NORMAL FORMS AND QUANTIZATION FORMULAE

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Abstract

We consider the Schrödinger operator $Q = -\hbar^2\Delta + V$ in $\mathbb{R}^n$ where $V(\mathbf{x})$ belongs to a Gevrey class, tends to $+\infty$ as $|x| \to +\infty$ and has a unique non-degenerate minimum. A quantization formula is obtained for all eigenvalues of Q belonging to any interval $[0, \varphi(\hbar)]$ up to an error of order $\hbar^\infty$. Here $\varphi(x)$ is any positive, increasing function on $]0,1[$ such that $\varphi^b(x)\ln x \to 0$ as $x \to 0$ and b an explicitly determined constant. The proof is based upon uniform Nekhoroshev estimates on the quantum normal form constructed quantizing the Lie transformation.

1 Introduction and statement of the results

Consider the Schrödinger operator $Q(\hbar) = -\hbar^2\Delta/2 + V(\mathbf{x})$. Assume:

(A1) $V \in C^\infty(\mathbb{R}^n),\ n \geq 1; \qquad V \geq 0, \quad V \sim |x|^\nu, \quad |\mathbf{x}| \to \infty, \quad$ for some $\nu > 0$;

(A2) V has a unique non-degenerate minimum, say at zero. Without loss we can take $V(0) = 0$ and $\mathrm{Hess}_V(0) = \mathrm{diag}[\omega_1^2, \ldots, \omega_n^2]$.

By (A1) Q defined on $D(Q) = H^2(\mathbb{R}^n) \cap D(V)$ is self-adjoint in $L^2(\mathbb{R}^n)$ with discrete spectrum (see e.g.[RS], §XIII.14). Finding an approximate quantization formula for its eigenvalues when the Hamiltonian function $q(\mathbf{x}, \xi) := \xi^2/2 + V(\mathbf{x})$ is non-integrable is a classical problem: it is known since the early days of quantum mechanics that the eigenvalues of Q close enough to 0 are approximated by those of the harmonic oscillator up to order $\hbar^2$. The mathematical justification of this and other properties of the spectrum "near the bottom of the well", including a quantization formula valid for all eigenvalues lying in an interval $[0, C\hbar]$, $C > 0$, has been obtained only much more recently (see e.g.[CDS, HS, KS, Si] and references therein). On the other hand, in analytical mechanics non-integrable systems of this type can be approximated by integrable ones via the normal form construction. Write $q = p_0(\mathbf{x}, \xi) + V_3(\mathbf{x})$ where

$$p_0(\mathbf{x}, \xi; \omega) := \frac{1}{2}\sum_{k=1}^{n}\left(\xi_k^2 + \omega_k^2 x_k^2\right), \quad V_3(\mathbf{x}) := V(\mathbf{x}) - \frac{1}{2}\sum_{k=1}^{n}\omega_k^2 x_k^2 \tag{1.1}$$

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Now p_0 is an integrable system (the harmonic oscillator), and $V_3(\mathbf{x},\xi) = O(|\mathbf{x}|^3)$ as $|\mathbf{x}| \to 0$. Restricting $(\mathbf{x},\xi)$ to the sphere $\Omega_R := \{(\mathbf{x},\xi)\} \mid \|\xi\|^2 + \|\mathbf{x}\|^2 \le R^2\}$ the "size" $\|V\|_R := \max_{\|\mathbf{x}\| \le R} V_3(\mathbf{x})$ of the perturbation V_3 can be made small: one then constructs on Ω_R a canonical transformation close to the identity, such that the image of q admits, to any given order, an expansion according to the powers of $\|V\|_R/\|p_0\|_R$ whose coefficients depend only on the integrable combinations $\xi_i^2 + \omega_i^2 x_i^2, i = 1,\dots,n$. As shown by Sjöstrand[Sj] through the construction of the Fourier integral operator "quantizing" the above canonical transformation, the microlocal analysis makes possible to transport this procedure in quantum mechanics. This yields a quantization formula mod $\hbar^\infty$ for the "semi-excited levels", i.e. there is a smooth real-valued function $F(t_1,\dots,t_n;\hbar)$ admitting a full asymptotic expansion in $\hbar$ such that, $\forall \delta > 0$, all eigenvalues of Q in $[0,\hbar^\delta]$ have the form

$$F((k_1+1/2)\hbar,\dots,(k_n+1/2)\hbar;\hbar) + O(\hbar^\infty), \qquad k \in \mathbb{N}^n \tag{1.2}$$

The term of order 0 is the classical Birkhoff normal form of q truncated at a suitable δ dependent order.

It is furthermore remarked in [Sj] that the Nekhoroshev technique ([Ne]; see also, e.g.,[Gi]) for estimating the remainder of the truncated normal form should be applicable in this context. To that effect recall that the normal form can be constructed measuring the "size" of the perturbation by a numerical parameter ϵ, considering the Hamiltonian family $p_\epsilon(\mathbf{x},\xi) = p_0(\mathbf{x},\xi;\omega) + \epsilon V_3(\mathbf{x})$. This generates expansions in powers of ϵ, and the canonical transformation mapping p_ϵ into the normal form can be defined on any bounded subset of $\mathbb{R}^{2n}$. If V is entire holomorphic and fulfills some growth conditions (for instance, any polynomial) this procedure admits order by order an exact quantization formula[DEGH, GP] yielding the quantum (Rayleigh-Schrödinger) perturbation expansions for the eigenvalues of $P_\epsilon := P_0 + \epsilon V_3$ near those of the harmonic oscillator $P_0(\hbar,\omega) := \dfrac{1}{2}\sum_{j=1}^{n} -\hbar^2\partial^2_{\xi_j} + \omega_j^2 x_j^2$. Moreover, if V_3 has a Hilbert-Schmidt (Weyl) symbol, the error of the quantum perturbation expansion can be estimated uniformly in $\hbar$ through the above-mentioned Nekhoroshev method[Ba1, BV].

We apply here this technique to generate a quantization formula mod $\hbar^\infty$: expansions in powers of $\hbar$ and of ϵ are indeed related by the unitary rescaling

$$U_\epsilon : L^2(\Re^n) \ni \psi(\mathbf{x}) \mapsto (U_\epsilon\psi)(\mathbf{x}) := \epsilon^n \psi(\epsilon \mathbf{x}) \in L^2(\Re^n)\ .$$

The image under U_ϵ of the operator $Q(\hbar)$ has clearly the form

$$S_\epsilon := U_\epsilon Q U_\epsilon^{-1} = \epsilon^2 P(\hbar',\epsilon), \qquad \hbar' = \frac{\hbar}{\epsilon^2} \tag{1.3}$$

$$P(\hbar',\epsilon) := P_0(\hbar',\omega) + \epsilon V_\epsilon(\mathbf{x}), \tag{1.4}$$

$$V_\epsilon(\mathbf{x}) := \epsilon^{-3} V_3(\epsilon \mathbf{x}) = \sum_{h+k+l=3} (\partial^3_{x_h x_k x_l} V)(0) x_h x_k x_l + \ldots \tag{1.5}$$

Hence $\mathrm{Spec}(Q(\hbar)) = \epsilon^2 \mathrm{Spec}(P(\hbar',\epsilon))$. On the other hand (details in Section 2) under modifications of the potential outside a compact the eigenvalues undergo variations of order $\hbar^\infty$; thus V_ϵ can be replaced by an operator with Hilbert-Schmidt (Weyl) symbol. Hence by uniform error estimates on perturbation theory (Section 3, Theorem 3.1) the eigenvalues $\lambda_k(\hbar',\epsilon), k \in (\mathbb{N} \cup \{0\})^n$ of $P(\hbar',\epsilon)$ will admit an approximate quantization formula. Actually, our first result is:

Theorem 1.1 *In addition to (A1), (A2) above assume*
(A3) There exist $\tau > n-1, \gamma > 0$ *such that*

$$\langle \omega, \mathbf{k} \rangle \geq \gamma |\mathbf{k}|^{-\tau}, \quad \forall \mathbf{k} \in \mathbb{Z}^n \setminus \{0\}, \quad |k| := |k_1| + \ldots + |k_n|, \ \omega := (\omega_1, \ldots, \omega_n) \tag{1.6}$$

(A4) There exist $1 < q < \infty$, $\alpha > 0$ *and a neighbourhood of zero* U *such that, such that,* $V \in \mathcal{G}^{q,\alpha}(U)$, *the Gevrey class of all* $f \in \mathbb{C}^\infty(\mathbb{R}^n;\mathbb{R})$ *fulfilling the estimates*

$$\sup_{\mathbf{x} \in U} \left| \frac{\partial^{|j|} f}{\partial z^j}(z) \right| \leq K(U) \frac{(|j|!)^q}{\alpha^{|j|}} \tag{1.7}$$

where $K(U) > 0$, $j = (j_1, \ldots, j_n)$ *is a multi-index, and* $|j| := |j|_1 + \ldots + |j|_n$.
Let furthermore $b := [(\tau+2)(2q+3)]^{-1}$. *Then:*

1. *There exist* $h_* > 0, \epsilon_* > 0, A > 0$ *(independent of* $\hbar$ *and* ϵ*) and a smooth function* $\mathcal{Z}(t_1, \ldots, t_n; \hbar, \epsilon) : \mathbb{R}^n \times [0, h_*] \times [0, \epsilon_*] \to \mathbb{R}$ *admitting a full asymptotic expansion in* $\hbar$:

   $$\mathcal{Z}(t_1, \ldots, t_n; \hbar, \epsilon) \sim \mathcal{Z}_0(t_1, \ldots, t_n; \epsilon) + \hbar \mathcal{Z}_1(t_1, \ldots, t_n; \epsilon) + \ldots \tag{1.8}$$

   such that, $\forall \epsilon \leq \epsilon_*, \hbar \leq \hbar_* \epsilon^2$ *the eigenvalues of* $Q(\hbar)$ *in* $[0, \epsilon^2]$ *have the representation*

   $$\langle (\mathbf{k}+1/2), \omega \rangle \hbar + \epsilon^3 \mathcal{Z}\left((\mathbf{k} + \frac{1}{2}) \frac{\hbar}{\epsilon^2}, \frac{\hbar}{\epsilon^2}, \epsilon \right) + \epsilon^2 O(\hbar^\infty) + O\left(\epsilon^3 \exp - \frac{A}{\epsilon^b} \right) \tag{1.9}$$

2. *For any fixed integer $M < N(\epsilon) := (\epsilon_*/\epsilon)^b$ we have*

$$\epsilon^3 \mathcal{Z}\left((\mathbf{k}+\frac{1}{2})\frac{\hbar}{\epsilon^2}, \frac{\hbar}{\epsilon^2}, \epsilon\right) = \mathcal{F}\left((\mathbf{k}+\frac{1}{2})\hbar, \hbar\right) + O_M(\epsilon^M) + O\left((\hbar/\epsilon^2)^\infty\right) \quad (1.10)$$

where $\mathcal{F}\left((\mathbf{k}+\frac{1}{2})\hbar, \hbar\right)$ is as in Theorem 0.1 of [Sj].

Remarks.

1. $\mathcal{Z}_0$ is the classical Birkhoff normal form computed up to order $N(\epsilon) = (\epsilon_*/\epsilon)^b$, i.e., $I_k = \dfrac{1}{2\omega_k}(\xi_k^2 + \omega_k^2 x_k^2)$ being the action variables of the harmonic oscillator:

$$\mathcal{Z}_0(I_1, \dots, I_n, \epsilon) = \sum_{k=1}^{n} \omega_k I_k + \sum_{s=0}^{N(\epsilon)} \zeta_s(I_1, \dots, I_n; \epsilon)\epsilon^s \quad (1.11)$$

   where the coefficients $\zeta_s : s = 1, \dots$ are constructed through the Birkhoff transformation (see e.g.[Ga], §5.12). The quantum corrections $\mathcal{Z}_j : j \geq 1$ are likewise recursively defined.

2. The dependence of (1.9) on ϵ comes from the modifications of the potential at infinity, which break the scale invariance of perturbation theory. Thus the normal form (1.9) is different from that of [Sj], Theorem 0.1. However, (Theorem 3.1 below) the difference is $O(\epsilon^{-2}\hbar)^\infty)$ on each term ζ_s. This will yield Assertion 2.

3. The accuracy of the approximation (1.9) for the eigenvalues increases exponentially approaching the bottom of the well ($\epsilon \to 0$) with $h/\epsilon^2 \to 0$. Hence (1.9) can be considered a local semiclassical version of the Nekhoroshev theorem.

