# Quantum Effects in Lattice Models of Vector Anharmonic Oscillators

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Abstract. A lattice model of $d$-dimensional quantum anharmonic oscillators with a polynomial anharmonicity and a ferroelectric interaction is considered. For arbitrary $d \in I\!N$, it is shown that the critical fluctuations of the position operator, peculiar to the critical point, are suppressed at all temperatures provided the particles are "strongly quantum". The latter means that for the corresponding one-dimensional model, the smallest distance between the energy levels of the isolated oscillator is large enough or that the reduced mass of the oscillator is less than some threshold value depending on the anharmonicity parameters. These results are the extension to the vector case of the statements obtained recently in [**2**] and [**3**] regarding such effects in similar lattice models.

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## 1. Posing of the problem and results.

The results described in this note were obtained in the framework of the approach in quantum statistical mechanics initiated by now classical paper [**1**] published by Sergio Albeverio and Raphael Høegh–Krohn almost 25 years ago.

We consider a countable system of interacting quantum particles (oscillators) with the reduced mass $m = \mu/\hbar^2$, performing $d$-dimensional oscillations around their equilibrium positions, the set of which is a lattice $\mathbb{L}$, e.g. $\mathbb{L} = \mathbb{Z}^\nu$ with certain $\nu \in \mathbb{N}$. The dynamics of the oscillations around the position $l$ is described by means of the momentum and displacement operators $\{p_l, q_l\}$ densely defined on the complex Hilbert space $\mathcal{H}_l = L^2(\mathbb{R}^d_l, dx_l)$. The whole system is described by the formal Hamiltonian

$$H = \frac{1}{2}\sum_{l,l' \in \mathbb{L}} d_{ll'}(q_l, q_{l'}) + \sum_{l \in \mathbb{L}} H_l; \tag{1.1}$$

$$H_l = \frac{p_l^2}{2m} + V(q_l), \quad p_l^2 \stackrel{\text{def}}{=} (p_l, p_l), \tag{1.2}$$

where $( \, . \, , \, . \, )$ stands for the scalar product in $\mathbb{R}^d$ and $d_{ll'}$ form the dynamical matrix. The one-particle potential $V$ has the form

$$V(x) = v((x,x)), \quad v(\xi) = \frac{1}{2}a\xi + \sum_{s=2}^{r} b_s \xi^s, \quad a \in \mathbb{R}, \ b_s \geq 0, \ b_r > 0. \tag{1.3}$$

The formal Hamiltonian cannot be defined directly and used is "represented" by its local versions – indexed by finite subsets $\Lambda \subset \mathbb{L}$ local Hamiltonians $H_\Lambda$. For such $\Lambda$, we set $\mathcal{H}_\Lambda = \bigotimes_{l \in \Lambda} \mathcal{H}_l$. Then each $H_\Lambda$ is defined as an essentially self-adjoint lower bounded operator acting in the Hilbert space $\mathcal{H}_\Lambda$. For our purposes, it is enough to consider these Hamiltonians indexed by boxes $\Lambda = \{l = (l_1, \, \ldots, \, l_\nu) \mid l_j^0 \leq l_j \leq l_j^1, \ j = 1, \, \ldots, \nu; \ l_j^0 < l_j^1, \ l_j^0, l_j^1 \in \mathbb{Z} \ \}$ with the periodic conditions on the boundaries of $\Lambda$. Let $\mathcal{L}$ stand for the family of all such boxes. For a box $\Lambda$, let $\mathcal{L}(\Lambda)$ denote the partition of $\mathbb{L}$ by the boxes which are obtained as the translations of $\Lambda$. We also denote by $\mathcal{G}$ the group of all translations of $\mathbb{L}$. Let $\mathcal{G}(\Lambda) \subset \mathcal{G}$ be the subgroup consisting of the translations which generate the partition $\mathcal{L}(\Lambda)$. This means $\mathcal{L}(\Lambda) = \{t(\Lambda), \ t \in \mathcal{G}(\Lambda)\}$, where $t(\Lambda) = \{t(l), \ l \in \Lambda\}$. Then for a box $\Lambda$, the dynamical matrix $(d^\Lambda_{ll'})_{l,l' \in \Lambda}$ corresponding to the periodic conditions on the boundary of $\Lambda$ is defined as follows

$$d^\Lambda_{ll'} = \min\{d_{lt(l')} \ : \ t \in \mathcal{G}(\Lambda)\}. \tag{1.4}$$

