

# Landauer-Büttiker formula and Schrödinger conjecture

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**Abstract.** We study the entropy flux in the stationary state of a finite one-dimensional sample  $\mathcal{S}$  connected at its left and right ends to two infinitely extended reservoirs  $\mathcal{R}_{l/r}$  at distinct (inverse) temperatures  $\beta_{l/r}$  and chemical potentials  $\mu_{l/r}$ . The sample is a free lattice Fermi gas confined to a box  $[0, L]$  with energy operator  $h_{\mathcal{S},L} = -\Delta + v$ . The Landauer-Büttiker formula expresses the steady state entropy flux in the coupled system  $\mathcal{R}_l + \mathcal{S} + \mathcal{R}_r$  in terms of scattering data. We study the behavior of this steady state entropy flux in the limit  $L \rightarrow \infty$  and relate persistence of transport to norm bounds on the transfer matrices of the limiting half-line Schrödinger operator  $h_{\mathcal{S}}$ . A natural conjecture is that the set of energies at which transport persists in the limit  $L \rightarrow \infty$  is precisely the essential support of the absolutely continuous spectrum of  $h_{\mathcal{S}}$ . We show that this conjecture is equivalent to the Schrödinger conjecture in spectral theory of one-dimensional Schrödinger operators, thus giving a physically appealing interpretation to the Schrödinger conjecture.

## 1 Introduction

This paper is part of the program initiated in [AJPP1] and concerns transport in the so called electronic black box model. This model describes a sample  $\mathcal{S}$  (e.g., a quantum dot or a more elaborate electronic device) coupled to several electronic reservoirs  $\mathcal{R}_j$ . These reservoirs are free Fermi gas in thermal equilibrium at given temperatures and chemical potentials. In the independent electron approximation,

the coupled system  $\mathcal{S} + \sum_j \mathcal{R}_j$  is a free Fermi gas with single particle Hamiltonian  $h = h_0 + h_T$ , where  $h_0$  is the single particle Hamiltonian of the decoupled system and  $h_T$  is the tunneling Hamiltonian describing the junctions coupling  $\mathcal{S}$  to the reservoirs. As time  $t$  goes to infinity, the coupled system approaches a steady state which carries a non-trivial entropy flux. The celebrated Landauer-Büttiker formula gives a closed expression for this steady state entropy flux in terms of the scattering data of the pair  $(h, h_0)$ . This formula was rigorously proven in the context of non-equilibrium quantum statistical mechanics relatively recently [AJPP1, N]<sup>1</sup>. Given the Landauer-Büttiker formula, the next natural question is the dependence of the steady state entropy flux on the structure of the sample  $\mathcal{S}$  (its geometry, its size, etc). This paper is the first step in this direction of research.

We consider the special case where  $\mathcal{S}$  is a finite one-dimensional structure described in the tight binding approximation by the single particle Hamiltonian  $h_{\mathcal{S},L} = -\Delta_L + v$  on the Hilbert space  $\ell^2([0, L] \cap \mathbb{Z})$ . There  $\Delta_L$  is the discrete Laplacian with Dirichlet boundary conditions and  $v : \mathbb{Z}_+ \rightarrow \mathbb{R}$  is a potential on the half line  $\mathbb{Z}_+ = \{0, 1, \dots\}$ . This finite sample is coupled to two infinitely extended reservoirs, one at each of its boundary point. The resulting steady state entropy flux may vanish in the limit  $L \rightarrow \infty$  and our goal is to characterize the persistence of transport in this limit in terms of the spectral data of the limiting half-line Schrödinger operator  $h_{\mathcal{S}} = -\Delta + v$  acting on  $\ell^2(\mathbb{Z}_+)$ .

We start with a precise description of the model and the problem we study.

## 1.1 Setup

The electronic black box (EBB) model we consider in this paper is a special case of the class of models studied in [AJPP1], where the reader can find the proofs of the results described in this introductory section. A pedagogical introduction to the topic can be found in the lecture notes [AJJP2].

Consider two free Fermi gases  $\mathcal{R}_l$  and  $\mathcal{R}_r$ , colloquially called left and right reservoir, with single particle Hilbert space  $\mathfrak{h}_l$  and  $\mathfrak{h}_r$  and Hamiltonian  $h_l$  and  $h_r$ . The single particle Hilbert space  $\mathfrak{h}_{\mathcal{S}}$  of the sample  $\mathcal{S}$  is finite dimensional and its single particle Hamiltonian is  $h_{\mathcal{S}}$ . Until the very end of this section we shall not need to further specify the structure of  $\mathcal{S}$ . The EBB model we shall study is a free Fermi gas with single particle Hilbert space

$$\mathfrak{h} = \mathfrak{h}_l \oplus \mathfrak{h}_{\mathcal{S}} \oplus \mathfrak{h}_r.$$

The identity operators on  $\mathfrak{h}$ ,  $\mathfrak{h}_l$ ,  $\mathfrak{h}_r$ ,  $\mathfrak{h}_{\mathcal{S}}$  will be denoted  $1$ ,  $1_l$ ,  $1_r$ ,  $1_{\mathcal{S}}$ . Whenever the meaning is clear within the context, vectors and operators of the form  $\psi \oplus 0$ ,  $A \oplus 0$ ,  $\dots$  will be simply denoted by  $\psi$ ,  $A$ ,  $\dots$ . Accordingly,  $1_l$ ,  $1_r$ ,  $1_{\mathcal{S}}$  will be identified with the corresponding orthogonal projections in  $\mathfrak{h}$ .

