ON THE SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NONLINEAR FREDHOLM OPERATORS WITH THE LOGARITHMIC LAPLACIAN

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Abstract: We address the solvability of certain nonlinear nonhomogeneous systems of equations in one dimension containing the logarithmic Laplacian and the drift terms. We establish that, under the reasonable technical conditions, the convergence in $L^1(\mathbb{R})$ of the integral kernels yields the existence and the convergence in $L^2(\mathbb{R}, \mathbb{R}^N)$ of the solutions. We emphasize that the study of the systems is more difficult than of the scalar case and requires to overcome more cumbersome technicalities.

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1. Introduction

Consider the problem

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and $V(x)$ is a function tending to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.1) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. We recall that elliptic equations with non-Fredholm operators, both

linear and nonlinear were studied extensively in recent years (see [16], [17], [18], [19], [30], [31], [32], [33], [34], [35], [36], [37], also [5]) along with their potential applications to the theory of reaction-diffusion problems (see [11], [12]). Fredholm structures, topological invariants and their application were discussed in [13]. The work [14] deals with the finite and infinite dimensional attractors for the evolution equations of mathematical physics. The large time behavior of the solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [15]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in the space of three dimensions was treated in [22]. The articles [23] and [29] are crucial for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were discussed in [24]. The article [34] is devoted to the Laplace operator with drift from the point of view of the non-Fredholm operators. Standing lattice solitons in the discrete NLS equation with saturation were studied in [1]. In the particular case when $a = 0$, the operator A mentioned above satisfies the Fredholm property in certain properly chosen weighted spaces (see [2], [3], [5], [7], [8]). But the situation when $a \neq 0$ is significantly different and the method developed in these articles cannot be applied.
The present article is our attempt to generalize these results by considering the solvability of the nonlinear system of equations containing in the left side the logarithmic Laplacian in one dimension, which can be defined via the spectral calculus along with the drift terms.
The logarithmic Laplacian $\ln(-\Delta)$ is the operator with the Fourier symbol $2\ln|p|$. It appears as formal derivative $\partial_s|_{s=0}(-\Delta)^s$ of fractional Laplacians at $s = 0$. The operator $(-\Delta)^s$ is extensively used, for example in the studies of the anomalous diffusion problems (see e.g. [37] and the references therein). Spectral properties of the logarithmic Laplacian in an open set of finite measure with Dirichlet boundary conditions were discussed in [28] (see also [10]). The studies of $\ln(-\Delta)$ are crucial for the understanding of the asymptotic spectral properties of the family of fractional Laplacians in the limit $s \to 0^+$. In [26] it has been demonstrated that this operator enables to characterize the s-dependence of solution to fractional Poisson equations for the full range of exponents $s \in (0, 1)$. A direct method of moving planes for the logarithmic Schrödinger operator was covered in [38]. The work [39] deals with the symmetry of positive solutions for Lane-Emden systems containing the Logarithmic Laplacian. The solvability of certain linear nonhomogeneous problems involving the logarithm of the sum of the two Schrödinger operators in higher dimensions was addressed in [20].
For the technical purposed we introduce the non self-adjoint operators with $1 \leq$

$k \le N,\ N \ge 2$, namely

$$L_{a,b,k} := \frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big) - b_k\frac{d}{dx} - a_k, \quad a_k, b_k \in \mathbb{R}, \quad b_k \neq 0, \quad x \in \mathbb{R}. \tag{1.2}$$