Choosing $\epsilon^2 = \varphi(\hbar)$ so as to maximize the interval of validity of the quantization formula for $Q(\hbar)$ within the requirements $\hbar/\epsilon^2 \to 0$ and $\exp -\dfrac{A}{\epsilon^b} = O(\hbar^\infty)$ as $\hbar \to 0$ we get our main result:

Corollary 1.1 *Let $\varphi :]0,1[\to \mathbb{R}_+$ be positive and increasing with $\lim_{x\to 0} \varphi(x)^b \ln x = 0$. Then all eigenvalues of $Q(\hbar)$ in any interval $[0, \varphi(\hbar)]$ admit the representation*

$$\langle(\mathbf{k}+1/2), \omega\rangle\hbar + \varphi(\hbar)^3 \mathcal{Z}\left(\frac{(k_1+1/2)\hbar}{\varphi(\hbar)^2}, \dots, \frac{(k_n+1/2)\hbar}{\varphi(\hbar)^2}, \frac{\hbar}{\varphi(\hbar)}, \frac{1}{\sqrt{\varphi(\hbar)}}\right) + O(\hbar^\infty) \quad (1.12)$$

Remarks.

1. Taking for instance $\varphi(x) = |\ln^{\frac{1}{b}+\eta}(x)|^{-1}$, $\eta > 0$, (1.12) yields a quantization formula more general than Theorem 0.1 of [Sj], because $[0, \hbar^\delta] \subset \left[0, |\ln^{1/b+\eta}(\hbar)|^{-1}\right]$ for any $\delta > 0, \eta > 0$. The result is stated for the non resonant, diophantine case (Assumption A3) and under more restrictive assumptions on V, but its generalization to the more general situations considered in [Sj] offers little difficulty.

2. Much stronger results, namely a quantization formula up to order $e^{-1/\hbar^\delta}$, $\delta < 1$, valid for eigenvalues in any interval $[0, M], M > 0$, have been recently stated by Popov[Po], by a direct quantization procedure of KAM theory. Its validity however requires restrictions on the quantum numbers, which are arbitrary here.

With a different choice of the function φ, by Assertion 2 of the theorem we recover Sjöstrand's result. Set indeed $\epsilon(\hbar) := h^\delta$. For ϵ so small that $1/\delta^2 < (\epsilon_*/\epsilon)^b$, and any $M > 1/\delta^2$ one has $O(\epsilon^M) = O(\hbar^{\sqrt{M}})$. This gives Theorem 0.1 of [Sj], namely:

Corollary 1.2 *Under the same assumptions of Theorem 1.1, fix* $0 < \delta < 1$. *Then all eigenvalues of* $Q(\hbar)$ *in* $[0, \hbar^\delta]$ *admit the representation*

$$\langle(\mathbf{k} + 1/2), \omega\rangle\hbar + \mathcal{F}((k_1 + 1/2)\hbar, \ldots, (k_n + 1/2)\hbar, \hbar) + O(\hbar^\infty) \tag{1.13}$$

2 Reduction to a trace-class perturbation

In this section we will estimate the variation undergone by the eigenvalues of the operator $P(\hbar', \epsilon)$ defined by (1.4,1.5) when the perturbation V_3 is replaced by an operator admitting a suitable Hilbert-Schmidt Weyl symbol.

Denote I_B the open interval $]0, B[, 0 < B \le +\infty$, and $\tilde{I}_B$ any compact subinterval of I_B. Let $H_1(\kappa), H_2(\kappa)$ be a pair of norm-resolvent continuous, self-adjoint operator families in a Hilbert space $\mathcal{H}$ ([Ka], §VII.1.2 and VII.1.3) for $\kappa \in I_B$, with compact resolvents. If $J \subset \mathbb{R}$ is a bounded interval, denote $\mathrm{Spec}_J(H_{1,2}) = \mathrm{Spec}(H_{1,2}) \cap J$.

Definition 2.1 *We say that* $\mathrm{Spec}_J(H_1) = \mathrm{Spec}_J(H_2) \pmod{\kappa^\infty}$ *as* $\kappa \to 0$ *in* I *if, given any eigenvalue* $\lambda_1(\kappa) \in J$ *of* $H_1(\kappa)$, *for any corresponding eigenvector* $\phi_1(\kappa)$ *we have* $\phi_1 \in D(H_2)$ *and* $\|(H_2(\kappa) - \lambda_1(\kappa))\phi_1(\kappa)\| = O(\kappa^\infty)$; *conversely, given any eigenvalue*

$\lambda_2(\kappa) \in J$ of $H_2(\kappa)$, for any corresponding eigenvector $\phi_2(\kappa)$ we have $\phi_2 \in D(H_1)$ and $\|(H_1(\kappa) - \lambda_2(\kappa))\phi_2(\kappa)\| = O(\kappa^\infty)$.

Remark. If $\mathrm{Spec}_J(H_1) = \mathrm{Spec}_J(H_2)$ (mod κ^∞), then for any $\lambda_1 \in \mathrm{Spec}_J(H_1)$ there exists $\lambda_2 \in \mathrm{Spec}_J(H_2)$ such that $\lambda_1(\kappa) = \lambda_2(\kappa) + O(\kappa^\infty)$ and conversely.

Let $V \in C^\infty(\mathbb{R}^n)$ fulfill Assumptions (A1-A4), and let once more $P(\hbar', \epsilon)$ be the Schrödinger operator family defined in $L^2(\mathbb{R}^n)$ by the action of $P_0(\hbar') + \epsilon V_\epsilon$ on $D(P_0) \cap D(V)$. As above, $\hbar' = \hbar/\epsilon^2$ and $P_0(\hbar')$ is the harmonic-oscillator operator

$$P_0(\hbar', \omega) := \frac{1}{2} \sum_{j=1}^{n} -\hbar'^2 \partial^2_{\xi_j} + \omega_j^2 x_j^2 \tag{2.1}$$

defined on $H^2(\mathbb{R}^n) \cap L^2_2(\mathbb{R}^n)$. We can define spheres in $\mathbb{R}^{2n}$ by means of p_0. Namely,

$$\Omega_R := \{(\mathbf{x}, \xi) \mid p_0(\mathbf{x}, \xi) \leq R^2\}$$

The desired result is represented by the following

Proposition 2.1 *There are $R > 0$ and $a_{\epsilon,R}(\mathbf{x}, \xi) \in C_0^\infty(\mathbb{R}^{2n}; \mathbb{R}_+)$ such that*

1. *$a_{\epsilon,R}(\mathbf{x}, \xi) \equiv V_\epsilon(\mathbf{x})$ for $(\mathbf{x}, \xi) \in \Omega_R$, $a_{\epsilon,R}(\mathbf{x}, \xi) \equiv 0$ for $(\mathbf{x}, \xi) \notin \Omega_{2R}$ and*

$$\max_{z \in \Re^{2n}} \left| \frac{\partial^{|j|} a_\epsilon}{\partial z^j}(z) \right| \leq \epsilon^{c-3} K \frac{(|j|!)^q}{\alpha^{|j|}} \tag{2.2}$$

Here $z = (\mathbf{x}, \xi)$, and $c \geq 3$ is the vanishing order of V_ϵ at $x = 0$.

2. *The operator family*

$$P_R(\hbar', \epsilon) := P_0 + \epsilon A_{\epsilon,R}, \qquad A_{\epsilon,R} = Op^W(a_{\epsilon,R}) \tag{2.3}$$

defined on $D(P_0)$ is positive self-adjoint in $L^2(\mathbb{R}^n)$, and norm-resolvent continuous with compact resolvents for $(\hbar', \epsilon)$ in a neighbourhood $\tilde{I}_B \times \tilde{I}_B$ of zero;

3. *For any $J \subset \mathbb{R}_+$, $\mathrm{Spec}_J(P) = \mathrm{Spec}_J(P_R)$ (mod $(\hbar')^\infty$).*

Remarks.

1. The estimates (2.2) imply that $a_{\epsilon,R} \in \mathcal{G}_R^{q,\alpha}(\mathbb{R}^{2n}) \subset C_0^\infty(\mathbb{R}^{2n})$ uniformly in $\epsilon \in \tilde{I}_B$, $\mathcal{G}_R^{q,\alpha}(\mathbb{R}^{2n})$ being the Gevrey class of all $f \in C^\infty(\mathbb{R}^{2n})$ with support in Ω_R.

2. $A = Op^W(a)$ denotes the operator in $L^2(\mathbb{R}^n)$ defined by the Weyl quantization of the symbol (equivalently, classical observable) $a(\mathbf{x}, \xi)$:
$$(Op^W(a)\psi)(x) = (2\pi\hbar)^{-n/2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a\left(\frac{\mathbf{x}+\mathbf{y}}{2}, \xi\right) e^{i\frac{\langle \mathbf{x}-\mathbf{y},\xi\rangle}{\hbar}} \psi(y)\, d\mathbf{y} d\xi \qquad (2.4)$$
(2.4) defines a continuous map from $\mathcal{S}(\mathbb{R}^n)$ into itself; in this case (see e.g.[Ro], Th.III.44) $A = Op^W(a)$ extends to a trace-class self-adjoint operator in $L^2(\mathbb{R}^n)$, realized as an $\hbar$- pseudodifferential operator (for this notion, see [Ro],II.4).

3. The Weyl quantization (2.4) still generates ([Ro], §II.3) a map from $\mathcal{S}$ to $\mathcal{S}$ if a belongs to the symbol class $S(m; \phi_1, \phi_2)$ defined as the set of all functions $a(\mathbf{x}, \xi) \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ such that
$$|\partial_x^\alpha \partial_\xi^\beta a(\mathbf{x}, \xi)| \le C_{\alpha,\beta} m(\mathbf{x}, \xi) \phi_1(\mathbf{x}, \xi)^{-\alpha/2} \phi_2(\mathbf{x}, \xi)^{-\beta/2} \qquad (2.5)$$
$$\phi_1(\mathbf{x}, \xi) = 1 + \|\mathbf{x}\|^2, \quad \phi_2(\mathbf{x}, \xi) = 1 + \|\xi\|^2 \qquad (2.6)$$
where $m(\mathbf{x}, \xi)$ is a tempered weight (equivalently, tempered scale function: see e.g.[He], §2.1). In this case (see e.g.[Ro],§III.1) the corresponding $\hbar$- pseudodifferential operator extends to a self-adjoint operator in $L^2(\mathbb{R}^n)$. Particular cases of (2.4) are thus the maximal differential operator $P_0(\hbar)$, with symbol $p_0(\mathbf{x}, \xi) = \frac{1}{2}\sum_{k=1}^n (\xi_k^2 + \omega_k^2 x_k^2)$, the maximal multiplication operator V, with symbol $v(\mathbf{x}, \xi) = V(x)$, and the maximal differential operator $P(\hbar, \epsilon) = P_0 + \epsilon V_\epsilon$, with symbol $p_\epsilon(\mathbf{x}, \xi) = p_0(\mathbf{x}, \xi) + \epsilon V_\epsilon(x)$. In case of a multiplication operator we shall indifferently use both notations.

