A remark on sharp estimates for high order nonresonant normal forms in Hamiltonian perturbation theory

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ABSTRACT. Using the method of majorants, we give an estimate of the rest for the nonresonant action-angle normal forms and exhibit a simple example suggesting the "optimality" of this estimate. Given an integer k, calling γ the size of the small denominators up to order k, we prove that the $k^{\rm th}$ order remainder is approximatively bounded by $O(\varepsilon_0^{-k})$ with $\varepsilon_0 = O(\gamma^2/k)$. Thus, if we disregard the dependence of γ upon k, we obtain a rest bounded by (const $k)^k$. These estimates are conjectured to be optimal: to support this idea we present a simplified model problem with no small denominators, formally related to the above calculations: this example exhibits a factorial divergence.

1 Introduction

We study the perturbation of an integrable action-angle system $H(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$ near a nonresonant action I_0. Fixed an integer k, by an averaging we can set the system in normal form up to the $k^{\rm th}$ order: this means that the system depends only on the actions until the $(k+1)^{\rm th}$ order in ε. We call the $(k+1)^{\rm th}$ order the "rest". We will also call

$$\gamma = \gamma(k,\rho) \equiv \min\{1, \min_{\substack{0<|n_1|+\ldots+|n_N|\le kP \\ |I-I_0|\le\rho}} |h'(I)\cdot n|\}$$

the size of the "small denominators".

The main statement of this paper is Theorem 3.6 in which we prove that

If $|\varepsilon| \le O(\gamma^2/k)$, the rest is bounded by $\big(O(k/\gamma^2)\big)^k$. So that, if we neglect the dependence of γ upon k we have that the rest is controlled by $(\,O(k)\,)^k$.

These estimates are probably not too far from sharpness, as a simplified example will show: we consider the Hamilton-Jacobi equation for the perturbation of an armonic oscillator $\omega\cdot\varphi + 2(1-I)\,\varepsilon\cos\varphi$; as one can sees, the effect of the operator $\omega\,\partial_\varphi$ in this equation is just multiplying for a small denominator, so that a reasonable example with no small denominator could be obtained substituting 1 in stead of $\omega\,\partial_\varphi$. We easily prove that this simplified model shows a factorial growth of the Birkhoff series.

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With more refined techniques[1] it is possible to extend the validity of these estimates even to the resonant case, providing the so called "analytic part" of the Theorem of Nekhoroshev[2]: anyhow, as remarked in [5], [6], [3] and [10], the estimates developed in our paper are enough to prove the *Theorem of Nekhoroshev in the case of harmonic oscillators*, obtaining exponentially long times of stability for systems of the type $H(I,\varphi) = \omega \cdot I + \varepsilon f(I,\varphi)$, being ω a diophantine vector.

2 Set up

We consider the nearly-integrable Hamiltonian[3] system $H(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$, where h and f are analytic for $I \in \mathbf{R}^N$ with $|I - I_0| \leq \rho^\star$ and $\varphi \in \mathbf{T}^N$ with $|\operatorname{Im}\varphi| \leq \xi$; moreover, we assume that f is a trigonometric polynomial in the φs of degree P. The Is and the φs are called action variables and angles respectively. We will briefly denote $\mathbf{T}^N_\xi$ the analiticity set of the angles. We also write B^N_r to mean the $N-$dimensional complex ball of radius r; $|\varepsilon| \leq \varepsilon_0$ must be considered as a small parameter.

Since from the equations of motion one can see that the "speed" of the actions is lower than the one of the angles, one hopes to be able to change the aforesaid system with a new one, obtained by taking the average of the fast angles. This naif idea has a precise statement in the following classic Theorem, due to Birkhoff (cfr., for example, [1]):

Consider the system above, and fix $k \in \mathbf{N}$, $k \geq 2$. Assume that $h'(I_0)$ is nonresonant up to the kP^{th} order, that means $h'(I_0) \cdot n \neq 0$ for all $n \in \mathbf{Z}^N$ with $0 < |n_1| + \ldots + |n_N| \leq kP$.

