqrt{d}\,|\omega\cdot\nu|\,.$$

This shows that if $Z \in \mathbb{I}^-\mathcal{A}'_r$ and $Y = [K,Z] = (\omega\cdot\nabla_x)Z$, then $\|Z\|_r \le \sigma^{-1}\|Y\|_r$ and

$$\sum_{j=1}^{d}\left\|\frac{\partial}{\partial x_j}Z\right\|_r \le \frac{1}{\sigma}\|Y\|_r\,, \qquad \sum_{j=1}^{\ell}\left\|\frac{\partial}{\partial y_j}Z\right\|_r \le \frac{\gamma+1}{\sigma r}\|Y\|_r\,. \tag{2.10}$$

These bounds imply (2.9). **QED**

This proposition allows us now to apply the results from Section 5, which describe a solution of equation (2.8). For convenience later on, let us first restate the assumptions of Proposition 2.4 in a slightly stronger form:

$$\|T\|, \|T^{-1}\| \le \frac{\chi}{2\sqrt{d}}\,, \quad \chi = \frac{\tau}{\sigma}\,, \qquad |T^*\xi| \le \frac{\tau}{2\sqrt{d}}|\xi|\,, \quad \xi \in \omega^\perp\,. \tag{2.11}$$

Lemma 2.5. *There exist positive constants C and C', such that the following holds, whenever (2.2) and (2.11) are satisfied. Denote by $\mathcal{D}$ the open ball in $\mathcal{A}_\varrho$ of radius $\varepsilon = C(\sigma/\gamma)^2$, centered at K. Then for every $X \in \mathcal{D}$, there exists an analytic change of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$, such that $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies equation (2.8). The map $X \mapsto \mathcal{U}_X$ is analytic from $\mathcal{D}$ to the affine space $\mathrm{I} + \mathcal{A}_{\rho'}$ and satisfies the bounds in Theorem 5.2, with $\kappa = C'\sigma/\gamma$.*

Proof. The image of $\mathbb{I}^-\mathcal{A}'_r$ under $Z \mapsto P[Z,K] = P(\omega \cdot \nabla_x)Z$ contains all nonresonant Fourier-Taylor polynomials, and thus it is dense in $\mathbb{I}^-\mathcal{A}_r$, for an $r > 0$. Assumption 5.1 regarding the spaces $\mathcal{A}_r$ is satisfied by Proposition 2.1, and the condition (5.3) on K holds by Proposition 2.4, with $\kappa = \rho'(1+\rho')\sigma/(2\gamma)$. The hypotheses (5.4) and (5.20) of Theorem 5.2 are clearly satisfied on $\mathcal{D}$, with $\varepsilon = C''\kappa^2$ and C'' some constant depending only on ρ and ρ'. The claims now follow from Theorem 5.2. **QED**

2.3. The transformation $\mathcal{R}$

Given $T \in \mathrm{SL}(d,\mathbb{R})$, a unit vector $\omega \in \mathbb{R}^d$, and a real number $\gamma \geq 1$, assume that there exists positive constants $\mu, \sigma, \tau < 1$ satisfying (2.2) and (2.11). In what follows, a quantity will be called *universal* if it is independent of the choice of T, ω, γ, μ, σ, and τ.

On the domain $\mathcal{D}$ described in Lemma 2.5, we can now define our RG transformation $\mathcal{R}$ according to equation (1.5). The normalization constant η is defined as $\eta = \|T^{-1}\omega\|$, so that $\widetilde{\omega} = \eta^{-1}T^{-1}\omega$ is again a unit vector. Notice that, by construction, $\mathcal{U}_X = \mathrm{I}$ whenever X is resonant. Thus, $\mathcal{R} \circ \mathbb{I}^+$ is linear, and so is $\mathcal{R} \circ \mathbb{E}$.

Let $\mathbb{P} = \mathbb{E}_{-1} + \mathbb{E}_0$. The subspace $\mathbb{P}\mathcal{A}_\varrho$ is spanned by vector fields of the form $Y(x,y) = (u, My+v)$ and is invariant under $\mathcal{R}$. The restriction of $\mathcal{R}$ to this subspace, which is linear, will be denoted by $\mathcal{L}$.

In the following theorem, $\mathcal{H}_\rho$ denotes either $\mathcal{A}_\rho$, or the subspace of Hamiltonian vector fields in $\mathcal{A}_\rho$, provided that $\ell = d$ and we choose for $\widehat{T}$ the inverse of the transpose of T.

Theorem 2.6. *There exist universal constants $R, C_0 > 0$, such that the following holds. Let $\mathcal{D}$ be the open ball in $\mathcal{H}_\varrho$ of radius $2R(\sigma/\gamma)^2$, centered at K. Then $\mathcal{R}$ is bounded and analytic on $\mathcal{D}$, satisfying*

$$
\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &\leq C_0\eta^{-1}(\gamma/\sigma)(C_0\tau/\sigma)^{\gamma+2}\mu^\gamma\|(\mathbb{I}-\mathbb{E})X\|_\varrho\,,\\
\|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\|_\varrho &\leq C_0\eta^{-1}(\gamma/\sigma)(\tau/\sigma)^3\mu\|(\mathbb{I}-\mathbb{P})X\|_\varrho\,,\\
\|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_\varrho &\leq C_0\eta^{-1}(\gamma/\sigma)^3(\tau/\sigma)\mu^{-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,,\\
\|\mathcal{L}^{-1}\| &\leq C_0\eta(\tau/\sigma)\,.
\end{aligned}
\tag{2.12}
$$

Proof. Let R be half the constant C from Lemma 2.5, so that we can apply the estimates from Theorem 5.2. Let X be some vector field in $\mathcal{D}$. By Lemma 2.3 we have

$$
\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &= \eta^{-1}\|\mathcal{T}^*\mathcal{S}_\mu^*(\mathbb{I}-\mathbb{E})\mathcal{U}_X^*X\|_\varrho\\
&\leq C_1\eta^{-1}(c\tau/\sigma)^{\gamma+2}\mu^\gamma\big[\|(\mathbb{I}-\mathbb{E})X\|_{\rho'} + \|\mathcal{U}_X^*X - X\|_{\rho'}\big]\,,
\end{aligned}
\tag{2.13}
$$

for $c = \exp(\rho'/2)\varrho/\rho'$ and some constant $C_1 > 0$. Here, and in what follows, $C_1, C_2, \ldots$ denote positive universal constants. Using the bound (5.5) on the norm of $\mathcal{U}_X^* X - X$, together with the fact that $P = P(\mathbb{I} - \mathbb{E})$, we obtain the first inequality in (2.12).

Similarly, Lemma 2.3 implies that

$$\|\mathbb{E}_k \mathcal{R}(X)\|_\varrho \le C_2 \eta^{-1} (\tau/\sigma)^3 \mu \big[\|\mathbb{E}_k X\|_{\rho'} + \|\mathbb{E}_k (\mathcal{U}_X^* X - X)\|_{\rho'} \big] , \tag{2.14}$$

for all $k \ge 1$. Summing over $k \ge 1$ to get a bound on $\|(\mathbb{E} - \mathbb{P})\mathcal{R}(X)\|_\varrho$, and then adding (2.13), yields a bound analogous to (2.14), but with $\mathbb{E}_k$ replaced by $\mathbb{I} - \mathbb{P}$. Applying again the estimate (5.5) on the norm of $\mathcal{U}_X^* X - X$, we obtain the second inequality in (2.12).

By Lemma 2.3, we also have

$$\begin{aligned} \|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_\varrho &= \eta^{-1} \|\mathcal{T}^* \mathcal{S}_\mu^* \mathbb{E}(\mathcal{U}_X^* X - X)\|_\varrho \\ &\le C_3 \eta^{-1} (\tau/\sigma) \mu^{-1} \|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'} . \end{aligned} \tag{2.15}$$

Using the bounds (5.5), the norm on the right hand side of this inequality can be estimated as follows:

$$\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'} \le C_4 (\gamma/\sigma)^3 \|(\mathbb{I} - \mathbb{E})X\|_\rho^2 + \|\mathbb{E}[Z, X]\|_{\rho'} , \tag{2.16}$$

where $Z = \mathbb{I}^- Z$ is the vector field described in Theorem 5.2. The fact that $\mathbb{E}Z = 0$ implies $\mathbb{E}[Z, \mathbb{E}X] = 0$. Thus,

$$\begin{aligned} \|\mathbb{E}[Z, X]\|_{\rho'} &= \|\mathbb{E}[Z, (\mathbb{I} - \mathbb{E})X]\|_{\rho'} \\ &\le C_5 \|Z\|_\rho \|(\mathbb{I} - \mathbb{E})X\|_\rho \le C_6 (\gamma/\sigma) \|(\mathbb{I} - \mathbb{E})X\|_\rho^2 . \end{aligned} \tag{2.17}$$

In the last step, we have used the bound on $\|Z\|_\rho$ from Theorem 5.2. Combining the last three equations yields the third inequality in (2.12).

The analyticity and boundedness of $\mathcal{R}$ on $\mathcal{D}$ follows from Lemma 2.5.

In order to bound the inverse of $\mathcal{L}$, let Y be a vector field in $\mathbb{P}\mathcal{H}_\rho$. Then Y can be written as $Y(x, y) = (u, My + v)$, and the last inequality in (2.12) now follows from the fact that $(\mathcal{L}^{-1} Y)(x, y) = \eta(Tu, My + \mu v)$. Here, we have used that $\widehat{T} = \mathrm{I}$, except (optionally) in the Hamiltonian case where M and v are zero. **QED**

Due to the potentially large factor μ^{-1} in the third inequality of (2.12), we will choose the domain of $\mathcal{R}$ to be of the form

$$\|\mathbb{P}(X - K)\|_\varrho < r , \qquad \|(\mathbb{I} - \mathbb{P})X\|_\varrho < r , \qquad \|(\mathbb{I} - \mathbb{E})X\|_\varrho < r\delta , \tag{2.18}$$

with $0 < r \le R(\sigma/\gamma)^2$, and with $\delta > 0$ small (to be determined later).

Definition 2.7. *Given $\gamma \ge 1$, we will call $(\mu, \sigma, \tau, r, \delta)$ proper RG parameters if $r \le R(\sigma/\gamma)^2$, and if (2.2) holds with $\chi = \tau/\sigma$. The parameters are also assumed to be positive, and $\mu, \sigma, \tau < 1$. We say that the pair (T, ω) is* compatible *with these parameters if the condition (2.11) is satisfied as well. The open subset $\mathcal{D}$ of $\mathcal{A}_\varrho$ defined by equation (2.18) will be referred to as the domain of $\mathcal{R}$.*

3. Infinitely renormalizable vector fields

Our goal now is to compose RG transformations of the type described above.