The first preliminary result we need is the following

Lemma 2.1 *There are $R_1 > 0$ and $\chi_1(\mathbf{x}) \in C_0^\infty(\mathbb{R}^n; \mathbb{R}_+)$ with $\chi_1 \equiv 1$ in $|\mathbf{x}| \le R_1$ and $\chi_1 \equiv 0$ outside $|\mathbf{x}| \ge 2R_1$ such the operator family*
$$P_1(\hbar', \epsilon) = P_0(\hbar') + \epsilon V_\epsilon \chi_1 \qquad (2.7)$$

defined on $D(P_0)$ is norm-resolvent continuous, positive self-adjoint in $L^2(\mathbb{R}^n)$ with compact resolvents for $(\hbar',\epsilon)\in \tilde I_B\times \tilde I_B$, and for any $J\subset \mathbb{R}_+$,

$$\mathrm{Spec}_J(P)=\mathrm{Spec}_J(P_1)\ (\mathrm{mod}\,\hbar'^{\infty}),\quad \text{as } \hbar'\to 0$$

Proof. Let $u\mapsto t_1(u)\in C^\infty([0,+\infty[,\mathbb{R}_+)$, $t_1(u)\equiv 1$ for $u\le R_1$, $t_1(u)\equiv 0$ for $u\ge 2R_1$ and set $\chi_1(x)=t_1(|x|)$. Since the maximal multiplication operator by $V_\epsilon\chi_1$ is bounded and self-adjoint in $L^2(\mathbb{R}^n)$, it is also bounded with respect to $P_0(\hbar')$ with relative bound zero. Hence the statement about the operator family $P_1(\hbar',\epsilon)$ follows by a well known perturbation argument (see e.g.[Ka],§VII.2). The symbol of $P_1(\hbar',\epsilon)$ is clearly the classical Hamiltonian

$$p^1_\epsilon(\mathbf{x},\xi)=\frac{1}{2}\sum_{k=1}^n(\xi_k^2+\omega_k^2x_k^2)+\epsilon\chi_1(x)V_\epsilon(x)$$

Under the present circumstancies any eigenvector $\psi_\epsilon(x;\hbar',\lambda)$ corresponding to any eigenvalue $0<\lambda(\epsilon)\le\overline{E}$ of $P(\hbar',\epsilon)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and fulfills the estimate

$$|\psi_\epsilon(x;\hbar',\lambda)|\le C\exp\left[-A\frac{|x|^{\frac{\nu}{2}+1}-B^{\frac{\nu}{2}+1}}{\hbar'}\right],\quad |x|>B,\ \epsilon\ge 0, \tag{2.8}$$

where A,B,C are suitable positive constants independent of $\epsilon,\hbar'$. The proof is obtained in Appendix by adapting the arguments of [BS], Theorem 3.3. Hence, on account of (2.8), choosing R_1 suitably large, and taking into account the normalization of the eigenvectors:

$$\|(1-\chi_1)V_\epsilon\psi_\epsilon(x;\hbar',\lambda)\|=O(\hbar'^{\infty})$$

and

$$\|(1-\chi_1)V_\epsilon\psi^1_\epsilon(x;\hbar',\mu)\|=O(\hbar'^{\infty})$$

the second estimate being clearly uniform with respect to ϵ. Therefore:

$$\begin{aligned}\|(P(\hbar',\epsilon)-\lambda)\psi_\epsilon(\hbar',\lambda)\|&=\|(P_0(\hbar')+\epsilon\chi_1V_\epsilon+\epsilon(1-\chi_1)V_\epsilon-\lambda)\psi_\epsilon(\hbar',\lambda)\|\\&=\|(P_1(\hbar',\epsilon)-\lambda)\psi_\epsilon(\hbar',\lambda)\|+O(\hbar'^{\infty})\end{aligned}$$

Conversely, any eigenvector $\psi^1_\epsilon(x;\hbar',\mu)$ corresponding to any eigenvalue $0<\mu(\epsilon)\le\overline{E}$ of $P_1(\hbar',\epsilon)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and fulfills the estimate (2.8) with $\nu=2$, possibly with different constants A and C. Hence:

$$\begin{aligned}\|(P_1(\hbar',\epsilon)-\mu)\psi^1_\epsilon(\hbar',\mu)\|&=\|(P_0(\hbar')+\epsilon V_\epsilon+\epsilon(\chi_1-1)V_\epsilon-\mu)\psi^1_\epsilon(\hbar,\mu)\|=\\&\|(P(\hbar',\epsilon)-\mu)\psi^1_\epsilon(\hbar,\mu)\|+O(\hbar'^{\infty})\end{aligned}$$

where $O(\hbar'^{\infty})$ is independent of ϵ. Since $\epsilon \to 0$ and $\hbar' \to 0$ this proves the Lemma.

Proof of Proposition 2.1. Let $u \mapsto t(u) \in C^{\infty}([0,+\infty[,\mathbb{R}_+)$, $t(u) \equiv 1$ for $u \leq R$, $t(u) \equiv 0$ for $u \geq 2R$ and set $\chi(\mathbf{x},\xi) = t(\sqrt{2p_0(\mathbf{x},\xi)})$, where, as above, $p_0(\mathbf{x},\xi) = \frac{1}{2}\sum_{k=1}^{n}(\xi_k^2+\omega_k^2 x_k^2)$. Denote

$$a_{\epsilon,R}(\mathbf{x},\xi) = V_\epsilon\chi_1\chi(\mathbf{x},\xi), \quad p_{\epsilon,R}(\mathbf{x},\xi) = \frac{1}{2}\sum_{k=1}^{n}(\xi_k^2+\omega_k^2 x_k^2) + \epsilon a_{\epsilon,R}(\mathbf{x},\xi) \tag{2.9}$$

Given $E>0$, denote furthermore $\Omega_1(E)$ the subgraph

$$\Omega_1(E) := \{(\mathbf{x},\xi) \in \mathbb{R}^{2n} \,|\, p^1_\epsilon(\mathbf{x},\xi) \leq E\} \tag{2.10}$$

with analogous definition for the subgraph $\Omega_{\epsilon,R}(E) := \{(\mathbf{x},\xi) \in \mathbb{R}^{2n} \,|\, p_{\epsilon,R}(\mathbf{x},\xi) \leq E\}$. Under our assumptions these sets are both compact.

Since $a_{\epsilon,R}(\mathbf{x},\xi) \in C_0^{\infty}(\mathbb{R}^{2n})$, the $\hbar'-$ pseudodifferential operator $A_{\epsilon,R} := Op^W(a_{\epsilon,R})$ is trace-class in L^2 and hence bounded (uniformly with respect to ϵ) as recalled above; consequently the pseudodifferential operator family $P_R(\hbar,\epsilon)$ in L^2 with Weyl symbol $p_{\epsilon,R}(\mathbf{x},\xi)$, defined by the action of $P_0 + A_{\epsilon,R}$ on $D(P_0)$ is norm-resolvent continuous with compact resolvents with respect to $(\hbar,\epsilon) \in \tilde{I}_B \times \overline{I}_B$. This proves Assertions (1) and (2) but for the estimate Gevrey which will be obtained below.

Writing as above $V_\epsilon\chi_1 = V_\epsilon\chi_1\chi(\mathbf{x},\xi) + V_\epsilon\chi_1(1-\chi(\mathbf{x},\xi))$, Assertion (3) will be an immediate consequence of the estimate

$$\|(1-X)\psi\| = 0(\hbar'^{\infty}), \qquad X := Op^W(\chi) \tag{2.11}$$

uniformly with respect to $\epsilon \in \overline{I}_B$, whenever $\psi = \psi_\epsilon(x,\hbar',\lambda)$ is either any eigenvector corresponding to any eigenvalue $\lambda_\epsilon < \overline{E}$ of $P_1(\hbar',\epsilon)$ or any eigenvector $\psi_\epsilon(x,\hbar',\mu)$ corresponding to any eigenvalue $\mu_\epsilon < \overline{E}$ of $P_R(\hbar',\epsilon)$. Since $\Omega^1_\epsilon(E) = \Omega_{a,\epsilon}(E)$ for $E \leq \overline{E}$ (with a suitable choice of $R(E)$) this estimate is essentially contained in the well known fact that the semiclassical wave front sets (or, equivalently, the semiclassical microlocal support; for these equivalent notions see e.g.[Ro],§IV.1) of the operators $P_1(\hbar',\epsilon)$ and $P_R(\hbar',\epsilon)$ are contained in $\Omega^1_\epsilon(E)$ and $\Omega_{\epsilon,R}(E)$, respectively. For the sake of completeness, however, we prove (2.11) here by adapting an argument of Helffer[He], §2.4.

First remark that, setting

$$m(\mathbf{x},\xi) = 1 + \|\mathbf{x}\|^2 + \|\xi\|^2, \quad \phi_1(\mathbf{x},\xi) = (1+|\mathbf{x}|^2)^{1/2}, \quad \phi_2(\mathbf{x},\xi) = (1+|\xi|^2)^{1/2} \tag{2.12}$$

m, ϕ_1, ϕ_2 are a triple of tempered weights ([Ro], §II.3 or [He], §2.1) and there are constants $C_{\alpha,\beta} > 0, C'_{\alpha,\beta} > 0$ independent of ϵ, R such that

$$|\partial_x^\alpha \partial_\xi^\beta p_{\epsilon,R}(\mathbf{x},\xi)| \leq C_{\alpha,\beta} m(\mathbf{x},\xi) \cdot \phi_1(\mathbf{x},\xi)^{-|\alpha|/2} \cdot \phi_2(\mathbf{x},\xi)^{-|\beta|2}$$
$$|\partial_x^\alpha \partial_\xi^\beta p_\epsilon^1(\mathbf{x},\xi)| \leq C'_{\alpha,\beta} m(\mathbf{x},\xi) \cdot \phi_1(\mathbf{x},\xi)^{-|\alpha|2} \cdot \phi_2(\mathbf{x},\xi)^{-|\beta|/2}$$

These estimates make $P_1(\hbar',\epsilon)$ and $P_a(\hbar',\epsilon)$ pseudodifferential operators with regular symbols in $S(m,\phi_1,\phi_2)$.

Let now $\Theta_1 \subset \mathbb{R}^{2n}$ be open with Θ_1^c compact, and such that $\Omega_1(E) \cap \overline{\Theta}_1 = \emptyset$. Let $\tau \in C^\infty(\mathbb{R}^{2n};\mathbb{R}_+)$ with $\tau = 1$ in a neighborhood of $\overline{\Theta}_1$ and $\mathrm{supp}\tau \cap \Omega_1(E) = \emptyset$. Let $\lambda(\hbar',\epsilon)$ be an eigenvalue of $P_1(\hbar',\epsilon)$ such that $\lambda(\hbar',\epsilon) \to E(\epsilon)$ as $\hbar' \to 0$, and $\psi(\hbar',\epsilon)$ the corresponding normalized eigenvector. Then there clearly exist $C(z) > 0$ and $\delta(z) > 0$ such that, for $z \in \mathbb{R}$, $|z - \lambda|$ suitably small,

$$\frac{1}{2}\sum_{k=1}^n \xi_k^2 + \omega_k^2 x_k^2 + \epsilon V_\epsilon \chi_1 - z + C(1-\tau(\mathbf{x},\xi)) \geq \delta m(\mathbf{x},\xi)$$

This makes the symbol $p_\epsilon^1 - z + C(1-\tau(\mathbf{x},\xi))$ uniformly elliptic. Hence (see once more ([Ro], §III.2 or [He], §2.3)) the operator family $P_1(\hbar',\epsilon) - z + C(1-T)$, $T := Op^W(\tau)$, which is once more norm-resolvent continuous in L^2 with compact resolvents, has an inverse mod$\hbar'^\infty$, i.e. there exists a continuous operator-valued family $E_{\hbar',\epsilon}(z)$, realized as an $\hbar'$- pseudodifferential operator, such that

$$E_{\hbar',\epsilon}(z) \cdot (P_1(\hbar',\epsilon) - z + C(1-T)) = I + R_{\hbar',\epsilon}(z)$$

with $\|R_{\hbar',\epsilon}(z)\| = 0(\hbar'^\infty)$ uniformly with respect to ϵ. Acting with this identity for $z = \lambda$ on the eigenvector $\psi(\hbar',\epsilon)$ we get

$$\psi(\hbar',\epsilon) = CE_{\hbar',\epsilon}(\lambda)(1-T)\psi(\hbar',\epsilon) - R_{\hbar',\epsilon}(\lambda)\psi(\hbar',\epsilon)$$

