OPERATORS WITH SINGULAR CONTINUOUS SPECTRUM, IV. HAUSDORFF DIMENSIONS, RANK ONE PERTURBATIONS, AND LOCALIZATION

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§1. Introduction

Although concrete operators with singular continuous spectrum have proliferated recently [7,11,13,17,34,35,37,39], we still don't really understand much about singular continuous spectrum. In part, this is because it is normally defined by what it isn't — neither pure point nor absolutely continuous. An important point of view, going back in part to Rodgers and Taylor [27,28], and studied recently within spectral theory by Last [22] (also see references therein), is the idea of using Hausdorff measures and dimensions to classify measures. Our main goal in this paper is to look at the singular spectrum produced by rank one perturbations (and discussed in [7,11,33]) from this point of view.

A Borel measure μ is said to have exact dimension $\alpha \in [0,1]$ if and only if $\mu(S)=0$ if S has dimension $\beta<\alpha$ and if μ is supported by a set of dimension α. If $0<\alpha<1$, such a measure is, of necessity, singular continuous. But, there are also singular continuous measures of exact dimension 0 and 1 which are "particularly close" to point and a.c. measures, respectively. Indeed, as we'll explain, we know of "explicit" Schrödinger operators with exact dimension 0 and 1, but, while they presumably exist, we don't know of any with dimension $\alpha \in (0,1)$.

While we're interested in the abstract theory of rank one perturbations, we're especially interested in those rank one perturbations obtained by taking a random Jacobi matrix and making a Baire generic perturbation of the potential at a single point. It is a disturbing fact that the strict localization (dense point spectrum with $\|xe^{-itH}\delta_0\|^2 = (e^{-itH}\delta_0, x^2e^{-itH}\delta_0)$ bounded in t), that holds a.e. for the random case, can be destroyed by arbitrarily small local perturbations [7,11]. We'll ameliorate this discovery in the present paper in three ways: First, we'll see that, in this case, the spectrum is always of dimension zero, albeit sometimes pure point and sometimes singular continuous. Second, we'll show that not only does the set of couplings with singular continuous spectrum has Lebesgue measure zero, it

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has Hausdorff dimension zero. Third, we'll also see that while $\|xe^{-itH}\delta_0\|$ may be unbounded after the local perturbation, it never grows faster than $C\ln(t)$.

Appendix 2 contains an example of a Jacobi matrix which sheds light on the proper definition of localization: It has a complete set of exponentially decaying eigenfunctions, but, nevertheless, $\overline{\lim\limits_{t\to\infty}}\, \|xe^{itH}\delta_0\|^2/t^\alpha = \infty$ for any $\alpha < 2$. Section 7 discusses further the connection between eigenfunction localization and transport.

In Section 2, we'll review some basic facts about Hausdorff measures that we'll use later. In Section 3, we relate these to boundary behavior of Borel transforms. In Section 4, we use these ideas to present relations between spectra produced by rank one perturbations and the behavior of the spectral measure of the unperturbed operator. In Section 5, we'll relate Hausdorff dimensions of some energy sets to the dimensions of some coupling constant sets. In Section 6, we use the results of Sections 4 and 5 to present examples (some related to those in [40]) that show that the Hausdorff dimension under perturbation can be anything.

In Section 7, we turn to systems with exponentially localized eigenfunctions, and show that under local perturbations the spectrum remains of Hausdorff dimension zero. Some of the lemmas in this section on the nature of localization are of independent interest. Finally, in Section 8, we prove that "physical" localization is "almost stable," that is, suitable decay of $(\delta_n, e^{-itH}\delta_m)$ in $|n-m|$ uniform in t implies that $\|x\exp(-it(H+\lambda\delta_0))\delta_0\|$ grows at worst logarithmically.

Appendix 1 provides a proof of a variant of a theorem of Aizenman relating Green's function estimates to dynamics and Appendix 2 is an example with interesting pathologies. Appendix 3 shows that our notion of "semi-uniform" localization introduced in Section 7 cannot be replaced by uniform localization for the Anderson model. Appendix 4 extends a lemma of Howland to allow consideration of dimension and Appendix 5 provides the technical details of one class of examples in Section 6.

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§2. Hausdorff Measures and Spectra

Given a Borel set S in $\mathbb{R}$ and $\alpha \in [0,1]$, we define

$$Q_{\alpha,\delta}(S) = \inf\left\{\sum_{\nu=1}^\infty |b_\nu|^\alpha \,\middle|\, |b_\nu| < \delta;\ S \subset \bigcup_{\nu=1}^\infty b_\nu\right\},$$

the inf over all δ-covers by intervals b_ν of size at most δ. Obviously, as δ decreases, Q increases since the set of covers becomes fewer, and

$$h^\alpha(S) = \lim_{\delta\downarrow 0} Q_{\alpha,\delta}(S)$$

is called *α-dimensional Hausdorff measure*. It is a non-sigma-finite measure on the Borel sets. Note that h^0 coincides with the counting measure (i.e., assigns to each set the number of points in it), and h^1 coincides with Lebesgue measure. Clearly, if $\beta < \alpha < \gamma$,

$$\delta^{\alpha-\gamma} Q_{\gamma,\delta}(S) \leq Q_{\alpha,\delta}(S) \leq \delta^{\alpha-\beta} Q_{\beta,\delta}(S),$$

so if $h^\alpha(S) < \infty$, then $h^\gamma(S) = 0$ for $\gamma > \alpha$ and if $h^\alpha(S) > 0$, then $h^\beta(S) = \infty$ for $\beta < \alpha$. Thus, for any S, there is a unique α_0, called its *Hausdorff dimension*, $\dim(S)$, so $h^\alpha(S) = 0$ if $\alpha > \alpha_0$ and $h^\alpha(S) = \infty$ if $\alpha < \alpha_0$. $h^{\alpha_0}(S)$ can be zero, finite, infinite, or so infinite S isn't even h^{α_0}-sigma-finite.

In what follows, we shall use Hausdorff measures and dimensions to classify measures. Unless pointed otherwise, by "a measure" (equivalently, "a measure on $\mathbb{R}$"; usually denoted by μ) we mean a positive sigma-finite Borel measure on $\mathbb{R}$. Note, however, that some parts of the paper only discuss more restricted classes of measures, such as finite measures.

Definition. A measure μ on $\mathbb{R}$ is said to be of exact dimension α for $\alpha \in [0,1]$ if and only if

(1) For any $\beta \in [0,1]$ with $\beta < \alpha$ and S a set of dimension β, $\mu(S) = 0$.
(2) There is a set S_0 of dimension α which supports μ in the sense that $\mu(\mathbb{R} \backslash S_0) = 0$.

Remarks. 1. One might think that the proper condition (2) is that for any $\beta > \alpha$, there is a set S_β of dimension β so $\mu(\mathbb{R} \backslash S_\beta) = 0$. But if so, then $S_0 \equiv \bigcap_{n=1}^{\infty} S_{\alpha+1/n}$ is of dimension α and supports μ.

2. Of special interest are the end points $\alpha = 0$ where only (2) is required, and $\alpha = 1$ where only (1) is required. Obviously, $\alpha = 0$ includes point measures and $\alpha = 1$ includes a.c. measures.

3. The definition is due to Rodgers-Taylor [27].

Not every measure is of some exact dimension; indeed, the sum of measures of exact distinct dimensions is not of any exact dimension. But in this paper, most of our examples will involve measures of some exact dimension. Last [22], following Rodgers-Taylor [27,28], discusses many different decompositions of any measure into a part of dimension less than α, equal to α, and larger than α. The piece of exact dimension α can be further decomposed in terms of its relation to h^α.

Definition. Given any measure μ and any $\alpha \geq 0$, we define

$$D_\mu^\alpha(x) = \overline{\lim_{\delta \downarrow 0}} \frac{\mu(x-\delta, x+\delta)}{\delta^\alpha}. \tag{2.1}$$

Note that if $D_\mu^{\alpha_0}(x_0) < \infty$ for some x, then $D_\mu^\beta(x_0) = 0$ for all $\beta < \alpha_0$ and if $D_\mu^{\alpha_0}(x_0) > 0$ for some x_0, then $D_\mu^\beta(x_0) = \infty$ for all $\beta > \alpha_0$. In particular, for each x_0, there is an $\alpha(x_0)$ so $D_\mu^\alpha(x_0) = 0$ if $\alpha < \alpha(x_0)$ and $= \infty$ if $\alpha > \alpha(x_0)$. Indeed,

$$\alpha(x_0) = \underline{\lim_{\delta \downarrow 0}} \frac{\ln \mu(x_0 - \delta, x_0 + \delta)}{\ln \delta}. \tag{2.2}$$

We'll sometimes write $\alpha_\mu(x_0)$ if we want to be explicit about the μ involved; and if we have a one-parameter family μ_λ, we'll use α_λ for α_{μ_λ}.

The following is a result of Rodgers-Taylor [27,28] (also see [26]):

Theorem 2.1. *Let μ be any measure and $\alpha \in [0,1]$. Let $T_\alpha = \{x \mid D_\mu^\alpha(x) = \infty\}$ and let χ_α be its characteristic function. Let $d\mu_{\alpha s} = \chi_\alpha \, d\mu$ and $d\mu_{\alpha c} = (1-\chi_\alpha)\, d\mu$. Then $d\mu_{\alpha s}$ is singular with respect to h^α (i.e., supported on a set of h^α-measure zero) and $d\mu_{\alpha c}$ is continuous with respect to h^α (i.e., gives zero weight to any set of h^α-measure zero).*

Remark. The following is also true: $\mu \restriction \{x \mid D_\mu^\alpha(x) > 0\}$ is supported on an h^α-sigma finite set, and $\mu \restriction \{x \mid D_\mu^\alpha(x) = 0\}$ gives zero weight to h^α-sigma-finite sets. Moreover, $\mu \restriction \{x \mid 0 < D_\mu^\alpha(x) < \infty\}$ is absolutely continuous with respect to h^α, in the sense that it is given by $f(x)\, dh^\alpha(x)$ for some $f \in L^1(\mathbb{R}, dh^\alpha)$.

Corollary 2.2. *A measure μ is of exact dimension $\alpha_0 \in [0,1]$ if and only if*

(1) *For any $\beta > \alpha_0$, $D_\mu^\beta(x) = \infty$ a.e. x w.r.t. μ.*
(2) *For any $\beta < \alpha_0$, $D_\mu^\beta(x) = 0$ a.e. x w.r.t. μ.*

(Equivalently, if $\alpha(x) = \alpha_0$ a.e. x w.r.t. μ). More generally, if (1) holds (equivalently, $\alpha(x) \le \alpha_0$ a.e. w.r.t. μ), then μ is supported on a set of dimension α and if (2) holds (equivalently, $\alpha(x) \ge \alpha_0$ a.e. w.r.t. μ), then μ gives zero weight to any set S of dimension $\beta < \alpha_0$.

Corollary 2.3. *Let μ be a measure on $\mathbb{R}$, let $S \subset \mathbb{R}$ be a Borel set with $\mu(S) > 0$, and suppose that $\alpha_0 \in [0,1]$ and*

$$D_\mu^{\alpha_0}(x) < \infty$$

for μ-a.e. x in S. Then $\dim(S) \ge \alpha_0$.

Remark. In fact, $h^{\alpha_0}(S) > 0$.

Proof. $\alpha_0 = 0$ is trivial, so suppose $\alpha_0 > 0$. Let ν be the measure $\mu(S \cap \cdot)$. Then, since $\nu \le \mu$, the hypothesis implies that

$$D_\nu^{\alpha_0}(x) < \infty$$

for a.e. x w.r.t. ν. Thus, by Theorem 2.1, ν gives zero weight to sets of h^{α_0}-measure zero, and so, since $\nu(S) \neq 0$, we must have $h^{\alpha_0}(S) > 0$, which implies $\dim(S) \ge \alpha_0$. $\square$

It is often easier to deal with power integrals, so we note:

Proposition 2.4. *Let μ be a finite measure, and let $\tilde{G}_\alpha(x_0) = \int \frac{d\mu(y)}{|x_0 - y|^\alpha}$. Then*

(i) *$\tilde{G}_\alpha(x_0) < \infty$ implies $D_\mu^\alpha(x_0) < \infty$.*
(ii) *$D_\mu^\alpha(x_0) < \infty$ implies $\tilde{G}_\beta(x_0) < \infty$ for any $0 \le \beta < \alpha$.*

Proof. (i) Looking at the contribution to the integral of the set where $|x_0 - y| < \delta$, we see that

$$\mu(x_0 - \delta, x_0 + \delta) \le \delta^\alpha \tilde{G}_\alpha(x_0)$$

so

$$D_\mu^\alpha(x_0) \le \tilde{G}_\alpha(x_0).$$

(ii) Let $M_\mu^\delta(x_0) = \mu(x_0 - \delta, x_0 + \delta)$. Then (with $\lambda =$ Lebesgue measure)

$$\begin{aligned}\tilde{G}_\beta(x_0) &= (\mu \otimes \lambda)((y,t) \mid 0 \leq t \leq |x_0 - y|^{-\beta}) \\ &= \int_0^\infty M_\mu^{t^{-1/\beta}}(x_0)\, dt \\ &= \beta \int_0^\infty M_\mu^\delta(x_0) \delta^{-\beta-1}\, d\delta.\end{aligned}$$

The integral always converges for δ large since M_μ^δ is bounded; and if $\beta < \alpha$, and $D_\mu^\alpha(x_0) < \infty$, then it converges for small δ. $\square$

Consider the set

$$W_\alpha = \left\{ x \,\middle|\, \overline{\lim_{\delta \downarrow 0}} \frac{\mu(x-\delta, x+\delta)}{\delta^\alpha} \neq \underline{\lim_{\delta \downarrow 0}} \frac{\mu(x-\delta, x+\delta)}{\delta^\alpha} \right\}. \tag{2.3}$$

For $\alpha = 0$, W_α is empty; and for $\alpha = 1$, the theorem of de la Vallée-Poussin (see [30] or Theorem 7.15 of [29]) says that $\mu(W_1) = 0$. For $0 < \alpha < 1$, however, the situation is quite different: A result going back to Besicovitch [5] (also see Theorem 5.2 of [10]) is that if μ is the restriction of h^α to a set of finite positive h^α-measure, then μ is supported on W_α. Moreover, there are even examples of μ's where for a.e. x w.r.t. μ,

$$\overline{\lim_{\delta \downarrow 0}} \frac{\ln \mu(x-\delta, x+\delta)}{\ln(\delta)} = 1 \quad \text{and} \quad \underline{\lim_{\delta \downarrow 0}} \frac{\ln \mu(x-\delta, x+\delta)}{\ln(\delta)} = 0.$$

Appendix 5 in this paper has such examples.

§3. Borel Transforms and Hausdorff Spectra

Given a measure μ with $\int (|x|+1)^{-1}\, d\mu(x) < \infty$, we define its Borel transform by

$$F_\mu(z) = \int \frac{d\mu(x)}{x - z}$$

for $\operatorname{Im} z > 0$. These play a crucial role in the theory of rank one perturbations as originally noticed by Aronszajn-Donoghue [3,9]; see [33] for their properties and this theory. In this section, we'll translate Theorem 2.1 into Borel transform language.

Definition. Fix $\gamma \leq 1$ and x. Let

$$\begin{aligned}Q_\mu^\gamma(x) &= \overline{\lim_{\epsilon \downarrow 0}}\, \epsilon^\gamma \operatorname{Im} F_\mu(x + i\epsilon) \\ R_\mu^\gamma(x) &= \overline{\lim_{\epsilon \downarrow 0}}\, \epsilon^\gamma\, |F_\mu(x + i\epsilon)|.\end{aligned}$$

Our goal in this section is to prove:

Theorem 3.1. *Fix μ and x_0. Fix $\alpha \in [0,1)$ and let $\gamma = 1-\alpha$. Then $D_\mu^\alpha(x_0)$, $Q_\mu^\gamma(x_0)$, and $R_\mu^\gamma(x_0)$ are either all infinite, all zero, or all in $(0,\infty)$.*

Remarks. 1. In particular, $Q_\mu^\gamma(x_0) = R_\mu^\gamma(x_0) = \infty$ if $\gamma < 1-\alpha_\mu(x_0)$ and $Q_\mu^\gamma(x_0) = R_\mu^\gamma(x_0) = 0$ if $\gamma > 1-\alpha_\mu(x_0)$ for any $\alpha_\mu(x_0) \in [0,1]$.

