SOME NONLOCAL FORMULAS
INSPIRED BY AN IDENTITY OF JAMES SIMONS

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ABSTRACT. Inspired by a classical identity proved by James Simons, we establish a new geometric formula in a nonlocal, possibly fractional, setting.

Our formula also recovers the classical case in the limit, thus providing an approach to Simons' work that does not heavily rely on differential geometry.

CONTENTS

1. INTRODUCTION

1.1. **Taking inspiration from Simons' work.** A classical identity proved by James Simons in [Sim68] states that at every point of a smooth hypersurface with vanishing mean curvature we have that

$$\Delta c^2 + 2c^4 = 2 \sum_{i,j,k=1}^{n-1} |\delta_k h_{ij}|^2. \tag{1.1}$$

Here above, δ_k denotes the tangential derivative in the k-th coordinate direction, h_{ij} the entries of the second fundamental form, c the norm of the second fundamental form, and Δ the Laplace-Beltrami operator on the hypersurface (see e.g. formula (2.16) in [CM11], or, equivalently, the seventh formula in display on page 123 of [Giu84], for further details on this classical formula).

Simons's Identity is pivotal, since it provides the essential ingredient to establish the regularity of stable minimal surfaces up to dimension 7.

In this note we speculate about possible generalizations of Simons' Identity to nonlocal settings. In particular, we will consider the case of boundary of sets and of level sets of functions. These

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cases are motivated, respectively, by the study of nonlocal minimal surfaces and nonlocal phase transition equations. The prototypical case of these problems comes from fractional minimal surfaces, as introduced in [CRS10], and we recall that the full regularity theory of the minimizers of the fractional perimeter is one of the main open problems in the field of nonlocal equations: up to now, this regularity is only known when the ambient space has dimension 2, see [SV13], or up to dimension 7 provided that the fractional parameter is sufficiently close to integer, see [CV13], or when the surface possesses a graphical structure, see [CC19] (see also [CSV19, CCS20, CDSV] for the case of stable nonlocal minimal surfaces, i.e. for surfaces of vanishing nonlocal mean curvature with nonnegative definite second variation of the corresponding energy functional).

The problem of nonlocal minimal surfaces can also be considered for more general kernels than the one of purely fractional type, see [MRT19, CSV19], and it can be recovered as the limit in the Γ-convergence sense of long-range phase coexistence problems, see [SV12]. In this regard, the regularity properties of nonlocal minimal surfaces are intimately related to the flatness of nonlocal phase transitions, which is also a problem of utmost importance in the contemporary research: up to now, these flatness properties have been established in dimension up to 3, or up to 8 for mildly nonlocal operators under an additional limit assumption, or in dimension 4 for the square root of the Laplace operator, see [CSM05, SV09, CC10, CC14, CS15, DFV18, Sav18, Sav19, FS20, DSV20], the other cases being widely open.

In this paper, we will not specifically address these regularity and rigidity problems, but rather focus on a geometric formula which is closely related to Simons' Identity in the nonlocal scenarios. The application of this formula for the regularity theory appears to be highly nontrivial, since careful estimates for the reminder terms are needed (in dimension 3, a reminder estimate has been recently put forth in [CDSV]).

An interesting by-product of the formula that we present here is that it recovers the classical Simons' Identity as a limit case. Therefore, our nonlocal formula also provides a new approach towards the original Simons' Identity, with a new proof which makes only very limited use of Riemannian geometry and relies instead on some clever use of the integration by parts.

Let us now dive into the technical details of our results.

1.2. **The geometric case.** Let K be a kernel satisfying

$$\begin{aligned} & K \in C^1(\mathbb{R}^n \setminus \{0\}), \\ & K(x) = K(-x), \\ & |K(x)| \leqslant \frac{C}{|x|^{n+s}} \\ \text{and} \qquad & |\omega \cdot \nabla K(x)| \leqslant \frac{C\,|\omega \cdot x|}{|x|^{n+s+2}} \qquad \text{for all } \omega \in S^{n-1}, \end{aligned} \tag{1.2}$$

for some $C > 0$ and $s \in (0,1)$.

Given a set E with smooth boundary, we consider the K-mean curvature of E at $x \in \partial E$ given by

$$H_{K,E}(x) := \frac{1}{2} \int_{\mathbb{R}^n} \big(\chi_{\mathbb{R}^n \setminus E}(y) - \chi_E(y)\big)\, K(x-y)\, dy. \tag{1.3}$$

Notice that the above integral is taken in the principal value sense.

The classical mean curvature of E will be denoted by H_E. We define

$$c_{K,E}(x) := \sqrt{\frac{1}{2} \int_{\partial E} \big(\nu_E(x) - \nu_E(y)\big)^2\, K(x-y)\, d\mathcal{H}^{n-1}_y}, \tag{1.4}$$

being $\nu_E = (\nu_{E,1}, \dots, \nu_{E,n})$ the exterior unit normal of E. The quantity $c_{K,E}$ plays in our setting the role played by the norm of the second fundamental form in the classical case, and we can consider it the K-total curvature of E.

We also define the (minus) K-Laplace-Beltrami operator along ∂E of a function f by

$$L_{K,E} f(x) := \int_{\partial E} \big(f(x) - f(y)\big)\, K(x-y)\, d\mathcal{H}^{n-1}_y. \tag{1.5}$$

As customary, we consider the tangential derivative

$$\delta_{E,i} f(x) := \partial_i f(x) - \nu_{E,i}(x)\, \nabla f(x) \cdot \nu_E(x) \tag{1.6}$$

and we recall that

$$\delta_{E,i}\nu_{E,j} = \delta_{E,j}\nu_{E,i}, \tag{1.7}$$

see e.g. formula (10.11) in [Giu84].

In this setting, our nonlocal formula inspired by Simons' Identity goes as follows:

Theorem 1.1. *Let K be as in (1.2). Let $E \subset \mathbb{R}^n$ with smooth boundary and $x \in \partial E$ with $\nu_E(x) = (0, \dots, 0, 1)$.*

Assume that there exist $R_0 > 0$ and $\beta \in [0, n+s)$ such that for all $R \geqslant R_0$ it holds that

$$\int_{\partial E \cap B_R(x)} \big(|H_E(y)| + 1\big)\, d\mathcal{H}^{n-1}_y \leqslant C R^\beta, \tag{1.8}$$

for some $C > 0$.

Then, for any $i, j \in \{1, \dots, n-1\}$ it holds that

$$\begin{aligned} \delta_{E,i}\delta_{E,j} H_{K,E}(x) \;=\; & - L_{K,E}\delta_{E,j}\nu_{E,i}(x) + c^2_{K,E}(x)\, \delta_{E,j}\nu_{E,i}(x) \\ & - \int_{\partial E} \Big(H_E(y) K(x-y) - \nu_E(y) \cdot \nabla K(x-y)\Big) \nu_{E,i}(y)\, \nu_{E,j}(y)\, d\mathcal{H}^{n-1}_y. \end{aligned} \tag{1.9}$$

The proof of Theorem 1.1 will be given in detail in Section 2.

It is interesting to remark that the result of Theorem 1.1 "passes to the limit efficiently and localizes": for instance, if one takes $\rho \in C^\infty_0([-1,1])$, $\varepsilon > 0$ and a kernel of the form $K_\varepsilon(x) := \varepsilon^{-n-2}\rho(|x|/\varepsilon)$, then, using Theorem 1.1 and sending $\varepsilon \searrow 0$, one recovers the classical Simons' Identity in [Sim68] (such passage to the limit can be performed e.g. with the analysis in [AV14] and Appendix C in [DdPW18]).

The details[1] on how to reconstruct the classical Simons' Identity in the appropriate limit are given in Section 1.3.

1.3. Back to the original Simons' Identity.

As mentioned above, our nonlocal formula (1.9) in Theorem 1.1 recovers, in the limit, the original Simons' Identity proved in [Sim68]. The precise result goes as follows:

Theorem 1.2. *Let $E \subset \mathbb{R}^n$ and $x \in \partial E$. Assume that there exist $R_0 > 0$ and $\beta \in [0, n+1)$ such that for all $R \geqslant R_0$ it holds that*

$$\int_{\partial E \cap B_R(x)} \big(|H_E(y)| + 1\big)\, d\mathcal{H}^{n-1}_y \leqslant C R^\beta, \tag{1.10}$$

for some $C > 0$. Then, the identity in (1.1) holds true as a consequence of formula (1.9).

[1] We also remark that condition (1.8) is obviously satisfied with $\beta := n-1$ when the set E is smooth and bounded. For minimizers, and, more generally, stable critical points, of the nonlocal perimeter functional, one still has perimeter estimates (see formula (1.16) in Corollary 1.8 of [CSV19]): however, in this general case, estimating the mean curvature, or, in greater generality, the "second derivatives" of the set, may be a demanding task, see [CDSV] for some results in this direction.

The proof of Theorem 1.2 is contained in Section 4.

We point out that Theorems 1.1 and 1.2 also provide a new proof of the original Simons' Identity. Remarkably, our proof relies less on the differential geometry structure of the hypersurface and it is, in a sense, "more extrinsic": these facts allow us to exploit similar methods also for the case of integrodifferential equations, as will be done in the forthcoming Section 1.4.

