ASYMPTOTICS OF RESONANCES INDUCED BY POINT INTERACTIONS

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Abstract. We consider the resonances of the self-adjoint three-dimensional Schrödinger operator with point interactions of constant strength supported on the set \( X = \{ x_n \}_{n=1}^N \). The size of \( X \) is defined by \( V_X = \max_{\pi \in \Pi_N} \sum_{n=1}^{N} |x_n - x_{\pi(n)}| \), where \( \Pi_N \) is the family of all the permutations of the set \( \{1, 2, \ldots, N\} \).

We prove that the number of resonances counted with multiplicities and lying inside the disc of radius \( R \) asymptotically behaves as \( W_X R + O(1) \) as \( R \to \infty \), where \( W_X \in [0, V_X] \) is the effective size of \( X \). Moreover, we show that there exist configurations of any number of points such that \( W_X = V_X \). Finally, we construct an example for \( N = 4 \) with \( W_X < V_X \), which can be viewed as an analogue of a non-Weyl quantum graph.

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1. Introduction

In this note we discuss the resonances of the three-dimensional Schrödinger operator \( H_{\alpha,X} \) with point interactions of constant strength \( \alpha \in \mathbb{R} \) supported on the discrete set \( X = \{ x_n \}_{n=1}^N \subset \mathbb{R}^3 \), \( N \geq 2 \). The corresponding Hamiltonian \( H_{\alpha,X} \) is associated with the formal differential expression

\[
-\Delta + \alpha \sum_{n=1}^{N} \delta(x - x_n), \quad \text{on } \mathbb{R}^3,
\]

where \( \delta(\cdot) \) stands for the \( \delta \)-distribution. The Hamiltonian \( H_{\alpha,X} \) can be rigorously defined as a self-adjoint extension of a certain symmetric operator in the Hilbert space \( L^2(\mathbb{R}^3) \); cf. Section 3 for details. Resonances of \( H_{\alpha,X} \) were discussed in the monograph [AGHH05] and in several more recent publications e.g. [AK17, BFT98, EGST96], see also the review [DFT08] and the references therein.

We define the size of \( X \) by

\[
V_X := \max_{\pi \in \Pi_N} \sum_{n=1}^{N} |x_n - x_{\pi(n)}|,
\]

where \( \Pi_N \) is the family of all the permutations of the set \( \{1, 2, \ldots, N\} \). This definition of the size is motivated by the condition on resonances for \( H_{\alpha,X} \) given in Section 4.1. As the main result of this note, we prove that the number \( N_{\alpha,X}(R) \) of the resonances of \( H_{\alpha,X} \) lying inside the disc \( \{ z \in \mathbb{C} : |z| < R \} \) and with multiplicities taken into account asymptotically behaves as

\[
N_{\alpha,X}(R) = \frac{W_X}{\pi} R + O(1), \quad R \to \infty,
\]
where $W_X \in [0, V_X]$ is the effective size of $X$, which does not depend on $\alpha$.

In the proof of (1.3) we use that the resonance condition for $H_{\alpha, X}$ acquires the form of an exponential polynomial, which can be obtained by a direct computation or alternatively using the pseudo-orbit expansion as explained in Section 4.3. Recall that an exponential polynomial is a sum of finitely many terms, each of which is a product of a rational function and an exponential; cf. the review paper [Lan31] and the monographs [BC63, BG95]. In order to obtain the asymptotics (1.3) we employ a classical result on the distribution of zeros of exponential polynomials, recalled in Section 2 for the convenience of the reader.

A configuration of points $X$ for which $W_X = V_X$ is said to be of Weyl-type. We show that for any $N \in \mathbb{N}$ there exist Weyl-type configurations consisting of $N$ points. For two and three points ($N \leq 3$), in fact, any configuration is of Weyl-type, as shown in Section 5.1. On the other hand, we present in Section 5.2 an example of a non-Weyl configuration for $N = 4$, for which strict inequality $W_X < V_X$ holds. We expect that such configurations can also be constructed for any $N > 4$. One can trace an analogy with non-Weyl quantum graphs studied in [DEL10, DP11]. Non-uniqueness of the permutation at which the maximum in (1.2) is attained, is a necessary condition for a configuration of points $X$ to be non-Weyl. Exact geometric characterization of non-Weyl-type point configurations remains an open question. Besides that a physical interpretation of this mathematical observation still needs to be clarified.