2. In particular,

$$\overline{\lim_{\epsilon\downarrow 0}} \ln(\operatorname{Im} F_\mu(x+i\epsilon)) / \ln(\epsilon^{-1}) = \overline{\lim_{\epsilon\downarrow 0}} \ln|F_\mu(x+i\epsilon)| / \ln(\epsilon^{-1}) = 1-\alpha_\mu(x).$$

so long as $\alpha_\mu(x) \le 1$.

3. The relation between $D_\mu^\alpha(x_0)$ and $Q_\mu^\gamma(x_0)$ also extends to the range $1 \le \alpha < 2$. This follows from Lemma 3.2 below along with Lemma 5.4 of Section 5.

4. J. Bellissard informed us that he, R. Mosseri, and J. Zhong also have related results.

Lemma 3.2. *For any $\gamma \le 1$,*

$$D_\mu^{1-\gamma}(x_0) \le 2Q_\mu^\gamma(x_0) \le 2R_\mu^\gamma(x_0).$$

Proof. Let $M_\mu^\delta(x_0) = \mu(x_0-\delta, x_0+\delta)$. Then looking at the contribution of $(x_0-\epsilon, x_0+\epsilon)$ to $\operatorname{Im} F_\mu(x_0+i\epsilon)$, we see that

$$\operatorname{Im} F_\mu(x_0+i\epsilon) = \epsilon \int_{-\infty}^{\infty} \frac{d\mu(y)}{(y-x_0)^2+\epsilon^2} \ge \frac{1}{2\epsilon} M_\mu^\epsilon(x_0), \tag{3.1}$$

so

$$\epsilon^\gamma \operatorname{Im} F_\mu(x_0+i\epsilon) \ge \frac{1}{2} \frac{1}{\epsilon^{1-\gamma}} M_\mu^\epsilon(x_0),$$

so the first inequality in the lemma holds. $Q_\mu^\gamma(x_0) \le R_\mu^\gamma(x_0)$ is, of course, trivial. $\square$

Lemma 3.3. *Let $\alpha < 1$. If $D_\mu^\alpha(x_0) < \infty$ (resp. $= 0$), $R_\mu^{1-\alpha}(x_0) < \infty$ (resp. $= 0$).*

Proof. Suppose first that $D_\mu^\alpha(x_0) < \infty$. Let $M_\mu^\delta(x_0) = \mu(x_0-\delta, x_0+\delta)$. The case $\alpha = 0$ is trivial so we'll suppose $\alpha > 0$. By hypothesis,

$$M_\mu^\delta(x_0) \le C\delta^\alpha, \tag{3.2}$$

so with $\gamma = 1 - \alpha$:

$$
\begin{aligned}
\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon^\gamma \, |F_\mu(x_0 + i\epsilon)| &\le \overline{\lim_{\epsilon\downarrow 0}}\, \epsilon^\gamma \int_{-\infty}^{\infty} \frac{d\mu(y)}{[(x_0 - y)^2 + \epsilon^2]^{1/2}} \\
&= \overline{\lim_{\epsilon\downarrow 0}}\, \epsilon^\gamma \int_0^1 \frac{1}{(\epsilon^2 + \delta^2)^{1/2}} \, [d_\delta M_\mu^\delta(x_0)] \\
&= \overline{\lim_{\epsilon\downarrow 0}}\, \epsilon^\gamma \int_0^1 \frac{\delta}{(\epsilon^2 + \delta^2)^{3/2}} \, M_\mu^\delta(x_0)\, d\delta \\
&\le \lim_{\epsilon\downarrow 0} C\epsilon^\gamma \int_0^1 \frac{\delta^{\alpha+1}}{(\epsilon^2 + \delta^2)^{3/2}} \, d\delta \\
&= \lim_{\epsilon\downarrow 0} C \int_0^{\epsilon^{-1}} \frac{\delta^{\alpha+1}}{(\delta^2 + 1)^{3/2}} \, d\delta \\
&< \infty.
\end{aligned}
$$

The first equality comes from noting that since $\gamma > 0$,

$$
\lim_{\epsilon\downarrow 0} \epsilon^\gamma \int_{|y - x_0| > 1} d\mu(y)/|x_0 - y - i\epsilon| = 0.
$$

The second equality is an integration by parts. The boundary term at zero vanishes since $\alpha > 0$. The term at 1 has a zero limit since $\gamma > 0$. The final equality comes by noting that since $\alpha < 1$, the integral is finite as $\epsilon^{-1} \to \infty$.

If $D_\mu^\alpha(x_0) = 0$, then (3.2) holds for $\delta \le \delta_0$ where C can be taken arbitrarily small (by taking δ_0 small). The above calculation (with 1 as the upper integrand replaced by δ_0) shows that

$$
R_\mu^{1-\alpha}(x_0) \le C \int_0^\infty \frac{\delta^{\alpha+1}}{(\delta^2 + 1)^{3/2}} \, d\delta.
$$

Since C is arbitrarily small, R is zero. $\square$

Proof. Theorem 3.1 is a direct consequence of Lemmas 3.2 and 3.3. $\square$

Corollary 3.4. *Let $\gamma \in [0, 1]$. Let $S \subset \mathbb{R}$ be a Borel set with $\mu(S) > 0$. Suppose $Q_\mu^\gamma(x) < \infty$ for μ-a.e. $x \in S$. Then,* $\dim(S) \ge 1 - \gamma$.

Remark. In fact, $h^{1-\gamma}(S) > 0$.

Proof. An immediate consequence of Corollary 2.3 and Lemma 3.2. $\square$

The following criterion won't be used in this paper but is an interesting result on its own.

Theorem 3.5. *Suppose that*

$$\sup_{\epsilon>0} \epsilon^s \int_a^b |\operatorname{Im} F_\mu(x+i\epsilon)|^2\, dx < \infty$$

for some $s<1$. Then $\mu \restriction (a,b)$ gives zero weight to sets of dimension less than $1-s$.

Remark. The $s=0$ result is stronger [36]; in that case μ is purely absolutely continuous on (a,b).

Proof. We'll prove that for any $\beta < 1-s$ and any closed interval $I \subset (a,b)$, we have

$$\int_{\substack{x\in I\\ y\in I}} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\beta} < \infty. \tag{3.3}$$

This implies $\tilde{G}_\beta(x) = \int \frac{d\mu(y)}{|x-y|^\beta} < \infty$ for μ-a.e. $x\in I$, and the theorem thus follows from Proposition 2.4 and Corollary 2.3.

Replacing μ by $\mu \restriction I$ and noting that $\operatorname{Im}(\int_{x\in I} \frac{d\mu(x)}{x-z}) \leq \operatorname{Im} F_\mu(z)$, we can suppose μ is supported in I. Since $I\subset (a,b)$ and $|\operatorname{Im} F_{\mu\restriction I}(z)| \leq \frac{C|\operatorname{Im} z|}{\operatorname{dist}(z,I)^2}$, we can suppose that

$$\sup_{\epsilon>0} \epsilon^s \int_{-\infty}^{\infty} |\operatorname{Im} F_\mu(x+i\epsilon)|^2\, dx < \infty. \tag{3.4}$$

By a straightforward calculation,

$$\int_{-\infty}^{\infty} |\operatorname{Im} F_\mu(x+i\epsilon)|^2\, dx = 2\pi\epsilon \int_{\substack{x\in I\\ y\in I}} \frac{d\mu(x)\, d\mu(y)}{(x-y)^2+4\epsilon^2}$$

so (3.4) says that

$$\int_{\substack{x\in I\\ y\in I}} \frac{d\mu(x)\, d\mu(y)}{(x-y)^2+\epsilon^2} \leq C\epsilon^{-1-s}\,. \tag{3.5}$$

Let

$$M_\mu^{(2)}(\delta) = \int_{\substack{|x-y|<\delta\\ x\in I\\ y\in I}} d\mu(x)\, d\mu(y).$$

Then (3.5) with $\epsilon=\delta$ says that

$$M_\mu^{(2)}(\delta) \leq 2C\delta^{1-s}.$$

Thus, if $\beta < 1 - s$,

$$\int\limits_{\substack{|x-y|\le 1\\ x\in I\\ y\in I}} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\beta} \le \sum_{n=0}^{\infty} M_\mu^{(2)}(2^{-n})2^{(n+1)\beta} < \infty$$

and (3.3) is proven. $\square$

§4. Rank One Perturbations: A General Criterion

Let μ be a normalized finite measure. Let A be the operator of multiplication by x on $L^2(\mathbb{R}, d\mu)$. Let φ be the unit vector $\varphi(x) \equiv 1$. Let $A_\lambda = A + \lambda(\varphi, \cdot\,)\varphi$, and let $d\mu_\lambda$ be the spectral measure for φ and the operator A_λ. Let $F_\lambda(z) = \int \frac{d\mu_\lambda(x)}{x-z}$, and denote $F(z)$ for $F_0(z)$. Then [33]

$$F_\lambda(z) = \frac{F(z)}{1+\lambda F(z)} \tag{4.1}$$

$$\operatorname{Im} F_\lambda(z) = \frac{\operatorname{Im} F(z)}{|1+\lambda F(z)|^2} \tag{4.2}$$

$$d\mu_\lambda(x) = \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im} F_\lambda(x+i\epsilon)\, dx \tag{4.3}$$

$$\mu_{\lambda,\text{sing}} \text{ is supported by } \{x \mid F(x+i0) = -\tfrac{1}{\lambda}\}. \tag{4.4}$$

Theorem 4.1. *Let $\alpha \in [0,1]$. Let $S_\alpha = \{x \mid \underline{\lim}\ \epsilon^{-(1-\alpha)} \operatorname{Im} F(x+i\epsilon) > 0\}$. If $\mu_\lambda([a,b]\backslash S_\alpha)$ $= 0$ for some $\lambda \neq 0$, then μ_λ gives zero weight to any subset of $[a,b]$ of dimension $\beta < \alpha$.*

Remarks. 1. The proof actually shows that $\mu_\lambda \restriction S_\alpha$ is continuous w.r.t. h^α (i.e., gives zero weight to sets of zero h^α-measure).

2. By a simple variant of the proof below and the remark to Theorem 2.1, one can also show that if $\check{S}_\alpha = \{x \mid \underline{\lim}\ \epsilon^{-(1-\alpha)} \operatorname{Im} F(x+i\epsilon) = \infty\}$, then $\mu_\lambda \restriction \check{S}_\alpha$ gives zero weight to h^α-sigma-finite sets.

Theorem 4.2. *Let $0 \le \alpha < 1$. Suppose μ is purely singular. Let $\widehat{S}_\alpha = \{x \mid \overline{\lim}\ \epsilon^{-(1-\alpha)}$ $\operatorname{Im} F(x+i\epsilon) < \infty\}$. If $\mu_\lambda(\mathbb{R}\backslash\widehat{S}_\alpha) = 0$ for some $\lambda \neq 0$, then μ_λ is supported on a set of dimension α.*

Remarks. 1. By the remark to Theorem 2.1, the proof below actually shows that $\mu_\lambda \restriction \widehat{S}_\alpha$ is supported on an h^α-sigma-finite set.

2. By a simple variant of the proof below, one can also show that if $\tilde{S}_\alpha = \{x \mid \overline{\lim}\ \epsilon^{-(1-\alpha)}$ $\operatorname{Im} F(x+i\epsilon) = 0\}$, then $\mu_\lambda \restriction \tilde{S}_\alpha$ is singular w.r.t. h^α (i.e., supported on a set of zero h^α-measure).

Proof of Theorem 4.1. Suppose $\underline{\lim}\ \epsilon^{-(1-\alpha)} \operatorname{Im} F(x_0 + i\epsilon) > 0$ (i.e., $x_0 \in S_\alpha$). By (4.2),

$$\operatorname{Im} F_\lambda(x_0 + i\epsilon) \le \frac{1}{\lambda^2 \operatorname{Im} F(x_0 + i\epsilon)}$$

so

$$Q^{1-\alpha}_{\mu_\lambda}(x_0) = \overline{\lim}\ \epsilon^{(1-\alpha)} \operatorname{Im} F_\lambda(x_0 + i\epsilon) < \infty.$$

Thus, the result follows from Corollary 3.4. $\square$

Proof of Theorem 4.2. Suppose $\overline{\lim}\ \epsilon^{-(1-\alpha)} \operatorname{Im} F(x_0 + i\epsilon) < \infty$ (i.e., $x_0 \in \widehat{S}_\alpha$) and that $F(x_0 + i0) = -\frac{1}{\lambda}$. By (3.1),

$$M^\epsilon_\mu(x_0) \le C\epsilon^{2-\alpha}$$

and

$$\begin{aligned}
|1 + \lambda \operatorname{Re} F(x_0 + i\epsilon)| &= |\lambda|\, |\operatorname{Re} F(x_0 + i\epsilon) - \operatorname{Re} F(x_0 + i0)| \\
&= |\lambda| \left| \int \left[\frac{(y - x_0)}{(y - x_0)^2} - \frac{(y - x_0)}{(y - x_0)^2 + \epsilon^2} \right] d\mu(y) \right| \\
&= |\lambda| \left| \int \frac{\epsilon^2}{(y - x_0)[(y - x_0)^2 + \epsilon^2]}\, d\mu(y) \right| \\
&\le |\lambda| \int \frac{\epsilon^2}{\delta(\delta^2 + \epsilon^2)}\, [d_\delta M^\delta_\mu(x_0)].
\end{aligned}$$

We can integrate by parts, use the bound on M^ϵ_μ, and integrate by parts again to bound this last integral by

$$|\lambda|(2 - \alpha) \int_0^\infty \frac{\epsilon^2 \delta^{1-\alpha}\, d\delta}{\delta(\delta^2 + \epsilon^2)} = |\lambda|(2 - \alpha)\epsilon^{1-\alpha} \int_0^\infty \frac{dy}{y^\alpha (y^2 + 1)}$$

and note the integrand is finite.