For  $f \in \mathfrak{h}$ , we denote by  $a(f)/a^*(f)$  the annihilation/creation operators on the antisymmetric (fermionic) Fock space  $\mathcal{H} = \Gamma_-(\mathfrak{h})$  over  $\mathfrak{h}$ . In the sequel,  $a^{\#}(f)$  stands for  $a(f)$  or  $a^*(f)$ . The Hamiltonian of the decoupled EBB system is  $H_0 = d\Gamma(h_0)$ , the second quantization of

$$h_0 = h_l \oplus h_{\mathcal{S}} \oplus h_r.$$

The Hamiltonians and the number operators of the reservoirs are  $H_{l/r} = d\Gamma(h_{l/r})$  and  $N_{l/r} = d\Gamma(1_{l/r})$ .

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<sup>1</sup>We refer the reader to these papers for additional information on the Landauer-Büttiker formula and for references to the vast physics literature on the subject.

The algebra  $\text{CAR}(\mathfrak{h})$  of canonical anticommutation relations over  $\mathfrak{h}$  is the  $C^*$ -algebra generated by the set of operators  $\{a^\#(f) \mid f \in \mathfrak{h}\}$ . To any self-adjoint operator  $k$  on  $\mathfrak{h}$  one associates the Bogoliubov group

$$b_k^t(A) = e^{itd\Gamma(k)} A e^{-itd\Gamma(k)},$$

of automorphisms of  $\text{CAR}(\mathfrak{h})$ . Note that

$$b_k^t(a^\#(f)) = e^{itd\Gamma(k)} a^\#(f) e^{-itd\Gamma(k)} = a^\#(e^{itk} f).$$

$\vartheta^t = b_1^t$  is the gauge group of the EBB model. We shall assume that the total charge  $N = d\Gamma(1)$  is conserved. The corresponding superselection rule distinguishes the gauge-invariant sub-algebra

$$\text{CAR}_\vartheta(\mathfrak{h}) = \{A \in \text{CAR}(\mathfrak{h}) \mid \vartheta^t(A) = A \text{ for all } t\},$$

as the algebra of observables of the EBB model. The Bogoliubov group  $\tau_0^t = b_{h_0}^t$  preserves  $\text{CAR}_\vartheta(\mathfrak{h})$  and describes the time evolution of the decoupled EBB model. The pair  $(\text{CAR}_\vartheta(\mathfrak{h}), \tau_0^t)$  is a  $C^*$ -dynamical system.

For any self-adjoint operator  $\varrho$  on  $\mathfrak{h}$  satisfying  $0 \leq \varrho \leq 1$  the formula

$$\omega_\varrho(a^*(f_n) \cdots a^*(f_1) a(g_1) \cdots a(g_n)) = \det\{\langle g_i, \varrho f_j \rangle\},$$

defines a unique state  $\omega_\varrho$  on  $\text{CAR}_\vartheta(\mathfrak{h})$ . It is called the quasi-free state of density  $\varrho$  and is completely determined by its two point function

$$\omega_\varrho(a^*(f) a(g)) = \langle g, \varrho f \rangle.$$

The initial state of the EBB model is the quasi-free state  $\omega_0$  of density

$$\varrho_l \oplus \varrho_S \oplus \varrho_r,$$

where  $\varrho_{l/r}$  denotes the Fermi-Dirac density at inverse temperature  $\beta_{l/r} > 0$  and chemical potential  $\mu_{l/r} \in \mathbb{R}$ ,

$$\varrho_{l/r} = \frac{1_{l/r}}{1_{l/r} + e^{\beta_{l/r}(h_{l/r} - \mu_{l/r} 1_{l/r})}}, \quad (1.1)$$

and  $\varrho_S = 1_S$  (none of our results depends on this particular choice of  $\varrho_S$ ).  $\omega_0$  describes the thermodynamic state in which the reservoirs  $\mathcal{R}_{l/r}$  are in thermal equilibrium at inverse temperatures  $\beta_{l/r}$  and chemical potentials  $\mu_{l/r}$ .

The coupling we will consider is specified by a choice of non-zero vectors  $\chi_{l/r} \in \mathfrak{h}_{l/r}$ ,  $\psi_{l/r} \in \mathfrak{h}_S$ . The left/right junction is described by the rank two operator

$$h_{T,l/r} = |\chi_{l/r}\rangle\langle\psi_{l/r}| + |\psi_{l/r}\rangle\langle\chi_{l/r}|.$$

The single particle Hamiltonian of the coupled EBB model is

$$h = h_0 + h_T = h_0 + h_{T,l} + h_{T,r},$$

and its Hamiltonian is

$$H = d\Gamma(h) = H_0 + a^*(\psi_l)a(\chi_l) + a^*(\chi_l)a(\psi_l) + a^*(\psi_r)a(\chi_r) + a^*(\chi_r)a(\psi_r).$$

The dynamics of the coupled EBB model is described by the Bogoliubov group  $\tau^t = b_h^t$ . It preserves  $\text{CAR}_\vartheta(\mathfrak{h})$  and the pair  $(\text{CAR}_\vartheta(\mathfrak{h}), \tau^t)$  is a  $C^*$ -dynamical system. The coupled EBB model is described by the quantum dynamical system  $(\text{CAR}_\vartheta(\mathfrak{h}), \tau^t, \omega_0)$ .

