ON A RELATION BETWEEN THE POSITIVE AND NEGATIVE SPECTRA OF SCHRÖDINGER OPERATORS

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ABSTRACT. We study the properties of Schrödinger operators $-\Delta \pm V$. We prove that if their negative spectra are discrete, then their positive spectra do not have gaps.

1. INTRODUCTION AND MAIN RESULTS

Consider the Schrödinger operator

$$-\Delta + V(x)$$

acting in the space $L^2(\mathbb{R}^d)$. If $V = 0$, the operator has purely absolutely continuous spectrum covering the interval $[0, \infty)$.

![Fig. 1. The spectrum of $-\Delta$](image)

However, if $V \neq 0$, then the spectrum might be different. In particular, it might have negative eigenvalues. In any case, if $V$ decays at infinity, then the spectrum of the Schrödinger operator typically looks like the set displayed on the picture below.

![Fig. 2. The spectrum of $-\Delta + V$](image)

It turns out, that there is a relation between the left and the right parts of this picture, i.e. a relation between the continuous spectra and the sets of negative eigenvalues of Schrödinger operators. However, in order to describe this relation, one has to consider two operators

$$H_{\pm} = -\Delta \pm V.$$  

The following result is one of the theorems proven in [6].

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Theorem 1.1. [cf. [6]] Let $V$ be a real-valued bounded function on $\mathbb{R}^d$. Suppose that the spectra of $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ in $(-\infty, 0)$ consist of eigenvalues $\lambda_j(V)$ and $\lambda_j(-V)$, satisfying
\[
\sum_j |\lambda_j(V)|^{1/2} < \infty, \quad \sum_j |\lambda_j(-V)|^{1/2} < \infty.
\] (1.1)
Then the absolutely continuous spectrum of each operator $H_+, H_-$ is essentially supported on the set $[0, \infty)$.

Remark. Although this theorem can be proven in any dimension $d$, the paper [6] proves it only for $d = 3$. The arguments in $d \neq 3$ require a small modification. The case $d = 1$ was considered by Damanik and Remling in [2]. For a similar result handling the case of finitely many negative eigenvalues see the articles [1] and [4].

It feels like the rate of accumulation of the eigenvalues $\lambda_j(\pm V)$ to zero determines the properties of the positive spectrum of each operator $H_+, H_-$. What happens in the case where $\lambda_j(\pm V) \to 0$ in an arbitrary way, when nothing is known about the rate of accumulation of the eigenvalues to 0?

The main result of the present paper is the following statement:

Theorem 1.2. Let $V$ be a real-valued bounded function on $\mathbb{R}^d$. Suppose that the spectra of $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ in $(-\infty, 0)$ are discrete. Then the spectrum of each operator $H_+, H_-$ contains the interval $[0, \infty)$.

In the picture below, $\lambda_j^\pm$ is just a different notation of $\lambda_j(\pm V)$.

Fig. 4. The left parts of the pictures imply the right parts

Theorem 1.2 was proven for $d = 1$ by Damanik and Remling [2] in 2007. We prove it in any dimension $d$. This theorem is especially interesting when $d \geq 3$, because its conditions do not imply that the difference of the resolvent operators
\[
(-\Delta - z)^{-1} - (H_\pm - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
is compact. So, without a modification, the method of the paper [2] relying on Weyl’s theorem does not work in this case.
2. THE MAIN TECHNICAL LEMMAS

By a Schrödinger operator on a domain $\Omega \subset \mathbb{R}^d$, we always mean an operator with the Dirichlet boundary conditions. We will sometimes denote such operators by the symbols $H_+|_{\Omega}$ or $H_-|_{\Omega}$. More often, we will use the symbols $H_+$ and $H_-$, but in this case, we will provide a verbal description mentioning the domain $\Omega$.

**Proposition 2.1.** Let $V \in L^\infty(\mathbb{R}^d)$. Assume that the negative spectra of the operators $H_+$ and $H_-$ are discrete. Then for any $\gamma > 0$, there exists an $R > 0$ such that the spectra of the operators $H_+$ and $H_-$ on the domain $\{x \in \mathbb{R}^d : |x| > R\}$ are situated to the right of the point $-\frac{\gamma^2}{2}$.