It is worth pointing out that $N_{\alpha,X}(R)$ is asymptotically linear similarly as the counting function for resonances of the one-dimensional Schrödinger operator $-\frac{d^2}{dx^2} + V$ with a potential $V \in C_0^\infty(\mathbb{R}; \mathbb{R})$; see [Zwo87]. The exact asymptotics of the counting function for resonances of the three-dimensional Schrödinger operator $-\Delta + V$ with a potential $V \in C_0^\infty(\mathbb{R}^3; \mathbb{R})$ is known only in some special cases, but for “generic” potentials this counting function behaves as $\sim R^3$, thus being not asymptotically linear; see [CH08] for details.

2. Exponential polynomials

In this section we introduce exponential polynomials and recall a classical result on the asymptotic distribution of their zeros. This result was first obtained by Pólya [Pol20] and later improved by many authors, including Schwengeler [Sch25] and Moreno [Mor73]. We refer the reader to the review [Lan31] by Langer and to the monographs [BC63, BG95].

**Definition 2.1.** An exponential polynomial $F: \mathbb{C} \rightarrow \mathbb{C}$ is a function of the form

\begin{equation}
F(z) = \sum_{m=1}^{M} z^{\nu_m} A_m(z) e^{i z \sigma_m},
\end{equation}
where $\nu_m \in \mathbb{R}$, $m = 1, 2, \ldots, M$, $A_m(z)$ are rational functions in $z$ not vanishing identically, and the constants $\sigma_m \in \mathbb{R}$ are ordered increasingly ($\sigma_{\min} := \sigma_1 < \sigma_2 < \cdots < \sigma_M =: \sigma_{\max}$).

For example, for the exponential polynomial

$$F(z) = \frac{z + i}{z - i} e^{iz} + z^2 \frac{z^2 + i}{z^2 + 1} e^{2iz},$$

we have $M = 2$, $\nu_1 = 1$, $\nu_2 = 2$, $\sigma_1 = 1$, $\sigma_2 = 2$, $A_1(z) = \frac{z + i}{z - i}$, $A_2(z) = \frac{z^2 + i}{z^2 + 1}$.

The zero set of an exponential polynomial $F$ is defined by

$$Z_F := \{z \in \mathbb{C} : F(z) = 0\}.$$  

For any $z \in Z_F$ we define its multiplicity $m_F(z) \in \mathbb{N}$ as the algebraic multiplicity of the root $z$ of the function (2.1). Moreover, we introduce the counting function for an exponential polynomial $F$ by

$$N_F(R) = \sum_{z \in Z_F \cap D_R} m_F(z),$$

where $D_R := \{z \in \mathbb{C} : |z| < R\}$ is the disc in the complex plane centered at the origin and having the radius $R > 0$. Thus, the value $N_F(R)$ equals the number of zeros of $F$ counted with multiplicities and lying inside $D_R$. Now we have all the tools at our disposal to state the result on the asymptotics of $N_F(R)$, proven in [Lan31, Thm. 6], see also [DEL10, Thm. 3.1].

**Theorem 2.2.** Let $F$ be an exponential polynomial as in (2.1) such that

$$\lim_{z \to \infty} A_m(z) = a_m \in \mathbb{C} \setminus \{0\}, \quad \forall m = 1, 2, \ldots, M.$$  

Then the counting function for $F$ asymptotically behaves as

$$N_F(R) = \frac{\sigma_{\max} - \sigma_{\min}}{\pi} R + O(1), \quad R \to \infty.$$  

