Converse KAM Theory for monotone positive symplectomorphisms

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Abstract. We apply variational methods to Converse KAM theory, being useful for symplectomorphisms in the annulus satisfying weaker hypothesis that usually one requires. For instance, we do not need the existence of a global Lagrangian generating function. The main object is the primitive function of an exact symplectomorphism, also known by many authors as generating function although in fact it does not generate the symplectomorphism. We introduce variational principles not only for the orbits of a symplectomorphism, but also for the so called invariant Lagrangian graphs. We introduce the non-degenerate ones and study the minimizing ones. Applications are also given for a broad class of examples.

AMS classification scheme numbers: 58F05

1. Introduction

A fundamental question in symplectic-Hamiltonian dynamics is which parts of the phase space contain invariant tori and which do not. Often one works *d-annulus* $\mathbb{A}^d$, that is, the cotangent bundle $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d = \mathbb{A}^d$ of the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, or in its universal covering space $\mathbb{R}^d \times \mathbb{R}^d$.

- On one hand, *KAM theory* (from Kolmogorov [17], Arnold [1] and Moser [25]) let us to obtain *many* invariant tori of dimension d for the exact symplectomorphisms which are close enough to completely integrable ones. Some non-degeneracy conditions are needed. In fact, this theory proves that the measure of the complement of the invariant tori in any bounded region is arbitrarily small when we make smaller the size of the perturbation. The dynamics on these tori are conjugated to ergodic translations on $\mathbb{T}^d$ satisfying Diophantine conditions. Herman has proven that these tori need to be Lagrangian [14].
- On the other hand, the Birkhoff theory [7] about the invariant rotational curves for area preserving maps in the annulus $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ is a first step to a global (non-perturbative) study on the existence of such curves. This theory gives several Lipschitzian inequalities for the invariant curves and first asserts that they must be graphs (under some non-degeneracy conditions). These theorems appear also

in works of Herman [13] and Mather [23]. Herman improved and generalized such results to higher dimensions along several papers [14, 15]. We shall refer to this theory as *BHM theory*.

- Finally, it would be useful to know conditions under which there are no invariant tori through a given point or region in phase space. The development of such criteria is known as *Converse KAM theory*. After the earlier work by Aubry and Mather, non-existence criteria were found for area preserving twist maps [19] and later for symplectic twist maps [20]. A summary of results appears in [8].

In order to study the existence of *invariant rotational tori* for symplectomorphisms defined on $\mathbb{T}^d \times \mathbb{R}^d$ (that is, invariant tori which are homotopic to the zero-section) we apply variational methods, trying that they be more effective in higher dimensions and with weaker hypotheses that usually one requires (say, that our symplectomorphism satisfies certain strong conditions of positivity). Although we still need certain positivity conditions and we only are able to work with those tori that are graphs, it could be possible that our results work in more general cases by using suitable coordinates, thanks to the local character of our variational principles.

As a summary, we obtain that, in $\mathbb{T}^d \times \mathbb{R}^d$:

the orbits on a monotone positive invariant Lagrangian horizontal torus are minimizing.

The definition of monotone positivity was given by Herman [14, 15] and the flavour of this result is like [20]. We obtain a similar *non-existence criterion* of invariant tori to the last reference:

If the orbit by z yields on a monotone positive region and has a non-minimizing segment, then it does not lie on any invariant Lagrangian horizontal torus included into such a region.

Notice that we do not require global conditions to our symplectomorphism because we test if a certain segment of orbit is a local minimum of the corresponding action. Hence, the existence of a global generating function, which involves global conditions for our symplectomorphism, is not necessary.

We have applied it to a broad class of examples, given by the family of generalized standard-like maps [18]. The examples that appear in this paper are examples of non-twist symplectic maps, but we can study the region on phase space where certain positivity conditions are satisfied.

2. Definitions

2.1. The symplectic group

We write any matrix M of $M_{2d}(\mathbb{R})$ by $d\times d$ blocks:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let J be the symplectic matrix

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},$$

where I_d is the $d{\times}d$-identity matrix, and $\mathrm{Sp}(2d)$ be the symplectic group of $\mathbb{R}^{2d}$ (subgroup of $\mathrm{GL}(2d,\mathbb{R})$), that is to say, the set of matrices $M \in M_{2d}(\mathbb{R})$ such that $M^\top J M = J$ (where $^\top$ means the transpose of a matrix).

Hence:

$$\begin{array}{ccccc} M \in \mathrm{Sp}(2d) & \Leftrightarrow & M^\top J M = J & \Leftrightarrow & A^\top C = C^\top A, B^\top D = D^\top B, A^\top D - C^\top B = I_d, \\ & & \Updownarrow & & \\ M^\top \in \mathrm{Sp(2d)} & \Leftrightarrow & M J M^\top = J & \Leftrightarrow & AB^\top = BA^\top, CD^\top = DC^\top, AD^\top - BC^\top = I_d. \end{array}$$

2.2. The primitive function of a symplectomorphism

We consider $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ endowed with the *position-momentum* coordinates

$$z = (x, y) = (x^1, \ldots, x^d, y^1, \ldots, y^d).$$

Definition: Let $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a diffeomorphism, that is, we write

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y). \end{cases}$$

It is a *symplectomorphism* iff $\forall z \in \mathbb{R}^{2d}$, $\mathrm{D}F(z) \in \mathrm{Sp}(2d)$. In such case, the *exactness equations* associated to F, which are the Pfaffian system

$$\begin{cases} \dfrac{\partial S}{\partial x}(x, y) = g(x, y)^\top \dfrac{\partial f}{\partial x}(x, y) - y^\top, \\[2ex] \dfrac{\partial S}{\partial y}(x, y) = g(x, y)^\top \dfrac{\partial f}{\partial y}(x, y), \end{cases}$$

are solvable.

Definition: The solution $S : \mathbb{R}^{2d} \to \mathbb{R}$ of the exactness equations is called the *primitive function* of F, and it is defined up to constants. Anyway, we write $\mathrm{pf}(F) = S$.

Remarks:

i) Recall that the standard symplectic structure on $\mathbb{R}^{2d}$ is given by $\boldsymbol{\omega} = \mathrm{d}y \wedge \mathrm{d}x$, and it is exact: $\boldsymbol{\alpha} = y\ \mathrm{d}x$ is the action form. The exactness equations are the translation to coordinates of the condition $F^*\boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathrm{d}S$. However, we shall not use this language along this paper.

ii) In the literature, the primitive function is often called generating function, although it does not generate the symplectomorphism! Because of this, we have followed the nomenclature used in [3]. Anyway, the primitive function and the different kinds of generating functions are related by certain p.d.e. [12].

iii) Symplectomorphism are closed by composition and inversion. Given two symplectomorphisms $F, G : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, with $\mathrm{pf}(F) = S$ and $\mathrm{pf}(G) = T$, we have the next formulae:

(i) $\mathrm{pf}(G \circ F) = S + T \circ F$,

(ii) $\mathrm{pf}(F^{-1}) = -S \circ F^{-1}$,

(iii) $\mathrm{pf}(G^{-1} \circ F \circ G) = S \circ G + T - T \circ G^{-1} \circ F \circ G$.

iv) Thanks to the topological properties of $\mathbb{R}^{2d}$, the primitive function of a symplectomorphism always exists. This is not necessary in more general contexts, and we need that our symplectic structure be exact (see, for instance [12]). $\triangleleft$

Examples:

1) Any diffeomorphism of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \phi(x) \\ \mathrm{D}\phi(x)^{-\top} y \end{pmatrix},$$

where $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism, is a symplectomorphism and has primitive function equal to zero ($^{-\top}$ means the transpose of the inverse of a matrix). This symplectomorphism is the *lift* of the diffeomorphism ϕ, and it is represented by $\hat{\phi}$. It is easy to see that if we compose on the left our initial symplectomorphism F with a lift $L = \hat{\phi}$ then we obtain another symplectomorphism $\bar{F} = L \circ F$, with the same primitive function.

2) A *(symplectic) vertical translation* $\tau = \tau_{\nabla l}$ is defined by means of a function $l : \mathbb{R}^d \to \mathbb{R}$ and it is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y + \nabla l(x) \end{pmatrix}.$$

It is a symplectomorphism and its primitive function is just l (it is a function which only depends on the x-variables). $\triangleleft$

2.3. Monotonicity

Let $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectomorphism. We write

$$\mathrm{D}F(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

Definition: We say that F is *monotone* iff $\forall z \in \mathbb{R}^{2d}$ $|B(z)| \neq 0$. In such a case, the matrices $B^{-1}(z)A(z)$ and $D(z)B^{-1}(z)$ are symmetric. Following [14, 15], we shall say that F is *monotone positive* iff some of the next two conditions is verified:

$(+_a)$ $\forall z \in \mathbb{R}^{2d}$ $B^{-1}(z)A(z)$ is positive definite;

$(+_d)$ $\forall z \in \mathbb{R}^{2d}$ $D(z)B^{-1}(z)$ is positive definite.

We shall distinguish the two types of monotone positivity writing $(+_a)$ or $(+_d)$. We define monotone negativity in the same way. We can perform analogous definitions for monotone regions, monotone points, etc.

The symmetric matrix

$$T(z) = \frac{1}{2}(B(z) + B(z)^\top)$$

is the *torsion* of F at the point z. If it is positive definite for all the points, we shall say that our diffeomorphism *has positive torsion.* If the torsion is uniformly positive definite, we shall say that F is a *twist map* and, as Avez proved in [6] (see also [20]), the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$$

is a diffeomorphism on $\mathbb{R}^{2d}$.

Remark: Notice that there exist symplectic matrices which are monotone positive but have not positive torsion, and other ones that have positive torsion and are not monotone positive. Easy examples, for $d = 1$, are given by

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

◁

2.4. Generating functions

It is well known that, sometimes, a symplectomorphism is given by a (Lagrangian) generating function. We need monotonicity. In fact, we need a stronger condition.

