ON A RELATION BETWEEN TWO DIFFERENT PARTS OF THE
SPECTRUM OF A DISCRETE SCHRÖDINGER OPERATOR

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ABSTRACT. We study the properties of a discrete Schrödinger operator. We prove that if its spectrum is discrete in the complement of \([-2d, 2d]\), then it contains \([-2d, 2d]\).

1. Introduction and main results

In this paper, we discuss spectral properties of the self-adjoint operator \(H\) defined by

\[
H\psi(n) = -\sum_{|n-m|=1} \psi(m) + V(n)\psi(n), \quad n \in \mathbb{Z}^d,
\]

and acting in the Hilbert space \(l^2(\mathbb{Z}^d)\). If \(V = 0\), the spectrum of this operator coincides with the set \([-2d, 2d]\) and is absolutely continuous.

\[
\sigma(H_0) -2d 2d
\]

Fig. 1. The spectrum of \(H_0 = H - V\)

However, the spectrum might look different if \(V \neq 0\). In particular, it might have eigenvalues outside of the interval \([-2d, 2d]\). Typically, if \(V\) decays at infinity, then the spectrum of the operator \(H\) looks like the set displayed on the picture below.

\[
\lambda_1^- \lambda_2^- \sigma(H) \lambda_2^+ \lambda_1^+
\]

Fig. 2. The spectrum of \(H = H_0 + V\)

It turns out, that there is a relation between the continuous spectra and the sets of discrete eigenvalues of such operators. The rate of accumulation of the eigenvalues \(\lambda_j^\pm\) to the points \(\pm 2d\) determines the properties of the spectrum of the operator \(H\) in the interval \([-2d, 2d]\). In particular, if \(d = 1\) and the spectrum of \(H\) in \(\mathbb{R} \setminus [-2, 2]\) consists of finitely many eigenvalues, then the remaining part of the spectrum in \([-2, 2]\) is absolutely continuous (see [3]). The main result of the present paper gives the answer to the following question: what happens in the case where \(\lambda_j^\pm \to \pm 2d\) as \(j \to \infty\).
in an arbitrary way, when nothing is known about the rate of accumulation of the eigenvalues to the edges of the interval $[-2d, 2d]$?

**Theorem 1.1.** Let $V$ be a real-valued bounded function on $\mathbb{Z}^d$. Suppose that the spectrum of $H$ in $\mathbb{R} \setminus [-2d, 2d]$ is discrete. Then the spectrum of the operator $H$ contains the interval $[-2d, 2d]$.

\[ \lambda_1^- \lambda_2^- \quad \text{?} \quad \lambda_2^+ \lambda_1^+ \quad \Rightarrow \quad \sigma_{\text{ess}}(H) \]

Fig. 3. The left picture implies the right one

Since we do not impose any restriction on the dimension $d$, Theorem 1.1 is a new mathematical result. The problem of finding the relation between different parts of spectra of multi-dimensional operators first appeared in 2002, when Theorem 1.1 was proved for $d = 1$ and $d = 2$ by Damanik, Hundertmark, Killip and Simon [1]. Perhaps, the discussion conducted in the paper [1] was more related to the conditions leading to compactness of the perturbation $V$ rather than to the properties of the spectrum itself. Nevertheless, one of the statements of [1] tells us what the main difficulty in proving Theorem 1.1 for $d \geq 3$ is: its conditions do not imply that the difference $H - H_0$ is a compact operator. That makes the proof for $d \geq 3$ different from the one in $d = 1$ and $d = 2$.

Another difficulty in proving Theorem 1.1 is related to the fact that we do not assume that $V$ is sign-definite. It is true that the problem would be much easier if $V$ did not change the sign. However, we do not know any method that allows us to reduce the general case to the one where $V$ is either positive or negative. Moreover, in $d \geq 4$, there is a real potential $V = V_+ - V_-$ (in this formula, $2V_\pm = |V| \pm V$) such that the spectrum of $H = H_0 + V$ is a subset of $[-2d, 2d]$ but the absolutely continuous spectrum of at least one of the two operators $H_0 + V_+$ or $H_0 - V_-$ is not contained in $[-2d, 2d]$. (Hint: the potential in this example is independent of one of the variables.)

