On the well-posedness of the magnetic, semi-relativistic Schrödinger-Poisson system

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Abstract. We prove global existence and uniqueness of strong solutions for the Schrödinger-Poisson system in the repulsive Coulomb case with relativistic, magnetic kinetic energy.

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1 Introduction

In the present work, we show the global well-posedness of the semi-relativistic, magnetic Schrödinger-Poisson system on a finite domain. Such system is relevant to the description of many-body semi-relativistic quantum particles in the mean-field limit (for example, in heated plasma), when the particles move with extremely high velocities and interact with an external magnetic field. Consider such semi-relativistic quantum particles localized in domain \( \Omega \subset \mathbb{R}^3 \) which is an open, bounded set with the Lebesgue measure \( |\Omega| < \infty \) and a \( C^2 \) boundary. The particles interact by the electrostatic field they collectively generate. In the mean-field limit, the density matrix \( \rho(t) \) that describes the mixed state of the system satisfies the Hartree-von Neumann equation

\[
\begin{align*}
 i\partial_t \rho(t) &= [H_{A,V}, \rho(t)], \quad x \in \Omega, \quad t \geq 0 \\
 -\Delta V &= n(t,x), \quad n(t,x) = \rho(t, x, x), \quad \rho(0) = \rho_0
\end{align*}
\]  

(1.1)
satisfying Dirichlet boundary conditions, \( \rho(t, x, y) = 0 \) if \( x \) or \( y \) \( \in \) \( \partial \Omega \), for \( t \geq 0 \). The Hamiltonian is given by

\[
H_{A,V} := T_{A,m} + V(t, x)
\]

with the magnetic, relativistic kinetic energy operator

\[
T_{A,m} := \sqrt{(-i \nabla + A)^2 + m^2} - m
\]
defined by means of the spectral calculus. Analogously to [6], we assume that \( A(x) \in C^1(\bar{\Omega}, \mathbb{R}^3) \) and \( \text{div} A = 0 \). In the present work \( (-i \nabla + A)^2 \) stands for the magnetic Dirichlet Laplacian on \( L^2(\Omega) \), and \( m > 0 \) is the particle mass; see [4, 5] for a derivation of such system of equations in the non-relativistic, non-magnetic case. Due to the fact that \( \rho(t) \) is a nonnegative, self-adjoint trace-class operator acting on \( L^2(\Omega) \), its kernel can, for every \( t \in \mathbb{R}_+ \), be decomposed with respect to an orthonormal basis of \( L^2(\Omega) \). The kernel of the initial data \( \rho_0 \) can be written in the form

\[
\rho_0(x, y) = \sum_{k \in \mathbb{N}} \lambda_k \psi_{0,k}(x) \overline{\psi_{0,k}(y)}.
\]

Here \( \{\psi_{0,k}\}_{k \in \mathbb{N}} \) stands for an orthonormal basis of \( L^2(\Omega) \), with \( \psi_k|_{\partial \Omega} = 0 \) for all \( k \in \mathbb{N} \), and coefficients

\[
\Delta := \{\lambda_k\}_{k \in \mathbb{N}} \in \ell^1, \; \lambda_k \geq 0, \; \sum_{k \in \mathbb{N}} \lambda_k = 1.
\]

As proven below, there exists a one-parameter family of complete orthonormal bases of \( L^2(\Omega) \), \( \{\psi_k(t)\}_{k \in \mathbb{N}} \), with \( \psi_k|_{\partial \Omega} = 0 \) for all \( k \in \mathbb{N} \), and for \( t \in \mathbb{R}_+ \), such that the kernel of the solution \( \rho(t) \) to (1.1) can be written as

\[
\rho(t, x, y) = \sum_{k \in \mathbb{N}} \lambda_k \psi_k(t, x) \overline{\psi_k(t, y)}.
\]

Notably, the coefficients \( \lambda \) are independent of \( t \), and thus the same as those in \( \rho_0 \), which is because the operators \( -iH_{A,V} \) and \( \rho(t) \) form a Lax pair in problem (1.1). When substituting (1.6) in (1.1), the one-parameter family of orthonormal vectors \( \{\psi_k(t)\}_{k \in \mathbb{N}} \) is seen to satisfy the semi-relativistic, magnetic Schrödinger-Poisson system

\[
\frac{\partial \psi_k}{\partial t} = T_{A,m} \psi_k + V[\Psi] \psi_k, \; k \in \mathbb{N},
\]

\[
-\Delta V[\Psi] = n[\Psi], \; \Psi := \{\psi_k\}_{k=1}^\infty,
\]

\[
n[\Psi](t, x) = \sum_{k=1}^\infty \lambda_k |\psi_k(t, x)|^2.
\]
with initial data
\[ \psi_k(t = 0, x) = \psi_{0,k}(x), \quad k \in \mathbb{N}. \tag{1.10} \]