Thus, $|1 + \lambda F(x_0 + i\epsilon)| \le C\epsilon^{1-\alpha}$ and so $\underline{\lim}\ \epsilon^{1-\alpha} |1 + \lambda F(x_0 + i\epsilon)|^{-1} > 0$. Thus, by (4.1), if $x_0 \in \widehat{S}_\alpha \cap \{x \mid F(x_0 + i\epsilon) = -\frac{1}{\lambda}\}$, $\overline{\lim}\ \epsilon^{(1-\alpha)} |F_\lambda(x_0 + i\epsilon)| > 0$. Since μ_λ is supported on $\{x \mid F(x_0 + i\epsilon) = -\frac{1}{\lambda}\}$, if $\mu_\lambda(\mathbb{R} \backslash \widehat{S}_\alpha) = 0$, then by Theorem 3.1, $\alpha_\lambda(x) \le \alpha$ a.e. and so by Corollary 2.2, μ is supported on a set of dimension α. $\square$

§5. Rank One Perturbations: Coupling Constant Dimensions

In addition to the functions $F_\lambda(z), F(z)$ of (4.1), an important role is played by

$$G(x) = \int \frac{d\mu(y)}{(x - y)^2} \tag{5.1}$$

in that

$$\{x \mid G(x) < \infty,\ F(x + i0) = -\lambda^{-1}\} = \text{set of eigenvalues of } A_\lambda. \tag{5.2}$$

Note that $G(x) = \lim \epsilon^{-1} \mathrm{Im}\, F(x+i\epsilon)$, so (5.2) follows from (4.4) and the $\alpha = 0$ case of the second remark to Theorem 4.1 and the first remark to Theorem 4.2. Moreover, if $\lambda < \infty$ (see [33]):

$$d\mu_\lambda^{\mathrm{pp}}(y) = \sum_{\{x \mid G(x)<\infty, F(x+i0)=-\lambda^{-1}\}} \frac{1}{\lambda^2 G(x)}\, d\delta_x(y). \tag{5.3}$$

Note that $G(x) < \infty$ implies $F(x+i\epsilon)$ has a real limit so

$$M = \{x \mid G(x) < \infty\} = \bigcup_{0<|\lambda|\le\infty} \{\text{eigenvalues of } A_\lambda\}.$$

In [7] del Rio, Makarov, and Simon prove that

$$M = \bigcup_{n=1}^{\infty} M_n$$

where M_n is such that there exists C_n with

$$C_n^{-1}(x-y) \le F(x+i0) - F(y+i0) \le C_n(x-y) \tag{5.4}$$

for all $x < y$ both in M_n. Let $L_n = \{\lambda \mid -\lambda^{-1} \in F[M_n]\}$. It follows from (5.4) that $\dim(M_n) = \dim(L_n)$. Thus, since $\dim(\overset{\infty}{\underset{n=1}{\cup}} A_n) = \sup \dim(A_n)$, we see that

Theorem 5.1. *Fix a Borel set I. Then the Hausdorff dimension of the set of λ's where A_λ has some eigenvalues in I is the same as the Hausdorff dimension of the set of $x \in I$ where $G(x) < \infty$.*

Remarks. 1. (5.4) actually implies the following stronger result: If, for some $\alpha \in [0,1]$, $\{x \mid G(x) < \infty\} \cap I$ has zero h^α-measure, or positive h^α-measure, or is h^α-sigma-finite, or is not h^α-sigma-finite, then the set of (nonzero) λ's where A_λ has some eigenvalues in I has the same property.

2. Examples in the next section show that $\{x \mid G(x) < \infty\}$ can have any dimension and illustrate the difference between some point spectrum and only point spectrum.

There is also a result on the other side:

Theorem 5.2. *Suppose μ is purely singular. Let $S = \{\lambda \mid A_\lambda$ has some continuous spectrum$\}$. Let $T = \{x \mid G(x) = \infty\}$. Then*

$$\dim(S) \le \dim(T).$$

In particular, if T has Hausdorff dimension zero, so does S.

Remarks. 1. The proof actually shows that for any $\alpha \in [0,1]$, $h^\alpha(S) > 0$ implies $h^\alpha(T) > 0$. In particular, this generalizes the known fact [33,40] that if $G(x) < \infty$ a.e. then for a.e. λ, A_λ has only pure point spectrum. Moreover, for $0 \le \alpha < 1$ we get the stronger result: $h^\alpha(S) > 0$ implies that T is not h^α-sigma finite. This shows that the inequality in Theorem 5.2 is, in some sense, strict. Note that for $\alpha = 0$ it becomes the obvious fact: $S \neq \emptyset$ implies T is uncountable.

2. While we have formulated Theorem 5.2 in a global way, the result is actually local. That is, fix a Borel set I and let $S(I) = \{\lambda \mid \mu_\lambda^{\rm sc}(I) > 0\}$, where $\mu_\lambda^{\rm sc}$ is the singular continuous part of μ_λ, then $h^\alpha(S(I)) > 0$ implies $h^\alpha(T \cap I) > 0$, and, in particular, $\dim(S(I)) \leq \dim(T \cap I)$. To prove this, just replace S by $S(I)$ and T_1 by $T_1 \cap I$ in the proof below.

3. Appendix 4 explores the relation between $\dim\{x \mid G(x) = \infty\}$ and the dimension of supports of μ.

We'll need a lemma that could have many other applications to the theory of rank one perturbations:

Lemma 5.3. *Let η be a finite measure on $\mathbb{R}$ and define a measure ν on $\mathbb{R}$ by*

$$\nu(A) = \int \mu_\lambda(A)\, d\eta(\lambda). \tag{5.4}$$

Let $F_\kappa(z) = \int d\kappa(x)/x - z$ be the Borel transform of the measure κ. Then

$$F_\nu(z) = F_\eta(-1/F_\mu(z)). \tag{5.5}$$

Proof. By the definition (5.4):

$$F_\nu(z) = \int d\eta(\lambda)\, F_{\mu_\lambda}(z).$$

Equation (4.1) implies the result. $\square$

We also need the following lemma:

Lemma 5.4. *Let $0 \leq \alpha < 2$ and let μ be a measure obeying $\mu(x-\delta, x+\delta) \leq C\delta^\alpha$ for some C and x and all $\delta > 0$. Then there exists C_1 so that $\operatorname{Im} F_\mu(x+i\epsilon) \leq C_1\epsilon^{-(1-\alpha)}$ for all $\epsilon > 0$. Moreover, if $\mu(x-\delta, x+\delta) \leq C\delta^\alpha$ holds for some fixed C and all x and $\delta > 0$, then there exists C_1 so that $\operatorname{Im} F_\mu(x+i\epsilon) \leq C_1\epsilon^{-(1-\alpha)}$ for all x and $\epsilon > 0$.*

Proof.

$$\begin{aligned}
\operatorname{Im} F_\mu(x+i\epsilon) &= \int \frac{\epsilon\, d\mu(y)}{(x-y)^2+\epsilon^2} \\
&= \int\limits_{|x-y|<\epsilon} \frac{\epsilon\, d\mu(y)}{(x-y)^2+\epsilon^2} + \sum_{n=0}^\infty \int\limits_{2^n\epsilon \leq |x-y| < 2^{n+1}\epsilon} \frac{\epsilon\, d\mu(y)}{(x-y)^2+\epsilon^2} \\
&\leq \frac{C\epsilon^\alpha}{\epsilon} + \sum_{n=0}^\infty \frac{\epsilon C(2^{n+1}\epsilon)^\alpha}{(2^n\epsilon)^2+\epsilon^2} \\
&\leq \frac{C\epsilon^\alpha}{\epsilon}\left(1 + 2^\alpha \sum_{n=0}^\infty 2^{n(\alpha-2)}\right)
\end{aligned}$$

so we see that the claim holds. □

Proof of Theorem 5.2. The $\alpha = 0$ case is trivial, so suppose $0 < \alpha \le 1$ and $h^\alpha(S) > 0$. Let $T_1 = \{x \mid G(x) = \infty, \lim_{\epsilon\downarrow 0} F(x+i\epsilon) \text{ exists and is finite and nonzero}\}$. We'll show $h^\alpha(T_1) > 0$, so we can conclude that $h^\alpha(T) > 0$. For each $\lambda \in S_1 \equiv S\backslash\{0, \pm\infty\}$, $\mu_\lambda^{\mathrm{sc}}$ is supported on T_1 so $\mu_\lambda(T_1) > 0$. Since $h^\alpha(S_1) > 0$, it is well known ([10], Proposition 4.11 and Corollary 4.12) that we can find a measure η so that η is supported by S_1, $\eta(S_1) > 0$, and

$$\eta(x-\delta, x+\delta) \le C\delta^\alpha \tag{5.6}$$

for all x and $\delta > 0$. Let ν be given by (5.4). Then $\nu(T_1) > 0$.

By (5.6) and Lemma 5.4 there exists C_1 so that

$$\operatorname{Im} F_\eta(x+i\epsilon) \le C_1 \epsilon^{-(1-\alpha)}$$

for all x and $\epsilon > 0$. It follows from (5.5) that for $x \in T_1$,

$$\overline{\lim_{\epsilon\downarrow 0}}\, \epsilon^{(1-\alpha)} \operatorname{Im} F_\nu(x+i\epsilon) \le C_1 \overline{\lim_{\epsilon\downarrow 0}}\, \epsilon^{(1-\alpha)} [\operatorname{Im}(-1/F_\mu(x+i\epsilon))]^{-(1-\alpha)}. \tag{5.7}$$

Since $G(x) = \infty$, we have

$$\lim_{\epsilon\downarrow 0} \frac{\operatorname{Im} F_\mu(x+i\epsilon)}{\epsilon} = G(x) = \infty$$

and since $\pm\infty \notin S_1$, $F_\mu(x+i\epsilon) \to -\lambda^{-1} \ne 0$ so $\epsilon\,[\operatorname{Im}(-1/F_\mu(x+i\epsilon))]^{-1} \to 0$. Thus, we see from (5.7) that for all $x \in T_1$,

$$Q_\nu^{1-\alpha}(x) < \infty$$

and if $\alpha < 1$, then $Q_\nu^{1-\alpha}(x) = 0$. Since $\nu(T_1) > 0$, Corollary 3.4 (along with its remark) implies that $h^\alpha(T_1) > 0$. The fact that in the $\alpha < 1$ case T_1 is not h^α-sigma finite follows from Lemma 3.2 and the remark to Theorem 2.1. □

Remark. To apply Proposition 4.11 and Corollary 4.12 of [10], we need that S is a Borel set. This follows, for example, by picking $\{\varphi_n\}_{n=1}^\infty$ an orthonormal basis for $L^2(\mathbb{R}, d\mu)$, letting $F(n \ge N)$ be the projection onto the span of $\{\varphi_n\}_{n=N}^\infty$ and noting that by the RAGE theorem [25]:

$$\mathbb{R}\backslash S = \left\{\lambda \,\middle|\, \forall m \lim_{N\to\infty} \lim_{K\to\infty} \frac{1}{K}\int_0^K \|F(n\ge N)e^{isA_\lambda}\varphi_m\|^2\, ds = 0\right\}.$$

§6. Rank One Perturbations: Some Examples

Rank one perturbations can be described by a measure μ given by

$$(\varphi, (A-z)^{-1}\varphi) = \int \frac{d\mu(x)}{x-z}$$

where $A + \lambda(\varphi, \cdot)\varphi$ is the rank one perturbation, so we'll phrase our examples in this section in terms of $d\mu$. To make things operator theoretic, one can always take $\mathcal{H} = L^2(\mathbb{R}, d\mu)$, $A =$ multiplication by x, and φ the function $\varphi(\lambda) \equiv 1$ (as in the last two sections).

We'll discuss four classes of examples in this section:

(i) Point measures with rapidly decreasing weights for which we'll show that the perturbed spectrum is supported by a set of Hausdorff dimension zero. This class is relevant for our study of localization in the next section.

(ii) Point measures where for a.e. λ, $d\mu_\lambda$ has exact dimension α_0. These are variants of the measures in [40].

(iii) A family of singular continuous measures where one can calculate many distinct dimensions. Details of the calculations are pushed to Appendix 5.

(iv) A set of examples that show $\{x \mid G(x) < \infty\}$ can have any dimension and that have point spectrum embedded in singular continuous spectrum.

Example 1. *Point spectrum with decaying weights*

Given a sequence of sets A_n, we call $A_\infty = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{m=n}^{\infty} A_m$, the $\limsup(A_n)$ consisting of points in infinitely many A_n's.

Lemma 6.1. *Suppose that for a family of intervals A_n, we have for each $j > 0$*

$$|A_n| \leq C_j n^{-j}. \tag{6.1}$$

Then $A_\infty = \limsup(A_n)$ is a set of Hausdorff dimension zero.

Proof. By (6.1), $|A_n| \to 0$ so given δ, choose N_0 so $|A_n| \leq \delta$ for $n \geq N_0$. Then for $m \geq N_0$, $\bigcup\limits_{n=m}^{\infty} A_n$ is a δ-cover of A_∞. Thus,

$$Q_{\alpha,\delta}(A_\infty) \leq C_j^\alpha \sum_{n=m}^{\infty} n^{-j\alpha}.$$

For a fixed $\alpha > 0$, pick j so $j\alpha > 1$. Then the sum is finite and clearly,

$$Q_{\alpha,\delta}(A_\infty) \leq C_j^\alpha \inf_{m \geq N_0} \sum_{n=m}^{\infty} n^{-j\alpha} = 0.$$

Thus, $h^\alpha(A_\infty) = 0$ if $\alpha > 0$ and so A_∞ has dimension zero as claimed. $\square$

Theorem 6.2. *Suppose $d\mu(E) = \sum\limits_{n=1}^{\infty} a_n \, d\delta_{E_n}(E)$ where a_n obeys the condition that for all j, there is a C_j with*

$$|a_n| \leq C_j n^{-j}. \tag{6.2}$$

Then for every λ, $d\mu_\lambda$ is supported on a set of Hausdorff dimension zero. Moreover, $d\mu_\lambda$ is pure point except for a set of λ's of Hausdorff dimension zero.

Remark. Equivalently, let A have a complete orthonormal set of eigenvectors

$$A\psi_n = E_n \psi_n$$

and let $\varphi = \sum\limits_n a_n \psi_n$, where a_n obeys (6.2), and $A_\lambda = A + \lambda(\varphi, \,\cdot\,)\varphi$. Then for *every* λ, the spectral measures of A_λ are all supported on a set of Hausdorff dimension zero. Moreover, A_λ has pure point spectrum except for a set of λ's of Hausdorff dimension zero.

Proof. Let $G(x)$ be defined by (5.1) and let $S = \{x \mid G(x) = \infty, x \notin \{E_i\}_{i=1}^\infty\}$. Then the Aronszajn-Donoghue theory [33] says that for any $\lambda \neq 0$, $d\mu_\lambda^{\rm sc}$, the singular continuous measure for A_λ is supported by S. Thus, the spectral measure $d\mu_\lambda$ is supported by $S \cup \{\text{eigenvalues of } A_\lambda\}$. Since the set of eigenvalues is a zero-dimensional set, it suffices to prove that S is zero-dimensional. The final assertion then follows from Theorem 5.2.

Let $b_n = \sqrt[3]{a_n}$ and let $A_n = [E_n - b_n, E_n + b_n]$. Then

$$|A_n| \leq 2C_j^{1/3} n^{-j/3}$$

for any j, so A_n obeys (6.1). Thus, $A_\infty \equiv \limsup(A_n)$ has dimension zero.

We claim $S \subset A_\infty$. To prove this, we need only show if $x \notin A_\infty$ and $x \notin \{E_i\}_{i=1}^\infty$, then $G(x) < \infty$. But if $x \notin A_\infty$, then for some N_0, $x \notin \bigcup\limits_{n=N_0}^\infty A_n$ so

$$\sum_{n=N_0}^\infty \frac{a_n}{|x - E_n|^2} \leq \sum_{n=N_0}^\infty \frac{a_n}{b_n^2} = \sum_{n=N_0}^\infty a_n^{1/3} < \infty$$

by (6.2). Since $x \notin \{E_i\}_{i=1}^\infty$, $\sum\limits_{n=1}^{N_0 - 1} \frac{a_n}{|x - E_n|^2} < \infty$ so $G(x) < \infty$ as required. □

Remarks. 1. In the next section, we'll apply this result to random Hamiltonians.

