Attracting invariant tori and analytic conjugacies

Hans Koch [1]

Abstract. We consider the problem of conjugating a flow on $\mathbb{T}^d\times\mathbb{R}$ to an "integrable" flow $\dot q=\omega$ and $\dot p=cp$, if possible, with $c\neq 0$. Our emphasis is on a constructive approach, using a KAM type iteration and analyticity. An application to the periodically forced Van der Pol oscillator is given.

1. Introduction and main results

Denote by $\mathbb{T}$ be the circle $\mathbb{R}/(2\pi\mathbb{Z})$. The flows considered here are of the type

$$\dot q = Q(q,p)\,,\quad \dot p = P(q,p)\,,\qquad q\in\mathbb{T}^d\,,\quad p\in\mathbb{R}^n\,, \tag{1.1}$$

with $d\geq 2$ and $n\geq 1$. Here $\dot q=\frac{dq}{dt}$ and $\dot p=\frac{dp}{dt}$ are derivatives with respect to time t. The vector field $X=(Q,P)$ is defined on some bounded domain $\mathcal{D}_X$ in $\mathbb{T}^d\times\mathbb{R}^n$ and takes values in $\mathbb{R}^{d+n}$. Our focus is on the existence of invariant d-tori that are either attracting or repelling. For simplicity we restrict to the case $n=1$. Possible extensions to $n>1$ will be discussed after Theorem 1.1.

The simplest vector fields that have invariant d-tori are given by

$$K_c^\omega(q,p)=(\omega,cp)\,,\qquad \omega\in\mathbb{R}^d\,,\quad c\in\mathbb{R}\,. \tag{1.2}$$

We will refer to such a vector field K_c^ω as being integrable, assuming $c\neq 0$ unless specified otherwise. The equation (1.1) in this case is simply $\dot q=\omega$ and $\dot p=cp$. The time-t map for $K=K_c^\omega$ is given by $\Phi_K^t(q,p)=\bigl(q+t\omega,e^{tc}p\bigr)$. So the flow Φ_K has $\mathbb{T}^d\times\{0\}$ as an invariant torus, which is attracting for $c<0$ and repelling for $c>0$.

In this paper, we consider analytic perturbations of such flows. In a weak sense mentioned below, if the perturbation is sufficiently small, then the resulting vector field X is conjugate to an integrable vector field. To be more precise, X is said to be conjugate to Y if there exists a diffeomorphism $V:\mathcal{D}_Y\to\mathcal{D}_X$, such that

$$Y=V^*X\,,\qquad V^*X\stackrel{\text{def}}{=}(DV)^{-1}X\circ V\,. \tag{1.3}$$

Formally, this is equivalent to a conjugacy $\Phi_Y^t=V^{-1}\circ\Phi_X^t\circ V$ for the associated flows.

Our focus is on flows and conjugacies that are real-analytic, and on perturbations that are small but not necessarily tiny. When investigation a specific flow $\dot x=X(x)$ that appears to have an attracting invariant d-torus, a natural first step is to perform a change of variables V that makes V^*X close to integrable. A method that is well suited for such a task is computer-assisted analysis, since analyticity allows for good approximations. After getting V^*X close to integrable, the goal would be to continue with a KAM type iteration. In this paper, we introduce such a procedure, which is constructive and can give usable bounds. A KAM theorem whose scope includes analytic near-integrable flows is stated in [20], but the methods give no information about the size of the allowed perturbation.

[1] Department of Mathematics, The University of Texas at Austin, Austin, TX 78712.

Applications of KAM methods to other dissipative systems can be found in [15,18,21,22,24] and references therein.

Attracting or repelling invariant tori are special cases of hyperbolic invariant manifolds. For known results in a C^r setting, see [4,7,9,19] and references therein. In the case of compact invariant manifolds, persistence under small perturbations is equivalent to normal hyperbolicity [5]. This includes the tori considered here. Conjugacy to the unperturbed vector field is in general only Hölder continuous. But under strong nonresonance conditions, one can guarantee C^{r-1} conjugacy [19]. Still, the size of the allowed perturbation is usually unknown and can tend to zero in the limit of large r.

For analytic perturbations of K_c^ω with $c \neq 0$, the smoothness of perturbed tori and conjugacies depends mainly on the frequency vector of the flow. The frequency vector for an invariant d-torus is a vector $\omega \in \mathbb{R}^d$, whose component ω_k describes the average amount of rotation per unit time around the k-th fundamental cycle on the torus. A precise definition will not be needed here. But for a flow that is conjugate to $(\dot{q}, \dot{p}) = (\omega, cp)$, the frequency vector is ω. (And the nonzero Lyapunov exponent is c.)

We restrict to frequency vectors ω that satisfy a Diophantine condition

$$|\omega \cdot \nu| \geq \zeta_0 |\nu|^{1-d-\chi}\,, \qquad \nu \in \mathbb{Z}^d \setminus \{0\}\,, \tag{1.4}$$

with χ and ζ_0 positive real numbers. Such vectors are common: for any fixed value $\chi > 0$, the set of vectors $\omega \in \mathbb{R}^d$ that violate the condition (1.4) approaches a set of measure zero as $\zeta_0 > 0$ tends to zero [2].

Given a vector field $X = (Q, P)$, we denote by $\alpha(X)$ and $\beta(X)$ the average over $\mathbb{T}^d$ of the function $q \mapsto Q(q, 0)$ and $q \mapsto P(q, 0)$, respectively. In Section 6, we will define a Banach space $\mathcal{X}$ of real-analytic vector fields on a domain $\mathcal{D}(\varrho) = \mathbb{T}^d \times (-\varrho, \varrho)$ that satisfy $\beta(X) = 0$. The subspace of all vectors $H \in \mathcal{X}$ that satisfy $\alpha(H) = 0$ is denoted by $\mathcal{X}^\circ$.

In order to give a concrete theorem, let us fix a vector $\bar{\omega} \in \mathbb{R}^d$ that satisfies a Diophantine condition (1.4). Let $\mathcal{C}$ be some non-empty open interval in $\mathbb{R}$ that is bounded away from the origin. Denote by $\mathcal{K}^{\bar{\omega}}$ the line of all vector fields $K_c^{\bar{\omega}}$ with $c \in \mathcal{C}$.

Theorem 1.1. *Let $0 < \varrho \leq 1$. Then there exists a positive $\varrho' < \varrho$, and a real-analytic codimension d manifold $\mathcal{W} \subset \mathcal{X}$, that includes $\mathcal{K}^{\bar{\omega}}$ and is transversal to $\mathcal{K}^{\bar{\omega}} + (\mathbb{R}^d, 0)$, such that the following holds. Every $X \in \mathcal{W}$ is real-analytically conjugate to some vector field in $\mathcal{K}^{\bar{\omega}}$ on $\mathcal{D}(\varrho')$. So in particular, the flow for $X \in \mathcal{W}$ has a real-analytic invariant torus with frequency vector $\bar{\omega}$.*

To be more specific, there exists a real-analytic function $\Omega : B \to \mathbb{R}^d$, defined on some open domain $B \subset \mathcal{X}^\circ$, such that $\mathcal{W}$ is the graph of the map $H \mapsto H + \big(\bar{\omega} + \Omega(H), 0\big)$.

Estimates will be given later. They are uniform in $\zeta_0, \chi, \varrho > 0$, as long as these parameters are bounded away from 0. The assumption $\varrho \leq 1$ is not really necessary but simplifies part of our proof. We note that the condition $\beta(X) = 0$ on our vector fields represents no true loss of generality. If a vector field close to $\mathcal{K}^{\bar{\omega}}$ has a nonzero average $\beta(X)$, then a suitable translation $\mathcal{T}$ in the variable p yields an average $\beta(\mathcal{T}^* X) = 0$.

As indicated above, our main goal in this paper is to develop a KAM type iteration that is constructive and can be combined with computer-assisted techniques. For Hamiltonian flows, such KAM techniques have been developed in [11,13,17,25]. The approach used here is similar in spirit to the one presented in [8,12,14] for elliptic tori, where a KAM step can be implemented as a transformation on a fixed space of vector fields.

The transformation $\mathcal{R}$ introduced in this paper does not define a dynamical system. But it is simpler and thus easier to control, except for the construction of the stable manifold $\mathcal{W}$. The estimates in Section 3 are non-perturbative and quite accurate. The same holds for much of Section 5. But the choices made in Section 4 are geared toward a purely qualitative result like Theorem 1.1.

Theorem 1.1 extends readily to cases where $n > 1$ in (1.1). The corresponding integrable vector fields are $K_C^\omega(q,p) = (\omega, Cp)$, with C a linear operator on $\mathbb{R}^n$. If we assume that the spectrum of C lies in a half-plane $\operatorname{Im} z < 0$ or $\operatorname{Im} z > 0$, then our proofs go through without essential changes. But for accurate estimates, this is best done in a concrete setting. A natural generalization replaces $\mathbb{R}^n$ by a Banach space and $t \mapsto e^{tC}$ by a suitable semigroup. Result on invariant manifolds for C^1 semiflows, and references, can be found in [9].

To give a simple application of Theorem 1.1, we consider the periodically driven Van der Pol (VdP) oscillator, described by the equation

$$\ddot{y} - \mu\bigl(1 - y^2\bigr)\dot{y} + y - \epsilon\cos(\omega_1\,\boldsymbol{.}) = 0\,, \tag{1.5}$$

for a real-valued function $y = y(t)$. Here $\mu > 0$ is the strength of the nonlinearity, while ω_1 and ϵ are the frequency and coupling constant, respectively, for the periodic driving term. The uncoupled system with $\epsilon = 0$ is the standard VdP oscillator. It was introduced in [1] to model relaxation-oscillations in electrical circuits. Some recent work and references can be found in [10,16,21,22,23].