We now describe the energy/charge/entropy flux observables. Although the self-adjoint operators  $H_{l/r}$  and  $N_{l/r}$  are not in  $\text{CAR}(\mathfrak{h})$ , the differences

$$\Delta H_{l/r}(t) = e^{itH} H_{l/r} e^{-itH} - H_{l/r}, \quad \Delta N_{l/r}(t) = e^{itH} N_{l/r} e^{-itH} - N_{l/r},$$

belong to  $\text{CAR}_\vartheta(\mathfrak{h})$  for any  $t \in \mathbb{R}$ , and one easily verifies the relations

$$\Delta H_{l/r}(t) = - \int_0^t \tau^s(\Phi_{l/r}) ds, \quad \Delta N_{l/r}(t) = - \int_0^t \tau^s(\mathcal{J}_{l/r}) ds,$$

where

$$\begin{aligned} \Phi_{l/r} &= -i[H, H_{l/r}] = d\Gamma(-i[h, h_{l/r}]) = a^*(ih_{l/r}\chi_{l/r})a(\psi_{l/r}) + a^*(\psi_{l/r})a(ih_{l/r}\chi_{l/r}), \\ \mathcal{J}_{l/r} &= -i[H, N_{l/r}] = d\Gamma(-i[h, 1_{l/r}]) = a^*(i\chi_{l/r})a(\psi_{l/r}) + a^*(\psi_{l/r})a(i\chi_{l/r}). \end{aligned} \tag{1.2}$$

The self-adjoint operators  $\Phi_{l/r}, \mathcal{J}_{l/r}$  belong to  $\text{CAR}_\vartheta(\mathfrak{h})$  and are observables describing, respectively, the energy and charge flux out of the reservoir  $\mathcal{R}_{l/r}$ . The associated entropy flux observable is

$$\sigma = -\beta_l(\Phi_l - \mu_l \mathcal{J}_l) - \beta_r(\Phi_r - \mu_r \mathcal{J}_r). \tag{1.3}$$

We recall the entropy balance equation [JP, Ru]

$$\text{Ent}(\omega_0 \circ \tau^t | \omega_0) = - \int_0^t \omega_0(\tau^s(\sigma)) ds, \tag{1.4}$$

where  $\text{Ent}(\cdot | \cdot)$  denotes Araki's relative entropy of two states [Ar]<sup>2</sup>. Since  $\text{Ent}(\cdot | \cdot) \leq 0$ , the balance equation ensures that for all  $t > 0$  the average entropy flux is non-negative,

$$\frac{1}{t} \int_0^t \omega_0(\tau^s(\sigma)) ds \geq 0, \tag{1.5}$$

in accordance with the second law of thermodynamics.

A basic characteristic of out of equilibrium physical systems is the presence of non-vanishing steady energy, charge and entropy fluxes. Sharp mathematical results concerning the existence and values of such fluxes can only be obtained in the idealization of the large time limit  $t \rightarrow \infty$ . To state the relevant result for the EBB model we need the assumption:

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<sup>2</sup>The entropy balance equation holds in a much wider context and is a very general structural property of non-equilibrium statistical mechanics.

(A1) The single particle Hamiltonian  $h$  has no singular continuous spectrum.

**Theorem 1.1** ([AJPP1]) *Suppose that (A1) holds. Then for all  $A \in \text{CAR}_\vartheta(\mathfrak{h})$  the limit*

$$\omega_+(A) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \omega_0(\tau^s(A)) ds,$$

*exists.*

The functional  $\omega_+$  is a state on  $\text{CAR}_\vartheta(\mathfrak{h})$  and is called Non-Equilibrium Steady State (NESS) of the EBB model. The entropy balance equation (1.5) ensures that  $\omega_+(\sigma) \geq 0$ . The existence of  $\omega_+$  is an open problem if  $h$  has some singular continuous spectrum.

Although the existence of a NESS for a given quantum dynamical system is generally a difficult analytical problem, the special quasi-free structure of the EBB model reduces the proof of Theorem 1.1 to the study of the spectral and scattering theory of the pair  $(h, h_0)$ . Moreover, the steady state expectation values  $\omega_+(\Phi_{l/r}), \omega_+(\mathcal{J}_{l/r}), \omega_+(\sigma)$ , can be expressed in closed form in terms of the scattering data of the pair  $(h, h_0)$ . The resulting expressions, the celebrated Landauer-Büttiker formulae, were rigorously proven in [AJPP1, N] and yield natural necessary and sufficient conditions for the strict positivity of  $\omega_+(\sigma)$ . We proceed to describe the Landauer-Büttiker formulae and the question we will study in this paper.

We start with some basic observations about the EBB model. Let  $\tilde{\mathfrak{h}}_{l/r} \subset \mathfrak{h}_{l/r}$  be the cyclic subspace generated by  $h_{l/r}$  and  $\chi_{l/r}$  (i.e., the smallest  $h_{l/r}$ -invariant subspace of  $\mathfrak{h}_{l/r}$  containing  $\chi_{l/r}$ ). The Hilbert space

$$\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_l \oplus \mathfrak{h}_S \oplus \tilde{\mathfrak{h}}_r,$$

is invariant under  $h$  and  $h_0$ , and  $\Phi_{l/r}, \mathcal{J}_{l/r}, \sigma \in \text{CAR}_\vartheta(\tilde{\mathfrak{h}})$ . Hence, for our purposes, w.l.o.g. we may replace  $\mathfrak{h}_{l/r}$  and  $\mathfrak{h}$  with  $\tilde{\mathfrak{h}}_{l/r}$  and  $\tilde{\mathfrak{h}}$  (we drop  $\tilde{\cdot}$  in the sequel). Let  $\nu_{l/r}$  be the spectral measure for  $h_{l/r}$  and  $\chi_{l/r}$ . By the spectral theorem we may assume that  $\mathfrak{h}_{l/r} = L^2(\mathbb{R}, d\nu_{l/r})$ ,  $\chi_{l/r}(E) = 1$  for all  $E \in \mathbb{R}$ , and that  $h_{l/r}$  is the operator of multiplication by the variable  $E$ . It follows that the density operator (1.1) acts by multiplication with the function

$$\varrho_{l/r}(E) = \frac{1}{1 + e^{\xi_{l/r}(E)}}, \quad \xi_{l/r}(E) = \beta_{l/r}(E - \mu_{l/r}).$$