2. One natural way that (6.2) can hold is if $|a_n| \leq Ce^{-\epsilon |n|}$ for some $\epsilon > 0$.

Example 2. *Perturbed measures of prescribed exact dimension*

Our second class of examples is intended to show that it can happen that for any $\alpha_0 \in [0,1]$, there is a rank one perturbation situation where $\mu_\lambda \restriction [0,1]$ is a measure of exact dimension α_0 for a.e. λ (w.r.t. Lebesgue measure). All our unperturbed measures in this example will live on $[0,1]$ and be point measures. The third set of examples will be similar but the unperturbed measures will be continuous. For each $n = 0, 1, 2, \dots$ let

$$d\mu_n = \frac{1}{2^n} \sum_{j=0}^{2^n} d\delta_{j/2^n}, \tag{6.3a}$$

and for $\alpha \in (0,1)$ define

$$d\nu_\alpha = \sum_{n=0}^\infty 2^{-n(1-\alpha)} d\mu_n. \tag{6.3b}$$

For any $x_0 \in [0,1]$ and n, there is $\frac{j}{2^n}$ within 2^{-n-1} of x_0, so $\nu_\alpha([x_0 - \frac{1}{2^{n+1}}, x_0 + \frac{1}{2^{n+1}}]) \geq 2^{-n(2-\alpha)}$. Thus for any $\epsilon < 1$, $\nu_\alpha(x_0 - \epsilon, x_0 + \epsilon) \geq \epsilon^{2-\alpha}$ so by (3.1), for $x_0 \in [0,1]$ and $0 < \epsilon$, $\operatorname{Im} F_{\nu_\alpha}(x_0 + i\epsilon) \geq \frac{1}{2}\epsilon^{1-\alpha}$. So the set S_α of Theorem 4.1 is all of $[0,1]$, and so (by Theorem 4.1):

Theorem 6.3. *Fix $0 < \alpha < 1$. Let $d\nu_\alpha$ be the measure (6.3) and let $d\nu_{\alpha;\lambda}$ be its rank one perturbations. Then for any $\lambda \neq 0$, $d\nu_{\alpha;\lambda}$ gives zero weight to any $S \subset [0,1]$ of dimension $\beta < \alpha$.*

On the other hand, suppose (for $\frac{j}{2^n}$ closest to x_0)

$$\left| x_0 - \frac{j}{2^n} \right| > \epsilon_n \equiv 2^{-n(1+\eta)}\delta_0 \tag{6.4}$$

for some η, $\delta_0 > 0$. Pick $1 < \gamma < (2-\alpha)/(1+\eta)$. Then

$$\begin{aligned} \int \frac{d\mu_n(y)}{|x_0 - y|^\gamma} &\leq {\epsilon_n}^{-\gamma} 2^{-n} + \int\limits_{2^{-n-1} \leq |x-y| \leq 1} \frac{dy}{|x-y|^\gamma} \\ &\leq C[{\epsilon_n}^{-\gamma} 2^{-n} + 2^{n(\gamma-1)}]. \end{aligned}$$

Thus, by (6.3)

$$\int \frac{d\nu_\alpha(y)}{|x_0 - y|^\gamma} \leq C\biggl(\sum_{n=0}^\infty 2^{-n(2-\alpha-\gamma)} + \sum_{n=0}^\infty {\delta_0}^{-\gamma} 2^{-n[-\gamma(1+\eta)+1+1-\alpha]} \biggr) < \infty$$

by the choice of γ and $\alpha + \gamma < 2$.

The measure of the set of $x_0 \in [0,1]$ where (6.4) fails is $\sum\limits_{n=0}^\infty 2^{-n\eta}\delta_0$ and is arbitrarily small if δ_0 gets small. Thus,

Lemma 6.4. *For any $\gamma < 2 - \alpha$ and a.e. $x_0 \in [0,1]$, $\int \frac{d\nu_\alpha(y)}{|x_0 - y|^\gamma} < \infty$.*

Since γ can be taken arbitrarily close to $2 - \alpha$, we see by Proposition 2.4 and Lemma 5.4 that the set $\widehat{S}_\beta$ of Theorem 4.2 has Lebesgue measure 1 if $\beta > \alpha$. Thus, $|[0,1] \backslash \bigcap\limits_{\beta > \alpha} \widehat{S}_\beta| = 0$. By the result of Simon-Wolff [40], $\mu_\lambda([0,1] \backslash \bigcap\limits_{\beta>\alpha} \widehat{S}_\beta) = 0$ for a.e. λ. Thus, by Theorem 4.2:

Theorem 6.5. *Fix $0 < \alpha < 1$. Then for a.e. λ, $\nu_{\alpha;\lambda}$ is supported on a set of dimension α. In particular, $\nu_{\alpha;\lambda} \restriction [0,1]$ is of exact dimension α.*

If we take $d\nu_1 = \sum\limits_{n=1}^\infty n^{-2}\, d\mu_n$, it is not hard to see that for all $\lambda \neq 0$, $\nu_{1;\lambda} \restriction [0,1]$ is of exact dimension one. Thus, we see that for any $\alpha \in [0,1]$ there are examples with singular spectrum of exact dimension α (in $[0,1]$) for a.e. λ (and for $\alpha = 0$, for all λ).

Example 3. *Some number theoretic examples*

Our third class of examples illustrates change of dimension from singular continuous to singular continuous spectrum. Details will be presented in Appendix 5.

These examples will depend critically on the base 2 decimal expansion of a number x in $[0,1]$. Given such an x, we can expand it, viz.

$$x = \sum_{n=1}^\infty \frac{a_n(x)}{2^n}. \tag{6.5}$$

We deal with the non-uniqueness for binary decimals (e.g., numbers of the form $\frac{j}{2^n}$) by requiring $a_m(x) = 0$ for m large for such x (except for $x = 1$). Thus, (6.5) defines a map of $\{0,1\}^{\mathbb{N}} \xrightarrow{F} [0,1]$, and $x \to \{a_n(x)\}$ defines a left inverse.

Any measure λ on $\{0,1\}^{\mathbb{N}}$ defines a measure μ on $[0,1]$ by $\mu(A) = \lambda(F^{-1}[A])$. For any p with $0 < p < 1$, let A_p be the product measure on $\{0,1\}^{\mathbb{N}}$ with each factor giving weights p to 0 and $(1-p)$ to 1, that is, the a_n's are i.i.d.'s with density $pd\delta_0 + (1-p)d\delta_1$. Let μ_p be the corresponding measure on $[0,1]$.

Two dimensions will arise below:

$$H(p) \equiv -\frac{p \ln p + (1-p)\ln(1-p)}{\ln 2} \tag{6.6}$$

$$L(p) \equiv 2 + \frac{\ln p(1-p)}{2\ln 2} \equiv 2 - \gamma(p). \tag{6.7}$$

We note that

$$L(p) < H(p) < 1 \qquad p \neq \frac{1}{2}$$

(but in fact $H(p) - L(p) \cong 0((p-\frac{1}{2})^4)$ for p near $\frac{1}{2}$ so they are very close for most p's). Notice also that $H(p) > 0$ and that

$$p \in \left(\frac{2-\sqrt{3}}{4}, \frac{2+\sqrt{3}}{4}\right) \equiv I_0 \Leftrightarrow L(p) > 0$$

(I_0 is about $(0.07, 0.93)$).

Theorem 6.6. (1) *$d\mu_p$ has exact dimension $H(p)$.*

(2) *Suppose $p \in I_0$. Then for a.e. λ w.r.t. Lebesgue measure, the restriction to $[0,1]$ of the rank one perturbation of $d\mu_p$ has exact dimension $L(p)$.*

(3) *If $p \notin \bar{I}_0$, then for a.e. λ, the rank one perturbation of $d\mu_p$ is pure point.*

(4) *If $p \in (\frac{1}{4}, \frac{3}{4})$, $p \neq \frac{1}{2}$, then for* all *λ, the restriction to $[0,1]$ of the rank one perturbation of $d\mu_p$ is purely singular continuous (so we have an example with singular continuous spectrum for all λ).*

Remarks. 1. (4) says for $p \in (\frac{1}{4}, \frac{3}{4})$, $G(x) = \infty$ for all $x \in [0,1]$.

2. We'll prove this theorem in Appendix 5.

Example 4. *Examples with pure point spectrum*

Our last class of examples will show $\{x \mid G(x) < \infty\}$ can have any Hausdorff dimension, and also provide examples where $d\mu_\lambda$ has a singular continuous component for all $\lambda \neq 0$ but sometimes mixed with embedded point spectrum. In this example, $d\mu$ will be a measure fixed once and for all with $\mathrm{supp}(\mu) = [0,1]$ and $G_\mu(x) \equiv \int \frac{d\mu(y)}{(x-y)^2} = \infty$ on $[0,1]$. Three possibilities to keep in mind are:

(1) $\chi_{[0,1]}(x)\,dx$ which is absolutely continuous.
(2) $d\mu_p$, the measure of Example 3, with $p \in (\frac{1}{4}, \frac{1}{2})$ where $G(x) = \infty$ by Theorem 6.6(4).
(3) Any of the point measures $d\nu_\alpha$ of Example 2 having $G(x_0) = \lim_{\epsilon \downarrow 0} \epsilon^{-1} \mathrm{Im}\, F_{\nu_\alpha}(x_0 + i\epsilon) = \infty$ for all $x_0 \in [0,1]$.

These show there are such μ with any spectral type.

Theorem 6.7. *Let C be an arbitrary closed nowhere dense set in $[0,1]$. Let μ be a Borel measure on $[0,1]$ with $G_\mu(x)=\infty$ on $[0,1]$ and $\int d\mu(x)=1$. Let*

$$d\nu(x)=\operatorname{dist}(x,C)^2\,d\mu(x).$$

Then, $\operatorname{supp}(\nu)=[0,1]$, *$G_\nu(x)=\infty$ on $[0,1]\backslash C$ and $G_\nu(x)\le 1$ on C.*

Proof. If $x\notin C$, $\operatorname{dist}(x,C)=\delta>0$ since C is closed. Thus, $G_\nu(x)\ge(\frac{\delta}{2})^2 \int\limits_{|x-y|\le\delta/2} \frac{d\mu(y)}{(x-y)^2}$ $=\infty$ since $G_\mu(x)=\infty$. On the other hand, if $x\in C$,

$$G_\nu(x)=\int\frac{\operatorname{dist}(y,C)^2}{\operatorname{dist}(x,y)^2}\,d\mu(y)\le\int d\mu(y)=1$$

since $\operatorname{dist}(x,y)\ge\operatorname{dist}(C,y)$. Finally, since $[0,1]\backslash C$ is dense, $\operatorname{supp}(d\nu)=[0,1]$. $\square$

Now let $\tilde\nu$ be $\nu/[\int d\nu]$. Then for every $x\in C$, $G_{\tilde\nu}(x)\le\frac{1}{N}$ for $N=\int d\nu$. Consider now the rank one perturbation $d\tilde\nu_\lambda$ of $d\tilde\nu$. ¿From (5.3), we see each pure point has weight at least $\frac{N}{\lambda^2}$ so there are at most $\frac{\lambda^2}{N}$ pure points (since $d\tilde\nu_\lambda$ is normalized in (5.3)). Thus,

Proposition 6.8. *If $N=\int d\nu(x)$ for the measure ν of Theorem 6.7, then $A_\lambda\equiv A+\lambda(\varphi,\,\cdot\,)\varphi$ has at most $\frac{\lambda^2}{N}$ eigenvalues in $[0,1]$. In particular, if $\lambda^2<N$, A_λ has purely singular continuous spectrum in $[0,1]$; and for any λ, $\sigma_{\rm sc}(A_\lambda)=[0,1]$.*

Remarks. 1. This shows the set in Theorem 5.1 can have any Hausdorff dimension since there are closed sets of any dimension. In addition, unlike the Simon-Wolff scenario, the s.c. spectrum need not ever be empty.

2. There exist nowhere dense C's of measure arbitrarily close to 1. So, to conclude $\sigma_{\rm sc}$ is empty for some λ, it is not enough $G(x)<\infty$ on a set of positive Lebesgue measure.

§7. Localization

One of our goals in this section is to prove that local perturbations of random Hamiltonians in the Anderson localization regime, while they may produce singular continuous spectrum, always produce zero-dimensional spectrum, in the sense that the spectral measures are all supported on a set of Hausdorff dimension zero. We'll use Theorem 6.2. Naively, one might confuse exponential decay of eigenfunctions in $\mathbb{Z}^\nu$ (as in $|\varphi_n(m)|\le C_n e^{-A|m|}$) with exponential decay in eigenfunction label (as in $|\varphi_n(0)|\le Ce^{-B|n|}$) which allows one to apply Theorem 6.2. In fact, they are distinct — indeed, if $\nu\ge 2$, we will not prove that $|\varphi_n(0)|\le Ce^{-B|n|}$ but only $|\varphi_n(0)|\le C\exp(-B|n|^{1/\nu})$; also see Appendix 2.

Throughout this section, n is an eigenvalue label and m is a $\mathbb{Z}^\nu$ point. It will be convenient to take the norm $|m|=\max\limits_{j=1,\dots,\nu}|m_j|$ on $\mathbb{Z}^\nu$.

Definition. Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$. We say that H has *semi-uniformly localized eigenfunctions* (SULE), pronounced "operators with a soul," if and only if H has a complete set $\{\varphi_n\}_{n=1}^\infty$ of orthonormal eigenfunctions, there is $\alpha > 0$ and $m_n \in \mathbb{Z}^\nu$, $n = 1, \dots$, and for each $\delta > 0$, a C_δ so that

$$|\varphi_n(m)| \leq C_\delta e^{\delta|m_n| - \alpha|m-m_n|} \tag{7.1}$$

for all $m \in \mathbb{Z}^\nu$ and $n = 1, 2, \dots$.

Thus, eigenfunctions are "localized about" points m_n. We use the "semi" in SULE because one can define ULE by requiring the bound with $\delta = 0$. The theory below extends to this case, but we'll restrict ourselves to the SULE case. In Appendix 3, we'll show that large classes of models, including the Anderson model in any dimension and the almost Mathieu operator, do not have ULE.

Below we'll first prove a result about the number of m_n in a box of side L, essentially proving that the number grows like L^ν as $L \to \infty$. This will show that local perturbations of SULE operators have zero-dimensional spectrum. Then, we'll relate SULE to dynamics and to Green's function localization; essentially, SULE always implies dynamical localization, and if the spectrum is simple, dynamical localization implies SULE. This will imply that Anderson-model Hamiltonians have SULE.

Appendix 2 has an example to show that a Jacobi matrix can have localized eigenfunctions which are not (semi) uniformly localized.

Theorem 7.1. *Suppose H has SULE. For each L, $\#\{n \mid |m_n| \leq L\}$ is finite and*

$$\lim_{L\to\infty} \frac{1}{(2L+1)^\nu} \#\{n \mid |m_n| \leq L\} = 1.$$

Remarks. 1. This says the density of centers of eigenfunctions is 1.

2. This will be a simple consequence of normalization and completeness, viz.

$$\sum_m |\varphi_n(m)|^2 = 1 \qquad n = 1, 2, \dots, \tag{7.2a}$$

$$\sum_n |\varphi_n(m)|^2 = 1 \qquad \text{each } m \in \mathbb{Z}^\nu. \tag{7.2b}$$

Lemma 7.2. *For each $\epsilon > 0$, there is a D_ϵ so that for each n and L:*

$$\sum_{|m-m_n| \geq \epsilon(|m_n|+L)} |\varphi_n(m)|^2 \leq D_\epsilon e^{-\alpha\epsilon L} e^{-\alpha\epsilon|m_n|/2}.$$

Proof. By hypothesis, we can find $C_\epsilon^{(1)}$ so

$$|\varphi_n(m)| \leq C_\epsilon^{(1)} e^{\alpha[\epsilon|m_n|/2 - |m-m_n|]}.$$

If $|m - m_n| \geq \epsilon(|m_n| + L)$, then $|m - m_n| \geq \frac{1}{2}|m - m_n| + \frac{\epsilon}{2}|m_n| + \frac{\epsilon}{2}L$ so in that regime

$$|\varphi_n(m)| \leq C_\epsilon^{(1)} e^{-\epsilon\alpha L/2} e^{-\alpha|m-m_n|/2}$$

so

$$\sum_{|m-m_n|\geq\epsilon(|m_n|+L)} |\varphi_n(m)|^2 \leq [C_\epsilon^{(1)}]^2 e^{-\alpha\epsilon L} \sum_{|k|\geq\epsilon|m_n|} e^{-\alpha|k|} \leq D_\epsilon e^{-\epsilon\alpha L} e^{-\alpha\epsilon|m_n|/2}$$

as claimed. $\square$

Proof of Theorem 7.1. To get the upper bound, we'll use the fact that functions localized in a box of side $2L$ contribute most of their norm to a box of side $2(1+\epsilon)L$. By the lemma, if $|m_n| \leq L$, then

$$\sum_{|m|\geq(1+2\epsilon)L} |\varphi_n(m)|^2 \leq \sum_{|m-m_n|\geq\epsilon(L+|m_n|)} |\varphi_n(m)|^2 \leq D_\epsilon e^{-\alpha\epsilon L}$$

and so by (7.2a),

$$\sum_{|m|\leq(1+2\epsilon)L} |\varphi_n(m)|^2 \geq 1 - D_\epsilon e^{-\alpha\epsilon L}\,.$$

Thus by (7.2b),

$$\begin{aligned}
[2(1+2\epsilon)L+1]^\nu &\geq \sum_{\substack{\text{all } n \\ |m|\leq(1+2\epsilon)L}} |\varphi_n(m)|^2 \\
&\geq \sum_{\substack{n \text{ so that } |m_n|\leq L \\ |m|\leq(1+2\epsilon)L}} |\varphi_n(m)|^2 \\
&\geq \#\{n \mid |m_n| \leq L\}(1 - D_\epsilon e^{-\alpha\epsilon L}).
\end{aligned}$$

Thus, $\#\{n \mid |m_n| \leq L\}$ is finite and

$$\overline{\lim}\,(2L+1)^{-\nu}\#\{n \mid |m_n| \leq L\} \leq 1. \tag{7.3}$$

In particular,

$$\#\{n \mid |m_n| \leq L\} \leq c_0 L^\nu \tag{7.4}$$

for some c_0 and all $L \geq 1$.