Our potential function \( V[\Psi] \) is a solution of the Poisson equation (1.8). On both \( V[\Psi] \) and \( \psi_k(t) \), for all \( k \in \mathbb{N} \), we impose Dirichlet boundary conditions
\[ \psi_k(t, x), \quad V(x, t) = 0, \quad t \geq 0, \quad \forall x \in \partial \Omega. \tag{1.11} \]

As we prove in Lemma 6 further down, solutions of (1.7)-(1.9) preserve the orthonormality of \( \{\psi_k(t)\}_{k \in \mathbb{N}} \). Let us introduce the magnetic Sobolev norms for functions:
\[ \|f\|_{H^1_2(\Omega)} := \|f\|_{L^2(\Omega)}^2 + \|(-i \nabla + A)f\|_{L^2(\Omega)}^2. \tag{1.12} \]
\[ \|f\|_{\overline{H^1_2}(\Omega)} := \|f\|_{L^2(\Omega)}^2 + \|((-i \nabla + A)f\|_{L^2(\Omega)}^2. \tag{1.13} \]

Here, \( | -i \nabla + A| \) denotes the operator \( \sqrt{(-i \nabla + A)^2} \), and has the meaning of the relativistic kinetic energy of a particle with zero mass in the presence of a magnetic field. The standard Sobolev norms \( \|f\|_{H^1_A(\Omega)} \) and \( \|f\|_{\overline{H^1_A}(\Omega)} \) will be used when the magnetic vector potential \( A(x) \) vanishes. The state space for our magnetic, semi-relativistic Schrödinger-Poisson system is defined as
\[ \mathcal{L} := \{ (\Psi, A) \mid \Psi = \{\psi_k\}_{k=1}^\infty \subset H^2_{0,A}(\Omega) \cap H^1_A(\Omega) \ is \ a \ complete \ orthonormal \ system \ in \ L^2(\Omega), \]
\[ \Delta = \left\{ \lambda_k \right\}_{k=1}^\infty \subset \ell^1, \quad \lambda_k \geq 0, \quad k \in \mathbb{N}, \quad \sum_{k=1}^\infty \lambda_k \int_\Omega |(-i \nabla + A)\psi_k|^2 \, dx < \infty \}. \]

For fixed \( \Delta \in \ell^1, \lambda_k \geq 0, \) and for sequences of square integrable functions \( \Phi := \{\phi_k\}_{k=1}^\infty \) and \( \Psi := \{\psi_k\}_{k=1}^\infty \), we define the inner product
\[ (\Phi, \Psi)_{X_\Omega} := \sum_{k=1}^\infty \lambda_k (\phi_k, \psi_k)_{L^2(\Omega)}, \]
which induces the norm
\[ \|\Phi\|_{X_\Omega} := (\sum_{k=1}^\infty \lambda_k \|\phi_k\|_{L^2(\Omega)}^2)^{1/2}. \]

Let us introduce the corresponding Hilbert space
\[ X_\Omega := \{ \Phi = \{\phi_k\}_{k=1}^\infty \mid \phi_k \in L^2(\Omega), \quad \forall k \in \mathbb{N}, \quad \|\Phi\|_{X_\Omega} < \infty \}. \]

Our main result is as follows.

**Theorem 1.** Let \( A(x) \in C^1(\Omega, \mathbb{R}^3) \) and \( \text{div} A = 0 \). For every initial state \( (\Psi(x, 0), \Delta) \in \mathcal{L} \), there is a unique mild solution \( \Psi(x, t), \ t \in [0, \infty), \) of (1.7)-(1.10) with \( (\Psi(x, t), \Delta) \in \mathcal{L} \). This is also a unique strong global solution in \( X_\Omega \), i.e. \( \Psi \in C([0, \infty); Z_{\Omega,A}) \cap C^1([0, \infty); X_\Omega) \).
Proving the global well-posedness of the Schrödinger-Poisson system plays a crucial role in establishing the existence and nonlinear stability of stationary states, i.e. the nonlinear bound states of the Schrödinger-Poisson system, which was done in the non-relativistic, non-magnetic case in [8, 12], in the magnetic, non-relativistic case in [6], in the semi-relativistic case without a magnetic field in [1] and [2]. The global well posedness of the non-relativistic, magnetic Schrödinger-Poisson system in the whole \( \mathbb{R}^3 \) was established in [7] on the assumption that the vector potential \( A(x) \) is smooth and bounded. The problem in one dimension was studied in [15]. The semiclassical limit of such system with the relativistic kinetic energy was treated in [3]. The global well-posedness for a single semi-relativistic Hartree problem in \( \mathbb{R}^3 \) was proved in [9]. In the present article, we study the infinite system of equations in a bounded set with Dirichlet boundary conditions, and, as distinct from [9], we do not use the regularization of the Poisson equation. Furthermore, both the results of [9] and Theorem 1 above do not rely on Strichartz type estimates. Note that operator (1.3) is crucial for the studies of the relativistic stability of matter in the presence of a magnetic field (see e.g. [11]).

