STATISTICAL PROPERTIES OF HYPERBOLIC SYSTEMS WITH TANGENTIAL SINGULARITIES

HENRY VAN DEN BEDEM

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA AT BIRMINGHAM
BIRMINGHAM, AL 35294 USA
vdbedem@math.uab.edu

OCTOBER 26, 1998.

Abstract. We consider piecewise smooth, uniformly hyperbolic systems on a Riemannian manifold, where we allow the angle between the unstable direction and the singularity manifolds to vanish. Under natural assumptions we prove that such systems exhibit Exponential Decay of Correlations and satisfy a Central Limit Theorem with respect to a mixing SRB-measure. These results have been shown previously for systems in which the angle between the singularity manifold and unstable direction is uniformly bounded away from zero.

1. Introduction

Anosov and Axiom A systems, together with their SRB (Sinai-Ruelle-Bowen) measures, are frequently used to model many particle statistical mechanical systems, both at equilibrium and at steady states not too far from equilibrium. In such applications, physicists are generally interested in strong statistical properties, like exponential decay of correlations (EDC) and central limit theorems (CLT). In the seventies already, exponential decay was shown for smooth, uniformly hyperbolic systems (e.g. [1]). Since many physical models exhibit nonuniform hyperbolic behavior and/or contain singularities (billiards and attractors being the most well-known examples of these), investigating statistical properties for such systems became an active research area. A CLT for billiards was shown by L. Bunimovich and Ya. Sinai [2] in the early eighties, but EDC proved to be much harder to obtain. The first result in this direction, obtained by C. Liverani [7], is an exponential upper bound for the decay of correlations for a class of two dimensional systems with singularities preserving a symplectic 2-form. L.-S. Young [11] only recently proved the existence of an exponential upper bound for more general two dimensional systems preserving an SRB measure that is not necessarily smooth. Both Liverani and Young assume that the singularity manifolds $\mathcal{S}$ and the unstable cones are uniformly transversal.

In some physical models however, the angle between the singularity manifolds and the unstable direction vanishes. This is for example the case in M. Wojtkowski's 'Falling Balls'-model [10] whenever the system comprises $n \geq 3$ balls. This motivates us to generalize Young's result by allowing the angle between the unstable direction and $\mathcal{S}$ to vanish. We will consider uniformly hyperbolic, piecewise smooth maps (the exact definitions can be found in Section 2) together with an SRB-measure (the existence of such a measure will be proved). We allow for tangencies of the singularity manifolds and the unstable directions, but exclude coincidence of the two on a set positive (induced) volume. Under natural conditions we prove EDC for a class of Hölder continuous functions and a CLT. We do so by showing that our systems meet the conditions Young uses in [11] to obtain EDC and CLT. We have included some examples.

2. Setting

Let M denote a compact, smooth Riemannian manifold, possibly with boundary. For simplicity we restrict ourselves to two dimensions. Assume that M is endowed with Riemannian metric d and Lebesgue measure m. The induced metric and measure on submanifolds γ of M are denoted by d_γ and m_γ, respectively. Throughout, we denote the tangent space to a (sub)manifold S at the point x by $\mathcal{T}_x S$. Let $\mathcal{S}_i \subset M$, $i = 1, \ldots, N$ denote a finite collection of compact, smooth curves, where $\partial M \subset \cup_i \mathcal{S}_i$. We may assume that for every $\mathcal{S}_i \not\subset \partial M$ we have $\partial \mathcal{S}_i \subset \cup_j \operatorname{Int} \mathcal{S}_j$, i.e. every component of $\mathcal{S}$ not in the boundary ∂M terminates in the interior of another component. This is not a restrictive assumption, since we can extend every component so that it satisfies this condition. The set $\mathcal{S} = \cup_i \mathcal{S}_i$ is the singularity set of T. The following six assumptions define the setting more precisely.

Hyp 1. T is a C^2-diffeomorphism from $M - \mathcal{S}$ onto its image $T(M - \mathcal{S})$. Both T and T^{-1} are twice differentiable up to the boundaries of their domains.

Note that the derivatives of T and T^{-1} are bounded since M is compact. We let $\mathcal{S}^n = \mathcal{S} \cup \ldots \cup T^{-n+1}\mathcal{S}$ denote the singularity set of T^n, the n-fold composition of T with itself.

Hyp 2. T is uniformly hyperbolic, i.e. for each $x \in M$ there exist cones $C_x^u, C_x^s \subset \mathcal{T}_x M$ such that $D_x T(C_x^u) \subset C_{Tx}^u$ and $D_x T^{-1}(C_x^s) \subset C_{T^{-1}x}^s$ whenever these derivatives exist and

$$\begin{aligned} \|D_x T(v)\| &\geq \lambda \|v\| && \text{for all } v \in C_x^u \\ \|D_x T^{-1}(v)\| &\geq \lambda \|v\| && \text{for all } v \in C_x^s \end{aligned}$$

for some constant $\lambda > 1$. These cones vary continuously with x and the angle between C_x^u and C_x^s has a positive lower bound.

The constant λ above thus is a lower bound on the expansion factor of unstable vectors and contraction factor of stable vectors.

We introduce some more notation. A *n-unstable curve* is a curve γ of finite diameter such that $T^{-n}\gamma$ consists of a single smooth component and for all $k = 0, \ldots, n$ and $z \in T^{-k}\gamma$ we have $\mathcal{T}_z T^{-k}\gamma \cap C_x^u \neq \{0\}$. A 0-unstable curve will just be called a u-curve. An *n-stable curve* is defined similarly. The diameter of a curve is defined as $\operatorname{diam}(\gamma) = \sup_{x,y \in \gamma} d_\gamma(x, y)$. Following [8], we define M^+ (M^-) to be the set of points $x \in M$ for which the future (past) orbit is defined, i.e.

$$M^+ = M - \bigcup_{n \geq 0} T^{-n}\mathcal{S}, \qquad M^- = \bigcap_{n \geq 1} T^n(M - \mathcal{S}^n).$$

The set $M^0 = M^+ \cap M^-$ consists of points for which the entire orbit is defined. Furthermore, let

$$\begin{aligned} E_x^s &= \bigcap_{n \geq 0} D_{T^n x} T^{-n}(C_{T^n x}^s) && \text{for } x \in M^+ \\ E_x^u &= \bigcap_{n \geq 0} D_{T^{-n} x} T^{n}(C_{T^{-n} x}^u) && \text{for } x \in M^-. \end{aligned}$$

Then $\mathcal{T}_x M = E_x^s \oplus E_x^u$ whenever $x \in M^0$ and the angle between E_x^s and E_x^u has a positive lower bound, uniformly on M^0. For a connected submanifold $\omega \subset M$ the smooth components of $T^{-n}(\omega - \cup_{k \geq 1}^n T^k \mathcal{S})$ are called the components of $T^{-n}\omega$. We call a submanifold ω a local unstable manifold (LUM) if $T^{-n}\omega$ has exactly one component for all $n \geq 0$ and for all $x, y \in \omega$ we have $d(T^{-n}x, T^{-n}y) \to 0$ as $n \to \infty$ exponentially fast. We denote the closed ϵ-ball in ω

centered at x (whenever it exists) by $W^u_\epsilon(x)$. Local stable manifolds (LSMs) and $W^s_\epsilon(x)$ are defined analogously for forward iterates of T. Define for $\kappa > 1$

$$\begin{aligned} M^-_{\kappa,\epsilon} =& \{x \in M^- : d(T^{-n}x, T\mathcal{S}) \geq \epsilon\kappa^{-n} \quad \forall n \geq 0\} \\ M^+_{\kappa,\epsilon} =& \{x \in M^+ : d(T^n x, \mathcal{S}) \geq \epsilon\kappa^{-n} \quad \forall n \geq 0\} \\ M^0_{\kappa,\epsilon} =& M^-_{\kappa,\epsilon} \cap M^+_{\kappa,\epsilon} \\ M^0_\kappa =& \bigcup_{\epsilon>0} M^0_{\kappa,\epsilon} \end{aligned}$$

Observe that $M^\pm_{\kappa,\epsilon}$ is a closed set for a fixed $\epsilon > 0$ and that M^0_κ is T-invariant. Proposition 4 of [8] implies that there exists a $\tilde{\kappa} > 1$ such that for all $x \in M^-_{\tilde{\kappa},\epsilon}(M^+_{\tilde{\kappa},\epsilon})$ and some $\delta(\epsilon) > 0$ the C^1-submanifold $W^u_\delta(x)(W^s_\delta(x))$ exists and is unique. At this point it is not clear that $M^-_{\tilde{\kappa},\epsilon} \neq \emptyset$. This will be discussed at the end of this section.

In showing statistical properties of systems satisfying Hyp 1 and 2, previous results ([11], [4]) assume that u-curves intersect the singularity manifolds transversally. We would like to relax that assumption and allow for tangencies. Intuitively, $x \in \mathcal{S}$ is a *point of tangency* if $\mathcal{T}_x\mathcal{S} \cap E^u_x \neq \{0\}$. We can not utilize this definition, since E^u_x may not exist. The following assumption allows singularity manifolds to intersect LUMs tangentially, provided they separate fast enough away from the point of intersection.

Hyp 3. There exists a $0 < p < \infty$ and a constant $C > 0$ such that for all $\epsilon > 0$ and all LUMs γ we have

$$m_\gamma(\gamma \cap B_\epsilon(\mathcal{S})) < C\epsilon^{\frac{1}{p}}.$$

We call the number p the *order of the tangency.*

A point $x \in \mathcal{S}^n$ is called *multiple* with multiplicity $l_{x,n}$ if it belongs to $l_{x,n} \geq 2$ smooth components of $\mathcal{S}^n$.

