Diffusive scaling of the spectral gap for the dilute Ising lattice gas dynamics below the percolation threshold

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Abstract. We consider a conservative stochastic lattice gas dynamics reversible with respect to the canonical Gibbs measure of the bond dilute Ising model on $\mathbb{Z}^d$ at inverse temperature β. When the bond dilution density p is below the percolation threshold we prove that for any particle density and any β, with probability one, the spectral gap of the generator of the dynamics in a box of side L centered at the origin scales like L^{-2}. Such an estimate is then used to prove a decay to equilibrium for local functions of the form $\frac{1}{t^{\alpha-\epsilon}}$ where ϵ is positive and arbitrarily small and $\alpha = \frac{1}{2}$ for $d = 1$, $\alpha = 1$ for $d \geq 2$. In particular our result shows that, contrary to what happens for the Glauber dynamics, there is no dynamical phase transition when β crosses the critical value β_c of the pure system.

Key Words: Kawasaki dynamics, random ferromagnet, spectral gap, equivalence of ensembles.

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1. Introduction.

In this paper we make a first attempt to analyze the relaxation time for a reversible stochastic spin exchange dynamics with random interactions. In other words we consider a simple model of a gas of interacting random walks on the lattice such that: *i)* at most one particle can sit at any given site; *ii)* the rate c_{xy} with which a particle at site x jumps to one of its nearest neighbors y, depends on the particle distribution around $x \cup y$ and on some external random field (the disorder) in such a way that the whole process is reversible w.r.t. the canonical Gibbs measure of a lattice gas with random interactions.

Our main interest is to analyze the dynamics in a finite box of side L centered at the origin as a function of L, when the thermodynamic parameters and the disorder distribution are such that one has simultaneously subsets of $\mathbb{Z}^d$ in which the jump rates are those of a gas in the high temperature phase, *i.e.* they depend very weakly on the particles configuration, and subsets where instead the jump rates are those of a gas in the phase coexistence region. In the physics literature this situation is sometimes referred to as the Griffiths phase (see e.g. [Gr] and [F]).

The simplest example of a system with the above properties is the bond diluted Ising lattice gas. In this case the grand canonical Gibbs measure takes the (formal) expression

$$\mu^{\beta,\lambda}(\sigma) = \frac{1}{Z} e^{-H^J(\sigma)+\lambda\sum_x \sigma(x)}$$

where λ is the chemical potential, the lattice gas variables $\{\sigma(x)\}_{x\in\mathbb{Z}^d}$ take values in $\{0,1\}$ and the energy function $H^J(\sigma)$ has the form $H^J(\sigma) = -\sum_{\langle x,y\rangle} J_{xy}(2\sigma(x)-1)(2\sigma(y)-1)$. Here $\langle x,y\rangle$ denotes a generic bond of the graph $\mathbb{Z}^d$ and the couplings $\{J_{xy}\}$ are i.i.d random variables taking only two values, $J_{xy}=0$ and $J_{xy}=\beta$ with probability $1-p$ and p respectively, where $\beta>0$ can be interpreted as the inverse temperature.