Let $\tau_1(\mathbf{x},\xi) \in C^\infty(\mathbb{R}^{2n})$ be bounded, with $\mathrm{supp}\,\tau_1 \subset \overline{\Theta}_1$, $\mathrm{supp}\,\tau_1 \cap \mathrm{supp}\,(1-\tau) = \emptyset$. Then $T_1 := Op^W(\tau_1)$ is once more a bounded pseudodifferential operator in L^2 with bound independent of $\hbar'$. Hence, writing

$$T_1\psi(\hbar',\epsilon) = CT_1E_{\hbar',\epsilon}(\lambda)(1-T)\psi(\hbar',\epsilon) - T_1R_{\hbar',\epsilon}(\lambda)\psi(\hbar',\epsilon)$$

we get

$$T_1\psi(\hbar',\epsilon) = CT_1E_{\hbar',\epsilon}(\lambda)(1-T)\psi(\hbar',\epsilon) + O(\hbar'^\infty)$$

where $O(\hbar'^{\infty})$ is uniform with respect to $\epsilon \in \overline{I}_B$. Now the product $T_1 E_{\hbar',\epsilon}(\lambda)(1-T)$ is again (see e.g.[Ro], Theorems II.32, III.9) an $h-$ admissibile pseudodifferential operator in L^2; the symbols τ_1 and $1-\tau$ of T_1 and $1-T$ are in S^0, while the symbol of $E_{\hbar',\epsilon}$ is ([He], Proposition 2.3.2) in $S\left(\frac{1}{m}, \phi_1, \phi_2\right)$; hence by the law of composition of symbols ([Ro], Theorem II.32 or [He], Theorem 2.3.1) the symbol of the product is bounded and hence the operator is continuous in L^2 by the Calderon-Vaillancourt theorem. On the other hand, the expansion in powers of $\hbar'$ of the symbol of $T_1 E_{\hbar',\epsilon}(\lambda)(1-T)$ is identically zero because τ_1 and $1-\tau$ have disjoint support. Hence $T_1\psi(\hbar',\epsilon) = O(\hbar'^{\infty})$, uniformly with respect to $\epsilon \in \overline{I}_B$. Now the support of τ_1 is arbitrary within $\overline{\Theta}_1$, and on the other hand also Θ_1 is arbitrary within the condition $\overline{\Theta}_1 \cap \Omega_1(E) = \emptyset$. This concludes the proof of the estimate (2.11) as far as the eigenvectors of $P_1(\hbar',\epsilon)$ are concerned. However the argument can be applied without change to the eigenvectors of $P_R(\hbar',\epsilon)$. This proves Assertion 3, i.e. the equivalence of the spectra mod $\hbar'^{\infty}$. We can now take $R > R_1$, $\chi_1(u) \in \mathcal{G}^{r,\beta}_{R_1}(\mathbb{R}_+)$, $\chi(u) \in \mathcal{G}^{r,\beta}_{R}(\mathbb{R}_+)$ $\forall r > 1$, $\beta > 0$. By (A4) there exist $q > 1, \alpha > 0$ such $V \in \mathcal{G}^{q,\alpha}(U)$ for any bounded open subset U of $\mathbb{R}^n$; since, by Taylor expansion near 0,

$$V_\epsilon(\mathbf{x}) = \frac{\epsilon^{c-3}}{3!}\sum_{h+k+l=3}(\partial^3_{x_h x_k x_l}V)(0)x_h x_k x_l + \frac{\epsilon^{c-2}}{4!}\sum_{h+k+l+m=4}(\partial^4_{x_h x_k x_l}V)(0)x_h x_k x_l x_m + \ldots$$

and the property

$$\frac{\partial^{|j|}V_\epsilon}{\partial x^j}(x) = \epsilon^{|j|-3}\left(\frac{\partial^{|j|}V}{\partial x^j}(\epsilon x)\right)$$

clearly imply also $V_\epsilon(\mathbf{x}) \in \mathcal{G}^{q,\alpha}(U)$ uniformly with respect to $\epsilon \in \tilde{I}_B$, i.e. $V_\epsilon(\mathbf{x})$ fulfills (2.2) for some $K = K(U)$. Now $\mathcal{G}^{r,\alpha}(U) \subset \mathcal{G}^{q,\alpha}(U)$ if $r < q$ (see e.g.[LM]), and therefore $a_\epsilon \in \mathcal{G}^{q,\alpha}(\mathbb{R}^{2n})$ with the same constants. This concludes the proof of the Proposition.

3 Perturbation theory estimates

Consider again the Schrödinger semiclassical pseudodifferential operator $P_R(\hbar',\epsilon) = P_0 + \epsilon A_{\epsilon,R}$ with Weyl symbol $p_{\epsilon,R}(\mathbf{x},\xi) = p_0(\mathbf{x},\xi;\omega) + \epsilon a_{\epsilon,R}(\mathbf{x},\xi)$ as in Proposition 2.1.

In this Section we will construct the quantum normal form for $P_R(\hbar',\epsilon)$ with uniform Nekhoroshev bound for the remainder. The result is (the dependence on R is dropped because the argument is carried out at R fixed):

Theorem 3.1 *There exist positive constants (independent of $\hbar', \epsilon$) $\epsilon_*, C_1, \ldots, C_6$ and, for $\epsilon < \epsilon_*$, a unitary transformation $U(\epsilon, \hbar') : L^2 \to L^2$ such that the Weyl symbol the unitary image $S(\hbar', \epsilon)$ of $P_R(\hbar', \epsilon)$ has the form*

$$p_0 + \epsilon \mathcal{Z}(\hbar', \epsilon) + \mathcal{R}(\hbar', \epsilon) \tag{3.1}$$

with the following additional properties:

1. *$\mathcal{Z}(\hbar', \epsilon)$ admits the representation*

$$\mathcal{Z}(\hbar', \epsilon) = \sum_{s=1}^{N(\epsilon)} \zeta_s(\mathbf{x}, \xi; \epsilon, \hbar')\epsilon^s \tag{3.2}$$

   Here $N(\epsilon) := (\epsilon_*/\epsilon)^b$ where $b := [(\tau + 2)(2q + 3)]^{-1}$ and $\zeta_s, s = 1, \ldots, N(\epsilon)$ is a smooth, bounded semiclassical symbol depending on $(\mathbf{x}, \xi)$ only through the actions $I_k = \dfrac{1}{2\omega_k}(\xi_k^2 + \omega_k^2 x_k^2), k = 1, \ldots, n$.

2. *The Weyl quantization of the symbols $\mathcal{Z}(\hbar', \epsilon), \mathcal{R}(\hbar', \epsilon)$ fulfill the estimates:*

$$\|Op^W(\mathcal{Z}(\hbar', \epsilon))\| \leq C_2 \tag{3.3}$$

$$\|Op^W(\mathcal{R}(\hbar', \epsilon))\| \leq C_4 \epsilon \exp\left[-\left(\frac{C_5}{\epsilon}\right)^b\right] \tag{3.4}$$

3. *For every fixed integer $k < N(\epsilon)$ let*

$$\mathcal{Z}^{(k)}(\mathbf{I}(\mathbf{x}, \xi); \epsilon, \hbar') = \sum_{s=1}^{k} \zeta_s(\mathbf{I}(\mathbf{x}, \xi); \epsilon, \hbar')\epsilon^s \tag{3.5}$$

   Then $\epsilon^3 \mathcal{Z}^{(k)}$ is independent of $\epsilon \bmod (\hbar')^\infty$, i.e. there is $\tilde{\mathcal{Z}}^{(k)}(\mathbf{I}(\mathbf{x}, \xi); \hbar)$ such that

$$\epsilon^3 \mathcal{Z}^{(k)}(\mathbf{I}(\mathbf{x}, \xi); \epsilon, \hbar/\epsilon^2) = \tilde{\mathcal{Z}}^{(k)}(\mathbf{I}(\mathbf{x}, \xi); \hbar) + O_k((\hbar')^\infty) \tag{3.6}$$

Remark. Denoting $Z = Op^W(\mathcal{Z}), R = Op^W(\mathcal{R})$ we have $S = P_0 + \epsilon Z + R$, where Z commutes with P_0. Hence by (3.2,3.4) the eigenvalues of $P_R(\hbar', \epsilon)$ have the expression

$$\sum_{s=1}^{N(\epsilon)} \zeta_s((l_1 + 1/2)\hbar'\omega_1, \ldots, (l_n + 1/2)\hbar'\omega_n; \epsilon, \hbar')\epsilon^s, \quad l_i = 0, 1, \ldots; \ i = 1, \ldots, n$$

up to an error of order $\exp[-(C_5/\epsilon)^b]$.

We prove this result first for an analytic perturbation f of p_0, and then we recover the conditions of the Theorem through an approximation procedure. To specify our assumptions on f we need some preliminary notions.

Define an analytic action Ψ of $\mathbb{T}^n$ into $\mathbb{R}^{2n}$ in the following way:

$$\Psi : \mathbb{T}^n \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \; (\phi, (\mathbf{x}, \xi)) \mapsto (\mathbf{x}', \xi') = \Psi_\phi(\mathbf{x}, \xi) \tag{3.7}$$

$$x'_k := \frac{\xi_k}{\omega_k} \sin \phi_k + x_k \cos \phi_k, \; \xi'_k := \xi_k \cos \phi_k - \omega_k x_k \sin \phi_k \tag{3.8}$$

This action is closely related to the Hamiltonian flow generated by p_0. Denoting $z := (\mathbf{x}, \xi)$, the motion of initial datum z_0 is $z(t) = \Psi_{\omega t}(z_0)$, $\omega t := (\omega_1 t, \ldots, \omega_n t)$.

If $g \in L^1_{loc}(\mathbb{R}^{2n})$, its angular Fourier coefficient of order k is defined by

$$\tilde{g}_k(z) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g(\Psi_\phi(z)) e^{-i\langle k, \phi \rangle} \, d\phi, \quad k \in \mathbb{Z}^n$$

If $g \in \mathbb{C}^1$ one has, as is well known

$$g(\Psi_\phi(z)) = \sum_{k \in \mathbb{Z}^n} \tilde{g}_k(z) e^{i\langle k, \phi \rangle} \Longrightarrow g(z) = \sum_{k \in \mathbb{Z}^n} \tilde{g}_k(z)$$

Remark furthermore that $g \equiv \tilde{g}_k$ for some fixed k if and only if

$$g(\Psi_\phi(z)) = e^{i\langle k, \phi \rangle} g(z) \tag{3.9}$$

Let now $g \in L^1(\mathbb{R}^{2n})$, and consider its space Fourier transform

$$\widehat{g}(s) := \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} g(z) e^{-i\langle s, z \rangle} \, dz \tag{3.10}$$

Given $\rho > 0, \sigma > 0$, denoting $\widehat{\tilde{g}_k}(s)$ the space Fourier transform of $\tilde{g}_k$, define the norm

$$\|g\|_{\rho,\sigma} := \sum_{k \in \mathbb{Z}^n} e^{\rho |k|} \int_{\mathbb{R}^{2n}} |\widehat{\tilde{g}_k}(s)| e^{\sigma \|s\|} \, ds \tag{3.11}$$

Definition 3.1 *Given $\rho > 0, \sigma > 0$, $\mathcal{A}_{\rho,\sigma}$ is the class of functions $g : \mathbb{R}^{2n} \to \mathbb{C}$ such that $\|g\|_{\rho,\sigma} < +\infty$.*

Remarks.

1. If $f \in \mathcal{A}_{\rho,\sigma}$ then it is analytic on $\mathbb{R}^{2n}$

2. As we have seen in the former Section $F = Op^W(f)$ is a trace-class, self-adjoint $\hbar'$-pseudodifferential operator in $L^2(\mathbb{R}^n)$; moreover (see e.g. [Ro], Cor.II.19)
$$\|F\|_{L^2\to L^2} \leq \int_{\mathbb{R}^{2n}} |\widehat{f}(s)|\, ds \equiv \|\widehat{f}\|_{L^1} \tag{3.12}$$
where $\widehat{f}(s)$ is the Fourier transform of f. Since $\|\widehat{f}\|_{L^1} \leq \|f\|_{\rho,\sigma}$, we have also
$$\|F\|_{L^2\to L^2} \leq \|f\|_{\rho,\sigma} \tag{3.13}$$

3. As above, since F is bounded the operator family
$$P_f = P_0(\hbar') + \epsilon F, \qquad D(P_f) = D(P_0) \tag{3.14}$$
of Weyl symbol $p_0(\mathbf{x}, \xi; \omega) + \epsilon f(\mathbf{x}, \xi)$ is norm-resolvent continuous, self-adjoint with compact resolvents in $L^2(\mathbb{R}^n)$.

Proposition 3.1 *For $\epsilon \in [0, \epsilon_*]$ let $f_\epsilon \in \mathcal{A}_{\rho,\sigma}$ for some positive $\rho > 0, \sigma > 0$; let also (1.6) be fulfilled. Set:*
$$E := \sup_{\epsilon\in[0,\epsilon_*]} \|f_\epsilon\|_{\rho,\sigma}$$
and assume
$$\epsilon < \epsilon_0 \rho^\tau \sigma^2, \qquad \epsilon_0 := \frac{\gamma}{2^{\tau+5} e^2 E \tau^\tau} \tag{3.15}$$
Then there exists a unitary transformation $T_\epsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $S(\hbar') := TP_f(\hbar')T^{-1}$ is a self-adjoint, $\hbar'-$ pseudodifferential operator family whose symbol has the expression
$$p_0(\mathbf{x}, \xi, \omega) + \epsilon \mathcal{Z}(\mathbf{I}(\mathbf{x}, \xi); \hbar', \epsilon) + \mathcal{R}(\mathbf{x}, \xi, \hbar', \epsilon) \tag{3.16}$$
where:

1. *$\mathcal{Z} \in A_{\rho/2,\sigma/2}$ and fulfills the estimate*
$$\|\mathcal{Z}\|_{\rho/2,\sigma/2} \leq 2E \tag{3.17}$$

2. *The principal symbol of $\mathcal{Z}$ is the normal form of the Hamiltonian $p_0(\mathbf{x}, \xi) + \epsilon f(\mathbf{x}, \xi)$ computed up to order $(\epsilon_0 \rho^\tau \sigma^2/\epsilon)^{\frac{1}{\tau+2}}$.*

3. *$\mathcal{R} \in A_{\rho/2,\sigma/2}$ is exponentially small, namely it fulfills the estimate*
$$\|\mathcal{R}\|_{\rho/2,\sigma/2} \leq e\epsilon E \exp\left[-(\tau+2)\left(\frac{\epsilon_0 \rho^\tau \sigma^2}{\epsilon}\right)^{\frac{1}{2+\tau}}\right] \tag{3.18}$$

The proof is based on the recursive construction and estimation of the quantum normal form, obtained via the quantum Lie transform[DEGH, Ba1].