Let $\lambda_0 = 1$. Given a sequence of matrices $P_0, P_1, P_2, \ldots$ in $\mathrm{SL}(d, \mathbb{Z})$, with P_0 the identity, and a unit vector ω in $\mathbb{R}^d$, we define $\omega_0 = \omega$ and

$$T_n = P_{n-1} P_n^{-1}, \qquad \lambda_n = \|P_n \omega_0\|, \qquad \omega_n = \lambda_n^{-1} P_n \omega_0, \tag{3.1}$$

for all $n \geq 1$. We also define $\lambda_0 = 1$. Assuming that each of the pairs (T_n, ω_{n-1}) is compatible with some proper set of RG parameters, we can define the corresponding RG transformation $\mathcal{R}_n : \mathcal{D}_{n-1} \to \mathcal{A}_\varrho$. Notice that the normalization constant η_n for $\mathcal{R}_n$ is given by $\eta_n = \lambda_n / \lambda_{n-1}$.

Let now $\widetilde{\mathcal{R}}_n = \mathcal{R}_n \circ \mathcal{R}_{n-1} \circ \ldots \circ \mathcal{R}_1$. The domain $\widetilde{\mathcal{D}}_n$ of the combined RG transformation $\widetilde{\mathcal{R}}_{n+1}$ is defined inductively as the set of all vector fields in the domain of $\widetilde{\mathcal{R}}_n$ that are mapped under $\widetilde{\mathcal{R}}_n$ into the domain $\mathcal{D}_n$ of $\mathcal{R}_{n+1}$. By Theorem 2.6, these domains are open and non-empty, and the transformations $\widetilde{\mathcal{R}}_n$ are analytic.

Theorem 3.1. *Let $\alpha > \beta$, $m > 2\alpha + 7$, and $\gamma \geq 2\alpha + 2$ be given. Then there exist real numbers $b, C > 0$, a decreasing sequence of proper RG parameters $(\mu_n, \sigma_n, \tau_n, r_{n-1}, \delta_{n-1})$ satisfying*

$$\sigma_{n+2} = \sigma_{n+1}^{1+\alpha}, \qquad \mu_n = \sigma_n^m, \qquad r_n = \tfrac{1}{5} \sigma_{n+1}^2 r_{n-1}, \qquad n = 1, 2, \ldots, \tag{3.2}$$

and for every every Diophantine vector $\omega \in \Omega$ a sequence of matrices $P_n \in \mathrm{SL}(d, \mathbb{Z})$ yielding pairs (T_n, ω_{n-1}) that are compatible with the RG parameters, and an open neighborhood B of $K = (\omega, 0)$ in $\mathcal{A}_\varrho$, such that the following holds. B contains a ball of radius b, centered at K. The set $\mathcal{W} = B \cap_n \widetilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})B \to \mathbb{P}B$, satisfying $W(0) = K$ and $DW(0) = 0$. For each $X \in \mathcal{W}$ and $n \geq 1$,

$$\begin{aligned} \big\|\widetilde{\mathcal{R}}_n(X) - K_n\big\|_\varrho &\leq C \sigma_n^{m-2\alpha-7} r_n \|(\mathbb{I} - \mathbb{P})X\|_\varrho, \\ \big\|\mathbb{P}[\widetilde{\mathcal{R}}_n(X) - K_n]\big\|_\varrho &\leq C \sigma_n^{2(m-2\alpha-7)} r_n^2 \|(\mathbb{I} - \mathbb{P})X\|_\varrho^2, \\ \big\|(\mathbb{I} - \mathbb{E})\widetilde{\mathcal{R}}_n(X)\big\|_\varrho &\leq C \sigma_n^{(m-1)\gamma-2\alpha-6} r_n \|(\mathbb{I} - \mathbb{E})X\|_\rho. \end{aligned} \tag{3.3}$$

A proof of this theorem will be given below. It uses a continued fractions expansion developed in [28,29,25], which we will now describe very briefly, and a stable manifold theorem given in Section 6. We note that the second bound in (3.3) is not strictly needed for our subsequent construction of invariant tori. The first bound, with a larger value of m, could be used instead.

Let F be a fundamental domain for the left action of $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ on $G = \mathrm{SL}(d, \mathbb{R})$. Consider the one-parameter subgroup of G, generated by the matrices

$$E^t = \mathrm{diag}\big(e^{-t}, \ldots, e^{-t}, e^{(d-1)t}\big), \qquad t \in \mathbb{R}. \tag{3.4}$$

Given a Diophantine vector $\omega \in \mathbb{R}^d$, define $W \in G$ to be the matrix obtained from the $d \times d$ identity matrix by replacing its last column vector by a constant multiple of ω whose last component is 1. Then, for every $t \in \mathbb{R}$, there exists a unique matrix $P(t) \in \Gamma$ such that $P(t)WE^t$ belongs to F. To a given sequence of "stopping times" $0 < t_1 < t_2 < \ldots$, we can now associate a sequence of matrices $P_n = P(t_n)$. The corresponding matrices T_n and vectors ω_n are defined as in (3.1).

Let $t_0 = 0$, and define $t_n' = t_n - t_{n-1}$ for all positive integers n. Let $\theta = \beta/(d+\beta)$.

Theorem 3.2. ([25]) *There exists $c_0 > 0$, depending only on the Diophantine constants β and ζ, such that for all $n > 0$, and for all vectors $\xi \in \mathbb{R}^d$ that are perpendicular to ω_{n-1},*

$$\begin{aligned} \|T_n\| &\le c_0 \exp\{(d-1)(1-\theta)t_n' + d\,\theta\, t_n\}, \\ \|T_n^{-1}\| &\le c_0 \exp\{(1-\theta)t_n' + d\,\theta\, t_n\}, \\ |T_n^*\xi| &\le c_0 \exp\{-(1-\theta)t_n' + d\,\theta t_{n-1}\}|\xi|\,. \end{aligned} \tag{3.5}$$

Proof of Theorem 3.1. Let $\alpha > \beta$ be fixed. We choose $t_n = c(1+\alpha)^n$ for each positive integer n, with $c > 0$ to be determined. Define $c_1 = 2c_0\sqrt{d}$ and

$$\sigma_n = \exp\{-dt_n'\}\,, \qquad \tau_n = c_1 \exp\{-(1-\theta)t_n' + d\,\theta t_{n-1}\}\,. \tag{3.6}$$

Then Theorem 3.2 guarantees that the conditions (2.11) are satisfied. By using that $t_1' = t_1$ and $t_n' = \frac{\alpha}{1+\alpha}t_n$ for $n > 1$, we obtain the bounds

$$\sigma_n \le \exp\{-d\tfrac{\alpha}{1+\alpha}t_n\}\,, \qquad \tau_n \le c_1 \exp\{-\epsilon t_n\}\,, \tag{3.7}$$

with $\epsilon = \frac{1-\theta}{1+\alpha}(\alpha-\beta) > 0$. Let now $\mu_n = \sigma_n^m$ with $m > 1$ fixed. Then it is clear that the conditions (2.2) are satisfied as well, for any $\gamma > 0$, provided that c is chosen sufficiently large. Here, and in what follows, any condition that is said to hold for large values of c is implicitly being satisfied by choosing c as large as necessary.

Next, let $r_0 = R(\sigma_1/\gamma)^2$, and define $r_1, r_2, \ldots$ as in (3.2). Then $r_{n-1} \le R(\sigma_n/\gamma)^2$, for all $n \ge 1$. Thus, we have shown that $(\mu_n, \sigma_n, \tau_n, r_{n-1}, \delta_{n-1})$ are proper RG parameters, in the sense of Definition 2.7, and that (T_n, ω_{n-1}) is compatible with these parameters. This is independent of the choice of $\delta_{n-1} > 0$, which we will describe below.

Consider now the rescaled RG transformation R_n, defined by the equation

$$R_n(Z) = r_n^{-1}\big[\mathcal{R}_n(K_{n-1} + r_{n-1}Z) - K_n\big]\,. \tag{3.8}$$

The domain of R_n is given by (2.18), with $r = 1$ and $\delta = \delta_{n-1}$, and with K replaced by the zero vector field. The restriction of R_n to $\mathbb{P}\mathcal{A}_\rho$, which is linear, will be denoted by L_n. By using the bound on $\mathcal{L}_n^{-1}$ from Theorem 2.6, we obtain $\|L_n^{-1}\| \le 1/5$, for large $c > 0$. Here, we have used also that $\sigma_n < \|T_n\|^{-1} \le \eta_n \le \|T_n^{-1}\| < \sigma_n^{-1}$. The same inequalities, and Theorem 2.6, also imply that

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})R_n(Z)\|_\varrho &\le \varepsilon_n \|(\mathbb{I}-\mathbb{E})Z\|_\varrho\,, \quad \varepsilon_n = 5C_0\gamma\sigma_n^{(m-1)\gamma-2\alpha-6}\,, \\ \|(\mathbb{I}-\mathbb{P})R_n(Z)\|_\varrho &\le \vartheta_n \|(\mathbb{I}-\mathbb{P})Z\|_\varrho\,, \quad \vartheta_n = 5C_0\gamma\sigma_n^{m-2\alpha-7}\,, \\ \|\mathbb{P}R_n(Z) - R_n(\mathbb{P}Z)\|_\varrho &\le \varphi_n\delta_{n-1}\|(\mathbb{I}-\mathbb{E})Z\|_\varrho\,, \quad \varphi_n = C_0\gamma^3\sigma_n^{-m-2\alpha-5}\,, \end{aligned} \tag{3.9}$$

for all Z in the domain of R_n. Assume now that $m > 2\alpha + 7$ and $\gamma \geq 2\alpha + 2$. Then $\varepsilon_n \leq \vartheta_n \leq 1/5$, if c is sufficiently large. Furthermore, by setting $\delta_{n-1} = (5\varphi_n)^{-1}$, it is easy to check that $\varepsilon_n \delta_{n-1} \leq \delta_n$, provided again that c has been chosen sufficiently large.

At this point we have verified the hypotheses of Theorem 6.1 with $\varepsilon = \vartheta = 1/5$. This includes the condition (6.4), since the third inequality in (3.9) remains true if δ_{n-1} is replaced by $\|(\mathbb{I} - \mathbb{E})Z\|_\varrho$. The assertions of Theorem 3.1 now follow from Theorem 6.1. **QED**

The above also proves Theorem 1.3, except for the reference to the statements in Theorem 1.1 that concern the symmetry properties of W, and invariant tori. The latter will be proved in the next section.