Definition: Let $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a monotone symplectomorphism, and $\mathrm{pf}(F) = S$. We say that it is *strongly monotone* iff

$$\forall x \in \mathbb{R}^d, f(x, \cdot) : \mathbb{R}^d \to \mathbb{R}^d \text{ is a diffeomorphism.}$$

Symplectic twist maps are examples of strongly monotone symplectomorphisms [6, 20].

In such case, let $\phi(x, \cdot)$ be the inverse of $f(x, \cdot)$, that is, $\forall x, x' \in \mathbb{R}^d$ $x' = f(x, \phi(x, x'))$. We define the function $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$L(x, x') = S(x, \phi(x, x')).$$

Therefore, applying the exactness equations, we reach to the relations

$$\begin{cases} y = -\nabla_x L(x, x'), \\ y' = \nabla_{x'} L(x, x'), \end{cases}$$

which define implicitly our symplectomorphism. The function L is called a *(global) Lagrangian generating function* of F, and defines implicitly our symplectomorphism F.

Hence, the relationship between the Lagrangian generating function and the primitive function is given by

$$S\left(x, -\nabla_x L(x,x')\right) = L(x,x').$$

Moreover, the second derivatives of L are given by

$$\frac{\partial^2 L}{\partial x^2} = B^{-1}A,\ \frac{\partial^2 L}{\partial x' \partial x} = -B^{-1},\ \frac{\partial^2 L}{\partial x'^2} = DB^{-1}.$$

Remarks:

i) Notice that this expressions appear in the definitions about monotonicity.

ii) If one asks for the inverse problem, that is, to obtain the generating function from the primitive function, one has to solve this p.d.e., and we need some additional information. ◁

An orbit $Z = (x_k, y_k)_{k\in\mathbb{Z}}$ is determined by the positions $(x_k)_{k\in\mathbb{Z}}$. Formally speaking, the orbits are in correspondence with the stationary bisequences $X = (x_k)_{k\in\mathbb{Z}}$ of the action

$$\boldsymbol{L}(X) = \sum_{k\in\mathbb{Z}} L(x_k, x_{k+1})$$

(see [5, 21]). Notice also that the value of this action along an orbit is $\sum_{k\in\mathbb{Z}} S(x_k, y_k)$.

3. Variational principles

Since generating functions are not always well defined, or they are difficult to compute, we shall perform variational principles without their aid. Last part of this paper is devoted to certain examples of symplectomorphisms which are not defined by means of generating functions.

3.1. Chains and connecting orbits

Let $F = (f,g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectomorphism with $\mathrm{pf}(F) = S$. Given two x-points $\mathbf{x}_m, \mathbf{x}_n \in \mathbb{R}^d$, where $n > m+1$ are two indices, we want to search for the *connecting orbits* from $\mathbf{x}_m$ to $\mathbf{x}_n$, that is, the (m,n)-segments of $\mathbb{R}^{2d}$

$$(x_m, y_m), (x_{m+1}, y_{m+1}), \dots (x_{n-1}, y_{n-1})$$

such that

- $x_m = \mathbf{x}_m,\ f(x_{n-1}, y_{n-1}) = \mathbf{x}_n,$
- $\forall i = m \div n-2\ F(x_i, y_i) = (x_{i+1}, y_{i+1}).$

We shall consider the (m,n)-*orbital action*

$$\boldsymbol{S}_{m,n}(x_m, y_m, x_{m+1}, y_{m+1}, \dots, x_{n-1}, y_{n-1}) = \sum_{i=m}^{n-1} S(x_i, y_i)$$

restricted to the (m,n)-segments satisfying

- $x_m = \mathbf{x}_m$, $f(x_{n-1}, y_{n-1}) = \mathbf{x}_n$,
- $\forall i = m \div n-2 \; f(x_i, y_i) = x_{i+1}$.

They are the *chains* from $\mathbf{x}_m$ to $\mathbf{x}_n$, and the corresponding set of chains, $\mathcal{C}_{\mathbf{x}_m,\mathbf{x}_n}$, is a $d(n-m-1)$-submanifold of $\mathbb{R}^{2d(n-m)}$, provided the rank of the matrix

$$\begin{pmatrix} B_m & -I & & & & & \\ & A_{m+1} & B_{m+1} & -I & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_{n-2} & B_{n-2} & -I & \\ & & & & & A_{n-1} & B_{n-1} \end{pmatrix}$$

be maximal ($= d(n-m)$) in all the chains. For instance, if F is monotone this *transversality condition* is satisfied. In general, if this condition is satisfied for a certain connecting orbit, we say that it is *transversal.* An orbit is transversal iff all its segments are transversal.

Proposition 1 *Let $F = (f,g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectomorphism, with $\mathrm{pf}(F) = S$. Given two x-points $\mathbf{x}_m, \mathbf{x}_n \in \mathbb{R}^d$, suppose that the corresponding set of chains, $\mathcal{C}_{\mathbf{x}_m,\mathbf{x}_n}$, satisfies the transversality condition, and consider the orbital action $\boldsymbol{S}_{m,n}$ on it. Then:*

- *The connecting orbits are critical chains of $\boldsymbol{S}_{m,n}$.*
- *If F is monotone, the critical chains of $\boldsymbol{S}_{m,n}$ are connecting orbits of F.*

Proof: By the Lagrange multiplier method, we have to search for the critical points of the function

$$L(y_m, x_{m+1}, y_{m+1}, \ldots, x_{n-1}, y_{n-1}, \lambda_{m+1}, \ldots, \lambda_n) = \sum_{i=m}^{n-1} (S(x_i, y_i) + \lambda_{i+1}(x_{i+1} - f(x_i, y_i))),$$

where all the variables belong to $\mathbb{R}^d$, and we have taken $x_m = \mathbf{x}_m$ and $x_n = \mathbf{x}_n$. The equations that we have to solve are

$$0 = \frac{\partial L}{\partial x_i} = (g(x_i, y_i) - \lambda_{i+1})^\top \frac{\partial f}{\partial x}(x_i, y_i) + (\lambda_i - y_i)^\top \qquad (i = m+1 \div n-1),$$

$$0 = \frac{\partial L}{\partial y_i} = (g(x_i, y_i) - \lambda_{i+1})^\top \frac{\partial f}{\partial y}(x_i, y_i) \qquad (i = m \div n-1),$$

$$0 = \frac{\partial L}{\partial \lambda_i} = (x_i - f(x_{i-1}, y_{i-1}))^\top \qquad (i = m+1 \div n),$$

where we have used the exactness equations (2.2). Therefore:

- If $(x_i, y_i)_{i=m \div n-1}$ is a connecting orbit from $\mathbf{x}_m$ to $\mathbf{x}_n$, then it is a critical chain of the function $\boldsymbol{S}_{m,n}$ (take $\forall i = m+1 \div n-1 \; \lambda_i = y_i$ and $\lambda_n = g(x_{n-1}, y_{n-1})$).
- Suppose that $\left|\frac{\partial f}{\partial y}\right| \neq 0$. For a critical chain of $\boldsymbol{S}_{m,n}$ (in particular, $\forall i = m \div n-1 \; x_{i+1} = f(x_i, y_i)$) the second equations gives $\forall i = m \div n-1 \; \lambda_{i+1} = g(x_i, y_i)$ and, therefore, the first ones gives $\forall i = m+1 \div n-1 \; \lambda_i = y_i$. Finally, we obtain $\forall i = m \div n-2 \; g(x_i, y_i) = \lambda_{i+1} = y_{i+1}$.

□

Remarks:

i) We can extend these ideas to any cotangent bundle of a manifold $\mathcal{M}$, $T^*\mathcal{M}$, where a canonical exact symplectic structure is defined [2]. Recall that the cotangent bundles are the *phase spaces* of classical mechanics, and the basic manifolds $\mathcal{M}$ are the *configuration spaces.* For further information about these variational principles of *discrete mechanics* see [12].

ii) From a physical point of view, if our symplectomorphism is a time-1 flow of a certain time-periodic Hamiltonian (with period 1), a F-chain correspond to an 'orbit' whose velocity is rudely changed every period (as in a maneuver). Hence, to extreme the action on the set of F-chains corresponds to smoothe the sharps. ◁

3.2. Extremal character of an orbit

We have seen that the connecting orbits from $\mathbf{x}_m$ to $\mathbf{x}_n$ are critical F-chains of a certain action. We wonder about their extremal character, i.e., about the second derivative $H_{m,n} = \mathrm{D}^2\boldsymbol{S}_{m,n}$.