1.4. **The case of integrodifferential equations.** The framework that we provide here is a suitable modification of that given in Section 1.2 for sets. The idea is to "substitute" the volume measure $\chi_E(x)\,dx$ with $u(x)\,dx$ and the area measure $\chi_{\partial E}(x)d\mathcal{H}^{n-1}_x$ with $|\nabla u(x)|\,dx$. However, one cannot really exploit the setting of Section 1.2 as it is also for integrodifferential equations, and it is necessary to "redo the computation", so to extrapolate the correct operators and stability conditions for the solutions.

The technical details go as follows. Though more general cases can be considered, for the sake of concreteness, we focus on a kernel K satisfying

$$\begin{aligned} & K \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \\ & |\nabla K| \in L^1(\mathbb{R}^n), \\ \text{and} \qquad & K(x) = K(-x). \end{aligned} \tag{1.11}$$

Given a function $u \in W^{1,\infty}(\mathbb{R}^n)$ whose level sets $\{u = t\}$ are smooth for a.e. $t \in \mathbb{R}$, we define the K-mean curvature of u at $x \in \mathbb{R}^n$ by

$$H_{K,u}(x) := C_K - \int_{\mathbb{R}^n} u(y)\,K(x-y)\,dy, \qquad \text{where } \; C_K := \frac{1}{2}\int_{\mathbb{R}^n} K(y)\,dy. \tag{1.12}$$

The setting in (1.12) has to be compared with (1.3) and especially with the forthcoming formula (2.2). The classical mean curvature of the level sets of u will be denoted by H_u (i.e., if $t_x := u(x)$, then $H_u(x)$ is the classical mean curvature of the set $\{u > t_x\}$ at x).

We also define the the K-total curvature of u as

$$c_{K,u}(x) := \sqrt{\frac{1}{2}\int_{\mathbb{R}^n} \big(\nu_u(x) - \nu_u(y)\big)^2\,K(x-y)\,d\mu_{u,y}}, \tag{1.13}$$

being $\nu_u(x)$ the exterior unit normal of the level set of u passing through x (i.e., if $t_x := u(x)$, then $\nu_u(x)$ is the exterior normal of the set $\{u > t_x\}$ at x). In (1.13), we also used the notation

$$d\mu_{u,y} := |\nabla u(y)|\,dy. \tag{1.14}$$

Of course, the definition in (1.13) has to be compared with that in (1.4). Moreover, by construction we have that

$$\nu_u(x) = -\frac{\nabla u(x)}{|\nabla u(x)|}, \tag{1.15}$$

the minus sign coming from the fact that the external derivative of $\{u > t_x\}$ points towards points with "decreasing values" of u.

We also define the K-Laplace-Beltrami operator induced by u acting on a function f by

$$L_{K,u}f(x) := \int_{\mathbb{R}^n} \big(f(x) - f(y)\big)\,K(x-y)\,d\mu_{u,y}. \tag{1.16}$$

Once again, one can compare (1.5) and (1.16). Also, we denote by $\delta_{u,i}$ the tangential derivatives along the level sets of u (recall (1.6)). This setting turns out to be the appropriate one to translate Theorem 1.1 into a result for solutions of integrodifferential equations, as will be presented in the forthcoming result:

Theorem 1.3. *Let K be as in (1.11). Let $u \in W^{1,\infty}(\mathbb{R}^n)$ and assume that $\{u = t\}$ is a smooth hypersurface with bounded mean curvature for a.e. $t \in \mathbb{R}$. For any $x \in \mathbb{R}^n$ with $\nu_u(x) = (0, \dots, 0, 1)$ and any i, $j \in \{1, \dots, n-1\}$, it holds that*

$$\begin{aligned} \delta_{u,i}\delta_{u,j}H_{K,u}(x) \;=&\; - L_{K,u}\delta_{u,j}\nu_{u,i}(x) + c^2_{K,u}(x)\,\delta_{u,j}\nu_{u,i}(x) \\ &- \int_{\mathbb{R}^n} \Big(H_u(y)K(x-y) - \nu_u(y)\cdot\nabla K(x-y)\Big)\nu_{u,j}(y)\,\nu_{u,i}(y)\,d\mu_{u,y}. \end{aligned} \tag{1.17}$$

The proof of Theorem 1.3 is a careful variation of that of Theorem 1.1, but, for the sake of clarity, we provide full details in Section 3. We also observe that the choice $u := \chi_E$ would formally allow one to recover Theorem 1.1 from Theorem 1.3.

1.5. **Stable sets.** In the study of variational problems, a special role is played by the “stable” critical points, i.e. those critical points at which the second derivative of the energy functional is nonnegative definite, see e.g. [CP18].

In this spirit, in the study of nonlocal minimal surfaces we say that E is a stable set in Ω if $H_{K,E}(x) = 0$ for any $x \in \Omega \cap \partial E$ and

$$\frac{1}{2}\int_{\partial E}\int_{\partial E} (f(x) - f(y))^2\,K(x-y)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y - \int_{\partial E} c^2_{K,E}(x)\,f^2(x)\,d\mathcal{H}^{n-1}_x \geqslant 0 \tag{1.18}$$

for any $f \in C^\infty_0(\Omega)$.

In connection with this, we set

$$B_{K,E}(u, v; x) := \frac{1}{2}\int_{\partial E} (u(x) - u(y))(v(x) - v(y))\,K(x-y)\,d\mathcal{H}^{n-1}_y,$$

where the integral is taken in the principal value sense, and

$$B_{K,E}(u, v) := \int_{\partial E} B_{K,E}(u, v; x)\,d\mathcal{H}^{n-1}_x.$$

In this notation, the first term in (1.18) takes the form $B_K(f, f)$.

Also, we consider the integrodifferential operator $L_{K,E}$ previously introduced in (1.5). When $K(x) = \frac{1}{|x|^{n+s}}$, this operator reduces to the fractional Laplacian, up to normalizing constants.

With this notation, we have:

Theorem 1.4. *Let $E \subset \mathbb{R}^n$ with smooth boundary. Then, for all $\eta \in C^\infty_0(\partial E)$,*

$$-\int_{\partial E} \left\{\frac{1}{2}L_{K,E}c^2_{K,E}(x) + B_{K,E}(c_{K,E}, c_{K,E}; x) - c^4_{K,E}(x)\right\}\eta^2(x)\,d\mathcal{H}^{n-1}_x \leqslant \int_{\partial E} c^2_{K,E}(x)B_{K,E}(\eta, \eta; x)\,d\mathcal{H}^{n-1}_x.$$

For the classical counterpart of the above in equality, see e.g. [CP18, equation (19)].

The rest of this paper contains the proofs of the results stated above. Before undertaking the details of the proofs, we mention that the idea of recovering classical results in geometry as a limit of fractional ones, thus providing a unified approach between different disciplines, can offer interesting perspectives (for instance, we will investigate the Ricci curvature from this point of view in the forthcoming article [DGTV]; see also [GH] for limit formulas related to trace problems and [HK] for a recovery technique of the Divergence Theorem coming from a nonlocal perspective).

2. Proof of Theorem 1.1

Up to a translation, we can suppose that $0 \in \partial E$ and prove Theorem 1.1 at the origin, hence we can choose coordinates such that

$$\nu_E(0) = (0, \dots, 0, 1). \tag{2.1}$$

We point out that assumption (1.8) guarantees that all the terms in (1.9) are finite, see e.g. the forthcoming technical calculation in (4.19).

Moreover, we take K to be smooth, compactly supported and nonsingular, so to be able to take derivatives inside the integral (the general case then follows by approximation, see e.g. [FFM+15]). In this way, we rewrite (1.3) as

$$H_{K,E}(x) = C_K - \int_E K(x-y)\,dy, \qquad \text{where } \; C_K := \frac{1}{2}\int_{\mathbb{R}^n} K(y)\,dy. \tag{2.2}$$

Also, this is a good definition for all $x \in \mathbb{R}^n$ (and not only for $x \in \partial E$), so we can consider the full gradient of such an expression. Moreover, for a fixed $x \in \mathbb{R}^n$, we use the notation

$$\phi(y) := K(x-y). \tag{2.3}$$

In this way, we have that, for any $\ell \in \{1, \dots, n\}$,

$$\partial_\ell K(x-y) = -\partial_\ell \phi(y). \tag{2.4}$$

Exploiting this, (2.2) and the Gauss-Green Theorem, we see that, for any $\ell \in \{1, \dots, n\}$,

$$\begin{aligned}\partial_\ell H_{K,E}(x) &= -\int_E \partial_\ell K(x-y)\,dy = \int_E \partial_\ell \phi(y)\,dy = \int_E \operatorname{div}\big(\phi(y)e_\ell\big)\,dy\\ &= \int_{\partial E} \nu_E(y)\cdot\big(\phi(y)e_\ell\big)\,d\mathcal{H}^{n-1}_y = \int_{\partial E} \nu_{E,\ell}(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y.\end{aligned}$$