The equation (1.5) can be written as a flow on $\mathbb{T} \times \mathbb{R}^2$, by setting $y_1 = y$, $y_2 = \dot{y}$, and $y_0(t) = \omega_1 t$, so that

$$\dot{y}_0 = \omega_1\,, \quad \dot{y}_1 = y_2\,, \quad \dot{y}_2 = Y_2(y) \stackrel{\text{def}}{=} \mu\bigl(1 - y_1^2\bigr)y_2 + \epsilon\cos(y_0)\,. \tag{1.6}$$

For $\epsilon = 0$, the VdP flow in the variables (y_1, y_2) has an invariant circle [6] that is attracting for all values $\mu > 0$. The period $T = T(\mu)$ takes the value 2π for $\mu = 0$ and tends to infinity as $\mu \to \infty$. Numerically, it seems to be strictly increasing.

If desired, the VdP vector field can be transformed into the form (1.1). One possibility is to change variables from $(y_1, y_2) \in \mathbb{R}^2$ to $(q_2, p) \in \mathbb{T} \times \mathbb{R}$ via $y_1 = e^p\cos(q_2)$ and $y_2 = -e^p\sin(q_2)$. Setting in addition $q_1 = y_0 = \omega_1 t$ yields the equation

$$\dot{q}_1 = \omega_1\,, \qquad \dot{q}_2 = 1 - Y_2(y)e^{-p}\cos(q_2)\,, \qquad \dot{p} = -Y_2(y)e^{-p}\sin(q_2)\,. \tag{1.7}$$

This form will not be needed here. But it illustrates that the deviation from an integrable flow $(\dot{q}, \dot{p}) = (\omega, cp)$ is not small. Still, for $\epsilon = 0$, one has conjugacy to an integrable flow. This will be described in Section 7. So for $\epsilon \neq 0$ close to zero, Theorem 1.1 applies after a change of variables.

To be more specific, we obtain the following.

Theorem 1.2. *Let $\bar\mu$ be a value of μ where $T'(\mu) \neq 0$. Let $\bar\omega_2 = 2\pi/T(\bar\mu)$ and $\bar\omega_1 \in \mathbb{R}$. Assume that $\bar\omega = (\bar\omega_1, \bar\omega_2)$ satisfies a Diophantine condition (1.4). Then there exists real-analytic functions $\mu = \mu(\epsilon)$ and $c = c(\epsilon)$, defined near $\epsilon = 0$, satisfying $\mu(0) = \bar\mu$ and $c(\epsilon) < 0$, such that the* VdP *vector field (1.6) with $\omega_1 = \bar\omega_1$ and $\mu = \mu(\epsilon)$ is real-analytically conjugate to $K^{\bar\omega}_{c(\epsilon)}$.*

In this application of Theorem 1.1, the manifold $\mathcal{W}$ is of codimension $d = 2$. The vector field (1.6) depends on the three parameters $(\omega_1, \mu, \epsilon)$. The condition $T'(\mu) \neq 0$ guarantees that the three-parameter VdP family intersects $\mathcal{W}$ transversally in a curve. This condition can fail only at a discrete set of values of μ, since T depends real-analytically on μ.

In fact, this is really a codimension 1 problem, since $\bar\omega_1$ is fixed. Our proof extends readily to quasiperiodic forcings $\sum_{i=1}^{n} \epsilon_i \cos(a_i t + b_i)$ with a Diophantine condition on $(a_1, \ldots, a_n, \omega_2)$. In this case, the invariant tori are of dimension $d = 1 + n$, but it is still just a codimension 1 problem.

The existence of invariant tori for the periodically forced VdP flow was proved in [23] for some parameter values that include non-small values of ϵ. Their proof uses a dynamics approach and computer-assisted estimates. A KAM theorem for coupled VdP oscillators with small nonlinearities is proved in [21,22].

The remaining part of this paper is organized as follows. Section 2 gives a rough description of our KAM procedure. The estimates needed are proved in Section 3. The setting allows vector fields to depend on parameters, as described in Section 5. This is used to construct the manifold $\mathcal{W}$. A proof Theorem 1.1 is given in Section 6, based on domain choices discussed in Section 4. Theorem 1.2 is proved in Section 7.

2. Informal description of the KAM procedure

2.1. The transformation $\mathcal{R}$

Consider a vector field X near $\mathcal{K}^{\bar\omega}$ with average $\beta(X) = 0$. The main step in any KAM approach is to perform a change of coordinates V, depending on X, in such a way that V^*X is closer to being integrable than is X. The easiest direction of improvement is along a subspace that we call nonresonant. Using a linear projection $\mathbb{I}^-$ onto this subspace, we determine V is such a way that

$$\mathbb{I}^- Y \simeq 0\,, \qquad Y \stackrel{\text{def}}{=} V^*X\,. \tag{2.1}$$

This condition will be made precise below. The intended effect of the change of variables $X \mapsto V^*X$ is to eliminate the nonresonant part of X, up to a small error. After this step, $\beta(Y)$ is nonzero in general. This is corrected via a translation $\mathcal{T}(q, p+\tau)$, with $\tau = \tau(Y)$ chosen in such a way that $\beta(\mathcal{T}^*Y) = 0$. Now we set

$$\mathcal{R}(X) = \mathcal{T}^*V^*X\,. \tag{2.2}$$

A similar transformation has been used in [14] to construct elliptic invariant tori for near-constant vector fields. There, the re-normalization $Y \mapsto \mathcal{T}^*Y$ includes a change of coordinates on $\mathbb{T}^d$ via a matrix in $\mathrm{SL}(d,\mathbb{Z})$. Solving $\mathbb{I}^- Y = 0$ exactly, for a proper choice of $\mathbb{I}^-$, results in a transformation $\mathcal{R}$ that involves no loss of domain (analyticity). In some special cases, $\mathcal{R}$ is in fact a dynamical system [8,11]. Then the analogue of the manifold $\mathcal{W}$ in Theorem 1.1 is the stable manifold of $\mathcal{R}$ at a fixed point.

The transformation $\mathcal{R}$ considered here does involve a loss of domain (analyticity). Still, we want to iterate $\mathcal{R}$. At the n-th step, we use a version $\mathcal{R}_n$ of $\mathcal{R}$ with a projection $\mathbb{I}_n^-$ that increases with n. This makes the above-mentioned elimination harder for large n, but, as in other quasi-Newton schemes, this is compensated for by the fact that the error (here the non-integrable part of the vector field) tends to zero quickly.

The manifold $\mathcal{W}$ described in Theorem 1.1 can be viewed as the local stable manifold at $\mathcal{K}^{\bar{\omega}}$ for the sequence $n \mapsto \mathcal{R}_n$. For its construction we first extend $\mathcal{R}_n$ to a transformation $\mathfrak{R}_n$ for parameterized vector fields $X = X(z)$. After a suitable re-parameterization S_n that depends on X, we set

$$\mathfrak{R}_n(X)(z) = \mathcal{R}_n(X(S_n(z)))\,. \tag{2.3}$$

For details we refer to Section 5.

2.2. Partial elimination of resonant modes

To make the condition $\mathbb{I}^- Y \simeq 0$ more precise, let us write $X = K_c^\omega + F$ and assume that F is small, say of order ε. We split $F = \mathbb{I}^- F + \mathbb{I}^+ F$, where $\mathbb{I}^+ = \mathbb{I} - \mathbb{I}^-$. By our choice of the projection $\mathbb{I}^-$, the resonant part $\mathbb{I}^+ F$ of F is typically of order ε^2, after a small reduction of the domain. So the goal is to find $V = \mathrm{I} + U$, with U of order ε, in such a way that $Y - K_c^\omega$ is of order ε^2. To this end, we consider a curve

$$\mathcal{Y}(z) = (\mathrm{I} + zDU)^{-1}(K_c^\omega + zF) \circ (\mathrm{I} + zU) \tag{2.4}$$

that interpolates between $\mathcal{Y}(0) = K_c^\omega$ and $\mathcal{Y}(1) = Y$. Then

$$Y = K_c^\omega + \mathcal{Y}'(0) + R\,, \qquad \mathcal{Y}'(0) = F - (DU)K_c^\omega + (DK_c^\omega)U\,, \tag{2.5}$$

where R is the second order Taylor remainder in the expansion of $\mathcal{Y}(z)$ in powers of z, evaluated at $z = 1$. The vague condition $\mathbb{I}^- Y \simeq 0$ is now made precise by requiring that $\mathbb{I}^- \mathcal{Y}'(0) = 0$. Using that K_c^ω is in the null space of $\mathbb{I}^-$, this leads to the equation

$$\mathbb{I}^- F - \big[(DU)K_c + (DK_c)\big]U = 0\,. \tag{2.6}$$

After solving this equation, the expression (2.5) for Y becomes

$$Y = K_c^\omega + \mathbb{I}^+ F + R\,. \tag{2.7}$$

Having eliminated the nonresonant part of X up to order ε, we expect R to be of order ε^2, which would make $Y - K_c^\omega$ of order ε^2 as well.

Remark 1. As mentioned above, it is possible to solve the equation $\mathbb{I}^- Y = 0$ exactly. This is very useful in a non-perturbative setting [11], since only $\mathbb{I}^- F$ needs to be small.