**Notations.** Below, \{e_1, e_2, \ldots, e_d\} denotes the standard orthonormal basis in $\mathbb{R}^d$. For a complex-valued function $u$ on $\mathbb{Z}^d$, the symbol $\nabla u(n)$ denotes the vector

\[ \nabla u(n) = \sum_{j=1}^d (u(n + e_j) - u(n)) e_j \]

For a vector-valued function $A : \mathbb{Z}^d \rightarrow \mathbb{R}^d$, the symbol $\text{div} A(n)$ denotes the number

\[ \text{div} A(n) = \sum_{j=1}^d (A_j(n) - A_j(n - e_j)), \quad A(n) = \sum_{j=1}^d A_j(n) e_j. \]

We also set

\[ \Delta u(n) = \text{div} \nabla u(n) \]
for any complex-valued function $u$ on $\mathbb{Z}^d$. Note that in these notations

$H_0 = -\Delta - 2d.$

Two classical results formulated below will be needed in the proof of Theorem 1.1. The first statement is the Weyl theorem. A sequence $u_n$ is called singular for a self-adjoint operator $A$ and $\lambda \in \mathbb{R}$, if

1) $u_n \in \text{Dom}(A), \quad \inf_n ||u_n|| > 0$;
2) $u_n$ converges to zero weakly;
3) $(A - \lambda)u_n$ converges to zero strongly (in the norm topology).

**Theorem 1.2. [H. Weyl]** Let $A$ be a self-adjoint operator in a separable Hilbert space. The condition that $\lambda \in \mathbb{R}$ is a point of the essential spectrum of $A$ is equivalent to existence of a singular sequence for $A$ and $\lambda$.

The next statement is a simple consequence of the mini-max principle.

**Proposition 1.3.** Let $A$ be a self-adjoint operator. Assume that the spectrum of the operator $A$ is situated to the right of the point $\lambda \in \mathbb{R}$. Then

$$(Au, u) \geq \lambda||u||^2$$

for all vectors $u$ from the domain of $A$.

2. **Proof of Theorem 1.1**

In order to proceed further we set

$$H_\pm = -\Delta \pm V$$

**Proposition 2.1.** The part of the spectrum of $H$ in $\mathbb{R} \setminus [-2d, 2d]$ is discrete if and only if the negative spectra of $H_\pm$ are discrete.

**Proof.** Since $H = H_+ - 2d$, it remains to prove that $-H$ is unitary equivalent to $H_- - 2d$. The latter fact is very well known: the corresponding unitary operator $U$ is defined by

$$Uv(n) = (-1)^{n_1+\cdots+n_d}v(n).$$

□

It is obvious that a potential $V$ satisfying the conditions of Theorem 1.1 can not be arbitrary. It is a function of a special type and the next statement gives us the idea of the properties that this function has.

**Lemma 2.2.** Let $V : \mathbb{Z}^d \rightarrow \mathbb{R}$ be a bounded potential. Assume that the negative spectra of the operators $H_+$ and $H_-$ are discrete. Then for any $\delta \in (0, 1)$ there exists an $R > 0$ such that $V$ is representable in the form

$$V(n) = \text{div} A(n) + |\tilde{A}(n)|^2 - \delta^2$$

(2.1)
for $|n| > R$. The bounded vector potentials $A : \mathbb{Z}^d \mapsto \mathbb{R}^d$ and $\tilde{A} : \mathbb{Z}^d \mapsto \mathbb{R}^d$ in (2.1) obey the condition
\[
\sup_{b > R} \left[ \sum_{b < |n| < b + 2 \delta^{-1} - 2} \frac{|A(n)|^2 + |\tilde{A}(n)|^2}{|n|^{d-1}} \right] < C \delta \tag{2.2}
\]
with a constant $C$ depending only on the dimension $d$ and the norm $||V||_\infty$. Moreover,
\[
\sup_{|n| > R} \left( |A(n)| + \frac{1}{2d} |\tilde{A}_j(n)|^2 \right) \leq ||V||_\infty + 2d + \delta^2. \tag{2.3}
\]
The vector components of $\tilde{A}$ are functions of the form $\tilde{A}_j(n) = \omega_j(n) A_j(n - e_j)$, where
\[
(||V||_\infty + 2d + \delta^2)^{-1} < \omega_j(n) < ||V||_\infty + 2d + \delta^2. \tag{2.4}
\]