This gives that, for any $x \in \partial E$,

$$\nabla H_{K,E}(x) = \int_{\partial E} \nu_E(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y. \tag{2.5}$$

In addition, from (1.4),

$$\begin{aligned} c^2_{K,E}(x) &= \frac{1}{2}\int_{\partial E} \big(\nu_E(x)-\nu_E(y)\big)^2 K(x-y)\,d\mathcal{H}^{n-1}_y\\ &= \int_{\partial E} K(x-y)\,d\mathcal{H}^{n-1}_y - \nu_E(x)\cdot\int_{\partial E} \nu_E(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y.\end{aligned} \tag{2.6}$$

Now, we fix the indices $i,\,j \in \{1, \dots, n-1\}$ and we make use of (1.6) and (2.5) to find that

$$\begin{aligned}\delta_{E,i} H_{K,E}(x) &= \partial_i H_{K,E}(x) - \nu_{E,i}(x)\,\nabla H_{K,E}(x)\cdot\nu_E(x)\\ &= \int_{\partial E} \nu_{E,i}(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y - \nu_{E,i}(x)\,\nu_E(x)\cdot\int_{\partial E} \nu_E(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y.\end{aligned} \tag{2.7}$$

We take another tangential derivative of (2.7) and evaluate it at the origin, recalling (2.1) (which, in particular, gives that $\nu_{E,i}(0) = 0 = \nu_{E,j}(0)$ for any $i,\,j \in \{1, \dots, n-1\}$). In this way, recalling (1.6), we obtain that

$$\begin{aligned}&\delta_{E,j}\delta_{E,i} H_{K,E}(0)\\ &= \partial_j \delta_{E,i} H_{K,E}(0)\\ &= \partial_j\left[\int_{\partial E} \nu_{E,i}(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y - \nu_{E,i}(x)\,\nu_E(x)\cdot\int_{\partial E}\nu_E(y)\,K(x-y)\,d\mathcal{H}^{n-1}_y\right]_{x=0}\\ &= \int_{\partial E} \nu_{E,i}(y)\,\partial_j K(-y)\,d\mathcal{H}^{n-1}_y - \partial_j \nu_{E,i}(0)\,\nu_E(0)\cdot\int_{\partial E}\nu_E(y)\,K(-y)\,d\mathcal{H}^{n-1}_y.\end{aligned} \tag{2.8}$$

Also, using the notation in (2.3) and (2.4) with $x := 0$ and (1.6), we see that

$$\begin{aligned}&\int_{\partial E} \nu_{E,i}(y)\,\partial_j K(-y)\,d\mathcal{H}^{n-1}_y\\ &= -\int_{\partial E} \nu_{E,i}(y)\,\partial_j \phi(y)\,d\mathcal{H}^{n-1}_y\\ &= -\int_{\partial E} \nu_{E,i}(y)\,\delta_{E,j}\phi(y)\,d\mathcal{H}^{n-1}_y - \int_{\partial E} \nu_{E,i}(y)\,\nu_{E,j}(y)\,\nabla\phi(y)\cdot\nu_E(y)\,d\mathcal{H}^{n-1}_y.\end{aligned} \tag{2.9}$$

Now we recall an integration by parts formula for tangential derivatives (see e.g. the first formula[2] in display on page 122 of [Giu84]), namely

$$\int_{\partial E} \delta_{E,j} f(y)\,d\mathcal{H}^{n-1}_y = \int_{\partial E} H_E(y)\,\nu_{E,j}(y)\,f(y)\,d\mathcal{H}^{n-1}_y, \tag{2.10}$$

being H_E the classical mean curvature of ∂E. Applying this formula to the product of two functions, we find that

$$\begin{aligned}\int_{\partial E} \delta_{E,j} f(y)\,g(y)\,d\mathcal{H}^{n-1}_y + \int_{\partial E} f(y)\,\delta_{E,j} g(y)\,d\mathcal{H}^{n-1}_y &= \int_{\partial E} \delta_{E,j}(fg)(y)\,d\mathcal{H}^{n-1}_y\\ &= \int_{\partial E} H_E(y)\,\nu_{E,j}(y)\,f(y)\,g(y)\,d\mathcal{H}^{n-1}_y.\end{aligned} \tag{2.11}$$

Using this and (2.3) (with $x := 0$ here), we see that

$$\begin{aligned}&-\int_{\partial E} \nu_{E,i}(y)\,\delta_{E,j}\phi(y)\,d\mathcal{H}^{n-1}_y\\ &= \int_{\partial E} \delta_{E,j}\nu_{E,i}(y)\,\phi(y)\,d\mathcal{H}^{n-1}_y - \int_{\partial E} H_E(y)\,\nu_{E,i}(y)\,\nu_{E,j}(y)\,\phi(y)\,d\mathcal{H}^{n-1}_y\\ &= \int_{\partial E} \delta_{E,j}\nu_{E,i}(y)\,K(-y)\,d\mathcal{H}^{n-1}_y - \int_{\partial E} H_E(y)\,\nu_{E,i}(y)\,\nu_{E,j}(y)\,K(-y)\,d\mathcal{H}^{n-1}_y.\end{aligned}$$

So, we insert this information into (2.9) and we conclude that

$$\begin{aligned}\int_{\partial E} \nu_{E,i}(y)\,\partial_j K(-y)\,d\mathcal{H}^{n-1}_y &= \int_{\partial E} \delta_{E,j}\nu_{E,i}(y)\,K(-y)\,d\mathcal{H}^{n-1}_y\\ &\quad - \int_{\partial E} H_E(y)\,\nu_{E,i}(y)\,\nu_{E,j}(y)\,K(-y)\,d\mathcal{H}^{n-1}_y + \int_{\partial E} \nu_{E,i}(y)\,\nu_{E,j}(y)\,\nabla K(-y)\cdot\nu_E(y)\,d\mathcal{H}^{n-1}_y.\end{aligned}$$

Plugging this into (2.8), we get that

$$\begin{aligned}\delta_{E,j}\delta_{E,i}H_{K,E}(0) = &\int_{\partial E} \delta_{E,j}\nu_{E,i}(y)\,K(-y)\,d\mathcal{H}^{n-1}_y - \int_{\partial E} H_E(y)\,\nu_{E,i}(y)\,\nu_{E,j}(y)\,K(-y)\,d\mathcal{H}^{n-1}_y\\ &+ \int_{\partial E} \nu_{E,i}(y)\,\nu_{E,j}(y)\,\nabla K(-y)\cdot\nu_E(y)\,d\mathcal{H}^{n-1}_y\\ &- \partial_j\nu_{E,i}(0)\,\nu_E(0)\cdot\int_{\partial E} \nu_E(y)\,K(-y)\,d\mathcal{H}^{n-1}_y.\end{aligned} \tag{2.12}$$

[2] We stress that the normal on page 122 of [Giu84] is internal, according to the distance setting on page 120 therein. This causes in our notation a sign change with respect to the setting in [Giu84]. Also, in the statement of Lemma 10.8 on page 121 in [Giu84] there is a typo (missing a mean curvature inside an integral). We also observe that formula (2.10) can also be seen as a version of the Tangential Divergence Theorem, see e.g. Appendix A in [Eck04].

In addition, from (2.6),

$$\partial_j \nu_{E,i}(0)\, c^2_{K,E}(0) = \int_{\partial E} \partial_j \nu_{E,i}(0)\, K(-y)\, d\mathcal{H}^{n-1}_y - \partial_j \nu_{E,i}(0)\, \nu_E(0) \cdot \int_{\partial E} \nu_E(y)\, K(-y)\, d\mathcal{H}^{n-1}_y.$$

Comparing with (2.12), we conclude that

$$\begin{aligned}\delta_{E,j}\delta_{E,i} H_{K,E}(0) = &\int_{\partial E} \Big(\delta_{E,j}\nu_{E,i}(y) - \delta_{E,j}\nu_{E,i}(0)\Big)\, K(-y)\, d\mathcal{H}^{n-1}_y \\ &- \int_{\partial E} H_E(y)\, \nu_{E,i}(y)\, \nu_{E,j}(y)\, K(-y)\, d\mathcal{H}^{n-1}_y \\ &+ \int_{\partial E} \nu_{E,i}(y)\, \nu_{E,j}(y)\, \nabla K(-y) \cdot \nu_E(y)\, d\mathcal{H}^{n-1}_y + \partial_j \nu_{E,i}(0)\, c^2_{K,E}(0).\end{aligned}$$

From this identity and the definition in (1.5), the desired result plainly follows. □

3. Proof of Theorem 1.3

The proof is similar to that of Theorem 1.1. Full details are provided for the reader's facility. Up to a translation, we can prove Theorem 1.3 at the origin and suppose that

$$\nu_u(0) = (0, \dots, 0, 1). \tag{3.1}$$

We observe that our assumptions on the kernel in (1.11) yield that all the terms in (1.17) are finite.