If F is monotone, or at least it is monotone in the region where the segment lives, then the function $\boldsymbol{S}_{m,n}$ can be locally written in variables $x_{m+1}, \ldots, x_{n-1}$, and we can compute the corresponding Hessian matrix. That is, the set of equations $\{f(x_i, y_i) = x_{i+1}\}_{i=m\div n-1}$ defines implicitly a set of functions $\{\eta_i = \eta_i(x, x')\}_{i=m\div n-1}$ such that $\forall i = m \div n-1$

$$f(x_i, \eta_i(x_i, x_{i+1})) = x_{i+1}.$$

Of course, these functions are defined on a neighborhood of our connecting orbit. Their derivatives are given from

$$0 = \frac{\partial f}{\partial x}(x_i, \eta_i(x_i, x_{i+1})) + \frac{\partial f}{\partial y}(x_i, \eta_i(x_i, x_{i+1}))\, \frac{\partial \eta_i}{\partial x}(x_i, x_{i+1})$$

and

$$I = \frac{\partial f}{\partial y}(x_i, \eta_i(x_i, x_{i+1}))\, \frac{\partial \eta_i}{\partial x'}(x_i, x_{i+1}).$$

Therefore, we have to compute the critical points of the function

$$\boldsymbol{S}_{m,n}(x_1, \ldots, x_{n-1}) = \sum_{i=m}^{n-1} S(x_i, \eta_i(x_i, x_{i+1})),$$

where we have taken $x_m = \mathbf{x}_m$ and $x_n = \mathbf{x}_n$. So then, $\forall i = m+1 \div n-1$

$$\begin{aligned}
\frac{\partial \boldsymbol{S}_{m,n}}{\partial x_i} &= \frac{\partial S}{\partial x}(x_i, \eta_i) + \frac{\partial S}{\partial y}(x_i, \eta_i)\, \frac{\partial \eta_i}{\partial x}(x_i, x_{i+1}) + \frac{\partial S}{\partial y}(x_{i-1}, \eta_{i-1})\, \frac{\partial \eta_{i-1}}{\partial x'}(x_{i-1}, x_i) \\
&= g_i^\top\, \frac{\partial f}{\partial x}(x_i, \eta_i) - \eta_i^\top + g_i^\top\, \frac{\partial f}{\partial y}(x_i, \eta_i)\, \frac{\partial \eta_i}{\partial x} + g_{i-1}^\top\, \frac{\partial f}{\partial y}(x_{i-1}, \eta_{i-1})\, \frac{\partial \eta_{i-1}}{\partial x'} \\
&= g_{i-1}^\top - \eta_i^\top,
\end{aligned}$$

where $g_i = g(x_i, \eta_i)$. As we see, our connecting orbit extremes the action. We are going to compute the second derivatives. We have written $A_i = \frac{\partial f}{\partial x}(x_i, y_i)$, etc. Then, $\forall i = m+1 \div n-1$

$$\frac{\partial^2 \boldsymbol{S}_{m,n}}{\partial x_i^2} = \frac{\partial g}{\partial y}(x_{i-1}, y_{i-1}) \; \frac{\partial \eta_{i-1}}{\partial x'}(x_{i-1}, x_i) - \frac{\partial \eta_i}{\partial x}(x_i, x_{i+1}) = D_{i-1} \; B_{i-1}^{-1} + B_i^{-1} \; A_i$$

and $\forall i = m+1 \div n-2$

$$\frac{\partial^2 \boldsymbol{S}_{m,n}}{\partial x_{i+1} \partial x_i} = -\frac{\partial \eta_i}{\partial x'}(x_i, x_{i+1}) = -B_i^{-1} \; .$$

In summary:

Proposition 2 *The Hessian matrix associated to a monotone (m,n)-segment of orbit is given by the block-tridiagonal symmetric matrix*

$$H_{m,n} = \begin{pmatrix} \hat{A}_{m+1} & \hat{B}_{m+1} & & & \\ \hat{B}_{m+1}^\top & \hat{A}_{m+2} & \hat{B}_{m+2} & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{B}_{n-3}^\top & \hat{A}_{n-2} & \hat{B}_{n-2} \\ & & & \hat{B}_{n-2}^\top & \hat{A}_{n-1} \end{pmatrix},$$

where the matrices $\hat{A}_i$ and $\hat{B}_i$ are given by

$$\hat{A}_i = D_{i-1} B_{i-1}^{-1} + B_i^{-1} A_i \; , \; \hat{B}_i = -B_i^{-1}.$$

Connecting orbits with positive definite Hessian matrix will be interesting in the sequel. All of their points must be monotone, because in other case the Hessian is degenerate.

3.3. Minimizing orbits

Definition: We say that a (m,n)-segment of orbit is *minimizing* iff it is monotone and the corresponding matrix $H_{m,n}$ is definite positive. Then, we say that an orbit is minimizing iff it is monotone and all its Hessian matrices $H_{m,n}$ are positive definite.

Remarks:

i) So then, minimizing means non-degenerate minimum.
ii) As the eigenvalues of a matrix depend continuously on its components, if we have a minimizing segment of orbit then another segment of orbit close enough to the first will be also minimizing.
iii) All the subsegments of a minimizing segment are also minimizing.
iv) A minimizing orbit for F is also minimizing orbit for a power of F, because in the second case the chains are defined with more constraints. This is due to the primitive function of a power of F is the sum of the primitive function on each points of the corresponding segment. ◁

3.3.1. The MMS method. In [20], MacKay, Meiss and Stark wrote a Hessian matrix using the Lagrangian generating function, but if we use 2.4 we see that the two matrices coincide, in the case that such a function exists. Then, although the Lagrangian generating function does not exist, we can define a (local) extremal behaviour of the orbits.

On the other side, they describe a method to block-diagonalize that matrix. They find that the diagonal blocks associated to $H_{0,n}$ are given by the recurrence

$$\begin{cases} \hat{D}_1 = \hat{A}_1, \\ \hat{D}_i = \hat{A}_i - \hat{B}_{i-1}^{\top}\hat{D}_{i-1}^{-1}\hat{B}_{i-1} \quad (i = 2 \div n-1), \end{cases}$$

provided $\hat{D}_{i-1}$ is invertible. Then, the matrix $H_{0,n}$ is positive definite iff all the symmetric matrices $\hat{D}_i$ are positive definite.

3.3.2. 1 degree of freedom. If $d = 1$, then we can obtain a recurrence for the characteristic polynomials of $H_{0,i}$ $(i > 1)$: $p_{i-1}(x) = \det(xI - H_{0,i})$. Then:

$$\begin{cases} p_0(x) = 1, p_1(x) = x - \hat{a}_1; \\ p_i(x) = (x - \hat{a}_i)p_{i-1}(x) - \hat{b}_{i-1}^2 p_{i-2}(x) \quad (i > 2), \end{cases}$$

where the lower-case letters mean the corresponding 1×1 matrices. The sequence of polynomials $\{p_i, p_{i-1}, \ldots, p_1, p_0\}$ is a Sturm sequence for the polynomial p_i. In particular, all the eigenvalues of $H_{0,i}$ are different (and real, of course) and we can compute the number of positive eigenvalues:

> If $p_{i-1}(0) \neq 0$, the number of positive eigenvalues of $H_{0,i}$ is equal to the number of changes of sign in the sequence $\{p_0(0), p_1(0), \ldots, p_{i-1}(0)\}$.

This is a particular case of the Sturm's theorem. Hence, we have to compute the number of changes of sign of the sequence

$$\begin{cases} r_0 = 1, \ r_1 = -\hat{a}_1; \\ \hat{r}_i = -\hat{a}_i r_{i-1} - \hat{b}_{i-1}^2 r_{i-2} \quad (i > 1). \end{cases}$$

This fit into the MMS method, by defining $\forall i \geq 1$ $\hat{d}_i = -\dfrac{r_i}{r_{i-1}}$.

3.4. Invariance of the extremal character of an orbit

We wonder if the extremal character of an orbit is independent of the variables in which we write our symplectomorphism. In fact, we shall see that they do not change under lifts and vertical translations, but it can change under conjugation with other symplectomorphisms. This is due to the concomitant distinction between x and y variables. For the sake of simplicity, we shall work in the monotone case. Otherwise, the orbits are degenerate as critical chains of the action.

Proposition 3 *Given a symplectomorphism $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, the extremal character of monotone orbits does not change by (symplectic) vertical translations and lifts.*

Proof: We shall prove the first point, the second being similar (see [12]). So then, let $G = \tau_{\nabla l}$ be the vertical translation induced by the function $l : \mathbb{R}^d \to \mathbb{R}$, which defines a change of variables

$$\begin{cases} x = \bar{x}, \\ y = \bar{y} + \nabla l(\bar{x}). \end{cases}$$

Our symplectomorphism F written in the new variables if $\bar{F} = G^{-1} \circ F \circ G$, and it is given by

$$\begin{cases} \bar{x}' = f(\bar{x}, \bar{y} + \nabla l(\bar{x})), \\ \bar{y}' = g(\bar{x}, \bar{y} + \nabla l(\bar{x})) - \nabla l(f(\bar{x}, \bar{y} + \nabla l(\bar{x}))). \end{cases}$$

Recall that if the primitive function of F is S, then the primitive function of $\bar{F}$ is $S \circ G + l \circ q - l \circ f \circ G$, where q is the projection on the x-variables.

We consider now two corresponding orbits by F and $\bar{F}$. Hence, let (x_0, y_0) the initial point of a F-orbit and $(\bar{x}_0, \bar{y}_0) = G^{-1}(x_0, y_0)$ the initial point of the corresponding $\bar{F}$-orbit. We know that the extremal character of a F-orbit is given by the recurrence

$$\begin{cases} \hat{D}_1 = \hat{A}_1, \\ \hat{D}_i = \hat{A}_i - \hat{B}_{i-1}^{\top} \hat{D}_{i-1}^{-1} \hat{B}_{i-1} \quad (i \geq 2), \end{cases}$$

where $\hat{A}_i = D_{i-1} B_{i-1}^{-1} + B_i^{-1} A_i$ and $\hat{B}_i = -B_i^{-1}$. We should write the same sequence for a $\bar{F}$-orbit putting bars in all the places, but we shall write $\check{A}_i$ rather than $\hat{\bar{A}}_i$, etc. We have to relate both sequences and see that the indexes of the matrices $\hat{D}_i$ and $\check{D}_i$ are the same. First of all, we relate $\mathrm{D}\bar{F}(\bar{z})$ and $\mathrm{D}F(z)$ by means of

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} A + BL_0 & B \\ \bar{C} & -L_1 B + D \end{pmatrix},$$

where $L_0 = \mathrm{D}^2 l(\bar{x})$, $L_1 = \mathrm{D}^2 l(f(\bar{x}, \bar{y}))$, etc. Notice that

monotonicity does not change under vertical translations.