Let W be a bounded self-adjoint operator in L^2 and $e^{tW/i\hbar'}$ the corresponding unitary group. If G is any bounded operator in L^2 its formal image under $e^{tW/i\hbar'}$ is

$$e^{-tW/i\hbar'}Ge^{tW/i\hbar'} = \sum_{l=0}^{\infty} t^l G_l \tag{3.19}$$

where $G_l : l = 0, 1, \dots$ is recursively defined as follows: $G_0 := G$, and

$$G_l = \frac{1}{l}\frac{1}{i\hbar'}[W, G_{l-1}], \qquad l \geq 1 \tag{3.20}$$

Definition 3.2 *Let the bounded self-adjoint operator W admit a smooth Weyl symbol $w : W = Op^W(w)$. Then we define quantum Lie transform generated by w the unitary operator* $\exp(W/i\hbar')$.

Recall that if a bounded operator G also admits a smooth Weyl symbol g, then the Weyl symbol of $\frac{1}{i\hbar'}[W, G]$ is the Moyal bracket $\{w, g\}_M$ (for the definition, see e.g. [Fo],§3.4). Hence by (3.20) the (formal) Weyl symbol of $e^{-W/i\hbar'}Ge^{W/i\hbar'}$ is $\sum_{k=0}^{\infty} g_l$, where

$$g_0 := g, \qquad g_l := \frac{1}{l}\{w, g_l\}_M, \qquad l \geq 1 \tag{3.21}$$

Consider now the Schrödinger operator P_f, with symbol $p_0 + \epsilon f$, and its image under the quantum Lie transform generated by a symbol w of order ϵ. Formally the symbol of the unitary image is

$$p_0 + \{w, p_0\}_M + \epsilon f + \mathcal{O}(\epsilon^2)$$

If w is such that $\{w, p_0\}_M + \epsilon f$ depends on $(\mathbf{x}, \xi)$ only through the actions $I_l = \dfrac{\xi_l^2 + \omega_l^2 x_l^2}{2\omega_l}$ the transformed operator commutes P_0 up to order ϵ^2.

A similar w can be determined and estimated by a purely classical method[Ba2] because, p_0 being a quadratic polynomial, the Moyal and Poisson brackets coincide. The main step in proving Proposition 3.1 is:

Proposition 3.2 *(Iterative Lemma). Let the function*

$$\mathcal{H}^{(k)} := p_0 + \epsilon \mathcal{Z}^{(k)} + f^{(k)} \tag{3.22}$$

be such that $\mathcal{Z}^{(k)}, f^{(k)} \in \mathcal{A}_{\rho-kd,\sigma-k\delta}$ *with* $d<\rho/(k+1)$ *and* $\delta<\sigma/(k+1)$*, and let* $\mathcal{Z}^{(k)}$ *be a function of* $(I_1,\ldots,I_n)$ *only. Assume that*

$$\|\mathcal{Z}^{(k)}\|_{\rho-kd,\sigma-k\delta} \leq \begin{cases} 0 & \text{if} \quad k=0 \\ E\sum_{l=0}^{k-1}\mu_s^l & \text{if} \quad k\geq 1 \end{cases} \tag{3.23}$$

$$\|f^{(k)}\|_{\rho-kd,\sigma-k\delta} \leq \epsilon E\mu_s^k \tag{3.24}$$

with

$$\mu_s := 8c_\Psi\frac{\epsilon E}{\delta^2 d^\tau}, \qquad c_\psi := (\tau/e)^\tau\gamma^{-1}$$

For $\hbar'>0$ *fixed, let* $H^{(k)}$ *be the Weyl quantization of* $\mathcal{H}^{(k)}$*. If*

$$\mu_s < \frac{1}{2} \tag{3.25}$$

there exists a unitary transformation T_k *such that Weyl symbol of the transformed operator* $T_kH^{(k)}T_k^{-1} := H^{(k+1)}$ *is given by (3.22) with* $k+1$ *in place of* k *(i.e.,* $H^{(k+1)} = Op^W(\mathcal{H}^{(k+1)})$*), and satisfies (3.23,3.24) with* $k+1$ *in place of* k*.*

Remarks.

1. In the notation of Theorem (3.1), $Op^W(\mathcal{Z}^{(k)}) = \sum_{s=0}^{k}\zeta_s(P_0,\hbar',\epsilon)\epsilon^s + O(\epsilon^{k+1})$. The eigenvalue $\zeta_s((l_1+1/2)\hbar'\omega_1,\ldots,(l_n+1/2)\hbar'\omega_n,\hbar',\epsilon)$, $l_i=0,1,\ldots,i=1,\ldots,n$ is the $k-$order coefficient of the quantum perturbation expansion of the eigenvalue of $P_R(\hbar')$ near the eigenvalue $(l_1+1/2)\hbar'\omega_1+\ldots+(l_n+1/2)\hbar'\omega_n$ of P_0.

2. The extra dependence on ϵ of the coefficients ζ_s is due to the ϵV_ϵ dependence of the perturbation. The ϵ multiplying V is the expansion parameter. The extra ϵ dependence disappears when the perturbation is a homogeneous polynomial.

The proof of this proposition consists in performing the necessary estimates on the formal algorithm recalled above, to be described in the following Lemmas.

Let us introduce also the space $\mathcal{A}_\sigma$ of all functions $g:\mathbb{R}^{2n}\to\mathbb{C}$ such that

$$\|g\|_\sigma := \int_{\mathbb{R}^{2n}} |\widehat{g}(s)|e^{\sigma\|s\|}\,ds < +\infty$$

and let us estimate the $\mathcal{A}_\sigma$ norm of the Moyal bracket as in Lemma 4.1 of [Ba1]; the argument is repeated here to make the paper self-contained.

Lemma 3.1 *Let $g \in \mathcal{A}_\sigma$ and $g' \in \mathcal{A}_{\sigma-\delta}$ be smooth functions. Then for any $0 < \delta' < \sigma - \delta$ one has*

$$\|\{g, g'\}_M\|_{\sigma-\delta-\delta'} \leq \frac{1}{e^2\delta'(\delta+\delta')}\|g\|_\sigma \|g'\|_{\sigma-\delta} \tag{3.26}$$

Proof. By definition of Moyal bracket and twisted convolution (see e.g.[Fo]):

$$(\{g, g'\}_M)^\wedge(s) =$$

$$\frac{2}{\hbar'}\int_{\mathbb{R}^{2n}} \widehat{g}(s^1)\widehat{g'}(s-s^1)\sin\left[\frac{\hbar'}{2}(s-s^1)\wedge s^1\right] ds^1 \tag{3.27}$$

Here, given two vectors $s = (v, w)$ and $s^1 = (v^1, w^1)$, $s \wedge s^1 := \langle w, v_1\rangle - \langle v, w_1\rangle$. Now $|s \wedge s^1| \leq \|s\|\,\|s^1\|$. Hence, by definition of $\mathcal{A}_\sigma-$norm and (3.27), we get

$$\begin{aligned}
\|\{g, g'\}_M\|_{\sigma-\delta-\delta'} &= 2\int_{\mathbb{R}^{2n}} e^{(\sigma-\delta-\delta')\|s\|}\,ds \int_{\mathbb{R}^{2n}} |\widehat{g}(s)||\widehat{g}'(s-s^1)|\frac{|\sin(\hbar'(s-s^1)\wedge s^1)/2|}{\hbar'}\,ds^1 \\
&\leq 2\int_{\mathbb{R}^{2n}} ds \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta-\delta')\|s\|+\|s^1\|}|\widehat{g}(s)||\widehat{g}'(s^1)|\frac{|\sin(\hbar' s\wedge s^1)/2|}{\hbar'}\,ds^1 \\
&\leq \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta-\delta')\|s\|}|\widehat{g}(s)|\,ds \cdot \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta-\delta')\|s^1\|}|\widehat{g}'(s^1)||s\wedge s^1|\,ds^1 \\
&\leq \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta-\delta')\|s\|}|\widehat{g}(s)|\,\|s\|\,ds \cdot \int_{\mathbb{R}^{2n}} e^{(\sigma-\delta-\delta')\|s^1\|}|\widehat{g}'(s^1)|\,\|s^1\|\,ds^1
\end{aligned}$$

whence the assertion because $e^{-\delta x}x \leq \frac{1}{e\delta}$, $\forall x > 0, \forall \delta > 0$.

To estimate the $\mathcal{A}_{\rho,\sigma}$ norm of the Moyal brackets we need to identify their angular Fourier coefficients.

Lemma 3.2 *For any $\phi \in \mathbb{T}^n$, and for any smooth symbol g, there exists a unitary operator $X_\phi : L^2 \to L^2$ with the property*

$$Op^W(g \circ \Psi_\phi) = X_\phi Op^W(g) X_\phi^{-1} \tag{3.28}$$

Proof. For any fixed ϕ the map Ψ_ϕ is a linear canonical transformation of $\mathbb{R}^{2n}$. Then (3.28) is a characteristic property of Weyl quantization (see e.g. [BS], §5.3).

This lemma entails that the $k-$th spatial Fourier coefficient of an $\hbar$ pseudodifferential symbol is also an $\hbar$ pseudodifferential symbol.

Lemma 3.3 *Let g and g' be smooth functions such that*

$$g \equiv \tilde{g}_k, \qquad g' \equiv \tilde{g}'_l$$

for some fixed $k, l \in \mathbb{Z}^n$*. Then, denoting* $h := \{g, g'\}_M$ *one has*

$$h \equiv \tilde{h}_{k+l}$$

Proof. By the former Lemma we can write

$$\begin{aligned} Op^W(h \circ \Psi_\phi) = X_\phi Op^W(h) X_\phi^{-1} &= \frac{1}{i\hbar'}[X_\phi Op^W(g) X_\phi^{-1}, X_\phi Op^W(g') X_\phi^{-1}] \\ &= Op^W(\{g \circ \Psi_\phi, g' \circ \Psi_\phi\}_M) \end{aligned}$$

Moreover, by the bilinearity of the Moyal bracket and (3.9):

$$h \circ \Psi_\phi = \{e^{i\langle k,\phi\rangle} g, e^{i\langle l,\phi\rangle} g'\}_M = e^{i\langle (k+l),\phi\rangle}\{g, g'\}_M = e^{i\langle (k+l,)\phi\rangle} h$$

This proves the assertion.

We can now estimate the Moyal brackets and the individual terms of the quantum Lie transform expansion.

Lemma 3.4 *Let* $g \in \mathcal{A}_{\rho,\sigma}$ *and* $g' \in \mathcal{A}_{\rho,\sigma-\delta}$*. Then, for any positive* $\delta' < \sigma - \delta$*:*

$$\|\{g, g'\}_M\|_{\rho,\sigma-\delta-\delta'} \le \frac{1}{e^2\delta'(\delta+\delta')}\|g\|_{\rho,\sigma}\,\|g'\|_{\rho,\sigma-\delta}$$

Proof. Denoting once more $h = \{g, g'\}_M$ we can write

$$\begin{aligned} \|h\|_{\rho,\sigma-\delta-\delta'} &= \sum_{k\in\mathbb{Z}^n} \|\tilde{h}_k\|_{\sigma-\delta-\delta'}\, e^{\rho|k|} \le \sum_{k,l\in\mathbb{Z}^n} e^{\rho|k+l|}\|\{\tilde{g}_k, \tilde{g'}_l\}_M\|_{\sigma-\delta-\delta'} \\ &\le \sum_{k,l\in\mathbb{Z}^n} e^{\rho|k|+\rho|l|}\frac{\|\tilde{g}_k\|_\sigma\|\tilde{g'}_l\|_{\sigma-\delta}}{e^2\delta'(\delta+\delta')} = \frac{\|\tilde{g}\|_{\rho,\sigma}\|\tilde{g'}\|_{\sigma-\delta}}{e^2\delta'(\delta+\delta')} \end{aligned}$$

by Lemma 3.3 and 3.26.