The absolutely continuous spectral subspace of  $h_0$  is

$$\mathfrak{h}_{\text{ac}}(h_0) = \mathfrak{h}_{\text{ac}}(h_l) \oplus \mathfrak{h}_{\text{ac}}(h_r) = L^2(\mathbb{R}, d\nu_{l,\text{ac}}) \oplus L^2(\mathbb{R}, d\nu_{r,\text{ac}}),$$

where  $\nu_{l/r,\text{ac}}$  is the absolutely continuous part of  $\nu_{l/r}$  (w.r.t. Lebesgue measure). To avoid discussion of trivialities we shall always assume that  $\nu_{l/r,\text{ac}}$  is non-zero (if either  $h_l$  or  $h_r$  has no absolutely continuous spectrum then  $\omega_+(\Phi_{l/r}) = \omega_+(\mathcal{J}_{l/r}) = \omega_+(\sigma) = 0$ , see [AJPP1]). The essential support of the measure  $\nu_{l/r,\text{ac}}$ , defined by,

$$\Sigma_{l/r} = \left\{ E \in \mathbb{R} \left| \frac{d\nu_{l/r,\text{ac}}}{dE}(E) > 0 \right. \right\},$$

is also called the essential support of the absolutely continuous spectrum of  $h_{l/r}$ . The intersection of the supports

$$\Sigma_{l \cap r} = \Sigma_l \cap \Sigma_r,$$

will play an important role in the sequel. As usual in measure theory,  $\Sigma_{l/r}$  is only specified up to a set of Lebesgue measure zero. More precisely, it is an equivalence class of the relation

$$B_1 \stackrel{\circ}{=} B_2 \Leftrightarrow |B_1 \triangle B_2| = 0,$$

where  $B_1, B_2$  are Borel sets and  $|B|$  is the Lebesgue measure of  $B$ . As usual in measure theory we shall refer to such classes as sets.

Denote by  $1_{\text{ac}}(h_0)$  the orthogonal projection on  $\mathfrak{h}_{\text{ac}}(h_0)$ . It follows from the trace class scattering theory that the wave operators

$$w_{\pm} = \text{s-}\lim_{t \rightarrow \pm\infty} e^{ith} e^{-ith_0} 1_{\text{ac}}(h_0),$$

exist. The scattering matrix  $s = w_+^* w_-$  is a unitary on  $\mathfrak{h}_{\text{ac}}(h_0)$  and acts as the operator of multiplication by a unitary  $2 \times 2$  matrix function  $s(E)$ . We shall write this on-shell scattering matrix as

$$s(E) = 1 + t(E)$$

where

$$t(E) = \begin{bmatrix} t_{ll}(E) & t_{lr}(E) \\ t_{rl}(E) & t_{rr}(E) \end{bmatrix},$$

is the so-called  $t$ -matrix. The entry  $t_{lr/rl}(E)$  is the transmission amplitude from reservoir  $\mathcal{R}_{l/r}$  to the reservoir  $\mathcal{R}_{r/l}$  at energy  $E$  and  $|t_{lr/rl}(E)|^2$  is the corresponding transmission probability. We recall that, as a consequence of unitarity,  $|t_{lr}(E)|^2 = |t_{rl}(E)|^2$ . We set  $\mathcal{T}(E) = |t_{lr}(E)|^2$  and notice that, as a consequence of formula (2.15)

$$\{E \mid \mathcal{T}(E) > 0\} \stackrel{\circ}{=} \Sigma_{l \cap r}. \quad (1.6)$$

**Theorem 1.2** ([AJPP1]) *Suppose that (A1) holds. The steady state energy and charge currents are given by the following Landauer-Büttiker formulae*

$$\omega_+(\Phi_{l/r}) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{l/r}(E) dE, \quad \omega_+(\mathcal{J}_{l/r}) = \frac{1}{2\pi} \int_{\mathbb{R}} j_{l/r}(E) dE, \quad (1.7)$$

where

$$\varphi_{l/r}(E) = \mathcal{T}(E)(\varrho_{l/r}(E) - \varrho_{r/l}(E))E, \quad j_{l/r}(E) = \mathcal{T}(E)(\varrho_{l/r}(E) - \varrho_{r/l}(E)). \quad (1.8)$$

Thus, one can identify the functions  $\varphi_{l/r}$  and  $j_{l/r}$  as the spectral densities of energy and charge current in the NESS  $\omega_+$ . They satisfy the conservation laws

$$\varphi_l(E) + \varphi_r(E) = 0, \quad j_l(E) + j_r(E) = 0.$$

By Eq. (1.3), the steady state entropy flux is given by

$$\omega_+(\sigma) = \frac{1}{2\pi} \int_{\mathbb{R}} \varsigma(E) dE, \quad (1.9)$$

where the spectral density

$$\begin{aligned}\varsigma(E) &= -\beta_l(\varphi_l(E) - \mu_l j_l(E)) - \beta_r(\varphi_r(E) - \mu_r j_r(E)) \\ &= \mathcal{T}(E)(\xi_r(E) - \xi_l(E))(\varrho_l(E) - \varrho_r(E)),\end{aligned}\tag{1.10}$$

is non-negative, and

$$\{E \mid \varsigma(E) > 0\} \doteq \{E \mid |\varphi_{l/r}(E)| > 0\} \doteq \{E \mid |j_{l/r}(E)| > 0\}.$$

If  $\beta_l = \beta_r$  and  $\mu_l = \mu_r$  (*the equilibrium case*), then  $\varphi_{l/r}$ ,  $j_{l/r}$ , and  $\varsigma$  are zero functions. If either  $\beta_l \neq \beta_r$  or  $\mu_l \neq \mu_r$  (*the non-equilibrium case*), then (1.6) implies

$$\{E \mid \varsigma(E) > 0\} \doteq \Sigma_{l \cap r}.$$

The functions  $\varphi_{l/r}$ ,  $j_{l/r}$  and  $\varsigma$  are well defined and all the above properties hold even if  $h$  has some singular continuous spectrum. However, the current state of the art results require Assumption (A1) to link these functions to steady state currents and prove the Landauer-Büttiker formulae (1.7).