**Corollary 2.3.** Let $V \in l^\infty(\mathbb{Z}^d)$ be a real valued function such that the negative spectra of $H_+$ and $H_-$ are discrete. Then
\[
\lim_{r \to \infty} \sup_{b > r} \left[ \sum_{b < |n| < b + 2 \delta^{-1} - 2} \frac{|V(n)|^2}{|n|^{d-1}} \right] = 0 \tag{2.5}
\]

**Proof.** According to the lemma, for any $\delta \in (0, 1)$, one can find $R$ such that $V$ is representable in the form (2.1) with $A$ and $\tilde{A}$ satisfy (2.2) and (2.3). Obviously,
\[
\sup_{b > R} \left[ \sum_{b + 1 < |n| < b - 3 + 2 \delta^{-1}} \frac{|\text{div } A(n)|^2}{|n|^{d-1}} \right] \leq c_0 \sup_{b > R} \left[ \sum_{b < |n| < b + 2 \delta^{-1} - 2} \frac{|A(n)|^2}{|n|^{d-1}} \right] \leq C \delta.
\]
Due to (2.3), we conclude also that
\[
\sup_{b > R} \left[ \sum_{b < |n| < b + 2 \delta^{-1} - 2} \frac{|\tilde{A}(n)|^4}{|n|^{d-1}} \right] \leq 2d(||V||_\infty + 2d + \delta^2) \sup_{b > R} \left[ \sum_{b < |n| < b + 2 \delta^{-1} - 2} \frac{|\tilde{A}(n)|^2}{|n|^{d-1}} \right] \leq 2d(||V||_\infty + 2d + \delta^2)C \delta.
\]
Therefore, if $0 < \delta < 1/4$, then
\[
\sup_{b > R} \left[ \sum_{b + 1 < |n| < b - 3 + 2 \delta^{-1}} \frac{|V(n)|^2}{|n|^{d-1}} \right] \leq C_V \delta,
\]
with a constant $C_V$ depending only on $||V||_\infty$ and the dimension $d$. \qed

In its turn, Lemma 2.2 follows from the two propositions formulated below.

**Proposition 2.4.** Assume that the negative spectra of the operators $H_+$ and $H_-$ are discrete. Then for any $\delta > 0$ there exists a bounded positive compactly supported function function $W \geq 0$ such that the spectra of the operators $H_+ + W$ on $\mathbb{Z}^d$ are situated to the right of the point $-\delta^2$.

The proof of Proposition 2.4 is similar to the proof of the corresponding proposition from the paper [6].
Proposition 2.5. Let $W : \mathbb{Z}^d \mapsto \mathbb{R}_+$ be a bounded positive function whose support is contained in the ball $\{ n \in \mathbb{Z}^d : |n| < R_0 \}$ of radius $R_0 > 2$. Let $\delta \in (0, 1)$. Suppose that the operators $H_+ + W$ and $H_- + W$ do not have spectra below $-\delta^2 < 0$. Then $V$ is representable in the form (2.1) in the region $|n| > R = 2\delta^{-1} + R_0$. The bounded vector potentials $A : \mathbb{Z}^d \mapsto \mathbb{R}_d$ and $\tilde{A} : \mathbb{Z}^d \mapsto \mathbb{R}_d$ in the representation (2.1) obey the conditions (2.2) and (2.3). The vector components of $\tilde{A}$ are functions of the form $\tilde{A}_j(n) = \omega_j(n)A_j(n - e_j)$, with $\omega_j(n)$ satisfying (2.4).

Proof. By adding a negative compactly supported function to $V$, one can always shift the bottom of the spectrum of $H_+ + W$ to the point $-\delta^2$. In this case, $-\delta^2$ is an eigenvalue of $H_+ + W$. Let $u$ be the corresponding eigenfunction. It is very well known that $u > 0$. It is also clear that

$$\max_{|n-m|=1} u(m) \leq -(H_0u)(n) = (W(n) + V(n) + 2d + \delta^2)u(n), \quad \forall n \in \mathbb{Z}^d. \quad (2.6)$$