Hyp 4. There exists a $K \geq 1$ and an $m \geq 1$ such that for all $x \in \mathcal{S}^m$ we have $l_{x,m} \leq K < \frac{\lambda^m - 1}{2}$.

Similar assumptions are standard (see for example [4]) and necessary to ensure that LUMs grow faster than they get shredded by singularity manifolds. Note that Hyp 4 implies that there exists an $\epsilon_0 > 0$ such that every ϵ_0-ball in M intersects at most K smooth components of $\mathcal{S}^m$.

Allowing for tangencies introduces a new problem to the study of decay of correlations: a smooth component of $\mathcal{S}$ may intersect a LUM infinitely many times. The following assumption excludes this possibility. Let $\mathcal{S}^m_i$ denote a smooth component of $\mathcal{S}^m$, with m as in Hyp 4.

Hyp 5. There exists an $\eta > 0$ such that for all $\epsilon > 0$ the intersection $\gamma \cap B_\epsilon(\mathcal{S}^m_i)$ consists of at most two disjoint subintervals of γ whenever γ is a LUM with $\operatorname{diam}(\gamma) < \eta$.

The number of subintervals in Hyp 5 is fixed at two for notational convenience; it can be replaced by any finite number (Hyp 4 should then be modified accordingly).

Without loss of generality we can set $m = 1$ in Hyps 4 and 5. Indeed, assumptions 1-3 still hold for T^m and it suffices to show our main theorem for T^m. Henceforth we will assume that $m = 1$. Note that Hyp 3 and Hyp 5 prevent the singularity manifolds from coinciding with a LUM γ on a set of positive m_γ-measure. The last two Hyps lead us to formulate the following lemma.

Lemma 1. Set $\bar{\epsilon} := \min\{\epsilon_0, \eta\}$. Then for all $\delta_1 > 0$ and for every LUM γ with $\operatorname{diam}(\gamma) < \bar{\epsilon}$ the set $\{x \in \gamma : d_\gamma(x, \mathcal{S}) > \delta_1)\}$ has $\leq 2K + 1 \stackrel{\text{def}}{=} K_0$ connected components.

As remarked above, it is not obvious that $M^0_{\tilde{\kappa}} \neq \emptyset$ in our setup. It is well known that systems satisfying Hyp 1, 2 and the following assumption

(H3). There exist a $C > 0$ and a $q > 0$ such that for all $\epsilon > 0$ and $n \geq 1$
$$m(T^{-n} B_\epsilon(\mathcal{S})) \leq C\epsilon^q,$$
have the property that $M^0_{\tilde{\kappa}} \neq \emptyset$ (see [8]). Chernov [4] uses u-curves to show that systems satisfying Hyps 1, 2, 4 and a transversality condition on the intersection of u-curves and singularity manifolds satisfy (H3), thus establishing the existence of LUMs. In the current setup we can not hope to use u-curves to obtain (H3), since we can not control the intersection of u-curves and neighborhoods of $\mathcal{S}$. Instead, we will assume that $M^-_{\tilde{\kappa},\epsilon}$ contains at least one point x, i.e

Hyp 6. $M^-_{\tilde{\kappa},\epsilon} \neq \emptyset$ for $\epsilon > 0$ small enough.

In Section 5 it will then be shown that there exists a T-invariant probability measure μ with $\mu(M^0_{\tilde{\kappa}}) = 1$.

It was observed by N. Chernov that Hyp 6 is necessary in the sense that it does not follow from the first five hypotheses. We adapt his example from Section 10 in [3].

Example 1. Let $R := \{(x, y) : 0 < x < 1, y > 1\}$ be an open strip in $\mathbb{R}^2$ and let $M' := \{(s, t) : 0 \leq s \leq 1, 0 \leq t \leq 1\}$ with the identification of $s = 0$ and $s = 1$ be a closed cylinder. Define $T_1 : R \to R$ given by $(x, y) \mapsto (\frac{x}{3}, 2y - 1)$ and $T_2 : R \to M'$ given by $s = y$ mod 1 and $t = \exp(-y) + x\big(\exp(-y-1) - \exp(-y)\big)$. Define $T = T_2 \circ T_1 \circ T_2^{-1}$. Obviously, Hyp 1 and 2 are satisfied for T. Since for each point at most finitely many pre-images are defined, no LUMs exist. Hence Hyp 3-5 are trivially satisfied, yet $M^-_{\tilde{\kappa},\epsilon} = \emptyset$, for all $\epsilon > 0$.

Definition 2. A T-invariant measure μ_{SRB}, supported on M^0, is called an SRB-measure if the conditional measures of μ_{SRB} on local unstable manifolds are absolutely continuous with respect to the Lebesgue measure on those manifolds.

Measures with the SRB property describe the asymptotics of observables in physical models: if μ_{SRB} is an ergodic SRB measure concentrated on a set $\Lambda \subset \overline{M^0}$, there exists a set $W^s(\Lambda)$ containing Λ with positive volume such that for any continuous $\psi : M \to \mathbb{R}$ we have $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ T^k z = \int_\Lambda \psi \, d\mu_{\text{SRB}}$ for m-almost every $x \in W^s(\Lambda)$. If the map T preserves a smooth measure m_0, it is automatically an SRB measure. In case the set M^0 has full volume and μ_{SRB} is singular, its support can still coincide with $\overline{M}^0$. This typically happens for transitive Anosov systems. If $m(M^0) < m(M)$ the support of μ_{SRB} may have zero volume and the set Λ above is sometimes called an *attractor for* (T, μ_{SRB}).

We conclude this section with an elementary example of a dynamical system satisfying Hyp 1-6.

Example 2. Consider a Hyperbolic Toral Automorphism $T_1 : \mathbb{T}^2 \to \mathbb{T}^2$ given by $(x, y) \mapsto (2x + y, x + y)$. Cut out an open disk S in $\mathbb{T}^2$ and define $T_2 : S \to S$ to be the map that rotates the disk over an angle π and leaves the rest of the torus invariant. The singularity manifold $\mathcal{S}$ for the map $T = T_1 \circ T_2$ is given by the boundary of the disk ∂S, which has two points of tangency with the unstable direction.

3. RESULTS

We state the main results of this paper. The proofs can be found in Sections 5-7.

Theorem 3. Let T satisfy Hyp 1-6. Then

(a) T admits an SRB measure μ_{SRB}, which is not necessarily unique.

(b) Any measure μ_{SRB} has a finite number of ergodic components on each of which it is, up to a finite cycle, mixing and Bernoulli.

Let $\mathcal{H}_\eta$ denote the class of Hölder continuous functions on M with Hölder exponent $\eta > 0$, i.e.

$$\mathcal{H}_\eta := \{\phi : M \to \mathbb{R} : |\phi(x) - \phi(y)| \leq C d(x,y)^\eta \text{ for all } x, y \in M\},$$

where $C > 0$ depends on the function ϕ. The time correlation function $C_{\phi,\psi}(n)$ is defined by

$$C_{\phi,\psi}(n) = \int (\phi \circ T^n)\psi \, d\mu - \int \phi \, d\mu \int \psi \, d\mu$$

Definition 4. The dynamical system (T, μ) exhibits exponential decay of correlations for Hölder continuous functions if for any $\eta > 0$ there exists a $0 < \theta < 1$ such that for all $\phi, \psi \in \mathcal{H}_\eta$

$$|C_{\phi,\psi}(n)| \leq C\theta^n \text{ for all } n \geq 1,$$

for some $C > 0$ depending on the functions ϕ, ψ.

Definition 5. Let $\phi \in \mathcal{H}_\eta$ such that $\int \phi d\mu = 0$. We say that ϕ satisfies a Central Limit Theorem with respect to (T, μ) if the random variables $\phi \circ T^n$ satisfy

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi \circ T^i \overset{\text{distr}}{\longrightarrow} N(0, \sigma),$$

where N denotes the normal distribution and σ depends on ϕ.

Theorem 6. Let T satisfy Hyp 1-6 and suppose that (T^n, μ_{SRB}) is ergodic for all $n \geq 1$. Then (T, μ_{SRB}) exhibits exponential decay of correlations and satisfies a central limit theorem.

4. Standard facts

We introduce some notation, definitions and standard results for dynamical systems with singularities, see also [6]. Let γ denote a u-curve. For $x \in \gamma$ let $J^u(x) = |\det D_x T \restriction T_x\gamma|$ denote the Jacobian at the point x, restricted to γ.

Distortion along u-curves. Let $x, y \in \gamma - \mathcal{S}^n$ and $T^n x, T^n y$ belong in the same connected component $\omega \subset T^n\gamma$. Then

$$\log \prod_{i=0}^{n-1} \frac{J^u(T^i x)}{J^u(T^i y)} \leq C' d_\omega(T^n x, T^n y) \tag{1}$$

for some $C'(T) > 0$.

Bounded curvature. The curvature of LUMs and LSMs is uniformly bounded by some constant $B > 0$.