The fact that W preserves the type of a vector field (Hamiltonian, divergence free, symmetric, or reversible) can be seen as follows. First, recall that $\mathbb{P}$ projects onto vector fields $Y(x, y) = (u, My + v)$ with with (u, v) a vector in $\mathbb{C}^d \times \mathbb{C}^\ell$ and M a complex $\ell \times \ell$ matrix. These vector fields are divergence free and symmetric, so W trivially preserves these two types. What remains to be shown is that M and v are zero if $Y = W(X)$ and X is either Hamiltonian or reversible.

The elimination step $X \mapsto \mathcal{U}_X$ is type-preserving by design; see the discussion at the end of Section 5. The scaling $\mathcal{T}^* \mathcal{S}_\mu^*$ is type-preserving as well, except in the case of Hamiltonian vector fields, if we renormalize with $\widehat{T}_n = \mathrm{I}$. In that case, $X_n = \widetilde{\mathcal{R}}_n(X_0)$ is not Hamiltonian with respect to the standard symplectic form $\sum_j dx_j \wedge dy_j$, but with respect to $\sum_j dx_j \wedge (Cdy)_j$, where C is the inverse of the transpose of P_n. Still, the second ($\mathbb{C}^\ell$) component of X_n has a zero torus average. The same is trivially true for reversible vector fields. Thus, our RG analysis could be restricted to vector fields with this property, replacing e.g. $\mathcal{R}_n$ by $\mathcal{R}_n' = E \circ \mathcal{R}_n$, where E is the canonical projection onto vector fields whose second component has a zero torus average. Since E does not affect Hamiltonian or reversible vector fields, the resulting stable manifold $\mathcal{W}'$ coincides with $\mathcal{W}$ in the corresponding subspaces. This shows that the restriction of W to vector fields of a specific type yields a vector field of the same type.

4. Construction of invariant tori

Following [24,22], our construction of invariant tori is based on the relation between an invariant torus of a vector field X and the corresponding torus of the renormalized vector field $\mathcal{R}(X)$. We start with an informal discussion of this relation. Then we prove Theorem 1.1 and Lemma 1.2.

4.1. Preliminaries

Let $X \in \mathcal{A}_\varrho$. Notice that $\mathcal{R}(X)$ is obtained from X by a change of coordinates (that depends on X), combined with a rescaling of time. Thus, the flow for $\mathcal{R}(X)$ is related to the flow for X by the equation

$$\Lambda_X \circ \Phi^t_{\mathcal{R}(X)} = \Phi_X^{\eta^{-1} t} \circ \Lambda_X \,, \qquad \Lambda_X = \mathcal{U}_X \circ \mathcal{S}_\mu \circ \mathcal{T} \,. \tag{4.1}$$

In particular, $\mathcal{T} \circ \Phi^t_{\mathcal{R}(K)} = \Phi_K^{\eta^{-1}t} \circ \mathcal{T}$ on D_0. The identity (4.1) can also be used to relate an invariant torus for X to an invariant torus for $\mathcal{R}(X)$. To this end, if F is any map from D_0 into the domain of Λ_X, define

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \mathcal{T}^{-1} . \tag{4.2}$$

Assume that $\mathcal{R}(X)$ has an invariant torus $\widetilde{\Gamma}$ with frequency vector $\widetilde{\omega} = \eta^{-1}T^{-1}\omega$, taking values in the domain of Λ_X, and define $\Gamma = \mathcal{M}_X(\widetilde{\Gamma})$. Then, by using (4.1), together with the fact that $\mathcal{R}(K) = (\widetilde{\omega}, 0)$, we obtain

$$\begin{aligned}\Gamma \circ \Phi_K^t &= \Lambda_X \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} \circ \Phi_K^t = \Lambda_X \circ \widetilde{\Gamma} \circ \Phi^{\eta t}_{\mathcal{R}(K)} \circ \mathcal{T}^{-1} \\ &= \Lambda_X \circ \Phi^{\eta t}_{\mathcal{R}(X)} \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} = \Phi_X^t \circ \Lambda_X \circ \widetilde{\Gamma} \circ \mathcal{T}^{-1} = \Phi_X^t \circ \Gamma .\end{aligned}$$

This shows that Γ is an invariant torus for X with frequency vector ω.

In order to make these identities more precise, we need to estimate the difference $Y(t) = \Phi_X^t - \Phi_K^t$ between the flow for a vector field X and the flow for $K = (\omega, 0)$. This can be done by solving the integral equation

$$Y(t) = \int_0^t [(X - K) \circ \Phi_K^s] \circ [\mathrm{I} + Y(s)]\, ds . \tag{4.3}$$

Notice that $\Phi_K^t = \mathrm{I} + tK$ and $\mathrm{I} + Y(t) = \Phi_K^{-t} \circ \Phi_X^t$.

Proposition 4.1. *Let τ be a positive real number and X a vector field in $\mathcal{A}_\rho$, such that $\tau\|X - K\|_\rho < r < \rho$. Then the equation (4.3) has a unique continuous solution $t \mapsto Y(t) \in \mathcal{A}_{\rho - r}$ on the interval $|t| \le \tau$, and*

$$\|\Phi_X^t - \Phi_K^t\|_{\rho - r} \le \|t(X - K)\|_\rho . \tag{4.4}$$

Proof. By using Proposition 2.1 and the contraction mapping principle, the equation (4.3) is easily seen to have a unique continuous solution $t \mapsto Y(t) \in \mathcal{A}_{\rho-r}$ for t near 0. The solution can be continued as usual, as long as $\|Y(t)\|_{\rho-r} < r$. But on any interval containing zero, where $\|Y(t)\|_{\rho-r} < r$, we have by (4.3) the bound

$$\|Y(t)\|_{\rho - r} \le \|t(X - K)\|_\rho . \tag{4.5}$$

Here, we have used also that $Z \mapsto Z \circ \Phi_K^s$ is an isometry on $\mathcal{A}_\rho$. Thus, equation (4.3) has a continuous solution Y for all times t satisfying $\|t(X - K)\|_\rho < r$. **QED**

4.2. Existence of tori

Consider now a fixed but arbitrary vector field X on the stable manifold $\mathcal{W}$ of our RG transformations $\mathcal{R}_n$. Let $X_0 = X$, and $X_n = \mathcal{R}_n(X_{n-1})$ for $n \ge 1$. In order to simplify notation, we will write $\mathcal{U}_k$ and $\mathcal{M}_{k+1}$ in place of $\mathcal{U}_{X_k}$ and $\mathcal{M}_{X_k}$, respectively. Our goal is to construct an appropriate sequence of functions $\Gamma_k : D_0 \to D_\varrho$, satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \mathcal{T}_n^{-1}\,, \qquad \Lambda_n = \mathcal{U}_{n-1} \circ \mathcal{S}_{\mu_n} \circ \mathcal{T}_n\,, \tag{4.6}$$

for all positive integers n. Then we will show that Γ_k is an invariant torus for X_k, with frequency vector ω_k, for each $k \geq 0$.

We assume that α, m, and γ satisfy the conditions given in Theorem 3.1. For every integer $n \geq 0$, define $\mathcal{B}_n$ to be the vector space $\mathcal{A}_0$, equipped with the norm

$$\|f\|'_n = a r_n^{-1} \|f\|_0 = a r_n^{-1} \sum_\nu \|f_\nu\|\,. \tag{4.7}$$

Here, a is some positive real number, to be specified later. Denote by B_n the unit ball in $\mathcal{B}_n$, centered at the identity function I.

Proposition 4.2. *Assume that $(m-1)\gamma > 3\alpha + 8$. If a has been chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, and a universal constant $C_1 > 0$, such that for every $X \in B \cap \mathcal{W}$, and for every $n \geq 1$, the map $\mathcal{M}_n$ is well defined and analytic, as a function from B_n to $\mathcal{B}_{n-1}$, and it takes values in $B_{n-1}/2$. Furthermore, $\|D\mathcal{M}_n(F)\| \leq C_1 \sigma_n$, for all $F \in B_n$.*

Proof. Clearly, $\mathcal{M}_n$ is well defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = \mathrm{I} + g + (\mathcal{U}_{n-1} - \mathrm{I}) \circ (\mathrm{I} + g)\,, \qquad g = \mathcal{S}_{\mu_n} \circ \mathcal{T}_n \circ f \circ \mathcal{T}_n^{-1}\,, \tag{4.8}$$

where $f = F - \mathrm{I}$. By Theorem 5.2 and Theorem 3.1, we have for $n > 1$ the bound

$$\begin{aligned} \|\mathcal{U}_{n-1} - \mathrm{I}\|_\rho &\leq C_2 \sigma_n^{-1} \|\mathbb{I}^- X_{n-1}\|_\varrho \leq C_3 \sigma_n^{-1} \sigma_{n-1}^{(m-1)\gamma - 2\alpha - 6} r_{n-1} \|(\mathbb{I} - \mathbb{E}) X\|_\varrho \\ &\leq C_4 \sigma_{n-1} r_{n-1} \|(\mathbb{I} - \mathbb{E}) X\|_\varrho \leq C_5 \sigma_n r_{n-1}\,, \end{aligned} \tag{4.9}$$

with $C_2, \ldots, C_5$ universal constants. The first inequality and the final bound in (4.9) also hold for $n = 1$, if the neighborhood B of K has been chosen sufficiently small.

Recall that $\rho' < \rho < \varrho$ have been fixed. The composition with $\mathrm{I}+g$ in equation (4.8) is controlled by Proposition 2.1, using that $\|g\|_0 \leq \sigma_n^{-1} a^{-1} r_n \|f\|'_n < \rho'$ independently of n, if a has been chosen sufficiently large. Here, and in what follows, we assume that $F \in B_n$.

By using that $r_n / r_{n-1} = \sigma_{n+1}^2 / 5$, we obtain $\|g\|'_{n-1} \leq \sigma_n / 5$. When combined with (4.9), this yields the bound $\|\mathcal{M}_n(F) - \mathrm{I}\|'_{n-1} \leq \sigma_n / 2$, if the neighborhood B of K has been chosen sufficiently small.

When restricting $\mathcal{U}_{n-1}$ to the domain $D_{\rho'}$, we obtain a bound analogous to (4.9) for the derivative of $\mathcal{U}_{n-1}$. This, together with the fact that the inclusion map from B_n into B_{n-1} is bounded in norm by $\sigma_{n+1}^2/5$, shows that $\|D\mathcal{M}_n(F)\| \leq C_1 \sigma_n$ for all $n \geq 1$, and for all $F \in B_n$, where C_1 is again a universal constant. This completes the proof of Proposition 4.2. **QED**

Denote by Φ_n and Ψ_n the flows for the vector fields X_n and K_n, respectively.

