Schrödinger Equation with Moving Point Interactions in Three Dimensions

G.F. Dell'Antonio, R. Figari, and A. Teta

Abstract. We consider the motion of a non relativistic quantum particle in R^3 subject to n point interactions which are moving on given smooth trajectories. Due to the singular character of the time-dependent interaction, the corresponding Schrödinger equation does not have solutions in a strong sense and, moreover, standard perturbation techniques cannot be used. Here we prove that, for smooth initial data, there is a unique weak solution by reducing the problem to the solution of a Volterra integral equation involving only the time variable. It is also shown that the evolution operator uniquely extends to a unitary operator in $L^2(R^3)$.

Dedicated to Sergio Albeverio

1. Introduction

We consider the Schrödinger equation in R^3 with an interaction supported by n points which are moving on preassigned smooth paths. More precisely let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a vector in R^n and let $y(t) = (y_1(t), \dots, y_n(t))$ be n given smooth non intersecting trajectories in R^3. For $t \in R$, let $H_{\alpha,y(t)}$ be the Schrödinger operator in $L^2(R^3)$ with point interactions supported at $y(t)$ and with strength α. We recall below the explicit definition of $H_{\alpha,y(t)}$.
We are interested in the non-autonomous evolution problem

$$i\,\frac{\partial \psi_s(t)}{\partial t} = H_{\alpha,y(t)}\psi_s(t), \qquad \psi_s(s) = f \tag{1}$$

where s is an arbitrary initial time and f is some (possibly smooth) initial datum. An existence theorem for the solution of problem (1) cannot be given using the standard theory of non-autonomous evolution problems (see e.g. [**9**]) because of the strong dependence on time of the operator domain, and in fact even of the form domain of $H_{\alpha,y(t)}$. Note that the case of point interactions at fixed positions with time-dependent strengths (see [**6**],[**8**]) is less singular since the form domain is constant.
As we shall see, problem (1) does not have solutions in a strong sense. The reason is that, even for very smooth initial datum, the solution exhibits an additional

singularity at the position of the moving points and then it does not belong to the operator domain.
In a previous paper ([**3**]) we studied the corresponding problem for the heat equation. For each $u_0 \in D(H_{\alpha,y(0)})$ we proved existence and uniqueness of a strong solution, i.e. of a function $u(t)$ belonging for each $t>0$ to $D(H_{\alpha,y(t)})$, satisfying in the L^2-sense the equation

$$\frac{\partial u(t)}{\partial t} = -H_{\alpha,y(t)}u(t), \quad u(0)=u_0 \tag{2}$$

The proof exploited the smoothing properties of the heat kernel and it cannot be generalized to the Schrödinger case.
In this paper we show that, when interpreted in a suitable weak sense, problem (1) has a unique solution.
More precisely, let $\mathcal{B}_{y(t)}(\cdot,\cdot)$ be the bilinear form associated to $H_{\alpha,y(t)}$ and let V_t be its domain (which depends on $y(t)$). Let

$$C^{\infty}_{y(t)} \equiv C^{\infty}_0(R^3 \setminus \{y(t)\}) \tag{3}$$

and notice that $C^{\infty}_{y(t)} \subset D(H_{\alpha,y(t)})$ (see (5)).
We shall prove that for all $f \in C^{\infty}_{y(s)}$ there is a unique solution of the equation

$$i\left(v(t), \frac{\partial \psi_s(t)}{\partial t}\right) = \mathcal{B}_{y(t)}(v(t)\psi_s(t)), \quad \psi_s(s)=f \tag{4}$$

for all $v(t) \in V_t$. Moreover $\psi_s(t)$ has a natural representation (see (14)).
The maps $f \to \psi_s(t)$, $s,t \in R$, are isometries and extend by continuity to unitary maps $U(t,s)$ in $L^2(R^3)$. The maps $U(t,s)$ are continuous in s,t in the strong operator topology, and therefore define a time-dependent dynamical system in $L^2(R^3)$, with generator $H_{\alpha,y(t)}$ at time t.
Notice that due to the assumptions on the initial data we do not define a flow on V_t. We conjecture however that indeed problem (4) defines a flow in V_t, continuous with respect to the Banach topology defined on V_t by the bilinear form $\mathcal{B}_{y(t)}$.
We consider the solution of problem (4) as the first step in the study of the motion of a quantum particle (e.g. a neutron) in a fluid, regarded as an assembly of n classical particles, each of which acts through a potential of very short range and therefore can be considered as a point interaction. The limit n going to infinity for the case of the heat equation was studied by us in [**2**].
The results presented here are also a preliminary step in the analysis of a class of nonlinear models in which the motion of the n classical particles is not preassigned but rather determined by the interaction with the quantum particle.