Finally, we obtain that $\check{B}_i = \hat{B}_i$ and

$$\check{A}_i = (-L_i B_{i-1} + D_{i-1}) B_{i-1}^{-1} + B_i^{-1}(A_i + B_i L_i) = D_{i-1} B_{i-1}^{-1} + B_i^{-1} A_i = \hat{A}_i,$$

and the corresponding sequences of diagonal blocks $\hat{D}_i$ and $\check{D}_i$ coincide. □

Remark: A physical interpretation of this result is that the laws of the *discrete mechanics* are independent from the coordinates on our configuration space and certain privileged observers. This is geometrically connected with the choice of a certain 1-form $\boldsymbol{\alpha} = y\, \mathrm{d}x$ in our phase space, and the distinction between x and y coordinates that it produces. ◁

4. Invariant Lagrangian graphs

A first step in order to understand the properties of invariant Lagrangian manifolds is to study the easier ones: the *invariant Lagrangian graphs* (in short, *i.L.g.*).

This section is devoted to extend some results due to Mather [22], Herman [14] and MacKay, Meiss and Stark [20], obtained by them by means of the use of a (global) Lagrangian generating function. In some sense our results are more local, because they do not use the existence of this global function, and they will let us to study different regions in our phase space where some positivity condition will be satisfied.

4.1. Characterization of i.L.g.

Definition: Given a function $l : \mathbb{R}^d \to \mathbb{R}$, it is well known that the immersion

$$\begin{array}{rccc} \nu: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \times \mathbb{R}^d \\ & x & \longrightarrow & (x, \nabla l(x)) \end{array}$$

defines a Lagrangian embedding of $\mathbb{R}^d$ into $\mathbb{R}^d \times \mathbb{R}^d$. The function l is known as the *generating function* of the *Lagrangian graph* $\mathcal{L}_{\nabla l} = \nu(\mathbb{R}^d) = \{y = \nabla l(x)\}$.

Definition: Given a symplectomorphism $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, we associate to the Lagrangian graph $\mathcal{L}_{\nabla l}$ a *critical function* $\hat{\Phi} : \mathbb{R}^{2d} \to \mathbb{R}$ by

$$\hat{\Phi}(x, y) = S(x, y) - (l(f(x, y)) - l(x)).$$

Its value on the graph is denoted by $\Phi(x) = \hat{\Phi}(x, \nabla l(x))$.

We have the next proposition of characterization of invariant Lagrangian graphs.

Proposition 4 *Let $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectomorphism with* $\mathrm{pf}(F) = S$ *and $\mathcal{L} = \mathcal{L}_{\nabla l}$ be the Lagrangian graph given by $l : \mathbb{R}^d \to \mathbb{R}$. We define the functions Φ and $\hat{\Phi}$ as above. Then:*

(i) $\mathcal{L}$ is F-invariant $\Rightarrow$ Φ is constant
(Conserved quantity on an i.L.g.).

(ii) $\mathcal{L}$ is F-invariant $\Leftrightarrow$ $\forall x \in \mathbb{R}^d$ $\mathrm{D}\hat{\Phi}(x, \nabla l(x)) = 0$
(Characterization of i.L.g.).

Proof: The *invariance condition* is $\forall x \in \mathbb{R}^d \quad g(x, \nabla l(x)) = \nabla l(f(x, \nabla l(x)))$. The derivatives of $\hat{\Phi}$ are:

$$\frac{\partial \hat{\Phi}}{\partial x}(x, y) = \left(g(x, y)^\top - \frac{\partial l}{\partial x}(f(x, y)) \right) \cdot \frac{\partial f}{\partial x}(x, y) - y^\top + \frac{\partial l}{\partial x}(x),$$

$$\frac{\partial \hat{\Phi}}{\partial y}(x, y) = \left(g(x, y)^\top - \frac{\partial l}{\partial x}(f(x, y)) \right) \cdot \frac{\partial f}{\partial y}(x, y).$$

So, if $\mathcal{L}$ is F-invariant the two derivatives vanish on the graph and, in particular, the function Φ is constant:

$$\frac{\partial \Phi}{\partial x}(x) = \frac{\partial \hat{\Phi}}{\partial x}(x, \nabla l(x)) + \frac{\partial \hat{\Phi}}{\partial y}(x, \nabla l(x)) \cdot \frac{\partial^2 l}{\partial x^2}(x) = 0.$$

Conversely, if the derivatives vanish at a point $(x, \nabla l(x))$, then we obtain

$$g(x, \nabla l(x)) = \nabla l(f(x, \nabla l(x))),$$

because the rank of the matrix $(A(x,y) \;\; B(x,y))$ is maximal at all points. □

Remark: First point can be easily generalized to any invariant exact Lagrangian manifolds of any exact symplectomorphisms. Other points could be also generalized to any invariant exact Lagrangian manifolds by means of Weinstein's theorems [29], which let us to identify a tubular neighborhood of a Lagrangian manifold with a tubular neighborhood of the zero section of its cotangent bundle. For more information about this subject see [12]. ◁

4.2. Extremal character of an i.L.g.

Following with the previous notation, a point of an i.L.g. $\mathcal{L} = \mathcal{L}_{\nabla l}$ is a fibered critical point of the critical function $\hat{\Phi}$, that is to say, for each point $x \in \mathbb{R}^d$

$$\frac{\partial \hat{\Phi}}{\partial y}(x, \nabla l(x)) = 0.$$

(The converse is true if F is monotone).

Definition: If all the points have the same character as critical points of the 'fibered' function, and then all the corresponding Hessian matrices have non-zero eigenvalues, we shall say that our graph is *non-degenerate.* In such a case, if all these matrices are positive definite we shall say that our graph is *minimizing*, and if all of them are negative definite we shall say that it is *maximizing.* Otherwise we shall say that it is *indefinite.*

4.3. Minimizing i.L.g.

Hence, the extremal character of the graph on each point $(x, \nabla l(x))$ is given by

$$\frac{\partial^2 \hat{\Phi}}{\partial y^2}(x, \nabla l(x)) = (D^\top(x) - B(x)^\top \, \mathrm{D}^2 l(\phi(x)))B(x),$$

where we write $\phi(x) = f(x, \nabla l(x))$, $A(x) = A(x, \nabla l(x))$, etc. If the graph is monotone, that is, it is included into a monotone region, the signature of such matrix does not change if we multiply it by B^{-1} and $B^{-\top}$:

$$B(x)^{-\top} \frac{\partial^2 \hat{\Phi}}{\partial y^2}(x, \nabla l(x)) B(x)^{-1} = D(x)B(x)^{-1} - \mathrm{D}^2 l(\phi(x)).$$

By deriving the equalities $\phi(x) = f(x, \nabla l(x))$ and $\nabla l(\phi(x)) = g(x, \nabla l(x))$ we obtain

$$\begin{aligned} \mathrm{D}\phi \quad\quad &= A + B \, \mathrm{D}^2 l, \\ \mathrm{D}^2 l \circ \phi \; \mathrm{D}\phi &= C + D \, \mathrm{D}^2 l. \end{aligned}$$

Hence, as $D^\top A - B^\top C = I_d$, $D^\top B = B^\top D$ and $\mathrm{D}^2 l$ is symmetric, we reach

$$\begin{aligned} B^{-1}A + \mathrm{D}^2 l \quad &= B^{-1}\mathrm{D}\phi, \\ DB^{-1} - \mathrm{D}^2 l{\circ}\phi &= (\mathrm{D}\phi)^{-\top}B^{-1}. \end{aligned}$$

Finally, we define the maps $E_1, E_2 : \mathbb{R}^d \to M_d(\mathbb{R})$ by $E_1(x) = B(x)^{-1}\mathrm{D}\phi(x)$ and $E_2(x) = (\mathrm{D}\phi(x))^{-\top}B(x)^{-1}$. They are symmetric and non-singular matrices, and they are related by

$$E_2(x) = B(x)^{-\top}E_1(x)^{-1}B(x)^{-1}.$$

Hence, the positive definiteness of one implies the positive definiteness of the other. Then, we obtain next lemma.

Lemma 1 *Let $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectomorphism and $\mathcal{L} = \mathcal{L}_{\nabla l}$ be a monotone F-invariant Lagrangian graph. Let $E_1, E_2 : \mathbb{R}^d \to M_d(\mathbb{R})$ be the maps defined as above. Then:*

$$\mathcal{L} \textit{ is minimizing} \Leftrightarrow \forall x \in \mathbb{R}^d \;\; E_1(x) \succ 0 \Leftrightarrow \forall x \in \mathbb{R}^d \;\; E_2(x) \succ 0.$$

4.4. Invariance of the extremal character of an i.L.g.

As we shall see, the extremal character of our graph does not change under vertical translations and lifts. In fact, we shall see a little less than this, but enough for us. We shall do two steps of normal form in order to simplify the dynamics around an i.L.g..

Proposition 5 *Let $F = (f, g) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectomorphism with $\mathrm{pf}(F) = S$ and $\mathcal{L} = \mathcal{L}_{\nabla l}$ be a F-invariant Lagrangian graph generated by $l : \mathbb{R}^d \to \mathbb{R}$. Hence, the extremal character of the graph does not change after the next to steps of normal form:*

(i) projection of the zero-section,

(ii) simplification of the dynamics on that zero-section, via conjugation by a lift.

Proof: We shall prove the first step, the second being similar (see [12]). So then, the extremal character of our graph is given by the symmetric matrices

$$(D^\top(x) - B(x)^\top \, \mathrm{D}^2 l(\phi(x)))B(x).$$

If we perform a change of variables by means of the vertical translation

$$\begin{cases} x = \bar{x}, \\ y = \bar{y} + \nabla l(\bar{x}), \end{cases}$$

then in the new variables $(\bar{x}, \bar{y})$ the zero-section $\{y = 0\}$ is fixed. After this projection, the character of the graph (the zero-section) is given by

$$\bar{D}(\bar{x})^\top \bar{B}(\bar{x}) = (D(x) - L_1(\bar{x}, 0)B(x))^\top B(x) = D(x)^\top B(x) - B(x)^\top \mathrm{D}^2 l(\phi(x))B(x),$$

and the character does not change (see the notation in 3.4). $\square$

4.5. Orbits on minimizing invariant Lagrangian graphs

In this section we shall see that the orbits on a minimizing invariant Lagrangian graph are minimizing. This is the key result of the paper.