2 Proof of global well-posedness

We make a fixed choice of \( \Delta = \{ \lambda_k \}_{k=1}^{\infty} \in \ell^1 \), with \( \lambda_k \geq 0 \) and \( \sum_{k \in \mathbb{N}} \lambda_k = 1 \), standing for the sequence of coefficients determined by the initial data \( \rho_0 \) of the Hartree-von Neumann equation (1.1) via (1.6), for \( t = 0 \).

Let us introduce the inner products \((\cdot, \cdot)_{Y_{\Omega,A}}\) and \((\cdot, \cdot)_{Z_{\Omega,A}}\) which induce the generalized inhomogenous magnetic Sobolev norms

\[
\|\Phi\|_{Y_{\Omega,A}} := \left( \sum_{k=1}^{\infty} \lambda_k \| \phi_k \|_{H^\frac{1}{2}_A(\Omega)}^2 \right)^{\frac{1}{2}} \text{ and }\|\Phi\|_{Z_{\Omega,A}} := \left( \sum_{k=1}^{\infty} \lambda_k \| \phi_k \|_{H^1_A(\Omega)}^2 \right)^{\frac{1}{2}},
\]

and define the corresponding Hilbert spaces

\[Y_{\Omega,A} := \{ \Phi = \{ \phi_k \}_{k=1}^{\infty} \mid \phi_k \in H^\frac{1}{2}_{0,A}(\Omega), \ \forall \ k \in \mathbb{N}, \ \|\Phi\|_{Y_{\Omega,A}} < \infty \}\]

and

\[Z_{\Omega,A} := \{ \Phi = \{ \phi_k \}_{k=1}^{\infty} \mid \phi_k \in H^1_{0,A}(\Omega) \cap H^1_A(\Omega), \ \forall \ k \in \mathbb{N}, \ \|\Phi\|_{Z_{\Omega,A}} < \infty \}\]

respectively. Let us also introduce the generalized homogenous Sobolev norms

\[
\|\Phi\|_{Y_{\Omega,A}} := \left( \sum_{k=1}^{\infty} \lambda_k \| -i\nabla + A \|_{L^2(\Omega)}^2 \phi_k \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \text{ and }\|\Phi\|_{Z_{\Omega,A}} := \left( \sum_{k=1}^{\infty} \lambda_k \| (i\nabla + A) \phi_k \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\]

The notations \(\|\Phi\|_{Y_{\Omega}}, \|\Phi\|_{Y_{\Omega}}, \|\Phi\|_{Z_{\Omega}}, \|\Phi\|_{Z_{\Omega}}\) will be used in the article when the magnetic vector potential \( A(x) \) vanishes in \( \Omega \), analogously to Section 3 of [8]. Let us note the following equivalence of magnetic and non magnetic norms.
Lemma 2. Assume that the vector potential $A(x) \in C^1(\bar{\Omega}, \mathbb{R}^3)$ and the Coulomb gauge is chosen, such that $\text{div} \, A = 0$.

a) Let $\Phi(x) \in Y_{\Omega,A}$. Then the norms $\|\Phi\|_{Y_{\Omega,A}}, \|\Phi\|_{Y_{\bar{\Omega}}}, \|\Phi\|_{\bar{\Omega}}$ and $\|\Phi\|_{\bar{\Omega}}$ are equivalent.

b) Let $\Phi(x) \in Z_{\Omega,A}$. Then the norms $\|\Phi\|_{Z_{\Omega,A}}, \|\Phi\|_{Z_{\bar{\Omega},A}}, \|\Phi\|_{Z_{\bar{\Omega}}} = 0$ and $\|\Phi\|_{\bar{\Omega}}$ are equivalent.