Note that in the non-equilibrium case  $\omega_+(\sigma) > 0$  iff  $|\Sigma_{l \cap r}| > 0$ , i.e.,  $\omega_+(\sigma) > 0$  iff there exists an open scattering channel between  $\mathcal{R}_l$  and  $\mathcal{R}_r$ . Note also that even if  $\omega_+(\sigma) > 0$ , it may happen that for some specific values of  $\beta_{l/r}$ ,  $\mu_{l/r}$  either  $\omega_+(\Phi_{l/r}) = 0$  or  $\omega_+(\mathcal{J}_{l/r}) = 0$ . However, in the non-equilibrium case,  $\omega_+(\Phi_{l/r})$  and  $\omega_+(\mathcal{J}_{l/r})$  cannot simultaneously vanish and generically they are both different from zero.

We now describe the question we shall study. Let  $v : \mathbb{Z}_+ \rightarrow \mathbb{R}$  be a given potential. Consider the finite lattice  $\Gamma_L = [0, L] \cap \mathbb{Z}_+$  and suppose that the single particle Hilbert space and Hamiltonian of the sample are  $\mathfrak{h}_{S,L} = \ell^2(\Gamma_L)$  and  $h_{S,L} = -\Delta_L + v_L$ , where  $(\Delta_L u)(x) = u(x-1) + u(x+1)$  is the discrete Laplacian on  $\Gamma_L$  with Dirichlet boundary conditions (i.e.,  $u(-1) = u(L+1) = 0$ ) and  $v_L$  is the restriction of the potential  $v$  to  $\Gamma_L$ . The reservoirs  $\mathcal{R}_{l/r}$  and the vector  $\chi_{l/r}$  are  $L$  independent. We take  $\psi_l = \delta_0$ ,  $\psi_r = \delta_L$  where  $\delta_x$  denotes the usual Kronecker delta at  $x \in \Gamma_L$ . We denote by  $h_{T,L}$  the corresponding tunneling Hamiltonian and set

$$h_L = h_{0,L} + h_{T,L}, \quad h_{0,L} = h_l \oplus h_{S,L} \oplus h_r.$$

Denote by  $\varphi_{l/r,L}$ ,  $j_{l/r,L}$  and  $\varsigma_L$  the spectral densities of the steady state fluxes and let

$$\begin{aligned}\overline{\mathfrak{X}} &= \{E \mid \limsup_{L \rightarrow \infty} \varsigma_L(E) > 0\}, \\ \underline{\mathfrak{X}} &= \{E \mid \liminf_{L \rightarrow \infty} \varsigma_L(E) > 0\}.\end{aligned}\tag{1.11}$$

Clearly,  $\underline{\mathfrak{X}} \subset \overline{\mathfrak{X}} \subset \Sigma_{l \cap r}$ . Note also that

$$\overline{\mathfrak{X}} = \{E \mid \limsup_{L \rightarrow \infty} |\varphi_{l/r,L}(E)| > 0\} = \{E \mid \limsup_{L \rightarrow \infty} |j_{l/r,L}(E)| > 0\},$$

and similarly for  $\underline{\mathfrak{X}}$ .

Let  $h_S = -\Delta + v$  be the limiting half-line Schrödinger operator acting on  $\ell^2(\mathbb{Z}_+)$ . If  $h_{S,L}$  is extended from  $\ell^2(\Gamma_L)$  to  $\ell^2(\mathbb{Z}_+)$  in the obvious way (by setting  $h_{S,L} = 0$  on  $\ell^2(\Gamma_L)^\perp$ ), then  $\lim_{L \rightarrow \infty} h_{S,L} = h_S$

in the strong resolvent sense.  $\delta_0$  is a cyclic vector for  $h_S$  and the corresponding spectral measure  $\nu_S$  contains the full spectral information about  $h_S$ . The set

$$\Sigma_S = \left\{ E \mid \frac{d\nu_{S,ac}}{dE}(E) > 0 \right\},$$

is the essential support of the absolutely continuous spectrum of  $h_S$ . Our work was motivated by the following

**Conjecture 1.** In the non-equilibrium case, for any choice of the reservoirs  $\mathcal{R}_{l/r}$  and for any potential  $v$ ,

$$\underline{\Sigma} \stackrel{\circ}{=} \overline{\Sigma} \stackrel{\circ}{=} \Sigma_S \cap \Sigma_{l \cap r}. \quad (1.12)$$

This is a natural conjecture on physical grounds. If (A1) holds for all  $L$  and the reservoirs are chosen so that  $\Sigma_S \subset \Sigma_{l \cap r}$ , then  $\Sigma_S$  is precisely the set of energies at which transport persists in the limit  $L \rightarrow \infty$ . Moreover, by Fatou's lemma, for any Borel set  $B \subset \Sigma_S$  of positive Lebesgue measure,

$$\liminf_{L \rightarrow \infty} \int_B \varsigma_L(E) dE > 0,$$

while the dominated convergence theorem implies

$$\lim_{L \rightarrow \infty} \int_{\mathbb{R} \setminus \Sigma_S} \varsigma_L(E) dE = 0.$$

Hence, the conjectured relation (1.12) would lead to a physically and mathematically very appealing characterization of the essential support of the absolutely continuous spectrum of half-line discrete Schrödinger operators.