Using (1.12), (1.14) and (1.15), we see that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\nabla H_{K,u}(x) &= \nabla \left(C_K - \int_{\mathbb{R}^n} u(x-y)\, K(y)\, dy \right) = - \int_{\mathbb{R}^n} \nabla u(x-y)\, K(y)\, dy \\ &= - \int_{\mathbb{R}^n} \nabla u(y)\, K(x-y)\, dy = \int_{\mathbb{R}^n} \nu_u(y)\, K(x-y)\, d\mu_{u,y}.\end{aligned} \tag{3.2}$$

In addition, from (1.13),

$$\begin{aligned}c^2_{K,u}(x) &= \frac{1}{2} \int_{\mathbb{R}^n} \big(\nu_u(x) - \nu_u(y)\big)^2\, K(x-y)\, d\mu_{u,y} \\ &= \int_{\mathbb{R}^n} K(x-y)\, d\mu_{u,y} - \nu_u(x) \cdot \int_{\mathbb{R}^n} \nu_u(y)\, K(x-y)\, d\mu_{u,y}.\end{aligned} \tag{3.3}$$

Also, in view of (1.6) and (3.2),

$$\begin{aligned}\delta_{u,i} H_{K,u}(x) &= \partial_i H_{K,u}(x) - \nu_{u,i}(x)\, \nabla H_{K,u}(x) \cdot \nu_u(x) \\ &= \int_{\mathbb{R}^n} \nu_{u,i}(y)\, K(x-y)\, d\mu_{u,y} - \nu_{u,i}(x)\, \nu_u(x) \cdot \int_{\mathbb{R}^n} \nu_u(y)\, K(x-y)\, d\mu_{u,y}.\end{aligned} \tag{3.4}$$

Consequently, using (3.1) and (3.4), for all i, $j \in \{1, \dots, n-1\}$,

$$\begin{aligned}&\delta_{u,j}\delta_{u,i} H_{K,u}(0) \\ &= \partial_j \delta_{u,i} H_{K,u}(0) \\ &= \partial_j \left[\int_{\mathbb{R}^n} \nu_{u,i}(y)\, K(x-y)\, d\mu_{u,y} - \nu_{u,i}(x)\, \nu_u(x) \cdot \int_{\mathbb{R}^n} \nu_u(y)\, K(x-y)\, d\mu_{u,y} \right]_{x=0} \\ &= \int_{\mathbb{R}^n} \nu_{u,i}(y)\, \partial_j K(-y)\, d\mu_{u,y} - \partial_j \nu_{u,i}(0)\, \nu_u(0) \cdot \int_{\mathbb{R}^n} \nu_u(y)\, K(-y)\, d\mu_{u,y}.\end{aligned} \tag{3.5}$$

Now, recalling the notation in (2.3) and (2.4) with $x := 0$ and (1.6), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^n} \nu_{u,i}(y)\,\partial_j K(-y)\,d\mu_{u,y}\\ &= -\int_{\mathbb{R}^n} \nu_{u,i}(y)\,\partial_j \phi(y)\,d\mu_{u,y}\\ &= -\int_{\mathbb{R}^n} \nu_{u,i}(y)\,\delta_{u,j}\phi(y)\,d\mu_{u,y} - \int_{\mathbb{R}^n} \nu_{u,i}(y)\,\nu_{u,j}(y)\,\nabla\phi(y)\cdot\nu_u(y)\,d\mu_{u,y}.\end{aligned} \tag{3.6}$$

Furthermore, exploiting the Coarea Formula twice and the tangential integration by parts identity in (2.11), we obtain that

$$\begin{aligned}-\int_{\mathbb{R}^n} \nu_{u,i}(y)\,\delta_{u,j}\phi(y)\,d\mu_{u,y} &= -\int_{\mathbb{R}^n} |\nabla u(y)|\,\nu_{u,i}(y)\,\delta_{u,j}\phi(y)\,dy\\ &= -\int_{\mathbb{R}}\int_{\{u(y)=t\}} \nu_{u,i}(y)\,\delta_{u,j}\phi(y)\,d\mathcal{H}^{n-1}_y\,dt\\ &= \int_{\mathbb{R}}\int_{\{u(y)=t\}} \delta_{u,j}\nu_{u,i}(y)\,\phi(y)\,d\mathcal{H}^{n-1}_y\,dt\\ &\qquad - \int_{\mathbb{R}}\int_{\{u(y)=t\}} H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,\phi(y)\,d\mathcal{H}^{n-1}_y\,dt\\ &= \int_{\mathbb{R}^n} |\nabla u(y)|\,\delta_{u,j}\nu_{u,i}(y)\,\phi(y)\,dy - \int_{\mathbb{R}^n} |\nabla u(y)|\,H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,\phi(y)\,dy\\ &= \int_{\mathbb{R}^n} \delta_{u,j}\nu_{u,i}(y)\,\phi(y)\,d\mu_{u,y} - \int_{\mathbb{R}^n} H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,\phi(y)\,d\mu_{u,y}\\ &= \int_{\mathbb{R}^n} \delta_{u,j}\nu_{u,i}(y)\,K(-y)\,d\mu_{u,y} - \int_{\mathbb{R}^n} H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,K(-y)\,d\mu_{u,y}.\end{aligned}$$

We can now insert this identity into (3.6) and we get that

$$\begin{aligned}\int_{\mathbb{R}^n} \nu_{u,i}(y)\,\partial_j K(-y)\,d\mu_{u,y} &= \int_{\mathbb{R}^n} \delta_{u,j}\nu_{u,i}(y)\,K(-y)\,d\mu_{u,y}\\ &\quad - \int_{\mathbb{R}^n} H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,K(-y)\,d\mu_{u,y} + \int_{\mathbb{R}^n} \nu_{u,i}(y)\,\nu_{u,j}(y)\,\nabla K(-y)\cdot\nu_u(y)\,d\mu_{u,y}.\end{aligned}$$

Plugging this into (3.5) we get that

$$\begin{aligned}\delta_{u,j}\delta_{u,i}H_{K,u}(0) &= \int_{\mathbb{R}^n} \delta_{u,j}\nu_{u,i}(y)\,K(-y)\,d\mu_{u,y} - \int_{\mathbb{R}^n} H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,K(-y)\,d\mu_{u,y}\\ &\quad + \int_{\mathbb{R}^n} \nu_{u,i}(y)\,\nu_{u,j}(y)\,\nabla K(-y)\cdot\nu_u(y)\,d\mu_{u,y} - \partial_j\nu_{u,i}(0)\,\nu_u(0)\cdot\int_{\mathbb{R}^n} \nu_u(y)\,K(-y)\,d\mu_{u,y}.\end{aligned} \tag{3.7}$$

Also, from (3.3), we have that

$$\partial_j\nu_{u,i}(0)\,c^2_{K,u}(0) = \int_{\mathbb{R}^n} \partial_j\nu_{u,i}(0)\,K(-y)\,d\mu_{u,y} - \partial_j\nu_{u,i}(0)\,\nu_u(0)\cdot\int_{\mathbb{R}^n} \nu_u(y)\,K(-y)\,d\mu_{u,y}.$$

Hence, from this and (3.7), we conclude that

$$\begin{aligned}\delta_{u,j}\delta_{u,i}H_{K,u}(0) &= \int_{\mathbb{R}^n} \Big(\delta_{u,j}\nu_{u,i}(y) - \delta_{u,j}\nu_{u,i}(0)\Big)\,K(-y)\,d\mu_{u,y} - \int_{\mathbb{R}^n} H_u(y)\,\nu_{u,i}(y)\,\nu_{u,j}(y)\,K(-y)\,d\mu_{u,y}\\ &\quad + \int_{\mathbb{R}^n} \nu_{u,i}(y)\,\nu_{u,j}(y)\,\nabla K(-y)\cdot\nu_u(y)\,d\mu_{u,y} + \partial_j\nu_{u,i}(0)\,c^2_{K,u}(0).\end{aligned}$$

This and (1.16) give the desired result. □

4. Proof of Theorem 1.2

For clarity, we denote by $\Delta_{\partial E}$ the Laplace-Beltrami operator on the hypersurface ∂E, by $\delta_{k,E}$ the tangential derivative in the kth coordinate direction, by ν_E the external derivative and by c_E the norm of the second fundamental form.

To obtain (1.1) as a limit of (1.9), we focus on a special kernel. Namely, given $\varepsilon > 0$, we let

$$K_\varepsilon(y) := \frac{\varepsilon}{|y|^{n+1-\varepsilon}}. \tag{4.1}$$

We now recall a simple, explicit calculation:

$$\int_{B_1} x_1^4\,dx = \frac{3\,\mathcal{H}^{n-1}(S^{n-1})}{n(n+2)(n+4)} \tag{4.2}$$

$$\text{and} \qquad \int_{S^{n-1}} \vartheta_1^4\,d\mathcal{H}^{n-1}_\vartheta = \frac{3\,\mathcal{H}^{n-1}(S^{n-1})}{n(n+2)}. \tag{4.3}$$

Not to interrupt the flow of the arguments, we postpone the proof of formulas (4.2) and (4.3) to Appendix A.