Then there exists a canonical transformation Φ_k, close to the identity for small ε, local in the actions but global in the angles, such that, denoting $\mathcal{H} \equiv H \circ \Phi_k$ the new Hamiltonian in the new variables (y,x), one has

$$\mathcal{H}(y,x) = h(y) + \sum_{j=1}^{k} \varepsilon^j g_j(y) + \varepsilon^{k+1} R_k(y,x) \tag{2.1}$$

for fitting analytic functions g_j. The representation (2.1) is often called "in normal forms".

We consider the following domain in the new variables (y,x): $y \in \mathbf{C}^N$ with $|y - y_0| \leq \rho$, for a suitable y_0, and $x \in \mathbf{T}^N_{\xi^\star}$, with ρ and $\xi^\star$ such that $\Phi_k\left(\{|y-y_0| \leq \rho\} \times \mathbf{T}^N_{\xi^\star}\right) \subseteq \{|I - I_0| \leq \rho^\star\} \times \mathbf{T}^N_\xi$.

The above statement can be made more precise, saying that g_1 is the average of f in the angles and that the transformation Φ_k is obtained by a generating function of the kind $\mathcal{G}(y,\varphi) = y \cdot \varphi + \sum_{j=1}^k \varepsilon^j \chi_j(y,\varphi)$. The χ_js can be described recursively: if we use the "hat" to mean the Fourier coefficient, one gets

$$\chi_1(y,\varphi) = \sum_{\substack{n \in \mathbf{Z}^N \\ 0<|n_1|+\ldots+|n_N|\leq P}} \frac{i\,\hat{f}_n(y)}{h'(y)\cdot n}\, e^{in\cdot\varphi}\,, \qquad \chi_j(y,\varphi) = \sum_{\substack{n \in \mathbf{Z}^N \\ 0<|n_1|+\ldots+|n_N|\leq jP}} \frac{i\,\hat{\Gamma}^{(j)}_n(y)}{h'(y)\cdot n}\, e^{in\cdot\varphi}$$

for all $2 \leq j \leq k$, where, denoting $[\cdot]_j$ the j^{th} order in power series of ε,

$$\Gamma^{(j)} \equiv [h(y + \sum_{m=1}^{j-1} \varepsilon^m \partial_\varphi \chi_m)]_j + [f(y + \sum_{m=1}^{j-1} \varepsilon^m \partial_\varphi \chi_m, \varphi)]_{j-1}\,.$$

[1] See for example: [10] with an iterative finite KAM algorithm, [2] and [7] which make use of Giorgilli transformation. We believe that the estimates developed in our paper are easier than the ones usually developed in the proofs of the Theorem of Nekhoroshev, but they remain essentially optimal. Unfortunately, it is not easy to extend our method near the resonances, since in this case differentiation with respect to the actions compares in the definition of the transformnation so that it is difficult to invent a good majoring problem.

We would also point out that this kind of estimates shows that the Birkhoff series are in the class of *Borel summability*: see [4].

[2] Perturbation theory is a basic instrument for most of the proofs of the Theorem of Nekhoroshev; an exception is [9], in which "the usual ingredients of canonical perturbation theory [...] do not enter the proof at all".

[3] The equations of motion are $\dot{I} = -\varepsilon\,\partial_\varphi f$ and $\dot{\varphi} = \partial_I H$, the dot standing for time derivative.

The function g_j, for $2 \leq j \leq k$, is the average of $\Gamma^{(j)}$ in the angles. Sending k to ∞, we obtain a formal[4] series $\sum_{j\geq 1} \varepsilon^j \chi_j$, called **Birkhoff series**.

The condition over ε requested by the Theorem of Birkhoff is $|\varepsilon| \leq \varepsilon_0$, with

$$\sum_{j=1}^{k} \varepsilon_0^j \sup_{\varphi\in\mathbf{T}^N} \left|\frac{\partial^2 \chi_j(y_0,\varphi)}{\partial y \partial \varphi}\right| < 1 . \tag{2.2}$$

This condition is to assure that the transformation Φ_k is a local diffeomorphism in the actions but a global diffeomorphism in the angles.