To get the lower bound, we'll use the fact that wave functions localized far outside a box of side $2L$ can't contribute much to the wave function sum inside that box. Explicitly, suppose that $|m_n| \geq \frac{1+\epsilon}{1-\epsilon}L$ and $|m| \leq L$. Then we claim

$$|m - m_n| \geq \epsilon(|m_n| + L)$$

for

$$|m - m_n| \geq |m_n| - L \geq |m_n|\left(1 - \frac{1-\epsilon}{1+\epsilon}\right) = \epsilon\left(1 + \frac{1-\epsilon}{1+\epsilon}\right)|m_n| \geq \epsilon(|m_n| + L).$$

Thus by Lemma 7.2, if $|m_n| \geq \frac{1+\epsilon}{1-\epsilon}L$, then

$$\sum_{|m|\leq L} |\varphi_n(m)|^2 \leq D_\epsilon e^{-\alpha\epsilon L} e^{-\alpha\epsilon|m_n|/2}$$

so

$$\sum_{\substack{n \text{ so that } |m_n| \geq \frac{1+\epsilon}{1-\epsilon} L \\ |m| \leq L}} |\varphi_n(m)|^2 \leq \sum_{k=0}^{\infty} \#\{n \mid |m_n| \leq (k+1)L\} D_\epsilon e^{-\alpha\epsilon L} e^{-\alpha\epsilon k L/2} \leq \tilde{D}_\epsilon e^{-\alpha\epsilon L/2}$$

by (7.4).

Thus by (7.2b),

$$(2L+1)^\nu = \sum_{\substack{\text{all } n \\ |m| \leq L}} |\varphi_n(m)|^2 \leq \#\left\{ n \,\middle|\, |m_n| < \frac{1+\epsilon}{1-\epsilon} L \right\} + \tilde{D}_\epsilon e^{-\alpha\epsilon L/2},$$

from which one immediately sees that

$$\underline{\lim}\, (2L+1)^{-\nu} \#\{n \mid |m_n| \leq L\} \geq 1.$$

Combining this with (7.3) yields the theorem. □

Corollary 7.3. *Suppose that H has* SULE. *Then there are C and D and a labeling of eigenfunctions so that*

$$|\varphi_n(0)| \leq C \exp(-D n^{1/\nu}). \tag{7.5}$$

Proof. Reorder the eigenfunctions so $|m_n|$ is increasing. By Theorem 7.1, $|m_n|/\frac{1}{2}n^{1/\nu} \to 1$ as $n \to \infty$ so $|m_n| \geq \frac{1}{3}n^{1/\nu} - C_0$ for some constant C_0. By (7.1), we get (7.5); indeed, we see D can be taken arbitrarily close to $\frac{1}{2}\alpha$. □

Combining this corollary with Theorem 6.2, we see:

Theorem 7.4. *Suppose that H has* SULE. *Let $H_\lambda = H + \lambda(\delta_0, \cdot)\delta_0$. Then for every λ, the spectral measures for H_λ are supported on a set of Hausdorff dimension zero. Moreover, H_λ has pure point spectrum except for a set of λ's of Hausdorff dimension zero.*

Next, we relate SULE to other conditions. We'll suppose H has simple spectrum, although one can easily extend this to examples with spectrum having a uniform finite upper bound on multiplicity.

Definition. Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$. We say that H has *semi-uniform dynamical localization* (SUDL) if and only if there is $\alpha > 0$ and for each $\delta > 0$, a C_δ so that for all $q, m \in \mathbb{Z}^\nu$:

$$\sup_t |(\delta_q, e^{-itH}\delta_m)| \leq C_\delta e^{\delta|m| - \alpha|q-m|}. \tag{7.6}$$

We say that H has *semi-uniformly localized projections* (SULP) if and only if H has a complete set of normalized eigenfunctions and there is $\alpha > 0$ and for each $\delta > 0$, a C_δ so that for all $q, m \in \mathbb{Z}^\nu$:

$$|(\delta_q, P_{\{E\}}\delta_m)| \leq C_\delta e^{\delta|m| - \alpha|q-m|}$$

for all spectral projections $P_{\{E\}}$ onto a single point (uniformly in E).

Theorem 7.5. *Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$ with simple spectrum. Then the following are equivalent:*

(i) *H has* SUDL.
(ii) *H has* SULP.
(iii) *H has* SULE.

Remarks. 1. The fact that dynamical localization implies point spectrum has a long history, going back at least to Kunz-Souillard [20]. Martinelli-Scoppola [23] used a variant of SULE, which they proved by analysis of eigenfunctions, to prove a restricted form of dynamical localization in the multi-dimensional Anderson model.

2. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) does not require simplicity of the spectrum. It is an interesting open problem whether (ii) $\Rightarrow$ (iii) can be extended to cases with unbounded multiplicity.

3. It is not claimed the α's are the same in the three statements. Indeed, (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) doesn't change α (by more than ϵ) but our proof of (ii) $\Rightarrow$ (iii) decreases α by a factor of 2.

Proof. (i) $\Rightarrow$ (ii): Follows immediately from $P_{\{E\}} = \operatorname*{s-lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{iEs} e^{-iHs}\, ds$.

(ii) $\Rightarrow$ (iii): Label the eigenvalues of H: $E_1, E_2, \dots$. For each $E_n \in \operatorname{spec}(H)$, pick an eigenfunction $\varphi_n(\,\cdot\,)$, unique up to phase. Then by (ii):

$$|\varphi_n(q)\varphi_n(m)| \le C_\epsilon e^{\delta|m|} e^{-\alpha|q-m|}\,. \tag{7.7a}$$

Since $\varphi_n \in \ell^2$, it takes its maximum value so choose m_n so that

$$|\varphi_n(m_n)| = \sup_m |\varphi_n(m)|. \tag{7.7b}$$

Then by (7.7),

$$\begin{aligned} |\varphi_n(q)|^2 &\le |\varphi_n(q)| \sup_m |\varphi_n(m)| \le |\varphi_n(q)|\,|\varphi_n(m_n)| \\ &\le C_\delta e^{\delta|m_n|} e^{-\alpha|q-m_n|} \end{aligned}$$

so H has SULE by taking square roots.

(iii) $\Rightarrow$ (i): Let φ_n be the eigenfunctions and E_n eigenvalues. Then

$$(\delta_q, e^{-itH}\delta_m) = \sum_n \overline{\varphi_n(q)}\, e^{-itE_n} \varphi_n(m),$$

so, assuming SULE,

$$\sup_t |(\delta_q, e^{-itH}\delta_m)| \le \sum_n |\,\overline{\varphi_n(q)}\,\varphi_n(m)| \le C_\delta^2 \sum_n e^{2\delta|m_n|} e^{-\alpha(|q-m_n|+|m-m_n|)}. \tag{7.8}$$

Now,

$$|q-m_n| + |m-m_n| \ge |q-m|$$

and

$$|q - m_n| + |m - m_n| \geq |m_n| - |m|.$$

Thus,

$$e^{-\alpha(|q-m_n|+|m-m_n|)} \leq e^{-3\delta|m_n|} e^{3\delta|m|} e^{-(\alpha-3\delta)|m-q|}.$$

So, by (7.8)

$$\sup_t |(\delta_q, e^{-itH}\delta_m)| \leq C_\delta^2 e^{3\delta|m|} e^{-(\alpha-3\delta)|m-q|} A_0$$

where

$$A_0 = \sum_n e^{-\delta|m_n|}.$$

By (7.4) which follows from SULE, A_0 is finite. □

One can prove by the above means a result that shows that if H has simple spectrum and $\sup_t |(\varphi, e^{-itH}\delta_n)| \leq Ce^{-\alpha|n|}$, then the spectral measure for φ can be written $\sum_{n=1}^{\infty} a_n \, d\delta_{E_n}$ where $|a_n| \leq De^{-\beta n^{1/\nu}}$ if the E_n's are properly labeled. That is, one can prove a result that requires less uniformity than the full-blown theory assumes.

Finally, we turn to when any, and hence all, of the conditions of Theorem 7.5 hold in the context of the Anderson model. We're dealing here with models depending on a random parameter so we first reduce SUDL to a requirement on expectations. General considerations [32] imply that the spectrum is simple in the localized regime.

Theorem 7.6. *Let (Ω, μ) be a probability measure space and $E(\,\cdot\,)$ its expectation. Let $\omega \to H_\omega$ be a strongly measurable map from Ω to the self-adjoint operators on $\ell^2(\mathbb{Z}^\nu)$ which is translation invariant in the sense that for each $m \in \mathbb{Z}^\nu$, there is a measure preserving $T_m : \Omega \to \Omega$ so $H_{T_m\omega} = U_m H_\omega U_m^{-1}$ where $(U_m\varphi)(q) = \varphi(q-m)$. Suppose that*

$$E\left(\sup_t |(\delta_q, e^{-itH_\omega}\delta_0)|\right) \leq C_1 e^{-\alpha|q|} \tag{7.9}$$

for some $\alpha > 0$ and that H_ω has simple spectrum for a.e. ω. Then for each $\beta < \alpha$, for a.e. ω, there is a $C_\omega < \infty$ so that for all $0 < \epsilon \leq 1$

$$\sup_t |(\delta_q, e^{-itH_\omega}\delta_m)| \leq \frac{C_\omega}{\epsilon^{\nu+1}} e^{\epsilon|m|} e^{-\beta(m-q)}.$$

In particular, a.e. H_ω has SULE.

Proof. Let

$$Q(\omega) = \sum_m (1 + |m|)^{-(\nu+1)} e^{\beta|m-q|} \sup_t |(\delta_q, e^{-itH_\omega}\delta_m)|.$$

Then by (7.9),

$$E(Q(\omega)) < \infty$$

so $Q(\omega) < \infty$ for a.e. ω. But for such ω,

$$\sup_t |(\delta_q, e^{-itH_\omega}\delta_m)| \leq C_\omega(1+|m|)^{\nu+1}e^{-\beta|m-q|}\,.$$

The result now follows from the trivial bound $(1+x)^\nu \leq \nu^\nu e^{\epsilon x}\epsilon^{-\nu}$ for $\epsilon \leq 1$. $\square$

So when does (7.9) hold? Delyon-Kunz-Souillard [8] have proven this bound for a general class of one-dimensional random potentials. In general, we have the following beautiful bound of Aizenman:

Theorem 7.7. (Aizenman's theorem) *Let $V_\omega(n)$ be a family of independent identically distributed random variables (indexed by $n \in \mathbb{Z}^\nu$; $\omega \in \Omega$ is the probability parameter). Suppose H_0 is an operator on $\ell^2(\mathbb{Z}^\nu)$ commuting with translations and $H_\omega = H_0 + V_\omega$ with V_ω viewed as a diagonal matrix. Suppose $V_\omega(n)$ has a distribution $g(\lambda)\,d\lambda$ with $g \in L^\infty$ and has compact support. Suppose*

$$E\biggl(\int_a^b |(\delta_n, (H_\omega-\lambda-i0)^{-1}\delta_0)|^s\,d\lambda\biggr) \leq Ce^{-\mu|n|} \tag{7.10}$$

for some $s \in (0,1)$. Then

$$E\biggl(\sup_t |(\delta_n, e^{-itH_\omega}P_{[a,b]}(H_\omega)\delta_0)|\biggr) \leq \tilde{C}e^{-\mu|n|/(2-s)} \tag{7.11}$$

where $\tilde{C}$ only depends on s and the distribution g.

Remarks. 1. In fact, as we'll see, one can take $\tilde{C} = \Delta^{s/(2-s)}\|g\|_\infty^{1/(2-s)}C^{1/(2-s)}$ where Δ is the diameter of the support of g.

2. The result as stated differs from [1] in several aspects. Most significantly, it hasn't a requirement of approximation by operators with discrete spectrum in (a,b). Moreover, we have a proof that, while it follows Aizenman [1] in the essentials, avoids a priori estimates on the distribution function of $|(\delta_0, (H-E-i0)^{-1}\delta_0)|$. For this reason, we provide this proof in Appendix 1.

3. We've stated a local (with $P_{[a,b]}$) result but one can take $[a,b]$ to be so big $\operatorname{spec}(H_\omega) \subset [a,b]$ to get the global result (7.9). Alternatively, we could localize the result earlier in this section.

4. Aizenman has neither a $\|g\|_\infty < \infty$ condition nor that g has compact support. We could mimic his technique to replace $\|g\|_\infty < \infty$ by $\|g\|_p < \infty$ for some $p > 1$. Moreover, we could replace the compact support assumption by the finiteness of some moment $\int |\lambda|^\alpha g(\lambda)\,d\lambda$ for some $\alpha > 0$.

Combining this result with those of Aizenman-Molchanov [2], we see that the strongly coupled multi-dimensional Anderson model has SULE.

§8. Semi-Stability of Dynamical Localization

Anderson localization (at least as proven in [1]) implies that if $\vec{x}$ is the operator

$$(x_i\psi)(m) = m_i\psi(m_i) \qquad i = 1,\ldots,\nu,$$

then in the localized regime,

$$\sup_t (e^{-itH}\delta_0, x^2 e^{-itH}\delta_0) < \infty. \tag{8.1}$$

It follows from the RAGE theorem (see, e.g., [22]) that (8.1) implies that H has pure point spectrum.

For operators H with dense pure point spectrum, it is proven in [7,11] that for a Baire generic set of λ, $H_\lambda = H + \lambda(\delta_0, \cdot\,)\delta_0$ has only singular continuous spectrum and so for such H_λ's, (8.1) must fail. Our purpose in this section is to show that the failure is only very mild. $\langle x^2\rangle(t) \equiv (e^{-itH}\delta_0, x^2 e^{-itH}\delta_0)$ is unbounded but grows at worst logarithmically!

Theorem 8.1. *Suppose that H is a self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$ with* SULE. *Let $H_\lambda = H + \lambda(\delta_0, \cdot\,)\delta_0$. Then*

$$\langle x^{2n}\rangle(t) \equiv (e^{-itH_\lambda}\delta_0, (x^2)^n e^{-itH_\lambda}\delta_0)$$

obeys

$$\langle x^{2n}\rangle(t) \le C_n(\ln|t|)^{2n}$$

for $|t|$ large.

Remarks. 1. The result is actually stronger since we only need dynamical localization in the sense that $\sup|(\delta_m, e^{-itH}\delta_0)| \le Ce^{-\alpha|m|}$. If this estimate holds, then so does the upper bound on $\langle x^{2n}\rangle(t)$, regardless of whether H has SULE, or even whether H has only pure point spectrum or not.