This proposition follows from the two lemmas below.

The statement of the first lemma can be proven by integration by parts.

**Lemma 2.2.** Let $a > 0$. Let $\varphi$ be a real-valued bounded function with bounded derivatives of first order. Suppose that $\psi$ is a real-valued solution of

$$-\Delta \psi \pm V \psi = \lambda \psi$$

and the product $\varphi \psi$ vanishes on the boundary of the domain $\{a < |x| < b\}$. Then

$$\int_{a < |x| < b} \left( |\nabla (\varphi \psi)|^2 \pm V |\varphi \psi|^2 \right) dx = \int_{a < |x| < b} \left( |\nabla \varphi|^2 \psi^2 + \lambda |\varphi \psi|^2 \right) dx$$

**Lemma 2.3.** Let $a > 0$. Assume that the lowest eigenvalue $-\gamma^2$ of $H_\pm$ on the domain $\{x \in \mathbb{R}^d : |x| > a\}$ is negative. Then there is a number $b \geq a$ such that the lowest eigenvalue of $H_\pm$ on $\Omega = \{b < |x| < b + 6\gamma^{-1}\}$ is not bigger than $-\frac{\gamma^2}{2}$.

**Proof.** Let $\psi$ be the eigenfunction corresponding to the eigenvalue $-\gamma^2$ for the problem on the domain $\{x \in \mathbb{R}^d : |x| > a\}$ with the Dirichlet boundary conditions. Put $L = \gamma^{-1}$ and find $c > 0$ that gives the maximum to the functional $\int_{c-L < |x| < c+L} |\psi|^2 dx$. The latter integral is a continuous positive function of $c$, tending to zero as $c \to \infty$, so it does have a maximum. Define

$$\varphi(x) = \begin{cases} 1, & \text{if } ||x| - c| < L, \\ 0, & \text{if } ||x| - c| \geq 3L, \\ 3/2 - ||x| - c|/(2L), & \text{otherwise.} \end{cases}$$

Now, the interesting fact is that

$$\int_{|x| > a} |\nabla \varphi|^2 \psi^2 dx \leq \frac{\gamma^2}{2} \int_{|x| > a} |\varphi \psi|^2 dx$$
Lemma 2.3, the operator spectra of the operators $H$ exists a bounded positive function $W$. 

Proposition 2.4. 

Proof of Proposition 2.1. It is sufficient to prove this statement only for the operator $H_\pm$. Assume the opposite, that all operators $H_\pm$ on the domains of the form $\{x \in \mathbb{R}^d : |x| > R\}$ have an eigenvalue below the level $-\gamma^2$. Then, according to Lemma 2.3, there are infinitely many disjoint domains $\Omega_n = \{b_n < |x| < b_n + 6\gamma^{-1}\}$ such that $H_\pm$ on $\Omega_n$ has an eigenvalue smaller than $-\gamma^2/2$. The latter implies that the spectrum of $H_\pm$ has a negative accumulation point, which contradicts the assumptions of the proposition. 

Proof of Proposition 2.4. Assume that the spectra of the operators $H_+$ and $H_-$ on the domain $\{x \in \mathbb{R}^d : |x| > R\}$ do not intersect the interval $(-\infty, -\gamma^2]$ with $\gamma > 0$. Then there exists a bounded positive function $W$ supported in $\{x \in \mathbb{R}^d : |x| \leq R\}$ such that the spectra of the operators $H_\pm + W$ on $\mathbb{R}^d$ are situated to the right of the point $-2\gamma^2$.

Proof. Again, it is sufficient to consider only the operator $H_+$. Let $W = t\chi$ where $t > 0$ is a sufficiently large number and $\chi$ is the characteristic function of the ball $\{x \in \mathbb{R}^d : |x| \leq R\}$. Assume the opposite, that the spectra of the operators $H_+ + t\chi$ intersect the interval $(-\infty, -2\gamma^2)$ for all values of $t$. We will use Lemma 2.3 which holds for $a = 0$ if one omits all inequalities of the form $|x| > 0$ in the domains description. Since $H_+$ on $\{x \in \mathbb{R}^d : |x| > R\}$ does not have any spectrum below $-\gamma^2$, according to Lemma 2.3, the operator $H_+ + t\chi$ on the domain $\Omega := \{x \in \mathbb{R}^d : |x| < R + 3\sqrt{2}\gamma^{-1}\}$ must have at least one eigenvalue to the left of this point. Let $\lambda(t) \leq -\gamma^2$ be lowest eigenvalue of $H_+ + t\chi$ on $\Omega$. Note that $\lambda(t)$ is a monotonically increasing function of $t$. Hence, it has a limit at infinity