In order to describe the projection $\mathbb{I}^-$, let us write a function g on $\mathbb{T}^d \times (-\rho, \rho)$ as a Fourier-Taylor series

$$g(q,p) = \sum_{(\nu,k)\in I} g_{\nu,k} e^{i\nu\cdot q} p^k\,, \qquad q \in \mathbb{T}^d\,, \quad p \in (-\rho,\rho)\,, \tag{2.8}$$

where $I = \mathbb{Z}^d \times \mathbb{N}$. In Section 3, we will define two index sets $I_0^- \subset I$ and $I_1^- \subset I$. Then $\mathbb{I}_0^- g$ is defined by restricting the sum in (2.8) to indices $(\nu, k) \in I_0^-$. And $\mathbb{I}_1^- g$ is defined analogously. Using the notation $X = (X_0, X_1)$ for a vector field, instead of $X = (Q, P)$, we define $\mathbb{I}^- X = (\mathbb{I}_0^- X_0, \mathbb{I}_1^- X_1)$. Given that $(DK_c^\omega)U = (0, cU_1)$, the equation (2.6) can be written as

$$\mathbb{I}_0^- F_0 = \bigl[\omega \cdot \nabla_q + cp\partial_p\bigr] U_0 \,, \qquad \mathbb{I}_1^- F_1 = \bigl[\omega \cdot \nabla_q + cp\partial_p - c\mathrm{I}\bigr] U_1 \,. \tag{2.9}$$

Notice that each of the modes $(q, p) \mapsto e^{i\nu\cdot q} p^k$ in the expansion (2.8) is an eigenvector of the operators $[\ldots]$ in the equation (2.9). The corresponding eigenvalues are

$$\lambda_{j,\nu,k} = i\omega \cdot \nu + c(k - j) \qquad \text{for} \;\; L_j = \omega \cdot \nabla_q + cp\partial_p - jc\mathrm{I} \,. \tag{2.10}$$

We will choose I_j^- in such a way that these eigenvalues are bounded away from 0 on the nonresonant subspace. Then the solution of (2.9) is given by

$$U_j = \bigl[\omega \cdot \nabla_q + cp\partial_p - cj\mathrm{I}\bigr]^{-1} \mathbb{I}_j^- F_j \,, \qquad j \in \{0, 1\} \,. \tag{2.11}$$

3. Controlling the transformation $\mathcal{R}$

In this section, we define and control the KAM transformation $\mathcal{R}$ described above.

3.1. Diophantine condition

For the the remaining part of this paper we fix three constants $\zeta, \chi > 0$ and $1 < \theta < 2$. The interval $\mathcal{C}$ mentioned before Theorem 1.1 is considered fixed as well. In order to leave some room for estimates, we choose another open interval $\mathcal{C}'$ that includes the closure of $\mathcal{C}$ and is still bounded away from 0. Then the Diophantine condition imposed on $\bar{\omega} \in \mathbb{R}^d$ is that

$$|i\bar{\omega} \cdot \nu + \bar{c}\mu| \geq 2\zeta\bigl(|\nu| + |\mu|\bigr)^{1-d-\chi} \,, \qquad (\nu, \mu) \in \mathbb{Z}^{d+1} \setminus \{(0,0)\} \,, \tag{3.1}$$

holds for all $\bar{c} \in \mathcal{C}'$. Notice that this is a standard Diophantine condition when restricted to $\mu = 0$, and it imposes a lower bound on $|\bar{c}|$ when restricted to $(\nu, \mu) = (0, 1)$. The remaining cases add nothing new to the condition (3.1).

In this section, we also fix an integer $n > 0$. The resulting transformation $\mathcal{R}$ will be denoted by $\mathcal{R}_n$ later. Later on, when referring ****

The vector fields considered are of the form $K_c^\omega + F$. Our process of eliminating nonresonant modes involves the following condition.

Definition 2. Our approximate Diophantine condition for a triple (ω, c, n) is

$$|i\omega \cdot \nu + c\mu| \geq \zeta\bigl(|\nu| + |\mu|\bigr)^{1-d-\chi} \,, \qquad (\nu, \mu) \in \mathbb{Z}^{d+1} \,, \quad 0 < |\nu| + |\mu| < \theta^n \,. \tag{3.2}$$

This condition is assumed to hold in this section. Besides a smallness condition on the function F, this determines the domain of the transformation $\mathcal{R}$. Notice that ω and c need not be real.

A more practical criterion that implies (3.2) is described in the following proposition. For $z \in \mathbb{C}^d$ define $\|z\| = \sup_i |z_i|$.

Proposition 3.1. *Assume that $\bar{c} \in \mathcal{C}'$, so that the Diophantine condition (3.1) holds. Let $m > 0$, and assume that $z \in \mathbb{C}^d$ and $v \in \mathbb{C}$ satisfy the bound*

$$\|z\|\,, |v| \le \zeta\theta^{-(d+\chi)m}\,. \tag{3.3}$$

Then (ω, c, m) with $\omega = \bar{\omega}+z$ and $c = \bar{c}+v$ satisfies the approximate Diophantine condition in the sense of Definition 2.

Proof. Consider $\nu \in \mathbb{Z}^d$ and $\mu \in \mathbb{Z}$ satisfying $0 < |\nu| + |\mu| < \theta^m$. Using the Diophantine condition (3.1), we obtain

$$\begin{aligned} |i\omega\cdot\nu + \mu c| &\ge |i\bar{\omega}\cdot\nu + \mu\bar{c}| - \max\bigl(\|z\|, |v|\bigr)\bigl(|\nu| + |\mu|\bigr) \\ &\ge 2\zeta\bigl(|\nu| + |\mu|\bigr)^{1-d-\chi} - \zeta\theta^{m(1-d-\chi)} \\ &\ge 2\zeta\bigl(|\nu| + |\mu|\bigr)^{1-d-\chi} - \zeta\bigl(|\nu| + |\mu|\bigr)^{1-d-\chi} = \zeta\bigl(|\nu| + |\mu|\bigr)^{1-d-\chi} \end{aligned} \tag{3.4}$$

This shows that (ω, c, m) satisfies an approximate Diophantine condition, as claimed. **QED**

Each of the $d+1$ components of one of our vector fields is a functions g that admits a Fourier-Taylor expansion of the type (2.8). Later on, we will need vector fields that depend on a parameter $z \in \mathbb{C}^d$. Then the Fourier-Taylor coefficients $g_{\nu,k}$ of g are function of z as well. The corresponding function space $\mathcal{B}$ will be specified in Section 5.

In what follows, $\mathcal{B}$ can be any commutative Banach algebra over $\mathbb{C}$ with a unit $\mathbf{1}$. An example would be $\mathcal{B} = \mathbb{C}$. The norm of an element $b \in \mathcal{B}$ will be denoted by $|b|$. We assume that $|\mathbf{1}| = 1$, and that $|ab| \le |a||b|$ whenever $a, b \in \mathcal{B}$. To simplify notation, a scalar multiple $s\mathbf{1}$ of $\mathbf{1}$ will be written as s.

3.2. Spaces and basic estimates

Given $\rho > 0$ define $\mathcal{A}_\rho(\mathcal{B})$ to be the space of all functions (2.8) with coefficients $g_{\nu,k} \in \mathcal{B}$, that have a finite norm

$$\|g\|_\rho = \sum_{(\nu,k)\in I} |g_{\nu,k}| e^{\rho|\nu|}\rho^k\,. \tag{3.5}$$

Here $|\nu|$ denotes the ℓ^1 norm of ν. Notice that every function $g \in \mathcal{A}_\rho(\mathcal{B})$ is analytic in the complex open neighborhood of $\mathbb{T}^d \times \{0\}$ defined by $|\operatorname{Im} q_i| < \rho$ and $|p| < \rho$.

When the choice of $\mathcal{B}$ does not matter, we simply write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(\mathcal{B})$. The norm in $\mathcal{A}_\rho^J$ for a finite set J is defined to be $\|g\|_\rho = \max_{j\in J} \|g(j)\|_\rho$. This applies e.g. to a vector field $F = (F_0, F_1)$ with $F_0 \in \mathcal{A}_\rho^d$ and $F_1 \in \mathcal{A}_\rho$.

We note that $\mathcal{A}_\rho$ is a Banach algebra under pointwise multiplication, in the sense that $\|gh\|_\rho \le \|g\|_\rho \|h\|_\rho$ for any $g, h \in \mathcal{A}_\rho$. Furthermore, $g \in \mathcal{A}_\rho$ is bounded on the above-mentioned complex domain and satisfies $|g(q,p)| \le \|g\|_\rho$. These properties imply the following. Given two vector fields $F = (F_0, F_1)$ and $U = (U_0, U_1)$, consider the composed map

$$G = F \circ (\mathrm{I} + U)\,, \qquad G(q,p) = F\bigl(q + U_0(q,p), p + U_1(q,p)\bigr)\,. \tag{3.6}$$

Proposition 3.2. *Let $0 < \rho' < \rho$. Assume that $U = (U_0, U_1)$ belongs to $\mathcal{A}_{\rho'}^{d+1}$, and that $F \in \mathcal{A}_\rho^{d+1}$. If*

$$\rho' + \|U_j\|_{\rho'} \le \rho\,, \qquad j = 0, 1\,, \tag{3.7}$$

then G belongs to $\mathcal{A}_{\rho'}^{d+1}$, and $\|G_j\|_{\rho'} \le \|F_j\|_\rho$ for $j = 0, 1$.