3 Estimate on the rest

We will make use of the method of **majorants** (see, for example, [8] and [11]):

Definition 3.1 *If* $f(z,\varphi) = \sum_{n\in\mathbf{Z}^N,\, m\in\mathbf{N}^M} f_{n,m}\, z^m\, e^{in\cdot\varphi}$ *and* $g(z,\varphi) = \sum_{n\in\mathbf{Z}^N,\, m\in\mathbf{N}^M} g_{n,m}\, z^m\, e^{in\cdot\varphi}$ *represent two analytic functions of* $\varphi \in \mathbf{T}^N$ *with* $z \in \mathbf{R}^M$ *near the origin, we say that* g *is a majorant of* f *(and we briefly write* $f \prec g$*) if* $|f_{n,m}| \leq g_{n,m}$.

Moreover, we will denote

$$\overline{f}(z,\varphi) \equiv \sum_{n\in\mathbf{Z}^N,\, m\in\mathbf{N}^M} |f_{n,m}|\, z^m\, e^{in\cdot\varphi} .$$

We will use afterwards the following easy results:

Lemma 3.2 *For all* $n, m \in \mathbf{N}$, $m \geq 2$,

$$\max\{j_1! \cdot \ldots \cdot j_p!,\ j_1,\ldots,j_p \in \mathbf{N},\ j_1+\ldots+j_p = n\} = n! \tag{3.3}$$

$$\max\{j_1! \cdot \ldots \cdot j_p!,\ j_1,\ldots,j_p \in \mathbf{N},\ 1 \leq j_1,\ldots,j_p \leq m-1,\ j_1+\ldots+j_p = m\} = (m-1)! . \tag{3.4}$$

The proof is elementary. The following Lemma will state the majoring problem. We will denote, for all $x \in \mathbf{C}$, $\vec{x} \equiv (\underbrace{x,\ldots,x}_{N\ \text{times}})$. We will call η_0 an auxiliary analicity parameter, and consider by now $|\varepsilon| \leq \eta_0$:

Lemma 3.3 *Fixed* $y \in \mathbf{C}^N$, $|y-y_0| \leq \rho/2$, *for all* $|z| \leq \rho/2$ *we define*

$$\begin{aligned} Q_y(z) &\equiv h(y+z) - h(y) - h'(y)\cdot z , \\ F_y(z,\varphi) &\equiv f(y+z,\varphi) . \end{aligned}$$

Denote $\mathcal{E} \equiv \{\varepsilon \in \mathbf{C},\ |\varepsilon| \leq \eta_0\}$. *Then there exists a unique* $\Psi = \Psi_y(\varphi,\varepsilon) \in \mathrm{C}^\omega(B^N_{\rho/2} \times \mathbf{T}^N_\xi \times \mathcal{E})$ *such that* $\Psi_y(\varphi,0) = 0$, $\sup|\Psi| \leq R$ *and*

$$\gamma\,\Psi - \overline{Q}_y(\vec{\Psi}) - \varepsilon\overline{F}_y(\vec{\Psi},\varphi) = 0 \tag{3.5}$$

provided that $\varepsilon \in \mathcal{E}$ *verifies*

$$\eta_0 \leq \min\left\{ \frac{\gamma R}{2 \sup\limits_{|y-y_0|\leq\rho/2,\varphi\in\mathbf{T}^N_\xi} |\overline{F}_y(0,\varphi)|},\ \frac{\gamma}{4 \sup\limits_{|y-y_0|\leq\rho/2,\ |z|_\infty\leq R,\ \varphi\in\mathbf{T}^N_\xi} |\partial_z \overline{F}|} \right\} \tag{3.6}$$

[4]It is well known that the series of Birkhoff are in general divergent: for example one can directly show that the two degrees of freedom Hamiltonian

$$\varpi I_1 + I_2 + \varepsilon\left(I_1 + \sum_{n\in\mathbf{Z}^2} e^{-|n_1|-|n_2|}\cos n\cdot\varphi\right)$$

has divergent Birkhoff series, standing ϖ for the golden number.

having fixed $R \le \rho/(2\sqrt{N})$ *such*[5] *that* $\sup_{|y-y_0|\le\rho/2,\,|z|_\infty\le R} |\partial_z \overline{Q}| \le \gamma/4$. *So one can choose* $R \le O(\gamma)$ *and* $\eta_0 \le O(\gamma^2)$.