2. Definitions, motivations and statement of the results

We have denoted by $H_{\alpha,y(t)}$ the Schrödinger operator in $L^2(R^3)$ with point interactions of strength $\alpha=(\alpha_1,\dots,\alpha_n)$ placed on the points with coordinates $y(t)=(y_1(t),\dots,y_n(t))$. For the sake of simplicity we have assumed that the strengths are constant, and we shall omit them in the labels from now on. The extension to the case where also the strength of the interactions depends on time is straightforward, since this dependence on time does not alter the form domain. The operator $H_{y(t)}$ is self-adjoint, bounded below, with domain and action given respectively for each value of t by

$$D(H_{y(t)}) = \Big\{ u(t) \in L^2(R^3) \mid u(t) = \phi(t) + \sum_{k=1}^{n} q_k(t) G(\cdot - y_k(t)),$$
$$\phi(t) \in H^2_{loc}(R^3),\ \Delta\phi(t) \in L^2(R^3),\ q_1(t), \dots, q_n(t) \in C,$$
$$\lim_{|x-y_k(t)|\to 0} [u(x,t) - q_k(t) G(x - y_k(t)] = \alpha_k q_k(t),\ k = 1, \dots n \Big\} \tag{5}$$

$$H_{\alpha, y(t)} u(t) = -\Delta\phi(t) \tag{6}$$

Here $H^m(R^3)$ is the standard Sobolev space, C denotes the set of complex numbers and G is the Green's function

$$G(x - x') = (-\Delta)^{-1}(x - x') = \frac{1}{4\pi|x - x'|}, \tag{7}$$

It is clear from (5) that the operator domain consists of functions with a regular part $\phi(t)$ plus the "potential" produced by the "point charges" $q_k(t)$. The limit in (5) is regarded as a boundary condition satisfied by $u(t)$ at $y(t)$.
We refer to [**1**] for a complete analysis of this kind of hamiltonians.
Denote by $\mathcal{F}_{y(t)}$,$D(\mathcal{F}_{y(t)})$ the closed and bounded below quadratic form associated to $H_{y(t)}$ and let $\mathcal{B}_{y(t)}$ the corresponding bilinear form.
One has (see [**7**] for details)

$$D(\mathcal{F}_{y(t)}) = \{ u(t) \in L^2(R^3) \mid u(t) = \phi(t) + \sum_{k=1}^{n} q_k(t) G(\cdot - y_k(t)),$$
$$\phi(t) \in H^1_{loc}(R^3),\ |\nabla\phi(t)| \in L^2(R^3) \} \tag{8}$$
$$\mathcal{F}_{y(t)}(u(t)) = \int_{R^3} dx |\nabla\phi(x,t)|^2 + \sum_{k=1}^{n} \alpha_k |q_k(t)|^2$$
$$- \sum_{k,j=1, k\neq j}^{n} G(y_k(t) - y_j(t)) \bar{q}_k(t) q_j(t) \tag{9}$$

where $\bar{z}$ denotes the complex conjugate of $z \in C$.
Notice that $|\nabla G| \notin L^2(R^3)$, and therefore the decomposition in (8) is unique. We also emphasize that $D(\mathcal{F}_{y(t)})$ is strictly larger than the form domain of the laplacian. To simplify the notation we denote by V_t the Hilbert space $D(\mathcal{F}_{y(t)})$ equipped with the scalar product

$$< v(t), u(t) >= \mathcal{B}_{y(t)}(v(t), u(t)) + \beta(v(t), u(t)) \tag{10}$$

where $\beta > -\inf \sigma(H_{y(t)})$.
We also introduce the dual space V_t^* of V_t with respect to the L^2-scalar product and denote by $(\xi(t), \eta(t))$ the corresponding duality, $\xi(t) \in V_t$, $\eta(t) \in V_t^*$.

Finally we define the following set of smooth curves in R^3

$$\begin{aligned}&\mathcal{M} = \{y \equiv (y_1, \dots, y_n) \mid y_j \ : \ R \to R^3,\ y_j \text{ is of class } C^3, \\ &j = 1, \dots n,\ \inf_{j \neq l} \inf_{t \in R} |y_j(t) - y_l(t)| \geq a > 0\}\end{aligned} \tag{11}$$

With these notation our main results are the following.

THEOREM 2.1. *Let $y \in \mathcal{M}$, $s \in R$ and $f \in C^\infty_{y(s)}$. Then there exists a unique $\psi_s(t) \in V_t$, $t \in R$, such that $\frac{\partial \psi_s(t)}{\partial t} \in V_t^*$ and*

$$i\left(v(t), \frac{\partial \psi_s(t)}{\partial t}\right) = \mathcal{B}_{y(t)}(v(t), \psi_s(t)) \qquad \forall v(t) \in V_t \tag{12}$$

$$\psi_s(s) = f \tag{13}$$

Moreover $\psi_s(t)$ has the following representation for $t > s$

$$\psi_s(t) = U_0(t-s)f + i\sum_{j=1}^n \int_s^t d\tau\, U_0(t-\tau; \cdot - y_j(\tau)) q_j(\tau) \tag{14}$$

where $U_0(t)$ is the free unitary group defined by the kernel

$$U_0(t; x - x') = e^{i\Delta t}(x - x') = \frac{e^{i\frac{|x-x'|^2}{4t}}}{(4\pi i t)^{3/2}} \tag{15}$$

and the charges $q_j(t)$ satisfy the Volterra integral equation

$$\begin{aligned}&q_j(t) + \alpha_j \frac{4\sqrt{\pi}}{\sqrt{-i}} \int_s^t d\tau \frac{q_j(\tau)}{\sqrt{t-\tau}} + \int_s^t d\tau q_j(\tau) C_j(t,\tau) \\ &+ \sum_{l=1, l\neq j}^n \int_s^t d\tau q_l(\tau) D_{jl}(t,\tau) = \frac{4\sqrt{\pi}}{\sqrt{-i}} \int_s^t d\tau \frac{(U_0(\tau - s)f)(y_j(\tau))}{\sqrt{t-\tau}}\end{aligned} \tag{16}$$