As mentioned earlier, $\theta \in (1, 2)$ is fixed globally, and $n > 0$ is fixed in this section. To motivate the definition below, we note that only $n \ge n_\circ$ will be needed later, for some large positive integer $n_\circ$. Recall that $I = \mathbb{Z}^d \times \mathbb{N}$.

Definition 3. Let $I_0^- = \bigl\{(\nu, k) \in I : 0 < |\nu| + k < \theta^n\bigr\}$. Then set $I_1^- = I_0^- \setminus \{(0, 1)\}$, and $I_j^+ = I \setminus I_j^-$ for $j \in \{0, 1\}$.

These index sets $I_j^\pm$ define linear projections $\mathbb{I}_j^\pm$ on the spaces $\mathcal{A}_\rho$ via

$$(\mathbb{I}_j^\pm g)(q, p) = \sum_{(\nu,k)\in I_j^\pm} g_{\nu,k} e^{i\nu\cdot q} p^k\,, \qquad j \in \{0, 1\}\,. \tag{3.8}$$

The subspaces $\mathbb{I}_j^- \mathcal{A}_\rho$ and $\mathbb{I}_j^+ \mathcal{A}_\rho$ of $\mathcal{A}_\rho$ will be referred to as the nonresonant and resonant subspaces, respectively. The corresponding projections $\mathbb{I}^\pm$ for vector fields $G = (G_0, G_1)$ are defined by $\mathbb{I}^\pm G = (\mathbb{I}_0^\pm G_0, \mathbb{I}_1^\pm G_1)$.

Recall from (2.2) that we want to define $\mathcal{R}(X) = \mathcal{T}^* Y$, where $Y = V^* X$, with $V = \mathrm{I} + U$ determined by the equation (2.11).

We start by decomposing $X = K_c^\omega + F$ in such a way that F is normalized.

Definition 4. We say that F is normalized if $\alpha(F) = \beta(F) = \gamma(F) = 0$, where

$$\alpha(F) = F_{0,0,0}\,, \qquad \beta(F) = F_{1,0,0}\,, \qquad \gamma(F) = F_{1,0,1}\,. \tag{3.9}$$

For α and β, this corresponds to the definition given in the introduction.

The following proposition describes an important feature of normalized resonant functions. Its proof is straightforward.

Proposition 3.3. *Let $0 < \rho' < \rho$. Assume that $F \in \mathcal{A}_\rho^{d-1}$ is normalized. Then*

$$\|\mathbb{I}_j^+ F_j\|_{\rho'} \le Q^{\theta^n} \|\mathbb{I}_j^+ F_j\|_\rho\,, \qquad Q = \max\bigl\{\rho'/\rho, e^{\rho'}/e^\rho\bigr\}\,. \tag{3.10}$$

For simplicity, we will restrict to $\rho \le 1$ later on. Writing $\rho' = \rho - \varepsilon$, we obtain

$$\frac{\rho'}{\rho} = 1 - \frac{\varepsilon}{\rho} \le e^{-\varepsilon/\rho} \le e^{-\varepsilon} = \frac{e^{\rho'}}{e^\rho}\,. \qquad (\rho \le 1) \tag{3.11}$$

So $Q = e^{\rho'-\rho}$ in this case.

3.3. First order elimination

In what follows, $\rho > 0$ and $F \in \mathcal{A}_\rho^{d+1}$ are considered fixed. Furthermore, we assume that the triple (ω, c, n) satisfies the approximate Diophantine condition in Definition 2.

Consider $U = L^{-1}\mathbb{I}^- F$ as defined in (2.11).

Proposition 3.4. *Let $0 < \rho' \leq \rho$. Then U belongs to $\mathcal{A}_{\rho'}^{d+1}$ and satisfies*

$$\theta^n \|U_j\|_{\rho'}\,, \|\partial_{q_i} U_j\|_{\rho'}\,, \rho' \|\partial_p U_j\|_{\rho'} \leq \zeta^{-1}\theta^{(d+\chi)n} \|\mathbb{I}_j^- F_j\|_{\rho'}\,, \tag{3.12}$$

for every $i \in \{1, 2, \ldots, d\}$ and $j \in \{0, 1\}$.

Proof. Let $j \in \{0, 1\}$. Recall from (2.10) that the eigenvectors and eigenvalues of L_j are

$$e_{\nu,k}(q,p) = e^{i\nu\cdot q} p^k\,, \qquad \lambda_{j,\nu,k} = i\omega\cdot\nu + c(k-j)\,. \tag{3.13}$$

Restrict now to $(\nu, k) \in I_j^-$. Then $j - k > 0$. Thus, if $|\nu| + (k-j) < \theta^n$, then

$$|\lambda_{j,\nu,k}| \geq \zeta\big(|\nu| + (k-j)\big)^{1-d-\chi} > \zeta\theta^{(1-d-\chi)n}\,, \tag{3.14}$$

by the condition (3.2). Using that $k \leq \theta^n$, we have

$$\begin{aligned} \|\partial_p L_j^{-1} e_{\nu,k}\|_{\rho'} &= |\lambda_{j,\nu,k}|^{-1} k \|e_{\nu,k-1}\|_{\rho'} \\ &= |\lambda_{j,\nu,k}|^{-1} k (\rho')^{-1} \|e_{\nu,k}\|_{\rho'} \leq (\rho')^{-1}\zeta^{-1}\theta^{(d+\chi)n} \|e_{\nu,k}\|_{\rho'}\,. \end{aligned} \tag{3.15}$$

Taking the supremum over $(\nu, k) \in I_j^-$ yields an upper bound $(\rho')^{-1}\zeta^{-1}\theta^{(d+\chi)n}$ on the operator norm of $\partial_p L_j \mathbb{I}_j^-$ acting on $\mathcal{A}_{\rho'}$. This implies the bound in (3.12) on $\partial_p U_j$. The remaining bounds in (3.12) are obtained similarly. **QED**

Corollary 3.5. *Let $0 < \rho' \leq \rho$. Then the operator norm of $DU : \mathcal{A}_{\rho'}^{d+1} \to \mathcal{A}_{\rho'}^{d+1}$ satisfies*

$$\|DU\|_{\rho'} \leq (d+1) B_n \|\mathbb{I}^- F\|_{\rho'}\,, \qquad B_n \stackrel{\text{def}}{=} \min\{1, \rho'\}^{-1}\zeta^{-1}\theta^{(d+\kappa)n}\,. \tag{3.16}$$

3.4. Change of coordinates

Our next goal is to estimate the vector field $Y = V^* X$. Let

$$Z = X \circ V = X \circ (\mathrm{I} + U)\,. \tag{3.17}$$

Then Y admits the decomposition

$$\begin{aligned} Y &= (\mathrm{I} + DU)^{-1} Z = Z - (\mathrm{I} + DU)^{-1}(DU)Z \\ &= K_c^\omega + (Z - K_c^\omega) - (\mathrm{I} + DU)^{-1}(DU)Z\,. \end{aligned} \tag{3.18}$$

Assumption 5. Given $0 < \rho' < \rho$ and $r \geq 1$, assume that $\mathcal{F} = rF$ satisfies

(*a*) $\rho' + B_n \|\mathbb{I}^- \mathcal{F}\|_{\rho'} \leq \rho$.

(*b*) $(d+1)B_n\|\mathbb{I}^-\mathcal{F}\|_{\rho'} \le \frac{1}{2}$.
(*c*) $|\omega| + |c|\rho' + \bigl(|c|B_n + 1\bigr)\|\mathcal{F}\|_\rho \le B$ for some fixed B.

Lemma 3.6. *Let $0 < \rho' < \rho$ and $r = 1$. Under Assumption 5, we have*

$$\|Z\|_{\rho'} \le B\,, \qquad \|Y - K_c^\omega\|_{\rho'} \le B\bigl(1 + 2(d+1)B_n\bigr)\|F\|_\rho\,. \tag{3.19}$$

Proof. First, we note that $K_c^\omega \circ (\mathrm{I} + U) = K_c^\omega + (DK_c^\omega)U$, and thus

$$\begin{aligned} Z = X \circ (\mathrm{I} + U) &= K_c^\omega \circ (\mathrm{I} + U) + F \circ (\mathrm{I} + U) \\ &= K_c^\omega + (DK_c^\omega)U + F \circ (\mathrm{I} + U)\,. \end{aligned} \tag{3.20}$$

So by Proposition 3.2, Proposition 3.4, and Assumption 5.a, we have

$$\begin{aligned} \|Z - K_c^\omega\|_{\rho'} &\le \|(DK_c^\omega)U\|_{\rho'} + \|F(\mathrm{I} + U)\|_{\rho'} \le |c|\|U\|_{\rho'} + \|F\|_\rho \\ &\le \bigl(|c|B_n + 1\bigr)\|F\|_\rho\,. \end{aligned} \tag{3.21}$$

Using Assumption 5.c and the fact that $\|K_c^\omega\|_{\rho'} = |\omega| + |c|\rho'$, this yields

$$\|Z\|_{\rho'} \le |\omega| + |c|\rho' + \bigl(|c|B_n + 1\bigr)\|F\|_\rho \le B\,. \tag{3.22}$$

Notice that Corollary 3.5 and Assumption 5.b imply that $\|DU\|_{\rho'} \le \frac{1}{2}$. Thus, using (3.18) and the last two inequalities, we obtain

$$\begin{aligned} \|Y - K_c^\omega\|_{\rho'} &\le \|Z - K_c^\omega\|_{\rho'} + \|(\mathrm{I} + DU)^{-1}(DU)Z\|_{\rho'} \\ &\le \|Z - K_c^\omega\|_{\rho'} + 2\|DU\|_{\rho'}\|Z\|_{\rho'} \\ &\le \bigl(|c|B_n + 1\bigr)\|F\|_\rho + 2(d+1)B_n\|\mathbb{I}^-F\|_{\rho'}B \\ &\le B\|F\|_\rho + 2B(d+1)B_n\|\mathbb{I}^-F\|_{\rho'}\,. \end{aligned} \tag{3.23}$$

This completes the proof of Lemma 3.6. **QED**

Recall from (2.5) that $Y = K_c^\omega + \mathbb{I}^+F + R$, where R is the second order Taylor remainder in the expansion of $\mathcal{Y}(z) = (\mathrm{I} + zDU)^{-1}(K_c^\omega + zF) \circ (\mathrm{I} + zU)$ in powers of z, evaluated at $z = 1$. Formally, $\|R\|_{\rho'}$ is of the order of $\|F\|_\rho^2$. The following lemma makes this more precise. It is intended to be used with r^{-1} somewhat larger than $\|F\|_\rho$.