Proposition 4.3. *Assume that $m > 2\alpha+7+p$ with $p > 0$. If a has been chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that the following holds, for every $X \in B \cap \mathcal{W}$, and for every $n \geq 1$. If $F \in B_n/2$ and $|s| \leq \sigma_n^{-p}$, then $\Phi_n^s \circ F \circ \Psi_n^{-s}$ belongs to B_n.*

Proof. We will use the identity

$$\Phi_n^s \circ F \circ \Psi_n^{-s} = \mathrm{I} + f \circ \Psi_n^{-s} + \bigl[\Phi_n^s \circ \Psi_n^{-s} - \mathrm{I}\bigr] \circ \bigl(\mathrm{I} + f \circ \Psi_n^{-s}\bigr) . \tag{4.10}$$

Let $\varepsilon = m - 2\alpha - 7 - p$. By Proposition 4.1 and Theorem 3.1, we have the bound

$$\bigl\|\Phi_n^s \circ \Psi_n^{-s} - \mathrm{I}\bigr\|_{\rho'} \leq \|s(X_n - K_n)\|_\rho \leq C\sigma_n^\varepsilon r_n \|(\mathbb{I} - \mathbb{P})X\|_\varrho \,, \qquad (4.11).$$

provided e.g. that the right hand side of this inequality is bounded by $\rho - \rho'$. This is certainly the case if ε is positive and $\|X - K\|_\varrho$ sufficiently small, independently of n. The composition by $\mathrm{I}+f\circ\Psi_n^{-s}$ in equation (4.10) is controlled the same way as the composition by $\mathrm{I} + g$ in the proof of Proposition 4.2, using also that $\|f \circ \Psi_n^{-s}\|_0 = \|f\|_0$. As a result, the third term on the right hand side of (4.10) belongs to $\mathcal{B}_n$ and is bounded in norm by $Ca\sigma_n^\varepsilon \|X - K\|_\varrho$, which is less than $1/2$ for any $n \geq 1$, if X is sufficiently close to K. QED

Assume that α, m, γ, and a have been chosen in such a way that the hypotheses of Theorem 3.1, Proposition 4.2, and Proposition 4.3 are satisfied, with $p = 1 + 1/\alpha$. Let $F_0, F_1, \ldots$ be a fixed but arbitrary sequence of functions in $\mathcal{A}_0$, such that $F_n \in B_n$ for all $n \geq 0$. Then we can define

$$\Gamma_{n,m} = \bigl(\mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_m\bigr)(F_m) \,, \qquad 0 \leq n < m \,. \tag{4.12}$$

Theorem 4.4. *Under the above-mentioned assumptions on α, m, and a, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$ such that the following holds. For every $X \in B \cap \mathcal{W}$, the limits $\Gamma_n = \lim_{m\to\infty} \Gamma_{n,m}$ exist in $\mathcal{B}_n$, are independent of the choice of $F_0, F_1, \ldots$, and satisfy the identities (4.6). Furthermore, Γ_0 is an elliptic invariant torus for X, and the map $X \mapsto \Gamma_0$ is analytic and bounded on $B \cap \mathcal{W}$.*

Proof. By Proposition 4.2 there exists $N > 0$ such that $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least $1/2$, if $n \geq N$. Thus, if $N \leq n < m < k$, then the difference $\Gamma_{n,k} - \Gamma_{n,m}$ is bounded in norm by 2^{n-m+1}. This shows that the sequence $m \mapsto \Gamma_{n,m}$ converges in $\mathcal{B}_n$ to a limit Γ_n, and that the limit is independent of the choice of the functions F_m. By choosing $F_m = \Gamma_m$ for all m, we obtain the identities (4.6). The analyticity of $X \mapsto \Gamma_0$ follows via chain rule from the analyticity of the maps used in our construction, and from uniform convergence.

In order to prove that Γ_0 is an invariant torus for X, let $t \in \mathbb{R}$, and define $t_n = \lambda_n t$ for all $n \geq 0$. Our goal is to apply the identity

$$\Phi_{n-1}^s \circ \mathcal{M}_n(F) \circ \Psi_{n-1}^{-s} = \mathcal{M}_n\bigl(\Phi_n^{\eta_n s} \circ F \circ \Psi_n^{-\eta_n s}\bigr) \,, \tag{4.13}$$

which follows from (4.1). To be more precise, given $t \in \mathbb{R}$, define $t_n = \lambda_n t$ for all $n \geq 0$. By using that $\lambda_n = \eta_1\eta_2 \cdot \ldots \cdot \eta_n$, together with the bound $\eta_n \leq \|T_n^{-1}\| < \sigma_n^{-1}$, and the recursion relation (3.2) satisfied by $\sigma_2, \sigma_3, \ldots$, we obtain a bound $\lambda_n \leq C_2^{-1}\sigma_n^{-p}$, for some universal constant $C_2 > 0$. Thus, if $|t| \leq C_2$, then $|t_n| \leq \sigma_n^{-p}$ for all $n \geq 1$. Proposition 4.3 now allows us to iterate (4.13), to get the identity

$$\Phi_0^t \circ \Gamma_{0,m} \circ \Psi_0^{-t} = \bigl(\mathcal{M}_1 \circ \ldots \circ \mathcal{M}_m\bigr)\bigl(\Phi_m^{t_m} \circ \Psi_m^{-t_m}\bigr)\,, \tag{4.14}$$

for all $m > 0$. As was shown above, the right hand side of this equation converges in $\mathcal{A}_0$ to Γ_0, and thus the left hand side converges to Γ_0 as well. In addition, $\Gamma_{0,m} \to \Gamma_m$ in $\mathcal{A}_0$, and since convergence in $\mathcal{A}_0$ implies pointwise convergence (see part (a) of Proposition 2.1), and the flow Φ_0^t is continuous, we have $\Phi_0^t \circ \Gamma_0 \circ \Psi_0^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$ by using the group property of the flow, together with the fact that composition with Ψ_0^s is an isometry on $\mathcal{A}_0$.

Finally, notice that by Theorem 3.1,

$$\lambda_n \|DX_n\|_\rho \leq C_3 \sigma_n^\varepsilon r_n \|(\mathbb{I} - \mathbb{P})X_0\|_\varrho\,, \tag{4.15}$$

where C_3 is some universal constant and $\varepsilon = m - 2\alpha - 7 - p$. The left (and thus right) hand side of this equation is an upper bound on the modulus of the Lyapunov exponents for the flow of $\lambda_n X_n$ on the the range of Γ_n. Since X_0 is obtained from $\lambda_n X_n$ by a change of coordinates, and Γ_0 is the corresponding invariant torus for X_0, the same upper bound applies to the flow for X_0 on the torus Γ_0. Taking $n \to \infty$ shows that this torus is elliptic. **QED**

4.3. Analyticity and families

In what follows, the torus Γ_0 associated with $X \in B \cap \mathcal{W}$ will be denoted by Γ_X. The domain parameter ρ used in the introduction is renamed to ϱ', to avoid notational conflicts.

The following theorem together with Theorem 3.1, and the discussion at the end of Section 3 (concerning the restriction of W to specific types), implies Theorem 1.1.

Theorem 4.5. *Let $\varrho' > \varrho + \delta$ with $\delta > 0$. Under the same assumptions as in Theorem 4.4, the map $X \mapsto \Gamma_X$ defines (via extension) a bounded analytic map from B' to $\mathcal{A}_\delta^0$, where B' is some open neighborhood of K in $\mathcal{A}_{\varrho'}$.*

Proof. For every $u \in \mathbb{R}^d$, define a translation R_u on $\mathbb{C}^d \times \mathbb{C}^\ell$ by setting $R_u(x,y) = (x+u,y)$. If X is a vector field on one of the domains D_r, then $R_u^* X$ denotes the pullback of X under R_u. And for functions $F : D_0 \to D_r$ we define $R_u^* F = R_u^{-1} \circ F \circ R_u$. An explicit computation shows that the RG transformation $\mathcal{R}$, and the maps $\mathcal{M}_X$ defined in (4.2) satisfy

$$\mathcal{R} \circ R_u^* = R_{T^{-1}u}^* \circ \mathcal{R}\,, \qquad \mathcal{M}_{R_u^* X} = R_u^* \circ \mathcal{M}_X \circ (R_{T^{-1}u}^*)^{-1}\,. \tag{4.16}$$

Here, we have used that the translations R_u^* are isometries on the spaces $\mathcal{A}_r$, and that the domain of $\mathcal{R}$ is translation invariant; see Definition 2.7. This also implies that the manifold $\mathcal{W}$ is invariant under translations R_u^*, which is used in the second identity in (4.16).

It is convenient to extend the function $X \mapsto \Gamma_X$ to an open neighborhood of K in $\mathcal{A}_\varrho$ by projecting X onto a point $X' \in \mathcal{W}$ and defining $\Gamma_X = \Gamma_{X'}$. More specifically, we take $X' = (\mathbb{I}+W)((\mathbb{I}-\mathbb{P})X)$, where W is the map defining $\mathcal{W}$, as described in Theorem 3.1. If restricted to a sufficiently small open ball $B \subset \mathcal{A}_\varrho$ centered at K, the map $X \mapsto \Gamma_X$ is now analytic and bounded on all of B.

The construction of Γ_0 in the proof of Theorem 4.4, together with the identities (4.16), and the invariance property $W = W \circ R_u^*$, shows that $\Gamma_{R_u^* X} = R_u^* \Gamma_X$, for all $X \in B$. Thus, if $u \in \mathbb{R}^d$ then

$$\Gamma_X(u,0) = \bigl(R_u \circ \Gamma_{R_u^* X}\bigr)(0,0)\,, \qquad X \in B. \tag{4.17}$$

The idea now is to extend the right hand side of (4.17) analytically to complex u, by using the analyticity of $X \mapsto \Gamma_X$. To this end, choose an open neighborhood B' of K in $\mathcal{A}_{\varrho'}$, such that $R_u^* B' \subset B$, for all $u \in \mathbb{C}^d$ of norm $r = \varrho' - \varrho$ or less. Then the right hand side of (4.17), regarded as a function of (X,u), is analytic and bounded on the product of B' with the strip $\|\mathrm{Im}\, u\| < r$. Denoting the this function by G, we clearly have $G(X,\,.) \in \mathcal{A}_\delta^0$ for all $X \in B'$. The analyticity of $X \mapsto G(X,\,.)$ is obtained e.g. by using a contour integral formula for $(g(t) - g(0) - zg'(0))/t^2$ with $g(t) = G(X+tZ,\,.)$. **QED**

Proof of Lemma 1.2. Let $\varrho + \delta < \varrho' < r$. (Recall that ρ has been renamed to ϱ'.) By Theorem 3.1 and Theorem 4.5, there exists an open ball B in $\mathcal{H}_{\varrho'}$, centered at K, such that $U = \mathbb{P} - W \circ (\mathbb{I} - \mathbb{P})$ defines an analytic map from B to H^u, and such that $X \in B$ has an invariant torus $\Gamma_X \in \mathcal{A}_\delta^0$ with frequency vector ω, whenever $U(X) = 0$. Furthermore, $X \mapsto \Gamma_X$ is analytic on $B \cap \mathcal{W}$.