Proof. Let us note that the statement b) of the lemma is the result of the part c) of Lemma A.2 of [6]. By means of the part a) of Lemma A.2 of [6], we have

$$C_1 \|\nabla f\|_{L^2(\Omega)} \leq \|(-i \nabla + A)f\|_{L^2(\Omega)} \leq C_2 \|\nabla f\|_{L^2(\Omega)}$$

for $f(x) \in H^1_{0,A}(\Omega)$, where $C_{1,2} > 0$ are constants. Hence, for the quadratic forms

$$C_1^2 (-\Delta f, f)_{L^2(\Omega)} \leq \|(-i \nabla + A)^2 f\|_{L^2(\Omega)} \leq C_2^2 (-\Delta f, f)_{L^2(\Omega)}.$$ 

This implies that for the square roots of these operators we have

$$C_1 (|p| f, f)_{L^2(\Omega)} \leq \|(-i \nabla + A) f\|_{L^2(\Omega)} \leq C_2 (|p| f, f)_{L^2(\Omega)},$$

where $|p|$ denotes the operator $\sqrt{-\Delta}$. Therefore, for the components of $\Phi = \{\phi_k\}_{k=1}^\infty$ we obtain

$$C_1 \|p|^{\frac{1}{2}} \phi_k\|_{L^2(\Omega)}^2 \leq \|(-i \nabla + A)^{\frac{1}{2}} \phi_k\|_{L^2(\Omega)}^2 \leq C_2 \|p|^{\frac{1}{2}} \phi_k\|_{L^2(\Omega)}^2, \quad k \in \mathbb{N}. \quad (2.1)$$

Without loss of generality, we assume that in inequality (2.1) we have $C_1 \leq 1$ and $C_2 \geq 1$. Then

$$C_1 \|\phi_k\|_{H^{\frac{1}{2}}(\Omega)}^2 \leq \|(-i \nabla + A)^{\frac{1}{2}} \phi_k\|_{L^2(\Omega)}^2 \leq C_2 \|\phi_k\|_{H^{\frac{1}{2}}(\Omega)}^2$$

yields

$$C_1 \|\phi_k\|_{H^{\frac{1}{2}}(\Omega)}^2 \leq \|\phi_k\|_{H^{\frac{1}{2}}(\Omega)}^2, \quad k \in \mathbb{N}. \quad (2.2)$$

Similarly, (2.1) gives us

$$\|(-i \nabla + A)^{\frac{1}{2}} \phi_k\|_{L^2(\Omega)}^2 \leq \|\phi_k\|_{L^2(\Omega)}^2$$

and therefore

$$\|\phi_k\|_{H^{\frac{1}{2}}(\Omega)}^2 \leq \|\phi_k\|_{H^{\frac{1}{2}}(\Omega)}^2, \quad k \in \mathbb{N}. \quad (2.3)$$

Let us multiply both sides of (2.2) and (2.3) by $\lambda_k$ and sum over $k \in \mathbb{N}$. Thus, we arrive at

$$\sqrt{C_1} \|\Phi\|_{Y_{\Omega}} \leq \|\Phi\|_{Y_{\bar{\Omega},A}} \leq \sqrt{C_2} \|\Phi\|_{\bar{\Omega}},$$

such that the norms $\|\cdot\|_{Y_{\Omega,A}}$ and $\|\cdot\|_{Y_{\bar{\Omega}}}$ are equivalent. The equivalence of $\|\cdot\|_{Y_{\Omega}}$ and $\|\cdot\|_{Y_{\bar{\Omega}}}$ norms was established in Lemma 2 of [1]. We multiply all sides of inequality (2.1) by $\lambda_k$ and add up over $k \in \mathbb{N}$. This yields

$$\sqrt{C_1} \|\Phi\|_{Y_{\Omega}} \leq \|\Phi\|_{Y_{\bar{\Omega},A}} \leq \sqrt{C_2} \|\Phi\|_{\bar{\Omega}},$$

such that the norms $\|\cdot\|_{Y_{\Omega,A}}$ and $\|\cdot\|_{Y_{\bar{\Omega}}}$ are equivalent as well. □
Let $\Psi = \{\psi_k\}_{k=1}^{\infty}$ be a wave function and the magnetic, relativistic kinetic energy operator acts on it $T_{A,m}\Psi = (\sqrt{(-i \nabla + A)^2 + m^2} - m)\Psi$ componentwise. We have the following two technical statements.

**Lemma 3.** The domain of the magnetic, semi-relativistic kinetic energy operator is given by $D(T_{A,m}) = Z_{\Omega,A} \subseteq X_{\Omega}$.