Lemma 3.7. *Let $0 < \rho' < \rho$ and $r \ge 2$. Under Assumption 5, we have*

$$\|R\|_{\rho'} \le \frac{B}{r-1}\bigl(1 + 2(d+1)B_n\bigr)\|F\|_\rho\,, \tag{3.24}$$

Proof. Recall that $Y = (\mathrm{I} + DU)^{-1}(K_c^\omega + F) \circ (\mathrm{I} + U)$. Replacing F by $\mathcal{F} = zF$ in this identity yields the vector field $\mathcal{Y}(z)$. Here we use that $F \mapsto U$ is linear. By assumption, $\mathcal{F}$

satisfies the bounds in Assumption 5 whenever $|z| \le r$. So we can apply Lemma 3.6 with $\mathcal{F}$ in place of F. Using the representation

$$R = \frac{1}{2\pi i}\oint_{|z|=r}\big[\mathcal{Y}(z) - K_c^\omega\big]\frac{dz}{z^2(1-z)}\,, \tag{3.25}$$

this yields the desired bound

$$\|R\|_{\rho'} \le \frac{1}{r(r-1)}\sup_{|z|=r}\|\mathcal{Y}(z) - K_c^\omega\|_{\rho'} \le \frac{1}{r(r-1)}B\big(1+2(d+1)B_n\big)\|rF\|_\rho\,.$$

QED

3.5. Normalization via translation

Recall that $X = K_c^\omega + F$, and that $Y = X \circ V$ admits a decomposition

$$Y = K_c^\omega + H\,, \qquad H \stackrel{\text{def}}{=} \mathbb{I}^+ F + R\,, \tag{3.26}$$

with R as described above. We expect that H is much smaller in $\mathcal{A}_{\rho'}^{p+1}$ than F is in $\mathcal{A}_\rho^{d+1}$. The reason is the factor Q^{θ^n} in Proposition 3.3 and the factor $(r-1)^{-1}$ in Lemma 3.7.

But $\beta(Y)$ is nonzero in general. To correct this, we choose a suitable p-translation $\mathcal{T}$ and then define $\mathcal{R}(X) = \mathcal{T}^*Y$. Here $\mathcal{T}$ is of the form $\mathcal{T}_\tau(q,p) = (q, p+\tau)$, and $\tau = \tau(Y)$ is determined by the condition $\beta(\mathcal{T}^*Y) = 0$. This is possible in the following sense.

Proposition 3.8. *Let $0 < \rho'' < \rho'$ and $\varepsilon = (\rho' - \rho'')/2$. Assume that H belongs to $\mathcal{A}_{\rho'}^{d+1}$ and satisfies $\|H\|_{\rho'} \le \min\{|c|,1\}\varepsilon/2$. Then there exists a unique solution τ of the equation $\beta(\mathcal{T}_\tau^*Y) = 0$ with $|\tau| \le \varepsilon$. Furthermore,*

$$|\tau| \le |c|^{-1}\|H\|_{\rho'}\,, \qquad \|\mathcal{T}_\tau^*Y - K_c^\omega\|_{\rho''} \le \|H\|_{\rho'}\,. \tag{3.27}$$

Proof. For $|\tau| \le \varepsilon$ we have $\|\mathcal{T}_\tau H\|_{\rho'-\varepsilon} \le \|H\|_{\rho'}$ by Proposition 3.2. Thus,

$$|\beta(\mathcal{T}_\tau H)| \le \|H\|_{\rho'} \le \frac{\varepsilon}{2}\,, \qquad \left|\frac{d}{d\tau}\beta(\mathcal{T}_\tau H)\right| \le \frac{1}{\varepsilon}\|H\|_{\rho'} \le \frac{|c|}{2}\,. \tag{3.28}$$

Here we have used a Cauchy bound for the derivative. So the map $\tau \mapsto -c^{-1}\beta(\mathcal{T}_\tau H)$ is a contraction on the disk $|\tau| \le \varepsilon$. For the (unique) fixed point τ we have $\beta(\mathcal{T}_\tau H) = -c\tau$. Given that $\beta(\mathcal{T}_\tau^* K_c^\omega) = c\tau$, this yields $\beta(\mathcal{T}_\tau^* Y) = 0$, as claimed.

Notice that $|c\tau| = |\beta(\mathcal{T}_\tau H)| \le \|\mathcal{T}_\tau H\|_{\rho''} \le \|H\|_{\rho'}$. This implies the first bound in (3.27). Furthermore, $\|\mathcal{T}_\tau^*Y - K_c^\omega\|_{\rho''} = \|\mathcal{T}_\tau^*H - (0,c\tau)\|_{\rho''} \le \|\mathcal{T}_\tau H\|_{\rho''} \le \|H\|_{\rho'}$. This proves the second bound in (3.27). **QED**

Summary 6. The transformation $\mathcal{R}$ is defined on a domain in $\mathcal{A}_\rho^\rho$ with $\rho > 0$. Consider a vector field $X = K_c^\omega + F$ with $F \in \mathcal{A}_\rho^{d+1}$ normalized. To check whether X belongs to the domain of $\mathcal{R}$ for a given $n > 0$, use Proposition 5.1 to verify that (ω, c, n) satisfies the approximate Diophantine condition (3.2). Then choose parameters $0 < \rho'' < \rho' < \rho$. Now check that F satisfies the smallness Assumption 5 for some (preferably large) value $r \geq 2$. Then $Y = V^* X$ is well-defined, and Lemma 3.7 yields a bound on the Taylor remainder R. Finally, verify that $H = \mathbb{I}^+ F + R$ satisfies the norm bound assumed in Proposition 3.8. Then $\mathcal{R}(X) = \mathcal{T}^* Y$ is well-defined and belongs to $\mathcal{A}^{d+1}_{\rho''}$.

In what follows, n is no longer fixed. So if necessary, we denote $\mathcal{R}$ by $\mathcal{R}_n$. And the projections $\mathbb{I}^\pm$ for vector fields will be be denoted by $\mathbb{I}^\pm_n$.

4. Choice of domains

The goal now is to iterate the sequence $n \mapsto \mathcal{R}_n$. This involves choosing parameters ρ_n, ρ'_n, r_n, and upper bounds $\|F^n\|_{\rho_n} \leq \varepsilon_n$ that propagate properly under iteration. Once F^n is very close to zero, maintaining accurate bounds is straightforward, independently of the values of $d, \zeta, \chi, \mathcal{C}$ and other constants. What is important for accurate bounds is to optimize the earlier steps. This is best done in a concrete setting. So we aim for simplicity at this point.

We start the iteration not with $n = 1$, but with $n = n_\circ$ for some (large) integer $n_\circ > 0$. Let $\varrho \in (0, 1]$ be fixed. In order to define our choice of domains, pick a positive integer $\bar{n}$ such that $\varrho - 1/\bar{n} > 0$. With $\rho_\infty \in [\varrho - 1/\bar{n}, \varrho)$ to be determined later, set

$$\rho_m = \rho_\infty + m^{-1}\,, \qquad m = \bar{n} + (0, 1, 2, \ldots)\,. \tag{4.1}$$

In what follows, we assume that $n \geq n_\circ$ with $n_\circ > \bar{n}$. After having chosen $n_\circ$ later on, we want our estimates to hold for $\rho_{n_\circ} = \varrho$. So our bounds will have to be uniform in the choice of $\rho_\infty \in [\varrho - 1/\bar{n}, \varrho]$. This requirement is easy to satisfy.

Notice that the sequence $m \mapsto \rho_m$ is decreasing, with gaps $\rho_{m-1} - \rho_m$ bounded from above and below by $(m-1)^{-2}$ and m^{-2}, respectively. The domain radii that have been used in Section 3 are related to the sequence $m \mapsto \rho_m$ via

$$\rho_n = \rho'' < \rho' = \rho_n + \tfrac{1}{2} n^{-2} < \rho = \rho_{n-1}\,. \tag{4.2}$$

So the domain of $\mathcal{R}_n$ is a subset of $\mathcal{A}^{d+1}_{\rho_{n-1}}$ and the range is included in $\mathcal{A}^{d+1}_{\rho_n}$, as described in Summary 6. Due to the choices made above, we have $\rho_n \leq 1$ for all $n \geq n_\circ$. This is just for convenience, so that we can use Proposition 3.3 with $Q = e^{\rho' - \rho}$. For the factor r that appears in Lemma 3.7, we use $r = r_n$ defined by

$$r_n = e^{\kappa \vartheta^n}\,, \qquad 1 < \vartheta < \theta < 2\,, \quad 1 - \vartheta^{-1} < \kappa < 1\,, \tag{4.3}$$

where, κ and ϑ are fixed parameters.