3. Rigorous definition of $H_{\alpha,X}$

The Schrödinger operator $H_{\alpha,X}$ associated with the formal differential expression (1.1) can be rigorously defined as a self-adjoint extension in $L^2(\mathbb{R}^3)$ of the closed, densely defined, symmetric operator

$$S_X u := -\Delta u, \quad \text{dom} \ S_X := \{u \in H^2(\mathbb{R}^3) : u|_{X} = 0\},$$

where the vector $u|_{X} = (u(x_1), u(x_2), \ldots, u(x_N))^\top \in \mathbb{C}^N$ is well-defined by the Sobolev embedding theorem [McL00, Thm. 3.26]. The self-adjoint extensions of $S_X$ with $N = 1$ have been first analyzed in the seminal paper [BF61]. For $N > 1$ the symmetric operator $S_X$ possesses a rich family of self-adjoint extensions, not all of which correspond to point interactions. The self-adjoint extensions of $S_X$ corresponding to point interactions are investigated in detail.
in the monographs [AGHH05, AK99], see also the references therein. Several alternative ways for parameterizing of all the self-adjoint extensions of \( S_X \) can be found in a more recent literature; see e.g. [GMZ12, Pos08, Tet90]. Below we follow the strategy of [GMZ12] and use some of notations therein. According to [GMZ12, Prop. 4.1], the adjoint of \( S_X \) can be characterized as follows

\[
\text{dom } S_X^* = \left\{ u = u_0 + \sum_{n=1}^{N} \left( \xi_0 n e^{-r_n} + \xi_1 n e^{-r_n} \right) : u_0 \in \text{dom } S_X, \xi_0, \xi_1 \in \mathbb{C}^N \right\},
\]

\[
S_X^* u = -\Delta u_0 - \sum_{n=1}^{N} \left( \xi_0 n e^{-r_n} + \xi_1 n e^{-r_n - 2e^{-r_n}} \right),
\]

where \( r_n : \mathbb{R}^3 \to \mathbb{R}^+, r_n(x) := |x - x_n| \) for all \( n = 1, 2, \ldots, N \) and \( \xi_0 = \{\xi_0 n\}_{n=1}^{N}, \xi_1 = \{\xi_1 n\}_{n=1}^{N} \). Next, we introduce the mappings \( \Gamma_0, \Gamma_1: \text{dom } S_X^* \to \mathbb{C}^N \) by

\[
(3.2) \quad \Gamma_0 u := 4\pi \xi_0 \quad \text{and} \quad \Gamma_1 u := \left\{ \lim_{x \to x_n} \left( u(x) - \frac{\xi_0 n}{r_n} \right) \right\}_{n=1}^{N}.
\]

Eventually, the operator \( H_{\alpha,X} \) is defined as the restriction of \( S_X^* \)

\[
(3.3) \quad H_{\alpha,X} u := S_X^* u, \quad \text{dom } H_{\alpha,X} := \{ u \in \text{dom } S_X^* : \Gamma_1 u = \alpha \Gamma_0 u \},
\]

cf. [GMZ12, Rem. 4.3]. Finally, by [GMZ12, Prop. 4.2], the operator \( H_{\alpha,X} \) is self-adjoint in \( L^2(\mathbb{R}^3) \). Note also that the operator \( H_{\alpha,X} \) is the same as the one considered in [AGHH05, Chap. II.1]. We remark that the usual self-adjoint free Laplacian in \( L^2(\mathbb{R}^3) \) formally corresponds to the case \( \alpha = \infty \).

### 4. Resonances of \( H_{\alpha,X} \)

The main aim of this section is to prove asymptotics of resonances given in (1.3). Apart from that we provide a condition on resonances through the pseudo-orbit expansion, which is of independent interest.

#### 4.1. A condition on resonances for \( H_{\alpha,X} \)

First, we recall the definition of resonances for \( H_{\alpha,X} \) borrowed from [AGHH05, Sec. II.1.1]. This definition provides at the same time a way to find them. To this aim we introduce the function

\[
(4.1) \quad F_{\alpha,X}(\kappa) := \text{det} \left[ \left\{ \left( \alpha - \frac{ik}{4\pi} \right) \delta_{nn'} - \tilde{G}_\kappa(x_n - x_{n'}) \right\}_{n,n'=1}^{N,N} \right],
\]

where \( \delta_{nn'} \) is the Kronecker symbol and \( \tilde{G}_\kappa(\cdot) \) is given by