They are considered on $L^2(\mathbb{R})$. By means of the standard Fourier transform, it can be easily obtained that the essential spectra of (1.2) are given by

$$\lambda_{a,b,k}(p) = \ln\Big(\frac{|p|}{e^{a_k}}\Big) - ib_k p, \quad a_k, b_k \in \mathbb{R}, \quad b_k \neq 0, \tag{1.3}$$

where $1 \le k \le N,\ N \ge 2$. Clearly, the estimate from below

$$|\lambda_{a,b,k}(p)| = \sqrt{\ln^2\Big(\frac{|p|}{e^{a_k}}\Big) + b_k^2 p^2} \ge C_{a,b} > 0, \quad p \in \mathbb{R} \tag{1.4}$$

is valid for $1 \le k \le N,\ N \ge 2$. Here $C_{a,b}$ is a constant. Hence, these operators (1.2) satisfy the Fredholm property. Let us recall the earlier article [19] dealing with the solvability of the linear nonhomogeneous problem involving the logarithmic Laplacian without the drift term. Thus, the operator contained in the left side of such equation was non-Fredholm.
Throughout the article we use the hat symbol to designate the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ipx}dx, \quad p \in \mathbb{R}. \tag{1.5}$$

Evidently, the upper bound

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\|f(x)\|_{L^1(\mathbb{R})} \tag{1.6}$$

holds. Let us use the norm for a vector function

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^{N}\|u_k\|^2_{L^2(\mathbb{R})}, \tag{1.7}$$

where

$$u(x) := (u_1(x), u_2(x), ..., u_N(x))^T. \tag{1.8}$$

Our work is devoted to the studies of the solvability of the system of nonlinear equations for $1 \le k \le N,\ N \ge 2$

$$\Big[-\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_k + b_k\frac{du_k}{dx} +$$

$$+a_k u_k + \int_{-\infty}^{\infty} G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \quad x \in \mathbb{R}, \tag{1.9}$$

where the constants $a_k, b_k \in \mathbb{R},\ b_k \neq 0$. The existence of solutions of the single equation analogical to (1.9) was discussed in [21]. In the problems of the Population Dynamics the integro-differential equations are actively used to describe the biological systems with the nonlocal consumption of resources and the intra-specific competition (see e.g. [4], [6], [25]). Solvability of the system analogous to (1.9) but with a standard Laplacian in the diffusion term was covered in [17]. Similarly to [17], we impose the following regularity conditions on the nonlinear part of problem (1.9).

Assumption 1.1. *Let* $1 \le k \le N,\ N \ge 2$. *Functions* $F_k(u, x) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *are satisfying the Caratheodory condition (see [27]), so that*

$$\sqrt{\sum_{k=1}^{N} F_k^2(u,x)} \le K|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \quad x \in \mathbb{R} \tag{1.10}$$

with a constant $K > 0$ *and* $h(x) : \mathbb{R} \to \mathbb{R}^+,\ h(x) \in L^2(\mathbb{R})$. *Furthermore, they are Lipschitz continuous functions, so that for any* $u^{(1),(2)} \in \mathbb{R}^N,\ x \in \mathbb{R}$:

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)}, x) - F_k(u^{(2)}, x))^2} \le L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N} \tag{1.11}$$

with a constant $L > 0$.

Here and below the norm of a vector function given by (1.8) is:

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

The solvability of a local elliptic problem in a bounded domain in $\mathbb{R}^N$ was discussed in [9]. The nonlinear function there was allowed to have a sublinear growth. In order to establish the existence of solutions of problem (1.9), we will use the auxiliary system of equations with $1 \le k \le N,\ N \ge 2$

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_k - b_k\frac{du_k}{dx} -$$

$$-a_k u_k = \int_{-\infty}^{\infty} G_k(x-y)F_k(v_1(y), v_2(y), ..., v_N(y), y)dy, \quad x \in \mathbb{R}, \tag{1.12}$$

where $a_k, b_k \in \mathbb{R},\ b_k \neq 0$ are the constants. Let us demonstrate that under the reasonable technical conditions system (1.12) defines a map $T_{a,b} : L^2(\mathbb{R}, \mathbb{R}^N) \to L^2(\mathbb{R}, \mathbb{R}^N)$, which is a strict contraction.