Consider now $X^{n_\circ} \in \mathcal{A}^{d+1}_{\rho_{n_\circ}}$. We would like to construct the sequence of vector fields

$$X^n = \mathcal{R}_n(X^{n-1})\,, \qquad n = n_\circ + (1, 2, 3, \ldots)\,. \tag{4.4}$$

If X^m is well-defined, then we set $\omega_m = \alpha(X^m)$ and $c_m = \gamma(X^m)$. This defines a decomposition $X^m = K^{\omega_m}_{c_m} + F^m$, with F^m normalized in the sense of Definition 4.

Lemma 4.1. *Let $C > 0$. If $n_\circ$ is chosen sufficiently large, then the following holds. Let $n > n_\circ$. Assume that $(\omega_{n-1}, c_{n-1}, n)$ satisfies an approximate Diophantine condition in the sense of Definition 2, and that the bounds*

$$\|X^m\|_{\rho_m} < C - m^{-1}\,, \qquad \|F^m\|_{\rho_m} < e^{-\vartheta^m}\,, \tag{4.5}$$

hold for $m = n-1$. Then $X^n = \mathcal{R}_n(X^{n-1})$ is well-defined and, together with the associated function F^n, satisfies the bounds (4.5) with $m = n$.

Proof. Let $n > n_\circ$, and assume that X^{n-1} satisfies the hypotheses of Lemma 4.1. Recall from (3.26) that

$$Y^n - K^{\omega_{n-1}}_{c_{n-1}} = H^n = \mathbb{I}^+_n F^{n-1} + R^{n-1}\,. \tag{4.6}$$

Let $\rho' = \rho_n + \frac{1}{2}n^{-2}$ and $\rho = \rho_{n-1}$. Then by Proposition 3.3, we have

$$\|\mathbb{I}^+_n F^{n-1}\|_{\rho'} \le e^{(\rho'_n - \rho)\theta^n}\|\mathbb{I}^+_n F^{n-1}\|_\rho < e^{-\frac{1}{2}n^{-2}\theta^n} e^{-\vartheta^{n-1}}\,. \tag{4.7}$$

And by Lemma 3.7, we have

$$\|R^{n-1}\|_{\rho'} \le C_1 r_n^{-1} B_n \|F^{n-1}\|_\rho < C_2 e^{-\kappa\vartheta^n}\theta^{(d+\chi)n} e^{-\vartheta^{n-1}}\,. \tag{4.8}$$

Here, and in what follows, $C_1, C_2, \ldots$ denote positive constants that depend only on the parameters $d, \zeta, \chi, \theta, \vartheta, \kappa, \mathcal{C}, \mathcal{C}'$. And we assume that $n_\circ$ has been chosen sufficiently large.

Then the bound in (4.7) on $\mathbb{I}^+_n F^{n-1}$ is smaller than the bound in (4.8) on R^{n-1}. Thus,

$$\|H^n\|_{\rho'} < C_3 e^{-\kappa\vartheta^n}\theta^{(d+\chi)n} e^{-\vartheta^{n-1}} = C_3\theta^{(d+\chi)n} e^{-(\kappa+\vartheta^{-1})\vartheta^n}\,. \tag{4.9}$$

By Proposition 3.8, we now have a bound

$$\begin{aligned} \|F^n\|_{\rho_n} &= \|X^n - K^{\omega_n}_{c_n}\|_{\rho_n} \le \big\|X^n - Y^n\big\|_{\rho_n} + \big\|Y^n - K^{\omega_n}_{c_n}\big\|_{\rho_n} \\ &\le \|F^n - H^n\|_{\rho_n} + \big\|Y^n - K^{\omega_{n-1}}_{c_{n-1}}\big\|_{\rho_n} \\ &\le 2\big(\|H^n\|_{\rho'} + \|F^{n-1}\|_{\rho_n}\big) \le C_4\theta^{(d+\chi)n} e^{-(\kappa+\vartheta^{-1})\vartheta^n} < e^{-\vartheta^n}\,. \end{aligned} \tag{4.10}$$

This proves that the second inequality in (4.5) holds for $m = n$. The first inequality is clearly satisfied as well. **QED**

A bound analogous to (4.10) applies to $\|X^n - K^{\omega_{n-1}}_{c_{n-1}}\|_{\rho_n}$. So

$$\|\omega_n - \omega_{n-1}\|\,,\ |c_n - c_{n-1}| < e^{-\vartheta^n}\,, \tag{4.11}$$

if $n_\circ$ has been chosen sufficiently large. This may suggest that we can iterate $\mathcal{R}$ indefinitely, getting $n \mapsto \omega_n$ and $n \mapsto c_n$ to converge, with limits close to $\alpha_{n_\circ}$ and $c_{n_\circ}$, respectively. The (only) problem is that the approximate Diophantine condition (3.2) could fail after a finite number of iterations.

5. Parameterized vector fields

We need to ensure that the sequence $n \mapsto \omega_n$ converges to $\bar{\omega}$. This can be achieved (only) by fine-tuning the initial value of $\omega_{n_\circ}$. So this value should depend on a parameter, that can be adjusted properly.

5.1. The transformation $\mathfrak{R}$

Our parameter $z \in \mathbb{C}^d$ will be restricted to some (small) ball $\mathbb{D} \subset \mathbb{C}^d$, centered at the origin. Then a parameterized vector field can be viewed as a function $X : \mathbb{D} \to \mathcal{A}_\rho^{d+1}$. Defining $\mathcal{R}$ pointwise via $\mathcal{R}(X)(z) = \mathcal{R}(X(z))$, we set

$$\mathfrak{R}(X) = \tilde{X} \circ S\,, \qquad \tilde{X} = \mathcal{R}(X)\,. \tag{5.1}$$

for some suitable re-parameterization map $S : \tilde{\mathbb{D}} \to \mathbb{D}$ that depends on X. We also write this as $\mathfrak{R} = S^* \circ \mathcal{R}$.

To be more precise, assume that $\alpha(X) = \omega$ with $\omega(z) = \bar{\omega} + z$. The goal is to choose S in such a way that $\alpha(\mathfrak{R}(X)) = \omega$ as well. Setting $\tilde{\omega} = \alpha\bigl(\tilde{X}\bigr)$, the equation for S is

$$\tilde{\omega} \circ S = \omega\,, \qquad \omega(z) \stackrel{\text{def}}{=} \bar{\omega} + z\,. \tag{5.2}$$

In order to solve this equation, consider the decomposition

$$\tilde{\omega}(z) = \bar{\omega} + z + w(z)\,, \qquad S(z) = z - \sigma(z)\,. \tag{5.3}$$

Then the equation $\tilde{\omega}(S(z)) = \bar{\omega} + z$ for S becomes $\bar{\omega} + z - \sigma(z) + w(z - \sigma(z)) = \bar{\omega} + z$, which simplifies to

$$\sigma(z) = w(z - \sigma(z))\,. \tag{5.4}$$

Given $\delta > 0$, denote by $\mathbb{D}_\delta$ the open disk in $\mathbb{C}^d$ defined by the condition $\|z\| < \delta$. The space $\mathcal{B}_\delta$ is defined as the vector space of all analytic functions $f : \mathbb{D}_\delta \to \mathbb{C}$ that extend continuously to the closure of $\mathbb{D}_\delta$, equipped with the sup-norm $|f|_\delta = \sup_{z \in \mathbb{D}_\delta^d} |f(z)|$. For $\omega \in \mathcal{B}_\delta^d$ we use the norm $\|\omega\|_\delta = \sup_i |\omega_i|_\delta$.

Proposition 5.1. *Let $0 < \delta'' < \delta$. Define $\delta' = (\delta + \delta'')/2$ and $\varepsilon = (\delta - \delta'')/2$. Assume that w belongs to $\in \mathcal{B}_\delta^d$ and satisfies $\|w\|_\delta \le \varepsilon/2$. Then the equation $\sigma = w \circ (\mathrm{I} - \sigma)$ has a unique solution $\sigma \in \mathcal{B}_{\delta'}^d$, satisfying $\|\sigma\|_{\delta'} \le \|w\|_\delta$. So $S = \mathrm{I} + \sigma$ maps $\mathbb{D}_{\delta''}$ into $\mathbb{D}_\delta$.*

Proof. We regard (5.4) as a fixed point problem for the map $\sigma \mapsto w \circ (\mathrm{I} + \sigma)$ on the disk $\|\sigma\|_{\delta'} \le \varepsilon$. On this disk we have

$$\|w\|_{\delta'} \le \varepsilon/2\,, \qquad \|Dw\|_{\delta'} \le \varepsilon^{-1}\|w\|_\delta \le \tfrac{1}{2}\,. \tag{5.5}$$

Here we have used a Cauchy bound for the derivative. So the map $\sigma \mapsto w \circ (\mathrm{I} + \sigma)$ is a contraction on the disk $\|\sigma\|_{\delta'} \le \varepsilon$. For the (unique) fixed point σ we have the bound $\|\sigma\|_{\delta'} = \|w \circ (\mathrm{I} + \sigma)\|_{\delta'} \le \|w\|_\delta$, as claimed. **QED**

At this point, we could characterize the domain of $\mathfrak{R}$ as was done for $\mathcal{R}$ in Summary 6. But the extra conditions on F are mild, so we refrain from doing this.