2. By a result of Last [22], which extends an idea originaly due to Guarneri [12], it follows that if the spectral measure of δ_0 (for H_λ) is not supported on a set of Hausdorff dimension zero, then for some $\beta > 0$, $\overline{\lim}\, t^{-2n\beta}\langle x^{2n}\rangle(t) > 0$. Thus, we get an alternative proof to the fact that SULE (for H) implies zero-dimensional spectrum for H_λ (for all λ's).

Proof. Write a DuHamel expansion:

$$(\delta_m, e^{-itH_\lambda}\delta_0) = (\delta_m, e^{-itH}\delta_0) - i\lambda\int_0^t (\delta_m, e^{-isH}\delta_0)(\delta_0, e^{-i(t-s)H_\lambda}\delta_0)\,ds. \tag{8.2}$$

Since H has SULE, by Theorem 7.5,

$$\sup_t |(\delta_m, e^{-itH}\delta_0)| \le Ce^{-\alpha|m|}$$

for suitable C and α. Plugging this into (8.2) and using $|(\delta_0, e^{-itH_\lambda}\delta_0)| \le 1$, we see that

$$|(\delta_m, e^{-itH_\lambda}\delta_0)| \le Ce^{-\alpha|m|}[1 + |\lambda|\,|t|]. \tag{8.3}$$

This would seem to give linear growth in t for $\langle x^{2m}\rangle^{1/2m}$ but we'll combine it with the trivial bound

$$\sum_m |(\delta_m, e^{-itH_\lambda}\delta_0)|^2 = 1. \tag{8.4}$$

Use (8.3) only if $|m| > 2\ln(1+|\lambda|\,|t|)/\alpha \equiv G(t)$. In that regime (8.3) says that

$$|(\delta_m, e^{-itH_\lambda}\delta_0)| \le Ce^{-\alpha|m|/2}.$$

Thus,

$$\sum_{|m|\ge G(t)} (m^2)^n |(\delta_m, e^{-itH_\lambda}\delta_0)|^2 \le C_n$$

and obviously by (8.4),

$$\sum_{|m|\le G(t)} (m^2)^n |(\delta_m, e^{-itH_\lambda}\delta_0)|^2 \le (G(t))^{2n},$$

so

$$\langle x^{2n}\rangle(t) \le (G(t))^{2n} + C_n$$

as claimed. □

In fact, the proof shows that

$$\varlimsup_{|t|\to\infty} (\ln|t|)^{-2n}\langle x^{2n}\rangle(t) \le \left(\frac{\alpha}{2}\right)^{-2n}.$$

Appendix 1: Aizenman's Theorem

Our goal here is to prove Theorem 7.7. We begin with a general fact about rank one perturbations. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $P = (\varphi, \cdot\,)\varphi$ a rank one projection onto a unit vector φ assumed cyclic for A. Let $A_\lambda = A + \lambda P$. Then φ is cyclic for A_λ. Let $d\mu_\lambda$ be the spectral measure of the pair φ, A_λ, so for example,

$$\int \frac{d\mu_\lambda(x)}{x-z} = (\varphi, (A_\lambda - z)^{-1}\varphi) \equiv F_\lambda(z).$$

By the spectral theorem, there is a natural map $U_\lambda : \mathcal{H} \to L^2(\mathbb{R}, d\mu_\lambda)$ so that $U_\lambda\varphi \equiv 1$ and $U_\lambda A U_\lambda^{-1}$ is multiplication by x. The point is that in the localized regime, there is an explicit formula for U_λ.

Recall that the function

$$G(x) = \int \frac{d\mu_0(y)}{(x-y)^2}$$

plays a critical role in situations where A_λ has point spectrum. Explicitly [33,40],

(1) A_λ has only pure point spectrum in $[a,b]$ for a.e. $\lambda \in \mathbb{R}$ if and only if $G(E) < \infty$ for a.e. $E \in (a,b)$.
(2) If $G(E) < \infty$, then $F(E + i0) = \alpha$ exists, is real, and E is an eigenvalue of A_λ if and only if $\lambda = -\alpha^{-1}$.

Our main preliminary is

Lemma A.1. *Suppose $G(E) < \infty$ for a.e. $E \in [a,b]$. Then for any such E,*

$$\lim_{\epsilon \downarrow 0} (A - E - i\epsilon)^{-1}\varphi = \varphi_E \tag{A.1}$$

exists. Moreover, if λ is such that $\mu_\lambda \restriction [a,b]$ is supported on $\{E \in [a,b] \mid G(E) < \infty\}$, then

$$(U_\lambda \psi)(E) = -\lambda(\varphi_E, \psi). \tag{A.2}$$

Proof. The general theory of rank one perturbations (see [33]) implies

$$\frac{(A_\lambda - z)^{-1}\varphi}{(\varphi, (A_\lambda - z)^{-1}\varphi)} = \frac{(A - z)^{-1}\varphi}{(\varphi, (A - z)^{-1}\varphi)} \tag{A.3}$$

for any z with $\operatorname{Im} z > 0$ and any λ. Given E with $G(E) < \infty$, $F(E + i0)$ exists and equals some $-\lambda^{-1}$. Pick that value of λ in (A.3). Then E is an eigenvalue of A_λ and the projection onto the corresponding eigenvector is

$$P_E = \operatorname*{s-lim}_{\epsilon \downarrow 0} [(-i\epsilon)(A_\lambda - E - i\epsilon)^{-1}].$$

Thus, multiplying the numerator and denominator of the left side of (A.3) by $(-i\epsilon)$ and taking ϵ to zero, we see that the limit in (A.1) exists, and by the fact that $F(E + i0) = -\lambda^{-1}$, that

$$(\varphi, -\lambda\varphi_E) = 1 \tag{A.4}$$

and that φ_E is a multiple of the eigenfunction for A_λ a.e. E.

Since $(U_\lambda\psi)(E)$ is precisely an inner product of ψ with that multiple of the eigenfunction that obeys $(\varphi, \cdot\,) = 1$, (A.4) implies (A.2). $\square$

Lemma A.2. *Suppose $G(E) < \infty$ for a.e. $E \in [a,b]$, that $\|\psi\| = 1$, and that λ is a random variable with distribution $g(\lambda)\,d\lambda$ where $g \in L^\infty$, with compact support. Then for any $\lambda_0 \in \operatorname{supp}(g)$ and $s \in (0,1)$:*

$$\mathbb{E}\biggl(\sup_t |(\psi, P_{[a,b]}(A_\lambda)e^{-itA_\lambda}\varphi)|\biggr) \leq \Delta^{s/(2-s)}\|g\|_\infty^{1/(2-s)} \biggl(\int_a^b |(\psi, (A_{\lambda_0} - E - i0)^{-1}\varphi)|^s\, dE\biggr)^{1/(2-s)} \tag{A.5}$$

where $\Delta = \operatorname{diam}(\operatorname{supp} g) = \max(|\lambda - \lambda'| \mid \lambda, \lambda' \in \operatorname{supp} g)$.

Proof. By the spectral theorem,

$$(\varphi, P_{[a,b]}(A_\lambda)e^{-itA_\lambda}\psi) = \int_a^b e^{-itE}(U_\lambda\psi)(E)\, d\mu_\lambda(E)$$

and by the unitarity of U,

$$\int |(U_\lambda\psi)(E)|^2\, d\mu_\lambda(E) = 1. \tag{A.6}$$

Hölder's inequality says that for $0 < s < 1$,

$$\int |g|\, d\mu \leq \left(\int |g|^2\, d\mu \right)^{(1-s)/(2-s)} \left(\int |g|^s\, d\mu \right)^{1/(2-s)} \tag{A.7}$$

so

$$\begin{aligned} \sup_t |(\varphi, P_{[a,b]}(A_\lambda) e^{-itA_\lambda}\psi)| &\leq \int_a^b |(U_\lambda \psi)(E)|\, d\mu_\lambda(E) \\ &\leq \left(\int_a^b |(U_\lambda \psi)(E)|^s\, d\mu_\lambda(E) \right)^{1/(2-s)} \end{aligned} \tag{A.8}$$

by (A.6) and (A.7). Since we can think of A_λ as a perturbation of A_{λ_0}, we can use Lemma A.1 to say that

$$(U_\lambda \psi)(E) = -(\lambda - \lambda_0)((A_{\lambda_0} - E - i0)^{-1}\varphi, \psi).$$

Thus, (A.8) implies

$$\sup_t |(\varphi, P_{[a,b]}(A_\lambda) e^{-itA_\lambda}\psi)| \leq \Delta^{s/(2-s)} \left(\int_a^b |(\psi, (A_{\lambda_0} - E - i0)^{-1}\varphi)|^s\, d\mu_\lambda(E) \right)^{1/(2-s)}.$$

Now take $\mathbb{E}$'s. Since $\frac{1}{2-s} < 1$, Hölder's inequality implies $\mathbb{E}(|f|^{1/(2-s)}) \leq (\mathbb{E}(|f|))^{1/(2-s)}$ and $\mathbb{E}(d\mu_\lambda(E)) \leq \|g\|_\infty \int d\lambda(d\mu_\lambda(E)) = \|g\|_\infty\, dE$ where the last equality is a result explicitly in Simon-Wolff [40] but obtained in related forms earlier by Javrjan [15] and Kotani [19]. □

Proof of Aizenman's Theorem (Theorem 7.7). The hypothesis (7.10) implies that for a.e. pairs $\omega, \lambda \in [a, b]$

$$|(\delta_n, (H_\omega - \lambda - i0)^{-1}\delta_m)| \leq C_{\omega,\lambda,m} e^{-\mu|n-m|/2}$$

so for a.e. such pairs,

$$\|(H_\omega - \lambda - i0)^{-2}\delta_m\| < \infty$$

and thus by the Simon-Wolff criterion [33,40], H_ω has pure point spectrum in $[a, b]$. Thus, for such ω, Lemma A.2 applies and we get (7.11) after averaging over λ_0 and then over ω. □

Remarks. 1. Independence of $\{v\}$ is not needed. It suffices that the conditional distribution of $v(0)$, conditioned on $\{v(n)\}_{n\neq 0}$ has a density $g_v(\lambda)\, d\lambda$ with $\|g_v\|_\infty$ bounded uniformly in v and with a uniform bound on $\operatorname{diam}(\operatorname{supp} g_v)$.

2. Relative to Aizenman's proof, we get a simplification by using $(\varphi, (A - E - i0)^{-1}\varphi) = -\lambda^{-1}$ and therefore not needing Boole's equality. We can turn this around and actually use the theory of rank one perturbations to prove Boole's equality in its natural setting.

Proposition A.3. *Let μ be a finite purely singular measure and let $F(E+i0) = \int \frac{d\mu(x)}{x-(E+i0)}$. Then for $t>0$,*

$$|\{E \mid F(E+i0) > t\}| = |\{E \mid F(E+i0) < -t\}| = t^{-1}\mu(\mathbb{R}).$$

Proof. Without loss, we can suppose $\mu(\mathbb{R}) = 1$. Let A_0 be the operator of multiplication by x on $L^2(\mathbb{R}, d\mu)$ and $(P\psi) = (1,\psi)1$. Let $d\mu_\lambda$ be the spectral measure for $A_0 + \lambda P$. As noted above:

$$\int d\lambda [d\mu_\lambda(E)] = dE$$

in the sense that for any measurable set S,

$$\int \mu_\lambda(S)\, d\lambda = |S|. \tag{A.9}$$

On the other hand, by the Aronszajn-Donoghue theory [33],

$$\mu_\lambda \text{ is supported on } \{E \mid F(E+i0) = -\lambda^{-1}\}. \tag{A.10}$$

Let $S_t = \{E \mid F(E+i0) < -t\}$. Then (A.10) says that

$$\mu_\lambda(S_t) = \begin{cases} 1, & 0 < \lambda < t^{-1} \\ 0, & \lambda < 0 \text{ or } \lambda > t^{-1} \end{cases}$$

so (A.9) implies $|S_t| = t^{-1}$. □

Remarks. 1. Boole's equality for μ, a measure with a finite number of pure points, was found in 1857 [6]. See [1,24] for more recent history.

2. Using this result in this form, it is not hard to show for any measure μ,

$$\lim_{t\to\infty} t|\{x \mid |F(x+i0)| > t\}| = 2\mu_{\text{sing}}(\mathbb{R})$$

the mass of the singular part of μ. Boole's equality applies explicitly only to μ purely singular.

3. This proof of Boole's equality was found independently by Poltoratski [24].

Appendix 2: A Pathological Example

Our goal in this appendix is to present a one-dimensional Jacobi matrix (i.e., potential $v(n)$ on $\mathbb{Z}_+$ and operator $(hu)(n) = u(n+1) + u(n-1) + v(n)u(n)$ on $\ell^2(\mathbb{Z}_+)$ with $\mathbb{Z}_+ = \{n \in \mathbb{Z}, n \geq 0\}$ and a Dirichlet boundary condition at $n=-1$) so that

(0) v is bounded.

(1) h has a complete set of normalized eigenfunctions.

(2) Each eigenfunction is exponentially decaying, that is,

$$|\varphi_n(m)| \leq C_n e^{-\alpha|m|}$$

for some fixed $\alpha > 0$.

(3) Let $F(t) = t^2/\ln(t)$. Then

$$\overline{\lim_{t\to\infty}}\, \|xe^{-ith}\delta_0\|^2/F(t) = \infty. \tag{B.1}$$

Thus, in spite of exponentially localized eigenfunctions, h doesn't have dynamical localization. This shows that proofs of "localization" that only show (1),(2) are only part of the story and that the SUDL shown by Aizenman in [1] and the SULE consideration in this paper are a significant desideratum. One can modify the proof to replace $F(t)$ by $t^2/f(t)$ for any monotone f with $\lim_{t\to\infty} f(t) = \infty$. Thus, this example also shows that the result of [31] that point spectrum implies

$$\lim_{t\to\infty} \|xe^{-ith}\delta_0\|^2/t^2 = 0$$

cannot be improved.

Our $v(n)$ will have the form

$$v(n) = 3\cos(\pi\alpha n + \theta) + \lambda\delta_{n0} \tag{B.2}$$

with α irrational. We'll prove that α can be constructed so that (B.1) holds for all θ and $\lambda \in [0,1]$. It is well known (e.g., [4]) that the Lyapunov exponent, which characterizes solutions of $(h-E)u = 0$ for a.e. E, θ, is everywhere larger than or equal to $\ln(\frac{3}{2})$. Thus, by the Simon-Wolff criterion [33,40], (1) and (2) hold for a.e. θ, λ and we only need to choose α so that (B.1) holds.

Let $P_{n>a}$ denote the projection onto those functions supported by $\{n \mid n > a\}$ and similarly for $P_{n\le a}$, etc. Let $f(t)$ be a monotone increasing function of t with $f(t) \to \infty$ at ∞ (we'll take $f(t) = [\ln(|t|+2)]^{1/5}$).