**Proof.** Let $\Psi \in Z_{\Omega,A}$. Hence

$$\|\Psi\|_{Z_{\Omega,A}} = \left( \sum_{k=1}^{\infty} \lambda_k [\|(-i \nabla + A)\psi_k\|^2_{L^2(\Omega)} + \|\psi_k\|^2_{L^2(\Omega)}] \right)^{\frac{1}{2}} \geq \left( \sum_{k=1}^{\infty} \lambda_k [\|\psi_k\|^2_{L^2(\Omega)}] \right)^{\frac{1}{2}} = \|\Psi\|_{X_{\Omega}}$$

such that $\Psi \in X_{\Omega}$. Let us estimate $\|T_{A,m}\psi_k\|^2_{L^2(\Omega)}$ as

$$\|(A, m^2)\psi_k\|_{L^2(\Omega)}^2 + m^2 \|\psi_k\|_{L^2(\Omega)}^2 - 2m \sqrt{(-i \nabla + A)^2 + m^2}\psi_k, \psi_k\|_{L^2(\Omega)} \leq \|(A, m^2)\psi_k\|_{L^2(\Omega)}^2 + 2m \|\psi_k\|_{L^2(\Omega)}^2 \leq c(m)\|\psi_k\|_{H^1(\Omega)}^2,$$

where $c(m)$ is a mass dependent constant. Thus

$$\|T_{A,m}\Psi\|^2_{X_{\Omega}} = \sum_{k=1}^{\infty} \lambda_k \|T_{A,m}\psi_k\|^2_{L^2(\Omega)} \leq c(m)\|\Psi\|_{Z_{\Omega,A}}^2 < \infty.$$

\[\blacksquare\]

**Lemma 4.** The operator $T_{A,m}$ generates the group $e^{-iT_{A,m}t}$, $t \in \mathbb{R}$, of unitary operators on $X_{\Omega}$.

Let us rewrite our magnetic, semi-relativistic Schrödinger-Poisson system for $x \in \Omega$ into the form

$$\Psi_t = -i T_{A,m} \Psi + F[\Psi(x,t)], \text{ where } F[\Psi] := i^{-1} V[\Psi] \Psi,$$

$$-\Delta V[\Psi] = n[\Psi], \text{ where } V|_{\partial \Omega} = 0,$$

$$n[\Psi] = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2$$

and establish the following technical result.

**Lemma 5.** The map defined in (2.4) $F : Z_{\Omega,A} \rightarrow Z_{\Omega,A}$ is locally Lipschitz continuous.
Let $\Psi, \Phi \in Z_{\Omega,A}$ with $\Psi = \{\psi_k\}_{k=1}^{\infty}$, $\Phi = \{\phi_k\}_{k=1}^{\infty}$ and $t \in [0, T)$. Then, according to the result of Lemma 5 of [1],

$$\| F[\Psi] - F[\Phi] \|_{Z_{\Omega}} \leq C (\| \Psi \|^2_{Z_{\Omega}} + \| \Phi \|^2_{Z_{\Omega}}) \| \Psi - \Phi \|_{Z_{\Omega}}.$$ 

The result of the lemma follows from the equivalence of magnetic and non magnetic norms established in Lemma 2 above.

From standard arguments (see for instance Theorem 1.7 of [13]) and Lemma 5 it follows that our magnetic, semi-relativistic Schrödinger-Poisson system admits a unique mild solution $\Psi$ belonging to $Z_{\Omega,A}$ on a time interval $[0, T)$, for a certain $T > 0$, satisfying the integral equation

$$\Psi(t) = e^{-iT_{A,m}} \Psi(0) + \int_0^t e^{-iT_{A,m}(t-s)} F[\Psi(s)] ds \quad (2.5)$$

in $Z_{\Omega,A}$. Furthermore,

$$\lim_{t \to T} \| \Psi(t) \|_{Z_{\Omega,A}} = \infty$$

in the case if $T$ is finite. Let us also note that $\Psi$ is a unique strong solution in $X_{\Omega}$, such that $\Psi \in C([0, T); Z_{\Omega,A}) \cap C^1([0, T); X_{\Omega})$. Below we will establish that this solution is in fact global in time. First let us prove the following proposition.

**Lemma 6.** Suppose for the unique mild solution (2.5) of the magnetic, semi-relativistic Schrödinger-Poisson system (1.7)-(1.10) at $t = 0$ the functions $\{\psi_k(x,0)\}_{k=1}^{\infty}$ form a complete orthonormal system in $L^2(\Omega)$. Then, for any $t \in [0, T)$, the set $\{\psi_k(x,t)\}_{k=1}^{\infty}$ remains a complete orthonormal system in $L^2(\Omega)$. Moreover, the $X_{\Omega}$-norm is preserved: $\| \Psi(x,t) \|_{X_{\Omega}} = \| \Psi(x,0) \|_{X_{\Omega}}$, $t \in [0, T)$.