Proof. Setting $\mathcal{B} \equiv \{u \in C^\omega(\mathbf{T}^N_\xi \times \mathcal{E})$ st $\sup|u| \le R\}$ and, $\forall u \in \mathcal{B}$,

$$\mathcal{F}_y[u](\varphi,\varepsilon) \equiv \gamma^{-1}\,\overline{Q}_y\,(\vec{u}(\varphi,\varepsilon)) + \gamma^{-1}\,\varepsilon\,\overline{F}_y\,(\vec{u}(\varphi,\varepsilon),\varphi)$$

one sees that $\mathcal{F}$ is a contraction over $\mathcal{B}$ endowed with the sup $-$norm. □

Note that, by definition, one has, $\forall r \ge 2$,

$$\Gamma^{(r)}(y,\varphi) = [Q_y(\sum_{j=1}^{r-1}\varepsilon^j \partial_\varphi \chi_j)]_r + [F_y(\sum_{j=1}^{r-1}\varepsilon^j \partial_\varphi \chi_j,\varphi)]_{r-1}\,. \tag{3.7}$$

We remark that in the previous lemma (as well as in the next Proposition) y is **fixed**[6]. So if

$$Q_y(z) = \sum_{p\in\mathbf{N}^N} Q_p(y)\; z^p\,, \qquad F_y(z,\varphi) = \sum_{p\in\mathbf{N}^N,\,n\in\mathbf{Z}^N} F_{n,p}(y)\, z^p\, e^{in\cdot\varphi}\,,$$

we have

$$\overline{Q}_y(z) = \sum_{p\in\mathbf{N}^N} |Q_p(y)|\; z^p\,, \qquad \overline{F}_y(z,\varphi) = \sum_{p\in\mathbf{N}^N,\,n\in\mathbf{Z}^N} |F_{n,p}(y)|\, z^p\, e^{in\cdot\varphi}\,.$$

In the same way, fixing y in the next Proposition, we look at χ and Ψ as functions of φ only.

Proposition 3.4 *Fixed* $y \in \mathbf{C}^N$, $|y-y_0| \le \rho/2$, *being* Ψ *the function defined in the previous Lemma, one has that, for all* $j \in \mathbf{N}$, $\chi_j \prec P^{2(j-1)}\,(j-1)!\,\Psi_j$.

Proof. Writing the $\varepsilon-$expansion of Ψ in (3.5) one obtains

$$\gamma\sum_{j\ge1}\varepsilon^j\,\Psi_j - \overline{Q}_y(\sum_{j\ge1}\varepsilon^j\,\vec{\Psi}_j) - \varepsilon\overline{F}_y(\sum_{j\ge1}\varepsilon^j\,\vec{\Psi}_j,\varphi) = 0\,. \tag{3.8}$$

Taking the first order of this, one has $\gamma\Psi_1 - \overline{f}(y,\varphi) = 0$; so, from the definition of χ_1, $\Psi_1 = \overline{f}/\gamma \succ \chi_1$. This is the inductive basis of our statement. For the inductive step we suppose to have proven the result for all $j \le r-1$ and we take the r^{th} order of (3.8):

$$\Psi_r = \frac{1}{\gamma}\left([\overline{Q}_y(\sum_{j=1}^{r-1}\varepsilon^j\,\vec{\Psi}_j)]_r + [\overline{F}_y(\sum_{j=1}^{r-1}\varepsilon^j\,\vec{\Psi}_j,\varphi)]_{r-1}\right)\,. \tag{3.9}$$