Lemma B.1. *Suppose there exists $T_m \to \infty$ so that*

$$\frac{1}{T_m}\int_{T_m}^{2T_m} \|P_{n\ge T_m/f(T_m)} e^{-ish}\delta_0\|^2 \, ds \ge \frac{1}{f(T_m)^2}. \tag{B.3}$$

Then

$$\overline{\lim_{t\to\infty}} \, \|xe^{-ish}\delta_0\|^2 f(t)^5/t^2 = \infty.$$

Proof. Under the hypothesis, there must be some $s_m \in [T_m, 2T_m]$ so

$$\begin{aligned} \|xe^{-is_m h}\delta_0\|^2 &\ge \left(\frac{T_m}{f(T_m)}\right)^2 \|P_{n\ge T_m/f(T_m)} e^{-is_m h}\delta_0\|^2 \\ &\ge T_m^2 f(T_m)^{-4}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{f(s_m)^5}{s_m^2}\|xe^{-is_m h}\delta_0\|^2 &\ge \left(\frac{T_m}{s_m}\right)^2 \left(\frac{f(s_m)}{f(T_m)}\right)^4 f(s_m) \\ &\ge \frac{1}{4} f(s_m) \to \infty \end{aligned}$$

as claimed. $\square$

We'll get the lower bound in (B.3) from the following:

Lemma B.2. *Let δ be a unit vector, P a projection, and h a self-adjoint operator. Suppose $\delta = \varphi + \psi$ with $(\varphi, \psi) = 0$. Then*

$$\frac{1}{T}\int_T^{2T} \|(1-P)e^{-ish}\delta\|^2\,ds \geq \|\psi\|^2 - 3\biggl(\frac{1}{T}\int_T^{2T} \|Pe^{-ish}\psi\|^2\,ds\biggr)^{1/2}. \tag{B.4}$$

Proof. Since $\|P\eta\|^2 + \|(1-P)\eta\|^2 = 1$ for any unit vector η, $\|\psi\|^2 + \|\varphi\|^2 = 1$ and $\|e^{-ish}\delta\|^2 = 1$, we see that

$$\text{LHS of (B.4)} \geq A + B$$

where

$$\begin{aligned} A &= \|\varphi\|^2 - (\varphi,\varphi)_\sim \\ B &= \|\psi\|^2 - (\psi,\psi)_\sim - 2\,\mathrm{Re}(\varphi,\psi)_\sim \end{aligned}$$

with

$$(\eta,\xi)_\sim = \frac{1}{T}\int_T^{2T} (Pe^{-ish}\eta, e^{-ish}\xi)\,ds.$$

Clearly, $(\varphi,\varphi)_\sim \leq 1$ and $(\psi,\psi)_\sim \leq 1$, so $A \geq 0$ and $B \geq \|\psi\|^2 - 3(\psi,\psi)_\sim^{1/2}$, which is the stated result. $\square$

We need to make a break-up so $(\psi,\psi)_\sim$ is small. This is what we turn to next.

Recall the notion of $|\!|\!|\cdot|\!|\!|$ introduced by Kato (see (X.4.17) of [18]). Let A be a self-adjoint operator. A vector φ is said to have finite triple norm if its spectral measure μ has the form $d\mu_\varphi^A = F(E)\,dE$ with $F \in L^\infty$. We set $|\!|\!|\varphi|\!|\!| \equiv |\!|\!|\varphi|\!|\!|_A \equiv \|F\|_\infty^{1/2}$. Given α, θ, λ, we set $h(\alpha,\theta,\lambda)$ to the Jacobi matrix with potential (B.2).

Lemma B.3. *Fix α rational. Then there exist $C_1 > 0$ and $C_2 < \infty$ and for each $\theta \in [0, 2\pi]$ and $\lambda \in [0,1]$, a breakdown*

$$\delta_0 = \varphi_{\theta,\lambda} + \psi_{\theta,\lambda}$$

so

$$(\varphi,\psi) = 0 \tag{B.4}$$

$$\|\psi_{\theta,\lambda}\| \geq C_1 \tag{B.5}$$

$$|\!|\!|\psi_{\theta,\lambda}|\!|\!|_{h(\alpha,\theta,\lambda)} \leq C_2. \tag{B.6}$$

Proof. Consider first $\lambda = 0$ and consider the periodic Jacobi matrix on $\ell^2(\mathbb{Z})$ which corresponds to the potential (B.2) (on $\mathbb{Z}$). It is a periodic Hamiltonian with a fixed Bloch Hamiltonian decomposition. If $\alpha = \frac{p}{q}$, the period is q and we can use a quasimomentum label that runs from 0 to $\frac{\pi}{q}$. Consider the lowest band and the quasimomenta range between $\frac{\pi}{3q}$ and $\frac{2\pi}{3q}$.

Let $E_\theta(k)$ denote the band function for the lowest band. E_θ is strictly monotone in k; indeed, $\frac{\partial E_\theta}{\partial k} > 0$, and jointly continuous in $\theta \in [0,\pi]$, $k \in [\frac{\pi}{3}, \frac{2\pi}{3q}]$. Thus, the width of the energy range, $E_\theta(\frac{2\pi}{3q}) - E_\theta(\frac{\pi}{3q}) \equiv \ell_\theta$ is uniformly bounded away from zero.

Let $\Phi_n^\theta(E)$ denote the 2×2 transfer matrix from 0 to n for the potential (B.2). That is, $\Phi_n^\theta(E) \equiv T_n^\theta(E) T_{n-1}^\theta(E) \ \dots \ T_0^\theta(E)$, where

$$T_n^\theta(E) \equiv \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix}$$

and $v(n)$ is given by (B.2) with $\lambda = 0$. By, for example, Lemma 3.1 of [21], we have the bound $\|\Phi_{mq-1}^\theta(E)\| \le 2q \left|\frac{\partial E_\theta}{\partial k}\right|^{-1}$ for any integer $m > 0$. (Remark: Lemma 3.1 of [21] is formulated for the transfer matrix over one period, but it is easy to see from its proof that the bound holds for any integer number of periods.) Thus, $\|\Phi_{mq-1}^\theta(E)\|$ is uniformly bounded for all θ's, $m > 0$, and $E \in [E_\theta(\frac{2\pi}{3q}), E_\theta(\frac{\pi}{3q})] \equiv I_\theta$.

Let $\tilde\Phi_n^{\theta,\lambda}(E)$ denote the transfer matrix for the potential (B.2) with $\lambda \in [0,1]$. Then we see that $\|\tilde\Phi_n^{\theta,\lambda}(E)\|$ must also be uniformly bounded. That is, $\|\tilde\Phi_n^{\theta,\lambda}(E)\| < C$ for all $n \ge 0$, $\lambda \in [0,1]$, $\theta \in [0,2\pi]$, and $E \in I_\theta$. By, for example, Theorem 2 of [38], this implies that the imaginary part of the m-function for $h(\alpha,\theta,\lambda)$, which is identical to the Borel transform $F_{\theta,\lambda}$ of the spectral measure of δ_0 (for $h(\alpha,\theta,\lambda)$), is uniformly bounded. Namely, $C_1^{-1} < \operatorname{Im} F_{\theta,\lambda}(E+i0) < C_1$ for some constant C_1 and for all $\lambda \in [0,1]$, $\theta \in [0,2\pi]$, and $E \in I_\theta$.

Let $\psi_{\theta,\lambda} = P_{I_\theta}^{\theta,\lambda}\delta_0$, where $P_{I_\theta}^{\theta,\lambda}$ is the spectral projection (for $h(\alpha,\theta,\lambda)$) on I_θ. Then the spectral measure of $\psi_{\theta,\lambda}$ is purely absolutely continuous and has the form $\pi^{-1}\operatorname{Im} F_{\theta,\lambda}(E+i0)\, dE$. Thus, we see that the claim holds. $\square$

As a final lemma, we need to control changes in the dynamics as we change α:

Lemma B.4.

$$\|(e^{-ish(\alpha,\theta,\lambda)} - e^{-ish(\alpha',\theta,\lambda)})\delta_0\| \le 3\pi s^2|\alpha - \alpha'| \tag{B.7}$$

Proof. $h(\theta,\alpha,\lambda) - h(\alpha',\theta,\lambda) = 3[\cos(\alpha\pi x + \theta) - \cos(\alpha'\pi x + \theta)]$ so

$$\|[h(\theta,\alpha,\lambda) - h(\alpha',\theta,\lambda)]\eta\| \le 3\pi|\alpha - \alpha'|\, \|x\eta\|$$

and so by a DuHamel formula,

$$\text{LHS of (B.7)} \le \int_0^s 3\pi|\alpha - \alpha'|\, \|xe^{-ith(\alpha',\theta,\lambda)}\delta_0\|\, dt.$$

But $x(t) = x + \int_0^t p(u)\, du$ where $p(u) = e^{iuh}pe^{-iuh}$ and $p = [x,h]$ has norm at most 2. Since $x\delta_0 = 0$,

$$\text{LHS of (B.7)} \le \int_0^s 3\pi|\alpha - \alpha'| 2t\, dt = 3\pi s^2|\alpha - \alpha'|$$

as claimed. $\square$

Theorem B.5. *α can be chosen irrational so that* (B.1) *holds for $h(\alpha,\theta,\lambda)$ and all $\theta\in[0,2\pi]$, $\lambda\in[0,1]$.*

Proof. Let $f(t)=(\ln(2+|t|))^{1/5}$. We'll pick α_m, T_m, Δ_m inductively starting with $\alpha_1=1$ so

(i) $\alpha_{m+1}-\alpha_m=2^{-k_m!}$ for some k_m.

(ii) $\frac{1}{T_m}\int\limits_{T_m}^{2T_m}\|P_{n\geq T_m/f(T_m)}e^{-ish(\alpha,\lambda,\theta)}\delta_0\|^2\,ds\geq\frac{1}{f(T_m)^2}$ for all $\theta\in[0,\pi]$, $\lambda\in[0,1]$ and α with $|\alpha-\alpha_m|\leq\Delta_m$.

(iii) $|\alpha_{m+1}-\alpha_k|<\Delta_k$ for $k=1,2,\dots,m$.

By (i), $\alpha_\infty=\lim\alpha_m$ is irrational, and by (ii), (iii), the bound holds for α_∞, and (B.1) holds by Lemma B.1.

Start with $\alpha_1=1$. We'll show how to pick $T_m,\Delta_m,\alpha_{m+1}$ given $\alpha_1,\dots,\alpha_m$, $T_1,\dots,$
T_{m-1}, $\Delta_1,\dots,\Delta_{m-1}$. Given α_m, let $\delta_0=\varphi+\psi$ be the decomposition given by Lemma B.3 and let C_1,C_2 be the corresponding constants. Choose $T_m\geq 2T_{m-1}$ (and $T_1\geq 2$ so $T_m\geq 2^m$) so that

$$C_1^2-3\sqrt{2\pi}C_2(f(T_m)^{-1}+T_m^{-1})^{1/2}\geq 2f(T_m)^{-1}. \tag{B.8}$$

Since C_1 and C_2 are fixed (given α_m) and $f(T_m)\to\infty$, it is certainly possible.

Notice that

$$\begin{aligned}(\psi,\psi)_\sim &\equiv \frac{1}{T}\int\limits_T^{2T}\|P_{n<T/f(T)}e^{-ish}\psi\|^2\,ds\\ &\leq\frac{2\pi}{T}\#\{n\mid n<T/f(T)\}\,\|\psi\|^2\end{aligned} \tag{B.9}$$

since for any unit vector η,

$$\int\limits_{-\infty}^{\infty}|(\eta,e^{-ish}\psi)|^2\,ds\leq 2\pi\|\psi\|^2\,\|\eta\|^2 \tag{B.10}$$

by the Plancherel theorem. Thus, by (B.8) and Lemma B.2,

$$\frac{1}{T_m}\int\limits_{T_m}^{2T_m}\|P_{n\geq T_m/f(T_m)}e^{-ish(\alpha_m,\lambda,\theta)}\delta_0\|^2\,ds\geq\frac{2}{f(T_m)}.$$

By Lemma B.4, we can pick Δ_m so $|\alpha-\alpha_m|<\Delta_m$ implies

$$\frac{1}{T_m}\int\limits_{T_m}^{2T_m}\|P_{n\geq T_m/f(T_m)}e^{-ish(\alpha,\lambda,\theta)}\delta_0\|^2\,ds\geq\frac{1}{f(T_m)}.$$

Finally, pick α_{m+1} so $|\alpha_n-\alpha_{m+1}|<\Delta_n$ for $n=1,\dots,m$. $\square$

Remarks. 1. (B.10) is the standard estimate for which $\|\!| \cdot |\!\|$ was introduced (see (X.4.18) of [18]). It is used here as the Strichartz estimate [41] is used in the proof of Theorem 6.1 of [22]. Indeed, the above proof is essentially a variant of the proof of a similar result in [22] (Theorem 7.2 of [22]).

2. One can similarly prove an analogous result for a corresponding operator on $\ell^2(\mathbb{Z})$. The main difference in this case is that δ_0 might not be cyclic, and thus, to assure pure point spectrum, we need to perturb the potential at two consecutive points. The proof is essentially unchanged except for Lemma B.3, the analog of which can be obtained by uniformly bounding the m-functions for the two "half-line" operators, from which one can construct the Borel transform of the spectral measure for the "line" problem.

Appendix 3: ULE Fails for Many Models

In analogy with SULE, we'd say that H on $\ell^2(\mathbb{Z}^\nu)$ has ULE if there are $C, \alpha > 0$ with

$$|\varphi_n(m)| \leq Ce^{-\alpha|m-m_n|} \tag{C.1}$$

for all eigenfunctions φ_n and suitable m_n.

Motivated by Jitomirskaya [16], we present a simple argument that many models do not have ULE: Let Ω be a topological space, $T_i : \Omega \to \Omega$, $i = 1, \dots, \nu$ commuting homeomorphisms, and let μ be an ergodic Borel measure on Ω. Let $f : \Omega \to \mathbb{R}$ be continuous and define $V_\omega(n) = f(T^n\omega)$ for $n \in \mathbb{Z}^\nu$ where $T^n = T_1^{n_1} \dots T_\nu^{n_\nu}$. Let H_ω be the operator on $\ell^2(\mathbb{Z}^\nu)$,

$$(H_\omega u)(n) = \sum_{|m-n|=1} u(m) + V_\omega(n)u(n).$$

Theorem C.1. *If H_ω has* ULE *for ω in a set of positive μ-measure, then H_ω has pure point spectrum for any $\omega \in \operatorname{supp}(\mu)$, where $\operatorname{supp}(\mu)$ is the complement of the largest open set $S \subset \Omega$ for which $\mu(S) = 0$.*

Proof. Define the function $F : \Omega \to [0, \infty]$ by

$$F(\omega) = \sup_{\substack{t \in \mathbb{Q} \\ n,m \in \mathbb{Z}^\nu}} [|(\delta_n, e^{-itH_\omega}\delta_m)|(1 + |n - m|)^\nu].$$

When ULE holds, the proof of Theorem 7.5 shows that

$$|(\delta_n, e^{-itH_\omega}\delta_m)| \leq C_\omega e^{-\alpha_\omega|n-m|}$$

and it follows that $F(\omega) < \infty$. F is clearly measurable and translation invariant so $F(\omega) < \infty$ on a set of positive measure shows that $F(\omega) = C < \infty$ for a.e. ω. Thus on a dense set in $\operatorname{supp}(\mu)$:

$$|(\delta_n, e^{-itH_\omega}\delta_m)| \leq C(1 + |n - m|)^{-\nu} \tag{C.2}$$

with C independent of ω. By continuity, (C.2) holds on *all* of $\operatorname{supp}(\mu)$ and so the RAGE theorem [25] implies that H_ω has pure point spectrum for any $\omega \in \operatorname{supp}(\mu)$. $\square$

Example 1. Let $d\lambda$ be a probability measure on R and let $S = \mathrm{supp}(\lambda)$. Let $\Omega = S^{\mathbb{Z}^\nu}$, $d\mu = \bigotimes_{n\in\mathbb{Z}^\nu} d\lambda(\omega_n)$, $(T^n\omega)_m = \omega_{m-n}$, and $f(\omega) = \omega_0$. Then $\{H_\omega\}$ is the Anderson model. If $\gamma \in S$, the constant potential $\omega_n = \gamma$ lies in $\mathrm{supp}(\mu)$ and the corresponding H_ω has purely a.c. spectrum. Thus, ULE cannot hold.

Example 2. Let $\Omega = S^1$, the circle, α irrational, $d\mu = d\theta/2\pi$ and $T\theta = \theta + \pi\alpha$. Let f be an even function (e.g., $\gamma\cos(\,\cdot\,)$). Then [17] shows there are θ's for which H_θ has no point spectrum and so again ULE cannot hold.

Appendix 4: The Dimension of the Set Where $G(x) = \infty$

In this appendix we consider a probability measure $d\mu$ on $[0,1]$ and the function $G(x)$ given by (5.1), and relate the dimension of supports of μ to the dimension of the set where $G(x)$ is infinite.