Theorem 1 *Let $F=(f,g):\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ be a symplectomorphism and $\mathcal{L}=\mathcal{L}_{\nabla l}$ be a F-invariant Lagrangian graph given by $l:\mathbb{R}^d\to\mathbb{R}$. Then:*

If $\mathcal{L}$ is minimizing, then all the orbits on $\mathcal{L}$ are minimizing.

Proof: Before doing a complete proof we shall see what is the key of our constructions. So, let S be the primitive function of F, $\hat{\Phi}$ be the function associated to the graph, and $C\in\mathbb{R}$ the corresponding conserved quantity: $\forall x\in\mathbb{R}^d$ $\hat{\Phi}(x,\nabla l(x))=C$. As $\mathcal{L}$ is minimizing, then in a tubular neighborhood of the graph we have $\hat{\Phi}(x,y)\geq\hat{\Phi}(x,\nabla l(x))=C$. Let $(\mathbf{x},\nabla l(\mathbf{x}))$ be any point on $\mathcal{L}$. Fix $n>1$, let $\mathcal{O}=((\mathbf{x}_i,\mathbf{y}_i)=F^i(\mathbf{x},\nabla l(\mathbf{x})))_{i=0\div n-1}$ be a segment of orbit and $\mathcal{C}=(x_i,y_i)_{i=0\div n-1}$ be any F-chain from $\mathbf{x}_0$ to $\mathbf{x}_n=f(\mathbf{x}_{n-1},\mathbf{y}_{n-1})$, close enough to the graph. So then:

$$\begin{aligned}
\boldsymbol{S}_{0,n}(\mathcal{C}) = &\sum_{i=0}^{n-1} S(x_i,y_i)\\
\overset{\text{chain}}{=} &\sum_{i=0}^{n-1}\left(S(x_i,y_i)-l(f(x_i,y_i))+l(x_i)\right)\ +l(\mathbf{x}_n)-l(\mathbf{x}_0)\\
= &\sum_{i=0}^{n-1}\hat{\Phi}(x_i,y_i)\ +l(\mathbf{x}_n)-l(\mathbf{x}_0)\\
\geq\ & nC+l(\mathbf{x}_n)-l(\mathbf{x}_0)\\
= &\sum_{i=0}^{n-1}\hat{\Phi}(\mathbf{x}_i,\mathbf{y}_i)\ +l(\mathbf{x}_n)-l(\mathbf{x}_0)\\
= &\ \boldsymbol{S}_{0,n}(\mathcal{O}).
\end{aligned}$$

Hence, our connecting orbit minimizes the action on the chains. Notice that the restriction of the action to the set of F-chains is fundamental.

We have to improve this result and obtain that any segment of orbit is, in fact, a non-degenerate minimum of the corresponding action. We do not need global conditions. Thanks to the invariance of the extremal character of orbits (3.4) and graphs (4.4) under vertical translations, we can restrict our attention to the case in which our graph is the zero-section. In such a case, our symplectomorphism is given by

$$\begin{cases} x'=\phi(x)+B(x)y+\dots,\\ y'=\mathrm{D}\phi(x)^{-\top}y+\dots,\end{cases}$$

where '…' means terms in upper orders in y. Hence we have

$$\mathrm{D}F(x,0)=\begin{pmatrix} A(x) & B(x)\\ 0 & D(x)\end{pmatrix},$$

where $A(x) = \mathrm{D}\phi(x)$, $D(x) = A(x)^{-\top}$, and $A(x)B(x)^\top = B(x)A(x)^\top$ (2.1). Therefore

$$\frac{\partial^2 \hat{\Phi}}{\partial y^2}(x,0) = \frac{\partial^2 S}{\partial y^2}(x,0) = D(x)^\top B(x) = A(x)^{-1}B(x).$$

Hence, our graph is minimizing iff $\forall x \in \mathbb{R}^d$ $A(x)^{-1}B(x) \succ 0$, where $\succ$ is the Loewner partial order (and $S \succ 0$ means that the symmetric matrix S is positive definite). In fact, as the symmetric matrices $A^{-1}B = B^\top A^{-\top}$ and $B^{-\top}A^{-1} = A^{-\top}B^{-1}$ have the same inertia, we have

$$A^{-1}B \succ 0 \Leftrightarrow B^{-1}A \succ 0 \Leftrightarrow A^{-\top}B^{-1} \succ 0 \Leftrightarrow AB^\top \succ 0,$$

applying that $M \succ 0 \Rightarrow M^{-1} \succ 0$, for any symmetric matrix M.

Now, let $(x, \nabla l(x))$ be any point on the graph. We shall write $A_i = A(\phi^i(x))$, $B_i = B(\phi^i(x))$ and $D_i = D(\phi^i(x)) = A_i^{-\top}$. We shall use the MMS method (3.3.1). In this case we have

$$\hat{A}_i = A_{i-1}^{-\top}B_{i-1}^{-1} + B_i^{-1}A_i \succ 0.$$

We shall prove by induction that $\hat{D}_i = B_i^{-1}A_i + K_i$, where $K_i \succ 0$, for any $i \geq 1$.

- For $i = 1$ we have $\hat{D}_1 = B_1^{-1}A_1 + K_1$, where $K_1 = A_0^{-\top}B_0^{-1} \succ 0$, and the property holds.
- Suppose the property holds for $i-1$. So then

  $$\hat{D}_{i-1} = B_{i-1}^{-1}A_{i-1} + K_{i-1} \succ B_{i-1}^{-1}A_{i-1} \succ 0$$

  and

  $$B_{i-1}\hat{D}_{i-1}B_{i-1}^\top \succ A_{i-1}B_{i-1}^\top \succ 0.$$

  On the other side, we have $\hat{D}_i = B_i^{-1}A_i + K_i$, where

  $$K_i = A_{i-1}^{-\top}B_{i-1}^{-1} - B_{i-1}^{-\top}\hat{D}_{i-1}^{-1}B_{i-1}^{-1} = A_{i-1}^{-\top}B_{i-1}^{-1} - (B_{i-1}\hat{D}_{i-1}B_{i-1}^\top)^{-1}.$$

  Finally,

  $$K_i \succ A_{i-1}^{-\top}B_{i-1}^{-1} - B_{i-1}^{-\top}A_{i-1}^{-1} = 0.$$

  In summary, all the matrices $\hat{D}_i$ are positive definite. □

5. Converse KAM theory

Some non-existence criteria of invariant tori are founded in some kind of positivity of our symplectomorphism. We have follow [14] rather than [20], that is, we have studied monotone positive cases rather than twist cases.

5.1. Set up

We shall work on the annulus. Let $F : \mathbb{A}^d \to \mathbb{A}^d$ be a symplectomorphism. We shall consider its lift $\tilde{F} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, which is a symplectomorphism with primitive function $S : \mathbb{R}^{2d} \to \mathbb{R}$.

Let $\mathcal{T}$ be a F-invariant Lagrangian horizontal torus, that is, an i.L.g.

$$y = a + \nabla \bar{l}(x)$$

where $a \in \mathbb{R}^d$ and $l : \mathbb{T}^d \to \mathbb{R}$. So, the generating function (on $\mathbb{R}^d$) of the graph is $l(x) = a \cdot x + \bar{l}(x)$. That is, $\mathcal{T} = \mathcal{L}_{\nabla l}$.

Let $\phi : \mathbb{T}^d \to \mathbb{T}^d$ be the dynamics on the torus, that is to say, the *diffeomorphism* given by $\phi(x) = f(x, \nabla l(x))$. We shall also write $A(x) = A(x, \nabla l(x)), B(x) = B(x, \nabla l(x))$,etc. Suppose also that $\mathcal{T}$ is monotone ($\forall x \in \mathbb{T}^d\ |B(x)| \neq 0$).

Remark: Along this paper we suppose that our invariant Lagrangian tori are graphs. Unfortunately, there is not an equivalent to higher dimensions of the next theorem due to Birkhoff, for $d = 1$:

Let $F : \mathbb{A} \to \mathbb{A}$ be a C^1 monotone symplectomorphism, satisfying

$$\sup_{z \in \mathbb{A}} (|B^{-1}(z)A(z)|, |D(z)B^{-1}(z)|) < \infty.$$

Then, any C^0 F-invariant torus homotopic to the circle $\{y = 0\}$, called *invariant rotational curve* (in short, *i.r.c.*), is the graph $\mathcal{L}_\psi$ of a certain continuous function $\psi \in C^0(\mathbb{T}^1, \mathbb{R})$.

For the proof of this theorem see [7, 13, 23]. Herman [14] has perturbative generalizations to higher dimension of this theorem. ◁

5.2. Monotone positivity implies minimizing

Next theorem completes the previous one, and gives a sufficient condition to an i.L.g. be minimizing.

Theorem 2 *Let $F : \mathbb{A}^d \to \mathbb{A}^d$ be a symplectomorphism, and $\mathcal{T} = \mathcal{L}_{\nabla l}$ be a F-invariant Lagrangian horizontal torus given by $l(x) = a \cdot x + \bar{l}(x)$ (using the notation above). Then:*

If $\mathcal{T}$ is monotone positive, then $\mathcal{T}$ is minimizing.

Proof: Suppose that the graph is monotone $(+_a)$, that is, $\forall x \in \mathbb{T}^d\ B^{-1}(x)A(x) \succ 0$ (2.3). Let x_0 be the minimum of the periodic function $\bar{l}$. Then:

$$\mathrm{D}^2 l(x_0) = \mathrm{D}^2 \bar{l}(x_0) \succeq 0.$$

Hence:

$$E_1(x_0) = B^{-1}(x_0)A(x_0) + \mathrm{D}^2 l(x_0) \succ 0.$$

Finally, since E_1 is non-singular at all, it is also definite positive at all, and the graph is minimizing.