Absolute continuity. Let $W^u_\eta(x)$ and $W^u_{\eta'}(x')$ denote two LUMs and let

$$W_\infty := \{y \in W^u_\eta(x) : W^s_{\delta(y)}(y) \cap W^u_{\eta'}(x') \neq \emptyset\}.$$

Assume that $\delta(y), \eta, \eta'$ are small enough such that any LSM intersects each of those LUMs at most once. Define a map $h : W_\infty \to W^u_{\eta'}(x')$ by $y \mapsto W^s_{\delta(y)}(y) \cap W^u_{\eta'}(x')$. Then h is absolutely continuous with respect to $m_{W^u_\eta(x)}, m_{W^u_{\eta'}(x')}$ and its Jacobian is bounded at any point of density of $m_{W^u_\eta(x)}$, i.e.

$$\frac{1}{C_0} \leq \frac{m_{W^u_{\eta'}(x')}\big(h(W_\infty)\big)}{m_{W^u_\eta(x)}\big(W_\infty\big)} \leq C_0 \tag{2}$$

for some $C_0 > 0$.

5. Existence of SRB measures

In this section we prove Theorem 3(a) by constructing an SRB-measure for T. In [8], Pesin proves the existence of an SRB measure (which he calls a Gibbs u-measure) for systems satisfying four assumptions, including our Hyps 1 and 2 and the additional (H3) mentioned in Section 2. Since we can not verify Pesin's hypotheses for our systems, we use a different approach to obtain an SRB-measure.

Although the class of systems Young studies in Section 7 of [11] does satisfy Pesin's assumptions, a direct argument for the existence of an SRB measure is given there. She proves the existence of some invariant probability measure μ satisfying (4) below and later proves the existence of an SRB-measure. We extend Young's approach to our systems and argue directly that the measure obtained from (3) below is SRB.

Choose $\epsilon > 0$ such that $M^-_{\tilde{\kappa},\epsilon} \neq \emptyset$. Let γ denote the LUM corresponding to one of the points in $M^-_{\tilde{\kappa},\epsilon}$. Define

$$\mu_N = \frac{1}{N}\sum_{k=0}^{N-1} T^k_* m_\gamma, \tag{3}$$

i.e. we iterate Lebesgue measure on the local unstable manifold γ. Any (normalized) limit point μ of $\{\mu_N\}$ in the weak*-topology is an invariant Borel probability measure on $\overline{M^0}$. Since the set M^0 is not compact, mass may escape from M^0 in the limit of (3). The following lemma prevents this from happening. Once we obtain (4) below, it easily follows that any limit point μ of (3) is supported on M^0.

Lemma 7. Let T satisfy Hyp 1 - 6 and let μ be a normalized limit point of (3). Then for all $\epsilon > 0$ we have

$$\mu(B_\epsilon(\mathcal{S})) < C\epsilon^{\frac{1}{p}} \tag{4}$$

for some $C > 0$.

Following L.-S. Young [11] we will use the idea of stopping times on LUMs in the proof of this lemma. Let γ denote an arbitrary LUM with $\operatorname{diam}(\gamma) < \bar{\epsilon}$. The idea is to keep record of the length of the connected components of $T^n\gamma$. We set $\gamma_0 := \gamma - \mathcal{S}$, which consists of finitely many connected components ω_i. In general, $\gamma_n := \gamma_{n-1} - T^{-n}\mathcal{S}$. The stopping times are defined by a sequence of mappings $s_1 < s_2 < \dots$ from subsets of γ into $\mathbb{N}$ as follows. For $x \in \gamma$ let

$$s_1(x) = \min_{n\geq 1}\{m_{T^n\omega_x}(T^n\omega_x) > \bar{\epsilon}\},$$

with ω_x the component of γ_{n-1} containing x. If no such n exists, we set $s_1(x) = \infty$. The n-dependence of ω_x will be tacitly understood in the remainder of this paper.

Lemma 8. Let γ and γ_n be defined as above. Then

$$m_\gamma\big(\gamma_n - \{x : s_1(x) \leq n\}\big) \leq \bar{\epsilon}\big(\frac{K_0}{\lambda}\big)^n. \tag{5}$$

Proof. Since $\operatorname{diam}\big(\gamma\big) < \bar{\epsilon}$, γ_0 consists of at most K_0 connected components γ_{0_i} with $\operatorname{diam}(\gamma_{0_i}) < \bar{\epsilon}$. For those components with $s_1(x) \restriction \gamma_{0_i} \neq 1$, $\operatorname{diam}(T\gamma_{0_i}) < \bar{\epsilon}$. At the n-th step we have that $\gamma_n - \{x : s_1(x) \leq n\}$ is the union of at most K_0^n components $\gamma_{n_k} \subset \gamma_n$ with $\operatorname{diam}(T^n\gamma_{n_k}) \leq \bar{\epsilon}$. Pulling back we obtain (5). □

In particular, $s_1(x) < \infty$ for m_γ-almost every $x \in \gamma$. On the full measure subset of γ for which $s_1(x) < \infty$ define

$$s_2(x) = \min_{n > s_1(x)} \{m_{T^n\omega_x}(T^n\omega_x) > \bar{\epsilon}\},$$

with ω_x the component of γ_{n-1} containing x. It is easily seen that s_2 is also defined on a subset of full measure of γ. Continuing this procedure we obtain that for any LUM γ with $\operatorname{diam}(\gamma) < \bar{\epsilon}$ and for any $k \geq 1$ the k-th stopping time $s_k(x)$ is defined m_γ-almost everywhere.

Proof of Lemma 7. Let $\epsilon > 0$ be given. Let γ be a LUM with $\operatorname{diam}(\gamma) < \bar{\epsilon}$ and μ a normalized limit point of the sequence given by (3). Our strategy is to estimate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$.

Fix an integer $N > 0$. Let s_i be defined on γ as above. Observe that $\{x : s_k(x) < N\}$ is a disjoint union of a finite number of connected components of γ for all $k < N$. We will evaluate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$ on the subsets $\{x : s_1(x) \geq N\}$, $S_k := \{x : s_k(x) < N\} \cap \{x : s_{k+1}(x) \geq N\}$ for $1 \leq k < N-1$ and $\{x : s_{N-1}(x) = N-1\}$ separately.

Consider the set $\{x : s_1(x) \geq N\}$. As in the proof of Lemma 8 we see that for each natural n the set $\{x : s_1(x) \geq n\}$ is a disjoint union of at most countably many components. Consider one of these components ω with $s_1 \restriction \omega \neq n$. Then $m_{T^n\omega}(T^n\omega) \leq \bar{\epsilon}$ and by Lemma 1 it intersects $B_\epsilon(\mathcal{S})$ in $\leq K_0$ components. Therefore, by Hyp 3

$$m_{T^n\omega}\big(T^n\omega \cap B_\epsilon(\mathcal{S})\big) < K_0 C \epsilon^{\frac{1}{p}}$$

Pulling back to ω and summing up to the $(N-1)$-st iterate over the components in $\{x : s_1(x) \geq N\}$ we obtain

$$\sum_{n=0}^{N-1} m_\gamma\big(T^{-n}B_\epsilon(\mathcal{S}) \cap \{x : s_1(x) \geq N\}\big) < K_0 C \epsilon^{\frac{1}{p}} \sum_{n=0}^{N-1} (\frac{K_0}{\lambda})^n \tag{6}$$

Next, we estimate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$ on the subsets S_k for $1 \leq k < N-1$. Let ω denote a component of γ_{j-1} and $s_k \restriction \omega = j$. Observe that $j \geq k$. Partition ω into $\omega_1 \cup \ldots \cup \omega_m$ with $\frac{\bar{\epsilon}}{2} \leq m_{T^j\omega}(T^j\omega_i) < \bar{\epsilon}$. We use the exact same procedure as above, choosing $T^j\omega$ as our starting point and ω replaced by $T^j\omega_i$.

It may happen that $T^{-i}B_\epsilon(\mathcal{S}) \cap \omega \neq \emptyset$, $0 \leq i < j$ for such a component ω. Observe that the measure of this set is already taken into account by the estimates up to the k-th stopping time. Indeed, for a segment in the set $\{x : s_l(x) < N\}$, $0 \leq l < k$ we in fact overestimate the measure covered by pre-images of $B_\epsilon(\mathcal{S})$ at the n-th iterate by assuming that all its K_0^n components have $\operatorname{diam}(T^n\omega) \leq \bar{\epsilon}$, regardless whether or not they are contained in $\{x : s_{l+1} < N\}$.