In what follows, n is no longer fixed. So if necessary, we denote $\mathfrak{R}$ by $\mathfrak{R}_n$.

5.2. Iterating $\mathfrak{R}$

Consider now iterating the sequence $n \mapsto \mathfrak{R}_n$. At the m-th step we choose for δ the value

$$\delta_m = e^{-m^3}\,, \tag{5.6}$$

so that $m \mapsto \delta_m/\delta_{m-1}$ decrease faster than exponential. This choice works for a proof of Theorem 1.1, but it should be reasonable in other applications as well.

In what follows, we use the notation $\mathcal{A}_{\rho,\delta} = \mathcal{A}_\rho(\mathcal{B}_\delta)$. The norm of $F \in \mathcal{A}_{\rho,\delta}$ will be denoted by $\|F\|_{\rho,\delta}$. Consider vector fields $X^m = K_{c_m}^{\omega_m} + F^m$ with F^m normalized. The analogue of the property (4.5) for families is

$$\|F^m\|_{\rho_m,\delta_m} \le e^{-\vartheta^m}\,. \tag{5.7}$$

Definition 7. For $m \ge n_\circ$ define $\mathfrak{D}_{m+1}$ to be the set of all vector fields $X^m \in \mathcal{A}_{\rho_m,\delta_m}$ with the following properties. Define $c_m^\circ = c_m(0)$ and $v_m = c_m - c_m^\circ$. Then
(a) X^m takes values in $\mathbb{R}^{d+1}$ for real arguments (q,p,z).
(b) F^m satisfies the bound (5.7).
(c) $\alpha(X^m) = \omega$, where $\omega(z) = \bar\omega + z$.
(d) $v = v_m$ satisfies the second bound in (3.3).

Theorem 5.2. *If $n_\circ > 0$ is chosen sufficiently large, then, with $c_{n_\circ}^\circ \in \mathcal{C}$ and for $n = n_\circ + (1,2,3,\ldots)$, the domain of $\mathfrak{R}_n$ includes $\mathfrak{D}_n$, and $\mathfrak{R}_n$ maps $\mathfrak{D}_n$ into $\mathfrak{D}_{n+1}$.*

Proof. First we note that, by construction, the reality condition (a) in Definition 7 propagates under our iteration.

For $n = n_\circ + (1,2,3,\ldots)$ we have the following. Assume that X^{n-1} belongs to $\mathfrak{D}_n$. So in particular, ω_{n-1} is the restriction of ω to $\mathbb{D}_{\delta_{n-1}}$. On this domain we have $|z| \le \delta_{n-1}$. Thus, both bounds in (3.3) are satisfied for $m = n-1$ and $v = v_m$. Furthermore, given that $c_{n_\circ} \in \mathcal{C}$, it is clear from (4.11) that $c_{n-1}^\circ \in \mathcal{C}'$. So by Proposition 3.1 with $\bar c = c_{n-1}^\circ$, the triple $(\omega_{n-1}, c_{n-1}, n)$ satisfies an approximate Diophantine condition in the sense of Definition 2. Then by Lemma 4.1, the vector field X^{n-1} belongs to the domain of $\mathcal{R}_n$. Here, and in what follows, we assume that $n_\circ$ has been chosen sufficiently large.

Let now $\tilde X = \mathcal{R}_n(X^{n-1})$. If we temporarily set $X^n = \tilde X$, then X^n satisfies property (b) in Definition 7 with $m = n$. The same holds after re-parameterization.

So it suffices to verify that (c) and (d) hold after a re-parameterization $\tilde X \mapsto X^n$. Notice that $w = \tilde\omega - \omega$ in (5.3). By (4.11) we have the bound

$$\|w\|_{\delta_{n-1}} = \|\tilde\omega - \omega\|_{\delta_{n-1}} < e^{-\vartheta^n}\,. \tag{5.8}$$

Choose $\delta'' = \delta_n$. Then Proposition 5.1 applies with $\delta = \delta_{n-1}$. As a result, $X^n = S^*\tilde X$ belongs to $\mathcal{A}^{d+1}_{\rho_n,\delta''}$. This shows that property (c) holds for $m = n$.

To verify property (d) for $n = m$, we can use that the same property holds for $n = m-1$. So we have $|v_{n-1}| \le \zeta\theta^{-(d+\chi)(n-1)}$. When combined with (4.11), this yields a bound

$$|v_n| \le 2\zeta\theta^{-(d+\chi)(n-1)}\,. \tag{5.9}$$

However, this estimate is not sufficient and has to be improved.

To this end, we repeat the arguments after (5.8) for the choice $\delta'' = (\delta_n\delta_{n-1})^{1/2}$. The function v_n is still analytic on $\mathbb{D}_{\delta''}$ and satisfies the bound (5.9). Given that v_n vanishes at the origin $z = 0$, its restriction to $\mathbb{D}_{\delta_n}$ satisfies $|v_n| \le 2\zeta\theta^{-(d+\chi)(n-1)}(\delta_n/\delta_{n-1})^{1/2}$ by Schwarz's lemma. This implies property (d) of Definition 7 for $m = n$. **QED**

6. Proof of Theorem 1.1

In what follows, $n_\circ$ is assumed to be sufficiently large to have $X^{n-1} \in \mathfrak{D}_n$ for all $n > n_\circ$, whenever $X^{n_\circ-1} \in \mathfrak{D}_{n_\circ}$. And we choose ρ_∞ in the definition (4.1) of the sequence $m \mapsto \rho_m$ in such a way that $\rho_{n_\circ}$ agrees with the parameter ϱ that has been chosen in Theorem 1.1.

Denote by $\mathcal{X}_\rho$ the real Banach space of all vector fields $X \in \mathcal{A}_\rho(\mathbb{C})^{d+1}$ that satisfy $\beta(X) = 0$, and that take values in $\mathbb{R}^{d+1}$ when restricted to real arguments (q,p). Define $\mathcal{X}_\rho^\circ$ to be the subspace of all $H \in \mathcal{X}_\rho$ that satisfy $\alpha(H) = 0$. And define $\mathcal{X}_\rho^{\circ\circ}$ to be the subspace of all $F \in \mathcal{X}_\rho^\circ$ that satisfy $\gamma(F) = 0$.

We start the iteration of $\mathfrak{R}$ with $n = n_\circ$ and a parameterized vector field X^n given by $X^n(z) = K_0^{\bar\omega+z} + H^n$ with $H^n \in \mathcal{X}_{\rho_n}^\circ$. Notice that $H^n = K_{c_n}^0 + F^n$, with $c_n = \gamma(H^n)$ and F^n normalized.

Assume that $X^{n_\circ}$ belongs to the domain $\mathcal{D}_{n_\circ+1}$ specified in Definition 7. By Proposition 5.1 and (5.8), the re-parameterization map S_n used in the step $\mathfrak{R}_n$ satisfies

$$\|S_n - \mathrm{I}\|_{\delta_n} = \|\sigma\|_{\delta_n} \le \|w\|_{\delta_{n-1}} < e^{-\vartheta^n}\,, \qquad n > n_\circ\,. \tag{6.1}$$

This bound guarantees the existence of the limits

$$z_n = \lim_{m\to\infty} z_{n,m}\,, \qquad z_{n,m} = \mathcal{S}_{n,m}(0)\,, \quad \mathcal{S}_{n,m} = S_{n+1}\circ\cdots\circ S_{m-1}\circ S_m\,, \tag{6.2}$$

for all $n \ge n_\circ$. Here we use that $\mathcal{S}_{n,m}$ maps $\mathbb{D}_{\delta_m}$ into $\mathbb{D}_{\delta_n}$, and that 0 belongs to all these balls. The sequence of limiting values satisfies $z_n = \mathcal{S}_{n,m}(z_m)$ for all $m > n \ge n_\circ$, and $z_n \to 0$. From the bounds (5.7), it is clear that $X^n(z_n) \to K_{c_\infty}^{\bar\omega}$ in $\mathcal{A}_{\rho_\infty}$ for some constant c_∞. Due to the uniform bounds used in our estimates of $\mathfrak{R}_n$, and due to the uniform convergence of $z_{n,m} \to z_n$, we know that c_∞ and each z_n depends analytically on $H^{n_\circ}$.

The function Ω mentioned after Theorem 1.1 is given by $\Omega(H^{n_\circ}) = z_{n_\circ}$. And the manifold $\mathcal{M}$ described in Theorem 1.1 is the graph of the map $H^{n_\circ} \mapsto X^{n_\circ}(z_{n_\circ})$ on a domain of vector fields $H = K_c^0 + F$ with $c \in \mathcal{C}$ and $F \in \mathcal{X}_{\rho_{n_\circ}}^{\circ\circ}$ close to zero.