Therefore, if we set $A = u^{-1}\nabla u$, then $A$ will be a bounded function on $\mathbb{Z}^d$, obeying

$$\sup_{|n|>R_0} |A(n)| \leq ||V||_{l^\infty} + 2d + \delta^2.$$

Moreover,

$$\text{div } A(n) = \sum_{j=1}^d \left( (u(n))^{-1}\nabla_j u(n) - (u(n-e_j))^{-1}\nabla_j u(n-e_j) \right) =$$

$$\sum_{j=1}^d \left( (u(n))^{-1}(u(n+e_j) - u(n)) - (u(n-e_j))^{-1}(u(n) - u(n-e_j)) \right) =$$

$$\sum_{j=1}^d \left( (u(n))^{-1}u(n+e_j) - (u(n-e_j))^{-1}u(n) \right) = (u(n))^{-1}\Delta u(n) + 2d -$$

$$\sum_{j=1}^d \left( (u(n))^{-1}u(n-e_j) + (u(n-e_j))^{-1}u(n) \right) = (u(n))^{-1}\Delta u(n) -$$

$$\sum_{j=1}^d \left( (u(n)u(n-e_j))^{-1}(u(n-e_j) - u(n))^2 \right) =$$

$$(u(n))^{-1}\Delta u(n) - \sum_{j=1}^d (u(n))^{-1}u(n-e_j)|A_j(n-e_j)|^2.$$

Consequently,

$$\text{div } A = u^{-1}\Delta u - |\tilde{A}|^2 = W + V + \delta^2 - |\tilde{A}|^2,$$

with $\tilde{A}_j(n) = ((u(n))^{-1}u(n-e_j))^{1/2}A_j(n-e_j)$. This proves (2.1).

The bound (2.4) is a simple implication of the two sided inequality

$$(||V||_{l^\infty} + 2d + \delta^2)^{-1} \leq u(m)/u(n) \leq ||V||_{l^\infty} + 2d + \delta^2, \quad |n-m| = 1, \quad |n|, |m| > R_0,$$
which follows from (2.6).

Note also that

$$|\tilde{A}_j(n)|^2 = (u(n))^{-1}u(n - e_j) + (u(n - e_j))^{-1}u(n) - 2,$$

which implies that \( \sup_{n > R} |\tilde{A}_j(n)|^2 \leq 2(||V||_{l_\infty} + 2d + \delta^2 - 1) \).

In order to prove (2.7), one has to use the fact that \( H_+ + W \geq -\delta^2 \). It means that for any \( \psi \in l^2(\mathbb{Z}^d) \),

$$\sum_{\mathbb{Z}^d} |\tilde{A}|^2|\psi(n)|^2 \leq \sum_{\mathbb{Z}^d} |\nabla \psi(n)|^2 - \sum_{\mathbb{Z}^d} (\text{div} A)|\psi(n)|^2 + \sum_{\mathbb{Z}^d} (W(n) + 2\delta^2)|\psi(n)|^2. \quad (2.7)$$

Summation by parts leads to the following estimate of the second term in the right hand side:

$$\left| \sum_{\mathbb{Z}^d} (\text{div} A)|\psi(n)|^2 \right| = \left| - \sum_{\mathbb{Z}^d} \sum_{j=1}^d A_j (|\psi(n + e_j)|^2 - |\psi(n)|^2) \right| \leq$$

$$\varepsilon_0 \sum_{\mathbb{Z}^d} \sum_{j=1}^d |A_j|^2 (|\psi(n + e_j)|^2 + |\psi(n)|^2) + \frac{2}{\varepsilon_0} \sum_{\mathbb{Z}^d} |\nabla \psi(n)|^2 \leq$$

$$2\varepsilon_0 \sum_{\mathbb{Z}^d} \sum_{j=1}^d \omega_j(n)^{-2}|\tilde{A}_j|^2|\psi(n)|^2 +$$

$$\varepsilon_0 \sum_{\mathbb{Z}^d} \sum_{j=1}^d |A_j|^2 (|\psi(n + e_j)| + |\psi(n)|)|\nabla \psi(n)| + \frac{2}{\varepsilon_0} \sum_{\mathbb{Z}^d} |\nabla \psi(n)|^2.$$