We compute the fraction of $T^j\omega_i$ covered by pre-images of $B_\epsilon(\mathcal{S})$ before the $k+1$-st stopping time and claim that this fraction approximately equals the fraction covering $S_k \cap \omega$. This claim is made precise in Lemma 9 below. We obtain that

$$\sum_{n=0}^{N-1-j} m_\omega\big(T^{-(n+j)}B_\epsilon(\mathcal{S}) \cap S_k\big) < m_\omega(\omega)\exp(C'\bar{\epsilon})\frac{2K_0 C \epsilon^{\frac{1}{p}}}{\bar{\epsilon}} \sum_{n=0}^{N-1-j} (\frac{K_0}{\lambda})^n$$

Summing over all components $\omega \in \{x : s_k(x) < N\}$ we obtain

$$\sum_{j=k}^{N-1} \sum_{n=0}^{N-1-j} m_\gamma\big(T^{-(n+j)}B_\epsilon(\mathcal{S}) \cap S_k\big) < \exp(C'\bar{\epsilon}) 2K_0 C \epsilon^{\frac{1}{p}} \sum_{n=0}^{N-1-k} (\frac{K_0}{\lambda})^n$$

Hence, summing up to $\{x : s_{N-2}(x) < N\}$, adding (6), dividing by N and writing $S := \cup^{N-2} S_k \cup \{x : s_1(x) \geq N\}$, we obtain for all $N \geq 0$

$$\frac{1}{N}\sum_{n=0}^{N-1} m_\gamma\big(T^{-n}B_\epsilon(\mathcal{S}) \cap S\big) < \exp(C'\bar{\epsilon})2K_0C\epsilon^{\frac{1}{p}} \sum_{n=0}^{N-1}(\frac{K_0}{\lambda})^n. \tag{7}$$

Finally, we estimate $\sum_{n=0}^{N-1} m_\gamma(T^{-n}B_\epsilon(\mathcal{S}))$ on $\{x : s_{N-1} = N-1\}$. For $\omega \subset \{x : s_{N-1} = N-1\}$ we have $m_{T^n\omega}(T^n\omega) > \bar{\epsilon}$ for all $n = 1, \dots, N-1$. Fix an n and partition ω into $\omega_1 \cup \dots \cup \omega_m$ with $\frac{\bar{\epsilon}}{2} \leq m_{T^n\omega}(T^n\omega_i) < \bar{\epsilon}$. Using the same estimates as above, we first sum over all components in Θ_{N-1}, and then over n. After dividing by N we obtain

$$\frac{1}{N}\sum_{n=1}^{N-1} m_\gamma\big(T^{-n}B_\epsilon(\mathcal{S}) \cap \{x : s_{N-1}(x) = N-1\}\big) < \frac{\exp(C'\bar{\epsilon})2K_0C\epsilon^{\frac{1}{p}}}{N}\sum_{n=0}^{N-1}(\frac{K_0}{\lambda})^n$$

Adding this last sum to (7) and letting $N \to \infty$ yields (4). □

In the proof we estimate the measure of the fraction of $\{x : s_k(x) < N\} \cap \{x : s_{k+1}(x) \geq N\}$ covered by pre-images of $B_\epsilon(\mathcal{S})$ by the corresponding fraction of $T^j\omega_i$. The following lemma justifies this estimate. The main idea is that the j-th iterate of Lebesgue measure on ω_i, denoted by $T^j_* m_{\omega_i}$, has an almost constant density with respect to $m_{T^j\omega_i}$.

Lemma 9. Let ω be as in the proof of Lemma 7 and $B \subset T^j\omega$ a measurable subset. Then

$$\exp(-C'\bar{\epsilon})\frac{m_{T^j\omega}(B)}{m_{T^j\omega}(T^j\omega)} \leq \frac{m_\omega(T^{-j}B)}{m_\omega(\omega)} \leq \exp(C'\bar{\epsilon})\frac{m_{T^j\omega}(B)}{m_{T^j\omega}(T^j\omega)} \tag{8}$$

Proof. Define a density $\rho(x)$ on a component $T^j\omega_{i_k}$ of $T^j\omega_i$ by

$$\rho(x) = \frac{dT^j_* m_{\omega_{i_k}}}{dm_{T^j\omega_{i_k}}}$$

On each of those $T^j\omega_{i_k}$ the distortion estimate (1) gives $\exp(-C'\bar{\epsilon}) \leq \frac{\rho(x)}{\rho(y)} \leq \exp(C'\bar{\epsilon})$, which yields (8). □

Next, we turn to the existence of local stable and unstable manifolds. Recall that for every $x \in M^0_{\tilde{\kappa}}$ both an LSM and a LUM exist. A standard application of the Borel-Cantelli Lemma gives that $\mu(M^0_{\tilde{\kappa}}) = 1$. In particular, the following corollary holds.

Corollary 10. Let μ be a normalized limit-point of (3). Then for μ-almost every $x \in M^0$ and some measurable functions $\eta(x), \eta'(x) > 0$ the LUM $W^u_{\eta(x)}(x)$ and the LSM $W^s_{\eta'(x)}(x)$ exist.

From the above corollary it does not immediately follow that for every $z \in W^u_{\eta(x)}(x)$ an LSM $W^s_{\eta'(z)}(z)$ exists. Indeed, $W^u_{\eta(x)}(x)$ may not be completely contained inside $M^0_{\tilde{\kappa}}$. We obtain $m_{W^u_{\eta(x)}}$-almost everywhere existence of LSMs on $W^u_{\eta(x)}(x)$ however from applying the Borel-Cantelli Lemma to the measure $m_{W^u_{\eta(x)}(x)}$, cf. [4].

Finally, we conclude that the invariant measure obtained from (3) is an SRB measure. Indeed, for almost every $x \in M^0$ a LUM exists. From the construction of μ (iteration of a smooth measure) it follows that on such a LUM the measure μ is absolutely continuous. This concludes the proof of Theorem 3(a). Henceforth we will denote limit points of (3) by μ_{SRB}.

6. Construction of the hyperbolic set Λ and return times

The key instrument in [11] to obtain exponential decay of correlations for hyperbolic dynamical systems is a set $\Lambda \subset M$ with hyperbolic product structure; a 'generalized horseshoe'. In this section we present a generic method to construct finitely many sets Λ_i with hyperbolic product structure such that $\Lambda = \cup_i \Lambda_i$ covers a 'large part' of M^0. The reader may think of this construction as a generalized Markov partition of Λ. However, a warning should be issued. Although we indeed construct finitely many 'rectangles' Λ_i, these rectangles do not necessarily cover the support of μ_{SRB} and are generally not disjoint. Once Λ is constructed, we define a return map $T^r : \cup_i \Lambda_i \to \cup_i \Lambda_i$.

6.1. **Construction of the hyperbolic set Λ.** Choose $4\delta_0 \leq \bar{\epsilon}$, with $\bar{\epsilon}$ as in Lemma 1, such that $A_{\delta_0} := \{x \in M^- : W^u_{\delta_0}(x) \text{ exists}\} \neq \emptyset$. Our choice of δ_0 is such that $\delta_0 \ll \frac{1}{B}$, with $\frac{1}{B}$ the minimum radius of curvature of LUMs and LSMs. For $x \in A_{\delta_0}$ we can not conclude that the LUM $W^u_{\frac{\delta_0}{3}}(x)$ exists uniquely. Therefore, let

$$W(x) := W^u_{\frac{\delta_0}{3}}(x) \in \{W^u_{\frac{\delta_0}{3}}(z)\}_{z \in A_{\delta_0}}.$$

denote an arbitrary LUM about $x \in A_{\delta_0}$. We will construct a subset $W_\infty(x) \subset W(x)$ with the property that for all $y \in W_\infty(x)$ a 'sufficiently long' LSM exists, i.e. for some $\delta_1 \ll \delta_0$ to be determined, $W^s_{\frac{\delta_1}{3}}(y)$ exists $\forall y \in W_\infty(x)$. We construct $W_\infty(x)$ as follows. Let ω_i denote the connected components of $W(x) \cap B_{\delta_1}(\mathcal{S})$. Delete ω_i from $W(x)$ only if the minimum distance between ω_i and $\mathcal{S}$ is less than $\frac{\delta_1}{2}$. Call the resulting set $W_0(x)$. Suppose $W_{n-1}(x)$ is defined inductively as above and let in the n-th step ω_i be the connected components of $\{y \in W_{n-1}(x) : d(T^n y, \mathcal{S}) < \delta_1 \lambda^{-n}\}$. Delete ω_i from $W_{n-1}(x)$ only if $d(T^n \omega_i, \mathcal{S}) < \frac{1}{2}\delta_1 \lambda^{-n}$. Now let $W_\infty(x) = \cap_n W_n(x)$. Note that $W_\infty(x)$ is a closed subset of $W(x)$.

Lemma 11. Let $0 < c < 1$ be arbitrary. Then there exists a $\delta_1 > 0$ such that

$$m_{W(x)}(W_\infty(x)) > c\, m_{W(x)}(W(x))$$

for all $W(x)$ as above.

Set $W := W(x)$ and $W_n = W_n(x)$. The proof closely follows that of Lemma 7; our strategy is to estimate $\sum_{n=0}^{\infty} m_W(T^{-n} B_{\delta_1 \lambda^{-n}}(\mathcal{S}))$. Define a sequence of stopping times $s_1 < s_2 < \dots$ on W as in Section 4 with the following difference. For $x \in W$ let

$$s_1(x) = \min_{n \geq 1} \{m_{T^n \omega_x}(T^n \omega_x) > \bar{\epsilon}\},$$

with ω_x the component of W_{n-1} containing x. If no such n exists, or x is deleted from W before the stopping time is reached, we agree that $s_1(x)$ is not defined. On $\Theta_1 := \{x \in W : s_1(x) \text{ is defined}\}$ define

$$s_2(x) = \min_{n > s_1(x)} \{m_{T^n \omega_x}(T^n \omega_x) > \bar{\epsilon}\},$$

with ω_x the component of W_{n-1} containing x. Set $\Theta_2 := \{x \in \Theta_1 : s_2(x) \text{ is defined}\}$, and so on. It is again easily seen that each Θ_k is a disjoint union of a countable number of components.

Proof of Lemma 11. Observe that whenever $x \in \Theta_k$ for all k, the point x never gets within an $\delta_1 \lambda^{-n}$-neighborhood of $\mathcal{S}$. Hence, it suffices to consider $\sum_{n=0}^{\infty} m_W(T^{-n} B_{\delta_1 \lambda^{-n}}(\mathcal{S}))$ on the subsets $W - \Theta_1$ and $\Theta_k - \Theta_{k+1}$ for all $k \geq 1$.