We will not be able to settle Conjecture 1 in this paper. Our main result gives sharp characterizations of the sets  $\overline{\Sigma}$  and  $\underline{\Sigma}$  in terms of the growth of the norms of the transfer matrices associated to  $h_S$ . This characterization yields a partial result regarding Conjecture 1 and, combined with the results of Last-Simon [LS], shows that Conjecture 1 is equivalent to the so called Schrödinger conjecture (Conjecture 2 in the next section), which is considered by many as the most outstanding open problem in the spectral theory of one-dimensional Schrödinger operators<sup>3</sup>. This equivalence, which came as a surprise to us, sheds a light on the the Schrödinger conjecture and on the mechanism of non-equilibrium transport in this class of EBB models.

## 1.2 Results

Since in the equilibrium case  $\varsigma_L$  is identically equal to zero, in what follows we assume the non-equilibrium case, *i.e.*, that either  $\beta_l \neq \beta_r$  or  $\mu_l \neq \mu_r$ .

The transfer matrix at energy  $E$  is defined by the product

$$T_L(E) = \begin{bmatrix} v(L) - E & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} v(0) - E & -1 \\ 1 & 0 \end{bmatrix}. \quad (1.13)$$

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<sup>3</sup>We have made Conjecture 1 before we were aware of the Schrödinger conjecture.

We denote by  $\mathfrak{L}$  the collection of all sequences  $(L_k)_{k \in \mathbb{N}}$  of positive integers such that  $L_k \uparrow \infty$ . Our main results is

**Theorem 1.3** *There is a set  $S$  in the equivalence class of  $\Sigma_{l \cap r}$  such that, for any  $E \in S$  and any  $(L_k)_{k \in \mathbb{N}} \in \mathfrak{L}$ , the following statements are equivalent.*

(1)

$$\lim_{k \rightarrow \infty} \varsigma_{L_k}(E) = 0.$$

(2)

$$\lim_{k \rightarrow \infty} \|T_{L_k}(E)\| = \infty.$$

Let

$$\mathfrak{S}_0 = \left\{ E \mid \sup_L \|T_L(E)\| < \infty \right\}, \quad \mathfrak{S}_1 = \left\{ E \mid \liminf_{L \rightarrow \infty} \|T_L(E)\| < \infty \right\}.$$

An immediate consequence of Theorem 1.3 is

**Corollary 1.4** (1)

$$\underline{\mathfrak{S}} \doteq \mathfrak{S}_0 \cap \Sigma_{l \cap r}.$$

(2)

$$\overline{\mathfrak{S}} \doteq \mathfrak{S}_1 \cap \Sigma_{l \cap r}.$$

(3) *For any Borel set  $B \subset \mathfrak{S}_0 \cap \Sigma_{l \cap r}$  of positive Lebesgue measure,*

$$\liminf_{L \rightarrow \infty} \int_B \varsigma_L(E) dE > 0.$$

(4)

$$\lim_{L \rightarrow \infty} \int_{\mathbb{R} \setminus (\mathfrak{S}_1 \cap \Sigma_{l \cap r})} \varsigma_L(E) dE = 0.$$

It follows from fundamental results of Last and Simon (Theorems 1.1 and 1.2 in [LS]) that

$$\mathfrak{S}_0 \subset \Sigma_S \subset \mathfrak{S}_1,$$

and so, by Corollary 1.4, Conjecture 1 is equivalent to

**Conjecture 2.**  $\mathfrak{S}_0 \doteq \Sigma_S \doteq \mathfrak{S}_1$ .

We have learned from Yoram Last that Conjecture 2 is a fundamental open problem in the theory of one dimensional Schrödinger operators, known as the Schrödinger Conjecture (see [MMG] and Appendix C4 in [S]). It is widely believed that this conjecture holds for all potentials  $v$ .

Theorem 1.3 and Corollary 1.4 reformulate the Schrödinger conjecture, via the Landauer-Büttiker formula, in terms of non-equilibrium transport and shed a new light on its physical interpretation.<sup>4</sup> The conjecture appears very natural from the point of view of transport theory and its failure would provide examples of models with strikingly singular non-equilibrium transport on a set of energies of positive Lebesgue measure.

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## 2 Proofs

### 2.1 Preliminaries

We will denote by  $\text{sp}(A)$  the spectrum of a Hilbert space operator  $A$ , and write  $\text{Im } A = (A - A^*)/2i$ . If  $A$  is self-adjoint, then  $\text{sp}_{\text{ac}}(A)$  denotes its absolutely continuous spectrum and we write  $A > 0$  whenever  $\text{sp}(A) \subset ]0, \infty[$ .

In the following, we shall use indices  $a, b, c, \dots \in \{l, r\}$ . We define

$$F_a(z) = \langle \chi_a, (h_a - z)^{-1} \chi_a \rangle,$$

and denote by  $F(z)$  the  $2 \times 2$  diagonal matrix with entries  $F_{ab}(z) = \delta_{ab} F_a(z)$ . We also introduce the  $2 \times 2$  Green matrices  $G_L^{(0)}(z)$  and  $G_L(z)$  with entries

$$G_{ab,L}^{(0)}(z) = \langle \psi_a, (h_{\mathcal{S},L} - z)^{-1} \psi_b \rangle, \quad G_{ab,L}(z) = \langle \psi_a, (h_L - z)^{-1} \psi_b \rangle.$$