Thereafter, the dynamical matrices $(d_{ll'})_{l,l' \in \mathbb{L}}$ and $(d^\Lambda_{ll'})_{l,l' \in \Lambda}$ are supposed to have the following properties

- **(D1):** $d_{ll'}$ is invariant under translations on $\mathbb{L}$;
- **(D2):** $d^\Lambda_{ll'} \leq 0$, $d_{ll'} \leq 0$ (ferroelectricity), $\quad d^\Lambda_{ll} = d_{ll} = 0$;
- **(D3):** the series $\sum_{l' \in \mathbb{L}} d_{ll'}$ converges for all $l \in \mathbb{L}$.

The mentioned local Hamiltonian $H_\Lambda$ corresponding to the periodic boundary conditions is

$$H_\Lambda = \frac{1}{2}\sum_{l,l'\in\Lambda} d^\Lambda_{ll'}(q_l, q_{l'}) + \sum_{l\in\Lambda} H_l. \tag{1.5}$$

For every $\Lambda \in \mathcal{L}$ and the inverse temperature $\beta = T^{-1}$, the periodic Gibbs state in $\Lambda$ is defined as a positive functional on appropriate $C^*$-algebra of observables as follows:

$$\gamma_{\beta,\Lambda}(A) = \frac{\mathrm{trace}(Ae^{-\beta H_\Lambda})}{\mathrm{trace}(e^{-\beta H_\Lambda})}. \tag{1.6}$$

By means of this state one can define temperature Green functions. For $\{\tau_0, \tau_1, \ldots, \tau_n\}$, $\tau_i \le \tau_{i+1}$, $\tau_i \in I_\beta := [0,\beta]$, and appropriate operators $\{A_0, A_1, \ldots, A_n\}$, the temperature Green function is

$$\begin{aligned} &\Gamma^{\beta,\Lambda}_{A_0;A_1;\ldots;A_n}(\tau_0,\tau_1,\ldots,\tau_n) \\ &\quad = \gamma_{\beta,\Lambda}\left(A_0 e^{-(\tau_1-\tau_0)H_\Lambda} A_1 e^{-(\tau_2-\tau_1)H_\Lambda} \ldots A_n e^{-(-\tau_n+\tau_0)H_\Lambda}\right). \end{aligned} \tag{1.7}$$

These functions can be obtained by using the analytic continuation of the usual Green functions corresponding to the unitary time evolution, see [**1**], [**9**]. Considering the limits $\Lambda \nearrow \mathbb{L}$ we will use collections of boxes $\mathcal{C}$, which are countable and completely ordered by inclusion. Our intention is to show that the large fluctuations of the displacements of particles in the model considered are absent (suppressed) at any temperature if these particles are "strongly quantum". The precise meaning of this will be clarified below. The first results in this direction (on the "physical level" of strictness) were published in [**10**]. Since that time the suppression of critical phenomena in lattice systems by quantum effects has been studied theoretically and experimentally in various physical publications (see [**8**], [**12**], [**13**]. The first mathematically strict results for the models with unbounded operators were obtained in [**2**], [**3**]. The research described in this note is a direct extension to the vector case of the statements from the latter paper.