$$\lambda_0 = \lim_{t \to \infty} \lambda(t) \leq -\gamma^2.$$ 

Let $\psi_0$ be the corresponding normalized eigenfunction of the operator $H_+ + t\chi$ on $\Omega$. Then

$$\int_{\Omega} (|\nabla \psi_0|^2 + (V + t\chi)|\psi_0|^2) dx = \lambda(t) < 0, \quad (2.1)$$
which implies that
\[ \int_{\Omega} |\nabla \psi_t|^2 dx \leq ||V||_{L^\infty}. \]
Thus, the \( H^1 \)-norms of the functions \( \psi_t \) are bounded by \( \sqrt{1 + ||V||_{L^\infty}} \). Consequently, there are numbers \( t_n \) such that the sequence of functions \( \psi_{t_n} \) converges in \( H^1(\mathbb{R}^d) \) weakly to a function \( \psi \). In order to prove that \( \psi \neq 0 \), we simply observe that \( \psi_{t_n} \) converges to \( \psi \) in \( L^2(\Omega) \) by the Sobolev embedding theorem and that \( ||\psi_{t_n}||_{L^2} = 1 \) for all \( n \). It also follows from the estimate (2.1) that
\[ \int_{|x|<R} |x|<R |\nabla \psi_t|^2 dx \leq 1 \]
which tells us that \( \psi(x) = 0 \) for all \( x \in \{ |x| < R \} \).

In order to get a contradiction, it is sufficient to show that \( \psi \) satisfies the equation
\[ -\Delta \psi + V \psi = \lambda_0 \psi \tag{2.2} \]
in the domain \( \{ x \in \mathbb{R}^d : R < |x| < R + 3\sqrt{2}\gamma^{-1} \} \). The latter relation follows from the equality
\[ \int_{\Omega} (\nabla \psi_t \nabla \varphi + V \psi_t \varphi) dx = \lambda(t) \int_{\Omega} \psi_t \varphi dx, \tag{2.3} \]
which holds for all \( \varphi \in C_0^\infty \left( \{ x \in \mathbb{R}^d : R < |x| < R + 3\sqrt{2}\gamma^{-1} \} \right) \)
Setting \( t = t_n \) and passing to the limit as \( n \to \infty \) in (2.3), we obtain that
\[ \int_{\Omega} (\nabla \psi \nabla \varphi + V \psi \varphi) dx = \lambda_0 \int_{\Omega} \psi \varphi dx, \]
which is equivalent to (2.2). Equation (2.2) contradicts the assumption that the spectrum of \( H_+ \) on \( \{|x| > R\} \) does not intersect the interval \( (-\infty, -\gamma^2] \). \( \square \)

In the proposition below, \( S_{d-1} \) is the unit sphere and \( |S_{d-1}| \) is its surface area.

**Proposition 2.5.** Let \( W : \mathbb{R}^d \to \mathbb{R}_+ \) be a bounded positive function whose support is contained in the ball \( \{ x \in \mathbb{R}^d : |x| < R_0 \} \) of radius \( R_0 > 0 \). Suppose that the operators \( H_+ + W \) and \( H_- + W \) do not have spectra below \( -\delta^2 < 0 \). Then there is a sufficiently large positive \( R \geq 2\delta^{-1} + R_0 \) such that \( V \) is representable in the form
\[ V(x) = \text{div} A(x) + |A(x)|^2 - \delta^2 \tag{2.4} \]
in the region \( |x| > R \). Moreover, the vector potential \( A : \mathbb{R}^d \to \mathbb{R}^d \) obeys the condition
\[ \sup_{b>R^1} \left[ \int_{b<|x|<b+2\delta^{-1}} \frac{|A|^2}{|x|^{d-1}} dx \right] < C \delta \tag{2.5} \]
with
\[ C = 28 + \frac{(d-1)^2}{2} |S_{d-1}|. \]
Proof. By adding a negative compactly supported function to $V$, one can always shift the bottom of the spectrum of $H_+ + W$ to the point $-\delta^2$. In this case, $-\delta^2$ is an eigenvalue of $H_+ + W$. Let $u$ be the corresponding eigenfunction. It is very well known that $u > 0$. Set $A = u^{-1}\nabla u$. Then
\[ \text{div} A = u^{-1}\Delta u - |A|^2 = W + V + \delta^2 - |A|^2. \]
This proves (2.4).