Let now $\beta_c = \beta_c(d)$ be the critical inverse temperature for the "pure" Ising lattice gas (*i.e.* the above model when $p=1$) and choose $\beta > \beta_c$ and $\lambda = 0$. Let also, for a given realization of the random couplings $\{J_{xy}\}$, C_x be the cluster of x, namely the set of all sites y such that there exists a path of nearest neighbor points joining x to y with the property that the coupling for each bond of the path is equal to β. Then clearly the grand canonical Gibbs measure factorizes over the different clusters and on each of them we have a grand canonical lattice gas measure with thermodynamic parameters λ and β which are in the low temperature part of the phase diagram for the "pure" infinite system. Thus in principle, depending on the geometry of the cluster, we could have the phenomenon of long range order and, in the canonical ensemble, the phenomenon of phase segregation. Concerning this problem we recall that, for the diluted Ising model, it has been proved in [G] (see also [HSS] for graphs other than $\mathbb{Z}^d$) that there exists a continuous decreasing function $\beta^* : (p_c, 1] \to [\beta_c, +\infty)$, $p_c = p_c(d)$ being the critical bond–percolation threshold in $\mathbb{Z}^d$, with $\lim_{p\to p_c^+} \beta^*(p) = +\infty$, such that if either $p \le p_c$ or $p > p_c$ and $\beta < \beta^*(p)$ then there exists a unique infinite volume grand canonical Gibbs measure, while if $p > p_c$, $\beta > \beta^*(p)$ and $\lambda = 0$ there exists more than one Gibbs state (phase transition). Actually one can show that there is a range of values of β between β_c and $\beta^*(p)$ such that not only the infinite volume Gibbs state is unique but it also has very good decay property of the covariances. It is natural to conjecture that these stronger properties hold for all $\beta < \beta^*(p)$. We refer to [ACCN] and references therein for a detailed discussion of behaviour of $\beta^*(p)$ close to p_c and to [ACCMM] and references therein for some upper bounds on $\beta^*(p)$. It should be remarked that, even when $p < p_c$ and no phase transition occurs, the presence of arbitrarily large (but finite) connected clusters of the pure system below its critical temperature affects the thermodynamics by destroying, for example, the analyticity of the free energy as a function of the chemical potential (see [Gr], [F] and also [BD], [DKP], [GM1], [GM2]).

Let us now examine some model of Markovian dynamics for the above lattice gas. There are basically two choices. Non conservative Glauber type (single spin flip) dynamics reversible w.r.t. the grand canonical Gibbs measure (see e.g. [M] for a general overview) and conservative Kawasaki dynamics reversible w.r.t. the canonical Gibbs measure.

In the non conservative case and $d = 2$ (in higher dimensions the results are somewhat less precise) it has been proved in [CMM1], [CMM2] and [ACCMM] that for the dilute Ising model described above there is a dynamical phase transition when β crosses the value β_c (but always remaining well above the critical value $\beta^*(p)$) in the following sense. If $p > p_c$ then the time–law with which the spin at the origin relaxes to its equilibrium value changes from a pure exponential e^{-t} to a slower decay of the form $e^{-te^{-(\log t)^{\frac{d-1}{d}}}}$. In the non–percolative regime $p < p_c$, for almost all coupling configurations, the relaxation to equilibrium is always exponentially fast but the rate in the exponential strongly depends on the observable one is measuring. Moreover, in both cases, the *average over the disorder* of the time auto–correlation function of the spin at the origin goes to zero as slow as $e^{-c(\log t)^{\frac{d}{d-1}}}$. One of the main reason behind such anomalous decay to equilibrium is the presence, due to the statistical fluctuations of the disorder, of cubic clusters whose relaxation time is exponentially large in the size of their boundary.

In the conservative case results were available up to now only for $\beta < \beta_c$, any λ and $p = 1$ (pure case). In this situation one of the central results is that the spectral gap of the generator of the dynamics in a box of side L and centered at the origin scales like L^{-2}. Such a result was a key input for the study of the hydrodynamical limit of the model and for the proof of the power law relaxation to equilibrium of local functions. We refer the reader to the basic references [LY], [Y], [VY], [JLQY] and, more recently, [CM2]. Actually the above result can be extended without effort also to the case $\beta < \beta_c$ and arbitrary p.
An interesting problem is whether the diffusive scaling of the spectral gap changes when $\beta \in (\beta_c, \beta^*(p))$ with either $p < p_c$ or $p > p_c$, since also for the Kawasaki dynamics the relaxation time of an isolated (that is with fixed number of particles) cubic cluster of side l may scale like $e^{cl^{d-1}}$ depending on the geometry of the cluster (see [CCM]). Let us examine the simpler case $p < p_c$. In the non conservative case it is clear that the leading contribution to the relaxation time of a local function is the largest among the relaxation times of the (finite) clusters that touch its support, simply because the Glauber dynamics factorizes over the clusters. In the conservative case this is no longer true due to the conservation of the number of particles and to the fact that different clusters exchange particles. In particular, even if a given cluster has a large relaxation time when its number of particles is kept fixed, its contribution to the global relaxation time could be not so large due to the fact that it is able to exchange particles with its complement. Moreover, when $p < p_c$, with large probability the largest cluster in a box Λ of side L and centered at the origin