By our non-degeneracy assumptions, the function $G_\varepsilon(z,v,s) = zK + (1+z)J_v^*(\varepsilon Z + s)$ defines a diffeomorphism $\phi = \mathbb{P} \circ G_\varepsilon$ between open neighborhoods of the origin in the spaces $\mathbb{C} \oplus V \oplus H_0^u$ and H^u. Let b be an open ball in H^u of radius $< R/2$, centered at the origin, where R is the radius of B. If $\varepsilon > 0$ and b are chosen sufficiently small, then $G'_\varepsilon = (\mathbb{I} - \mathbb{P})(G_\varepsilon \circ \phi^{-1})$ is a family in $\mathcal{F}(b)$ of norm $< R/2$, and the equation $F(\phi(z,v,s)) = (1+z)J_v^* f(s)$ defines an analytic map $f \mapsto F$ from some open neighborhood B_2 of f_ε in $\mathcal{F}(b_2)$, to $\mathcal{F}(b)$. The image of f_ε under this map is the family F_ε given by $F_\varepsilon(\sigma) = K + \sigma + G'_\varepsilon(\sigma)$. By using that $\mathbb{P} \circ G'_\varepsilon = 0$ and $W \circ G'_\varepsilon = K$, we see that $U \circ F_\varepsilon$ is the identity map on b. Thus, by the implicit function theorem, the equation $(U \circ F)(\sigma) = 0$ has a unique solution $\sigma = \sigma_F$, for any family F sufficiently close to F_ε in $\mathcal{F}(b)$, and this solution depends analytically on F. The assertion now follows, with $(z_f, v_f, s_f) = \phi^{-1}(\sigma_F)$ and $c_f = (1 + z_f)$. **QED**

5. A normal form theorem

Here we state and prove a normal form theorem that can be applied e.g. to the problem (2.8) of eliminating nonresonant modes in the renormalization of vector fields on $\mathbb{T}^d \times \mathbb{R}^\ell$. The changes of variables are chosen in such a way as to preserve certain symmetries. This aspect will be discussed at the end of this section.

Let D_0 be a non-empty set in some complex Banach space $\mathcal{X}$. For every r in some fixed interval $[\rho', \rho]$ of positive real numbers, consider the set D_r of all points $x \in \mathcal{X}$ whose distance from (any point in) D_0 is less than r, and let $\mathcal{A}_r$ be a Banach space of vector fields $Y : D_r \to \mathcal{X}$, that contains the inclusion map I from D_r into $\mathcal{X}$.

Assumption 5.1. *The following is assumed to hold, whenever $\rho' \le r < s \le \rho$. Let B be the open ball in $\mathcal{A}_r$ of radius $s-r$, centered at the origin. If X is any vector field in $\mathcal{A}_s$, then $X \mapsto X\circ(\mathrm{I}+Z)$ defines an analytic function from B to $\mathcal{A}_r$, and $\|X\circ(\mathrm{I}+Z)\|_r \le \|X\|_s$.*

This implies in particular that $\mathcal{A}_s \subset \mathcal{A}'_r$ whenever $r < s$, where $\mathcal{A}'_r$ denotes space of vector fields $X \in \mathcal{A}_r$ whose derivative DX defines a continuous linear operator $Z \mapsto (DX)Z$ on $\mathcal{A}_r$. The norm of this linear operator will be denoted by $\|DX\|_r$.

Let now P be a fixed but arbitrary projection operator on $\mathcal{A}_{\rho'}$, whose restriction to each of the spaces $\mathcal{A}_r$ is a partial isometry on that space. Our aim is to find conditions under which vector fields satisfying $PX = 0$ can be considered normal forms. To be more precise, let K be a fixed vector field in $\mathcal{A}'_\rho$ satisfying $PK = 0$. Given a vector field X near K in $\mathcal{A}'_\rho$, we are looking for a change of variables $\mathcal{U}_X : D_{\rho'} \to D_\rho$, such that the pullback $\mathcal{U}_X^* X$ of X under $\mathcal{U}_X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies

$$P\mathcal{U}_X^* X = 0\,. \tag{5.1}$$

In order to see what conditions may be needed, consider writing $\mathcal{U}_X$ to first order in $X-K$ as the time-one flow Φ_Z^1 for some vector field $Z = PZ$. In this approximation, equation (5.1) becomes

$$P(X + [Z,X]) = 0\,, \tag{5.2}$$

where $[Z,X] = (DX)Z - (DZ)X$. This motivates the following condition on K. In addition to $PK = 0$, we assume that the image of $P\mathcal{A}'_r$ under $Z \mapsto P[Z,K]$ is dense in $P\mathcal{A}_r$, and that there exists a positive real number $\kappa < 1$, such that

$$\big\|P[Z,K]\big\|_r \ge \kappa\|Z\|'_r\,, \qquad Z \in P\mathcal{A}'_r\,, \tag{5.3}$$

whenever $\rho' \le r \le \rho$. Here, $\|Z\|'_r = \|DZ\|_r + \|Z\|_r$, which we consider from now on to be the norm on $\mathcal{A}'_r$. The main result of this section is the following.

Theorem 5.2. (normal form) *Let $r = \rho - \rho'$. If X is a vector field in $\mathcal{A}'_\rho$ such that*

$$\|X-K\|'_\rho \le 2^{-3}\kappa\,, \qquad \|PX\|_\rho \le \varepsilon\,, \tag{5.4}$$

with $\varepsilon > 0$ satisfying the condition (5.20) given below, then there exists an analytic change of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$, such that $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$ and satisfies equation (5.1). The map $X \mapsto \mathcal{U}_X - \mathrm{I}$ takes values in $\mathcal{A}_{\rho'}$, is continuous in the region defined by equation (5.4), analytic in the interior of this region, and satisfies the bounds

$$\begin{aligned}
\|\mathcal{U}_X - \mathrm{I}\|_{\rho'} &\le \frac{3}{\kappa}\|PX\|_\rho\,,\\
\|\mathcal{U}_X^* X - X\|_{\rho'} &\le 32R\frac{e^r}{\kappa r}\|PX\|_\rho\,,\\
\big\|\mathcal{U}_X^* X - X - [Z,X]\big\|_{\rho'} &\le \Big(2^{11}\frac{e^r}{\kappa r}+1\Big)28R\frac{e^r}{(\kappa r)^2}\|PX\|_\rho^2\,.
\end{aligned} \tag{5.5}$$

Here, $Z \in P\mathcal{A}'_\rho$ is defined by (5.2) and satisfies the bound $\|Z\|_\rho \le \frac{2}{\kappa}\|PX\|_\rho$.

We start with some basic estimates on flows. The flow $t \mapsto \Phi_X^t$ associated with a vector field $X \in \mathcal{A}_\rho$ is obtained by solving $\frac{d}{dt}\Phi_X^t = X \circ \Phi_X^t$ with initial condition $\Phi_X^0 = \mathrm{I}$. Writing $\Phi_X^t = \mathrm{I} + Y(t)$, this amounts to solving the integral equation

$$Y(t) = \int_0^t X \circ [\mathrm{I} + Y(s)]\, ds\,. \tag{5.6}$$

In what follows, any reference to a space $\mathcal{A}_\varrho$ implicitly assumes that $\rho' \le \varrho \le \rho$.

Proposition 5.3. *Let ϱ', ϱ and τ be positive real numbers, such that $\varrho' + \tau\|X\|_\varrho < \varrho$. Then the equation (5.6) has a unique continuous solution $t \mapsto Y(t) \in \mathcal{A}_{\varrho'}$ on the interval $|t| \le \tau$, and*

$$\|\Phi_X^t - \mathrm{I}\|_{\varrho'} \le \|tX\|_\varrho\,. \tag{5.7}$$

We will omit the proof of this proposition since it is standard: First, equation (5.6) is solved locally, using the contraction mapping principle. Due to the uniqueness of these local solutions, they combine into a continuous solution on all of $[-\tau, \tau]$. What makes things straightforward is that, by Assumption 5.1, the derivative of $X \mapsto X \circ (\mathrm{I} + Z)$ admits a uniform bound on the entire domain needed. As a result, the intervals for the local proof can be taken of uniform size.

Proposition 5.4. *Let $0 < r < \varrho$ and $t \in \mathbb{R}$. Let Z and X be two vector fields in $\mathcal{A}_\varrho$, satisfying $\|tZ\|_\varrho \le r\varepsilon$ and $\|tDZ\|_\varrho \le s\varepsilon$, with $\varepsilon \le 1/6$. Then $(\Phi_Z^t)^*X$ belongs to $\mathcal{A}_{\varrho - r}$, and*

$$\begin{aligned} \big\|(\Phi_Z^t)^*X - X\big\|_{\varrho - r} &\le 3e^s\|X\|_\varrho\,\varepsilon\,,\\ \big\|(\Phi_Z^t)^*X - X - t[Z,X]\big\|_{\varrho - r} &\le 7e^s\|X\|_\varrho\,\varepsilon^2\,. \end{aligned} \tag{5.8}$$

Proof. It suffices to consider $t = 1$, since we can always rescale time. Let n be a fixed positive integer. By using Proposition 2.1, and Cauchy's formula with contour $|z| = 1$, to estimate

$$(DX)Z = n\varepsilon \frac{d}{dz}\Big[X \circ \Big(\mathrm{I} + \frac{z}{n\varepsilon}Z\Big)\Big]_{z=0}\,, \tag{5.9}$$

we obtain the bound

$$\|\widehat{Z}X\|_{\varrho' - r/n} \le (n\varepsilon + s\varepsilon)\|X\|_\varrho\,, \tag{5.10}$$

where $\varrho' = \varrho$. Here, and in what follows, $\widehat{Z}$ denotes the map $Y \mapsto [Z, Y]$. This bound can be iterated n times, with ϱ' decreasing by r/n after each step, and we find

$$\begin{aligned} \frac{1}{n!}\big\|\big(\widehat{Z}\big)^n X\big\|_{\varrho - r} &\le \frac{1}{n!}(n+s)^n\varepsilon^n\|X\|_\varrho\\ &\le \frac{n^n}{n!}e^s\varepsilon^n\|X\|_\varrho \le \frac{1}{2}(e\varepsilon)^n e^s\|X\|_\varrho\,. \end{aligned} \tag{5.11}$$

In the last inequality, we have used Stirling's formula. Now

$$\bigl\|(\Phi_Z^1)^*X - X\bigr\|_{\varrho-r} = \left\|\sum_{n=1}^{\infty}\frac{1}{n!}\bigl(\widehat{Z}\bigr)^n X\right\|_{\varrho-r} \le \frac{\varepsilon}{2}\cdot\frac{e^{s+1}}{1-e\varepsilon}\|X\|_\varrho\,, \tag{5.12}$$

and the first bound in (5.8) follows. The second bound is obtained analogously, with the sum in (5.12) starting at $n = 2$. **QED**

Returning to the main problem (5.1), our first step is to solve this equation to first order in the size of $X - K$.