To complete the proof of Theorem 1.2, without loss of generality, we assume that $0 = x \in \partial E$ and that $\partial E \cap B_{r_0}$ is the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with vertical normal, hence $f(0) = 0$ and $\partial_i f(0) = 0$ for all $i \in \{1, \dots, n-1\}$. We can also diagonalize the Hessian matrix of f at 0, and obtain that the mean curvature H_E at the origin coincides with the trace[3] of such matrix, namely

$$H_E(0) = -\big(\partial_1^2 f(0) + \dots + \partial_{n-1}^2 f(0)\big). \tag{4.4}$$

The sign convention here is inferred by the assumption that E is locally the subgraph of f and the normal is taken to point outwards. Consequently, for every $y = (y', f(y')) \in \partial E \cap B_{r_0}$,

$$\begin{aligned} f(y') &= \frac12 \sum_{i=1}^{n-1} \partial_i^2 f(0)\,y_i^2 + O(|y'|^3),\\ \nabla f(y') &= (\partial_1 f(y'), \dots, \partial_{n-1} f(y')) = \big(\partial_1^2 f(0)\,y_1, \dots, \partial_{n-1}^2 f(0)\,y_{n-1}\big) + O(|y'|^2) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \nu_E(y) &= \frac{(-\nabla f(y'), 1)}{\sqrt{1 + |\nabla f(y')|^2}} = (-\nabla f(y'), 1) + O(|y'|^2)\\ &= \big(-\partial_1^2 f(0)\,y_1, \dots, -\partial_{n-1}^2 f(0)\,y_{n-1}, 1\big) + O(|y'|^2)\big). \end{aligned} \tag{4.6}$$

Here, the notation $g = O(h(|y'|))$ means that $|g| \leqslant C|h(|y'|)|$ for $|y'|$ sufficient close to 0 with C independent of ε, that is, g is *uniformly in* ε big O of h as $|y'| \to 0$. As a consequence, for every $y = (y', f(y')) \in \partial E \cap B_{r_0}$,

$$|y|^2 = |y'|^2 + |f(y')|^2 = |y'|^2 + O(|y'|^4) = |y'|^2\big(1 + O(|y'|^2)\big), \tag{4.7}$$

and, for any $i, j \in \{1, \dots, n-1\}$,

$$\nu_{E,j}(y)\nu_{E,i}(y) = \partial_j^2 f(0) \partial_i^2 f(0) y_j y_i + O(|y'|^3). \tag{4.8}$$

Thus, using (4.7), we see that, for any fixed $\alpha \in \mathbb{R}$,

$$|y|^\alpha = |y'|^\alpha \big(1 + O(|y'|^2)\big)^{\alpha/2} = |y'|^\alpha \big(1 + O(|y'|^2)\big). \tag{4.9}$$

[3] We stress that we are not dividing the quantity in (4.4) by $n-1$, to be consistent with the notation in formula (10.12) in [Giu84].

Then, from (4.6) and (4.9), we obtain that, for any $\ell \in \{1, \dots, n-1\}$ and $y \in \partial E \cap B_{r_0}$,

$$\begin{aligned}\nu_{E,\ell}(y)\partial_\ell K_\varepsilon(-y) &= -\frac{(n+1-\varepsilon)\varepsilon\partial_\ell^2 f(0)\, y_\ell^2}{|y|^{n+3-\varepsilon}} + \varepsilon O\left(|y'|^{\varepsilon-n}\right) \\ &= -\frac{(n+1-\varepsilon)\varepsilon\partial_\ell^2 f(0)\, y_\ell^2}{|y'|^{n+3-\varepsilon}} + \varepsilon O\left(|y'|^{\varepsilon-n}\right)\end{aligned}$$

and also, recalling (4.5),

$$\begin{aligned}\nu_{E,n}(y)\partial_n K_\varepsilon(-y) &= \frac{(n+1-\varepsilon)\varepsilon y_n(1+O(|y'|^2))}{|y|^{n+3-\varepsilon}} \\ &= \frac{1}{2}\sum_{\ell=1}^{n-1} \frac{(n+1-\varepsilon)\varepsilon\partial_\ell^2 f(0)\, y_\ell^2}{|y'|^{n+3-\varepsilon}} + \varepsilon O(|y'|^{\varepsilon-n}).\end{aligned}$$

Accordingly, we have that

$$\nu_E(y) \cdot \nabla K_\varepsilon(-y) = -\frac{1}{2}\sum_{\ell=1}^{n-1} \frac{(n+1-\varepsilon)\varepsilon\partial_\ell^2 f(0)\, y_\ell^2}{|y'|^{n+3-\varepsilon}} + \varepsilon O(|y'|^{\varepsilon-n}).$$

We thereby deduce from the latter identity and (4.8) (and exploiting an odd symmetry argument) that, for any $r \in (0, r_0]$,

$$\begin{aligned}&-\frac{2}{(n+1-\varepsilon)\varepsilon}\int_{\partial E\cap B_r} \nu_E(y)\cdot\nabla K_\varepsilon(-y)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y \\ &= \sum_{\ell=1}^{n-1}\int_{\partial E\cap B_r}\left(\frac{\partial_\ell^2 f(0)\,\partial_j^2 f(0)\,\partial_i^2 f(0)\,y_\ell^2\,y_j\,y_i}{|y'|^{n+3-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,d\mathcal{H}^{n-1}_y \\ &= \sum_{\ell=1}^{n-1}\int_{\{|y'|<r\}}\left(\frac{\partial_\ell^2 f(0)\,\partial_j^2 f(0)\,\partial_i^2 f(0)\,y_\ell^2\,y_j\,y_i}{|y'|^{n+3-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,\sqrt{1+|\nabla f(y')|^2}\,dy' \\ &= \sum_{\ell=1}^{n-1}\int_{\{|y'|<r\}}\left(\frac{\partial_\ell^2 f(0)\,\partial_j^2 f(0)\,\partial_i^2 f(0)\,y_\ell^2\,y_j\,y_i}{|y'|^{n+3-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,dy' \\ &= \sum_{\ell=1}^{n-1}\int_{\{|y'|<r\}}\left(\frac{\partial_\ell^2 f(0)\,(\partial_j^2 f(0))^2\,y_\ell^2\,y_j^2\,\delta_{ji}}{|y'|^{n+3-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,dy'.\end{aligned} \tag{4.10}$$

Furthermore, exploiting again (4.8) and (4.9), we see that

$$\begin{aligned}&\int_{\partial E\cap B_r} H_E(y)K_\varepsilon(-y)\,\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y \\ &= \varepsilon\int_{\partial E\cap B_r}\left(\frac{H_E(y)\partial_j^2 f(0)\partial_i^2 f(0)y_j y_i}{|y'|^{n+1-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,d\mathcal{H}^{n-1}_y \\ &= \varepsilon\int_{\{|y'|<r\}}\left(\frac{H_E(0)\partial_j^2 f(0)\partial_i^2 f(0)y_j y_i}{|y'|^{n+1-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,dy' \\ &= \varepsilon\int_{\{|y'|<r\}}\left(\frac{H_E(0)\,(\partial_j^2 f(0))^2\,y_j^2\,\delta_{ji}}{|y'|^{n+1-\varepsilon}} + O(|y'|^{2+\varepsilon-n})\right)\,dy'.\end{aligned} \tag{4.11}$$

Now we use polar coordinates in $\mathbb{R}^{n-1}$ to observe that

$$\int_{\{|y'|<r\}} |y'|^{2+\varepsilon-n}\,dy' = \mathcal{H}^{n-2}(S^{n-2})\int_0^r \rho^{2+\varepsilon-n}\,\rho^{n-2}\,d\rho = \frac{C\,r^{1+\varepsilon}}{1+\varepsilon}, \tag{4.12}$$

for some $C > 0$.

Moreover, for any fixed index $j \in \{1, \dots, n-1\}$,

$$
\begin{aligned}
\varepsilon \int_{\{|y'|<r\}} \frac{y_j^2}{|y'|^{n+1-\varepsilon}}\, dy' &= \frac{\varepsilon}{n-1} \sum_{k=1}^{n-1} \int_{\{|y'|<r\}} \frac{y_k^2}{|y'|^{n+1-\varepsilon}}\, dy' \\
&= \frac{\varepsilon}{n-1} \int_{\{|y'|<r\}} \frac{dy'}{|y'|^{n-1-\varepsilon}} = \frac{\varepsilon\, \mathcal{H}^{n-2}(S^{n-2})}{n-1} \int_0^r \rho^{\varepsilon-1}\, d\rho = \varpi\, r^{\varepsilon},
\end{aligned} \tag{4.13}
$$

where

$$
\varpi := \frac{\mathcal{H}^{n-2}(S^{n-2})}{n-1}. \tag{4.14}
$$

Now, we compute the term $\varepsilon \int_{\{|y'|<r\}} \frac{y_\ell^2\, y_j^2}{|y'|^{n+3-\varepsilon}}\, dy'$. For this, first of all we deal with the case $\ell = j$: in this situation, we have that

$$
\varepsilon \int_{\{|y'|<r\}} \frac{y_j^4}{|y'|^{n+3-\varepsilon}}\, dy' = \varepsilon \int_{\{|y'|<r\}} \frac{y_1^4}{|y'|^{n+3-\varepsilon}}\, dy' = C_\star\, r^{\varepsilon}, \tag{4.15}
$$

where

$$
\begin{aligned}
C_\star := \varepsilon \int_{\{|y'|<1\}} \frac{y_1^4}{|y'|^{n+3-\varepsilon}}\, dy' &= \varepsilon \iint_{(\rho,\vartheta)\in(0,1)\times S^{n-2}} \rho^{\varepsilon-1}\, \vartheta_1^4\, d\rho\, d\mathcal{H}^{n-2}_\vartheta \\
&= \int_{\vartheta\in S^{n-2}} \vartheta_1^4\, d\mathcal{H}^{n-2}_\vartheta = \frac{3\,\mathcal{H}^{n-2}(S^{n-2})}{(n-1)(n+1)} = \frac{3\varpi}{n+1},
\end{aligned} \tag{4.16}
$$

thanks to (4.3) (applied here in one dimension less).