Starting with $W-\Theta_1$ we obtain, using the same methods as in Lemma 7, that the total measure deleted from $W-\Theta_1$ is at most

$$\sum_{n=0}^{\infty} m_W\big(T^{-n}B_{\delta_1\lambda^{-n}}(\mathcal{S})\cap(W-\Theta_1)\big) < K_0C\delta_1^{\frac{1}{p}}\sum_{n=0}^{\infty}\lambda^{-\frac{n}{p}}(\frac{K_0}{\lambda})^n.$$

Using the second part of the proof of Lemma 7 we obtain for the total measure deleted from $\Theta_k-\Theta_{k+1}$

$$\sum_{n=0}^{\infty} m_W\big(T^{-n}B_{\delta_1\lambda^{-n}}(\mathcal{S})\cap(\Theta_k-\Theta_{k+1})\big) < \exp(C'\bar{\epsilon})\lambda^{-\frac{k}{p}}2K_0C\delta_1^{\frac{1}{p}}\sum_{n=0}^{\infty}\lambda^{-\frac{n}{p}}(\frac{K_0}{\lambda})^n$$

Summing over k we see that the total measure deleted can be made arbitrarily small by choosing δ_1 accordingly. □

We proceed with the construction of the set $\Lambda\subset M$. For the remainder we fix the constant c from Lemma 11 close to 1 and set $\delta=\frac{\delta_1}{3}$. Note that $W^s_\delta(y)$ exists uniquely for all $y\in W_\infty$. For each $W_\infty(x)$, $x\in A_{\delta_0}$, set

$$\Gamma^s\big(W_\infty(x)\big)=\{W^s_\delta(y):y\in W_\infty(x)\}.$$

Since the angle between C^u_x and C^s_x has a positive lower bound and LSMs have bounded curvature uniformly on M, there exists an $\eta>0$ depending only on this angle and the curvature such that for all $W^u_{\frac{\delta_0}{3}}(x)\in\{W^u_{\frac{\delta_0}{3}}(z)\}_{z\in A_{\delta_0}}$

$$d(\partial W^s_\delta(y),W^u_{\frac{\delta_0}{3}}(x))>\eta \tag{9}$$

for all $W^s_\delta(y)\in\Gamma^s\big(W_\infty(x)\big)$. For the remainder of this paper, fix $0<\eta<\frac{\delta}{4}$ satisfying (9).

We let $\mathcal{K}(M)$ denote the collection of all nonempty compact subsets of the manifold M, endowed with the Hausdorff metric

$$H(K_1,K_2)=\inf\{\delta:K_1\subset B_\delta(K_2)\text{ and }K_2\subset B_\delta(K_1)\}.$$

Since M is compact, the space $\mathcal{K}(M)$ is compact.

Let $\{W^u_{\delta_0}(x_i)\}_{i=1}^N$ be the smallest finite $\frac{\eta}{2}$-dense subset of $\{W^u_{\delta_0}(z)\}_{z\in A_{\delta_0}}\subset\mathcal{K}(M)$, distinguishing the x_i by their index. For each x_i set

$$\hat{\Gamma}^u(x_i):=\{W^u_{\delta_0}(z):H\big(W^u_{\frac{\delta_0}{3}}(z),W^u_{\frac{\delta_0}{3}}(x_i)\big)<\frac{\eta}{2}\},$$

and define a set with hyperbolic product structure by $\hat{\Lambda}_i:=\hat{\Lambda}(x_i)=\Gamma^s(x_i)\cap\hat{\Gamma}^u(x_i)$, where $\Gamma^s(x_i):=\Gamma^s\big(W_\infty(x_i)\big)$. With each $\hat{\Lambda}_i$ we associate a slightly larger set $\Lambda_i:=\Gamma^s(x_i)\cap\Gamma^u(x_i)$ where

$$\Gamma^u(x_i)=\{W^u_{\delta_0}(z):H\big(W^u_{\frac{\delta_0}{3}}(z),W^u_{\frac{\delta_0}{3}}(x_i)\big)<\eta\}.$$

We will call the sets Λ_i rectangles and refer to $W^u_{\frac{\delta_0}{3}}(x_i)$ as *the central* LUM *of* Λ_i. Note that rectangles are not necessarily disjoint; they may overlap. Observe that each $W^u_{\delta_0}(z)\in\Gamma^u(x_i)$ intersects every LSM $W^s_\delta(y)\in\Gamma^s(x_i)$. Indeed, $H\big(W^u_{\frac{\delta_0}{3}}(z),W^u_{\frac{\delta_0}{3}}(x_i)\big)<\eta$ for all such $W^u_{\frac{\delta_0}{3}}(z)$. In fact, if in rectangle Λ_i we let $[z,y]_i=W^u_{\delta_0}(z)\cap W^s_\delta(y)$ ($W^u_{\delta_0}(z)$ denotes a single element of $\{W^u_{\delta_0}(z)\}$: $[z,y]_i$ may not be unique.), then $W^u_{\frac{\delta_0}{3}}([z,y]_i)\subset W^u_{\delta_0}(z)$ for all $W^s_\delta(y)\in\Gamma^s(x_i)$. This follows from $\eta<\delta\ll\delta_0$.

u**-crossings.** We say that a LUM W^u u-crosses a rectangle Λ_i if $W^u\cap\Gamma^u(x_i)\supset W^u_{\frac{\delta_0}{3}}(z)$ for some $W^u_{\frac{\delta_0}{3}}(z)\in\Gamma^u(x_i)$. Every LUM W^u with $\operatorname{diam}(W^u)\geq 2\delta_0$ u-crosses one of the $\hat{\Lambda}_i$. Indeed,

$W^u = W^u_{\delta_0}(z)$ for some $z \in A_{\delta_0}$ and hence there is an integer i such that $W^u_{\delta_0}(z) \in \Gamma^u(x_i)$. Again, we conclude that $W^u_{\frac{\delta_0}{3}}([z,y]_i) \subset W^u_{\delta_0}(z)$ for all $W^s_\delta(y) \in \Gamma^s(x_i)$.

Subrectangles. For later use, we define the notion of u- and s-subrectangles. A subrectangle $\Lambda^u_i \subset \Lambda_i$ is called a u-subrectangle if $W^u(x) \cap \Lambda^u_i = W^u(x) \cap \Lambda_i$ for all $x \in \Lambda^u_i$. An s-subrectangle is defined similarly. Obviously, $\hat{\Lambda}_i$ is a u-subrectangle of Λ_i for each i.

Finally, we set $\Lambda = \cup^N \Lambda_i$. We will regard this union to be disjoint and construct a map $T^r : \cup^N \Lambda_i \to \cup^N \Lambda_i$.

6.2. **Return times.** Here we define a return map $T^r : \cup^N \Lambda_i \to \cup^N \Lambda_i$. Fix a point x_i with its rectangle Λ_i and let $W_n = W_n(x_i)$. Let ω denote a component of W_n. Intuitively, we think of a point $x \in \omega$ returning to Λ at time $r(x) = n$ if $x \in \omega \cap T^{-n}\Lambda_j$ for some n and j and $T^n\omega$ u-crosses Λ_j. The idea is to first partition W_n for each n into a countable collection of subsets ω^k_n with $k \geq 0$, to keep control of the length of their images. Then we define return times on W_∞, and finally we show that whenever $W_\infty \cap \omega \cap T^{-n}\hat{\Lambda}_j \neq \emptyset$ for some n and j and $T^n\omega$ u-crosses $\hat{\Lambda}_j$, we in fact have that an entire s-subrectangle of Λ_i returns (to Λ_j).

We say that a set $\Lambda^\omega_i \subset \Lambda_i$ is an s-subrectangle *based on* $\omega \subset W$ if $\Lambda^\omega_i := \{x \in \Lambda_i : W^s(x) \cap W \in \omega\}$. We do not allow returns to occur before a certain iterate N of the map T. This *minimum return time* N is chosen so that any s-subrectangle under T^N has contracted sufficiently in the stable direction to return as a u-subrectangle. More precisely,

Let $\omega \subset W_n$ be a connected component and let Λ^ω_i be the s-subrectangle based on ω. The minimal return time N is the smallest integer N with the property that if $T^n\omega$ u-crosses $\hat{\Lambda}_j$ for some j, then $T^n\Lambda^\omega_i$ is a u-subrectangle of Λ_j whenever $n \geq N$.

Denote by $\tilde{W}_n$ the points of W_n that have not yet returned at time n, i.e.

$$\tilde{W}_n = W_n - \{x \in W : r(x) \leq n\}.$$

We define a partition $\tilde{\mathcal{P}}_n$ of $\tilde{W}_n$ inductively in n on the connected components $\tilde{\omega}_n$, distinguishing between times before and after N. The k-th element of the partition $\tilde{\mathcal{P}}_n \restriction \tilde{\omega}_n$ of $\tilde{\omega}_n$ will be denoted by ω^k_n.

The case $n < N$**.** Suppose $\omega^{k'}_{(n-1)}$ is an element of the partition on level $n-1$ and let ω^k_n be a component of $W_n \cap \omega^{k'}_{(n-1)}$. Since we do not allow returns yet, $\omega^k_n \subset \tilde{W}_n$ and we define $\tilde{\mathcal{P}}_n \restriction \omega^k_n$ as follows

1. If $\mathrm{diam}(T^n\omega^k_n) \leq 4\delta_0$, then $\omega^k_n \in \tilde{\mathcal{P}}_n$.
2. If $\mathrm{diam}(T^n\omega^k_n) > 4\delta_0$, we partition ω^k_n into components $\omega^{k_l}_n$ with $2\delta_0 \leq \mathrm{diam}(T^n\omega^{k_l}_n) < 4\delta_0$ for every l and put $\omega^{k_l}_n \in \tilde{\mathcal{P}}_n$.