Next, we recall several basic facts regarding the boundary values of the resolvent and their role in spectral theory. A pedagogical introduction to this topic, including complete proofs, can be found in [J]. Let  $A$  be a self-adjoint operators on Hilbert space  $\mathfrak{H}$  and  $\psi_1, \psi_2 \in \mathfrak{H}$ . For Lebesgue a.e.  $E \in \mathbb{R}$  the boundary values

$$\langle \psi_1, (A - E - i0)^{-1} \psi_2 \rangle = \lim_{\epsilon \downarrow 0} \langle \psi_1, (A - E - i\epsilon)^{-1} \psi_2 \rangle, \quad (2.14)$$

exists and are finite. In the sequel, whenever we write  $\langle \psi_1, (A - E - i0)^{-1} \psi_2 \rangle$ , we will always assume that the limit exists and is finite. If the spectral measure  $\nu_{\psi_1, \psi_2}$  for  $A$  and  $\psi_1, \psi_2$  is real-valued, then either  $\psi_1$  is orthogonal to the cyclic subspace spanned by  $A$  and  $\psi_2$  and  $\nu_{\psi_1, \psi_2}$  is the zero measure or

<sup>4</sup>We remark that to link Corollary 1.4 with transport in non-equilibrium statistical mechanics one needs that the Landauer-Büttiker formulae hold for all  $L$  and hence that the coupled single particle Hamiltonian  $h_L$  has no singular continuous spectrum for all  $L$ . A concrete example of reservoirs where this is the case for any potential  $v$  is  $\mathfrak{h}_{l/r} = \ell^2(\mathbb{Z}_+)$ ,  $h_{l/r} = -k\Delta$ ,  $k > 0$ . For other examples and general results regarding this point we refer the reader to [GJW].

$\langle \psi_1, (A - E - i0)^{-1} \psi_2 \rangle \neq 0$  for Lebesgue a.e.  $E \in \mathbb{R}$ . If  $\psi \in \mathfrak{H}$  then  $\text{Im} \langle \psi, (A - E - i0)^{-1} \psi \rangle \geq 0$  and if  $\nu_\psi$  is the spectral measure for  $A$  and  $\psi$ , then

$$d\nu_{\psi,ac}(E) = \frac{1}{\pi} \text{Im} \langle \psi, (A - E - i0)^{-1} \psi \rangle dE,$$

so that the set  $\{E \mid \text{Im} \langle \psi, (A - E - i0)^{-1} \psi \rangle > 0\}$  is an essential support of  $\nu_{\psi,ac}$ .

In particular, one has

$$d\nu_{l/r,ac}(E) = \frac{1}{\pi} \text{Im} F_{l/r}(E + i0) dE,$$

and, with a slight abuse of notation, we may denote the following concrete representative of the class  $\Sigma_{l \cap r}$  by the same letter

$$\{E \mid \text{Im} F(E + i0) > 0\} = \Sigma_{l \cap r}.$$

In words,  $\Sigma_{l \cap r}$  consists of  $E$ 's for which the boundary values  $F_{l/r}(E + i0)$  exist, are finite, and have strictly positive imaginary part.

## 2.2 Green's and transfer matrices

It follows from stationary scattering theory (see [Y], Chap. 5) that the  $t$ -matrix  $t_L$  can be expressed in terms of the Green matrix  $G_L$  by

$$t_L(E) = 2i(\text{Im} F(E + i0))^{1/2} G_L(E + i0) (\text{Im} F(E + i0))^{1/2}. \quad (2.15)$$

The formulae (2.15) can be also proven directly by elementary means following the arguments in [JKP]. The unitarity of the on shell scattering matrix  $s_L(E) = 1 + t_L(E)$  implies that for Lebesgue a.e.  $E \in \mathbb{R}$ ,

$$t_L^*(E) t_L(E) + t_L(E) + t_L^*(E) = 0. \quad (2.16)$$

It follows that

$$\mathfrak{R} = \bigcap_L \{E \in \Sigma_{l \cap r} \mid \text{Eqs. (2.15) and (2.16) hold}\},$$

satisfies

$$\Sigma_{l \cap r} \doteq \mathfrak{R}.$$

The following lemma relates the Green matrices  $G_L^{(0)}$  and  $G_L$ .

**Lemma 2.1** *For  $E \in \mathfrak{R} \setminus \text{sp}(h_{S,L})$ , one has  $G_L^{(0)}(E) = (I - G_L^{(0)}(E)F(E + i0))G_L(E + i0)$ .*

**Proof.** For  $z \in \mathbb{C} \setminus \mathbb{R}$ , the second resolvent formula

$$(h_L - z)^{-1} - (h_{0,L} - z)^{-1} = -(h_{0,L} - z)^{-1} h_{T,L} (h_L - z)^{-1},$$

yields

$$G_{ab}(z) - G_{ab}^{(0)}(z) = - \sum_c G_{ac}^{(0)}(z) \langle \chi_c, (h_L - z)^{-1} \psi_b \rangle,$$

and

$$\langle \chi_c, (h_L - z)^{-1} \psi_b \rangle = -F_c(z) G_{cb}(z),$$

which combine to give the desired formula.  $\square$

We proceed to relate the Green matrix  $G_L^{(0)}$  with the transfer matrix (1.13).

**Lemma 2.2** For  $E \in \mathbb{R} \setminus \text{sp}(h_{S,L})$  and any  $x, y, u, v \in \mathbb{C}$  one has

$$G_L^{(0)}(E) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \iff T_L(E) \begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} y \\ v \end{bmatrix}.$$

In other words, the permutation matrix  $P^{(0)} : (x, y, u, v) \mapsto (u, x, y, v)$  maps the graph of  $G_L^{(0)}(E)$  into that of  $T_L(E)$ .