Moreover, the number of different indices k, $m \in \{1, \dots, n-1\}$ is equal to $(n-1)(n-2)$ and so, for each $j \neq \ell \in \{1, \dots, n-1\}$,

$$
\begin{aligned}
\varepsilon \int_{\{|y'|<r\}} \frac{y_\ell^2\, y_j^2}{|y'|^{n+3-\varepsilon}}\, dy' &= \varepsilon \int_{\{|y'|<r\}} \frac{y_1^2\, y_2^2}{|y'|^{n+3-\varepsilon}}\, dy' = \frac{\varepsilon}{(n-1)(n-2)} \sum_{k\neq m=1}^{n-1} \int_{\{|y'|<r\}} \frac{y_k^2\, y_m^2}{|y'|^{n+3-\varepsilon}}\, dy' \\
&= \frac{\varepsilon}{(n-1)(n-2)} \left[\sum_{k,m=1}^{n-1} \int_{\{|y'|<r\}} \frac{y_k^2\, y_m^2}{|y'|^{n+3-\varepsilon}}\, dy' - \sum_{k=1}^{n-1} \int_{\{|y'|<r\}} \frac{y_k^4}{|y'|^{n+3-\varepsilon}}\, dy' \right] \\
&= \frac{\varepsilon}{(n-1)(n-2)} \int_{\{|y'|<r\}} \frac{dy'}{|y'|^{n-1-\varepsilon}} - \frac{C_\star\, r^{\varepsilon}}{n-2} = \frac{\varpi\, r^{\varepsilon}}{n-2} - \frac{C_\star\, r^{\varepsilon}}{n-2} = \frac{\varpi\, r^{\varepsilon}}{n+1}.
\end{aligned}
$$

From this and (4.15), we obtain that

$$
\varepsilon \int_{\{|y'|<r\}} \frac{y_\ell^2\, y_j^2}{|y'|^{n+3-\varepsilon}}\, dy' = \frac{(1+2\delta_{\ell j})\, \varpi\, r^{\varepsilon}}{n+1}.
$$

Substituting this identity and (4.12) into (4.10), and recalling also (4.4), we conclude that

$$\begin{aligned}
&-\frac{2}{n+1-\varepsilon}\int_{\partial E\cap B_r}\nu_E(y)\cdot\nabla K_\varepsilon(-y)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\\
&=\varepsilon\sum_{\ell=1}^{n-1}\int_{\{|y'|<r\}}\frac{\partial^2_\ell f(0)\,(\partial^2_j f(0))^2\,y_\ell^2\,y_j^2\,\delta_{ji}}{|y'|^{n+3-\varepsilon}}\,dy'+o(1)\\
&=\frac{\varpi\,r^\varepsilon}{n+1}\sum_{\ell=1}^{n-1}\partial^2_\ell f(0)\,(\partial^2_j f(0))^2\,(1+2\delta_{\ell j})\,\delta_{ji}+o(1)\\
&=\;-\frac{\varpi\,r^\varepsilon\,H_E(0)}{n+1}\,(\partial^2_j f(0))^2\,\delta_{ji}+\frac{2\varpi\,r^\varepsilon}{n+1}\,(\partial^2_j f(0))^3\,\delta_{ji}+o(1)\\
&=\;-\frac{\varpi\,H_E(0)}{n+1}\,(\partial^2_j f(0))^2\,\delta_{ji}+\frac{2\varpi}{n+1}\,(\partial^2_j f(0))^3\,\delta_{ji}+o(1),
\end{aligned}\tag{4.17}$$

as $\varepsilon\searrow 0$. Similarly, substituting (4.12) and (4.13) into (4.11), we obtain that, as $\varepsilon\searrow 0$,

$$\begin{aligned}
&\int_{\partial E\cap B_r}H_E(y)K_\varepsilon(-y)\,\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\\
&=\varepsilon\int_{\{|y'|<r\}}\frac{H_E(0)\,(\partial^2_j f(0))^2\,y_j^2\,\delta_{ji}}{|y'|^{n+1-\varepsilon}}\,dy'+o(1)\\
&=\varpi\,r^\varepsilon\,H_E(0)\,(\partial^2_j f(0))^2\,\delta_{ji}+o(1)\\
&=\varpi\,H_E(0)\,(\partial^2_j f(0))^2\,\delta_{ji}+o(1).
\end{aligned}$$

From this and (4.17) it follows that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_{\partial E\cap B_r}\Big(H_E(y)K_\varepsilon(-y)-\nu_E(y)\cdot\nabla K_\varepsilon(-y)\Big)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\\
&=\frac{\varpi}{2}\,H_E(0)\,(\partial^2_j f(0))^2\,\delta_{ji}+\varpi\,(\partial^2_j f(0))^3\,\delta_{ji}.
\end{aligned}\tag{4.18}$$

Now we exploit (1.10) and we see that

$$\begin{aligned}
&\left|\int_{\partial E\setminus B_r}\Big(H_E(y)K_\varepsilon(-y)-\nu_E(y)\cdot\nabla K_\varepsilon(-y)\Big)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\right|\\
&\leqslant C\varepsilon\int_{\partial E\setminus B_r}\left(\frac{|H_E(y)|}{|y|^{n+1-\varepsilon}}+\frac{1}{|y|^{n+2-\varepsilon}}\right)d\mathcal{H}^{n-1}_y\\
&\leqslant C\varepsilon\int_{\partial E\setminus B_r}\frac{1}{|y|^{n+1-\varepsilon}}\left(|H_E(y)|+r^{-1}\right)d\mathcal{H}^{n-1}_y\\
&\leqslant C\left(1+r^{-1}\right)\varepsilon\int_{\partial E\setminus B_r}\frac{|H_E(y)|+1}{|y|^{n+1-\varepsilon}}\,d\mathcal{H}^{n-1}_y\\
&= C\left(1+r^{-1}\right)\varepsilon\sum_{k=0}^{+\infty}\int_{\partial E\cap(B_{2^{k+1}r}\setminus B_{2^k r})}\frac{|H_E(y)|+1}{|y|^{n+1-\varepsilon}}\,d\mathcal{H}^{n-1}_y\\
&\leqslant \frac{C\left(1+r^{-1}\right)\varepsilon}{r^{n+1-\varepsilon}}\sum_{k=0}^{+\infty}\frac{1}{2^{k(n+1-\varepsilon)}}\int_{\partial E\cap(B_{2^{k+1}r}\setminus B_{2^k r})}\big(|H_E(y)|+1\big)\,d\mathcal{H}^{n-1}_y\\
&\leqslant \frac{C\left(1+r^{-1}\right)\varepsilon}{r^{n+1-\varepsilon}}\sum_{k=0}^{+\infty}\frac{(2^{k+1}r)^\beta}{2^{k(n+1-\varepsilon)}}\\
&\leqslant \frac{2^\beta\,C\left(1+r^{-1}\right)\varepsilon}{r^{n+1-\varepsilon-\beta}}\sum_{k=0}^{+\infty}\frac{1}{2^{k(n+1-\varepsilon-\beta)}}\\
&\leqslant \frac{2^\beta\,C\left(1+r^{-1}\right)\varepsilon}{r^{n+1-\varepsilon-\beta}}\sum_{k=0}^{+\infty}\frac{1}{2^{\frac{k(n+1-\beta)}{2}}}\\
&= o(1),
\end{aligned}\tag{4.19}$$

for small ε, up to renaming C line after line, and consequently

$$\lim_{\varepsilon\searrow0}\int_{\partial E\setminus B_r}\Big(H_E(y)K_\varepsilon(-y)-\nu_E(y)\cdot\nabla K_\varepsilon(-y)\Big)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y=0.$$

This and (4.18) give that

$$\begin{aligned}
&\lim_{\varepsilon\searrow0}\int_{\partial E}\Big(H_E(y)K_\varepsilon(-y)-\nu_E(y)\cdot\nabla K_\varepsilon(-y)\Big)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\\
&=\frac{\varpi}{2}\,H_E(0)\,(\partial^2_j f(0))^2\,\delta_{ji}+\varpi\,(\partial^2_j f(0))^3\,\delta_{ji}.
\end{aligned}\tag{4.20}$$

In addition, from Lemma A.2 of [DdPW18], we have that

$$\lim_{\varepsilon\searrow0}L_{K_\varepsilon,E}=-\frac{\varpi}{2}\,\Delta_{\partial E},\tag{4.21}$$

where the notation in (4.14) has been used. Similarly, from Lemma A.4 of [DdPW18],

$$\lim_{\varepsilon\searrow0}c^2_{K_\varepsilon,E}=\frac{\varpi}{2}\,c^2_E,\tag{4.22}$$

being c_E the norm of the second fundamental form of ∂E.