If we suppose that the graph is monotone $(+_d)$ the proof is similar. We have to take the inverse image by ϕ of the maximum of $\bar{l}$ to prove that E_2 is positive definite at all. □

Remarks:

i) We have that $\forall x \in \mathbb{T}^d$

$E_1(x) + E_2(\phi^{-1}(x)) = D(\phi^{-1}(x))B^{-1}(\phi^{-1}(x)) + B^{-1}(x)A(x) = \hat{A}(x).$

Hence, if our graph is minimizing (for instance, if our graph is monotone positive) then these matrices are all positive definite. These matrices appear on the diagonal of the second derivative of the action, and we had already obtained this result. In particular, if $\hat{A}(x,y) \preceq 0$ then there is no minimizing invariant graph through (x,y). For $d=1$, this coincides with the first step in the *Lipschitz criterion* for non-existence of invariant graphs [19, 24].

ii) Herman [14, 15] obtained an upper bound for the slope of the monotone positive i.L.g.. $y = \psi(x) = \nabla l(x)$. From the inequality

$$-B^{-1}(x)\ A(x) \prec \mathrm{D}\psi(x) \prec D(\phi^{-1}(x))\ B^{-1}(\phi^{-1}(x)),$$

which is satisfied $\forall x \in \mathbb{T}^d$, he got

$$\|\rho(\mathrm{D}\psi)\|_\infty \leq \max(\|\rho(B^{-1}A)\|_\infty, \|\rho(DB^{-1})\|_\infty)$$

where $\|\cdot\|_\infty$ means the sup-norm of a function defined on $\mathbb{T}^d$ and ρ means the spectral radius of a matrix. This is the main result of BHM theory, and we have used its proof to relate it with Converse KAM theory.

This upper bound is a bit stronger that the previous one, because we have stronger hypothesis. Herman also proved that if our exact symplectomorphism is $\mathcal{C}^1$ and monotone globally positive and the invariant torus is $\mathcal{C}^0$-Lagrangian (see [14, 15] for definitions and proofs). It is a generalization to higher dimension of a theorem due to Birkhoff for $d=1$ [7], which gives us bounds of the slope of i.r.c..

In the case $d=1$, such bounds give Lipschitz cones, and inside them there is the i.r.c.. This is the heart of the *cone-crossing* criterion for non-existence of i.r.c. performed by MacKay and Percival [19], and used firstly by Herman [13] and Mather [23]. On the other side, Newman and Percival [26] and, independently, Aubry and coworkers [4, 5] used criteria connected with action principles. In [19], they prove that both methods are equivalent.

iii) The proposition was also proved by Herman [14, 15] and MacKay, Meiss and Stark [20] using different assumptions. They need the generating function (and impose twist conditions on the symplectomorphism). We think that many results can be proven without using the existence of a global generating function satisfying some strong conditions of positivity.

iv) Notice that E_1 is $B^{-1}A$ and E_2 is DB^{-1} *after projection* of the graph on the zero-section. So then, the graph is minimizing iff when we project it on the zero-section then it is monotone positive in the two senses. $\lhd$

5.3. Non-existence criterion

In [20], they derived a *variational criterion* for the non-existence of invariant Lagrangian graphs. We can write it in the next way. We shall use the same notation as in the proposition.

> *If the orbit by z yields on a monotone positive region and has a non-minimizing segment, then it does not lie on any invariant Lagrangian horizontal torus included into such a region.*

In order to check if a segment is minimizing we shall use the MMS method (3.3.1). Of course, if we take segments of length 1 we obtain rather rude estimates.

In [20], they applied the test to the Froeschlé map [10], which is a 4D twist map (given by a Lagrangian generating function). However, we have seen that we must not be so restrictive, and we can apply their methods to other examples.

6. Examples

6.1. Generalized standard-like maps

A *generalized standard-like map* [18] is a symplectomorphism in $\mathbb{T}^d \times \mathbb{R}^d$ given by two *potentials*, $V : \mathbb{T}^d \to \mathbb{R}$ and $W : \mathbb{R}^d \to \mathbb{R}$, by means of

$$\left\{ \begin{array}{l} y' = y - \nabla V(x), \\ x' = x + \nabla W(y') \pmod 1. \end{array} \right.$$

Its inverse is given by

$$\left\{ \begin{array}{l} y = y' + \nabla V(x), \\ x = x' - \nabla W(y') \pmod 1. \end{array} \right.$$

Its primitive function is

$$S(x, y) = (y - \nabla V(x)) \cdot \nabla W(y - \nabla V(x)) - W(y - \nabla V(x)) - V(x)$$

(notice that it is 1-periodic in all its x-variables). The Jacobian matrix is

$$\mathrm{D}F(x, y) = \begin{pmatrix} I_d - \mathrm{D}^2W(y')\,\mathrm{D}^2V(x) & \mathrm{D}^2W(y') \\ -\mathrm{D}^2V(x) & I_d \end{pmatrix}.$$

Moreover, $\hat{A}(x, y) = \mathrm{D}^2W(y)^{-1} + \mathrm{D}^2W(y')^{-1} - \mathrm{D}^2V(x)$ and $\hat{B}(x, y) = -\mathrm{D}^2W(y')^{-1}$.

Examples:

1) If we choose $V(x) = 0$ we get an integrable map, and all the tori $\{y = y_0\}$ are invariant. The corresponding frequency vectors are $\omega(y_0) = \nabla W(y_0)$. The extremal character of the orbits depends on $\mathrm{D}^2W(y_0)$. If $\mathrm{D}^2W(y_0) \succ 0$ the orbits are minimizing and if $\mathrm{D}^2W(y_0) \prec 0$ the orbits are maximizing. Indefinite orbits appear when $\mathrm{D}^2W(y_0)$ is indefinite.

2) If we take $W(y) = \frac{1}{2}y^2$ we obtain a *standard-like map*, which is generated by the Lagrangian generating function
$$L(x, x') = \frac{1}{2}(x' - x)^2 - V(x)$$
and it is given by
$$\begin{cases} y' = y - \nabla V(x), \\ x' = x + y - \nabla V(x) \pmod 1. \end{cases}$$
The standard-like maps are monotone $(+_d)$, and twist. They are a discrete model of the second Newton's law, because
$$x'' - 2x' + x = -\nabla V(x').$$
Hence, an orbit is determined by its sequence of angles. $\triangleleft$

If the potential W is a strictly convex function ($\mathrm{D}^2 W(y) \succ 0$ for any point $y \in \mathbb{R}^d$) then our symplectomorphism is monotone $(+_d)$. If this is not our case, we can study the monotone positive region:
$$\mathcal{M}_+ = \{(x, y) \in \mathbb{T}^d \times \mathbb{R}^d \mid \mathrm{D}^2 W(y - \nabla V(x)) \succ 0\}.$$
Of course, we can do the same with the monotone negative region $\mathcal{M}_-$.

Applying the first step in the variational criterion, if there is an i.L.g. inside $\mathcal{M}_+$ through a point $(x, y) \in \mathcal{M}_+$, then the corresponding orbit must be minimizing and, in particular, the matrix $\hat{A}(x, y)$ must be positive definite. As any i.L.g. intersects each fiber $\{x = x_0\}$, we can pay attention in a particular one. For instance, we can take x_0 as a critical point of V, and then $\hat{A}(x_0, y) = 2\ \mathrm{D}^2 W(y)^{-1} - \mathrm{D}^2 V(x_0)$.

Of course, stronger results may be obtained by iterating, with the aid of a computer. For instance, suppose our map be positive definite. Then, we can throw out the pieces of the phase space where the points are not minimizing after a finite number of iterations. We are sure that in these pieces there are not invariant tori. On the other side, minimizing orbits not only correspond to invariant tori, but also to minimizing periodic orbits, cantori, etc. This is the philosophy in [20]. Using, for instance, techniques of interval analysis it could be possible to obtain rigorous results [19].

We shall apply the method to different generalized standard maps.

6.2. Some 2D examples

We shall consider $d = 1$, that is, an *area preserving map* (in short, *a.p.m.*), and the potential V given by
$$V(x) = -\frac{K}{(2\pi)^2}\cos(2\pi x),$$
where K is a positive perturbative parameter.

In these examples we shall apply:

- the first step in the variational criterion, in order to get rather rude estimates of the critical value of K in which all the i.r.c. have broken;

- the MMS method to a region of the phase space, in order to check in which parts of the phase space do not exist invariant tori and in which parts such existence is possible.

In order to show this second point, we have taken different values of K to see the minimizing and maximizing regions. We have taken the piece of the cylinder, $C_{[-1,1]} = \mathbb{T} \times [-1, 1]$, and we have compared:

- dynamics, taking 1024 points and iterating all of them 1024 times;
- extremal character of the orbits, applying the MMS method to segments of length 128 and drawing the initial points with a colour which depends on the number of positive eigenvalues of the corresponding Hessian matrix (the scale appears in *figure 1*);
- minimizing and maximizing regions, by choosing the corresponding points of the previous picture.

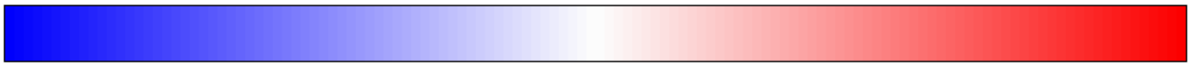

Figure 1. Scale from minimizing to maximizing character. The middle of the scale means minimaximizing character.

Examples:

1) *The standard map.* The *standard map* or *Taylor-Chirikov map* [9], has $W(y) = \frac{1}{2}y^2$ and it is monotone $(+_d)$. Then:

$$\hat{A}(x, y) = 2 - V''(x) = 2 - K\cos(2\pi x).$$

As $K > 0$, $A(x, y)$ takes his smaller value for $x = 0$, being $\hat{A}(0, y) = 2 - K$. So then, as in the first step in Mather's calculations [23]:

If $K \geq 2$, there are not any i.L.g..