Theorem 1.2. *Let* $1 \leq k \leq N,\ N \geq 2$, *Assumption 1.1. holds, the functions* $G_k(x) : \mathbb{R} \to \mathbb{R}$, *so that* $G_k(x) \in L^1(\mathbb{R})$ *and* $\dfrac{GL}{C_{a,b}} < 1$.
Then the map $T_{a,b}v = u$ *on* $L^2(\mathbb{R}, \mathbb{R}^N)$ *defined by system (1.12) has a unique fixed point* $v^{(a,b)}$, *which is the only solution of of problem (1.9) in* $L^2(\mathbb{R}, \mathbb{R}^N)$.
This fixed point $v^{(a,b)}$ *does not vanish identically in* $\mathbb{R}$ *provided that for a certain* $1 \leq k \leq N$ *the intersection of supports of the Fourier images of functions* *supp*$\widehat{F_k(0,x)}(p) \cap$ *supp*$\widehat{G_k}(p)$ *is a set of nonzero Lebesgue measure on the real line.*

Here and further down we will use the auxiliary, positive quantity

$$G := \max_{1 \leq k \leq N} \|G_k(x)\|_{L^1(\mathbb{R})}. \tag{1.13}$$

Related to problem (1.9), we study the sequence of the approximate systems of equations for $m \in \mathbb{N},\ 1 \leq k \leq N,\ N \geq 2$, namely

$$\Big[-\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_k^{(m)} + b_k\frac{du_k^{(m)}}{dx} + a_k u_k^{(m)} +$$

$$+\int_{-\infty}^{\infty} G_{k,m}(x-y)F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y)dy = 0, \quad x \in \mathbb{R} \tag{1.14}$$

with the constants $a_k, b_k \in \mathbb{R},\ b_k \neq 0$. Each sequence of kernels $\{G_{k,m}(x)\}_{m=1}^{\infty}$ tends to $G_k(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$. We demonstrate that, under the certain technical assumptions, each system (1.14) admits a unique solution $u^{(m)}(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$, limiting system of equations (1.9) has a unique solution $u(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$, and $u^{(m)}(x) \to u(x)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences*. The significance of Theorem 1.3 below is the continuous dependence of the solution with respect to the integral kernels. The solvability in the sense of sequences for the equations containing the Schrödinger type non-Fredholm operators was addressed in [16], [32], [36], [37]. Similarly to (1.13), we introduce the positive technical quantities

$$G^{(m)} := \max_{1 \leq k \leq N} \|G_{k,m}(x)\|_{L^1(\mathbb{R})}, \quad m \in \mathbb{N}. \tag{1.15}$$

Theorem 1.3. *Let* $m \in \mathbb{N},\ 1 \leq k \leq N,\ N \geq 2$, *Assumption 1.1 holds, the functions* $G_{k,m}(x) : \mathbb{R} \to \mathbb{R}$ *are such that* $G_{k,m}(x) \in L^1(\mathbb{R})$ *and* $G_{k,m}(x) \to G_k(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$. *Furthermore, we assume that*

$$\frac{G^{(m)}L}{C_{a,b}} \leq \frac{1-\varepsilon}{\sqrt{2}} \tag{1.16}$$

holds for each $m \in \mathbb{N}$ *with a certain fixed* $0 < \varepsilon < 1$.
Then each system of equations (1.14) has a unique solution $u^{(m)}(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$, *limiting system (1.9) admits a unique solution* $u(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$, *and* $u^{(m)}(x) \to u(x)$ *in* $L^2(\mathbb{R}, \mathbb{R}^N)$ *as* $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system (1.14) is nontrivial provided that for some $1 \leq k \leq N$ the intersection of supports of the Fourier images of functions $supp\widehat{F_k(0,x)}(p) \cap supp\widehat{G_{k,m}}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. Analogously, the unique solution $u(x)$ of limiting system (1.9) does not vanish identically in $\mathbb{R}$ if $supp\widehat{F_k(0,x)}(p) \cap supp\widehat{G_k}(p)$ is a set of nonzero Lebesgue measure on the real line for a certain $1 \leq k \leq N$.