Thus we restrict ourselves to the consideration of the operators which describe the mentioned fluctuations, namely

$$A^{(k)}(\Lambda) = \frac{1}{\sqrt{|\Lambda|}}\sum_{l\in\Lambda} q_l^{(k)}, \quad k = 1, 2, \ldots, d. \tag{1.8}$$

We set

$$\Gamma^{(k)}_\Lambda(\tau,\tau') = \Gamma^{\beta,\Lambda}_{A^{(k)}(\Lambda);A^{(k)}(\Lambda)}(\tau,\tau'). \tag{1.9}$$

The $O(d)$-symmetry of the model implies that such functions with different $k$ coincide with each other. Further, we set

$$U_\Lambda = \sum_{k=1}^d U^{(k)}_\Lambda, \quad U^{(k)}_\Lambda = \frac{1}{\beta}\int\int_{I^2_\beta} \Gamma^{(k)}_\Lambda(\tau,\tau') d\tau d\tau'. \tag{1.10}$$

This parameter $U_\Lambda$ describes, in some sense, variance of the fluctuations. It depends on $\beta$ and is known as local (for $|\Lambda| < \infty$) static ferroelectric susceptibility (see [**8**], [**12**]).

Definition 1.1. For given inverse temperature $\beta$ and arbitrary choice of the sequence of boxes $\mathcal{C}$, let the sequence $\{U_\Lambda, \Lambda \in \mathcal{C}\}$ be bounded. Then the critical fluctuations of the displacements of particles are absent at this temperature.

In case this sequence is convergent, one may establish analyticity properties for thermodynamic potentials describing this model. Now we are in a position to state our main result.

Theorem 1.2. *For any positive integer $d$, there exists a positive threshold value of the reduced mass $m_*$ such that for all $m < m_*$, the critical fluctuations of the displacement of particles in the $d$-dimensional model described above are absent at all temperatures.*

The proof of this statement is based upon the comparison of our vector model with the model of one–dimensional quantum oscillators described by the following local Hamiltonians

$$\tilde{H}_\Lambda = \frac{1}{2}\sum_{l,l'\in\Lambda} d^\Lambda_{ll'} q^{(1)}_l q^{(1)}_{l'} + \sum_{l\in\Lambda} \tilde{H}_l, \tag{1.11}$$

$$\tilde{H}_l = \frac{1}{2m}\left(p^{(1)}_l\right)^2 + v\left((q^{(1)}_l)^2\right), \tag{1.12}$$

where the canonical operators $p^{(1)}_l, q^{(1)}_l$ act in the space $L^2(I\!\!R_l, dx_l)$. They can be viewed as the components of the momentum and displacement operators introduced above. In the same sense we consider the operators $A^{(1)}(\Lambda)$. The other parameters of the Hamiltonians $\tilde{H}_\Lambda$, $\tilde{H}_l$ are the same as in (1.5), (1.2), and (1.3). Let the state $\tilde{\gamma}_{\beta,\Lambda}$ be defined by (1.6) with $\tilde{H}_\Lambda$ instead of $H_\Lambda$. For $0 \le \tau \le \tau' \le \beta$, we set

$$\tilde{\Gamma}^{(1)}_\Lambda(\tau,\tau') = \tilde{\gamma}_{\beta,\Lambda}\left(A^{(1)}(\Lambda)e^{-(\tau'-\tau)\tilde{H}_\Lambda}A^{(1)}(\Lambda)e^{-(\tau-\tau')\tilde{H}_\Lambda}\right), \tag{1.13}$$

$$\tilde{U}^{(1)}_\Lambda = \frac{1}{\beta}\int\int_{I^2_\beta} \tilde{\Gamma}^{(k)}_\Lambda(\tau,\tau')d\tau d\tau'. \tag{1.14}$$

Lemma 1.3. *For every $\beta$, finite $\Lambda$, and any $k = 1, 2, \ldots, d$*

$$U^{(k)}_\Lambda \le \tilde{U}^{(1)}_\Lambda. \tag{1.15}$$

In what follows, in order to have the critical $d$-dimensional fluctuations suppressed it is enough to suppress these fluctuations at the same temperature in the corresponding one–dimensional model. This conclusion is evident from the physical point of view but its mathematical justification was impossible so far, mainly due of the fact that the tool of proving here – Griffiths and Lebowitz inequalities – works only in one dimension (although the Griffiths inequality holds also for $d = 2$, it is not enough to obtain such effective estimates like that described below in Proposition 1.4, which was proved in [**3**]). Having the bound (1.15) we may use the results obtained in [**3**] for the one–dimensional quantum oscillators described by the local Hamiltonians (1.11).