Lemma 3.5 *Let* $g \in \mathcal{A}_{\rho,\sigma}$ *and* $w \in \mathcal{A}_{\rho,\sigma}$ *be two smooth symbols, and define*

$$g_r := \frac{1}{r}\{w, g_{r-1}\}_M, \qquad r \ge 1;\ \ g_0 := g$$

Then $g_r \in \mathcal{A}_{\rho,\sigma-\delta}$ *for any* $0 < \delta < \sigma$*, and the following estimate holds*

$$\|g_r\|_{\rho,\sigma-\delta} \le \left(\delta^{-2}\|w\|_{\rho,\sigma}\right)^r \|g\|_{\rho,\sigma} \tag{3.29}$$

Proof. Fix r, denote $\tilde{\delta} := \delta/r$, and look for a numerical sequence $C_l^{(r)}$ such that

$$\|g_l\|_{\rho,\sigma-\tilde{\delta}l} \leq C_l^{(r)}$$

By (3.26) a similar sequence can be recursively defined as follows:

$$C_l^{(0)} = \|g\|_{\rho,\sigma}, \quad C_l^{(r)} = \frac{r^2}{l^2\delta^2 e^2} C_{l-1}^{(r)} \|w\|_{\rho,\sigma}$$

Therefore we get

$$C_r^{(r)} = \frac{1}{(r!)^2} \left(\frac{r^2 \|w\|_{\rho,\sigma}}{\delta^2 e^2} \right)^r \|g\|_{\rho,\sigma}$$

Denoting $C_r := C_r^{(r)}$, clearly one has $\|g_r\|_{\rho,\sigma-\delta} < C_r$. Moreover:

$$C_r = \left(\frac{r}{r-1} \right)^{2r} \frac{(r-1)^{2(r-1)}}{r^2[(r-1)!]^2} e^{-2r} \frac{\|w\|_{\rho,\sigma}^r}{\delta^{2r}} \leq \frac{\|w\|_{\rho,\sigma}}{\delta^2} C_{r-1}$$

so that the sequence $C_l^{(r)}$ majorizes C_r, and this proves the Lemma.

We now construct and estimate the solution of the homological equation.

Lemma 3.6 *Let $g \in \mathcal{A}_{\rho,\sigma}$ be a smooth symbol. Then the "quantum" homological equation*

$$\{p_0, w\}_M + \mathcal{Z} = g \tag{3.30}$$

admits analytic solutions $w, \mathcal{Z}$ such that $\mathcal{Z} \circ \Psi_\phi = \mathcal{Z}$ and fulfilling, for any $d < \rho$, the following estimates

$$\|\mathcal{Z}\|_{\rho,\sigma} \leq \|g\|_{\rho,\sigma}; \qquad \|w\|_{\rho-d,\sigma} \leq c_\Psi \frac{\|g\|_{\rho,\sigma}}{d^\tau}$$

where

$$c_\Psi := \left(\frac{\tau}{e} \right)^\tau \frac{1}{\gamma}$$

Proof. Define

$$\mathcal{Z} := \tilde{g}_0; \qquad w := \sum_{k \neq 0} \frac{\tilde{g}_k}{i\langle \omega, k \rangle} \tag{3.31}$$

Since $p_0(\mathbf{x}, \xi)$ is quadratic in $z = (\mathbf{x}, \xi)$ one has:

$$\{p_0, w\}_M(z) = \{p_0, w\}(z) = \left.\frac{d}{dt}\right|_{t=0} w(\Psi_{\omega t}(z)) =$$

$$\left.\frac{d}{dt}\right|_{t=0} \sum_{k \neq 0} \frac{\tilde{g}_k(\Psi_{\omega t}(z))}{i\langle \omega, k \rangle} = \left.\frac{d}{dt}\right|_{t=0} \sum_{k \neq 0} \frac{\tilde{g}_k(z)}{i\langle \omega, k \rangle} e^{i\langle \omega t, k \rangle} = \sum_{k \neq 0} \tilde{g}_k(z)$$

Thus (3.31) solves the homological equation (3.30). The estimate of $\mathcal{Z}$ is trivial, while that of w is obtained as follows:

$$\begin{aligned}\|w\|_{\rho-d,\sigma} &= \sum_{k\neq 0}\frac{\|\tilde{g}_k\|_\sigma}{|\langle\omega,k\rangle|}e^{\rho|k|-d|k|} \leq \|g\|_{\rho,\sigma}\mathrm{sup}_{k\neq 0}\frac{e^{-d|k|}}{|\langle\omega,k\rangle|}\\ &\leq \|g\|_{\rho,\sigma}\mathrm{sup}_{m\neq 0}\frac{e^{-dm}m^\tau}{\gamma} = \frac{\|g\|_{\rho,\sigma}}{\gamma}\left(\frac{\tau}{ed}\right)^\tau\end{aligned}$$

and this proves the Lemma.

Our final Lemma deals with $\mathcal{A}_{\rho,\sigma}$ estimates of the individual terms in the expansion of the quantum Lie transform of p_0 (remark that $p_0 \notin \mathcal{A}_{\rho,\sigma}$) v

Lemma 3.7 *Let $g \in \mathcal{A}_{\rho,\sigma}$ be a smooth symbol, and w be the solution of the homological equation (3.30). Define the sequence $p_{r0} : r = 0, 1, \ldots$ as follows:*

$$p_{00} := p_0; \qquad p_{r0} := \frac{1}{r}\{w, p_{r-10}\}_M,\ r \geq 1$$

Then, for any $0 < d < \rho, 0 < \delta < \sigma$, $p_{r0} \in \mathcal{A}_{\rho-d,\sigma-\delta}$ and fulfills the following estimate

$$\|p_{r0}\|_{\rho-d,\sigma-\delta} \leq 2\left(\delta^{-2}\|w\|_{\rho-d,\sigma}\right)^{r-1}\|g\|_{\rho-d,\sigma},\ \ k \geq 1$$

Proof. It is enough to notice that, by (3.30)

$$\{w, p_0\}_M = \mathcal{Z} - g$$

whence

$$\|p_{10}\|_\sigma \leq 2\|g\|_\sigma$$

The argument of Lemma 3.5 can then be taken over to the present case.

We are now in position to prove Proposition 3.2.

Proof of Proposition 3.2. Consider the quantum Lie transform generated by the solution w of the homological equation

$$\{p_0, w\}_M + \epsilon\mathcal{Z}_k = f^{(k)}$$

where $\mathcal{Z}_k$ depends on $z = (\mathbf{x}, \xi)$ only through $I_1, \ldots, I_n$. By Lemma 3.6 and (3.24) w exists and fulfills the estimate

$$\|w\|_{\rho-(k+1)d,\sigma-k\delta} \leq \frac{c_\Psi}{d^\tau}\epsilon E\mu_s^k$$

while $\mathcal{Z}_k$ is such that

$$\|\mathcal{Z}_k\|_{\rho-kd,\sigma-k\delta} \le E\mu_s^k$$

Define $\mathcal{Z}^{(k+1)} := \mathcal{Z}^{(k)} + \mathcal{Z}_k$ and

$$f^{(k+1)} := \epsilon \sum_{l\ge 1} \mathcal{Z}_l^{(k)} + \sum_{l\ge 1} f_l^{(k)} + \sum_{l\ge 2} p_{l0}$$

where:

$$\mathcal{Z}_0^{(k)} := \mathcal{Z}^{(k)}; \qquad \mathcal{Z}_l^{(k)} = \frac{1}{l}\{w, \mathcal{Z}_{l-1}^{(k)}\}_M, \quad l \ge 1$$

and analogous definitions for the other two sequences. Then the symbol of the transformed Schrödinger operator has the form (3.22) with $k+1$ in place of k. To obtain the estimate, assume $k \ge 1$. Denote

$$\eta := \delta^{-2}\|w\|_{\rho-(k+1)d,\sigma-k\delta} \le \frac{c_\Psi}{d^\tau \delta^2}\epsilon E\mu_s^k < \frac{1}{2}$$

By Lemmas 3.6,3.7, 3.5 we can write

$$\sum_{l\ge 1} \|f_l^{(k)}\|_{\rho-(k+1)d,\sigma-(k+1)\delta} \le \epsilon E\mu_s^k \sum_{l\ge 1} \eta^l = \frac{\epsilon E\mu_s \eta}{1-\eta} \le 2\eta\epsilon E\mu_s^k$$
$$\sum_{l\ge 1} \|\mathcal{Z}_l^{(k)}\|_{\rho-(k+1)d,\sigma-(k+1)\delta} \le \|\mathcal{Z}^{(k)}\|_{\rho-kd,\sigma-k\delta}\frac{\eta}{1-\eta} \le 4E\eta$$
$$\sum_{l\ge 2} \|p_{l0}\|_{\rho-(k+1)d,\sigma-(k+1)\delta} \le 4\epsilon\mu_s^k \eta E$$

whence the assertion in a straightforward way. This proves Proposition 3.2.

Proof of Proposition 3.1. Apply k times Proposition 3.2 to the Schrödinger operator $P_f(\hbar')$ with symbol $p_0 + \epsilon f$, taking $d := (\rho/2k)$ and $\delta := (\sigma/2k)$. This yields a unitary map T in L^2 such that the symbol of the unitary image $TP_f(\hbar')T^{-1}$ has the form (3.22), with $\mathcal{Z}^{(k)}$ and $f^{(k)}$ fulfilling the estimates

$$\|\mathcal{Z}^{(k)}\|_{\rho/2,\sigma/2} \le 2E, \quad \|f^{(k)}\|_{\rho/2,\sigma/2} \le \epsilon E\left(\frac{2^{\tau+5}}{\rho^\tau\sigma^2}\epsilon E c_\Psi k^{\tau+2}\right)^k \tag{3.32}$$

The choice

$$k := \left[\left(\frac{\epsilon_0 \rho^\tau \sigma^2}{\epsilon}\right)^{1/(\tau+2)}\right]$$

yields the estimate (3.4); $k > 1$ by (3.15). This proves Proposition 3.1.

Let us finally prove Theorem 3.1 approximating a by a function fulfilling the conditions of Proposition 3.1. To this end we first prove the existence $\mu > 0$ such that

$$\langle \| a_\epsilon \| \rangle := \sum_{k \in \mathbb{Z}^n} e^{\mu |k|^\beta} \int_{\mathbb{R}^{2n}} |\widehat{a}_{\epsilon,k}(s)| e^{\mu |s|^\beta} \, ds < +\infty \tag{3.33}$$

where $\beta := (1/2(q+1))$ The proof of this estimate is divided in two Lemmas.

Lemma 3.8 *Under the assumptions of Theorem 3.1 there exist constants $\overline{C}_1$ and $\overline{C}$ such that the following estimate holds:*

$$\sup_{z \in \Re^{2n}, \phi \in \mathbb{T}^n} \left| \partial_{\zeta^j}^{|j|} a_\epsilon(\Psi_\phi(z)) \right| \leq \overline{C}_1 \overline{C}^{-|j|} (|j|!)^{q+1}$$

Here $j = (j_1, \ldots, j_n)$ is a multi-index, $|j| = j_1 + \ldots + j_n$, q is one of the Gevrey constants of a_ϵ and ζ_i denotes indifferently a phase variable z_i or an angle ϕ_i.