So that, invoking (3.3), (3.4), (3.7), the definition of χ_r and the fact that $\partial_\varphi\chi_j \prec jP\vec{\overline{\chi}}_j$:

$$\begin{aligned}
\chi_r \;&\prec\; \frac{1}{\gamma}\overline{\Gamma}^{(r)} \prec \frac{1}{\gamma}\left([\overline{Q}_y(\sum_{j=1}^{r-1}\varepsilon^j\,\overline{\frac{\partial\chi_j}{\partial\varphi}})]_r + [\overline{F}_y(\sum_{j=1}^{r-1}\varepsilon^j\,\overline{\frac{\partial\chi_j}{\partial\varphi}},\varphi)]_{r-1}\right) \prec \\
&\prec\; \frac{1}{\gamma}\left([\overline{Q}_y(\sum_{j=1}^{r-1}\varepsilon^j Pj\vec{\overline{\chi}}_j)]_r + [\overline{F}_y(\sum_{j=1}^{r-1}\varepsilon^j Pj\vec{\overline{\chi}}_j,\varphi)]_{r-1}\right) \prec \\
&\prec\; \frac{1}{\gamma}\left([\overline{Q}_y(\sum_{j=1}^{r-1}\varepsilon^j P^{2j-1} j!\vec{\Psi}_j)]_r + [\overline{F}_y(\sum_{j=1}^{r-1}\varepsilon^j P^{2j-1} j!\vec{\Psi}_j,\varphi)]_{r-1}\right) \prec
\end{aligned}$$

[5] We need the condition $R \le \rho/(2\sqrt{N})$ to remain in the analyticity domain: if $z = \vec{g}$ with $\sup|g| \le R$, it follows in this way that $|z| \le \rho/2$.

[6] But the dependence of Ψ upon y is analytical, since the estimates are uniform in y.

$$
\begin{aligned}
&\prec \frac{1}{\gamma}\left(P^{2(r-1)} \max_{\substack{1\le j_1,\dots,j_p\le r-1\\ j_1+\dots+j_p=r\\ 2\le p\le r-1}} j_1!\,\dots\, j_p!\,[\overline{Q}_y(\sum_{j=1}^{r-1}\varepsilon^j\vec{\Psi}_j)]_r+\right.\\
&\left.+P^{2(r-1)-1} \max_{\substack{1\le j_1,\dots,j_p\le r-1\\ j_1+\dots+j_p=r-1\\ 1\le p\le r-1}} j_1!\,\dots\, j_p!\,[\overline{F}_y(\sum_{j=1}^{r-1}\varepsilon^j\vec{\Psi}_j)]_{r-1}\right)\prec\\
&\prec \frac{1}{\gamma}P^{2(r-1)}(r-1)!\,\left([\overline{Q}_y(\sum_{j=1}^{r-1}\varepsilon^j\vec{\Psi}_j)]_r+[\overline{F}_y(\sum_{j=1}^{r-1}\varepsilon^j\vec{\Psi}_j,\varphi)]_{r-1}\right)=P^{2(r-1)}(r-1)!\,\Psi_r
\end{aligned}
$$

ending the proof. □

Proposition 3.5 *There exists a constant c such that*

$$\sup_{|y-y_0|\le\rho/2,\,\varphi\in\mathbf{T}^N_{\xi/2}}|\chi_j|\le c\,\frac{P^{2(j-1)}(j-1)!\,\gamma}{\eta_0^j}\,.$$

Proof. From the Cauchy Estimate and Lemma 3.3, $\forall|y-y_0|\le\rho/2$, $\varphi\in\mathbf{T}^N_\xi$,

$$|\Psi_j(y,\varphi)|=\frac{1}{j!}\left|\,\frac{\partial^j}{\partial\varepsilon^j}\right|_{\varepsilon=0}\Psi\,\Big|\le\frac{\sup_{|y-y_0|\le\rho/2,\,\varphi\in\mathbf{T}^N_\xi}|\Psi|}{\eta_0^j}\le\frac{R}{\eta_0^j}\le\frac{O(\gamma)}{\eta_0^j}\,. \tag{3.10}$$