The dynamical matrix $(d_{ll'})$ can be defined as a linear bounded operator $D$ on the Hilbert space $l^2(\mathbb{L})$. Since it is translation invariant (D1) and ferroelectric (D2), the series mentioned in (D3) does not depend on $l$. Thus

$$\|D\| = -\sum_{l' \in \mathbb{L}} d_{ll'}. \tag{1.16}$$

Define

$$T = \tilde{H}_l + \|D\| \left(q_l^{(1)}\right)^2, \quad l \in \mathbb{L}, \tag{1.17}$$

where $\tilde{H}_l$ is given by (1.12). The operator $T$ has purely discrete nondegenerate spectrum. We set

$$T\psi_n = E_n \psi_n, \quad n \in \mathbb{Z}_+, \tag{1.18}$$

where the eigenvalues $E_n$ are numbered in increasing order. Denote

$$\Delta = \min\{E_{n+1} - E_n \mid n \in \mathbb{Z}_+\}. \tag{1.19}$$

Proposition 1.4. [**3**] Let the mass $m$, the spectral parameter $\Delta$, and the interaction parameter $\|D\|$ satisfy the condition

$$m\Delta^2 > 2\|D\|. \tag{1.20}$$

Then for every $\beta > 0$ and arbitrary $\Lambda \in \mathcal{L}$, the following estimate holds

$$\tilde{U}_\Lambda^{(1)} \leq \frac{1}{m\Delta^2 - 2\|D\|}. \tag{1.21}$$

Thus in order to have the critical one-dimensional fluctuations suppressed for all temperatures it is enough to satisfy (1.20). In this case both one-dimensional and $d$-dimensional models may be called "strongly quantum" and the parameter appeared in the left hand side of (1.20) may be used to characterize this property quantitatively. It can be set large by means of strong hydrostatic pressure (see e.g. [**12**] or [**13**]). For the harmonic oscillator (i.e. if in (1.3) $a > 0$ and all $b_s = 0$), this parameter does not depend on $m$ (one has $m\Delta^2 = a$ which is growing with pressure), but for the anharmonic oscillator described by the Hamiltonian (1.12), (1.3) with $r \geq 2$, it does depend.

Proposition 1.5. [**3**] For the parameter $m\Delta^2$ defined by (1.19), (1.18), (1.17), (1.12), and (1.3), there exists $K > 0$ such that

$$\lim_{m \to 0+} m^{(r-1)/(r+1)} \left(m\Delta^2\right) = K. \tag{1.22}$$

Therefore, there exists $m_* > 0$ such that for $m < m_*$, the condition (1.20) is satisfied.

The proof of Theorem 1.2 easily follows from the latter three statements. The above results may be extended to the models with dynamical matrices, which are not translation invariant. For such matrices, we write $\hat{D} \leq D$ if for their elements one has $\hat{d}_{ll'} \leq d_{ll'}$ for every pair $l, l' \in \mathbb{L}$.

COROLLARY 1.6. For the model described by the Hamiltonian (1.1) - (1.3), let conditions (D2), (D3) be satisfied. Let also there exist a dynamical matrix $\hat{D}$ obeying (D1) - (D3) and such that $\hat{D} \leq D$. Then Theorem 1.2 holds for this model as well.

The proof of this statement follows from the Griffiths inequality, that will be explained in the next section. It is also worth to mention the following property of the one-dimensional models, relevant to the problems discussed here (see [**4**], [**6**]).

PROPOSITION 1.7. The local susceptibility $\tilde{U}_{\Lambda}^{(1)}$ defined by (1.14) is a monotone increasing continuous function of the reduced mass $m$.