where

$$\begin{aligned}&C_j(t,\tau) = -\frac{1}{\pi}\int_\tau^t d\sigma \frac{1}{\sqrt{t-\sigma}\sqrt{\sigma-\tau}} \left(i A_{jj}(\sigma,\tau) + \frac{dB_{jj}}{d\tau}(\sigma,\tau) \right. \\ &\left. + \frac{B_{jj}(\sigma,\tau) - 1}{2(\sigma-\tau)} \right)\end{aligned} \tag{17}$$

$$A_{jl}(t,\tau) = \frac{(y_j(t) - y_l(\tau)) \cdot \dot{y}_l(\tau)}{2(t-\tau)} \frac{1}{w_{jl}^3(t,\tau)} \int_0^{w_{jl}(t,\tau)} dz\, z^2 e^{iz^2} \tag{18}$$

$$B_{jl}(t,\tau) = \frac{1}{w_{jl}(t,\tau)} \int_0^{w_{jl}(t,\tau)} dz\, e^{iz^2} \tag{19}$$

$$w_{jl}(t,\tau) = \frac{|y_j(t) - y_l(\tau)|}{2\sqrt{t-\tau}}, \quad t > \tau \tag{20}$$

$$D_{jl}(t,\tau) = \frac{\sqrt{-2i}}{\pi} \int_\tau^t d\sigma \frac{1}{\sqrt{t-\sigma}} U_0(\sigma - \tau; y_j(\sigma) - y_l(\tau)) \tag{21}$$

A similar representation for the solution holds for $t < s$ (see Sect. 4).

Using the representation of the solution we can moreover prove

THEOREM 2.2. *The map $f \to \psi_s(t)$, $s, t \in R$, extends uniquely to a unitary map $U(t,s)$ in $L^2(R^3)$.*

The conditions we impose on the smoothness of the curves are not optimal. Optimal conditions can be found analyzing in detail the representation of the solution. We do not discuss further this problem here but notice that it may be relevant in the coupled case.

3. Some auxiliary lemmas

We shall construct the solution of (12),(13) for $t \geq s$. The case $t \leq s$ is obtained following the same steps and it is outlined in Sect. 4. We start considering $\psi_s(t)$ given by (14) for some functions $q_j(t)$.
In the following we shall drop the dependence on the initial time s.
We show first that if $q_j(t)$ and $y(t)$ are sufficiently smooth, e.g. $y \in \mathcal{M}$ and $q_j \in W^{1,1}_{loc}(R)$, then $\psi(t)$ belongs to the form domain V_t. It will also be clear that $\psi(t)$ does not belong to the operator domain even for an arbitrarly smooth $q_j(t)$.
In the second step, using the representation (14) for $\psi(t)$, we reduce the solution of (12),(13) to an integro-differential equation for $q_j(t)$.
In the third step we show that the resulting equation is in fact equivalent to the integral equation (16), which has a unique solution with the required regularity.
The first result is summarized in the following lemma

LEMMA 3.1. *Assume $y \in \mathcal{M}$ and $q_j \in W^{1,1}_{loc}(R)$, with $q_j(s) = 0$ and $f \in C^\infty_{y(s)}$. Then $\psi(t) \in V_t$, where $\psi(t)$ is given by (14).*

Proof. Expression (14) has a simpler form in the Fourier space

$$\tilde{\psi}(k,t) = e^{-ik^2(t-s)} \tilde{f}(k) + \frac{i}{(2\pi)^{3/2}} \sum_{j=1}^n \int_s^t d\tau e^{-ik^2(t-\tau)} e^{-ik \cdot y_j(\tau)} q_j(\tau) \tag{22}$$

We prove first that $\psi(t) \in L^2(R^3)$. Due to the regularity assumptions on y, q_j and f, it is sufficient to prove that

$$\int_{|k|>1} dk |\zeta_j(k,t)|^2 < \infty \tag{23}$$

where

$$\zeta_j(k,t) = \frac{i}{(2\pi)^{3/2}} \int_s^t d\tau e^{-ik^2(t-\tau)} e^{-ik\cdot y_j(\tau)} q_j(\tau) \tag{24}$$

An integration by parts yields

$$\begin{aligned} \zeta_j(k,t) &= -\frac{1}{(2\pi)^{3/2}k^2} \int_s^t d\tau e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)} \dot{q}_j(\tau) \\ &+ \frac{i}{(2\pi)^{3/2}k^2} \int_s^t d\tau k\cdot \dot{y}_j(\tau) e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)} q_j(\tau) + \frac{e^{-ik\cdot y_j(t)} q_j(t)}{(2\pi)^{3/2}k^2} \end{aligned} \tag{25}$$