In order to prove (2.5), one has to use the fact that $H_- + W \geq -\delta^2$. It means that for any $\psi \in H^1(\mathbb{R}^d)$,
\[ \int_{\mathbb{R}^d} |A|^2|\psi|^2 \, dx \leq \int_{\mathbb{R}^d} |\nabla \psi|^2 \, dx - 2 \int_{\mathbb{R}^d} (\text{div} A)|\psi|^2 \, dx + \int_{\mathbb{R}^d} (W(x) + \delta^2)|\psi|^2 \, dx. \tag{2.6} \]
Integration by parts leads to the estimate of the second term in the right hand side:
\[ \left| \int_{\mathbb{R}^d} (\text{div} A)|\psi|^2 \, dx \right| \leq \frac{1}{2} \int_{\mathbb{R}^d} |A|^2|\psi|^2 \, dx + 2 \int_{\mathbb{R}^d} |\nabla \psi|^2 \, dx. \]
Combining this inequality with the relation (2.6), we obtain that
\[ \int_{\mathbb{R}^d} |A|^2|\psi|^2 \, dx \leq 6 \int_{\mathbb{R}^d} |\nabla \psi|^2 \, dx + 2 \int_{\mathbb{R}^d} (W(x) + \delta^2)|\psi|^2 \, dx. \tag{2.7} \]
Let $\zeta$ be the $H^1(\mathbb{R})$-function defined by
\[ \zeta(t) = \begin{cases} 0, & \text{if } t < -2; \\ t + 2, & \text{if } -2 < t < -1; \\ 1, & \text{if } -1 < t < 1; \\ 2 - t, & \text{if } 1 < t < 2; \\ 0, & \text{if } t > 2. \end{cases} \]
Setting $\psi(x) = |x|^{-(d-1)/2}\zeta(\delta|x| - a)$ in (2.7), we get the bound
\[ \int_{-\delta^{-1} < |x| - a < \delta^{-1}} \frac{|A|^2}{|x|^{d-1}} \, dx \leq 6 \left( 4\delta + \frac{(d-1)^2}{2} \int_{|x| > a - \delta^{-1}} \frac{1}{|x|^{d+1}} \, dx \right) + 4\delta. \tag{2.8} \]
The estimate (2.5) follows from (2.8) once we assume that $a \geq 2\delta^{-1} + R_0$. \qed

Proposition 2.6. Let $u \in H^1(\mathbb{R}^d)$ be a spherically symmetric function and let $n > 1$. Then
\[ \sup_{n < r < n+1} \left( |u(r)|^2 r^{d-1} \right) \leq C \int_{n < |x| < n+1} (|\nabla u|^2 + |u|^2) \, dx \tag{2.9} \]
with a constant $C$ depending only on the dimension $d$.

Proof. Set $\varphi(r) = r^{(d-1)/2}u(r)$. According to the Sobolev embedding theorem,
\[ \sup_{n < r < n+1} |\varphi(r)|^2 \leq C_1 \int_{n}^{n+1} (|\varphi'(r)|^2 + |\varphi(r)|^2) \, dr. \]
On the other hand, \( \varphi'(r) = \frac{(d-1)}{2r} r^{(d-1)/2} u(r) + r^{(d-1)/2} u'(r) \). Therefore,

\[
|\varphi'(r)|^2 \leq c_0 (|\varphi(r)|^2 + r^{d-1} |u'(r)|^2), \quad \text{for } r > 1.
\]

Consequently,

\[
\int_n^{n+1} (|\varphi'(r)|^2 + |\varphi(r)|^2) dr \leq C_2 \int_{n<|x|<n+1} (|\nabla u|^2 + |u|^2) dx,
\]

which implies (2.9). \( \square \)

**Proposition 2.7.** Let \( u \) be a spherically symmetric function of the class \( \mathcal{H}^2(\mathbb{R}^d) \) and let \( n > 1 \). Then

\[
\sup_{n < |x| < n+1} \left( |\nabla u|^2 |x|^{d-1} \right) \leq C \int_{n<|x|<n+1} (|\Delta u|^2 + |\nabla u|^2 + |u|^2) dx \tag{2.10}
\]

with a constant \( C \) depending only on the dimension \( d \).