\[
\tilde{G}_\kappa(x) := \begin{cases} 0, & x = 0, \\ \frac{e^{ix|x|}}{4\pi|x|}, & x \neq 0. \end{cases}
\]
We say that $\kappa_0 \in \mathbb{C}$ is a resonance of $H_{\alpha, X}$ if
\begin{equation}
F_{\alpha, X}(\kappa_0) = 0,
\end{equation}
holds. The multiplicity of the resonance $\kappa_0$ equals the multiplicity of the zero of $F_{\alpha, X}(\cdot)$ at $\kappa = \kappa_0$. In our convention true resonances and negative eigenvalues of $H_{\alpha, X}$ correspond to $\text{Im} \, \kappa_0 < 0$ and $\text{Im} \, \kappa_0 > 0$, respectively. According to [AGHH05, Thm. II.1.1.4] the number of negative eigenvalues of $H_{\alpha, X}$ is finite and in the end it does not contribute to the asymptotics of the counting function for resonances of $H_{\alpha, X}$.

It is not difficult to see using standard formula for the determinant of a matrix that $F_{\alpha, X}$ is an exponential polynomial as in Definition 2.1 with the coefficients dependent on $\alpha$ and on the set $X$.

### 4.2. Asymptotics of the number of resonances

Recall the definition of the counting function for resonances of $H_{\alpha, X}$.

**Definition 4.1.** We define the counting function $N_{\alpha, X}(R)$ as the number of resonances of $H_{\alpha, X}$ with multiplicities lying inside the disc $D_R$.

Now, we have all the tools to provide a proof for the asymptotics of resonances (1.3) stated in the introduction.

**Theorem 4.2.** The counting function for resonances of $H_{\alpha, X}$ asymptotically behaves as
\begin{equation}
N_{\alpha, X}(R) = \frac{W_X}{\pi} R + O(1), \quad R \to +\infty,
\end{equation}
with $W_X \in [0, V_X]$, where $V_X$ is the size of $X$ defined in (1.2). In addition, $W_X$ is independent of $\alpha$.

**Proof.** The argument relies on the resonance condition (4.2). Note that the element of the matrix under the determinant in (4.1) located in the $n$-th row and the $n'$-th column is a product of a polynomial in $\kappa$ and the exponential $\exp(i \kappa \ell_{nn'})$ with $\ell_{nn'} = |x_n - x_{n'}|$. Hence, expanding $F_{\alpha, X}$ by means of a standard formula for the determinant, we get that each single term in $F_{\alpha, X}$ is a product of a polynomial in $\kappa$ and the exponential $\exp(i \kappa \sum_{n=1}^{N} \ell_{n\pi(n)})$, where $\pi \in \Pi_N$ is a permutation of the set $\{1, 2, \ldots, N\}$.

The term with the lowest multiple of $i \kappa$ in the exponential is $(\alpha - \frac{i \kappa}{2\pi})^N$, i.e. there is no exponential at all and hence $\sigma_{\text{min}} = 0$. The largest possible multiple of $i \kappa$ in the exponentials of $F_{\alpha, X}$ is $V_X$. Hence, we get $\sigma_{\text{max}} \leq V_X$. The equality $\sigma_{\text{max}} = V_X$ is not always satisfied. If the coefficient by $\exp(i \kappa V_X)$ vanishes, we have strict inequality $\sigma_{\text{max}} < V_X$. Finally, Theorem 2.2 yields
\begin{equation}
N_{\alpha, X}(R) = N_{F_{\alpha, X}}(R) = \frac{W_X}{\pi} R + O(1), \quad R \to \infty,
\end{equation}
with some $W_X \in [0, V_X]$. 

The term with the largest multiple of $\kappa$ in the exponent can be represented as a product $P \left( \alpha - \frac{ik}{4\pi} \right) \exp(ik\sigma_{\text{max}})$, where $P$ is a polynomial with real coefficients of degree $< N$. For simple algebraic reasons, if this term does not identically vanish as a function of $\kappa$ for some $\alpha = \alpha_0 \in \mathbb{R}$, then it does not identically vanish in the same sense for all $\alpha \in \mathbb{R}$. Hence, we obtain by Theorem 2.2 that $W_X$ is independent of $\alpha$. □

Remark 4.1. The proof of Theorem 4.2 gives slightly more, namely the case $W_X < V_X$ can occur only if the maximum in the definition (1.2) of the size $V_X$ of $X$ is attained at more than one permutation, as otherwise cancellation of the principal term in the exponential polynomial $F_{\alpha,X}$ can not occur.