Proof. Denote $h_\epsilon(z, \phi) := a_\epsilon(\Psi_\phi(z)), N = |j|$ and consider

$$\frac{\partial^N h_\epsilon}{\partial \zeta^N}$$

where ζ is a fixed variable; we can write

$$\frac{\partial^N h_\epsilon}{\partial \zeta^N} = \sum_{k=2}^{N+2} C_k^N O_k \tag{3.34}$$

where C_k^N are positive integers, and $C_k^N O_k$ denotes a sum of C_k^N terms O_k of the form

$$O_k = (d^{k-1} a_\epsilon)(\Psi_\phi(z)) \mathcal{G}_1(z, \phi) \cdots \mathcal{G}_{k-1}(z, \phi), \qquad k \leq N$$

Here $(d^k a_\epsilon)(\Psi_\phi(z))$ denotes the $k-$th differential of a evaluated at the point $\Psi_\phi(z)$, and $\mathcal{G}_i$ is for $i = 1, \ldots, k-1$ a partial derivative of order not exceeding N of $\Psi_\phi(z)$. Given the form (3.7,3.8) of $\Psi_\phi(z)$ one clearly has

$$\sup_{\|z\| \leq 2R, \phi \in \mathbb{T}^n} \|\mathcal{G}_i(z, \phi)\| \leq R_2$$

with a suitable R_2 whence

$$\sup_{z \in \mathbb{R}^{2n}, \phi \in \mathbb{T}^n} |O_k| \leq \sum_{|j|=k} \sup_{\|z\| \leq 2R} \left| \partial_{z^j}^k a_\epsilon(z) \right| R_2^{k-1} \leq \tag{3.35}$$

$$\leq K(R) \alpha^{-k} k^{2n} R_2^{k-1} (k!)^q \leq K(R) \alpha^{-k} N^{2n} R_2^{k-1} (k!)^q$$

Hence to estimate (3.34) we just have to estimate the coefficients C_k^N. To this end, remark that we can symbolically write

$$\frac{\partial O_k}{\partial \zeta} = O_{k+1} + (k-1)O_k$$

The l.h.s. can be estimated through the r.h.s. via triangular inequality, because in the estimate of O_k each derivative of $\mathcal{G}_j$ clearly fulfills the same estimate as $\mathcal{G}_j$ itself. From this it is easy to check that

$$\begin{cases} C_{N+1}^N = 1 \\ C_k^N = C_k^{N-1}(k-1) + C_{k-1}^{N-1}, \quad 3 \leq k \leq N \\ C_2^N = 1 \end{cases}$$

Hence, defining $S_N := \sum_{k=2}^{N+1} C_k^N$ one has

$$S_N := \sum_{k=2}^{N+1} k C_k^{N-1} < N S_{N-1}$$

whence $S_N < N!$. This proves the result when all derivatives are performed with respect to the same variable; however the general case can be dealt with by exactly the same argument, and this proves the Lemma.

Lemma 3.9 *Under the assumption of Theorem 3.1 there exist $\tilde{\alpha}$ and C_7 such that*

$$\left|\hat{\hat{a}}_{\epsilon,k}(s)\right| \leq C_7 \exp - \left[(\tilde{\alpha}|k|)^{1/2(q+1)} + (\tilde{\alpha}|s|)^{1/2(q+1)}\right] \tag{3.36}$$

Proof. Performing $l \geq 1$ partial integrations, the above Lemma and the uniform Gevrey property (2.2) of $a_\epsilon \circ \Psi$ yield, for some $C_8 > 0$, $C_9 > 0$:

$$\left|\hat{\hat{a}}_{\epsilon,k}(s)\right| \leq C_8 \left(\frac{1}{C_9|(k,s)|}\right)^l (l!)^{q+1}$$

where: $|(k,s)| := \sum_{p=1}^{n} |k_p| + \sum_{p=1}^{2n} |s_p|$. Choosing $l = l(|(k,s)|)$ as follows

$$l := \left[\left(\frac{C_9|(k,s)|}{e}\right)^{1/(q+1)}\right]$$

we get

$$\left|\hat{\hat{a}}_{\epsilon,k}(s)\right| \leq C_{10} \exp - \left(\frac{C_9|(k,s)|}{e}\right)^{1/(q+1)}$$

Now remark that $|(k,s)| = |k| + |s|$ and that, given any pair of real numbers (x,y) both larger than 2, and any $l \in \mathbb{N}$, one has

$$x^{2l} + y^{2l} > (x+y)^l$$

Therefore, for $\min(|k|,|s|) > 2^{2(q+1)}$:

$$(|k|+|s|)^{\frac{1}{q+1}} > |k|^{\frac{1}{2(q+1)}} + |s|^{\frac{1}{2(q+1)}} \tag{3.37}$$

This yields the Lemma for $\min(|k|,|s|) > 2^{2(q+1)}$; its validity when (3.37) does not hold is ensured by suitably choosing the constants C_7. This proves the Lemma.

Taking μ suitably small, it follows immediately from this Lemma:

Corollary 3.1 *There is $C_{11} > 0$ independent of a such that*

$$\langle \|a_\epsilon\| \rangle \le C_{11} \tag{3.38}$$

Proof of Theorem 3.1. For $\Lambda > 0$ and $K > 0$ fixed, approximate a_ϵ by

$$a_\epsilon^{\Lambda,K}(z) := \sum_{|k|\le K} \int_{\|s\|<\Lambda} \hat{\hat{a}}_{\epsilon,k}(s) e^{i\langle s,z\rangle}\, ds \tag{3.39}$$

We have:

$$a_\epsilon^{\Lambda,K}(z) - a_\epsilon^{\Lambda,\infty}(z) = \sum_{|k|>K} \int_{\|s\|<\Lambda} \hat{\hat{a}}_{\epsilon,k}(s) e^{i\langle s,z\rangle}\, ds$$

Therefore, denoting $A_\epsilon^{\Lambda,K} = Op^W(a_\epsilon^{\Lambda,K})$, $A_\epsilon^{\Lambda,\infty} = Op^W(a_\epsilon^{\Lambda,\infty})$ by (3.12) we can write

$$\|A_\epsilon^{\Lambda,K} - A_\epsilon^{\Lambda,\infty}\|_{L^2\to L^2} \le \sum_{|k|>K} \int_{\|s\|<\Lambda} |\hat{\hat{a}}_{\epsilon,k}(s)|\, ds \le C_{11} e^{-\mu K^\beta} \tag{3.40}$$

$$\|A_\epsilon^{\Lambda,\infty} - A_\epsilon\|_{L^2\to L^2} \le C_{11} e^{-\mu \Lambda^\beta} \tag{3.41}$$

Now choose

$$\rho = \sigma = \epsilon^\alpha, \qquad K = \Lambda = \epsilon^{-\alpha}$$

with a still undetermined α. On account of (3.38,3.40,3.41) this yields:

$$\|A_\epsilon^{\Lambda,K} - A_\epsilon\|_{L^2\to L^2} \le 2C_{11} \exp\left[-\frac{\mu}{\epsilon^{\alpha\beta}}\right],$$

Let us now apply Proposition 3.1 to the operator $P_0 + \epsilon A_\epsilon^{\Lambda,K}$ of symbol $p_0 + \epsilon a_\epsilon^{\Lambda,K}$. We have

$$E := \|a^{\Lambda,K}\|_{\rho,\sigma} \le e^{\rho K + \sigma\Lambda} \langle \|a\| \rangle = e^2 \langle \|a\| \rangle$$

and hence, by (3.38):

$$\epsilon E \leq \epsilon e C_{11} \leq \epsilon_0 \rho^\tau \sigma^2 = \epsilon_0 \epsilon^{\alpha(\tau+2)} \tag{3.42}$$

holds provided $\alpha < 1/(\tau+2)$ and ϵ is suitably small. Hence, considering the symbol $p_0 + \epsilon \mathcal{Z}_\epsilon^{\Lambda,K} + \mathcal{R}_\epsilon^{\Lambda,K}$ of the trasformed operator, by (3.17) and (3.12) we have

$$\|Op^W(\mathcal{Z}_\epsilon^{\Lambda,K})\|_{L^2\to L^2} \leq 2e^2 C_{11}$$

while by Assertion 1 of Proposition 3.1 the principal symbol of $\mathcal{Z}_\epsilon^{\Lambda,K}$ coincides with the classical normal form computed up to order

$$N(\epsilon) := \frac{\epsilon_0^{1/(\tau+2)}}{\epsilon^{\frac{1}{\tau+2}-\alpha}} \tag{3.43}$$

Moreover, since the cut off operator $P_0 + \epsilon A_\epsilon^{\Lambda,K}$ differs form the original one only by exponentially small terms, also the normal form computed on the original system differs from $\mathcal{Z}_\epsilon^{\Lambda,K}$ only by exponentially small terms which can therefore be put in the remainder. On the other hand, by (3.18):

$$\|Op^W(\mathcal{R}_\epsilon^{\Lambda,K})\|_{L^2\to L^2} \leq e^3 \epsilon C_{11} \exp\left[-(\tau+2){\epsilon_0}^{1/(\tau+2)} \epsilon^{\alpha - \frac{1}{\tau+2}}\right]$$

We choose α in order to optimize the remainder, i.e. in such a way that

$$\alpha\beta = \frac{1}{\tau+2} - \alpha \qquad \Longrightarrow \qquad b := \alpha\beta = \frac{1}{(\tau+2)(2q+3)}$$

Setting

$$\epsilon_* := \epsilon_0^{1/(\tau+2)} \quad C_3 = 2e^2 C_{11} \quad C_4 = (4+e^3) \quad C_{11}, \quad C_5 = \min(\mu, (\tau+2)\epsilon_*)$$

we conclude the proof of Assertions 1 and 2 of Theorem 3.1.

Assertion 3 will be a consequence of the property

$$\epsilon^3 a_\epsilon\left(\frac{z}{\epsilon}\right) = V_3(z) \ , \quad \forall |z| \leq R\epsilon$$

Looking in more detail into the normal form construction we will show that this property holds (mod $((\hbar')^\infty)$, and on a sphere of radius smaller than $R\epsilon$) for the symbols recursively defined by the perturbative algorithm.

Let $w_\epsilon, g_\epsilon \in \Sigma_0^1(\mathbb{R}^{2n})$ (notations of [He]) be two smooth bounded semiclassical symbols with the property that

$$\epsilon^3 w_\epsilon\left(\frac{z}{\epsilon}, \frac{\hbar}{\epsilon^2}\right) = \bar{w}(z,\hbar) \ , \epsilon^3 g_\epsilon\left(\frac{z}{\epsilon}, \frac{\hbar}{\epsilon^2}\right) = \bar{g}(z,\hbar) \ , \quad \forall |z| \leq R\epsilon$$

for some $\bar{w}, \bar{g}$ (independent of ϵ). Let $M > 0$ be fixed. Then the following properties are easily verified:

1. Denote $d_\epsilon := \{g_\epsilon ; w_\epsilon\}_M$. Then $\forall \delta > 0$ one has, as $\hbar' \to 0$:
$$\epsilon^3 d_\epsilon \left(\frac{z}{\epsilon}; \frac{\hbar}{\epsilon^2} \right) = \bar{d}(z;\hbar) + O((\hbar')^M) \ , \quad \forall \, |z| < \epsilon(R-\delta)$$

2. Let $\tilde{g}_{\epsilon,k}$ be the angular Fourier coefficients of g_ϵ. Then:
$$\epsilon^3 \tilde{g}_{\epsilon,k} \left(\frac{z}{\epsilon}; \frac{\hbar}{\epsilon^2} \right) = \bar{\tilde{g}}_k(z;\hbar) \ , \quad \forall \, |z| < \epsilon R$$

3. Let φ_t the flow generated by the Hamiltonian ϵw_ϵ. Then (as $\epsilon \to 0$)
$$\epsilon^3 \varphi_t \left(\frac{z}{\epsilon}; \frac{\hbar}{\epsilon^2} \right) = \bar{\varphi}_t(z;\hbar) + O((\hbar')^M) \ , \quad \forall \, |z| < \epsilon(R-\delta)$$

4. Denote $W := Op^W(w_\epsilon)$ and g_ϵ^w the symbol of $e^{iW\epsilon/\hbar'} Op^W(g_\epsilon) e^{-iW\epsilon/\hbar'}$. By the semiclassical Egorov theorem ([Ro], IV.10) one has (as $\hbar' \to 0$ and $\epsilon \to 0$)
$$\epsilon^3 g_\epsilon^w \left(\frac{z}{\epsilon}; \frac{\hbar}{\epsilon^2} \right) = \bar{g}^w(z;\hbar) + O((\hbar')^M) \ , \quad \forall \, |z| < \epsilon(R-\delta)$$

Then Assertion 3 is an immediate consequence ofthe following lemma.

Lemma 3.10 *Let $M > 0$ be fixed and suitably large, and let the function*
$$\mathcal{H}^{(k)} := p_0 + \epsilon \mathcal{Z}^{(k)} + \epsilon^{k+1} \mathcal{S}_1^{(k)} + \epsilon \mathcal{S}_2^{(k)} \tag{3.44}$$
be such that $\mathcal{Z}^{(k)}$ depends on $(I_1, \dots, I_n)$ only. Assume that
$$\epsilon^3 \mathcal{Z}^{(k)} \left(\frac{z}{\epsilon}, \frac{\hbar}{\epsilon^2} \right) = \bar{\mathcal{Z}}^{(k)}(z,\hbar) \quad \epsilon^3 \mathcal{S}_1^{(k)} \left(\frac{z}{\epsilon}, \frac{\hbar}{\epsilon^2} \right) = \bar{\mathcal{S}}_1^{(k)}(z,\hbar) \tag{3.45}$$
for $|z| \le \epsilon(R - \delta k)$. Assume furthermore $\mathcal{Z}^{(k)}, \mathcal{S}_1^{(k)} \in \Sigma_0^1(\mathbb{R}^{2n})$ and $\mathcal{S}_2^{(k)} = O_k((\hbar')^M)$. Then there exists a unitary transformation T_k such that the transformed Schrödinger operator $T_k H^{(k)} T_k^{-1} := H^{(k+1)}$ has a Weyl symbol given by (3.44) with $k+1$ in place of k: $H^{(k+1)} = Op^W(\mathcal{H}^{(k+1)})$.