has volume smaller than $c \log L$; thus its relaxation time at fixed particle number is smaller than $e^{c'(\log L)^{\frac{d-1}{d}}} << L^2$ (see theorem 5.1 below). Since we cannot hope, and this indeed the case if one takes a careful choice of a test function into the variational characterization of the spectral gap, to get a relaxation time shorter than that of the free simple exclusion process in Λ (which corresponds to take $p = 0$) which is of order L^2, we are led to the conjecture that, with large probability, the spectral gap of our lattice gas dynamics when $p < p_c$ is likely to scale as L^{-2} exactly as in the high temperature case and that a local function with support containing the origin should relax with a power law even if we average over the disorder. In particular, at least for this particular aspect of the problem, the conservative dynamics would not show any dynamical phase transition as the non conservative one. This is actually the case and its proof is the main goal of this work (see theorems 2.1, 2.2). Although we have worked out only the non percolative regime, we think that one could also cover the percolative case for suitable values of β between β_c and $\beta^*(p)$.
Before briefly discussing our approach to the proof of the main result a comment on the "neglected" parameter λ or its canonical counterpart the particle density ρ is in order. It is clear that slow relaxation to equilibrium related to the phenomenon of phase segregation can occur only when the particle density is in a certain range. In particular, for very low values of ρ depending on β, the relaxation time should scale like L^2 even when $\beta > \beta^*(p)$. Unfortunately all the existing technologies to prove such a scaling law require good mixing properties of the grand canonical Gibbs measure uniformly in the chemical potential and the problem of removing such an obstruction, related to the dynamics of anomalous fluctuations of the density profile, does not seem to be an easy one.

We conclude with a short discussion of our approach to the problem. As in the basic reference for the non conservative dynamics [CMM1], we first prove that, given $p < p_c$, $0 < \epsilon \ll 1$ and a box Λ of side L centered at the origin, with high probability the configuration of the random couplings is such that in any sub–box of Λ of side $l \in [L^\epsilon, L]$ the system has certain homogeneity properties that make it quite indistiguishable from a usual, translation invariant high temperature lattice gas. We can then apply the techniques developed in [CM1], [CM2] to get that

$$\mathrm{gap}(\Lambda) L^2 \geq \left[k + L^{-\alpha}\, \mathrm{gap}(\Lambda, L^\epsilon)^{-1}\right]^{-1}$$

where $\mathrm{gap}(\Lambda, L^\epsilon)$ is the largest among the spectral gaps in sub–boxex of Λ of side L^ϵ and k, α are suitable positive constants independent of ϵ. We then show that, if the largest cluster in Λ has size $O(\log(L))$ then $\mathrm{gap}(\Lambda, L^\epsilon)^{-1} \leq L^{10\epsilon}$ and the sought bound follows.

2. Notation and results.

In this section we first define the setting in which we will work (spin model, Gibbs measure, dynamics), and then state the main theorem on this work.