Proposition 5.5. *Let $0 < r < \varrho$. Let X be a vector field in $\mathcal{A}'_\varrho$, satisfying*

$$\|X - K\|'_\varrho \le \tfrac{1}{4}\kappa\,, \qquad \|PX\|_\varrho \le \tfrac{1}{12}\kappa r\,. \tag{5.13}$$

Then the equation (5.2) has a unique solution $Z \in P\mathcal{A}'_\varrho$. The vector field Z satisfies $\|Z\|_\varrho \le \frac{2}{\kappa}\|PX\|_\varrho$. Furthermore, $(\Phi_Z^1)^*X$ belongs to $\mathcal{A}_{\varrho-r}$ and satisfies

$$\begin{aligned}\bigl\|(\Phi_Z^1)^*X - X\bigr\|_{\varrho-r} &\le \frac{6e^r}{\kappa r}\|PX\|_\varrho\|X\|_\varrho\,,\\ \bigl\|(\Phi_Z^1)^*X - X - [Z,X]\bigr\|_{\varrho-r} &\le \frac{28e^r}{(\kappa r)^2}\|PX\|_\varrho^2\|X\|_\varrho\,.\end{aligned} \tag{5.14}$$

Proof. The first condition in (5.13) implies that

$$\|[Z, X-K]\|_\varrho \le 2\|Z\|'_\varrho\|X-K\|'_\varrho \le \frac{\kappa}{2}\|Z\|'_\varrho\,, \tag{5.15}$$

for every $Z \in \mathcal{A}'_\varrho$. As a consequence, we have

$$\bigl\|P[Z,X]\bigr\|_\varrho \ge \bigl\|P[Z,K]\bigr\|_\varrho - \bigl\|P[Z,X-K]\bigr\|_\varrho \ge \frac{\kappa}{2}\|Z\|'_\varrho\,, \tag{5.16}$$

whenever Z belongs to $P\mathcal{A}'_\varrho$. These inequalities, together with our assumption that the range of $P\widehat{X}P$ is dense in $P\mathcal{A}_\varrho$, implies that the linear operator $P\widehat{X} : P\mathcal{A}'_\varrho \to P\mathcal{A}_\varrho$ has a bounded inverse, and in particular, that the equation (5.2) has a unique solution $Z \in P\mathcal{A}'_\varrho$. The bound (5.16) also shows that this solution satisfies

$$\|Z\|_\varrho \le \frac{2}{\kappa}\|PX\|_\varrho\,, \qquad \|DZ\|_\varrho \le \frac{2}{\kappa}\|PX\|_\varrho\,. \tag{5.17}$$

The remaining claims now follow from Proposition 5.4, setting $\varepsilon = \frac{2}{\kappa r}\|PX\|_\varrho$ and $s = r$. **QED**

Now we iterate the map $X \mapsto (\Phi_Z^1)^*X$ described in Proposition 5.5, by starting with a vector field $X = X_0$ and setting

$$X_{n+1} = (\Phi_{Z_n}^1)^*X_n\,, \qquad P(X_n + [Z_n, X_n]) = 0\,, \tag{5.18}$$

for $n = 0, 1, \ldots$. The expectation is that the sequence of maps

$$U_n = \Phi^1_{Z_0} \circ \Phi^1_{Z_1} \circ \ldots \circ \Phi^1_{Z_{n-1}} \tag{5.19}$$

converges to a solution $\mathcal{U}_X$ of equation (5.1), as n tends to infinity.

Proof of Theorem 5.2. Setting $r = \rho - \rho'$ and $R = \|K\|_\rho + \kappa$, choose $\varepsilon > 0$ such that

$$\varepsilon \le 2^{-6}\kappa r\,, \qquad \varepsilon \le 2^{-9}\kappa^2 e^{-r}(1+r)^{-1}R^{-1}\,. \tag{5.20}$$

Let $\rho_0 = \rho$, and for $m = 0, 1, \ldots$ define $\rho_{m+1} = \rho_m - 2r_m$, where $r_m = 2^{-m-2}r$. Our first goal is to prove that (5.18) defines a sequence of vector fields $X_m \in \mathcal{A}'_{\rho_m}$, satisfying

$$\|X_m - X_{m-1}\|'_{\rho_m} \le 2^{-m-3}\kappa\,, \qquad \|PX_m\|_{\rho_m} \le 8^{-m}\varepsilon\,. \tag{5.21}$$

If we define $X_{-1} = K$ and $X_0 = X$, then these bounds hold for $m = 0$ by (5.4). Assume now that (5.21) holds for $m \le n$. Then, by summing up the bounds on $X_m - X_{m-1}$ for $m \le n$, we obtain the first inequality in

$$\|X_n - K\|'_{\rho_n} \le \frac{1}{4}\kappa\,, \qquad \|PX_n\|_{\rho_n} \le 4^{-n-2}\kappa r_n\,. \tag{5.22}$$

The second inequality follows from (5.21), by substituting the first bound in (5.20) on ε. Thus, Proposition 5.5 guarantees a unique solution to (5.18), and it yields the bounds

$$\|X_{n+1} - X_n\|_{\rho_n - r_n} \le 6\frac{e^r}{\kappa r}4^{-n+1}R\varepsilon\,, \qquad \|PX_{n+1}\|_{\rho_n - r_n} \le 7\frac{e^r}{(\kappa r)^2}4^{-2n+3}R\varepsilon^2\,. \tag{5.23}$$

Here, we have used also that $\|X_n\|_{\rho_n} \le R$, which follows from the first inequality in (5.22). By using the second condition in (5.20), together with the fact that $\|F\|'_{\rho_n - 2r_n} \le r_n^{-1}\|F\|_{\rho_n - r_n}$, we now obtain (5.21) for $m = n+1$ from the bounds (5.23).

Next, consider the functions $\phi_k = \Phi^1_{Z_k} - \mathrm{I}$. By Proposition 4.1 and Proposition 5.5,

$$\|\phi_k\|_{\rho_{k+1}} \le \|Z_k\|_{\rho_k} \le \frac{2}{\kappa}\|PX_k\|_{\rho_k} < r_k\,. \tag{5.24}$$

This shows that $U_{m,n} = \Phi^1_{Z_m} \circ \Phi^1_{Z_{m+1}} \circ \ldots \circ \Phi^1_{Z_{n-1}}$ defines a function in $\mathrm{I} + \mathcal{A}_{\rho_n}$ that takes values in D_{ρ_m}. Here, and in what follows, it is assumed that $0 \le m < n$. Setting $U_{k,k} = \mathrm{I}$, we have the bound

$$\begin{aligned} \|U_n - U_m\|_{\rho'} &= \left\| \sum_{k=m}^{n-1} \phi_k \circ U_{k+1,n} \right\|_{\rho'} \\ &\le \sum_{k=m}^{n-1} \|\phi_k\|_{\rho_{k+1}} \le \sum_{k=m}^{n-1} \frac{2}{\kappa} 8^{-k}\varepsilon\,. \end{aligned} \tag{5.25}$$

This shows that $n \mapsto U_n$ converges in $\mathrm{I} + \mathcal{A}_{\rho'}$ to a limit $\mathcal{U}_X$ that takes values in D_ρ, and that satisfies the first inequality in (5.5) if we set $\varepsilon = \|PX\|_\rho$. Clearly, $X_n \to \mathcal{U}_X X$ in $\mathcal{A}_{\rho'}$. The second inequality in (5.5) is now obtained by using the first bound in (5.23).

Since $\mathcal{U}_X^* X = \mathcal{U}_{X_1}^* X_1$ with $X_1 = (\Phi_Z^1)^* X$, we can write

$$\mathcal{U}_X^* X - X - [Z, X] = \bigl(\mathcal{U}_{X_1}^* X_1 - X_1\bigr) + \bigl((\Phi_Z^1)^* X - X - [Z, X]\bigr) . \tag{5.26}$$

The first term on the right hand side of this equation can be estimated in the same way as $\mathcal{U}_X^* X - X$, which yields the bound

$$\bigl\|\mathcal{U}_{X_1}^* X_1 - X_1\bigr\|_{\rho'} \le 32R \frac{e^r}{\kappa r_1} \|PX_1\|_{\rho_1} . \tag{5.27}$$

Since $\|PX_1\|_{\rho_1} \le \|(\Phi_Z^1)^* X - X - [Z, X]\|_{\rho_1}$, the third inequality in (5.5) now follows from the second inequality in (5.14).

The analyticity of the map $X \mapsto \mathcal{U}_X$ follows from the uniform convergence of $U_n \to \mathcal{U}_X$. **QED**

Some special cases

We conclude this section with a discussion of invariance properties of $\mathcal{U}_X$ that result from choosing projections of the type described in Definition 2.2, acting on the spaces $\mathcal{A}_r$ defined in Section 2.

Consider a projection $P(J)$, with J chosen in such a way that the condition (5.3) holds for $P = P(J)$. A possible choice for J is the set I^- introduced in Definition 2.2, as Proposition 2.4 shows. It is easy to check that P maps Hamiltonian vector fields to Hamiltonian vector fields. (Similarly for divergence free vector fields.) This includes vector fields that are Hamiltonian with respect to the pullback of the standard symplectic form under linear transformations $\mathcal{C}(x, y) = (x, Cy)$, where C can be any real nonsingular $\ell \times \ell$ matrix. This follows from the fact that C^* commutes with P. Thus, since $[Z, X]$ is Hamiltonian whenever Z and X are, the entire analysis in this section could be restricted to Hamiltonian vector fields. Since the solution of (5.2) is unique, we find that the change of coordinates $\mathcal{U}_X$ is symplectic whenever K and X are Hamiltonian. The same type of argument shows that $\mathcal{U}_X$ is volume preserving if the vector fields K and X are divergence free.