## 2. The Proofs

For given inverse temperature $\beta$, let

$$C_\beta = \{\omega \in C(I_\beta \to \mathbb{R}^d) | \omega(0) = \omega(\beta)\},$$

and

$$\Omega_{\beta,\Lambda} = \{\omega_\Lambda = (\omega_l)_{l\in\Lambda} | \omega_l \in C_\beta\}, \quad \Lambda \in \mathcal{L}.$$

Since $\Lambda$ is finite, $\Omega_{\beta,\Lambda}$ may be equipped with the usual Banach space structure. Let $\mathcal{B}_{\beta,\Lambda}$ stand for the Borel $\sigma$–algebra of its subsets. The space $\Omega_{\beta,\Lambda}$ may be embedded densely into the following real Hilbert space

$$\mathcal{X}_{\beta,\Lambda} = \bigotimes_{l\in\Lambda} \mathcal{X}_l, \quad \mathcal{X}_l \cong \mathcal{X} = L^2(I_\beta \to \mathbb{R}^d).$$

The scalar product in the space $\mathcal{X}$ is

$$< \omega, \omega' > = \int_{I_\beta} (\omega(\tau), \omega'(\tau)) d\tau. \tag{2.1}$$

Consider the following strictly positive trace class operator on $\mathcal{X}$

$$S = (-m\Delta_\beta + 1)^{-1} \mathbf{1}, \tag{2.2}$$

where $\Delta_\beta$ is the Laplace operator in $L^2(I_\beta)$, $m$ is the reduced mass, and $\mathbf{1}$ is the identity operator in $\mathbb{R}^d$. The operator $S$ determines on $\mathcal{X}$ an isotropic Gaussian measure $\chi_\beta$ having the following Laplace transform

$$\int_{\mathcal{X}} \exp\left(< \varphi, \omega >\right) \chi_\beta(d\omega) = \exp\left(< S\varphi, \varphi >\right).$$

One has $\chi_\beta(C_\beta) = 1$ (see e.g. [**9**]), thus we can define on $\mathcal{X}$ the measure $\sigma_\beta$ which is supported on $C_\beta$ and has there the following representation

$$\sigma_\beta(d\omega) = \frac{1}{Y_\beta} \exp\left( - \int_{I_\beta} V(\omega(\tau)) d\tau \right) \chi_\beta(d\omega), \tag{2.3}$$

where $V$ is given by (1.3) and $Y_\beta$ is the normalization constant.

In the framework of the approach mentioned at the beginning of this note, the states in the boxes $\Lambda$ are defined by means of Euclidean Gibbs measures. The periodic Euclidean Gibbs measure which corresponds to the $d$–dimensional model we consider in this work is the following probability measure on the Hilbert space $\mathcal{X}_{\beta,\Lambda}$

$$\mu_{\beta,\Lambda}(d\omega_\Lambda) = \frac{1}{Z_{\beta,\Lambda}} \exp\left\{ \frac{1}{2} \sum_{l,l' \in \Lambda} d^\Lambda_{ll'} < \omega_l, \omega_{l'} > \right\} \bigotimes_{l \in \Lambda} \sigma_\beta(d\omega_l), \tag{2.4}$$

where $Z_{\beta,\Lambda}$ is the normalization constant and the periodic dynamical matrix $(d^\Lambda_{ll'})_{l,l' \in \Lambda}$ is defined by (1.4). Just in the same way one introduces the objects which describe the one-dimensional version of the model considered here. We denote them by $\tilde{C}_\beta$, $\tilde{\mathcal{X}}$, $\tilde{\Omega}_{\beta,\Lambda}$, $\tilde{\mathcal{X}}_{\beta,\Lambda}$, $\tilde{S}_\beta$, $\tilde{\chi}_\beta$. Obviously

$$C_\beta = \tilde{C}^d_\beta, \quad \mathcal{X} = \tilde{\mathcal{X}}^d, \quad \Omega_{\beta,\Lambda} = \tilde{\Omega}^d_{\beta,\Lambda}, \tag{2.5}$$

and

$$\gamma_\beta(d\omega_l) = \bigotimes_{k=1}^{d} \tilde{\gamma}_\beta(d\omega_l^{(k)}). \tag{2.6}$$