The case $n \geq N$**.** Suppose again that $\omega^{k'}_{(n-1)}$ is an element of the partition on level $n-1$ and let ω^k_n be a component of $W_n \cap \omega^{k'}_{(n-1)}$.

1. As above put $\omega^k_n \in \tilde{\mathcal{P}}_n$ whenever $\mathrm{diam}(T^n\omega^k_n) \leq 4\delta_0$.
2. If $\mathrm{diam}(T^n\omega^k_n) > 4\delta_0$, partition ω^k_n into components $\omega^{k_l}_n$ with $2\delta_0 \leq \mathrm{diam}(T^n\omega^{k_l}_n) < 4\delta_0$ for every l. For each l choose a Λ_j such that $T^n\omega^{k_l}_n$ u-crosses $\hat{\Lambda}_j$. For $x \in \omega^{k_l}_n \cap T^{-n}\Lambda_j$ we set $r(x) = n$. Now, $\omega^{k_l}_n - \{x : r(x) = n\} \subset \tilde{W}_n$ and we let $\tilde{\mathcal{P}}_n$ on this set be the partition into its connected components.

Note that although it is not hard to see that whenever $y \in T^n\omega^{k_l}_n \cap \Lambda_j$ one has $T^{-n}y \in W_\infty \cap \omega^{k_l}_n \cap T^{-n}\Lambda_j$, some portion of $T^n(W_\infty \cap \omega^{k_l}_n)$ may fall through gaps of Λ_j. Next, we need to extend the definition of a return time to the entire rectangle Λ_i. It turns out that whenever part of the central LUM of Λ_i returns, an entire s-subrectangle of Λ_i returns. Young proves the following lemma (Sublemma 3, Section 7 in [11]), which also holds in our setup.

Lemma 12. Let $\omega_n^{k_l} \subset W_n$ be a connected component such that $\operatorname{diam}(T^n\omega_n^{k_l}) > 2\delta_0$ for some $n \geq N$ and let Λ_i^s denote the smallest s-subrectangle of Λ_i containing $W_\infty \cap \omega_n^{k_l} \cap T^{-n}\hat{\Lambda}_j$. Then $T^n\Lambda_i^s$ is a u-subrectangle of Λ_j.

7. Proofs of the Theorems

7.1. **Proof of Theorem 3.** The existence of an SRB-measure was already established in Section 5. It is well-known (see for example [8]) that any SRB-measure has at most countably many ergodic components M_l^0, on each of which it is, up to a finite cycle, mixing and Bernoulli. We will argue that the number of ergodic components is finite in our setup. It is well known that μ_{SRB}-a.e. LUM (LSM) in M^- is almost surely contained in one ergodic component. From a standard argument one then gets that each rectangle Λ_i is almost surely contained in one ergodic component M_l^0. So, it suffices to show that μ_{SRB}-almost every LUM in M^- grows under iteration until some component reaches a length $> 2\delta_0$ for some n and lands on a typical unstable fiber of some Λ_i.

Replacing $\bar{\epsilon}$ with $2\delta_0$, the first statement follows from Lemma 8 which says that for m_γ-almost every $x \in \gamma$ there exists a natural n such that $\operatorname{diam}(T^n\omega_x) > 2\delta_0$, with ω_x the component containing x. Therefore, μ_{SRB}-almost every LUM returns to one of the Λ_i, but possibly lands on a non-typical fiber. Let A denote the set of non-typical LUMs in $\cup_i\Gamma^u(x_i)$. Then $\mu_{\text{SRB}}(M^- - \cup_{n\geq 0}T^{-n}A) = 1$, and so μ_{SRB}-almost every unstable fiber in M^- returns to a typical fiber in some Λ_i. It follows that the number of ergodic components is finite.

Remark 13. In fact, Lemma 8 states that the measure of the set of points not having a long LUM at the n-th iterate is exponentially small. This is a much stronger statement than needed here. As was shown above, to obtain finitely many ergodic components it suffices to know that there exists a natural n such that $\operatorname{diam}(T^n\omega_x) > 2\delta_0$.

7.2. **Proof of Theorem 6.** The abstract class of systems Young studies in [11] satisfies conditions she labels (P1)-(P5). For such systems she establishes the existence of an SRB-measure ν, supported on $\cup_{n\geq 0}T^n\Lambda_i$. The main result of that paper is that whenever (T^n, ν) is ergodic for all $n \geq 0$ and

$$m_W\big(x \in W_\infty : r(x) > n\big) \leq C\theta^n, \tag{10}$$

for some $C > 0$ and $\theta < 1$, then (T, ν) has exponential decay of correlations and satifies a central limit theorem.

We prove Theorem 6 by showing that Hyp 1-6 imply (P1)-(P5) and that $\mu_{\text{SRB}} = \nu$, whenever μ_{SRB} is ergodic. Theorem 6 then follows from (10), which is shown in Section 9.

We verify that our system satisfies (P1)-(P5). For x and y in the same element of $\Gamma^u(x_i)$, define a separation time by $\operatorname{sep}(x, y) = n$, with n the largest integer such that x and y still belong to the s-subrectangle based on a connected component $\omega \subset \tilde{\mathcal{P}}_n$. With this separation time, (P3)-(P5) are standard. Furthermore, instead of one set $\Lambda \subset M$ with hyperbolic product structure, in Section 6 we obtained finitely many of those sets Λ_i. From Lemma 11 we have $m_{W(x_i)}(W_\infty(x_i)) > c\, m_{W(x_i)}(W(x_i))$, which in conjunction with absolute continuity of Γ^s gives $m_{W^u}(W^u \cap \Lambda_i) > 0$ for all $W^u \in \Gamma^u$. This is (P1); (P2) immediately follows from the construction of the Λ_i's.

Next, let μ_{SRB} be an ergodic SRB-measure such that $\mu_{\text{SRB}}(\Lambda_i) > 0$. It is well-known that every SRB-measure is a linear combination of (unique) ergodic SRB-measures μ_i supported on the finitely many (disjoint) components M_i^0 (see again [8]). Uniqueness immediately gives $\mu_{\text{SRB}} = \nu$. To finish the proof, it remains to show (10).

8. Discussion and further remarks

It is easily checked that the dynamical system from Example 2 preserves volume measure m_0, which is an SRB measure. From Theorem 3(b) it then follows that (T, m_0) has at most finitely many ergodic components. In fact, from Chernov's local ergodicity theorem [5] we obtain ergodicity of (T^n, m_0) for all $n \geq 1$ and therefore (T, m_0) satisfies EDC and CLT.

The present results do not allow us to conclude that the falling balls satisfy EDC and CLT. First of all, the results in this paper are limited to two dimensions, whereas tangential intersections only occur for $n \geq 3$ balls. This does not seem to be a major obstacle, since we expect that techniques developed by Chernov (see [4] and [3], in particular Chernov's Z-function) can be employed to extend our results to high dimensions. Second, the derivatives of the system of falling balls are not bounded. This problem can easily be overcome by applying the techniques described in Section 8 of [11].

The absence of uniform hyperbolicity poses a more serious problem. Although recently the system of falling balls was shown to have all relevant Lyapunov exponents nonzero [9], it is not uniformly hyperbolic. In general there is very little hope for exponential decay of correlations in systems where the expansion factor is not bounded away from unity. In this particular case however, careful analysis of the cross section map may lead to some results. Lastly we mention that as of yet nothing can be said about the nature of the intersections of LUMs and E^u_x for this model: coincidence can not be ruled out *a priori*. This also stands in the way of proving ergodicity for the system. This is a rich area of future research.

9. Tail Estimate

In this section we show (10). The techniques we use are standard; we will be following [11] closely in this section. Fix an integer $k \geq 0$, and let ω be a connected component of $T^k W_k$, with W_k as defined in Section 6.2. We introduce some more notation. Besides $\tilde{W}_n$ and $\tilde{\mathcal{P}}_n$ from Section 6.2 we introduce for $n = 0, 1, \dots,$

$$\omega_n = \omega \cap T^k W_{k+n} \qquad \tilde{\omega}_n = \omega \cap T^k \tilde{W}_{k+n} \qquad \tilde{\mathcal{P}}^{\omega}_n = T^k \tilde{\mathcal{P}}_{k+n} \restriction \tilde{\omega}_n.$$

Note that ω_n consist of finitely many connected components for each n, whereas $\tilde{\omega}_n$ may contain countably many. We write ω_∞ to denote $\omega \cap T^k W_\infty$. Abusing notation, we denote the collection of components $\cup_{\omega^{\tilde{\mathcal{P}}} \in \tilde{\mathcal{P}}^{\omega}_{n-1}} (\omega^{\tilde{\mathcal{P}}} \cap \omega_n)$ by $\tilde{\mathcal{P}}^{\omega}_{n-1} \cap \omega_n$.