The only delicate term in r.h.s. of (25) is the second. The explicit computation of its L^2-norm gives

$$\begin{aligned} &\frac{1}{(2\pi)^3} \int_{R^3} dk \frac{1}{k^4} \int_s^t d\tau \int_s^t d\sigma q_j(\tau) \bar{q}_j(\sigma) k\cdot \dot{y}_j(\tau) k \cdot \dot{y}_j(\sigma) \\ &\times e^{-ik^2(\sigma-\tau)-ik\cdot(y_j(\tau)-y_j(\sigma))} \\ &= \frac{1}{(2\pi)^3} \int_s^t d\tau \int_s^t d\sigma q_j(\tau)\bar{q}_j(\sigma) |\dot{y}_j(\tau)||\dot{y}_j(\sigma)| \int_{S^2} d\Omega(\theta,\phi) \cos\xi_j^\tau \cos\xi_j^\sigma \\ &\times \int_0^\infty dk e^{-ik^2(\sigma-\tau)-ik|y_j(\tau)-y_j(\sigma)|\cos\theta} \end{aligned} \tag{26}$$

where we have denoted by ξ_j^ν the angle between k and $\dot{y}_j(\nu)$. The last integral in (26) can be written as

$$\begin{aligned} &\int_0^\infty dk e^{-ik^2(\sigma-\tau)-ik|y_j(\tau)-y_j(\sigma)|\cos\theta} \\ &= \frac{e^{i\gamma^2}}{\sqrt{\sigma-\tau}} \int_\gamma^\infty dz e^{-iz^2}, \quad \gamma \equiv \frac{|y_j(\tau)-y_j(\sigma)|\cos\theta}{2\sqrt{\sigma-\tau}}, \quad \sigma > \tau \end{aligned} \tag{27}$$

and similarly for $\tau > \sigma$.
Using (27) one easily sees that the l.h.s. of (26) is finite and hence one concludes that $\psi(t) \in L^2(R^3)$.
Now we have to show that

$$|\nabla\phi(t)| \equiv \left| \nabla \left(\psi(t) - \sum_{j=1}^n q_j(t) G(\cdot - y_j) \right) \right| \in L^2(R^3) \tag{28}$$

¿From (22),(25) we have

$$\tilde{\phi}(k,t) = e^{-ik^2(t-s)}\tilde{f}(k) - \frac{1}{(2\pi)^{3/2}k^2}\sum_{j=1}^{n}\int_s^t d\tau \dot{q}_j(\tau)e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)}$$

$$+\frac{i}{(2\pi)^{3/2}k^2}\sum_{j=1}^{n}\int_s^t d\tau q_j(\tau)k\cdot\dot{y}_j(\tau)e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)}$$

$$\equiv \tilde{\phi}_1(k,t)+\tilde{\phi}_2(k,t)+\tilde{\phi}_3(k,t) \tag{29}$$

The smoothness of f guarantees that $|\nabla\phi_1(t)| \in L^2(R^3)$. Concerning $\phi_2(t)$ we have

$$\int_{R^3} dk k^2 |\tilde{\phi}_2(k,t)|^2$$

$$\leq c\sup_j \int_{R^3} dk k^2 \left|\frac{1}{k^2}\int_s^t d\tau \dot{q}_j(\tau)e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)}\right|^2$$

$$= c\sup_j \int_s^t d\tau \int_s^t d\sigma \dot{q}_j(\tau)\bar{\dot{q}}_j(\sigma)\int_{R^3} dk\frac{1}{k^2}e^{-ik^2(\sigma-\tau)+ik\cdot(y_j(\sigma)-y_j(\tau))} \tag{30}$$

The last integral can be explicitely computed. Using spherical coordinates and the position (20), for $\sigma > \tau$ one has

$$\int_{R^3} dk\frac{1}{k^2}e^{-ik^2(\sigma-\tau)+ik\cdot(y_j(\sigma)-y_l(\tau))}$$

$$= \frac{2\pi}{\sqrt{\sigma-\tau}w_{jl}(\sigma,\tau)}\int_0^\infty dp\frac{e^{-ip^2}}{p}\sin\left(2w_{jl}(\sigma,\tau)p\right)$$

$$= \frac{2\pi^{3/2}}{\sqrt{i}\sqrt{\sigma-\tau}}B_{jl}(\sigma,\tau) \tag{31}$$

(See e.g. [**4**]). An analogous computation holds for $\sigma < \tau$. The function $B_{jl}(t,s)$, $t > s$, has been defined in (19) and it is continuous in both variables and differentiable in the second one.
¿From (30) and (31) one easily gets the estimate for ϕ_2.
It remains to estimate $\phi_3(t)$. A further integration by parts yields

$$(2\pi)^{3/2}\tilde{\phi}_3(k,t) = \frac{i}{k^4}\sum_{j=1}^{n}\int_s^t d\tau \dot{q}_j(\tau)k\cdot\dot{y}_j(\tau)e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)}$$

$$+\frac{i}{k^4}\sum_{j=1}^{n}\int_s^t d\tau q_j(\tau)k\cdot\ddot{y}_j(\tau)e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)}$$

$$+\frac{1}{k^4}\sum_{j=1}^{n}\int_s^t d\tau q_j(\tau)\left(k\cdot\dot{y}_j(\tau)\right)^2 e^{-ik^2(t-\tau)-ik\cdot y_j(\tau)}$$

$$-\frac{i}{k^4}\sum_{j=1}^{n} q_j(t)k\cdot\dot{y}_j(t)e^{-ik\cdot y_j(t)} \tag{32}$$

The only delicate term in the r.h.s. of (32) is the third one. Proceeding as in (30), one easily sees that its gradient has a finite L^2-norm and this concludes the proof of the lemma.

Remark. From (29), (32) we get the following representation for $\tilde{\phi}(k,t)$

$$\tilde{\phi}(k,t) = \tilde{\chi}(k,t) - \frac{i}{(2\pi)^{3/2}k^4}\sum_{j=1}^{n} q_j(t) k \cdot \dot{q}_j(t) e^{-ik\cdot y_j(t)}$$

Assuming further regularity on y, q_j and using again integration by parts one easily sees that $\Delta\chi \in L^2(R^3)$ which implies $\Delta\phi \notin L^2(R^3)$.
This means that $\psi(t)$ given by (22) does not belong to $D(H_{y(t)})$, i.e. problem (1) does not have strong solutions.