Therefore, using (4.20), (4.21) and (4.22), we obtain that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\Big[-L_{K_\varepsilon,E}\delta_{E,j}\nu_{E,i}(0)+c^2_{K_\varepsilon,E}(0)\,\delta_{E,j}\nu_{E,i}(0)\\
&\qquad -\int_{\mathbb{R}^n}\Big(H_E(y)K_\varepsilon(-y)-\nu_E(y)\cdot\nabla K_\varepsilon(-y)\Big)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\Big]\\
&=\frac{\varpi}{2}\,\Delta_{\partial E}\delta_{E,j}\nu_{E,i}(0)+\frac{\varpi}{2}\,c^2_E(0)\,\delta_{E,j}\nu_{E,i}(0)-\frac{\varpi}{2}\,H_E(0)\,(\partial^2_j f(0))^2\,\delta_{ji}-\varpi\,(\partial^2_j f(0))^3\,\delta_{ji}.
\end{aligned}\tag{4.23}$$

Now, given two functions ψ, ϕ, we exploit (2.11) twice to obtain that

$$\begin{aligned}
&\int_{\partial E}\delta_{E,i}\delta_{E,j}\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\\
&=-\int_{\partial E}\delta_{E,j}\psi(x)\,\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x+\int_{\partial E}H_E(x)\,\nu_{E,i}(x)\,\delta_{E,j}\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\\
&=\int_{\partial E}\psi(x)\,\delta_{E,j}\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x-\int_{\partial E}H_E(x)\,\nu_{E,j}(x)\,\psi(x)\,\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad+\int_{\partial E}H_E(x)\,\nu_{E,i}(x)\,\delta_{E,j}\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x.
\end{aligned}\tag{4.24}$$

On the other hand, applying (2.11) once again, we see that

$$\begin{aligned}
&\int_{\partial E}H_E(x)\,\nu_{E,i}(x)\,\delta_{E,j}\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\\
=\;&-\int_{\partial E}\delta_{E,j}\big(H_E(x)\,\nu_{E,i}(x)\,\phi(x)\big)\,\psi(x)\,d\mathcal{H}^{n-1}_x+\int_{\partial E}H^2_E(x)\,\nu_{E,i}(x)\,\nu_{E,j}(x)\,\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x.
\end{aligned}$$

Plugging this information into (4.24), we find that

$$\begin{aligned}
&\int_{\partial E}\delta_{E,i}\delta_{E,j}\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\\
&=\int_{\partial E}\psi(x)\,\delta_{E,j}\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x-\int_{\partial E}H_E(x)\,\nu_{E,j}(x)\,\psi(x)\,\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad-\int_{\partial E}\delta_{E,j}\big(H_E(x)\,\nu_{E,i}(x)\,\phi(x)\big)\,\psi(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad+\int_{\partial E}H^2_E(x)\,\nu_{E,i}(x)\,\nu_{E,j}(x)\,\psi(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x.
\end{aligned}\tag{4.25}$$

Applying (4.25) (twice, at the beginning with $\psi:=H_{K_\varepsilon,E}(x)$ and at the end with $\psi:=H_E(x)$) and considering ϕ as a test function, the convergence of $H_{K_\varepsilon,E}$ to $\frac{\varpi H_E}{2}$ (see Theorem 12 in [AV14]) gives that

$$
\begin{aligned}
&-\lim_{\varepsilon\searrow 0}\int_{\partial E}\delta_{E,i}\delta_{E,j}H_{K_\varepsilon,E}(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\\
=\;&\lim_{\varepsilon\searrow 0}\Bigg[-\int_{\partial E}H_{K_\varepsilon,E}(x)\,\delta_{E,j}\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x+\int_{\partial E}H_E(x)\,\nu_{E,j}(x)\,H_{K_\varepsilon,E}(x)\,\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad+\int_{\partial E}\delta_{E,j}\big(H_E(x)\,\nu_{E,i}(x)\,\phi(x)\big)\,H_{K_\varepsilon,E}(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad-\int_{\partial E}H^2_E(x)\,\nu_{E,i}(x)\,\nu_{E,j}(x)\,H_{K_\varepsilon,E}(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\Bigg]\\
=\;&\frac{\varpi}{2}\Bigg[-\int_{\partial E}H_E(x)\,\delta_{E,j}\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x+\int_{\partial E}H_E(x)\,\nu_{E,j}(x)\,H_E(x)\,\delta_{E,i}\phi(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad+\int_{\partial E}\delta_{E,j}\big(H_E(x)\,\nu_{E,i}(x)\,\phi(x)\big)\,H_E(x)\,d\mathcal{H}^{n-1}_x\\
&\qquad-\int_{\partial E}H^2_E(x)\,\nu_{E,i}(x)\,\nu_{E,j}(x)\,H_E(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x\Bigg]\\
=\;&-\frac{\varpi}{2}\int_{\partial E}\delta_{E,i}\delta_{E,j}H_E(x)\,\phi(x)\,d\mathcal{H}^{n-1}_x.
\end{aligned}
$$

This says that $\delta_{E,i}\delta_{E,j}H_{K_\varepsilon,E}$ converges to $\frac{\varpi}{2}\delta_{E,i}\delta_{E,j}H_E$ in the distributional sense as $\varepsilon\searrow 0$: since, by the Ascoli-Arzelà Theorem, we know that $\delta_{E,i}\delta_{E,j}H_{K_\varepsilon,E}$ converges strongly up to a subsequence, the uniqueness of the limit gives that $\delta_{E,i}\delta_{E,j}H_{K_\varepsilon,E}$ converges also pointwise to $\frac{\varpi}{2}\delta_{E,i}\delta_{E,j}H_E$.

Combining this with (4.23), we obtain that

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\Bigg[\delta_{E,i}\delta_{E,j}H_{K_\varepsilon,E}(0)+L_{K_\varepsilon,E}\delta_{E,j}\nu_{E,i}(0)-c^2_{K_\varepsilon,E}(0)\,\delta_{E,j}\nu_{E,i}(0)\\
&\qquad+\int_{\mathbb{R}^n}\Big(H_E(y)K_\varepsilon(-y)-\nu_E(y)\cdot\nabla K_\varepsilon(-y)\Big)\nu_{E,i}(y)\,\nu_{E,j}(y)\,d\mathcal{H}^{n-1}_y\Bigg]\\
&=\frac{\varpi}{2}\delta_{E,i}\delta_{E,j}H_E(0)-\frac{\varpi}{2}\,\Delta_{\partial E}\delta_{E,j}\nu_{E,i}(0)-\frac{\varpi}{2}\,c^2_E(0)\,\delta_{E,j}\nu_{E,i}(0)\\
&\qquad+\frac{\varpi}{2}\,H_E(0)\,(\partial^2_j f(0))^2\,\delta_{ji}+\varpi\,(\partial^2_j f(0))^3\,\delta_{ji}.
\end{aligned}
\tag{4.26}
$$

By formula (1.9), we know that the left hand side of (4.26) is equal to zero. Therefore, if H_E also vanishes identically, we obtain that

$$\Delta_{\partial E}h_{ij}(0)+c^2_E(0)\,h_{ij}(0)+2\,h^3_{jj}(0)\,\delta_{ji}=0.$$

Recall that h_{ij} are the entries of the second fundamental form. Multiplying by h_{ij} and summing up over i, $j\in\{1,\dots,n-1\}$, we infer that

$$\sum_{i,j=1}^{n-1}h_{ij}(0)\Delta_{\partial E}h_{ij}(0)+c^2_E(0)\sum_{i,j=1}^{n-1}h^2_{ij}(0)+2\sum_{j=1}^{n-1}h^4_{jj}(0)=0. \tag{4.27}$$

Also, by (4.6), we have that $h_{in}(0)=\delta_{E,i}\nu_{E,n}(0)=0$ for all $i\in\{1,\dots,n-1\}$, and also $h_{nn}(0)=\delta_{E,n}\nu_{E,n}(0)=0$ by (1.6), and so (4.27) becomes

$$\sum_{i,j=1}^{n-1}h_{ij}(0)\Delta_{\partial E}h_{ij}(0)+c^4_E(0)+2\sum_{j=1}^{n-1}h^4_{jj}(0)=0. \tag{4.28}$$

On the other hand,

$$\Delta_{\partial E} h_{ij}^2(0) = 2h_{ij}(0)\Delta_{\partial E} h_{ij}(0) + 2\sum_{k=1}^{n-1} |\delta_{E,k} h_{ij}(0)|^2.$$