Then we set

$$\tilde{\sigma}_\beta(d\omega^{(1)}) = \frac{1}{\tilde{Y}_\beta} \exp\left\{ - \int_{I_\beta} v\left( (\omega^{(1)}(\tau))^2 \right) d\tau \right\} \tilde{\chi}_\beta(d\omega^{(1)}), \tag{2.7}$$

where $\tilde{Y}_\beta$ is the normalization constant and $v$ is given by (1.3). Thus the periodic Euclidean Gibbs measure of the one-dimensional model is

$$\tilde{\mu}_{\beta,\Lambda}(d\omega^{(1)}_\Lambda) = \frac{1}{\tilde{Z}_{\beta,\Lambda}} \exp\left\{ \frac{1}{2} \sum_{l,l' \in \Lambda} d^\Lambda_{ll'} \int_{I_\beta} \omega_l^{(1)}(\tau) \omega_{l'}^{(1)}(\tau) d\tau \right\} \bigotimes_{l \in \Lambda} \tilde{\sigma}_\beta(d\omega_l^{(1)}). \tag{2.8}$$

In our approach the Green functions (1.7), (1.9), (1.13) are written as expectations with respect to the above introduced Euclidean Gibbs measures. That is

$$\Gamma^{(k)}_\Lambda(\tau,\tau') = \frac{1}{|\Lambda|} \sum_{l,l' \in \Lambda} \mathbb{E}_{\mu_{\beta,\Lambda}} \left\{ \omega_l^{(k)}(\tau) \omega_{l'}^{(k)}(\tau') \right\}, \quad k = 1, 2, \ldots, d, \tag{2.9}$$

$$\tilde{\Gamma}^{(1)}_\Lambda(\tau,\tau') = \frac{1}{|\Lambda|} \sum_{l,l' \in \Lambda} \mathbb{E}_{\tilde{\mu}_{\beta,\Lambda}} \left\{ \omega_l^{(1)}(\tau) \omega_{l'}^{(1)}(\tau') \right\}. \tag{2.10}$$

**Proof of Lemma 1.3.** The proof is based on an infinite dimensional version of the Griffiths-Kelly-Sherman (GKS) inequality (see e.g. [**11**] p..119-124). A detailed study of this and other correlation inequalities for Euclidean Gibbs measures will be given in our forthcoming paper [**6**]. Let us consider

$$\mathbb{E}_{\mu_{\beta,\Lambda}}\left\{\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right\} = \int_{\mathcal{X}_{\beta,\Lambda}} \omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\mu_{\beta,\Lambda}(d\omega_\Lambda) =$$

$$\frac{1}{Z_{\beta,\Lambda}Y_\beta^{|\Lambda|}}\int_{\Omega_{\beta,\Lambda}} \omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\exp\{-\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'}^{\Lambda}\sum_{k=1}^{d}\int_{I_\beta}\omega_l^{(k)}(t)\omega_{\lambda'}^{(k)}(t)dt -$$

$$\frac{a}{2}\sum_{\lambda\in\Lambda}\sum_{k=1}^{d}\int_{I_\beta}\left(\omega_\lambda^{(k)}(t)\right)^2 dt - \sum_{\lambda\in\Lambda}\sum_{s=1}^{r} b_s\int_{I_\beta}\left(\sum_{k=1}^{d}(\omega_\lambda^{(k)}(t))^2\right)^s dt\}\bigotimes_{\lambda\in\Lambda}\chi(d\omega_\lambda).$$