In the next lemma we reduce the evolution problem to the solution of an integro-differential equation for $q_j(t)$.

Lemma 3.2. *Assume $y \in \mathcal{M}$, $q_j \in W^{1,1}_{loc}(R)$, with $q_j(s) = 0$, and $f \in C^{\infty}_{y(s)}$. Then $\psi(t)$ given by (14) solves problem (12) (13) if $q_j(t)$ solves the equation*

$$\begin{aligned} 4\pi(U_0(t-s)f)(y_j(t)) &= 4\pi\alpha_j q_j(t) - \sum_{l=1, l\neq j}^{n} \frac{q_l(t)}{|y_j(t) - y_l(t)|} \\ &+ \frac{1}{\sqrt{i\pi}}\sum_{l=1}^{n}\int_s^t d\tau \dot{q}_l(\tau)\frac{B_{jl}(t,\tau)}{\sqrt{t-\tau}} - \frac{\sqrt{i}}{\sqrt{\pi}}\sum_{l=1}^{n}\int_s^t d\tau q_l(\tau)\frac{A_{jl}(t,\tau)}{\sqrt{t-\tau}} \end{aligned} \tag{33}$$

where $B_{jl}(t,\tau)$ and $A_{jl}(t,\tau)$ are given in (19),(18).

Proof. From lemma 3.1 we know that $\psi(t) \in V_t$ and then the r.h.s. of (12) is well defined. Now we check that $\frac{\partial\psi(t)}{\partial t} \in V_t^*$. A direct computation yields (see (22),(25))

$$\begin{aligned} \frac{\partial\tilde{\psi}(k,t)}{\partial t} &= -ik^2 e^{-ik^2(t-s)}\tilde{f}(k) + i\sum_{j=1}^{n}\int_s^t d\tau e^{-ik^2(t-\tau)}\frac{d}{d\tau}\left(q_j(\tau)\frac{e^{-ik\cdot y_j(\tau)}}{(2\pi)^{3/2}}\right) \\ &= -ik^2\left[e^{-ik^2(t-s)}\tilde{f}(k) - \frac{1}{(2\pi)^{3/2}k^2}\sum_{j=1}^{n}\int_s^t d\tau e^{-ik^2(t-s)}\frac{d}{d\tau}\left(q_j(\tau)e^{-ik\cdot y_j(\tau)}\right)\right] \\ &= -ik^2\tilde{\phi}(k,t) \end{aligned} \tag{34}$$

For $v(t) \in V_t$, we write

$$v(t) = \xi^v(t) + \sum_{j=1}^{n} q_j^v(t) G(\cdot - y_j(t)) \ , \quad \xi^v(t) \in H^1(R^3) \tag{35}$$

Using (34) we have

$$\left(\xi^v(t), \frac{\partial\psi(t)}{\partial t}\right) = -i\int_{R^3} dx \nabla\bar{\xi^v}(x,t)\cdot\nabla\phi(x,t) \tag{36}$$

which is obviously finite. Moreover

$$\begin{aligned}
&\left(G(\cdot - y_j(t)), \frac{\partial\psi(t)}{\partial t}\right) = \int_{R^3} dk \frac{e^{ik\cdot y(t)}}{(2\pi)^{3/2}k^2}\frac{\partial\tilde{\psi}(k,t)}{\partial t} \\
&= \int_{R^3} dk \frac{e^{ik\cdot y_j(t)}}{(2\pi)^{3/2}k^2} \\
&\times\left[-ik^2 e^{-ik^2(t-s)}\tilde{f}(k) + \frac{i}{(2\pi)^{3/2}}\sum_{l=1}^n \int_s^t d\tau e^{-ik^2(t-\tau)}\frac{d}{d\tau}\left(q_l(\tau)e^{-ik\cdot y_l(\tau)}\right)\right] \\
&= -i(U_0(t-s)f)(y_j(t)) \\
&+\frac{i}{(2\pi)^3}\sum_{l=1}^n\int_s^t d\tau \dot{q}_l(\tau)\int_{R^3} dk\frac{1}{k^2}e^{-ik^2(t-\tau)+ik\cdot(y_j(t)-y_l(\tau))} \\
&+\frac{1}{(2\pi)^3}\sum_{l=1}^n\int_s^t d\tau q_l(\tau)\int_{R^3} dk\frac{k\cdot\dot{y}_l(\tau)}{k^2}e^{-ik^2(t-\tau)+ik\cdot(y_j(t)-y_l(\tau))}
\end{aligned} \tag{37}$$