**Proof.** Fix  $L$  and  $E \in \mathbb{R} \setminus \text{sp}(h_{S,L})$ . For  $f \in \ell^2(\Gamma_L)$ , the function  $\psi(x) = \langle \delta_x, (h_{S,L} - E)^{-1} f \rangle$  satisfies the finite difference equation

$$(-\Delta + v - E)\psi = f, \quad (2.17)$$

with boundary conditions  $\psi(-1) = \psi(L+1) = 0$ . Using the transfer matrix

$$T(x, y) = T_x T_{x-1} \cdots T_{y+1}, \quad T_j = \begin{bmatrix} v(j) - E & -1 \\ 1 & 0 \end{bmatrix},$$

the solution of the initial value problem for Equ. (2.17) can be written as

$$\begin{bmatrix} \psi(x+1) \\ \psi(x) \end{bmatrix} = T(x, -1) \begin{bmatrix} \psi(0) \\ \psi(-1) \end{bmatrix} - \sum_{z=0}^x T(x, z) \begin{bmatrix} f(z) \\ 0 \end{bmatrix}.$$

Setting  $x = L$  and taking the boundary conditions into account yields

$$T_L(E) \begin{bmatrix} \psi(0) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \psi(L) \end{bmatrix} = \sum_{z=0}^L T(L, z) \begin{bmatrix} f(z) \\ 0 \end{bmatrix},$$

which is an equation for the unknown  $\psi(0)$  and  $\psi(L)$ . Setting  $f = \delta_0$  and  $f = \delta_L$ , we obtain the following equations for the entries of the matrix  $G_L^{(0)}(E)$ ,

$$T_L(E) \begin{bmatrix} G_{ul,L}^{(0)}(E) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ G_{rl,L}^{(0)}(E) \end{bmatrix}, \quad T_L(E) \begin{bmatrix} G_{lr,L}^{(0)}(E) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ G_{rr,L}^{(0)}(E) \end{bmatrix}.$$

Thus, the two linearly independent vectors  $(G_{ul,L}^{(0)}(E), 1, 0, G_{rl,L}^{(0)}(E))$  and  $(G_{lr,L}^{(0)}(E), 0, 1, G_{rr,L}^{(0)}(E))$  span the graph of  $T_L(E)$ . One easily checks that they are the images by the permutation matrix  $P^{(0)}$  of the two vectors  $(1, 0, G_{ul,L}^{(0)}(E), G_{rl,L}^{(0)}(E))$  and  $(0, 1, G_{lr,L}^{(0)}(E), G_{rr,L}^{(0)}(E))$  which span the graph of  $G_L^{(0)}(E)$ .  $\square$

Combining the two previous lemmata, we obtain the connection between the transfer matrix and the Green matrix  $G(E + i0)$ .

**Lemma 2.3** For  $E \in \mathfrak{R} \setminus \text{sp}(h_{S,L})$  and any  $x, y, u, v \in \mathbb{C}$  one has

$$G_L(E + i0) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \iff T_L(E) \begin{bmatrix} u \\ x + F_l(E + i0)u \end{bmatrix} = \begin{bmatrix} y + F_r(E + i0)v \\ v \end{bmatrix}.$$

In other words, the automorphism  $P : (x, y, u, v) \mapsto (u, x + F_l(E + i0)u, y + F_r(E + i0)v, v)$  of  $\mathbb{C}^4$  maps the graph of  $G_L(E + i0)$  into that of  $T_L(E)$ .

### 2.3 Proof of Theorem 1.3

Formulas (1.10) and (2.15) imply that Theorem 1.3 follows from

**Theorem 2.4** Let  $E \in \mathfrak{R} \setminus (\cup_L \text{sp}(h_{S,L})) \stackrel{\circ}{=} \Sigma_{l \cap r}$  and  $(L_k)_{k \in \mathbb{N}} \in \mathfrak{L}$  be given. Then the following statements are equivalent.

(1)

$$\lim_{k \rightarrow \infty} G_{lr, L_k}(E + i0) = 0.$$

(2)

$$\lim_{k \rightarrow \infty} \|T_{L_k}(E)\| = \infty.$$

**Proof.** (1)  $\Rightarrow$  (2). We start with the observation that the unitarity relation (2.16) implies  $\|t_L(E)\| \leq 2$ . It follows from (2.15) that the sequence  $\|G_{L_k}(E + i0)\|$  is bounded. Writing

$$G_{L_k}(E + i0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \end{bmatrix},$$

we conclude that the sequences  $u_k$  and  $v_k$  are bounded while (1) implies  $u_k = G_{lr, L_k}(E + i0) \rightarrow 0$ . It follows from Lemma 2.3 that

$$T_{L_k}(E) \begin{bmatrix} 1 \\ F_l(E + i0) \end{bmatrix} = \frac{1}{u_k} \begin{bmatrix} 1 + F_r(E + i0)v_k \\ v_k \end{bmatrix},$$

which clearly implies (2).

(2)  $\Rightarrow$  (1). There exists bounded sequences  $u_k$  and  $x_k$  such that, writing

$$T_{L_k}(E) \begin{bmatrix} u_k \\ x_k + F_l(E + i0)u_k \end{bmatrix} = \begin{bmatrix} y_k + F_r(E + i0)v_k \\ v_k \end{bmatrix},$$

the sequence  $|v_k| + |y_k|$  diverges to infinity. By Lemma 2.3, one has

$$G_{L_k}(E + i0) \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \end{bmatrix},$$

and the boundedness of  $\|G_{L_k}(E + i0)\|$  implies that  $|v_k| \leq A + B|y_k|$  for some positive constants  $A$  and  $B$ . We conclude that  $|y_k| \rightarrow \infty$  and (1) follows from

$$G_{lr, L_k}(E + i0) = \frac{u_k - G_{ll, L_k}(E + i0)x_k}{y_k}.$$

□

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