Therefore, (4.28) becomes

$$\frac{1}{2}\sum_{i,j=1}^{n-1} \Delta_{\partial E} h_{ij}^2(0) = \sum_{i,j,k=1}^{n-1} |\delta_{E,k} h_{ij}(0)|^2 - c_E^4(0) - 2\sum_{j=1}^{n-1} h_{jj}^4(0). \tag{4.29}$$

We observe now that, in light of (4.6),

$$\nu_{E,n}(y) = 1 - \frac{1}{2}\sum_{j=1}^{n-1} \left(\partial_j^2 f(0)\right)^2 y_j^2 + O(|y'|^3) \tag{4.30}$$

and so, by (1.6),

$$h_{nn}(y) = \delta_{E,n}\nu_{E,n}(y) = \partial_n \nu_{E,n}(y) - \nu_{E,n}(y)\nabla\nu_{E,n}(y)\cdot\nu_E(y) = -\sum_{j=1}^{n-1} \left(\partial_j^2 f(0)\right)^3 y_j^2 + O(|y'|^3).$$

This gives that $h_{nn}^2(y) = O(|y|^4)$ and therefore

$$\Delta_{\partial E} h_{nn}^2(0) = 0. \tag{4.31}$$

Furthermore, by (1.6) and (4.30), for any $i \in \{1, \ldots, n-1\}$,

$$h_{in}(y) = \delta_{E,i}\nu_{E,n}(y) = \partial_i \nu_{E,n}(y) - \nu_{E,i}(y)\nabla\nu_{E,n}(y)\cdot\nu_E(y) = -\left(\partial_i^2 f(0)\right)^2 y_i + O(|y|^2),$$

which gives that

$$h_{in}^2(y) = \left(\partial_i^2 f(0)\right)^4 y_i^2 + O(|y|^3).$$

As a consequence,

$$\Delta_{\partial E} h_{in}^2(0) = 2\left(\partial_i^2 f(0)\right)^4.$$

This and (4.31) give that

$$\Delta_{\partial E} c_E^2(0) = \sum_{i,j=1}^{n-1} \Delta_{\partial E} h_{ij}^2(0) + 2\sum_{i=1}^{n-1} \Delta_{\partial E} h_{in}^2(0) + \Delta_{\partial E} h_{nn}^2(0) = \sum_{i,j=1}^{n-1} \Delta_{\partial E} h_{ij}^2(0) + 4\sum_{i=1}^{n-1} \left(\partial_i^2 f(0)\right)^4.$$

Plugging this information into (4.29) we conclude that

$$\frac{1}{2}\Delta_{\partial E} c_E^2(0) = \frac{1}{2}\sum_{i,j=1}^{n-1} \Delta_{\partial E} h_{ij}^2(0) + 2\sum_{i=1}^{n-1} \left(\partial_i^2 f(0)\right)^4 = \sum_{i,j,k=1}^{n-1} |\delta_{E,k} h_{ij}(0)|^2 - c_E^4(0),$$

which is (1.1). □

5. Proof of Theorem 1.4

Let $\eta \in C_0^\infty(\partial E)$ be arbitrary. Using $f := c_K\eta$ as a test function in (1.18), we have that

$$0 \leqslant B_K(c_K\eta, c_K\eta) - \int_{\partial E} c_K^4 \eta^2 \, d\mathcal{H}^n.$$

Next, for all $x, y \in \partial E$, we have that

$$\begin{aligned}(c_K(x)\eta(x) - c_K(y)\eta(y))^2 &= \left(c_K(x)(\eta(x)-\eta(y)) + \eta(y)(c_K(x)-c_K(y))\right)^2 \\ &= c_K^2(x)(\eta(x)-\eta(x))^2 + \eta^2(y)(c_K(x)-c_K(y))^2 \\ &\qquad +2c_K(x)\eta(y)(c_K(x)-c_K(y))(\eta(x)-\eta(y)),\end{aligned}$$

so it follows that

$$B_K(c_K\eta, c_K\eta) = \int_{\partial E} c_K^2(x) B_K(\eta,\eta;x)\,d\mathcal{H}^n_x + \int_{\partial E} \eta^2(x) B_K(c_K,c_K;x)\,d\mathcal{H}^n_x + I,$$

where

$$I := \int_{\partial E}\int_{\partial E} c_K(x)\eta(y)(c_K(x)-c_K(y))(\eta(x)-\eta(y))\,K(x-y)\,d\mathcal{H}^n_y\,d\mathcal{H}^n_x.$$

Next, by symmetry of x and y, we have that

$$I = \frac{1}{2}\int_{\partial E}\int_{\partial E} (c_K(x)\eta(y)+c_K(y)\eta(x))(c_K(x)-c_K(y))(\eta(x)-\eta(y))\,K(x-y)\,d\mathcal{H}^n_y\,d\mathcal{H}^n_x.$$

Moreover, by a simple algebraic manipulation,

$$\begin{aligned}
&(c_K(x)\eta(y)+c_K(y)\eta(x))(c_K(x)-c_K(y))(\eta(x)-\eta(y))\\
&\quad = \frac{1}{2}(\eta^2(x)-\eta^2(y))(c_K^2(x)-c_K^2(y)) - \frac{1}{2}(c_K(x)-c_K(y))^2(\eta(x)-\eta(y))^2\\
&\quad \leqslant \frac{1}{2}(\eta^2(x)-\eta^2(y))(c_K^2(x)-c_K^2(y))
\end{aligned}$$

and accordingly

$$\begin{aligned}
I &\leqslant \frac{1}{4}\int_{\partial E}\int_{\partial E} (\eta^2(x)-\eta^2(y))(c_K^2(x)-c_K^2(y))\,K(x-y)\,d\mathcal{H}^n_y\,d\mathcal{H}^n_x\\
&= \frac{1}{2}\int_{\partial E}\int_{\partial E} \eta^2(x)(c_K^2(x)-c_K^2(y))\,K(x-y)\,d\mathcal{H}^n_y\,d\mathcal{H}^n_x\\
&= \frac{1}{2}\int_{\partial E} \eta^2(x)\mathcal{L}_K c_K^2(x)\,d\mathcal{H}^n_x.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&B_K(c_K\eta, c_K\eta)\\
&\quad \leqslant \int_{\partial E} c_K^2(x)B_K(\eta,\eta;x)\,d\mathcal{H}^n_x + \int_{\partial E}\left\{B_K(c_K,c_K;x)+\frac{1}{2}\int_{\partial E}\eta^2(x)\mathcal{L}_K c_K^2(x)\right\}\eta^2(x)\,d\mathcal{H}^n_x
\end{aligned}$$

and the result follows. □

Appendix A. Proof of formulas (4.2) and (4.3)

Let

$$Q := \int_{B_1} x_1^4\,dx \qquad \text{and} \qquad D := \int_{B_1} x_1^2 x_2^2\,dx.$$

We consider the isometry $x \mapsto X \in \mathbb{R}^n$ given by

$$X_1 := \frac{x_1 - x_2}{\sqrt{2}}, \qquad X_2 := \frac{x_1+x_2}{\sqrt{2}}, \qquad X_i := x_i \quad \text{for all } \ i \in \{3,\dots,n\}.$$

We notice that

$$4X_1^2X_2^2 = (2X_1X_2)^2 = \big((x_1-x_2)(x_1+x_2)\big)^2 = (x_1^2-x_2^2)^2 = x_1^4+x_2^4-2x_1^2x_2^2$$

and therefore, by symmetry,

$$4D = \int_{B_1} 4X_1^2X_2^2\,dX = \int_{B_1}\left(x_1^4+x_2^4-2x_1^2x_2^2\right)dx = 2Q-2D,$$

which gives

$$D = \frac{Q}{3}. \tag{A.1}$$

On the other hand

$$|x|^4 = (|x|^2)^2 = \left(\sum_{i=1}^n x_i^2\right)^2 = \sum_{i,j=1}^n x_i^2 x_j^2 = \sum_{i=1}^n x_i^4 + \sum_{i\neq j=1}^n x_i^2 x_j^2.$$

Therefore, by polar coordinates and symmetry,

$$\begin{aligned}\frac{\mathcal{H}^{n-1}(S^{n-1})}{n+4} &= \mathcal{H}^{n-1}(S^{n-1})\int_0^1 \rho^{n+3}\,d\rho = \int_{B_1} |x|^4\,dx \\ &= \sum_{i=1}^n \int_{B_1} x_i^4\,dx + \sum_{i\neq j=1}^n \int_{B_1} x_i^2 x_j^2\,dx = nQ + n(n-1)D.\end{aligned}$$

From this and (A.1) we deduce that

$$\frac{\mathcal{H}^{n-1}(S^{n-1})}{n+4} = \frac{n(n+2)\,Q}{3},$$

hence

$$\frac{3\,\mathcal{H}^{n-1}(S^{n-1})}{n(n+2)(n+4)} = Q = \int_0^1 \int_{S^{n-1}} \rho^{n+3}\vartheta_1^4\,d\mathcal{H}_\vartheta^{n-1}\,d\rho = \frac{1}{n+4}\int_{S^{n-1}} \vartheta_1^4\,d\mathcal{H}_\vartheta^{n-1},$$

which gives (4.2) and (4.3), as desired.

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