For $\eta\in[0,1]$, we set

$$\Xi_{l,l'}^{(k)}(\eta|\tau,\tau') = \mathbb{E}_{\rho_\eta}\left\{\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right\}, \tag{2.11}$$

where the expectation is taken with respect to such measure on $\tilde{\Omega}_{\beta,\Lambda}$

$$\begin{aligned}
\rho_\eta(d\omega_\Lambda^{(k)}) &= \frac{1}{\Upsilon^{(k)}(\eta)}\exp\{-\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'}^{\Lambda}\int_{I_\beta}\omega_\lambda^{(k)}(t)\omega_{\lambda'}^{(k)}(t)dt \\
&- \frac{a}{2}\sum_{\lambda\in\Lambda}\int_{I_\beta}\left(\omega_\lambda^{(k)}(t)\right)^2 dt - \sum_{\lambda\in\Lambda}\sum_{s=1}^{r} b_s\int_{I_\beta}\left(\omega_\lambda^{(k)}(t)\right)^{2s} dt \qquad (2.12)\\
&- \eta\sum_{\lambda\in\Lambda}\sum_{n=1}^{r-1}\int_{I_\beta}\left[\left(\omega_\lambda^{(k)}(t)\right)^{2n} P_{\lambda,n}^{(k)}(t)\right]dt\}\bigotimes_{\lambda\in\Lambda}\tilde{\gamma}(d\omega_\lambda^{(k)}).
\end{aligned}$$

Here

$$\begin{aligned}
(2.13)\quad P_{\lambda,n}^{(k)} &:= \sum_{s=n+1}^{r} b_s\left(\sum_{k'\neq k}(\omega_\lambda^{(k')}(t))^2\right)^{s-n} \geq 0, \\
\Upsilon^{(k)}(\eta) &:= \int_{\tilde{\Omega}_{\beta,\Lambda}}\exp\{-\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'}^{\Lambda}\int_{I_\beta}\omega_\lambda^{(k)}(t)\omega_{\lambda'}^{(k)}(t)dt \\
&- \frac{a}{2}\sum_{\lambda\in\Lambda}\int_{I_\beta}\left(\omega_\lambda^{(k)}(t)\right)^2 dt - \sum_{\lambda\in\Lambda}\sum_{s=1}^{r} b_s\int_{I_\beta}\left(\omega_\lambda^{(k)}(t)\right)^{2s} dt \\
&- \eta\sum_{\lambda\in\Lambda}\sum_{n=1}^{r-1}\int_{I_\beta}\left\{\left(\omega_\lambda^{(k)}(t)\right)^{2n} P_{\lambda,n}^{(k)}(t)\right\}dt\}\bigotimes_{\lambda\in\Lambda}\tilde{\gamma}(d\omega_\lambda^{(k)}).
\end{aligned}$$

Taking into account (2.5) and (2.6) one obtains from (2.11)–(2.13)

$$\mathbb{E}_{\mu_{\beta,\Lambda}}\left\{\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right\} = \frac{1}{Z_{\beta,\Lambda}Y_\beta^{|\Lambda|}}\int_{\tilde{\Omega}_{\beta,\Lambda}^{d-1}} \Xi_{ll'}^{(k)}(1|\tau,\tau')\Upsilon^{(k)}(1) \tag{2.14}$$

$$\exp\{-\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'}^{\Lambda}\sum_{k'\neq k}\int_{I_\beta}\omega_\lambda^{(k')}(t)\omega_{\lambda'}^{(k')}(t)dt - \frac{a}{2}\sum_{\lambda\in\Lambda}\sum_{k'\neq k}\int_{I_\beta}\left(\omega_\lambda^{(k')}(t)\right)^2 dt$$

$$-\sum_{\lambda\in\Lambda}\sum_{s=1}^{r} b_s\int_{I_\beta}\left(\sum_{k'\neq k}(\omega_\lambda^{(k')}(t))^2\right)^s dt\}\bigotimes_{\lambda\in\Lambda}\bigotimes_{k'\neq k}\tilde{\chi}(d\omega_\lambda^{(k')}).$$