**Proof.** By means of (1.7), we obtain

$$\frac{d}{dt} (\psi_k, \psi_l)_{L^2(\Omega)} = -i((T_{A,m} + V)\psi_k, \psi_l)_{L^2(\Omega)} + i(\psi_k, (T_{A,m} + V)\psi_l)_{L^2(\Omega)} = 0.$$ 

This gives us

$$(\psi_k(x,t), \psi_l(x,t))_{L^2(\Omega)} = (\psi_k(x,0), \psi_l(x,0))_{L^2(\Omega)} = \delta_{k,l}, \quad k, l \in \mathbb{N}.$$ 

Here $\delta_{k,l}$ denotes the Kronecker symbol. Thus, for $k \in \mathbb{N}$

$$\| \psi_k(.,t) \|^2_{L^2(\Omega)} = \| \psi_k(.,0) \|^2_{L^2(\Omega)}.$$ 

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Therefore, for \( t \in [0, T) \), we have the preservation of the \( X_\Omega \)-norm, namely
\[
\|\Psi(\cdot, t)\|_{X_\Omega} = \left( \sum_{k=1}^{\infty} \lambda_k \|\psi_k(x, t)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} = \left( \sum_{k=1}^{\infty} \lambda_k \|\psi_k(x, 0)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} = \|\Psi(\cdot, 0)\|_{X_\Omega}.
\]

For the unique given solution \( \Psi(t) \) of our magnetic, semi-relativistic Schrödinger-Poisson system on \([0, T)\), we arrive at the time-dependent one-particle Hamiltonian
\[
H_{A,V}(t) = T_{A,m} + V(\Psi(t, x)),
\]
where the potential \( V \) solves \(-\Delta V(\Psi(t, x)) = n[\Psi(t)]\) with Dirichlet boundary conditions, see (1.2). The properties of \( V \) are discussed in more detail in Lemma 8 below. Accordingly, the components of \( \Psi(t) \) satisfy the non-autonomous magnetic, semi-relativistic Schrödinger equation
\[
i\partial_t \psi_k(t, x) = H_{A,V}(t)\psi_k(t, x), \quad k \in \mathbb{N},
\]
on the time interval \([0, T)\). Therefore, by means of Theorem X.71 of [14], there exists a propagator, denoted as \( e^{-i\int_0^t H_{A,V}(\tau)d\tau} \) such that for \( t \in [0, T) \),
\[
\psi_k(x, t) = e^{-i\int_0^t H_{A,V}(\tau)d\tau} \psi_k(x, 0), \quad k \in \mathbb{N}.
\]
Consider an arbitrary function \( f(x) \in L^2(\Omega) \). Obviously, we have the expansion
\[
f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y, 0))_{L^2(\Omega)} \psi_k(x, 0)
\]
and analogously
\[
e^{i\int_0^t H_{A,V}(\tau)d\tau} f(x) = \sum_{k=1}^{\infty} (e^{i\int_0^t H_{A,V}(\tau)d\tau} f(y), \psi_k(y, 0))_{L^2(\Omega)} \psi_k(x, 0).
\]

Therefore, by virtue of (2.6) we derive the expansion
\[
f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y, t))_{L^2(\Omega)} \psi_k(x, t)
\]
for \( t \in [0, T) \).

Moreover, we establish the conservation of energy for the solutions to the magnetic, semi-relativistic Schrödinger-Poisson system in the following sense.

**Lemma 7.** For the unique mild solution (2.5) of the Schrödinger-Poisson system (1.7)-(1.10) and for any value of time \( t \in [0, T) \) we have the identity
\[
\sum_{k \in \mathbb{N}} \lambda_k \|T_{A,m}^{\frac{1}{2}} \psi_k(x, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla V[\Psi(x, t)]\|_{L^2(\Omega)}^2
\]
\[
= \sum_{k \in \mathbb{N}} \lambda_k \|T_{A,m}^{\frac{1}{2}} \psi_k(x, 0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla V[\Psi(x, 0)]\|_{L^2(\Omega)}^2.
\]

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Proof. The complex conjugation of the Schrödinger-Poisson system (1.7) gives us
\[-i \frac{\partial \bar{\psi}_k}{\partial t} = T_{A,m} \bar{\psi}_k + V[\psi] \bar{\psi}_k, \quad k \in \mathbb{N}.
\] (2.7)