### 4.3. Pseudo-orbit expansion for the resonance condition.

The resonance condition (4.2) can be alternatively expressed by contributions of the irreducible pseudo-orbits similarly as for quantum graphs [BHJ12, Lip15, Lip16]. This expression is just yet another way how to write the determinant. However, in some cases one can easier find the terms of the determinant by studying pseudo-orbits on the corresponding directed graph and, eventually, verify their cancellations.

Consider a complete metric graph $G$ having $N$ vertices identified with the respective points in the set $X$ and connected by $\frac{N(N-1)}{2}$ edges of lengths $\ell_{nn'} = |x_n - x_{n'}|$. To this graph we associate its oriented $G'$ counterpart, which is obtained from $G$ by replacing each edge $e$ of $G$ ($e$ is the edge between the points with indices $n$ and $n'$) by two oriented bonds $b, \hat{b}$ of lengths $|b| = |\hat{b}| = \ell_{nn'}$. The orientation of the bonds is opposite; $b$ goes from $x_n$ to $x_{n'}$, whereas $\hat{b}$ goes from $x_{n'}$ to $x_n$.

Definition 4.3. With the graph $G'$ we associate the following concepts.

(a) A periodic orbit $\gamma$ in the graph $G'$ is a closed path, which begins and ends at the same vertex, we label it by the oriented bonds, which it subsequently visits $\gamma = (b_1, b_2, \ldots, b_n)$.

(b) A pseudo-orbit $\tilde{\gamma}$ is a collection of periodic orbits $\tilde{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. The number of periodic orbits contained in the pseudo-orbit $\tilde{\gamma}$ will be denoted by $|\tilde{\gamma}|_o \in \mathbb{N}_0$.

(c) An irreducible pseudo-orbit $\bar{\gamma}$ is a pseudo-orbit which does not contain any bond more than once. Furthermore, we define

$$B_{\bar{\gamma}}(\kappa) = \prod_{b_j \in \bar{\gamma}} \left( -\frac{e^{ik|b_j|}}{4\pi |b_j|} \right).$$

For $|\gamma|_o = 0$ we set $B_{\gamma} := 1$. We denote by $\mathcal{O}_m$ the set of all irreducible pseudo-orbits in $G'$ containing exactly $m \in \mathbb{N}_0$ bonds.
The resonance condition

Proposition 4.2. whose proof is inspired by the proof of [BHJ12, Thm. 1].

and of non-Weyl-type if

where

in its decomposition, satisfying

disjoint cycles [Bon04, Sec. 3.1]

Expanding the determinant in the definition of

Proof. Expanding the determinant in the definition of $F_{\alpha,X}$ we get

According to [BC09, Sec 4.1], we have sign $\pi = (-1)^{N+m(\pi)}$. Substituting this formula for sign $\pi$ into (4.5), making use of the correspondence between irreducible periodic orbits and permutations, the formula $m(\pi) = n(\pi) + |\gamma(\pi)|_{\mathcal{O}}$, and performing some simple rearrangements, we find

\[
F_{\alpha,X}(\kappa) = (-1)^N \sum_{\pi \in \Pi_N} \text{sign} \pi \prod_{n=0}^{N} \prod_{s=1}^{n(\pi) \leq N-n} \left( \alpha - \frac{i\kappa}{4\pi} \right) \delta_{n\pi(n)} - \tilde{G}_\kappa(x_n - x_{\pi(n)})
\]

\[
= (-1)^N \sum_{\pi \in \Pi_N} \text{sign} \pi \prod_{n=0}^{N} \prod_{s=1}^{n(\pi) \leq N-n} \left( \alpha - \frac{i\kappa}{4\pi} \right) \delta_{n\pi(n)} - \tilde{G}_\kappa(x_n - x_{\pi(n)})
\]

5. Point configurations of Weyl- and non-Weyl-types

Recall that a configuration of points is said to be of Weyl-type if $W_X = V_X$ and of non-Weyl-type if $W_X < V_X$. In this section we provide examples for
both types of point configurations and discuss related questions. For the sake of convenience, for a configuration of points \( X = \{ x_n \}_{n=1}^N \) and a permutation \( \pi \in \Pi_N \) we define
\[
V_X(\pi) := \sum_{n=1}^N |x_n - x_{\pi(n)}|.
\]

### 5.1. Weyl-type configurations

First, we show that for low number of points non-Weyl configurations do not exist.