As a result of the above-mentioned iteration, we get a sequence of vector fields $X^n(z_n)$ that are related via changes of coordinates,

$$X^m(z_m) = \mathcal{V}_{n,m}^* X^n(z_n)\,, \qquad \mathcal{V}_{n,m} = V_{n+1}\circ\mathcal{T}_{n+1}\circ\cdots\circ V_m\circ\mathcal{T}_m\,.$$

By construction, the domain of $\mathcal{V}_{n,m}$ includes the complex open neighborhood A_{ρ_n} of $\mathbb{T}^d \times \{0\}$, where $|\operatorname{Im} q_i| < \rho_n$ and $|p| < \rho_n$. Furthermore, the maps $\mathcal{T}_k - \mathrm{I}$ and $V_k - \mathrm{I}$ are bounded in norm by $Ce^{-\vartheta^k}$ on their domains $A_{\rho_k'}$, where C is some fixed constants. Their derivatives satisfy an analogous bound by Corollary 3.5. So it is clear that $\mathcal{V}_{n,m} \to \mathcal{V}_n$ uniformly on A_{ρ_n}, and that $\mathcal{V}_n^* X^n(z_n) = K_{c_\infty}^{\bar\omega}$, for all $n \ge n_\circ$.

This completes our proof of Theorem 1.1.

The following will be used in our application of Theorem 1.1 to the VdP flow.

Remark 8. Consider the restriction of our analysis to vector fields $X = (X_1, X_0)$ with the property that X_1 has a constant component, say X_1^i. Then X_1^i is resonant at each iteration of $\mathcal{R}$ and thus remains unchanged. And the re-parameterization in $\mathfrak{R}$ does not change the parameter z_i. So in the affine subspace of such vector fields, the manifold $\mathcal{W}$ in Theorem 1.1 is trivially of codimension $d-1$.

7. Proof of Theorem 1.2

We start by linearizing the VdP flow (1.6) in the plane $y = (y_1, y_2)$, using the following lemma. A proof of this lemma can likely be found elsewhere, but we include it here for completeness.

Lemma 7.1. *Let Y be a real-analytic vector field in some open domain in $\mathbb{R}^2$ that has an attracting periodic orbit $\mathcal{C}$ in this domain. Let T be its period and $\alpha = 2\pi/T$. Then the flow Φ_Y for Y is real-analytically conjugate near $\mathcal{C}$ to a linear flow $\dot{q} = \alpha$ and $\dot{p} = cp$ for some $c < 0$. The conjugacy depends real-analytically on the vector field Y, in the sense described below.*

Proof. Consider a vector field $\dot{y} = Y(y)$ with the indicated properties. As a first step, we perform a real-analytic change of coordinates $y = V(q, r)$ from an annulus $A_\varepsilon = \mathbb{T} \times (-\varepsilon, \varepsilon)$ to an open neighborhood in $\mathbb{R}^2$ of the circle $\mathcal{C}$. Subsequent coordinate changes map one such annulus into another, with the size $\varepsilon > 0$ decreasing at each step. The vector $Z = V^*Y$ belongs to one of our spaces $\mathcal{X}_\rho$. It will be clear from the description below that our estimates are uniform for suitable choices of $\varepsilon, \rho > 0$. Pulling back $\mathcal{X}_\rho$ via V defines uniformity of bounds for the original vector field Y.

Defining $V(q, 0) = \Phi_Y^{\alpha q}(y)$ yields a map $q \mapsto y$ from $\mathbb{T}$ to $\mathcal{C}$. Notice that every point $y = V(q, 0)$ is a fixed point of Φ_Y^T. Since the torus is attracting, the derivative $D\Phi_Y^T$ has a stable subspace E_q with eigenvalue $\lambda \in (-1, 1)$. Pick a continuous family $y \mapsto \ell_q$ of linear isometries $\ell_q : \mathbb{R} \mapsto E_q$. The choice is unique up to a sign, so this family is necessarily real-analytic. Now consider the change of coordinates V defined by $V(q, r) = V(q, 0) + \ell_q r$.

In these new coordinates, we have a vector field $Z = V^*Y$, and a flow of the form

$$\Phi_Z^t(q, r) = \left(q + t\alpha, \phi_q^t(r)\right), \tag{7.1}$$

with $D\Phi_Z^T(q, 0) = \operatorname{diag}(1, \lambda)$. Our goal is to conjugate Φ_Z to the linearized flow Ψ,

$$\Psi^t(q, r) = \left(q + t\alpha, D\phi_q^t(0)r\right). \tag{7.2}$$

Let us start with $t = T$, where $D\phi_q^t(0) = \lambda$. By standard results on invariant manifolds, the map Φ_Z^T has a one-dimensional stable manifold $\mathcal{W}$ that passes through the point $(0, 0)$ and is tangent to the eigenspace $(0, \mathbb{R})$ for the eigenvalue λ. Furthermore, $\mathcal{W}$ is the range of a function W from some open neighborhood of 0 in $\mathbb{R}$ to an open neighborhood of $(0,0)$ in $\mathbb{T}\times\mathbb{R}$, satisfying

$$\Phi_Z^T \circ W = W \circ \Psi^T \quad \text{at} \quad q = 0\,. \tag{7.3}$$

In other words, W linearizes Φ_Z^T at $q=0$. The identity (7.3) holds if T is replaced by nT with $n \ge 0$. And it extends to any fixed $n<0$, if restricted to A_ε with $\varepsilon>0$ sufficiently small.

We note that W is real-analytic. This can be seen e.g. by construction W via Hadamard's graph transform method: the curve W that linearizes the unstable manifold for the map Φ_Z^{-T} at $q=0$ is the unique fixed point of a contraction. The contraction property is due to the fact that $W(0)=0$ and $W'(0)=(0,1)$ can be fixed, so that the equation is really for W''. This yields a factor λ^2 from the re-parameterization that appears in the graph transform.

The goal now is to extend the conjugacy of Φ_Z and Ψ to arbitrary times $t\ge 0$. The following is rather standard; see also [3]. To simplify the description, let us lift our flows from $q\in\mathbb{T}$ to $q\in\mathbb{R}$ as 2π-periodic functions. For q in some open neighborhood of $[-\pi,\pi]$, define

$$H(q,r) = \Phi_Z^{\tau(q)} \circ W \circ \Psi^{-\tau(q)}(q,r)\,, \qquad \tau(q) = q/\alpha\,. \tag{7.4}$$

Using the relation (7.3) and its counterpart where T is replaced by $-T$, we see that $q\mapsto H(q,r)$ defines a real-analytic function on A_ε, for $\varepsilon>0$ sufficiently small. And a straightforward computation shows that $H\circ\Psi^t = \Phi_Z^t\circ H$ for all $t\ge 0$.

After the two changes of variables defined by V and H, we now have a flow Ψ that is generated by

$$\dot q = \alpha\,, \qquad \dot r = g(q)r\,, \tag{7.5}$$

for some real-analytic function $g:\mathbb{T}\to\mathbb{R}$. If c denotes the average value of g, then $g-c=G'$ for some function $G:\mathbb{T}\to\mathbb{R}$. A straightforward computation shows that

$$\dot q = \alpha\,, \qquad \dot p = cp\,, \qquad p \stackrel{\text{def}}{=} e^{-G(q)/\alpha} r\,. \tag{7.6}$$

Obviously we must have $c<0$ for the flow to have an attracting periodic orbit. This completes our proof of Lemma 7.1. **QED**

Consider now the VdP flow (1.6) for $\epsilon=0$ in the variables $y=(y_1,y_2)$. It is of the form $\dot y = Y(y)$, with $Y_1(y)=y_2$, and with Y_2 as defined in (1.6). The flow for this vector field has an attracting periodic orbit. Its period T depends real-analytically on the the parameter μ. For $\mu=0$ we have $T=2\pi$, and it is well-known that $T(\mu)\to\infty$ as $\mu\to\infty$. So $\omega_2=2\pi/T$ can take on any value in $(0,1)$ as $\mu>0$ is being varied.

By Lemma 7.1, there exists a real-analytic change of variables $(q_2,p)\mapsto(y_1,y_2)$ that yields the flow $(\dot q_2,\dot p)=(\omega_2,cp)$. If we include the circle $\dot y_0=\omega_1$ in our flow and set $Y_0=\omega_1$, then we have a vector field Y on $\mathbb{T}\times\mathbb{R}^2$ with an attracting invariant torus. Renaming y_0 to q_1, this flow is real-analytically conjugate to the flow

$$\dot q = \omega\,, \qquad \dot p = cp\,, \tag{7.7}$$

for the integrable vector field K_c^ω. To be more precise, the conjugacy is known to hold only near the invariant torus. Denote by $\mathcal{V}$ the change of coordinates that conjugates Y to K_c^ω in the case $\epsilon = 0$. For $\epsilon \neq 0$ close to zero, we perform the same change of coordinates to get a vector field $X = \mathcal{V}^* Y$ close to K_c^ω.

As was mentioned above, in the case $\epsilon = 0$ we can fix any desired frequency vector $\omega \in (0,1) \times \mathbb{R}$ and find parameter values (ω_1, μ) for which the VdP vector field is conjugate to K_c^ω for some $c < 0$.

Consider now vectors ω close to a vector $\bar\omega$ that satisfies a Diophantine condition (1.4). In fact, let us restrict to a codimension 1 situation where ω_1 takes the fixed value $\bar\omega_1$. As mentioned in Remark 8, the manifold $\mathcal{W}$ is of codimension $d - 1 = 1$ in this restricted setting. Under the assumption that $T'(\bar\mu) \neq 0$, the frequency ω_2 is a strictly monotone function of μ near $\bar\mu$. And the curve $\omega_2 \mapsto K_c^\omega$ crosses $\mathcal{W}$ transversally. Now it suffices to note that transversality persists under small perturbations, which includes varying ϵ near zero.

This concludes our proof of Theorem 1.2.

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