Proof. We construct the generator w of the transformation exactly as in the proof of Lemma 3.2, with $\epsilon^{k+1}\mathcal{S}_1^{(k)}$ in place of $f^{(k)}$. Remark that, since $\mathcal{S}_1^{(k)} \in C^\infty(\mathbb{R}^{2n})$, its angular Fourier coefficient $\tilde{\mathcal{S}}_{1,l}^{(k)}$ decays faster than any inverse power of $|l|$ as $|l| \to \infty$.

Therefore by (1.6) w is well defined and $C^\infty(\mathbb{R}^{2n})$ uniformly with respect to ϵ. Moreover, by Property 2 one has

$$\epsilon^3 w\left(\frac{z}{\epsilon}, \frac{\hbar}{\epsilon^2}\right) = \bar{w}(z,\hbar)$$

Then in the notations of remark 4 above, we have

$$\mathcal{S}_1^{(k),w} = \bar{\mathcal{S}}_1^{(k),w} + O(\hbar'^M)\ , \quad \mathcal{Z}^{(k),w} = \bar{\mathcal{Z}}^{(k),w} + O(\hbar'^M)$$

for $|z| \leq (R-(k+1)\delta)$. Moreover, using the homological equation it is easy to see that

$$p_0^w = \bar{p}_0^w + O(\hbar'^M)\ .$$

Hence let us set:

$$\epsilon\mathcal{S}_1^{(k+1)} := \bar{\mathcal{S}}_1^{(k),w} - \mathcal{S}_1^{(k)} + \bar{p}_0^w - p_0 - \epsilon^{k+1}\overline{\{w, p_0\}_M} + \bar{\mathcal{Z}}^{(k),w} - \mathcal{Z}^{(k)}, \tag{3.46}$$

$$\mathcal{Z}^{(k+1)} := \mathcal{Z}^{(k)} + \epsilon^{k+1}\tilde{\mathcal{S}}_{1,0}^{(k)}\ , \tag{3.47}$$

It is now easy to see, using the explicit formula for $\bar{\mathcal{S}}_1^{(k),w}$ and the other transformed symbols (see once [Ro], Theorem IV.10) that the l.h.s. of (3.46) is actually of order ϵ. Then the assertion is a direct consequence of the above listed properties. This proves the Lemma, and concludes the proof of Theorem 3.1.

Proof of Theorem 1.1. By Theorem 2.1 and Theorem 3.1 the eigenvalues $\lambda_k(\hbar',\epsilon)$, $k \in (\mathbb{N}\cup\{0\})^n$ of $P(\hbar',\epsilon)$ admit the approximate quantization formula

$$\lambda_s(\hbar',\epsilon) = \mathcal{Z}((k_1+1/2)\hbar', \ldots, (k_n+1/2)\hbar', \hbar', \epsilon) + \tilde{\mathcal{R}}(\epsilon,\hbar') \tag{3.48}$$

$$\tilde{\mathcal{R}}(\epsilon,\hbar') = 0(C_4\epsilon e^{-(C_5/\epsilon)^b}) + O(\hbar'^\infty) \tag{3.49}$$

where $\mathcal{Z}(\epsilon) = \mathcal{Z}^{N(\epsilon)}$ is given by (3.2) and the constants have been determined above. On the other hand, $\mathrm{Spec}[P(\hbar')] = \epsilon^2\mathrm{Spec}[P(\hbar',\epsilon)]$. Hence, for any $\Gamma > 0$

$$\lambda \in \mathrm{Spec}[P(\hbar)] \cap [0,\Gamma\epsilon^2] \Longleftrightarrow \frac{\lambda}{\epsilon^2} \in \mathrm{Spec}[P(\hbar',\epsilon)] \cap [0,\Gamma]$$

This yields (1.9) and hence Assertion (1). To prove formula (1.10), for any fixed $M < (\epsilon_*/\epsilon)^b$ write:

$$\mathcal{Z}\left((k+\frac{1}{2})\frac{\hbar}{\epsilon^2}, \frac{\hbar}{\epsilon^2}, \epsilon\right) = \sum_{s=1}^{M} \zeta_s((k+\frac{1}{2})\frac{\hbar}{\epsilon^2}, \hbar/\epsilon^2)\epsilon^s + \sum_{s=M}^{N(\epsilon)} \zeta_s((k+\frac{1}{2})\frac{\hbar}{\epsilon^2}, \hbar/\epsilon^2)\epsilon^s$$

The second addend is $O(\epsilon^M)$ by construction. By Assertion 3 of 3.1,

$$\epsilon^3\sum_{s=1}^{M}\zeta_s((k+\frac{1}{2})\frac{\hbar}{\epsilon^2},\hbar/\epsilon^2)\epsilon^s=\tilde{\mathcal{Z}}^M\left((k_1+\frac{1}{2})\hbar,\dots,(k_n+\frac{1}{2})\hbar,\hbar\right)+O_M((\hbar/\epsilon^2)^\infty)$$

To identify $\tilde{\mathcal{Z}}$, note that by [Sj], Theorem 1.4 the quantum normal form is constructed as a h pseudodifferential operator commuting with P_0 by quantization of the classical Lie transform as in Sect.3. Hence by Remark 2 after Proposition 3.2 and Assertion 3, formula (3.6), $\tilde{\mathcal{F}}=\mathcal{F}$. This concludes the proof of Theorem 1.1.

Proof of Corollary 1.1. Consider now the scaling parameter ϵ, which can be chosen as an arbitrary function $\epsilon(\hbar)$ vanishing as $\hbar\to 0$ subject to the only requirement

$$\lim_{\hbar\to 0}\hbar\epsilon(\hbar)^{-2}\to 0 \tag{3.50}$$

Look now for a positive, increasing function $\epsilon=\varphi(\hbar):]0,1[:\mathbb{R}_+$, such that

$$\varphi(\hbar)\exp\left[-(C_5/\varphi(\hbar))^b\right]=O(\hbar^\infty)$$

This condition is satisfied by any positive, increasing function $\varphi(\hbar)$ on $]0,1]$, such that

$$\lim_{\hbar\to 0}\varphi(\hbar)^b\ln\hbar\to 0 \tag{3.51}$$

This choice of ϵ fulfills (3.50) in both cases, and hence (1.12) is proved.

Appendix: eigenfunction estimates

The purpose of this Appendix is to prove the estimate (2.8), namely:

Lemma A.1 *Let $\psi_\epsilon(x;\hbar',\lambda)$ be any eigenvector corresponding to any eigenvalue $0<\lambda(\epsilon)\leq\overline{E}$ of $P(\hbar',\epsilon)$. Then there are positive constants A,B,R_1, uniform with respect to $(\lambda,\epsilon)\in[0,\overline{E}]\times[0,1]$ such that*

$$|\psi_\epsilon(x;\hbar',\lambda)|\leq C\exp\left[-A\frac{|x|^{\frac{\nu}{2}+1}-R_1^{\frac{\nu}{2}+1}}{\hbar'}\right],\quad |x|\geq R_1,\quad \epsilon\geq 0, \tag{A0}$$

First recall some known estimates on solutions of the Schrödinger equation ([BS], §3.2).

1. Let $\mathcal{E}(\mathbf{x})$be a solution of the Schrödinger equation

$$(-\Delta + b|\mathbf{x}|^\alpha)\mathcal{E}(\mathbf{x}) = 0, \quad b>0,\, \alpha>0$$

Then $\exists\, K_1 > 0$ such that, for $|\mathbf{x}| > K_1$:

$$\frac{1}{2}\exp\left[-\frac{2\sqrt{b}}{\alpha}|\mathbf{x}|^{\frac{\alpha}{2}+1}\right] \le \mathcal{E}(\mathbf{x}) \le 2\exp\left[-\frac{2\sqrt{b}}{\alpha}|\mathbf{x}|^{\frac{\alpha}{2}+1}\right] \tag{$A1$}$$

2. Let $V(\mathbf{x}) \in C^\infty(\mathbb{R}^n)$ be such that $V(\mathbf{x}) \ge c|\mathbf{x}|^\alpha$ for some $K > 0$ and any $b < 2c$. Let $\psi(\mathbf{x})$ be eigenfunction of the Schrödinger operator $-\Delta + V$ corresponding to the eigenvalue λ. Set:

$$R := \max\left\{K, K_1, \left(\frac{2\lambda}{2c-b}\right)^{1/\alpha}\right\}$$

Then if there is $M > 0$ such that $\psi^2(\mathbf{x}) - M\mathcal{E}(\mathbf{x}) < 0$ for $|\mathbf{x}| = R$ one has

$$\psi^2(\mathbf{x}) \le 2M\exp\left[-\frac{2\sqrt{b}}{\alpha}|\mathbf{x}|^{\frac{\alpha}{2}+1}\right] \tag{$A2$}$$

Remark that a similar M always exists: it is enough to take

$$M > 2\left(\max_{|\mathbf{x}|=R}\psi^2(\mathbf{x})\right)\exp\left[\frac{2\sqrt{b}}{\alpha}R^{\frac{\alpha}{2}+1}\right]$$

Proof of Lemma A1. Let $\psi(\mathbf{x})$ be eigenfunction of the Schrödinger operator $Q(\hbar) = -\hbar^2\Delta/2 + V(\mathbf{x})$. By Assumption (A1) there are $c_1 > 0, N > 0$ such that $V(\mathbf{x}) \ge c_1|\mathbf{x}|^\nu$ for $|\mathbf{x}| > N$. It is easy to verify that, setting

$$R_1 := \max\left\{N, \frac{1}{\sqrt{2}}\left(\frac{2\lambda}{\sqrt{2}c - b_1}\right)^{1/\nu}\right\}$$

for $b_1 < \sqrt{2}c_1$ and $\hbar < K_1/R_1$ one has:

$$\psi^2(\mathbf{x}) \le \left[\max_{|\mathbf{x}|=R_1}\psi^2(\mathbf{x})\right]\cdot\exp\left[-\frac{\gamma}{\nu}\frac{|\mathbf{x}|^{\frac{\nu}{2}+1} - R_1^{\frac{\nu}{2}+1}}{\hbar}\right] \tag{$A3$}$$

for $|\mathbf{x}| > R_1$, where $\gamma := 2\sqrt{b_1}(\sqrt{2})^{\frac{\nu}{2}+1}$. To see this just perform the rescaling $\mathbf{x} = \dfrac{\hbar\mathbf{y}}{\sqrt{2}}$ and apply the above results, item 2.

We can now prove (A0). Recall that by (1.3,1.4) if $\lambda \le \overline{E}$ is eigenvalue of $P(\hbar', \epsilon)$ then $\epsilon^2\lambda$ is eigenvalue of $Q(\hbar)$ corresponding to the rescaled eigenfunction. Hence we consider an eigenvector of Q corresponding to $\epsilon^2\lambda$, and distinguish two cases:

1) $\nu > 2$. Then obviously there is $c_1 > 0$ such that $V > c_1\|\mathbf{x}\|^2$ $\forall x \in \mathbb{R}^n$. Apply now (A3) to $\epsilon^2\lambda$, we have $R_1 = \epsilon R_2$ with

$$R_2 := \frac{1}{\sqrt{2}}\left(\frac{2\lambda}{\sqrt{2}c_1 - b_1}\right)^{1/2}$$

and a simple computation shows that the corresponding eigenvector of $P_R(\hbar')$ satisfies the estimate, for $|\mathbf{x}| > R_2(\lambda)$,

$$\psi^2(\mathbf{x}) \le C\exp-\left(\frac{\gamma}{2}\frac{|\mathbf{x}|^2 - R_2^2}{\hbar'}\right)$$

2) $\nu \le 2$. Then, given R there exists C such that

$$V(x) \ge C\epsilon^{2-\nu}|x|^\nu\ , \quad \forall|x| > R\epsilon\ ,$$

Hence applying (A3) once more one we get the result.

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