**Proof.** Set again \( \varphi(r) = r^{(d-1)/2} u(r) \). Then

\[
r^{d-1} |\nabla u|^2 \leq c_0 \left[ |\varphi'(r)|^2 + \frac{1}{r} |\varphi(r)|^2 \right] \leq c_0 \left[ |\varphi'(r)|^2 + |\varphi(r)|^2 \right], \quad \text{for } r > 1. \tag{2.11}
\]

By the Sobolev embedding theorem,

\[
\sup_{n < |x| < n+1} \left( |\varphi'(r)|^2 + |\varphi(r)|^2 \right) \leq C_1 \int_n^{n+1} (|\varphi''(r)|^2 + |\varphi'(r)|^2 + |\varphi(r)|^2) dr. \tag{2.12}
\]

On the other hand, \( r^{(d-1)/2} \Delta u = \varphi'' - \frac{\kappa}{d} \varphi \), where \( 4\kappa_d = (d-1)(d-3) \). Therefore,

\[
|\varphi''(r)|^2 \leq c_0 \left( r^{d-1} |\Delta u|^2 + |\varphi|^2 \right), \quad \text{for } r > 1. \tag{2.13}
\]

Moreover,

\[
|\varphi'(r)|^2 \leq c_0 \left( r^{d-1} |\nabla u|^2 + |\varphi|^2 \right), \quad \text{for } r > 1. \tag{2.14}
\]

Combining the inequalities (2.11)-(2.14), we obtain (2.10). \( \square \)

**Corollary 2.8.** Let \( \Omega_n = \{ x \in \mathbb{R}^d : n < |x| < n+1 \} \) where \( n > 1 \). Assume that \( W \geq 0 \) is a locally integrable function on \( \mathbb{R}^d \). Let also \( u \in \mathcal{H}^1(\mathbb{R}^d) \) and \( v \in \mathcal{H}^2(\mathbb{R}^d) \) be two spherically symmetric functions. Then

\[
\int_{|x| > 1} W |u|^2 dx \leq C_d \sup_{n > 1} \int_{\Omega_n} \frac{W}{|x|^{d-1}} dx \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2) dx, \tag{2.15}
\]

\[
\int_{|x| > 1} W |\nabla v|^2 dx \leq \tilde{C}_d \sup_{n > 1} \int_{\Omega_n} \frac{W}{|x|^{d-1}} dx \int_{\mathbb{R}^d} (|\Delta v|^2 + |\nabla v|^2 + |v|^2) dx \tag{2.16}
\]

with constants \( C_d \) and \( \tilde{C}_d \) depending only on the dimension \( d \).
Proof. The first inequality follows from the estimate
\[
\int_{\Omega_n} W|u|^2 \, dx \leq C_d \int_{\Omega_n} \frac{W}{|x|^{d-1}} \, dx \int_{\Omega_n} \left( |\nabla u|^2 + |u|^2 \right) \, dx.
\]
The second inequality follows from the bound
\[
\int_{\Omega_n} W|\nabla u|^2 \, dx \leq \tilde{C}_d \int_{\Omega_n} \frac{W}{|x|^{d-1}} \, dx \int_{\Omega_n} \left( |\Delta u|^2 + |\nabla u|^2 + |u|^2 \right) \, dx. \quad \Box
\]

Proposition 2.9. Let \( W : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) be a bounded positive function whose support is contained in the ball \( \{ x \in \mathbb{R}^d : |x| < R_0 \} \) of radius \( R_0 > 0 \). Suppose that the operators \( H_+ + W \) and \( H_- + W \) do not have spectra below \( -\delta^2 < 0 \), where \( \delta < 1 \). Let also \( R \) be the same as in Proposition 2.5. If a spherically symmetric function \( u \in H^2(\mathbb{R}^d) \) and \( v \in H^1(\mathbb{R}^d) \) both vanish in the ball \( \{|x| \leq R\} \), then
\[
\left| \int_{\mathbb{R}^d} V(x) \bar{u} \, v \, dx \right| \leq C\delta ||u||_{H^2} ||v||_{H^1} \quad (2.17)
\]
with a constant \( C \) depending only on the dimension \( d \).