We add the $k$-th equation of the original system (1.7) multiplied by $\frac{\partial \bar{\psi}_k}{\partial t}$, and the $k$-th equation of (2.7) multiplied by $\frac{\partial \psi_k}{\partial t}$ and arrive at
\[\frac{\partial}{\partial t} \|T_{A,m} \psi_k\|_{L^2(\Omega)}^2 + \int_{\Omega} V[\psi] \frac{\partial}{\partial t} |\psi_k|^2 dx = 0, \quad k \in \mathbb{N}.
\]

Hence, multiplying by $\lambda_k$, and summing over $k$, we obtain
\[\frac{\partial}{\partial t} \sum_{k \in \mathbb{N}} \lambda_k \|T_{A,m} \psi_k(x,t)\|_{L^2(\Omega)}^2 + \int_{\Omega} V[\Psi(x,t)] \frac{\partial}{\partial t} n[\Psi(x,t)] dx = 0.
\] (2.8)

It can be easily verified that
\[\frac{\partial}{\partial t} \|\nabla V[\Psi(x,t)]\|_{L^2(\Omega)}^2 = 2 \int_{\Omega} V[\Psi(x,t)] \frac{\partial}{\partial t} n[\Psi(x,t)] dx.
\]

By substituting this identity into (2.8), we complete the proof of the lemma.

With the auxiliary statements proven above at our disposal, we may now prove our main result, Theorem 1.

Proof of Theorem 1. The proof follows from the blow-up alternative and the conservation laws. Let us recall that the mild solution of the Schrödinger-Poisson system (1.7)-(1.10) is given by (2.5). We apply the norm $\|\cdot\|_{\dot{Z}_{\Omega,A}}$ to both sides of (2.5), which yields
\[\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|e^{-iT_{A,m}t}\Psi(0)\|_{\dot{Z}_{\Omega,A}} + \int_0^t \|e^{-iT_{A,m}(t-s)} F[\Psi(s)]\|_{\dot{Z}_{\Omega,A}} ds.
\]

Hence,
\[\|\Psi(t)\|_{\dot{Z}_{\Omega,A}} \leq \|\Psi(0)\|_{\dot{Z}_{\Omega,A}} + \int_0^t \|F[\Psi(s)]\|_{\dot{Z}_{\Omega,A}} ds.
\]

In the proof of Theorem 1 of [1], it was shown that
\[\|F[\Psi]\|_{\dot{Z}_\Omega} \leq C \|\Psi\|_{\dot{Y}_{\Omega,A}}^2 \|\Psi\|_{\dot{Z}_\Omega}.
\]

By virtue of the equivalence of the magnetic and non magnetic norms established in our Lemma 2 above, we obtain
\[\|F[\Psi]\|_{\dot{Z}_{\Omega,A}} \leq C \|\Psi\|_{\dot{Y}_{\Omega,A}}^2 \|\Psi\|_{\dot{Z}_{\Omega,A}}.
\] (2.9)
The energy conservation established in Lemma 7 above gives us the boundedness of the quantity
\[ \sum_{k \in \mathbb{N}} \lambda_k \| T_{A,m}^{\frac{1}{2}} \psi_k(x,t) \|_{L^2(\Omega)}^2, \quad t \in [0,T) \]
by the amount of the initial energy of our magnetic, semi-relativistic Schrödinger-Poisson system. Since the \( \| \cdot \|_{X_0} \) norm is conserved for our unique mild solution according to Lemma 6, we have the boundedness of the quantity
\[ \sum_{k \in \mathbb{N}} \lambda_k (\sqrt{(-i\nabla + A)^2 + m^2} \psi_k(x,t), \psi_k(x,t))_{L^2(\Omega)} \]
for \( t \in [0,T) \). Because \( \sqrt{(-i\nabla + A)^2 + m^2} \geq | -i\nabla + A | \) in the sense of the quadratic forms, we derive the boundedness of
\[ \| \Psi \|_{\dot{Y}_{\Omega,A}}^2 = \sum_{k \in \mathbb{N}} \lambda_k \| | -i\nabla + A |^{\frac{1}{2}} \psi_k \|_{L^2(\Omega)}^2, \quad t \in [0,T) \]
for our unique mild solution. Then (2.9) yields
\[ \| F[\Psi] \|_{Z_{\Omega,A}} \leq C_0 \| \Psi \|_{\dot{Z}_{\Omega,A}}, \]
where \( C_0 \) is a positive constant obtained by virtue of our conservation laws discussed above. This gives us
\[ \| \Psi(t) \|_{\dot{Z}_{\Omega,A}} \leq \| \Psi(0) \|_{\dot{Z}_{\Omega,A}} + C_0 \int_0^t \| \Psi(s) \|_{\dot{Z}_{\Omega,A}} ds. \]
By the Gronwall’s lemma,
\[ \| \Psi(t) \|_{\dot{Z}_{\Omega,A}} \leq \| \Psi(0) \|_{\dot{Z}_{\Omega,A}} e^{C_0 t}, \quad t \in [0,T). \]
Therefore, by virtue of the blow-up alternative, our magnetic, semi-relativistic Schrödinger-Poisson system is globally well-posed in \( Z_{\Omega,A} \).