9.1. **Growth of Partition Elements.** Once more we will introduce a stopping time $s(x)$. On the connected component $\omega \in T^k \tilde{\mathcal{P}}_k$ define

$$s(x) = \min_{n \geq 1} \{\operatorname{diam}(T^n \omega_x) > 2\delta_0\},$$

where $\omega_x \in \tilde{\mathcal{P}}^{\omega}_{n-1} \cap \omega_n$ contains x. If no such n exists, or x is deleted from ω before this stopping time is reached, we agree that $s(x)$ is not defined. Observe that returns do not play a part, since the stopping time is reached before a return is possible. From Lemma 8 (replacing γ_n by ω_n) we therefore see

$$m_\omega\big(\omega_n - \{x : s(x) \leq n\}\big) \leq 2\delta_0 \big(\frac{K_0}{\lambda}\big)^n. \tag{11}$$

As an immediate corollary we have that $s(x)$ is finite for m_ω-almost every $x \in \omega_\infty$.

9.2. **Growth of Gaps.** Let $\omega_0 \in T^k\tilde{\mathcal{P}}_{k-1}$ be a segment u-crossing Λ_j for some j at time k and let $\omega \subset \omega_0$ denote the smallest subsegment containing $\omega_0 \cap \Gamma_j^s$. The set $\omega^c = \omega - \Gamma_j^s$ consists of countably many connected components ω', corresponding to the gaps of $W_\infty := W_\infty(x_j)$. The components ω' that 'fall through' the gaps of Λ_j are of small size and need to grow in order to return as well. Our goal is to show that these gaps grow exponentially fast in the number of iterations n.

Formally, a gap in W_∞ is a connected component $\hat{\omega}' \subset W - W_\infty$. By sliding along the stable manifolds of W_∞ at the edges of $\hat{\omega}'$ these gaps transfer to gaps ω' in all $W^u_{\frac{\delta_0}{3}}(z) \in \Gamma_j^u$. We say that ω' is a *gap of generation* q if q is the smallest nonnegative integer such that $\hat{\omega}' \cap W_{q-1} - W_q \neq \emptyset$. So, a gap ω' is of generation q, written $\mathrm{Gen}(\omega') = q$, if q is the first time (part of) the corresponding gap $\hat{\omega}'$ is removed in the construction of W_∞. Define $G_q := \{\omega' : \mathrm{Gen}(\omega') = q\}$.

Define a stopping time $s(x)$ on ω^c as in the previous subsection, replacing ω with ω', where ω' is a gap in the T^k-image of an element of $\tilde{\mathcal{P}}_{k-1}$. Write $\omega_n^c = \omega^c \cap T^k W_{k+n}$. In the next step we show that gaps grow exponentially fast, which is the content of Lemma 16. We will use that for all $\epsilon > 0$

$$m_\omega\big(\omega_n^c - \{x : s(x) \le n\}\big) \le \sum_{q=1}^{\epsilon n} \sum_{\omega' \in G_q} m_\omega\big(\omega' - \{x : s(x) \le n\}\big) + \sum_{q > \epsilon n} \sum_{\omega' \in G_q} m_\omega(\omega'). \tag{12}$$

We need an additional lemma to show that the length of a gap is bounded below.

Lemma 14. Let ω as above and $\omega' \subset \omega$ be a gap of generation q. Then $T^q\omega'$ has only one connected component and $m_{T^q\omega}(T^q\omega') > \frac{\delta_1}{4\lambda^q}$.

Proof. Since $\omega' \subset \omega$ is a gap of generation q, we have $\hat{\omega}' \subset W_{q-1}$, which means that $T^q\hat{\omega}'$ has one component. Moreover, gap of generation q does not intersect gaps of previous generations. By construction of W_∞, there exists an $\hat{x} \in \hat{\omega}'$ such that $d(T^q\hat{x}, \mathcal{S}) < \frac{\delta_1}{2\lambda^q}$, so $d(T^q\hat{x}, \partial T^q\hat{\omega}') > \frac{\delta_1}{2\lambda^q}$ and therefore $\frac{\delta_1}{2\lambda^q} < m_{T^q\hat{\omega}'}(T^q\hat{\omega}') < C(\frac{\delta_1}{\lambda^q})^{\frac{1}{p}}$, where the upper bound is obtained from Hyp 3.

Since $d(T^i\hat{\omega}', \mathcal{S}) > \frac{\delta_1}{2\lambda^i}$ for all $i = 0, \dots, q-1$, we can choose a continuous family of q-stable curves ς_x on $\hat{\omega}'$ such that $d(\partial\varsigma_x, \hat{\omega}') = \eta$, with $0 < \eta < \frac{\delta}{4}$ as in Section 6. So, in particular, $T^q\omega'$ has one component. Also, $T^q\hat{\omega}'$ is almost flat and hence any LUM γ intersecting all curves $T^q\varsigma_x$ satisfies $\mathrm{diam}(\gamma) > \frac{1}{2}\mathrm{diam}(T^q\hat{\omega}')$. Recalling the construction of Λ_i we obtain $m_{T^q\omega}(T^q\omega') > \frac{\delta_1}{4\lambda^q}$. □

Remark 15. The proof above reveals that at the time of creation of a gap also $m_{T^q\omega'}(T^q\omega') < 2m_{T^q\hat{\omega}'}(T^q\hat{\omega}')$. Since T^q is smooth on the family of q-stable curves on $\hat{\omega}'$, the bounds on distortion (1) give $m_{\omega'}(\omega') < Cm_{\hat{\omega}'}(\hat{\omega}')$ for some $C > 0$.

Lemma 16. There exists a constant $C_2 > 0$ and a $\theta_2 < 1$ such that for all $n \ge 1$

$$m_\omega\big(\omega_n^c - \{x : s(x) \le n\}\big) \le C_2\theta_2^n, \tag{13}$$

for all ω.

Proof. We use (12). For $\epsilon > 0$ to be determined we estimate

$$\sum_{q=1}^{\epsilon n} \sum_{\omega' \in G_q} m_\omega\big(\omega' - \{x : s(x) \le n\}\big).$$

Consider a gap $\omega' \in G_q$ for some fixed $q \le n$. Then $T^q\omega'$ has only one component. If $\mathrm{diam}(T^i\omega') > 2\delta_0$ for some $i < q$, then $s \restriction \omega' < q$ and we do not consider ω'. Apply (11) to

$T^{q-1}\omega'$ and pull back to obtain

$$m_\omega\big(\omega'_n - \{x : s(x) \le n\}\big) \le \frac{\exp(C'\bar\epsilon) m_{\omega'}(\omega') 2\delta_0}{m_{T^{q-1}\omega'}(T^{q-1}\omega')} \big(\frac{K_0}{\lambda}\big)^{n-q+1}.$$

By the previous lemma $m_{T^{q-1}\omega'}(T^{q-1}\omega') > \frac{\delta_1}{4\lambda^{q-1}}$, so that summing over all components in $\omega' \in G_q$ yields

$$\sum_{\omega' \in G_q} m_\omega\big(\omega'_n - \{x : s(x) \le n\}\big) \le C\exp(C'\bar\epsilon) m_\omega(\omega) \big(\frac{K_0}{\lambda}\big)^{n-q} \lambda^q.$$

Choose $\epsilon > 0$ small enough such that $\frac{K_0}{\lambda}\big(\frac{\lambda^2}{K_0}\big)^{2\epsilon} < 1$. Then

$$\sum_{q=1}^{\epsilon n} \sum_{\omega' \in G_q} m_\omega\big(\omega'_n - \{x : s(x) \le n\}\big) \le C\exp(C'\bar\epsilon) m_\omega(\omega) \left(\frac{K_0}{\lambda}\big(\frac{\lambda^2}{K_0}\big)^{2\epsilon}\right)^n.$$

Next, we turn to the second sum in (12). From the remark above and the proof of Lemma 11 one obtains

$$\sum_{q > \epsilon n} \sum_{\omega' \in G_q} m_\omega(\omega') < C\lambda^{\frac{-\epsilon n}{p}}$$

Finally, set $\theta_2 = \max\{\frac{K_0}{\lambda}\big(\frac{\lambda^2}{K_0}\big)^{2\epsilon}, \lambda^{-\frac{\epsilon}{p}}\}$ to obtain (13). □

9.3. **Return Rate.** Define stopping times $s_i : \Theta_i \to \mathbb{N}$, $i = 1, 2, \cdots$, with $s_1(x) < s_2(x) < \cdots$ on subsets Θ_i of W by

$$s_1(x) = \min_{n \ge N}\{\mathrm{diam}(T^n \omega_x) > 2\delta_0\},$$

where ω_x is the component of $\tilde{\mathcal{P}}^\omega_{n-1} \cap \omega_n$ containing x. If no such n exists or x is deleted from W before the stopping time is reached, we agree that $s_1(x)$ is not defined. Let $\Theta_1 = \{x \in W : s_1(x) \text{ is defined}\}$. From (11) we obtain that Θ_1 is defined almost everywhere on W_∞. Note that $s_1(x)$ is precisely the first time part of W returns to Λ. For $x \in \big(\Theta_1 - \{x : r(x) = s_1(x)\}\big)$ define

$$s_2(x) = \min_{n > s_1}\{\mathrm{diam}(T^n \omega_x) > 2\delta_0\},$$

where, as above, ω_x is the component of $\tilde{\mathcal{P}}^\omega_{n-1} \cap \omega_n$ containing x. Again, we say that s_2 is not defined whenever x is deleted from W before s_2 is reached or if no such n exists. We let $\Theta_2 = \{x \in \Theta_1 : s_2(x) \text{ is defined}\}$. Note that Θ_2 is defined almost everywhere on $W_\infty \cap \big(\Theta_1 - \{x : r(x) = s_1(x)\}\big)$. This follows from applying (11) to each of the (countably many) components of $\Theta_1 - \{x : r(x) = s_1(x)\}$. In general, set $\Theta_k = \{x \in \Theta_{k-1} : s_k(x) \text{ is defined}\}$, define $s_{k+1}(x)$ for $x \in \big(\Theta_k - \{x : r(x) = s_k(x)\}\big)$ and obtain that Θ_{k+1} is defined almost everywhere on $W_\infty \cap \big(\Theta_k - \{x : r(x) = s_k(x)\}\big)$. From this construction it obviously follows that for m_W-almost every $x \in W_\infty$ either $r(x) < \infty$ or $x \in \Theta_k$ for all k.