Proof. We already know that \( V \) is representable in the form (2.4) with \( A \) obeying (2.5). Integrating by parts, we obtain
\[
\left| \int_{\mathbb{R}^d} \text{div} A u \bar{v} \, dx \right|^2 = \left| \int_{\mathbb{R}^d} A (\nabla u \bar{v} + u \nabla \bar{v}) \, dx \right|^2 \leq
\]
\[
2 \int_{\mathbb{R}^d} |A|^2 (|\nabla u|^2 + |u|^2) \, dx \cdot \int_{\mathbb{R}^d} (|\nabla v|^2 + |v|^2) \, dx \leq
\]
\[
C_d \delta \int_{\mathbb{R}^d} (|\Delta u|^2 + |\nabla u|^2 + |u|^2) \, dx \cdot \int_{\mathbb{R}^d} (|\nabla v|^2 + |v|^2) \, dx. \quad (2.18)
\]
Similarly,
\[
\left| \int_{\mathbb{R}^d} |A|^2 u \bar{v} \, dx \right|^2 = \left( \int_{\mathbb{R}^d} |A|^2 |u|^2 \, v \, dx \right) \left( \int_{\mathbb{R}^d} |A|^2 |\bar{v}|^2 \, dx \right) \leq
\]
\[
C_d \delta \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2) \, dx \cdot \left( \int_{\mathbb{R}^d} |A|^2 |\bar{v}|^2 \, dx \right). \quad \text{Note that inequality (2.7) holds for } \psi = v \text{ as well. It implies the estimate}
\]
\[
\int_{\mathbb{R}^d} |A|^2 |v|^2 \, dx \leq 6 ||v||_{H^1} \quad \text{Consequently,}
\]
\[
\left| \int_{\mathbb{R}^d} |A|^2 u \bar{v} \, dx \right|^2 = \left( \int_{\mathbb{R}^d} |A|^2 |u|^2 \, v \, dx \right) \left( \int_{\mathbb{R}^d} |A|^2 |\bar{v}|^2 \, dx \right) \leq 6 C_d \delta ||u||_{H^1} ||v||_{H^1}. \quad (2.19)
\]
Combining (2.18) and (2.19) with (2.4) we obtain (2.20). \( \Box \)

The next result follows from Propositions 2.1, 2.4 and 2.9.
Theorem 2.10. Let $V \in L^\infty(\mathbb{R}^d)$. Assume that the negative spectra of $H_+$ and $H_-$ are discrete. Then for any $\varepsilon > 0$ there exists an $R > 0$ such that if a spherically symmetric function $u \in \mathcal{H}^2(\mathbb{R}^d)$ and $v \in \mathcal{H}^1(\mathbb{R}^d)$ both vanish in the ball $\{|x| \leq R\}$, then
\[
\left| \int_{\mathbb{R}^d} V(x) u \bar{v} \, dx \right| \leq \varepsilon \|u\|_{\mathcal{H}^2} \|v\|_{\mathcal{H}^1}
\] (2.20)

In order to proceed further, we choose a function $\zeta \in C^\infty(\mathbb{R}^d)$ equal to 1 in the domain $\{x \in \mathbb{R}^d : |x| > 2\}$ and vanishing in the unit ball $\{x \in \mathbb{R}^d : |x| < 1\}$. Set $\zeta_R(x) = \zeta(x/R)$.

Corollary 2.11. Let the conditions of Theorem 1.2 be fulfilled. Let $P$ be the orthogonal projection onto the subspace of spherically symmetric functions in $L^2(\mathbb{R}^d)$. Then the operator-norm
\[
\|(-\Delta + I)^{-1/2} \zeta_R V \zeta_R (-\Delta + I)^{-1} P \|
\]
tends to zero as $R \to \infty$.