Combining this inequality with the relation (2.7), we obtain that

$$\left( 1 - 2\varepsilon_0(||V||_{l_\infty} + 2d + \delta^2) \right) \sum_{\mathbb{Z}^d} |\tilde{A}|^2|\psi(n)|^2 \leq \left( 1 + \frac{2}{\varepsilon_0} \right) \sum_{\mathbb{Z}^d} |\nabla \psi(n)|^2 +$$

$$\sum_{\mathbb{Z}^d} (W(n) + 2\delta^2)|\psi(n)|^2 + \varepsilon_0 \sum_{\mathbb{Z}^d} \sum_{j=1}^d |A_j|^2 (|\psi(n + e_j)| + |\psi(n)|)|\nabla \psi(n)| \leq$$

$$\left( 1 + \frac{2}{\varepsilon_0} \right) \sum_{\mathbb{Z}^d} |\nabla \psi(n)|^2 + \sum_{\mathbb{Z}^d} (W(n) + 2\delta^2)|\psi(n)|^2 +$$

$$\varepsilon_0 \sum_{\mathbb{Z}^d} \sum_{j=1}^d |A_j|^2 |\nabla \psi(n)|^2 + \varepsilon_0 \sum_{\mathbb{Z}^d} \sum_{j=1}^d |A_j|^2 (|\psi(n + e_j)|^2 + |\nabla \psi(n)|^2).$$

Consequently,

$$\left( 1 - 3\varepsilon_0(||V||_{l_\infty} + 2d + \delta^2) \right) \sum_{\mathbb{Z}^d} |\tilde{A}|^2|\psi(n)|^2 \leq$$

$$\left( 1 + \frac{2}{\varepsilon_0} + 2\varepsilon_0(||V||_{l_\infty} + 2d + \delta^2) \right) \sum_{\mathbb{Z}^d} |\nabla \psi(n)|^2 + \sum_{\mathbb{Z}^d} (W(n) + 2\delta^2)|\psi(n)|^2. \quad (2.8)$$
Let \( \zeta \) be the \( L^\infty(\mathbb{R}) \)-function defined by

\[
\zeta(t) = \begin{cases} 
0, & \text{if } t < -2; \\
t + 2, & \text{if } -2 < t < -1; \\
1, & \text{if } -1 < t < 1; \\
2 - t, & \text{if } 1 < t < 2; \\
0, & \text{if } t > 2.
\end{cases}
\] (2.9)

Obviously, the graph of this function looks as displayed on the following picture:

Fig. 4. The graph of \( \zeta \)

Setting \( \psi(n) = |n|^{-(d-1)/2} \zeta(|n| - a) \) in (2.8) and assuming that \( a \geq 3\delta^{-1} + R_0 \), we get the bound

\[
\sum_{-\delta^{-1} < |n| - a < \delta^{-1}} \frac{|\tilde{A}|^2}{|n|^{d-1}} \leq C_{d,V}\left(\delta + \sum_{|n| > a - 2\delta^{-1}} \frac{1}{|n|^{d+1}}\right),
\] (2.10)

where the constant \( C_{d,V} \) depends only on the dimension \( d \) and the norm \( ||V||_\infty \). The estimate (2.2) follows from (2.10) once we observe that

\[
\sum_{-\delta^{-1}+1 < |n| - a < \delta^{-1} - 1} \frac{|\tilde{A}|^2}{|n|^{d-1}} \leq (||V||_\infty + 2d + \delta^2) \sum_{-\delta^{-1} < |n| - a < \delta^{-1}} \sum_{j=1}^d \frac{|\tilde{A}_j|^2}{|n - e_j|^{d-1}}.
\]

The proof is completed. \( \square \)

The following result of Vainberg and Shaban [7] will be used in the proof of Theorem 1.1 to construct a singular sequence of functions for the operator \( H \).

**Theorem 2.6.** [cf. [7]] For every non-integer \( \lambda \in (0, 4d) \) there exists a fast decaying function \( f : \mathbb{Z}^d \mapsto \mathbb{C} \) and a non-empty subset \( S = \{ \theta \in \mathbb{R}^d : |\theta| = 1, |\theta - \theta_0| \leq \varepsilon \} \) of the unit sphere \( S_{d-1} \) such that the equation

\[-\Delta \psi = \lambda \psi + f\] (2.11)

has a solution with the asymptotics

\[
\psi(n) = e^{i\mu(n/|n|)|n|} \Psi\left(\frac{n}{|n|}\right) + O(|n|^{-(d+1)/2}), \quad \text{as } |n| \to \infty.
\] (2.12)

The function \( \Psi \) in this formula is different from zero and is smooth in a \( S_{d-1} \)-neighborhood of \( S \). The function \( \mu \) is real. This asymptotics is uniform in \( n/|n| \in S \). Moreover, \( f \)
in (2.11) satisfies
\[ |f(n)| \leq C_l (1 + |n|)^{-l}, \quad \forall l \in \mathbb{N}, \]
with constants \( C_l > 0 \) independent of \( n \).