The last integral in the k-variable can be explicitely computed. We introduce spherical coordinates $k = (r,\theta,\phi)$ with polar axis directed along $y_j(t) - y_l(\tau)$ and $\dot{y}_l(\tau) = (|\dot{y}_l(\tau)|, \hat{\theta}, 0)$. Using the formula

$$k\cdot\dot{y}_l(\tau) = r|\dot{y}_l(\tau)|\left(\cos\theta\cos\hat{\theta} + \sin\theta\sin\hat{\theta}\cos\phi\right) \tag{38}$$

we have

$$\begin{aligned}
&\int_{R^3} dk\frac{k\cdot\dot{y}_l(\tau)}{k^2}e^{-ik^2(t-\tau)+ik\cdot(y_j(t)-y_l(\tau))} \\
&= \frac{4\pi|\dot{y}_l(\tau)|\cos\hat{\theta}}{i\sqrt{t-\tau}|y_j(t)-y_l(\tau)|}\left[\int_0^\infty dp e^{-ip^2}\cos(2w_{jl}(t,\tau)p)\right. \\
&\left.-\frac{1}{2w_{jl}(t,\tau)}\int_0^\infty dp\frac{e^{-ip^2}}{p}\sin(2w_{jl}(t,\tau)p)\right] \\
&= \frac{2(\pi)^{3/2}|\dot{y}_l(\tau)|\cos\hat{\theta}}{i\sqrt{i}\sqrt{t-\tau}|y_j(t)-y_l(\tau)|}\left[e^{iw_{jl}(t,\tau)^2} - \frac{1}{w_{jl}(t,\tau)}\int_0^{w_{jl}(t,\tau)} dv e^{iv^2}\right] \\
&\frac{4(\pi)^{3/2}|\dot{y}_l(\tau)|\cos\hat{\theta}}{\sqrt{i}\sqrt{t-\tau}|y_j(t)-y_l(\tau)|}\frac{1}{w_{jl}(t,\tau)}\int_0^{w_{jl}(t,\tau)} dv v^2 e^{iv^2} \\
&= \frac{2\pi^{3/2}}{\sqrt{i}\sqrt{t-\tau}}A_{jl}(t,\tau)
\end{aligned} \tag{39}$$

where the function $A_{jl}(t,\tau)$ has been defined in (18). ¿From (37), (39) we find

$$\left(G(\cdot - y_j(t)), \frac{\partial \psi(t)}{\partial t}\right) = -i(U_0(t-s)f)(y_j(t)$$
$$+\frac{\sqrt{i}}{4\pi^{3/2}}\sum_{l=1}^{n}\int_s^t d\tau \dot{q}_l(\tau)\frac{B_{jl}(t,\tau)}{\sqrt{t-\tau}} + \frac{1}{4\sqrt{i}\pi^{3/2}}\sum_{l=1}^{n}\int_s^t d\tau q_l(\tau)\frac{A_{jl}(t,\tau)}{\sqrt{t-\tau}} \tag{40}$$

which implies $\frac{\partial \psi(t)}{\partial t} \in V_t^*$. Using (36),(40), it is now easy to check that the evolution equation (12) reduces to the integro-differential equation (33) which is satisfied by hypotheses. This concludes the proof of lemma 3.2.

4. Proof of theorem 2.1

We shall use the results of lemma 3.1 and 3.2 to complete the proof of theorem 2.1. Let us fix $y \in \mathcal{M}$ and the initial datum $f \in C^\infty_{y(s)}$. For $t > s$ we consider the equation (16) for $q_j(t)$.
It is a Volterra integral equation containing the Abel operator

$$(Lq_j)(t) \equiv \frac{1}{\sqrt{-i\pi}}\int_s^t d\tau \frac{q_j(\tau)}{\sqrt{t-\tau}} \tag{41}$$

and the integral operators

$$(C_j q_j)(t) \equiv \int_s^t d\tau q_j(\tau) C_j(t,\tau) \tag{42}$$

$$(D_{jl} q_l) \equiv \int_s^t d\tau q_l(\tau) D_{jl}(t,\tau) \qquad j \neq l \tag{43}$$

The datum of the equation

$$h_j(t) \equiv \frac{4\sqrt{\pi}}{\sqrt{-i}}\int_s^t d\tau \frac{(U_0(\tau-s)f)(y_j(\tau))}{\sqrt{t-\tau}} \tag{44}$$

is the result of the application of the Abel operator L to the function $4\pi(U_0(t-s)f)(y_j(t))$. Due to the smoothness of f it is obviously true that $h_j \in W^{1,1}_{loc}(R)$ and $h_j(s) = 0$.
By direct inspection of (17)-(20), one verifies that if $y \in \mathcal{M}$ then $C_j(t,\tau)$ is continuous in both variables and it is differentiable as a function of t.
By a detailed analysis of the expression (21) (which we omit for brevity), one can also show that D_{jl} is a bounded operator in $W^{1,1}_{loc}(R)$.
The same is true for the Abel operator (see e.g. [**5**]) and we conclude that equation (16) has a unique solution $q_j \in W^{1,1}_{loc}(R)$, with $q_j(s) = 0$.
Now we apply the Abel operator to equation (16), and make use of the fact that for η differentiable with $\eta(s) = 0$ one has

$$\frac{d}{dt}[(L)^2\eta] = i\eta, \quad \frac{d}{dt}(L\eta)(t) = (L\dot{\eta})(t) \tag{45}$$

The resulting equation reads

$$\begin{aligned} &-i(L\dot{q}_j)(t) + 4\pi\alpha_j q_j(t) - i\frac{d}{dt}(LC_jq)(t) - i\sum_{l=1,l\neq j}^{n}\frac{d}{dt}\left(LD_{jl}q_l\right)(t) \\ &= 4\pi(U_0(t-s)f)(y_j(t)) \end{aligned} \tag{46}$$