Next, consider a linear map G on $\mathbb{C}^d \times \mathbb{C}^\ell$ that leaves the domains D_r invariant. Assume that G^* is an isometry on each of the spaces $\mathcal{A}_r$, and that it commutes with P. An example of such a map is $G(x, y) = (x, -y)$. If Z and X are symmetric with respect to G, then so is $[Z, X]$. Thus, the analysis of this section can be restricted to symmetric vector fields. This shows that $\mathcal{U}_X$ commutes with G^* whenever K and X are symmetric.

Assume in addition that $G \circ G = \mathrm{I}$, and that K, X are reversible with respect to G. Then $[Z, X]$ is reversible whenever Z is symmetric (both with respect to G). Thus, the operator $P\widehat{X}$ maps the symmetric subspace of $P\mathcal{A}_\rho'$ to the reversible subspace of $P\mathcal{A}_\rho$. As the proof of Proposition 5.5 shows, this operator has (under the given assumptions) a bounded inverse, so the solution Z of equation (5.2) is symmetric. Consequently, the flows $\Phi_{Z_k}^1$ defining $\mathcal{U}_X$ commute with G, and the same is true for $\mathcal{U}_X$.

In summary, if X is of the type described above (Hamiltonian, divergence free, symmetric, or reversible), then the same is true for $\mathcal{U}_X^* X$.

6. A stable manifold theorem

Here we state and prove a local stable manifold theorem for sequences of maps of the type encountered e.g. in renormalization. It allows for a description of two different contraction rates.

For every integer $n \geq 0$ let $\mathcal{X}_n$ be a complex Banach space, and let $\mathbb{E}_n$, $\mathbb{P}_n$ be continuous linear projections on $\mathcal{X}_n$, satisfying $\mathbb{P}_n\mathbb{E}_n = \mathbb{E}_n\mathbb{P}_n = \mathbb{P}_n$ and $\|\mathbb{E}_n\| = \|\mathbb{I} - \mathbb{E}_n\| = 1$. For each $n > 0$, let R_n be a bounded analytic map, from an open neighborhood D_{n-1} of the origin in $\mathcal{X}_{n-1}$, to $\mathcal{X}_n$, with the following properties: $R_n\mathbb{P}_{n-1}$ is linear, and the restriction L_n of this linear operator to $\mathbb{P}_{n-1}\mathcal{X}_{n-1}$ is invertible. Furthermore, there exist real numbers $\vartheta_n \leq \vartheta < 1$ and $\varepsilon_n \leq \varepsilon = (1-\vartheta)/4$, such that for all $x \in D_{n-1}$,

$$
\begin{aligned}
\|(\mathbb{I} - \mathbb{E}_n)R_n(x)\| &\leq \varepsilon_n \|(\mathbb{I} - \mathbb{E}_{n-1})x\|\,,\\
\|(\mathbb{I} - \mathbb{P}_n)R_n(x)\| &\leq \vartheta_n \|(\mathbb{I} - \mathbb{P}_{n-1})x\|\,,\\
\|\mathbb{P}_n R_n(x) - L_n\mathbb{P}_{n-1}x\| &\leq \varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1})x\|\,,\\
\|L_n^{-1}\| &\leq \vartheta\,.
\end{aligned}
\tag{6.1}
$$

Consider now the composed maps $\widetilde{R}_n = R_n \circ R_{n-1} \circ \ldots \circ R_1$. The domain of $\widetilde{R}_1$ is taken to be $\widetilde{D}_0 = D_0$, and for $n = 1, 2, \ldots$, the domain $\widetilde{D}_n$ of $\widetilde{R}_{n+1}$ is defined inductively as the subset of $\widetilde{D}_{n-1}$ that is mapped into D_n by $\widetilde{R}_n$.

We will assume that the domain D_{n-1} of R_n is given by conditions

$$
\|\mathbb{P}_{n-1}x\| < 1\,, \qquad \|(\mathbb{I} - \mathbb{P}_{n-1})x\| < 1\,, \qquad \|(\mathbb{I} - \mathbb{E}_{n-1})x\| < \delta_{n-1}\,,
\tag{6.2}
$$

where $\{\delta_k\}$ is a sequence of positive real numbers, such that $\delta_k \geq \varepsilon_k\delta_{k-1}$ for all $k > 0$.

Theorem 6.1. (local stable manifold) *Let $R_1, R_2, \ldots$ be a sequence of maps with the properties described above. Then $\mathcal{W}_0 = \bigcap_{n=0}^{\infty} \widetilde{D}_n$ is the graph of an analytic function $W_0 : (\mathbb{I} - \mathbb{P}_0)D_0 \to \mathbb{P}_0 D_0$, satisfying $W_0(0) = 0$. For every $x \in \mathcal{W}_0$,*

$$
\begin{aligned}
\|\widetilde{R}_m(x)\| &\leq \left[\vartheta^{(m)} + \varepsilon^{(m)}\right] \|(\mathbb{I} - \mathbb{P}_0)x\|\,,\\
\|(\mathbb{I} - \mathbb{E}_m)\widetilde{R}_m(x)\| &\leq \varepsilon^{(m)} \|(\mathbb{I} - \mathbb{E}_0)x\|\,,
\end{aligned}
\tag{6.3}
$$

where $\vartheta^{(m)} = \vartheta_1\vartheta_2\cdots\vartheta_m$ and $\varepsilon^{(m)} = \varepsilon_1\varepsilon_2\cdots\varepsilon_m$. Furthermore, if the third condition in (6.1) is strengthened to

$$
\|\mathbb{P}_n R_n(x) - L_n\mathbb{P}_{n-1}x\| \leq \varphi_n \|(\mathbb{I} - \mathbb{E}_{n-1})x\|^2\,,
\tag{6.4}
$$

with $\varphi_n\delta_{n-1} \leq \varepsilon$, then $DW_0(0) = 0$, and

$$
\|\mathbb{P}_m\widetilde{R}_m(x)\| \leq \left[\vartheta^{(m)}\right]^2 \|(\mathbb{I} - \mathbb{P}_0)x\|^2\,.
\tag{6.5}
$$

Notice that, by our assumptions (6.1), if x belongs to the domain of R_n , and if $\mathbb{P}_n R_n(x)$ has norm less than one, then $R_n(x)$ belongs to the domain of R_{n+1} . This shows that

$$\widetilde{D}_n = \{x \in \widetilde{D}_{n-1} : \|\mathbb{P}_n \widetilde{R}_n(x)\| < 1\}\,, \qquad n = 1, 2, \ldots . \tag{6.6}$$

Let $S_n = \mathbb{P}_n \mathcal{X}_n$, and denote by b_n the open unit ball in S_n , centered at the origin. Define $\mathcal{F}_n$ to be the space of analytic functions $f : b_n \to \mathcal{X}_n$, equipped with the sup-norm $\|f\| = \sup_{s\in b_n} \|f(s)\|$. Denote by I_n the inclusion map of b_n into $\mathcal{X}_n$. Notice that, if $f \in \mathcal{F}_{n-1}$ satisfies

$$\mathbb{P}_{n-1} f = I_{n-1}\,, \qquad \|f - I_{n-1}\| < 1\,, \qquad \|(\mathbb{I} - \mathbb{E}_{n-1}) \circ f\| < \delta_{n-1}\,, \tag{6.7}$$

then $f(s)$ belongs to the domain of R_n , for all $s \in b_{n-1}$. For such functions f, define

$$Y_{n,f} = \mathbb{P}_n (R_n \circ f)\,. \tag{6.8}$$

Proposition 6.2. *Assume that $f \in \mathcal{F}_{n-1}$ satisfies (6.7). Then $Y_{n,f} : b_{n-1} \to S_n$ has a unique right inverse $Y_{n,f}^{-1} : b_n \to b_{n-1}$. Both $Y_{n,f}$ and its inverse depend analytically on f, on the domain defined by (6.7). Furthermore,*

$$\begin{aligned} \|Y_{n,f} - \vartheta_n\| &\le \varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1}) \circ f\|\,, \\ \|Y_{n,f}^{-1} - L_n^{-1}\| &\le \vartheta\varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1}) \circ f\|\,. \end{aligned} \tag{6.9}$$

Proof. Let $U = Y_{n,f} - L_n$. By the third condition in (6.1) we have

$$\|U(s)\| = \|\mathbb{P}_n R_n(f(s)) - L_n \mathbb{P}_{n-1} f(s)\| \le \varepsilon \|(\mathbb{I} - \mathbb{E}_{n-1}) f(s)\|\,, \tag{6.10}$$

for all $s \in b_{n-1}$ This implies the first bound in (6.9).

By our assumption on f and ε, we have $\|U\| \le \varepsilon \le r/2$, where $r = (1 - \vartheta)/2$. If $s \in S_{n-1}$ is of norm $\le \vartheta$ and $h \in S$ of norm one, then by Cauchy's formula

$$\|DU(s)h\| \le r^{-1} \sup_{|z|=r} \|U(s + zh)\| \le r^{-1} \|U\| \le 1/2\,. \tag{6.11}$$

The equation for a right inverse $L_n^{-1} + V$ of $L_n + U$ can be written as $\psi(V) = V$, with ψ defined by $\psi(V) = -L_n^{-1} U \circ (L_n^{-1} + V)$. Consider the space of analytic functions $V : b_n \to S_{n-1}$, equipped with the sup-norm. Denote by B the closed ball of radius r in this space, centered at the origin. Then ψ is analytic on B, with derivative given by

$$D\psi(V)h = -L_n^{-1} \bigl((DU) \circ (L_n^{-1} + V)\bigr) h\,. \tag{6.12}$$

By equation (6.11), we see that $\|D\psi(V)\| < 1/2$, for all $V \in B$. Since $\|\psi(0)\| \le r/2$, the map ψ is a contraction on B, and thus has a (unique) fixed point in B. This fixed point V satisfies $\|V\| = \|\psi(V)\| \le \|L_n^{-1} U\|$, which implies the second inequality in (6.9). The

analyticity of $U \mapsto V$ follows form the uniform convergence of $\psi^n(0) \to V$ for $\|U\| \le r/2$.
QED

This proposition allows us to define the maps

$$\mathfrak{R}_n(f) = R_n \circ f \circ Y_{n,f}^{-1}\,, \qquad \widetilde{\mathfrak{R}}_n = \mathfrak{R}_n \circ \mathfrak{R}_{n-1} \circ \ldots \circ \mathfrak{R}_1\,. \tag{6.13}$$

Notice that $\mathbb{P}_n \mathfrak{R}_n(f) = I_n$. In particular, since $R_n \circ \mathbb{P}_{n-1} = \mathbb{P}_n \circ R_n \circ \mathbb{P}_{n-1}$ by the second condition in (6.1), we have $\mathfrak{R}_n(I_{n-1}) = I_n$. The domain of $\mathfrak{R}_n$ is the set of all $f \in \mathcal{F}_{n-1}$ satisfying (6.7).