Proof. Let us show that for any $\varepsilon > 0$ there exists an $R_0 > 0$ such that
\[
\left| \left( (-\Delta + I)^{-1/2} \zeta_R V \zeta_R (-\Delta + I)^{-1} P, g \right) \right| \leq \varepsilon \|f\| \cdot \|g\|
\]
for all $f \in L^2(\mathbb{R}^d)$, $g \in L^2(\mathbb{R}^d)$ and $R > R_0$. The latter statement follows from Theorem 2.10 with $u = \zeta_R (-\Delta + I)^{-1} P f$ and $v = \zeta_R (-\Delta + I)^{-1/2} g$. Additionally, one needs to note that
\[
\|u\|_{\mathcal{H}^2} \leq C \|f\|, \quad \text{and} \quad \|v\|_{\mathcal{H}^1} \leq C \|g\|.
\]

Corollary 2.12. Let the conditions of Theorem 1.2 be fulfilled. Let $P$ be the orthogonal projection onto the subspace of spherically symmetric functions in $L^2(\mathbb{R}^d)$. Then
\[
(-\Delta + I)^{-1/2} V (-\Delta + I)^{-1} P
\]
is a compact operator.

Proof. It is clear that
\[
(-\Delta + I)^{-1/2} V (-\Delta + I)^{-1} P = (-\Delta + I)^{-1/2} (1 - \zeta_R) V (-\Delta + I)^{-1} P +
\]
\[
(-\Delta + I)^{-1/2} \zeta_R V (1 - \zeta_R) (-\Delta + I)^{-1} P + (-\Delta + I)^{-1/2} \zeta_R V \zeta_R (-\Delta + I)^{-1} P.
\]
It remains to note that the last term in the right hand side is small when $R \to \infty$ and the other two terms are compact operators. □
3. The end of the proof of Theorem 1.2

Let $P$ be the orthogonal projection onto the subspace of spherically symmetric functions in $L^2(\mathbb{R}^d)$. Let $P_1 = I - P$. Note that Corollary 2.12 implies that the operators

\[ (-\Delta + I)^{-1}V(-\Delta + I)^{-1}P, \quad P(-\Delta + I)^{-1}V(-\Delta + I)^{-1} \tag{3.1} \]

are compact.

To prove Theorem 1.2, it is sufficient to show that the essential spectrum of $H_+$ contains the interval $[0, \infty)$. Set $\tilde{H} = -\Delta + P_1VP_1$. We will prove that the difference of the resolvent operators

\[ (H_+ - z)^{-1} - (\tilde{H} - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.2} \]

is a compact. This would imply that the essential spectra of $H_+$ and $\tilde{H}$ coincide. Note that

\[ (H_+ - z)^{-1} - (\tilde{H} - z)^{-1} = \]

\[ -(H_+ - z)^{-1}(P_1VP + PVP + PV_{P_1})(\tilde{H} - z)^{-1} = \]

\[ T_1(-\Delta + I)^{-1}(P_1VP + PVP + PV_{P_1})(-\Delta + I)^{-1}T_2 \tag{3.3} \]

where $T_1 = -\left((-\Delta + I)(H_+ - z)^{-1}\right)^* \quad$ and $T_1 = (-\Delta + I)(\tilde{H} + -z)^{-1}$ are bounded operators. On the other hand, due to (3.1), the middle factor of the product in the right hand side of (3.3)

\[ (-\Delta + I)^{-1}(P_1VP + PVP + PV_{P_1})(-\Delta + I)^{-1} P = -(\Delta + I)^{-1}V(-\Delta + I)^{-1}P \]

is a compact operator. Consequently, the operator (3.2) is compact as well.

By the Weyl theorem, $H_+$ and $\tilde{H} = -\Delta + P_1VP_1$ have the same essential spectrum. It enough to show now that the essential spectrum of $\tilde{H}$ contains the interval $[0, \infty)$. The latter follows from the fact that the set of all spherically symmetric functions is an invariant subspace of the operator $\tilde{H}$. The part of $\tilde{H}$ in this subspace is an operator that is unitary equivalent to the operator $Ay(r) = -y''(r) + \frac{n\pi}{r}y(r)$. It remains to note that $\sigma(A) = [0, \infty)$. □

References


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