The integral operator C_j can be rewritten as

$$\begin{aligned} &(C_jq_j)(t) \\ &= -\frac{1}{\pi}\int_s^t d\sigma\frac{1}{\sqrt{t-\sigma}}\int_s^\sigma d\tau q_j(\tau)\left(\frac{iA_{jj}(\sigma,\tau)}{\sqrt{\sigma-\tau}} + \frac{d}{d\tau}\left(\frac{B_{jj}(\sigma,\tau)-1}{\sqrt{\sigma-\tau}}\right)\right) \\ &= -\frac{i}{\pi}\int_s^t d\sigma\frac{1}{\sqrt{t-\sigma}}\int_s^\sigma d\tau q_j(\tau)\frac{A_{jj}(\sigma,\tau)}{\sqrt{\sigma-\tau}} \\ &+\frac{1}{\pi}\int_s^t d\sigma\frac{1}{\sqrt{t-\sigma}}\int_s^\sigma d\tau \dot{q}_j(\tau)\frac{B_{jj}(\sigma,\tau)-1}{\sqrt{\sigma-\tau}} \end{aligned} \tag{47}$$

Using again the first equation in (45) we have

$$\begin{aligned} &\frac{d}{dt}(LC_jq_j)(t) = \frac{\sqrt{-i}}{\sqrt{\pi}}\int_s^t d\tau q_j(\tau)\frac{A_{jj}(t,\tau)}{\sqrt{t-\tau}} + i\frac{\sqrt{-i}}{\sqrt{\pi}}\int_s^t d\tau\dot{q}_j(\tau)\frac{B_{jj}(t,\tau)}{\sqrt{t-\tau}} \\ &-i\frac{\sqrt{-i}}{\sqrt{\pi}}\int_s^t d\tau\frac{\dot{q}_j(\tau)}{\sqrt{t-\tau}} \end{aligned} \tag{48}$$

Concerning the integral operator D_{jl} we have

$$\begin{aligned} &\left(LD_{jl}q_l\right)(t) = \sqrt{\frac{2}{\pi}}\int_s^t d\tau q_l(\tau)\int_\tau^t d\sigma U_0(\sigma-\tau;y_j(\sigma)-y_l(\tau)) \\ &= \frac{1}{2\pi^2}\int_s^t d\sigma\int_{R^3}dk\int_s^\sigma d\tau q_l(\tau)e^{-ik^2(\sigma-\tau)+i(y_j(\sigma)-y_l(\tau))} \\ &= \frac{i}{2\pi^2}\int_s^t d\sigma\int_s^\sigma d\tau\int_{R^3}dk\frac{1}{k^2}\frac{d}{d\tau}\left(q_l(\tau)e^{-ik\cdot y_l(\tau)}\right)e^{-ik^2(\sigma-\tau)+ik\cdot y_j(\sigma)} \\ &-\frac{i}{2\pi^2}\int_s^t d\sigma q_l(\sigma)\int_{R^3}dk\frac{1}{k^2}e^{ik\cdot(y_j(\sigma)-y_l(\sigma))} \\ &= \frac{i}{\sqrt{i}\sqrt{\pi}}\int_s^t d\sigma\int_s^\sigma d\tau\dot{q}_l(\tau)\frac{B_{jl}(\sigma,\tau)}{\sqrt{\sigma-\tau}} + \frac{1}{\sqrt{i}\sqrt{\pi}}\int_s^t d\sigma\int_s^\sigma d\tau q_l(\tau)\frac{A_{jl}(\sigma,\tau)}{\sqrt{\sigma-\tau}} \\ &-i\int_s^t d\sigma q_l(\sigma)\frac{1}{|y_l(\sigma)-y_j(\sigma)|} \end{aligned} \tag{49}$$

If we substitute (49),(48) into equation (46) we find that the charges $q_j(t)$ satisfy the integro-differential equation (33). By lemma 3.2 this means that if the $q_j(t)$ solve equation (16) then $\psi(t)$ given in (14) solves the evolution problem (12),(13) for $t > s$.

We now briefly consider the case of the backward evolution. More precisely, given the initial time $t \in R$ and $g \in C^{\infty}_{y(t)}$, we want to find $\psi_t(s) \in V_s$, for $s < t$, satisfying the equation

$$
\begin{aligned}
& i \left(v(s), \frac{\partial \psi_t(s)}{\partial s} \right) = \mathcal{B}_{y(s)}(v(s), \psi_t(s)) \quad \forall v(s) \in V_s \\
& \psi_t(t) = g
\end{aligned}
\tag{50}
$$

Again we start representing $\psi_t(s)$ as

$$
\psi_t(s) = U_0^*(t-s)g - i \sum_{j=1}^{n} \int_s^t d\tau U_0^*(\tau - s; \cdot - y_j(\tau))\tilde{q}_j(\tau) \tag{51}
$$

for some functions $\tilde{q}_j$ and then we determine $\tilde{q}_j$ in such a way that (51) solves (50). The steps are similar to the case of the forward evolution and will be omitted. We only write the integral equation which is satisfied by $\tilde{q}_j$

$$
\begin{aligned}
& \tilde{q}_j(s) + \alpha_j \frac{4\sqrt{\pi}}{\sqrt{i}} \int_s^t d\tau \frac{\tilde{q}_j(\tau)}{\sqrt{\tau - s}} + \int_s^t d\tau \tilde{q}_j(\tau) \bar{C}_j(\tau, s) \\
& + \sum_{l=1, l \neq j}^{n} \int_s^t d\tau \tilde{q}_l(\tau) \bar{D}_{jl}(\tau, s) = \frac{4\sqrt{\pi}}{\sqrt{i}} \int_s^t d\tau \frac{(U_0^*(t-\tau)g)(y_j(\tau))}{\sqrt{\tau - s}}
\end{aligned}
\tag{52}
$$

Finally the uniqueness of the solution of problem (12), (13) easily follows from the fact that for any solution of (12) the L^2-norm is conserved.