Lemma 6.3. *If f_0 belongs to the domain of $\mathfrak{R}_1$, then $\widetilde{\mathfrak{R}}_n(f_0)$ is well defined for all $n \ge 1$, and*

$$\|\widetilde{\mathfrak{R}}_n(f_0) - I_n\| \le \vartheta^{(n)} \|f_0 - I_0\|\,. \tag{6.14}$$

Proof. Let $n \ge 1$. Let f be an arbitrary function in the domain of $\mathfrak{R}_n$, and define $f' = \mathfrak{R}_n(f)$. Consider a fixed but arbitrary $s \in b_n$ and define $s' = Y_{n,f}^{-1}(s)$. By Proposition 6.2, s' belongs to b_{n-1}. Thus, the second condition in (6.1) implies that

$$\begin{aligned}\|f'(s) - s\| &= \|(\mathbb{I} - \mathbb{P}_n) R_n(f(s'))\| \\ &\le \vartheta_n \|(\mathbb{I} - \mathbb{P}_{n-1}) f(s')\| = \vartheta_n \|f(s') - s'\|\,.\end{aligned}$$

This shows that $\|f' - I_n\| \le \vartheta_n \|f - I_{n-1}\|$. In addition, we have $\mathbb{P}_n f' = I_n$ by the definition of $\mathfrak{R}_n$, and $\|(\mathbb{I} - \mathbb{E}_n) \circ f'\| \le \varepsilon_n \delta_{n-1}$ by the first inequality in (6.1). Thus, since $\vartheta_n < 1$ and $\varepsilon_n \delta_{n-1} \le \delta_n$, the function f' belongs to the domain of $\mathfrak{R}_{n+1}$. This proves Lemma 6.3.
QED

This lemma shows that the domain of $\widetilde{\mathfrak{R}}_n$ can be taken to be the domain of $\mathfrak{R}_1$. If f_0 is any function in this domain, define

$$f_n = \widetilde{\mathfrak{R}}_n(f_0)\,, \qquad Y_n = Y_{n,f_{n-1}}\,, \qquad Z_{m,n} = Y_{m+1}^{-1} \circ \ldots \circ Y_{n-1}^{-1} \circ Y_n^{-1}\,, \tag{6.15}$$

whenever $0 \le m < n$.

Proposition 6.4. *For every f in the domain of $\mathfrak{R}_1$, there exists a unique sequence $m \mapsto z_m \in b_m$ satisfying*

$$z_{m-1} = Y_m^{-1}(z_m)\,, \qquad m = 1, 2, \ldots\,, \tag{6.16}$$

and this sequence is given by the limits $z_m = \lim_{n\to\infty} Z_{m,n}(0)$. The maps $f \mapsto z_m$ are analytic on the domain of $\mathfrak{R}_1$.

Proof. First, we note that it suffices to prove the claims for $m \ge N$, where N is fixed but arbitrary positive integer.

Let $f_0 = f$. Since $\|Y_n^{-1} - L_n^{-1}\| \le \vartheta^n$ by Proposition 6.2 and Lemma 6.3, there exist an integer $N > 0$, and two positive real numbers $r, r' < 1$, both independent of f_0, such that Y_n^{-1} maps b_n into to $r' b_{n-1}$ and contracts distances by a factor $\le r$, whenever $n \ge N$. In what follows, we assume that $N \le m < n$.

Consider now an arbitrary sequence $n \mapsto s_n \in b_n$, with the property that s_n belongs to the closure of $r'b_n$. Notice that if a sequence $n \mapsto z_n \in b_n$ satisfies (6.16), then it automatically has this property. Define $s_{m,n} = Z_{m,n}(s_n)$. Then $\|s_{m,k} - s_{m,n}\| < 2r^{n-m}$ whenever $k > n$. This shows that $n \mapsto s_{m,n}$ converges as $n \to \infty$, and that the limit $\hat{s}_m$ is independent of the sequence $\{s_n\}$. In particular, we see that $\hat{s}_m = z_m$ by choosing $s_n = 0$ for all n. The identities (6.16) are obtained by choosing $s_n = z_n$ for all n.

By Proposition 6.2, the maps $f \mapsto s_{m,n} = Z_{m,n}(0)$ are analytic on the domain of $\mathfrak{R}_1$. The analyticity of $f \mapsto z_m$ now follows from the uniform convergence of $s_{m,n} \to z_m$. **QED**

Corollary 6.5. *Let f be a family in the domain of $\mathfrak{R}_1$, and let $s \in b_0$. Then $f(s)$ belongs to $\mathcal{W}_0$ if and only if $s = z_0(f)$.*

Proof. Consider first $x = f(z_0)$. Then $x \in D_0$, since f belongs to the domain of $\mathfrak{R}_1$, and the following holds for $n = 1, 2, \ldots$. Set $x_n = f_n(z_n)$. By the definition of $\mathfrak{R}_n$, and by Proposition 6.4, we have $x_n = R_n(x_{n-1}) = \widetilde{R}_n(x)$. Furthermore, $\mathbb{P}_n x_n = \mathbb{P}_n f_n(z_n) = z_n \in b_n$, and thus x belongs to the set $\widetilde{D}_n$ described in (6.6). This shows that $x \in \mathcal{W}_0$.

Consider now a fixed $s = s_0$ in b_0, and assume that $x_0 = f(s_0)$ belongs $\mathcal{W}_0$. Then we can define $x_n = \widetilde{\mathfrak{R}}_n(x)$ for all $n > 0$, and $s_n = \mathbb{P}_n x_n$ belongs to b_n. Set $f_0 = f$. Proceeding by induction, let $n > 0$, and assume that $x_{n-1} = f_{n-1}(s_{n-1})$. Since $s_n = Y_n(s_{n-1})$, and since Y_n has a unique right inverse on b_n by Proposition 6.2, we have $s_{n-1} = Y_n^{-1}(s_n)$. As a result, $x_n = f_n(s_n)$. This shows that $s_{n-1} = Y_n^{-1}(s_n)$ holds for all $n > 0$, and thus $s_n = z_n$ by Proposition 6.4. **QED**

Proof of Theorem 6.1. Denote by B_0' the unit ball in $(\mathbb{I} - \mathbb{P}_0)\mathcal{X}_0$, centered at the origin. To a point $x \in B_0'$ we associate the family $f : s \mapsto s + x$. This family belongs to the domain of $\mathfrak{R}_1$. Now define $W_0(x) = z_0(f)$. By Corollary 6.5, $x + s = f(s)$ belongs to $\mathcal{W}_0$ if and only if $s = W_0(x)$. This shows that $\mathcal{W}_0$ is the graph of W_0 over B_0'. The analyticity of W_0 follows from the analyticity of z_0. Furthermore, we have $W_0(0) = z_0(I_0) = 0$.

The second bound in (6.3) follows from the first condition in (6.1). In order to prove the first bound, consider the family $f_0(s) = s + (\mathbb{I} - \mathbb{P}_0)x$, the associated functions f_n and Y_n defined in (6.15), and the parameters z_n described in Proposition 6.4. Then $\widetilde{R}_n(x) = f_n(z_n)$ for all $n \geq 0$. By Lemma 6.3 we have

$$\|f_m(z_m) - z_m\| \leq \vartheta^{(m)} \|(\mathbb{I} - \mathbb{P}_0)x\|,$$

and by Proposition 6.2 and the second inequality in (6.3),

$$\begin{aligned}
\|z_m - L_{m+1}^{-1} \cdots L_n^{-1} z_n\| &= \left\| \sum_{k=m}^{n-1} L_{m+1}^{-1} \cdots L_k^{-1} \big[Y_{k+1}^{-1} - L_{k+1}^{-1}\big](z_{k+1}) \right\| \\
&\leq \sum_{k=m}^{n-1} \vartheta^{k-m} \vartheta\varepsilon\, \varepsilon^{(k)} \|(\mathbb{I} - \mathbb{E}_0)x\| \\
&\leq \frac{\vartheta\varepsilon}{1 - \vartheta\varepsilon} \varepsilon^{(m)} \|(\mathbb{I} - \mathbb{E}_0)x\|,
\end{aligned}$$

whenever $0 \le m < n$. These two inequalities, together with the fact that $L_{m+1}^{-1} \cdots L_n^{-1} z_n$ tends to zero as $n \to \infty$, imply the first bound in (6.3).

Next, assume that (6.4) holds. Then the equation (6.10) shows that for each $n > 0$, the map $f \mapsto Y_{n,f}$ has a vanishing derivative at $f = \mathrm{I}_{n-1}$. By the definition of W_0, this implies that $DW_0(0) = 0$.

Let now $x_0 \in \mathcal{W}_0$. Then $x_0 = u + W_0(u)$ with $u = (\mathbb{I} - \mathbb{P}_0)x_0$. Assume that $u \ne 0$. Let ℓ be a continuous linear functional on $\mathcal{X}_0$ of norm one, such that $\ell(W_0(u)) = \|W_0(u)\|$. Define $g(z) = \ell(W_0(zu/\|u\|))$ for all z in the complex unit disk $|z| < 1$. Since W_0 and DW_0 vanish at the origin, $z \mapsto z^{-2} g(z)$ defines an analytic function on the unit disk, and by Schwarz's lemma, this function is bounded in modulus by 1. Here, we have used that W_0 has norm less than one on its domain. This shows that $\|W_0(u)\| = g(\|u\|) \le \|u\|^2$, or in other words, that $\|\mathbb{P}_0 x_0\| \le \|(\mathbb{I} - \mathbb{P}_0)x_0\|^2$.

Finally, let $m > 0$ and consider the stable manifold $\mathcal{W}_m$ for the shifted sequence of maps $R_m, R_{m+1}, \ldots$. Clearly, $x_m = \widetilde{R}_m(x_0)$ belongs to $\mathcal{W}_m$. The same arguments as above show that $\|\mathbb{P}_m x_m\| \le \|(\mathbb{I} - \mathbb{P}_m)x_m\|^2$. The bound (6.5) now follows from the second condition in (6.1). **QED**

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