For every $n \in \mathbb{N}^+$ and $k \le n$

(14)
$$\{x \in W_\infty : r(x) > n\} \subset \{x \in W_\infty : s_k(x) > n\} \cup \{x \in W_\infty : s_k(x) \le n \text{ and } r(x) > s_k(x)\}.$$

We first show that the measure of the second set at the right hand side decreases exponentially fast in k. The bound on the Jacobian of the holonomy map (2) and Lemma 11 give that at a return at time n, a fraction

$$\frac{m_W(\omega \cap \{x : r(x) = n\})}{m_W(\omega)} > \frac{c\exp(-C'\bar\epsilon)}{6C_0}$$

of a component $\omega \subset \tilde{\mathcal{P}}_{n-1} \cap W_n$ is absorbed by Λ. Since $\Theta_{k+1} \subset \big(\Theta_k - \{x : r(x) = s_k(x)\}\big)$,

$$\frac{m_W(\Theta_{k+1})}{m_W(\Theta_k)} < 1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0},$$

and hence

$$m_W(\Theta_k) < m_W(\Theta_1)(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0})^{k-1}.$$

In particular

$$m_W\{x \in W_\infty : r(x) > s_k(x)\} < m_W(\Theta_1)(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0})^{k-1},$$

so that for the second term on the right hand side of (14) we obtain

$$m_W\big(\{x \in W_\infty : s_k(x) < n \text{ and } r(x) > s_k(x)\}\big) < m_W(\Theta_1)(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0})^{k-1}. \tag{15}$$

It remains to show

Lemma 17. There exists a $D_3 > 0$, a $\theta_3 < 1$ and a $\beta > 0$ such that

$$m_W\big(\{x \in \tilde{W}_n : s_{[\beta n]}(x) > n\}\big) \leq D_3\theta_3^n, \tag{16}$$

for all $n \geq 0$.

Indeed, combined with (15) above, this lemma yields (10) with $\theta_0 = \big(\max\{(1 - \frac{c\exp(-C'\bar{\epsilon})}{6C_0}), \theta_3\}\big)^{\beta}$.

Proof of Lemma 17. Fix a sequence $N < k_1 < \cdots < k_j \leq n$, with N the minimum return time. Define, for $N < k \leq n$

$$A_k := \{x \in \tilde{W}_k : s_i(x) = k_i \text{ for all } k_i \leq k\}.$$

We estimate the measure of A_k. Note that the A_k's form a decreasing sequence of sets.

We start with estimating $m_W(A_{k_1-1})$. Let $\omega \in \tilde{\mathcal{P}}_N$, apply (11) to $T^N\omega$ and pull back to obtain

$$m_W(\omega \cap A_{k_1-1}) \leq \frac{\exp(C'\bar{\epsilon})m_\omega(\omega)2\delta_0}{m_{T^N\omega}(T^N\omega)}\big(\frac{K_0}{\lambda}\big)^{k_1-N-1}.$$

Summation over the finitely many components $\omega \in \tilde{\mathcal{P}}_N$ yields,

$$m_W(A_{k_1-1}) \leq C\exp(C'\bar{\epsilon})\lambda^{-N}\big(\frac{K_0}{\lambda}\big)^{k_1-N-1}.$$

Next we estimate $m_W(A_{k_2-1})$. Let $\omega \subset A_{k_1-1} \cap W_{k_1}$ with $s_1(x) \restriction \omega = k_1$. Write $\omega - \{x : r(x) = k_1\} = (\cup\omega_r') \cup (\cup\omega_r'')$, where $T^{k_1}(\cup\omega_r')$ is the union of gaps of some Λ_j and $\cup\omega_r''$ consists of the two components $\omega_{r_1,r_2} \in \omega - \big(\text{smallest subsegment containing } \cup\,\omega_r'\big)$. Observe $\frac{2\delta_0}{3} < m_{T^{k_1}\omega_{r_1,r_2}}(T^{k_1}\omega_{r_1,r_2}) < \frac{5\delta_0}{3}$. Applying Lemma 16 to $T^{k_1}(\cup\omega_r')$ and pulling back we obtain

$$m_W\big((\cup\omega_r') \cap A_{k_2-1}\big) \leq \frac{C_2\exp(C'\bar{\epsilon})m_\omega(\bar{\omega}_r')}{m_{T^{k_1}\omega}(T^{k_1}\bar{\omega}_r')}\theta_2^{k_2-k_1-1} \leq C_2'\exp(C'\bar{\epsilon})m_\omega(\bar{\omega}_r')\theta_2^{k_2-k_1-1}$$

with $\bar{\omega}' \subset W_{k_1}$ the smallest subsegment of ω containing $\cup\omega_r' \cap A_{k_2-1}$. The last inequality follows from the fact that we have finitely many rectangles.

For the components $T^{k_1}\omega_{r_1,r_2}$ we obtain using (11)

$$\begin{aligned} m_W\big(\omega_{r_1,r_2} \cap A_{k_2-1}\big) &\leq C_1\frac{\exp(C'\bar{\epsilon})m_\omega(\omega_{r_1,r_2})2\delta_0}{m_{T^{k_1}\omega_{r_1,r_2}}(T^{k_1}\omega_{r_1,r_2})}\big(\frac{K_0}{\lambda}\big)^{k_2-k_1-1} \\ &\leq C_1'\exp(C'\bar{\epsilon})m_\omega(\omega_{r_1,r_2})\big(\frac{K_0}{\lambda}\big)^{k_2-k_1-1}. \end{aligned}$$

Since $\omega \subset A_{k_1-1}$, summing over all those components ω we obtain

$$m_W(A_{k_2-1}) \leq C_3 \theta_3^{k_2-k_1-1} m_W(A_{k_1-1})$$

for some $C_3 > 0$ and $\theta_3 = \max\{\frac{K_0}{\lambda}, \theta_2\}$ independent of k_i.

Repeating the steps above for k_{i-1} and k_i, $i = 2, \ldots, j$ we have

$$m_W(A_n) = \frac{m_W(A_n)}{m_W(A_{k_j-1})} \cdot \frac{m_W(A_{k_j-1})}{m_W(A_{k_{j-1}-1})} \cdots \frac{m_W(A_{k_2-1})}{m_W(A_{k_1-1})} \cdot m_W(A_{k_1-1}) \leq C\big(\frac{C_3}{\theta_3}\big)^j \theta_3^n.$$

and hence

$$\sum_{j=1}^{[\beta n]} \sum_{N<k_1<\cdots<k_j\leq n} m\big(A_n(k_1, \ldots, k_j)\big) \leq C \sum_{j=1}^{[\beta n]} \binom{n}{j} \big(\frac{C_3}{\theta_3}\big)^j \theta_3^n.$$

Choosing $\beta < \frac{1}{2}$ we can rewrite the above expression using Stirling's formula as

$$m_W\big(\{x \in \tilde{W}_n : s_{[\beta n]}(x) > n\}\big) \leq C\Big[\frac{\big(\frac{C_3}{\theta_3}\big)^{2\beta}\theta_3}{\beta^\beta(1-\beta)^{1-\beta}}\Big]^n.$$

Choosing $\beta < \frac{1}{2}$ small enough such that the expression in brackets is less than one finishes the proof.

□

Acknowledgements

The author thanks his advisor N. Chernov for many illuminating discussions and careful reading of the manuscript.

References

[1] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Number 470 in Lecture Notes in Math. Springer-Verlag, Berlin, 1975.

[2] L.A. Bunimovich and Ya.G. Sinai. Statistical properties of a Lorentz gas with periodic configuration of scatterers. *Commun. Math. Phys.*, 78:479–497, 1981.

[3] N.I. Chernov. Decay of correlations and dispersing billiards. To appear in *J. Stat. Phys.*

[4] N.I. Chernov. Statistical properties of piecewise smooth hyperbolic systems in high dimensions. To appear in *Discrete and Continuous Dynamical Systems.*

[5] N.I. Chernov. Local ergodicity in hyperbolic systems with singularities. *Functional Analysis and its Applications*, 27:51–54, 1993.

[6] A. Katok and J.-M. Strelcyn. *Invariant manifolds, entropy and billiards; smooth maps with singularities.* Number 1222 in Lecture Notes in Math. Springer-Verlag, Berlin, 1986.

[7] C. Liverani. Decay of correlations. *Ann. Math.*, 142:239–301, 1995.

[8] Ya. B. Pesin. Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties. *Ergod. Th. & Dyn. Sys.*, 12:123–151, 1992.

[9] N. Simányi. The characteristic exponents of the falling ball model. *Commun. Math. Phys.*, 182:457–468, 1996.

[10] M.P. Wojtkowski. A system of one dimensional balls with gravity. *Commun. Math. Phys.*, 126:507–533, 1990.

[11] L.-S. Young. Statistical properties of systems with some hyperbolicity including certain billiards. *Ann. Math.*, 147:585–650, 1998.