It is not difficult to check that for any choice of $k, l, l'$, and $\tau, \tau'$, $\Xi_{l,l'}^{(k)}(\eta|\tau,\tau')$ given by (2.11) is a continuous function of $\eta$ on $[0,1]$, which is differentiable on $(0,1)$, where its derivative is

$$\begin{aligned}\frac{\partial}{\partial\eta}\Xi_{l,l'}^{(k)}(\eta|\tau,\tau') = &-\sum_{\lambda\in\Lambda}\sum_{n=1}^{r-1}\int_{I_\beta} P_{\lambda,n}^{(k)}(t)\\ &\{\mathbb{E}_{\rho_\eta}\left[\left(\omega_\lambda^{(k)}(t)\right)^{2n}\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right]\\ - \; &\mathbb{E}_{\rho_\eta}\left[\left(\omega_\lambda^{(k)}(t)\right)^{2n}\right]\mathbb{E}_{\rho_\eta}\left[\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right]\}dt.\end{aligned} \tag{2.15}$$

The measure $\rho_\eta$ is such that, for every $\eta\in[0,1]$ and for any value of $P_{\lambda,n}^{(k)}(t)\geq 0$ (2.13), its moments obey the GKS-inequality

$$\mathbb{E}_{\rho_\eta}\left[\left(\omega_\lambda^{(k)}(t)\right)^{2n}\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right]\geq\mathbb{E}_{\rho_\eta}\left[\left(\omega_\lambda^{(k)}(t)\right)^{2n}\right]\mathbb{E}_{\rho_\eta}\left[\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right],$$

for every $\tau, \tau'$, and $t$. Taking into account that $P_{\lambda,n}^{(k)}(t)\geq 0$ one deduces from (2.15)

$$\Xi_{l,l'}^{(k)}(1|\tau,\tau')\leq\Xi_{l,l'}^{(k)}(0|\tau,\tau') = \mathbb{E}_{\tilde{\mu}_{\beta,\Lambda}}\left\{\omega_l^{(k)}(\tau)\omega_{l'}^{(k)}(\tau')\right\}.$$

We use the latter estimate in (2.14) and obtain for every $\tau,\tau'\in[0,\beta]$

$$\mathbb{E}_{\mu_{\beta,\Lambda}}\left\{\omega_l^{(1)}(\tau)\omega_{l'}^{(1)}(\tau')\right\}\leq\mathbb{E}_{\tilde{\mu}_{\beta,\Lambda}}\left\{\omega_l^{(1)}(\tau)\omega_{l'}^{(1)}(\tau')\right\}\int_{\tilde{\Omega}_{\beta,\Lambda}^{d-1}}\frac{\Upsilon^{(1)}(1)}{Z_{\beta,\Lambda}Y_\beta^{|\Lambda|}}$$

$$\exp\{-\frac{1}{2}\sum_{\lambda,\lambda'\in\Lambda} d_{\lambda\lambda'}^{\Lambda}\sum_{k=2}^{d}\int_{I_\beta}\omega_\lambda^{(k)}(t)\omega_{\lambda'}^{(k)}(t)dt - \frac{a}{2}\sum_{\lambda\in\Lambda}\sum_{k=2}^{d}\int_{I_\beta}\left(\omega_\lambda^{(k)}(t)\right)^2 dt -$$

$$\sum_{\lambda\in\Lambda}\sum_{s=1}^{r} b_s\int_{I_\beta}\left(\sum_{k=2}^{d}(\omega_\lambda^{(k)}(t))^2\right)^s dt\}\bigotimes_{\lambda\in\Lambda}\bigotimes_{k=2}^{d}\tilde{\chi}(d\omega_\lambda^{(k)}) = \mathbb{E}_{\tilde{\mu}_{\beta,\Lambda}}\left\{\omega_l^{(1)}(\tau)\omega_{l'}^{(1)}(\tau')\right\}.$$

Now we use the representation (2.9), (2.10) and the definitions (1.10), (1.14) and obtain the estimate (1.15). $\square$

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