5. Unitary evolution

In this section we give the proof of theorem 2.2 following the idea developped in [**6**] for the case of point interactions at fixed positions with time-dependent strengths.
We fix $s, t \in R$ and, without loss of generality, we take $s \leq t$.
By theorem 2.1 we have existence and uniqueness of the forward evolution $\psi_s(t)$ and the backward evolution $\psi_t(s)$ for smooth initial data, denoted respectively by f and g.
Moreover the linear maps

$$
f \to \psi_s(t), \quad g \to \psi_t(s) \tag{53}
$$

are both defined on a dense set of $L^2(R^3)$ and are isometries. Then they can be uniquely extended to isometries on $L^2(R^3)$.
We shall denote them respectively by $U(t,s)$ and $U(s,t)$.
In order to prove that they are unitary maps, we have to show that the adjoints $U^*(t,s)$ and $U^*(s,t)$ are also isometries. This fact will follow from the equalities

$$U^*(t,s) = U(s,t), \quad U^*(s,t) = U(t,s) \tag{54}$$

Here we prove the first equality in (54) (the second is obtained in the same way). For the sake of clarity, it is convenient to rewrite $U(t,s)$ using various integral operators. In particular we use the Abel operator L on $L^2([s,t])$ defined in (41) and its adjoint L^*, the operator $T : L^2(R^3) \to \oplus_{j=i}^n L^2([s,t])$ defined by

$$(Th)_j(\tau) = 2\sqrt{\pi} \int_{R^3} dx U_0(\tau; x - y_j(\tau)) h(x) \tag{55}$$

and its adjoint T^*. Moreover, from (17),(21), we have

$$C_j = LR_j, \quad D_{jl} = LS_{jl} \tag{56}$$

where R_j and S_{jl} are the integral operators defined by

$$(R_j q_j)(\tau) = -\frac{1}{\sqrt{i\pi}} \int_s^\tau d\sigma \frac{q_j(\sigma)}{\sqrt{\tau-\sigma}} \left(iA_{jj}(\tau,\sigma) + \frac{dB_{jj}}{d\tau}(\tau,\sigma) + \frac{B_{jj}(\tau,\sigma)-1}{2(\tau-\sigma)} \right) \tag{57}$$

$$(S_{jl} q_j)(\tau) = -\frac{i}{\sqrt{\pi}} \int_s^\tau d\sigma q_j(\sigma) U_0(\tau-\sigma; y_j(\tau) - y_l(\sigma)) \tag{58}$$

We also introduce the corresponding operators on the space of vector valued functions $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))$, i.e.

$$(R\mathbf{q})_j(\tau) = (R_j q_j)(\tau) \tag{59}$$

$$(S\mathbf{q})_j(\tau) = \sum_{l=1, l\neq j}^n (S_{jl} q_l)(\tau) \tag{60}$$

$$(\Lambda \mathbf{q})_j(\tau) = 4\pi\alpha_j q_j(\tau) \tag{61}$$

Using the above notation we rewrite equation (16) in the form

$$\mathbf{q} + L(\Lambda + R + S)\mathbf{q} = 2\sqrt{\pi} LTU_0^*(s) f \tag{62}$$

¿From (14), (55) and (62) we obtain the following representation of $U(t,s)f$, for $s \le t$ and $f \in C^\infty_{y(s)}$

$$U(t,s)f = U_0(t-s)f + iU_0(t)T^* \left[I + L(\Lambda + R + S) \right]^{-1} LTU_0^*(s) f \tag{63}$$

The same procedure can also be applied to obtain the representation of $U(s,t)$, for $s \le t$ and $g \in C^\infty_{y(t)}$

$$U(s,t)g = U_0^*(t-s)g - iU_0(s)T^* \left[I + L^*(\Lambda + R^* + S^*) \right]^{-1} L^* TU_0^*(t) g \tag{64}$$

A straightforward computation gives

$$
\begin{aligned}
&(U^*(t,s)g,f) = (g, U(t,s)f)\\
&= (U_0^*(t-s)g,f) + i\left(U_0(s)T^*L^*\left[(I+L(\Lambda+R+S))^{-1}\right]^* TU_0^*(t)g,f\right)\\
&= (U_0^*(t-s)g,f) + i\left(U_0(s)T^*\left[I+L^*(\Lambda+R^*+S^*)\right]^{-1} L^*TU_0^*(t)g,f\right)
\end{aligned}
\tag{65}
$$

Thus, from (64), (65), we have $U^*(t,s) = U(s,t)$ on a dense set and then on $L^2(R^3)$ and this concludes the proof of the theorem.

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Dipartimento di Matematica, Universita' di Roma "La Sapienza", Italy and Laboratorio Interdisciplinare, S.I.S.S.A., Trieste, Italy.
E-mail address: `dellantonio@mat.uniroma1.it`

Dipartimento di Scienze Fisiche, Universita' di Napoli, Italy.
E-mail address: `figari@na.infn.it`

Dipartimento di Matematica, Universita' di Roma "La Sapienza", Italy.
E-mail address: `teta@mat.uniroma1.it`