**Proposition 5.1.** For \( N = 2, 3 \), \( W_X = V_X \) holds for any \( X = \{ x_n \}_{n=1}^N \).

**Proof.** For \( N = 2 \), we have \( V_X = 2\ell_{12} \). From (4.1) and (4.2) we obtain the resonance condition
\[
\left( \frac{i\kappa}{4\pi} - \alpha \right)^2 - \frac{e^{2i\kappa\ell_{12}}}{(4\pi \ell_{12})^2} = 0.
\]
Obviously, the coefficient at \( e^{i\kappa V_X} \) does not identically vanish and the claim follows from Theorem 2.2.

Let \( N = 3 \). Without loss of generality we assume that \( \ell_{12} \geq \ell_{23} \geq \ell_{13} \). By triangle inequality we have \( \ell_{12} + \ell_{23} + \ell_{13} \geq 2\ell_{12} \). The equality is attained only if all three points belong to a straight line. Hence, we have \( V_X = \ell_{12} + \ell_{23} + \ell_{13} \), which is attained at the cyclic shift, having the decomposition \( \pi = (1, 2, 3) \).

From (4.2) we obtain the resonance condition
\[
\left( \frac{i\kappa}{4\pi} - \alpha \right)^3 - \left( \frac{i\kappa}{4\pi} - \alpha \right) f(\kappa) + g(\kappa) = 0,
\]
where
\[
f(\kappa) := \frac{1}{(4\pi)^2} \left( \frac{e^{2i\kappa\ell_{12}}}{(\ell_{12})^2} + \frac{e^{2i\kappa\ell_{23}}}{(\ell_{23})^2} + \frac{e^{2i\kappa\ell_{13}}}{(\ell_{13})^2} \right), \quad g(\kappa) := \frac{2e^{i\kappa(\ell_{12} + \ell_{23} + \ell_{13})}}{(4\pi)^3 \ell_{12}\ell_{23}\ell_{13}}.
\]

For simple algebraic reasons, in both cases \( \ell_{12} + \ell_{23} + \ell_{13} > 2\ell_{12} \) and \( \ell_{12} + \ell_{23} + \ell_{13} = 2\ell_{12} \) the coefficient at \( e^{i\kappa V_X} \) does not vanish identically and the claim also follows from Theorem 2.2. \( \square \)

Next, we show that Weyl-type configurations are not something specific for low number of points and they can be constructed for any number of them.

**Theorem 5.1.** For any \( N \geq 2 \) there exist a configuration of points \( X = \{ x_n \}_{n=1}^N \) such that \( W_X = V_X \).

**Proof.** We provide two different constructions for the cases of even and odd number of points in the set \( X \).

For \( N = 2m, m \in \mathbb{N} \), we choose the configuration \( X = \{ x_n \}_{n=1}^{2m} \) as follows. First, we fix arbitrary distinct point \( x_1, x_2, \ldots, x_m \) on the unit sphere \( S^2 \subset \mathbb{R}^3 \), so that none of them is diametrically opposite to the other. Second, we select the point \( x_{m+k} \in S^2, k = 1, \ldots, m \) to be diametrically opposite to \( x_k \). For simple geometric reasons, we have \( V_X = 4m \) and this maximum is attained at the unique permutation \( \pi \) having the following decomposition into cycles
\[ \pi = (1, m + 1)(2, m + 2) \ldots (m, 2m). \] In view of Remark 4.1, we conclude that \( W_X = V_X. \)