2. Proofs of the main results

Proof of Theorem 1.2. Let us first suppose that for a certain $v(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ of system (1.12). Clearly, the vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ satisfies the homogeneous system of equations

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]w_k - b_k\frac{dw_k}{dx} - a_k w_k = 0, \quad 1 \leq k \leq N, \quad N \geq 2.$$

The operators $L_{a,b,k}$ on $L^2(\mathbb{R})$ are defined in (1.2). Obviously, they do not have any nontrivial zero modes, just the essential spectra (1.3). Therefore, the vector function $w(x)$ is trivial on $\mathbb{R}$.

We choose arbitrarily $v(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ and apply the standard Fourier transform (1.5) to both sides of system (1.12). This yields

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{\varphi_k}(p)}{\ln\Big(\frac{|p|}{e^{a_k}}\Big) - ib_k p}, \quad p \in \mathbb{R}, \quad 1 \leq k \leq N, \quad N \geq 2. \tag{2.1}$$

Here $\widehat{\varphi_k}(p)$ denotes the Fourier image of $F_k(v(x), x)$. Let us recall inequalities (1.4) and (1.6). Hence,

$$|\widehat{u_k}(p)| \leq \frac{\|G_k(x)\|_{L^1(\mathbb{R})}|\widehat{\varphi_k}(p)|}{C_{a,b}}, \quad 1 \leq k \leq N, \quad N \geq 2,$$

so that

$$\|u_k(x)\|_{L^2(\mathbb{R})} \leq \frac{\|G_k(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\|F_k(v(x), x)\|_{L^2(\mathbb{R})}, \quad 1 \leq k \leq N, \quad N \geq 2. \tag{2.2}$$

We use bound (1.10) of Assumption 1.1. Thus, all $F_k(v(x), x) \in L^2(\mathbb{R})$ for $v(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$. Therefore, for an arbitrary $v(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ there exists a unique solution $u(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ of the system of equations (1.12), so that its Fourier image is given by (2.1). This means that the map $T_{a,b} : L^2(\mathbb{R}, \mathbb{R}^N) \to L^2(\mathbb{R}, \mathbb{R}^N)$ is well defined.

This enables us to choose arbitrarily the vector functions $v^{(1),(2)}(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$, so that their images $u^{(1),(2)} := T_{a,b}v^{(1),(2)} \in L^2(\mathbb{R}, \mathbb{R}^N)$. Evidently, (1.12) gives us

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_k^{(1)} - b_k\frac{du_k^{(1)}}{dx} -$$

$$-a_k u_k^{(1)} = \int_{-\infty}^{\infty} G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y)dy, \quad x \in \mathbb{R}, \tag{2.3}$$

$$\Big[\frac{1}{2}\ln\Big(-\frac{d^2}{dx^2}\Big)\Big]u_k^{(2)} - b_k\frac{du_k^{(2)}}{dx} -$$

$$-a_k u_k^{(2)} = \int_{-\infty}^{\infty} G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y)dy, \quad x \in \mathbb{R}, \tag{2.4}$$

where $1 \leq k \leq N,\ N \geq 2$ and $a_k, b_k \in \mathbb{R},\ b_k \neq 0$ are the constants. Let us apply the standard Fourier transform (1.5) to both sides of the equations of systems (2.3), (2.4). We obtain for $1 \leq k \leq N,\ N \geq 2$

$$\widehat{u_k^{(1)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{\varphi_k^{(1)}}(p)}{\ln\Big(\frac{|p|}{e^{a_k}}\Big) - ib_k p}, \quad \widehat{u_k^{(2)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{\varphi_k^{(2)}}(p)}{\ln\Big(\frac{|p|}{e^{a_k}}\Big) - ib_k p}, \quad p \in \mathbb{R}. \tag{2.5}$$