Theorem D.1. *If $A = \{x \mid G(x) = \infty\}$ is a set of dimension α, then μ is supported on a set of dimension α.*

Proof. μ is obviously supported on A.

There is no inequality in the other direction for all μ, since there are point measures (obviously supported on a set of dimension 0) with $G(x) = \infty$ on $[0,1]$. However, if we are willing to replace μ by an equivalent measure, there is a complementary result:

Theorem D.2. *Let μ be a probability measure on $[0,1]$ and suppose μ is supported on a set of dimension α. Then, there is a measure ν equivalent to μ so that $A = \{x \mid G_\nu(x) = \infty\}$ has dimension at most α.*

Remark. The proof follows the strategy in Howland [14]; more precisely, it follows the strategy in [14] with some errors corrected.

Proof. Let S be a set of dimension α which supports μ. By inner regularity, we can find $\{C_n\}_{n=1}^\infty$ closed sets so $C_1 \subset C_2 \subset \cdots \subset S$, and μ is supported on $\bigcup_{n=1}^\infty C_n$. Since $C_n \subset S$ has dimension at most α, we can find a δ-cover $\bigcup_{m=1}^\infty B_m^{(n)}$ of C_n so that

$$\text{(i)} \qquad |B_m^{(n)}| \le 2^{-n}, \qquad B_m^{(n)} \text{ is an open interval}$$

$$\text{(ii)} \qquad C_n \subset \bigcup_{m=1}^\infty B_m^{(n)}$$

$$\text{(iii)} \qquad \sum_{m=1}^\infty |B_m^{(n)}|^{\alpha+2^{-n}} \le 2^{-n}\,. \tag{D.1}$$

Let $O_n = \bigcup_{m=1}^\infty B_m^{(n)}$ and $K_n = [0,1]\backslash O_n$. Since O_n is open, K_n is closed and so $d_n = \mathrm{dist}(K_n, C_n) > 0$. Let

$$\nu(\,\cdot\,) = \sum_{n=1}^\infty 2^{-n} d_n^2 \mu(\,\cdot \cap C_n) \equiv \sum_{n=1}^\infty \nu_n(\,\cdot\,).$$

Then, $\nu(A) = 0 \Leftrightarrow \mu(A \cap C_n) = 0$ for all $n \Leftrightarrow \mu(A) = 0$ so ν is equivalent to μ. Let

$$K_\infty = \underline{\lim}\, K_n = \bigcup_{m=1}^{\infty} \biggl(\bigcap_{n=m}^{\infty} K_n \biggr).$$

We claim that $G_\nu(x) < \infty$ for $x \in K_\infty$ and that

$$\begin{aligned} O_\infty &= [0,1] \backslash K_\infty = [0,1] \bigcap \overline{\lim}\, O_n \\ &= [0,1] \bigcap \biggl[\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} O_n \biggr] \end{aligned}$$

has dimension at most α which proves the desired result.

If $x \in K_\infty$, then eventually $x \in K_n$ and so $x \notin \bigcup_{n=1}^{\infty} C_n$. Thus,

(i) $\int \frac{d\nu_n(y)}{|x-y|^2} < \infty$ for all n (since $x \notin C_n$).

(ii) $\int \frac{d\nu_n(y)}{|x-y|^2} \le 2^{-n}$ for large n (since $x \in K_n$ for n large).

It follows that $G_\nu(x) < \infty$ as promised.

Given $\tilde\alpha > \alpha$, pick n_0 so $\alpha + 2^{-n_0} \le \tilde\alpha$. Then for each $n \ge n_0$, $\bigcup_{k=n}^{\infty} \bigcup_{m=1}^{\infty} B_m^{(k)}$ is a 2^{-n}-cover of O_∞ and by (D.1), its $|\cdot|^{\tilde\alpha}$ power sum is at most $2^{-(n-1)}$. Thus, O_∞ has $h^{\tilde\alpha}$ measure zero and so O_∞ has dimension at most α. $\square$

Appendix 5: Analysis of the Measures μ_p

Here we analyze Example 3 from Section 6.

We'll need information on the weight that μ_p gives to intervals. For any x, let

$$\Delta_n^{(1)}(x) = \{y \mid a_j(y) = a_j(x) \text{ for } j = 1, \dots, n\}. \tag{E.1}$$

$\Delta_n^{(1)}(x)$ is a dyadic interval of length 2^{-n} containing x uniquely determined by that except for certain dyadic rationals. Clearly,

$$\delta > 2^{-n} \Rightarrow \Delta_n^{(1)}(x) \subset (x-\delta, x+\delta) \tag{E.2}$$

and so, if $\delta > 2^{-n}$

$$\mu_p(x-\delta, x+\delta) \ge p^{N_n(x)}(1-p)^{n-N_n(x)} \tag{E.3a}$$

where

$$N_n(x) = \#\{j \le n \mid a_j(x) = 0\}. \tag{E.3b}$$

In particular, if $p < \frac{1}{2}$ ($\overline{\lim}$ occurs because $\log\delta < 0$)

$$\overline{\lim_{\delta\downarrow 0}}\, \frac{\ln[\mu_p(x-\delta, x+\delta)]}{\ln(2\delta)} \le -\frac{f(x)\ln p + (1-f(x))\ln(1-p)}{\ln 2} \tag{E.4}$$

where

$$f(x) = \varlimsup_{n\to\infty} N_n(x)/n. \tag{E.5}$$

If $p > \frac{1}{2}$, we replace f by $\varliminf N_n(x)/n$.

In particular, for any x, p:

$$\varlimsup_{\delta\downarrow 0} \frac{\ln[\mu_p(x-\delta, x+\delta)]}{\ln(2\delta)} \le -\frac{\ln(\min(p, 1-p))}{\ln 2} \tag{E.6a}$$

and

$$\int \frac{d\mu_p(y)}{|x-y|^\alpha} = \infty \text{ for all } x \in [0,1] \text{ if } 2^\alpha \min(p, 1-p) > 1. \tag{E.6b}$$

To get an upper bound let

$$\tilde{\Delta}_n^1(x) = \begin{cases} \Delta^{(1)}(x + \frac{1}{2^n}) & \text{if } a_{n+1}(x) = 1 \\ \Delta^{(1)}(x - \frac{1}{2^n}) & \text{if } a_{n+1}(x) = 0 \end{cases} \tag{E.7}$$

so $\tilde{\Delta}_1(x)$ is the next nearest dyadic interval (with the convention that we take $\Delta^{(1)}(x + \frac{1}{2^n})$ if x is at the midpoint of $\Delta^{(1)}(x)$). Define

$$\Delta_n^{(2)}(x) = \Delta^{(1)}(x) \cup \tilde{\Delta}^{(1)}(x).$$

Then

$$\delta < 2^{-n-1} \Rightarrow (x-\delta, x+\delta) \subset \Delta_n^{(2)}(x). \tag{E.8a}$$

Normally, $\mu_p(\tilde{\Delta}^{(1)}(x))$ and $\mu_p(\Delta^{(1)}(x))$ are of the same magnitude; the exception when $p < \frac{1}{2}$ (resp. $p > \frac{1}{2}$) is when a long string of 0's (resp. 1's) starts before position n and includes position $n+1$. For then subtracting $\frac{1}{2^n}$ from x changes many 0's into 1's. Explicitly, if $a_{n-\ell}(x) = \cdots = a_n(x) = a_{n+1}(x) = 0$ but $a_{n-\ell-1}(x) = 1$, then

$$\mu_p(\tilde{\Delta}^{(1)}(x)) = \left[\frac{(1-p)}{p}\right]^\ell \mu_p(\Delta^{(1)}(x)). \tag{E.8b}$$

For example, if x_0 is defined by

$$a_n(x_0) = \begin{cases} 1 & N! \le n < (N+1)! \quad N \text{ even} \\ 0 & N! \le n < (N+1)! \quad N \text{ odd} \end{cases}$$

then

$$\begin{aligned} \varlimsup \frac{\ln[\mu_p(x_0-\delta, x_0+\delta)]}{\ln(2\delta)} &= -\frac{\ln(\min(p, 1-p))}{\ln 2} \\ \varliminf \frac{\ln[\mu_p(x_0-\delta, x_0+\delta)]}{\ln[2\delta]} &= -\frac{\ln(\max(p, 1-p))}{\ln 2} . \end{aligned}$$

Fortunately, as we'll see, this behavior is very atypical of any of the μ_p's. For $p < \frac{1}{2}$, let $C_n(x)$ be defined by

$$C_n(x) = \sup\{\ell \mid a_n(x) = a_{n-1}(x) = \cdots = a_{n-\ell}(x) = 0\}$$

where we set $C_n(x) = 0$ if $a_n(x) = 1$. Then (6.12a,b) imply

$$\varliminf_{\delta\downarrow 0} \frac{\ln \mu_p(x-\delta, x+\delta)}{\ln(2\delta)} \geq -\frac{g(x)\ln p + (1-g(x))\ln(1-p)}{\ln 2} \tag{E.9a}$$

where

$$g(x) = \varliminf_{n\to\infty} \left[\frac{N_n(x) - C_n(x)}{n}\right]. \tag{E.9b}$$

Both N_n and C_n are functions of the sequence $\{a_i\}$ and so their behavior is well known. The law of large numbers says that a.e. w.r.t. μ_p, $\lim \frac{N_n(x)}{n} = p$ and a standard Borel-Cantelli argument shows that a.e. w.r.t. μ_p, $\varlimsup \frac{C_n(x)}{\ln n} = \frac{-1}{\ln p}$ so $\lim \frac{C_n(x)}{n} = 0$. Thus:

Proposition E.1. *Fix $p, q \in (0,1)$. Then a.e. x w.r.t. $d\mu_q$, we have*

$$\lim_{\delta\downarrow 0} \frac{\ln \mu_p(x-\delta, x+\delta)}{\ln(2\delta)} = \frac{-q\ln p - (1-q)\ln(1-p)}{\ln 2}.$$

Recall the definition of $H(p)$, $L(p)$ in (6.6)/(6.7) and of I_0.

Theorem E.2 ($\equiv$ Theorem 6.6). (1) *$d\mu_p$ has exact dimension $H(p)$.*

(2) *Suppose $p \in I_0$. Then for a.e. λ w.r.t. Lebesgue measure, the restriction to $[0,1]$ of the rank one perturbation of $d\mu_p$ has exact dimension $L(p)$.*

(3) *If $p \notin \bar{I}_0$, then for a.e. λ, the rank one perturbation of $d\mu_p$ is pure point.*

(4) *If $p \in (\frac{1}{4}, \frac{3}{4})$, $p \neq \frac{1}{2}$, then for* all *λ, the restriction to $[0,1]$ of the rank one perturbation of $d\mu_p$ is purely singular continuous (so we have an example with singular continuous spectrum for all λ).*

Proof. (1) By the last proposition, the quantity $\alpha(x)$ given by (2.2) is $H(p)$ for a.e. x w.r.t. $d\mu_p$ so by Corollary 2.2, $d\mu_p$ has dimension $H(p)$.

(2) By the last proposition with $q = \frac{1}{2}$ (recall $d\mu_{1/2}$ is Lebesgue measure) and Lemma 5.4 for a.e. x w.r.t. Lebesgue measure

$$\lim_{\epsilon\downarrow 0} \epsilon^{-(1-\alpha)} \operatorname{Im} F(x+i\epsilon) = 0 \qquad (\text{resp. } \infty)$$

if $\alpha > L(p)$ (resp. $\alpha < L(p)$). By Simon-Wolff [33,40], $(d\mu_p)_\lambda$ is supported on this Lebesgue typical set for a.e. λ. Thus by Theorems 4.1 and 4.2, the rank one perturbation has dimension $L(p)$ for a.e. λ.

(3) By the last proposition and Proposition 2.4, for a.e. x w.r.t. Lebesgue measure

$$\int \frac{d\mu_p(y)}{|x-y|^\alpha} < \infty$$

if $\alpha < \gamma(p)$ (and is infinite a.e. if $\alpha > \gamma(p)$). $\frac{2\pm\sqrt{3}}{4}$ are precisely the points where $\gamma(p) = 2$ and so $p \notin \bar{I}_0$ means $\int \frac{d\mu_p(y)}{|x-y|^2} < \infty$ for a.e. x so Simon-Wolff [33,40] implies the rank one perturbations are pure point for a.e. λ.

(4) By (E.6), if $p \in (\frac{1}{4}, \frac{3}{4})$, then $\int \frac{d\mu_p(y)}{|x-y|^2} = \infty$ for all $x \in [0,1]$ and so by the Aronszajn-Donoghue theory [33], there is no point spectrum for any λ. $\square$

Remark. By using Theorem 5.1, one can show if $\frac{2-\sqrt{3}}{4} < p < \frac{1}{4}$, then the dimension of the set of λ for which $(\mu_p)_\lambda$ has some pure point spectrum is

$$D(p) = -\frac{q \ln q + (1-q)\ln(1-q)}{\ln 2}$$

where

$$q = \frac{-2\ln 2 - \ln(1-p)}{\ln p - \ln(1-p)} .$$

As a final variant in this class of examples, we'll give an example of a measure supported by the set W_α defined in (2.3) (examples of this kind go back to Besicovitch [5]). Fix $0 < p_1 < p_2 < \frac{1}{2}$. Define a measure $d\mu_{p_1 p_2}$ on $[0,1]$ as follows: The variables $a_n(x)$ will be independent for different n but not identically distributed. Rather

$$\text{Prob}(a_n(x) = 0) = \begin{cases} p_1 & N! \le n < (N+1)! \quad N \text{ odd} \\ p_2 & N! \le n < (N+1)! \quad N \text{ even.} \end{cases}$$

Then by the law of large numbers, one easily sees that for a.e. x w.r.t. $d\mu_{p_1 p_2}$,

$$\overline{\lim_{n\to\infty}} \frac{N_n(x)}{n} = p_2$$
$$\underline{\lim_{n\to\infty}} \frac{N_n(x)}{n} = p_1$$
$$\overline{\lim_{n\to\infty}} \frac{C_n(x)}{\ln n} < \infty$$

so that by analogs of (E.4), (E.5), and (E.9):

$$\overline{\lim_{\delta\downarrow 0}} \frac{\ln \mu_{p_1 p_2}(x-\delta, x+\delta)}{\ln 2\delta} = -\frac{p_2 \ln p_2 + (1-p_2)\ln(1-p_2)}{\ln 2} = H(p_2)$$
$$\underline{\lim_{\delta\to\infty}} \frac{\ln \mu_{p_1 p_2}(x-\delta, x+\delta)}{\ln 2\delta} = -\frac{p_1 \ln p_1 + (1-p_1)\ln(1-p_1)}{\ln 2} = H(p_1).$$

It follows that

$$H(p_1) < \alpha < H(p_2) \Rightarrow \text{for } \mu_{p_1 p_2} \text{ a.e. } x,$$
$$\overline{\lim_{\delta\downarrow 0}} \frac{\mu(x-\delta, x+\delta)}{\delta^\alpha} = \infty; \quad \underline{\lim_{\delta\downarrow 0}} \frac{\mu(x-\delta, x+\delta)}{\delta^\alpha} = 0.$$

Remarks. 1. One can modify this contradiction to find a measure $d\nu$ so that a.e. $\overline{\lim_{\delta\downarrow 0}}$ $\frac{\ln \nu(x-\delta,x+\delta)}{\ln(2\delta)} = 1$, $\underline{\lim_{\delta\downarrow 0}} \frac{\ln \nu(x-\delta,x+\delta)}{\ln(2\delta)} = 0$ so that only if $\alpha = 0, 1$ does $\nu(x \mid \lim_{\delta\downarrow 0} \frac{\nu(x-\delta,x+\delta)}{\delta^\alpha}$ exists$) = 1$.

2. One can further analyze $d\mu_{p_1p_2}$ to prove that for a.e. λ, the rank one perturbed measure $(d\mu_{p_1p_2})_\lambda$ has dimension $L(p_1)$.

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