For \( N = 2m + 1, m \in \mathbb{N}, \) we choose the configuration \( X = \{ x_n \}_{n=1}^{2m+1} \), as follows. First, we distribute the points \( \{ x_n \}_{n=1}^{2m+1} \) on \( S^2 \) as in the case of even \( N. \) Second, we put the point \( x_{2m+1} \) into the center of \( S^2. \) If a permutation \( \pi \in \Pi_{2m+1} \) does not contain the cycle \((2m + 1), \) then we have \( V_X(\pi) \leq 4m \) and the case of equality occurs only for the permutations

\[
\begin{align*}
\pi_1 &= (1, m + 1)(2, m + 2) \ldots (m - 1, 2m - 1)(m, 2m, 2m + 1), \\
\pi_2 &= (1, m + 1)(2, m + 2) \ldots (m - 1, 2m - 1)(m, 2m + 1, 2m), \\
\pi_3 &= (1, m + 1)(2, m + 2) \ldots (m - 2, 2m - 2)(m - 1, 2m - 1, 2m + 1)(m, 2m), \\
\pi_4 &= (1, m + 1)(2, m + 2) \ldots (m - 2, 2m - 2)(m - 1, 2m + 1, 2m - 1)(m, 2m), \\
\ldots \ldots
\end{align*}
\]

\[ \pi_{2m-1} = (2, m + 2) \ldots (m - 1, 2m - 1)(m, 2m)(1, m + 1, 2m + 1), \]
\[ \pi_{2m} = (2, m + 2) \ldots (m - 1, 2m - 1)(m, 2m)(1, 2m + 1, m + 1). \]

If a permutation \( \pi \in \Pi_{2m+1} \) contains the cycle \((2m + 1), \) then we again have \( V_X(\pi) \leq 4m \) and the case of equality happens for the unique permutation

\[ \pi_{2m+1} = (1, m + 1)(2, m + 2) \ldots (m, 2m)(2m + 1). \]

Hence, we obtain that \( V_X = 4m. \) Moreover, the exponential polynomial \( F_{a,X} \)
in (4.1) can be written as

\[ F_{a,X}(\kappa) = (-1)^m \frac{4m + 4\pi a - i\kappa}{2^{m+1}(4\pi)^2} e^{i(4m-2\kappa)} + g_0(\kappa) + \sum_{l=1}^{L} g_l(\kappa) e^{ln\kappa}, \]

where \( \sigma_l \in (0, 4m) \) and \( g_0, g_l \) are polynomials, \( l = 1, 2, \ldots, L. \) Finally, by Theorem 2.2 we get \( W_X = V_X = 4m. \)

5.2. An example of a non-Weyl-type configuration. Eventually, we provide an example of a configuration of points \( X = \{ x_n \}_{n=1}^{4} \) for which \( W_X < V_X \) in Theorem 4.2, since there will be a significant cancellation of some terms.

For \( a, b, c > 0, \) we consider a configuration of points \( X = \{ x_n \}_{n=1}^{4}, \) where

\[
\begin{align*}
x_1 &= (0, 0, 0)^\top, & x_2 &= (a, -b, 0)^\top, & x_3 &= (a, b, 0)^\top, & x_4 &= (c, 0, 0)^\top;
\end{align*}
\]

see Figure 5.1. Notice that

\[ \ell_{12} = \sqrt{a^2 + b^2}, \quad \ell_{23} = 2b, \quad \ell_{34} = \sqrt{(a-c)^2 + b^2}, \quad \ell_{14} = c. \]

Let us assume that \( b \) and \( c \) are sufficiently small in comparison to \( a, \) being more precise \( 2b + c < \sqrt{a^2 + b^2} + \sqrt{(a-c)^2 + b^2}. \) Let us first write down the general resonance condition (4.2) for four points.

\[
\begin{align*}
c_0^2 &- c_0^2(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2) + 2c_0(c_1c_2c_4 + c_1c_3c_5 + c_2c_3c_6 + c_4c_5c_7) \\
&+ c_1c_6^2 + c_2c_5^2 + c_3c_4^2 - 2(c_1c_2c_5c_6 + c_1c_3c_4c_6 + c_2c_3c_4c_5) = 0,
\end{align*}
\]
Using the above decompositions of permutations and (5.2) we find (5.3)