The end of the proof of Theorem 1.1. Let \( \lambda, \psi, \Psi, S, \varepsilon > 0 \) and \( \theta_0 \in S_{d-1} \) be the same as in Theorem 2.6. Let \( \eta \) be a smooth function on the unit sphere with a support contained in the set \( \{ \theta \in S_{d-1} : |\theta - \theta_0| < \varepsilon \} \) and having the property that \( \eta \Psi \neq 0 \). Let \( \zeta \) be the function defined by (2.9). Set
\[ \zeta_j(n) = \eta(n/|n|) \zeta(j^{-1}(|n| - j^2)), \quad j \in \mathbb{N} \setminus \{0, 1\}, \ n \in \mathbb{Z}^d. \]
It is easy to see that
\[ \sup_n \left( |\nabla \zeta_j(n)| + |\Delta \zeta_j(n)| \right) \to 0, \quad \text{as} \ j \to \infty. \]
Define now
\[ \psi_j(n) := \frac{\zeta_j(n) \psi(n)}{||\zeta_j\psi||_2}, \quad n \in \mathbb{Z}^d. \quad (2.13) \]

**Proposition 2.7.** Under conditions of Theorem 1.1, the sequence \( \psi_j \) is singular for the operator \( H \) and the point \( \lambda' = \lambda - 2d \).

**Proof.** Indeed, since \( H \) is bounded, \( \psi_j \in \text{Dom}(H) = l^2(\mathbb{Z}^d) \) for all \( j \). Moreover, \( ||\psi_j|| = 1 \).

Since the support of \( \psi_j \) does not intersect the ball \( \{ n \in \mathbb{Z}^d : |n| < j^2 - 2j \} \), the sequence \( \psi_j \) converges to zero in the weak topology.

Finally, due to (2.5) and (2.12),
\[ ||V \psi_j||_2 \to 0, \quad \text{as} \ j \to \infty. \]
Therefore, it remains to prove that
\[ ||-\Delta \psi_j - \lambda \psi_j||_2 \to 0, \quad \text{as} \ j \to \infty. \quad (2.14) \]
For that purpose we conduct a very simple computation
\[
(\Delta \zeta_j \psi)(n) = \sum_{|m-n|=1} (\zeta_j(m) \psi(m) - \zeta_j(n) \psi(n)) = \sum_{|m-n|=1} ((\zeta_j(m) - \zeta_j(n)) \psi(m)) + \sum_{|m-n|=1} (\zeta_j(n)(\psi(m) - \psi(n))) = \sum_{|m-n|=1} ((\zeta_j(m) - \zeta_j(n)) \psi(m)) + \zeta_j(n)(f(n) - \lambda \psi(n)),
\]
which implies the estimate
\[ ||(-\Delta - \lambda) \zeta_j \psi||_2 \leq 2d ||\nabla \zeta_j||_\infty \sqrt{j} + C_l (1 + j)^{-l}, \quad \forall j, l \in \mathbb{N} \setminus \{0, 1\}. \]
According to the definition (2.13), that already leads to (2.14), since
\[ \liminf_{j \to \infty} \frac{||\zeta_j \psi||_2}{\sqrt{j}} > 0. \]
We see now that every non-integer point $\lambda \in [-2d, 2d]$ is a point of the spectrum of $H$. Consequently $[-2d, 2d] \subset \sigma(H)$. That completes the proof of Theorem 1.1. □

We conclude this paper by drawing the reader’s attention to the papers [2], [4]-[6] related to similar questions for “continuous” Schrödinger operators on $\mathbb{R}^d$. We also mention the paper [3] which deals with close but still somewhat different problems for the discrete operator.

REFERENCES


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