So, if $|y-y_0|\le\rho/2$ and $\varphi\in\mathbf{T}^N_{\xi/2}$, using the previous Proposition,

$$
\begin{aligned}
|\chi_j(y,\varphi)| &\le \sum_{n\in\mathbf{N}^N}|\widehat{(\chi_j)}_n|\,e^{|n|\,\xi/2}\le\sum_{n\in\mathbf{N}^N}P^{2(j-1)}\,(j-1)!\,|\widehat{(\Psi_j)}_n|\,e^{|n|\,\xi/2}\le\\
&\le C_\xi\,P^{2(j-1)}\,(j-1)!\sup_{|y-y_0|\le\rho/2,\,\varphi\in\mathbf{T}^N_\xi}|\Psi_j|\le C_\xi\,P^{2(j-1)}\,(j-1)!\,\frac{O(\gamma)}{\eta_0^j}\,,
\end{aligned}
$$

where $C_\xi\equiv\sum_{n\in\mathbf{N}^N}e^{-|n|\,\xi/2}$. □

Theorem 3.6 *Standing $c^\star$ for a suitable constant, set*

$$
\begin{aligned}
\gamma &\equiv \min\{\min_{\substack{0<|n_1|+\dots+|n_N|\le kP\\ |I-I_0|\le\rho}}|h'(I)\cdot n|,\ 1\}\\
\varepsilon_0 &\equiv \frac{c^\star\,\gamma^2}{k}
\end{aligned}
\tag{3.11}
$$

Then, $\forall|\varepsilon|\le\varepsilon_0/2$, we have the following bound on the rest R_k of Birkhoff averaging (2.1):

$$\sup_{|y-y_0|\le\rho,\,x\in\mathbf{T}^N_{\xi^\star}}|R_k|\le\sup_{|I-I_0|\le\rho^\star,\,\varphi\in\mathbf{T}^N_\xi}|H|\cdot\left(\frac{2}{\varepsilon_0}\right)^{k+1}\,. \tag{3.12}$$

Moreover, with the same assumption as above, we have the following bound on the size of the j^{th} order in ε of the new Hamiltonian:

$$\sup_{|y-y_0|\le\rho,\,x\in\mathbf{T}^N_{\xi^\star}}|g_j|\le\sup_{|I-I_0|\le\rho^\star,\,\varphi\in\mathbf{T}^N_\xi}|H|\cdot\left(\frac{1}{\varepsilon_0}\right)^j\,. \tag{3.13}$$

Proof. Fixed $|\varepsilon| \le \varepsilon_0/2$, writing the $\varepsilon-$expansion of $\mathcal{H}$ we have that there exist $\varepsilon^\star$, $|\varepsilon^\star| \le \varepsilon_0/2$ such that

$$\mathcal{H}(y,x;\varepsilon) = \sum_{j=1}^{k} \varepsilon^j \mathcal{H}_j(y,x) + \frac{\varepsilon^{k+1}}{(k+1)!}\frac{\partial^{k+1}}{\partial\varepsilon^{k+1}}\mathcal{H}(y,x;\varepsilon^\star)\,.$$

From the Theorem of Birkhoff one has $\mathcal{H}_0 = h$ and $\mathcal{H}_j = \dfrac{1}{j!}\dfrac{\partial^j}{\partial\varepsilon^j}\mathcal{H}(y,x;0) = g_j$ for $1 \le j \le k$ and $R_k(y,x) \equiv \dfrac{1}{(k+1)!}\dfrac{\partial^{k+1}}{\partial\varepsilon^{k+1}}\mathcal{H}(y,x;\varepsilon^\star)$. So from the Cauchy Estimate

$$\sup_{|y-y_0|\le\rho,\, x\in\mathbf{T}^N_{\xi^\star},\, |\varepsilon|\le\varepsilon_0/2} |R_k| \le \frac{\sup_{|y-y_0|\le\rho,\, x\in\mathbf{T}^N_{\xi^\star},\, |\varepsilon|\le\varepsilon_0} |\mathcal{H}|}{(\varepsilon_0/2)^{k+1}} \le \sup_{|I-I_0|\le\rho^\star,\, \varphi\in\mathbf{T}^N_{\xi},\, |\varepsilon|\le\varepsilon_0} |H|\cdot\left(\frac{2}{\varepsilon_0}\right)^{k+1}.$$