Moreover, using (5.1) we get

\[ \ell_{12} + \ell_{23} + \ell_{34} + \ell_{14} = 2b + c + \sqrt{a^2 + b^2} + \sqrt{(a - c)^2 + b^2} \]

\[ < 2\sqrt{a^2 + b^2} + 2\sqrt{(a - c)^2 + b^2} \]

\[ = \ell_{12} + \ell_{34} + \ell_{13} + \ell_{24} . \]

The elements of the group \( \Pi_4 \) can be decomposed into disjoint cycles as

\[ \pi_1 = (1)(2)(3)(4), \quad \pi_9 = (1, 2, 3)(4), \quad \pi_{17} = (1, 3)(2, 4), \]
\[ \pi_2 = (3, 4)(1, 2), \quad \pi_{10} = (1, 2, 3, 4), \quad \pi_{18} = (1, 3, 2, 4), \]
\[ \pi_3 = (2, 3)(1, 4), \quad \pi_{11} = (1, 2, 4, 3), \quad \pi_{19} = (1, 4, 3, 2), \]
\[ \pi_4 = (2, 3, 4)(1), \quad \pi_{12} = (1, 2, 4)(3), \quad \pi_{20} = (1, 4, 2)(3), \]
\[ \pi_5 = (2, 4, 3)(1), \quad \pi_{13} = (1, 3, 2)(4), \quad \pi_{21} = (1, 4, 3)(2), \]
\[ \pi_6 = (2, 4)(1, 3), \quad \pi_{14} = (1, 3, 4, 2), \quad \pi_{22} = (1, 4)(2)(3), \]
\[ \pi_7 = (1, 2)(3, 4), \quad \pi_{15} = (1, 3)(2)(4), \quad \pi_{23} = (1, 4, 2, 3), \]
\[ \pi_8 = (1, 2)(3, 4), \quad \pi_{16} = (1, 3, 4)(2), \quad \pi_{24} = (1, 4)(2, 3). \]

Using the above decompositions of permutations and (5.2), (5.3) we find

\[ V_X(\pi_8) = V_X(\pi_{11}) = V_X(\pi_{14}) = V_X(\pi_{17}) \]
\[ > V_X(\pi_{10}) = V_X(\pi_{18}) = V_X(\pi_{19}) = V_X(\pi_{23}) > \cdots > V_X(\pi_1) = 0 . \]

Hence, \( V_X = V_X(\pi_8) = V_X(\pi_{11}) = V_X(\pi_{14}) = V_X(\pi_{17}) \) and in view of (5.2)

the leading term corresponding to \( \exp(\imath V_X) \) in the resonance condition (4.2) cancels

\[ \frac{e^{2\imath \ell_{12} + \ell_{34} \imath}}{(4\pi)^4 \ell_{12}^{12} \ell_{34}^{34}} + \frac{e^{2\imath \ell_{13} + \ell_{24} \imath}}{(4\pi)^4 \ell_{13}^{13} \ell_{24}^{24}} - \frac{2e^{2\imath \ell_{12} + \ell_{34} + \ell_{13} + \ell_{24}}}{(4\pi)^4 \ell_{12}^{12} \ell_{34}^{34} \ell_{13}^{13} \ell_{24}^{24}} = 0 . \]
However, the succeeding term in the condition (4.2) corresponding to the exponent \( \exp(i\kappa V_X(\pi_{10})) \) does not cancel
\[
- \frac{2}{(4\pi)^4} \left( \frac{e^{i\kappa(\ell_{12}+\ell_{23}+\ell_{34}+\ell_{14})}}{\ell_{12}\ell_{23}\ell_{34}\ell_{14}} + \frac{e^{i\kappa(\ell_{13}+\ell_{23}+\ell_{24}+\ell_{14})}}{\ell_{13}\ell_{23}\ell_{24}\ell_{14}} \right) \neq 0.
\]
Finally, we end up with
\[
W_X = V_X(\pi_{10}) = V_X(\pi_{18}) = V_X(\pi_{19}) = V_X(\pi_{23}) < V_X.
\]

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