2.1 The lattice and the configuration space

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = (x_1, \ldots, x_d)$ and norms

$$|x|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p} \quad p \geq 1 \quad \text{and} \qquad |x| = |x|_\infty = \max_{i \in \{1,\ldots,d\}} |x_i| \, .$$

The associated distance functions are denoted by $d_p(\cdot, \cdot)$ and $d(\cdot, \cdot)$. By Q_L we denote the cube of all $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \ldots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball (w.r.t $d(\cdot, \cdot)$) of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \ldots, -L))$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. $[x, y]$ is the *closed segment* with endpoints x and y. The *bonds* of $\mathbb{Z}^d$ are those $e = [x, y]$ with x, y nearest neighbors in $\mathbb{Z}^d$. By an abuse of notation we will still denote by $\mathbb{Z}^d$ the associated graph. $\hat{\mathbb{F}}$ is the set of all nonempty finite subgraphs of the graph $\mathbb{Z}^d$. Given $A \in \hat{\mathbb{F}}$ we write A_v and A_b for the set of vertices and the set of bonds of A respectively. On the other hand, given Λ in $\mathbb{F}$, we will always identify Λ with the unique element $\hat{\Lambda}$ of $\hat{\mathbb{F}}$ with vertices the sites of Λ and bonds the set of all bonds of $\mathbb{Z}^d$ such that both endpoints are in Λ.

Given $\Lambda \subset \mathbb{Z}^d$ we define its interior and exterior boundaries as respectively, $\partial^- \Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq 1\}$ and $\partial^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq 1\}$, and more generally we define the boundaries of width n as $\partial_n \Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq n\}$, $\partial_n^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq n\}$.
For a fixed small positive number $\epsilon \in (0, 1)$ we define $\mathcal{R}_L^\epsilon$ be the class of parallelepipeds inside Q_L with sides parallel to the coordinate axes, longest side greater than L^ϵ and ratio between the shortest and the longest side greater than ϵ, $\mathcal{R}_L^\epsilon(l)$ be the class of all those parallelepipeds in $\mathcal{R}_L^\epsilon$ such that the longest side is smaller than l, and $\tilde{\mathcal{R}}_L^\epsilon(l)$ the class of all parallelepipeds in $\mathcal{R}_L^\epsilon(l)$ such that the shortest side is greater than L^ϵ.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{0, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \tau_V$ is the configuration on $U \cup V$ which is equal to σ on U and τ on V. Given $V \in \mathbb{F}$ we define the *number of particles* $N_V : \Omega \mapsto \mathbb{N}$ as $N_V(\sigma) = \sum_{x \in V} \sigma(x)$ while the *density* is given by $\rho_V = N_V / |V|$.
If f is a function on Ω, Δ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Δ_f}. f is called *local* if Δ_f is finite. $\mathcal{F}_\Lambda$ stands for the $\sigma-$algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{0, 1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x. Finally $\mathrm{Osc}(f) = \sup_{\sigma,\eta} |f(\sigma) - f(\eta)|$.

2.2 The dilute Ising lattice gas.

We consider an abstract probability space $(\Theta, \mathcal{B}, \mathbb{P})$ and a set of i.i.d. real valued random variables indexed by the bonds of $\mathbb{Z}^d$, $J = \{J_{xy}\}_{[x,y]\in\mathbb{Z}^d}$. $\mathbb{E}(\cdot)$ stands for the expectation with respect to $\mathbb{P}$. We assume that the couplings J_{xy} take only two values, $\beta > 0$ and 0, with probability p and $1-p$ respectively.

Given a disorder configuration we declare a bond $[x,y]$ *open* if $J_{xy} = \beta$ and *closed* otherwise. We denote by C_x the cluster of the site x, namely the set of all sites in $\mathbb{Z}^d$ which are connected to x by a path of open bonds, and by $\hat{C}_x$ the connected subgraph of $\mathbb{Z}^d$ whose vertices are the sites in C_x and whose bonds are the open bonds with endpoints in C_x. Notice that $\hat{C}_x = \{x\}$ if all the bonds with x as one endpoint are closed.