The conditions that ε_0 must satisfy are: (2.2) in order to apply the Averaging Theorem of Birkhoff, and $\varepsilon_0 \le \eta_0$ with η_0 satisfying (3.6), which means $\varepsilon_0 \le \eta_0 = O(\gamma^2)$, in order to use our estimates on χ_j. From the previous Proposition we get

$$\sup_{\varphi\in\mathbf{T}^N} |\partial^2_{y\varphi}\chi_j(y_0,\varphi)| \le \frac{P^{2(j-1)}\,(j-1)!\,O(\gamma)}{\eta_0^j\,\rho\,\xi}\,,$$

showing that there exists a constant $\tilde{c}$ such that

$$\sum_{j=1}^{k} \varepsilon_0^j \sup_{\varphi\in\mathbf{T}^N} |\partial^2_{y\varphi}\chi_j(y_0,\varphi)| \le \sum_{j=1}^{k}\frac{\gamma\,\tilde{c}^j\,j^j\,\varepsilon_0^j}{\eta_0^j} \le \gamma\sum_{j=1}^{k}\left(\frac{\tilde{c}\,k\,\varepsilon_0}{\eta_0}\right)^j < \gamma \le 1\,,$$

having chosen $c^\star$ to make $\tilde{c}\,k\,\varepsilon_0/\eta_0 \le 1/2$. This proves (3.12). In the same way one can prove (3.13). $\square$

4 A simple model with no small denominators but with diverging Birkhoff series, showing the "sharpness" of the previous estimates

The estimates in the Proposition 3.4 states that the divergence of the formal series $\sum\varepsilon^j\chi_j$ is not only due to the small divisors (but it is, in the worst cases, factorial): as a matter of fact we can control $|\chi_j|$ only with $(j-1)!$ in general. To understand this, we present a simplified model: if we consider the perturbation of an armonic oscillator

$$H(I,\varphi) = \omega\,I + \varepsilon\,f(I,\varphi)\,,$$

with $I \in \mathbf{R}$, $\varphi \in \mathrm{S}^1$ and $f = 2\cos\varphi\cdot(1-I)$, we obtain the Hamilton-Jacobi equation

$$\omega\,\partial_\varphi\chi + \varepsilon\,f(y+\partial_\varphi\chi,\varphi) = 0\,. \tag{4.14}$$

Since the effect of the operator $\omega\,\partial_\varphi$ is "multiplying for a small denominator", instead of (4.14) we consider the following simplified problem

$$u + \varepsilon\,f(\partial_\varphi u,\varphi) = 0\,, \tag{4.15}$$

being $u = \sum_{j\ge1}\varepsilon^j\,u_j(I,\varphi)$ the unknown function, in which we have suppressed the small divisors. The formal solution of (4.15) is

$$u_j = \begin{cases} \sigma_j \displaystyle\sum_{k=0}^{(j-1)/2} c_{j,k}\cos\left((j-2k)\varphi\right) & \text{if } j \text{ is odd} \\ \\ \sigma_j \displaystyle\sum_{k=0}^{(j-2)/2} c_{j,k}\sin\left((j-2k)\varphi\right) & \text{if } j \text{ is even} \end{cases}\,,$$

for fitting $c_{j,k} > 0$ with $c_{j,0} = 2\,(j-1)!$ and $\sigma_j = \pm 1$. This shows that, if j is odd, u_j evaluated in $\varphi = 0$ grows up factorially.

This example shows that, even in problems with no small divisors, the series of Birkhoff can diverge; the rate of divergence given by Proposition 3.4 is "sharp", in the sense that this simplified example shows just the factorial growth.

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