If we take a segment of length 2, then we must take into account $\hat{D}_1(0, y) = 2 - K$ and

$$\hat{D}_2(0, y) = 2 - K\cos(2\pi y) - \frac{1}{2 - K}.$$

If $0 < K < 2$ then $\hat{D}_1 > 0$ and there are not any i.L.g.. if $\hat{D}_2 \leq 0$. The maximum value of $\hat{D}_2$ is taken for $y = \frac{1}{2}$ and it is $2 + K - \frac{1}{2-K}$. Finally, we improve the previous bound and we get that there are not i.L.g. if $K \geq \sqrt{3}$. We could improve the bounds taking account segments of higher length.

A better bound is obtained by improving the Lipschitz cone. For instance, Mather [23] obtained the bound $K \geq 4/3$ taking account segments of length 2 (this kind of bound was generalized by Herman [14] to higher dimensions). Later, MacKay and Percival [19] refined it to obtain $K \geq 63/64$. This refinement is an example of computer assisted proof. Finally, Jungreis [16] also performed a method for proving (computer assisted) that the standard map has no invariant circles for $K \geq 0.9718$. These bounds are in according with the result of Greene [11], who estimated (not rigorously) the bound $K > 0.971635406$, by means of the residue criterion.

Figure 2 shows how the invariant tori disappear when we increase the parameter K.

2) *The quadratic standard map.* We consider $W(y) = \frac{1}{3}y^3$. It is an example of non-monotone a.p.m., because the monotonicity condition fails on the curve $y = V'(x)$. As $K > 0$, the points over $y = V'(x)$ are monotone $(+_d)$, and the points below $y = V'(x)$ are monotone $(-_d)$. The a.p.m. has not Lagrangian generating function. As

$$\hat{A}(0, y) = \frac{1}{y} - K, \ \hat{A}(\frac{1}{2}, y) = \frac{1}{y} + K$$

then

there is no monotone $(+_d)$ i.L.g. through any point $(0, y)$ with $y \geq \frac{1}{K}$, and
there is no monotone $(-_d)$ i.L.g. though any point $(\frac{1}{2}, y)$ with $y \leq -\frac{1}{K}$.

Notice in *figure 3* that the non-monotone i.r.c. are more robust that the monotone ones, which seem to be graphs. Moreover, the non-monotone curves, which have folds, forbid the mixing between the monotone positive and monotone negative regions. When they break, the mixing is possible. Then, we see that these curves are minimaximizing, in the sense that 'half' of the eigenvalues of the Hessian matrix are positive. Almost all i.r.c. are, in fact, monotone, choosing suitable coordinates (cf. [28]). We should adapt the coordinates to dynamics.

Notice that the set of the points which go to a non-monotone one, after iteration, has measure zero. $\triangleleft$

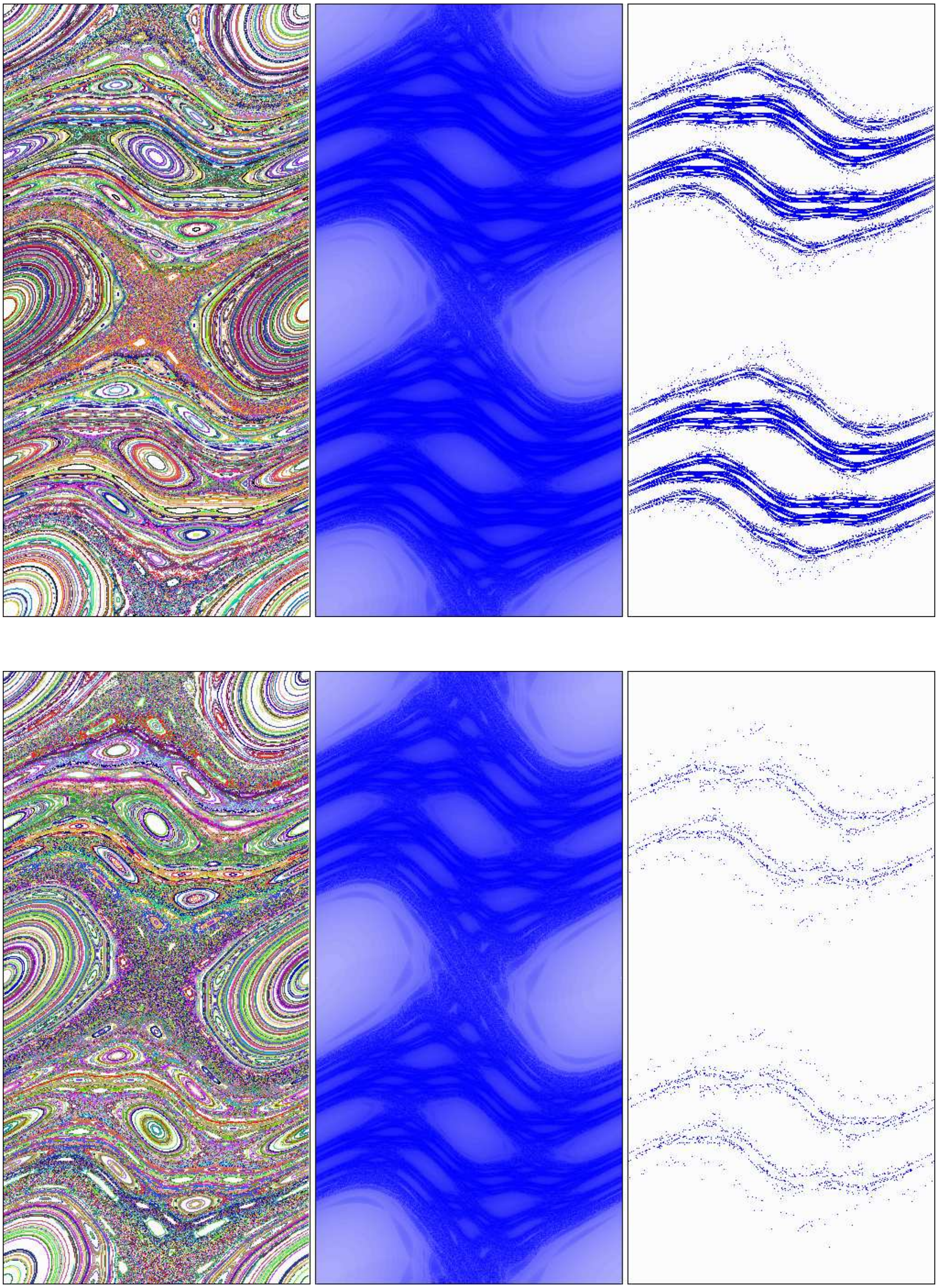

Figure 2. Destruction of the i.r.c. in the standard map, when we increase the parameter K. The values of K are $K = 0.9$ and $K = 1$, respectively. The windows mean the cylinder $\{y \in [-1, 1]\}$, where $x \in [0, 1]$ is the angle variable. Recall that the last i.r.c. breaks for $K \simeq 0.971635406$.

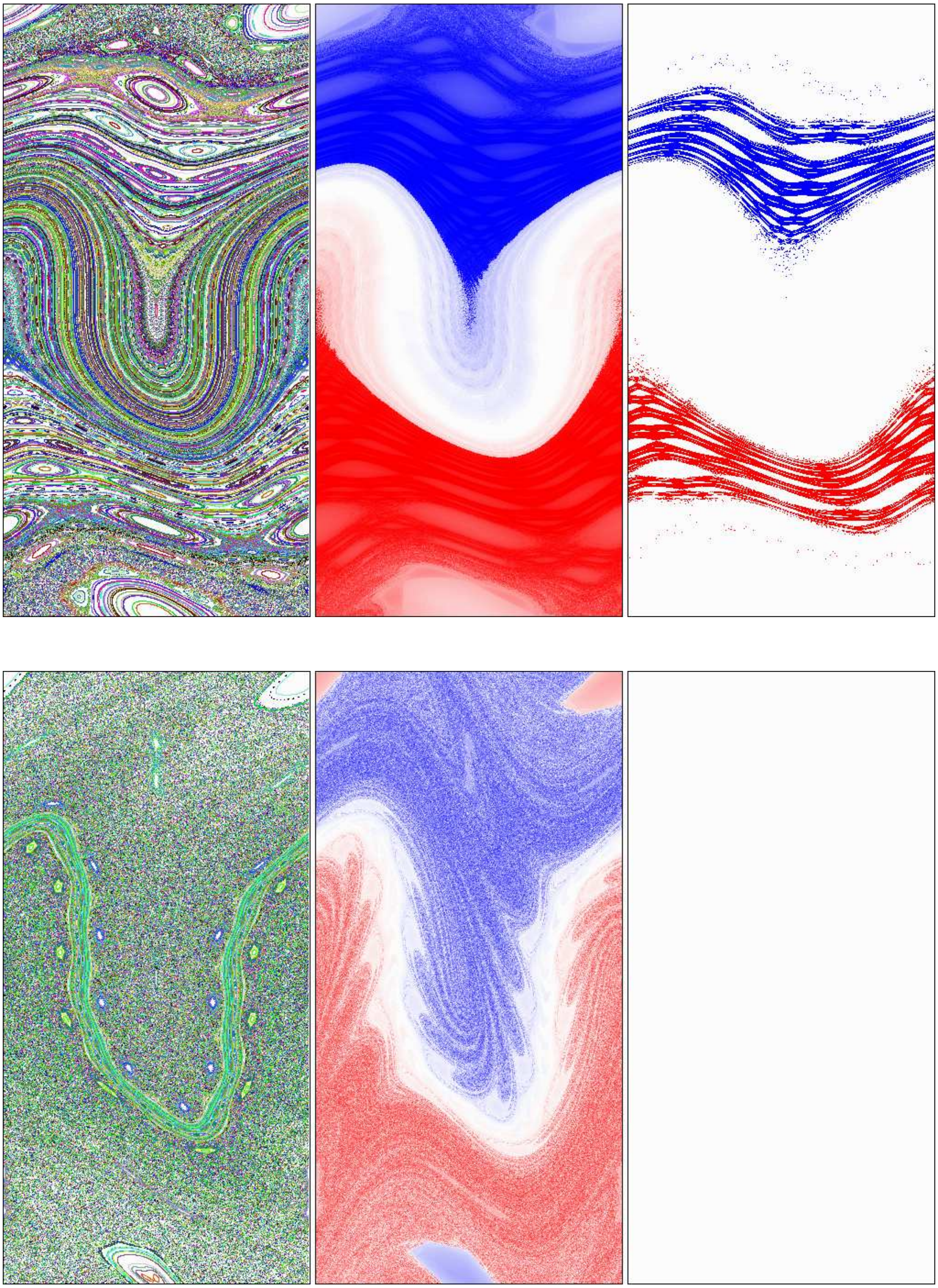

Figure 3. Evolution of i.r.c for the quadratic standard map. The parameters are $K = 0.7$ and $K = 1.5$. The windows mean the cylinder $\{y \in [-1, 1]\}$, where $x \in [0, 1]$ is the angle variable. Non-monotone i.r.c. are stronger than monotone ones, and they are 'meandering'.