Here $\widehat{\varphi_k^{(1),(2)}}(p)$ are the Fourier images of $F_k(v^{(1),(2)}(x), x)$. By virtue of (2.5) along with (1.6) and (1.4), we arrive at the estimate from above

$$\Big|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\Big| \leq \frac{\|G_k(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\Big|\widehat{\varphi_k^{(1)}}(p) - \widehat{\varphi_k^{(2)}}(p)\Big|, \quad 1 \leq k \leq N, \quad N \geq 2.$$

Hence, for $1 \leq k \leq N,\ N \geq 2$

$$\|u_k^{(1)}(x) - u_k^{(2)}(x)\|_{L^2(\mathbb{R})} \leq$$

$$\leq \frac{\|G_k(x)\|_{L^1(\mathbb{R})}}{C_{a,b}}\|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|_{L^2(\mathbb{R})}. \tag{2.6}$$

By means of (1.7) along with (1.13) and (2.6),

$$\|u^{(1)}(x) - u^{(2)}(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \frac{G^2}{C_{a,b}^2}\sum_{k=1}^N \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R})}.$$

Let us recall condition (1.11) of Assumption 1.1 above. Thus,

$$\sum_{k=1}^N \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R})} \leq L^2\|v^{(1)}(x) - v^{(2)}(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)},$$

so that

$$\|T_{a,b}v^{(1)}(x) - T_{a,b}v^{(2)}(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \frac{GL}{C_{a,b}} \|v^{(1)}(x) - v^{(2)}(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}. \qquad (2.7)$$

The constant in the right side of (2.7) is less than one as assumed. By virtue of the Fixed Point Theorem, there exists a unique vector function $v^{(a,b)} \in L^2(\mathbb{R}, \mathbb{R}^N)$, such that $T_{a,b}v^{(a,b)} = v^{(a,b)}$. This is the only solution of system (1.9) in $L^2(\mathbb{R}, \mathbb{R}^N)$. Let us suppose that $v^{(a,b)}$ is trivial on the real line. This will contradict to the stated assumption that for some $1 \leq k \leq N$ the intersection of the supports of the Fourier transforms supp$\widehat{F_k(0,x)}(p) \cap$ supp$\widehat{G_k}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. ∎

We conclude the article by discussing the issue of the solvability in the sense of sequences for our nonlinear system of equations.

Proof of Theorem 1.3. By means of the result of Theorem 1.2 above, under the given conditions, each system of equations (1.14) has a unique solution $u^{(m)}(x) \in L^2(\mathbb{R}, \mathbb{R}^N),\ m \in \mathbb{N}$.
Clearly, $\|G_{k,m}(x)\|_{L^1(\mathbb{R})} \to \|G_k(x)\|_{L^1(\mathbb{R})}$ as $m \to \infty$ due to the standard triangle inequality with $1 \leq k \leq N,\ N \geq 2$. Let us use (1.13), (1.15) and (1.16) to obtain that

$$\frac{GL}{C_{a,b}} \leq 1 - \varepsilon$$

via the simple limiting argument. By virtue of Theorem 1.2, limiting system (1.9) admits a unique solution $u(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ as well.
We apply the standard Fourier transform (1.5) to both sides of the systems of equations (1.9) and (1.14). Hence, for $1 \leq k \leq N,\ N \geq 2,\ m \in \mathbb{N}$

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k}(p)}{\ln\left(\frac{|p|}{e^{a_k}}\right) - ib_kp}, \quad \widehat{u_k^{(m)}}(p) = \sqrt{2\pi}\frac{\widehat{G_{k,m}}(p)\widehat{f_k^{(m)}}(p)}{\ln\left(\frac{|p|}{e^{a_k}}\right) - ib_kp}, \quad p \in \mathbb{R}. \quad (2.8)$$