We conclude the article with addressing the properties of the scalar potential function involved in our magnetic, semi-relativistic Schrödinger-Poisson system.

**Lemma 8.** For \( (\Psi, \lambda) \in \mathcal{L} \) we have
\[ n_{\Psi,\lambda} = \sum_{k=1}^{\infty} \lambda_k | \psi_k |^2 \in L^2(\Omega). \]
Let \( V_{\Psi,\lambda} \) denote the Coulomb potential induced by \( n_{\Psi,\lambda} \), such that
\[ -\Delta V_{\Psi,\lambda}(x) = n_{\Psi,\lambda}(x), \quad x \in \Omega; \quad V_{\Psi,\lambda}(x) = 0, \quad x \in \partial \Omega. \]
Then \( V_{\Psi,\lambda}(x) \in H_0^1(\Omega) \cap H^2(\Omega) \).
Proof. Clearly, by virtue of the Schwarz inequality we have the upper bound

$$\|n_{\psi,\lambda}\|_{L^2(\Omega)}^2 = \sum_{k,s \in \mathbb{N}} \lambda_k \lambda_s \int_{\Omega} |\psi_k(x)|^2 |\psi_s(x)|^2 dx \leq \left( \sum_{k \in \mathbb{N}} \lambda_k \sqrt{\int_{\Omega} |\psi_k(x)|^4 dx} \right)^2.$$ 

By means of the Hölder’s inequality,

$$\int_{\Omega} |\psi_k(x)|^4 dx \leq \left( \int_{\Omega} |\psi_k(x)|^6 dx \right)^{\frac{2}{3}} |\Omega|^\frac{1}{3}.$$ 

Let us use the standard Sobolev inequality (see e.g. p.186 of [10])

$$\|\nabla f\|_{L^2(\Omega)} \geq c_s \|f\|_{L^6(\Omega)},$$

where $c_s > 0$ is a constant. This along with the Diamagnetic inequality

$$\int_{\Omega} |(-i \nabla + A)f|^2 dx \geq \int_{\Omega} |\nabla f|^2 dx$$

(see e.g. p.179 of [10]) gives us

$$\sqrt{\int_{\Omega} |\psi_k(x)|^4 dx} \leq \frac{|\Omega|^\frac{1}{3}}{c_s^2} \int_{\Omega} |\nabla \psi_k|^2 dx \leq \frac{|\Omega|^\frac{1}{3}}{c_s^2} \|(-i \nabla + A)\psi_k\|_{L^2(\Omega)}^2.$$ 

Therefore,

$$\|n_{\psi,\lambda}\|_{L^2(\Omega)}^2 \leq \frac{|\Omega|^\frac{4}{3}}{c_s^4} \left( \sum_{k \in \mathbb{N}} \lambda_k \|(-i \nabla + A)\psi_k\|_{L^2(\Omega)}^2 \right)^2 = \frac{|\Omega|^\frac{4}{3}}{c_s^4} \|\Psi\|_{L^2(\Omega)}^4 < \infty,$$

such that $n_{\psi,\lambda}(x) \in L^2(\Omega)$. Then, by means of our Poisson equation we have $\Delta V_{\psi,\lambda} \in L^2(\Omega)$.

Let $\{\mu_k^0\}_{k \in \mathbb{N}}$ denote the set of the Dirichlet eigenvalues for the negative Laplace operator on $L^2(\Omega)$, such that $\mu_k^0 > 0$, $k \in \mathbb{N}$ and $\mu_1^0$ is the lowest eigenvalue. Thus, since $V_{\psi,\lambda} = (-\Delta)^{-1}n_{\psi,\lambda}$, we have the estimate

$$\|V_{\psi,\lambda}\|_{L^2(\Omega)} \leq \frac{1}{\mu_1^0} \|n_{\psi,\lambda}\|_{L^2(\Omega)} < \infty.$$ 

Furthermore, since $V_{\psi,\lambda}$ vanishes on the Lipschitz boundary of our bounded set $\Omega$ as assumed, $V_{\psi,\lambda}$ is a trace zero function in $H^1(\Omega)$. Let us also note that $V_{\psi,\lambda}(x) \in L^\infty(\Omega) \subset H^2(\Omega)$ by virtue of the Sobolev embedding. 

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References