6.3. Around an elliptic fixed point

We can also apply these methods to the study of a neighborhood of an elliptic fixed point, by means of suitable changes of variables. For the sake of simplicity we shall consider the 2D case.

Suppose we have a symplectomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$, being the origin an elliptic fixed point. We consider the polar symplectic change of variables

$$\begin{array}{rccl} P: & \mathbb{S}\times\mathbb{R}^*_+ & \longrightarrow & \mathbb{R}^2 \\ & (\theta, I) & \longrightarrow & (x = \sqrt{2I}\cos\theta, y = \sqrt{2I}\sin\theta). \end{array}$$

In order to do the calculations, we have to consider the differential of $P^{-1}\circ F\circ P$. Note that it is not necessary to perform the change of variables and it is enough to consider the matrices

$$M(x,y) = \begin{pmatrix} \frac{-\bar{y}}{\bar{x}^2+\bar{y}^2} & \frac{\bar{x}}{\bar{x}^2+\bar{y}^2} \\ \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} \begin{pmatrix} -y & \frac{x}{x^2+y^2} \\ x & \frac{y}{x^2+y^2} \end{pmatrix},$$

being $(\bar{x}, \bar{y}) = F(x, y)$.

Example: In *figure 4*, we consider the standard map with $K = 1$. We also show dynamics, extremal character and minimizing regions, as in the previous subsection. The extremal character and the minimizing segments of orbits (of length 128) is computed in symplectic polar coordinates.

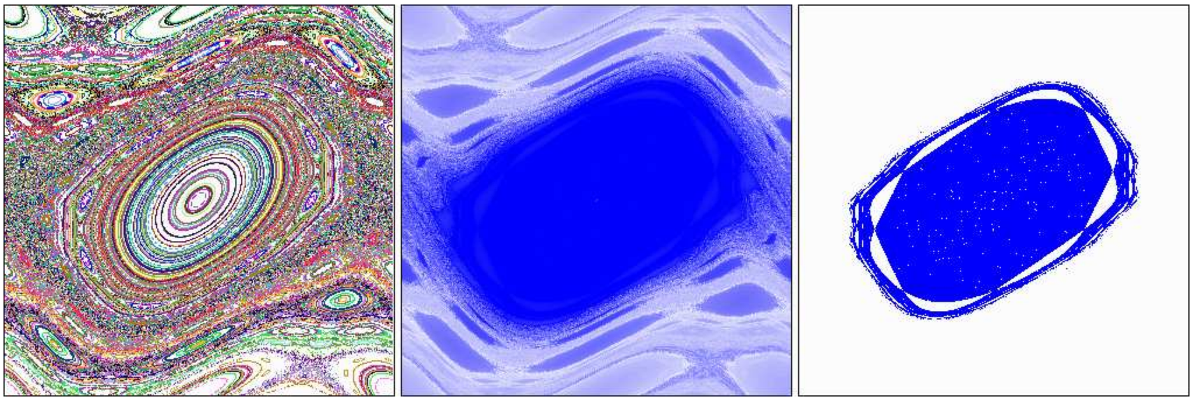

Figure 4. Around the origin, which is an elliptic fixed point of the standard map for $K = 1$. The window means the square $[-0.5, 0.5] \times [-0.5, 0.5]$.

In these pictures, the resonance zone associated to the elliptic fixed point has been roughly bounded. The resonance zone, whose points go around the elliptic one and do not escape, is bounded by an invariant curve. For a more accurate study of this 'last' invariant curve see the paper by Simó and Treschev [27]. ◁

6.4. A 4D example

Now, we consider $d = 2$ and the potential V given by

$$V(x_1, x_2) = -\frac{1}{(2\pi)^2}(K_1 \cos(2\pi x_1) + K_2 \cos(2\pi x_2) + \lambda \cos(2\pi(x_1 + x_2)).$$

We can consider different potentials W corresponding to couple (via λ) two generalized standard maps. The Froeschlé map [10] corresponds to $W(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$. For the next and other examples see [12].

Example: We have consider a *standard×exponential Froeschlé map*, which is given by $W(y_1, y_2) = \frac{1}{2}y_1^2 + \exp(y_2)$. It is a monotone positive symplectomorphism, but it is not generated by a Lagrangian generating function. Since

$$\hat{A}(0, 0, y_1, y_2) = \begin{pmatrix} 2 - (K_1 + \lambda) & -\lambda \\ -\lambda & 2e^{-y_2} - (K_2 + \lambda) \end{pmatrix},$$

there are not invariant tori if $K_1 + \lambda \geq 2$. Otherwise, there are not invariant tori through any point $(0, 0, y_1, y_2)$ with

$$y_2 \geq \log\left(\frac{2(2 - (K_1 + \lambda))}{2(K_2 + \lambda) - K_1K_2 - (K_1 + K_2)\lambda}\right).$$

In *figure 5* we show the extremal character of the points of a piece of the symmetry plane $\{x_1 = 0, x_2 = 0\}$ and we have extracted those which are minimizing.

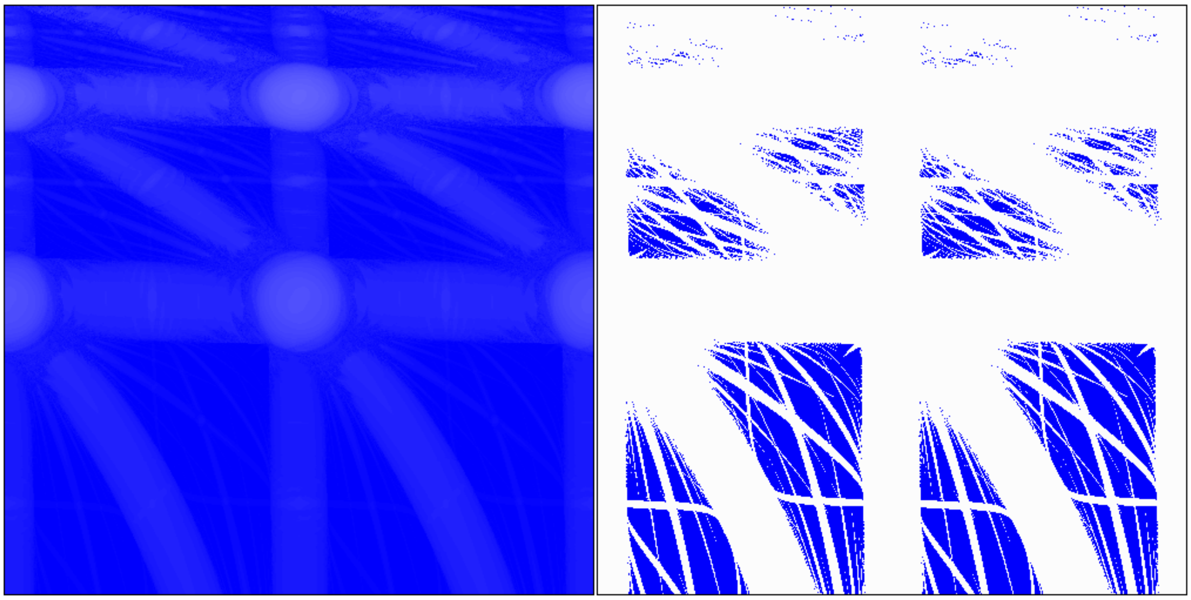

Figure 5. Box $\{(y_1, y_2) \in [-1, 1] \times [-1, 1]\}$ in the slice $\{(x_1, x_2) = (0, 0)\}$, for the standard×exponential Froeschlé map. The parameters are $K_1 = 0.1$, $K_2 = 0.2$ and $\lambda = 0.1$. Notice that the number of invariant tori decrease when we increase y_2.

The channels in second picture are similar to those obtained in [20] for the Froeschlé map. They are related to resonances associated to elliptic periodic orbits. ◁

Conclusions and perspectives

We have performed variational principles for the orbits of any symplectomorphism defined on $\mathbb{R}^d \times \mathbb{R}^d$ (or, in general, for exact symplectomorphisms defined on the cotangent bundle of a manifold). The key point is to use the primitive function and define the action on the set of chains. We have used them in order to give non-existence criteria of monotone positive invariant tori, because their orbits must be minimizing. These methods also work for a broader class of maps: the symplectic skew-products.

In order to study the break down of a certain invariant torus, we can perform some criteria similar to Greene's one [11], but instead of using the dynamical character of periodic orbits (given by their Floquet multipliers) we can use their extremal character.

For further information about these subjects see [12].

Acknowledgements

I specially thank C. Simó for his suggestions and encouragement. I would like also to thank R.S. MacKay for helpful discussions.

The research has been supported by DGICYT grant PB 94–0215 (Spain). Partial support of the EC grant ERBCHRXCT940460, and the catalan grant CIRIT 1996S0GR-00105 also is acknowledged.

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