Here $\widehat{f_k}(p)$ and $\widehat{f_k^{(m)}}(p)$ are the Fourier images of $F_k(u(x),x)$ and $F_k(u^{(m)}(x),x)$ respectively. Using (2.8), we easily derive

$$\left|\widehat{u_k^{(m)}}(p) - \widehat{u_k}(p)\right| \leq$$

$$\leq \sqrt{2\pi}\frac{|\widehat{G_{k,m}}(p)|}{\sqrt{\ln^2\left(\frac{|p|}{e^{a_k}}\right) + b_k^2p^2}}|\widehat{f_k^{(m)}}(p) - \widehat{f_k}(p)| + \sqrt{2\pi}\frac{|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)|}{\sqrt{\ln^2\left(\frac{|p|}{e^{a_k}}\right) + b_k^2p^2}}|\widehat{f_k}(p)|.$$

By means of inequalities (1.4) and (1.6), we arrive at

$$\left|\widehat{u_k^{(m)}}(p) - \widehat{u_k}(p)\right| \leq$$

$$\leq \frac{\|G_{k,m}(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} |\widehat{f_k^{(m)}}(p) - \widehat{f}_k(p)| + \frac{\|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} |\widehat{f}_k(p)|,$$

such that

$$\|u_k^{(m)}(x) - u_k(x)\|_{L^2(\mathbb{R})} \leq \frac{\|G_{k,m}(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(\mathbb{R})} +$$

$$+ \frac{\|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})}}{C_{a,b}} \|F_k(u(x), x)\|_{L^2(\mathbb{R})}.$$

Obviously,

$$\|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R}, \mathbb{R}^N)} \leq$$

$$\leq \sum_{k=1}^{N} \frac{2\|G_{k,m}(x)\|^2_{L^1(\mathbb{R})}}{C^2_{a,b}} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|^2_{L^2(\mathbb{R})} +$$

$$+ \sum_{k=1}^{N} \frac{2\|G_{k,m}(x) - G_k(x)\|^2_{L^1(\mathbb{R})}}{C^2_{a,b}} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})}.$$

Let us recall bound (1.11) of Assumption 1.1. above. Thus,

$$\sum_{k=1}^{N} \|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|^2_{L^2(\mathbb{R})} \leq L^2 \|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R}, \mathbb{R}^N)}. \quad (2.9)$$

This allows us to obtain the estimate from above

$$\left[1 - \frac{2[G^{(m)}]^2 L^2}{C^2_{a,b}}\right] \|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R}, \mathbb{R}^N)} \leq$$

$$\leq \frac{2}{C^2_{a,b}} \sum_{k=1}^{N} \|G_{k,m}(x) - G_k(x)\|^2_{L^1(\mathbb{R})} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})}.$$

By virtue of (1.16), we have

$$\|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R}, \mathbb{R}^N)} \leq$$

$$\leq \frac{2}{\varepsilon(2 - \varepsilon) C^2_{a,b}} \sum_{k=1}^{N} \|G_{k,m}(x) - G_k(x)\|^2_{L^1(\mathbb{R})} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})}.$$

We use inequality (1.10) of Assumption 1.1. Hence, all $F_k(u(x), x)$ belong to $L^2(\mathbb{R})$ for $u(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$. Therefore, under the stated assumptions

$$u^{(m)}(x) \to u(x), \quad m \to \infty \quad (2.10)$$

in $L^2(\mathbb{R}, \mathbb{R}^N)$. Let us suppose that $u^{(m)}(x)$ is trivial in $\mathbb{R}$ for a certain $m \in \mathbb{N}$. This will contradict to the given condition that for some $1 \leq k \leq N$ the intersection of the supports of the Fourier images supp$\widehat{F_k(0,x)}(p) \cap$ supp$\widehat{G_{k,m}}(p)$ is a set of nonzero Lebesgue measure on the real line. The analogous argument holds for the solution $u(x)$ of the limiting system of equations (1.9). ■

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