Given a disorder configuration J, for each $\sigma \in \Omega$ and $\Lambda \in \hat{\mathbb{F}}$ the Hamiltonian or energy function of the particle configuration σ in the graph Λ is given by

$$H^J_\Lambda(\sigma) = -\sum_{[x,y]\in\Lambda_b} J_{xy}(2\sigma(x)-1)(2\sigma(y)-1)$$

Given a collection of real numbers $\underline{\lambda} = \{\lambda_x\}_{x\in\mathbb{Z}^d}$ that in the sequel will be referred to as generalized chemical potential, we define $H^{J,\underline{\lambda}}_\Lambda(\sigma)$ as

$$H^{J,\underline{\lambda}}_\Lambda(\sigma) = H^J_\Lambda(\sigma) - \sum_{x\in\Lambda_v} \lambda_x \sigma(x)$$

Finally, given $\tau \in \Omega$, we let

$$H^{\tau,J,\underline{\lambda}}_\Lambda(\sigma) = H^{J,\underline{\lambda}}_\Lambda(\sigma) - \sum_{\substack{[x,y]\in\mathbb{Z}^d \\ x\in\Lambda_v,\ y\notin\Lambda_v}} J_{xy}(2\sigma(x)-1)(2\tau(y)-1)$$

and τ is called the *boundary condition.*

For each $\Lambda \in \hat{\mathbb{F}}$ and $\tau \in \Omega$ the (finite volume) grand canonical conditional Gibbs measure on $(\Omega, \mathcal{F})$, is given by

$$d\mu^{\tau,J,\underline{\lambda}}_\Lambda(\sigma) = \begin{cases} \left(Z^{\tau,J,\underline{\lambda}}_\Lambda\right)^{-1} \exp[-H^{\tau,J,\underline{\lambda}}_\Lambda(\sigma)] & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in \Lambda^c_v \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

where $Z^{\tau,J,\underline{\lambda}}_\Lambda$ is the proper normalization factor called partition function.

Warning. *In most notation we will drop the superscript J if that does not generate confusion and the superscript $\underline{\lambda}$ if $\underline{\lambda} = 0$. Moreover, for any $\Lambda \subset \mathbb{Z}^d$ we will always write $\mu_\Lambda^{\tau,\underline{\lambda}}$ instead of the more precise notation $\mu_{\hat{\Lambda}}^{\tau,\underline{\lambda}}$. Finally, if the couplings J_{xy} are constant and equal to β for all $[x,y] \in \Lambda_b$ and zero if either x or y are not in Λ_v, then we will write $\mu_\Lambda^{\beta,\underline{\lambda}}$ for the corresponding Gibbs measure. In other words $\mu_\Lambda^{\beta,\underline{\lambda}}$ is the Gibbs measure for the standard Ising model in Λ with inverse temperature β, chemical potentials $\underline{\lambda}$ and free boundary conditions.*

We will sometimes refer to this model as the grand canonical dilute Ising model with parameters β, $\underline{\lambda}$ and p.

We finally introduce the *canonical Gibbs measures* on $(\Omega, \mathcal{F})$ defined as

$$\nu_{\Lambda,N}^\tau = \mu_\Lambda^\tau(\cdot \mid N_\Lambda = N) \qquad N \in \{0, 1, \ldots, |\Lambda|\} \tag{2.2}$$

where N_Λ is the number of particles in Λ.

2.3 The dynamics

We consider the so–called Kawasaki dynamics in which particles ($\sigma(x) = 1$) can jump to nearest neighbor empty ($\sigma(x) = 0$) locations. For $\sigma \in \Omega$, let σ^{xy} be the configuration obtained from σ by exchanging the spins $\sigma(x)$ and $\sigma(y)$. Let $t_{xy}\sigma = \sigma^{xy}$ and define $(T_{xy}f)(\sigma) = f(t_{xy}\sigma)$ The stochastic dynamics we want to study is determined by the Markov generators L_Λ, Λ a connected finite subgraph of $\mathbb{Z}^d$, defined by

$$(L_\Lambda f)(\sigma) = \sum_{[x,y]\in\Lambda_b} c_{xy}(\sigma)\,(\nabla_{xy} f)(\sigma) \qquad \sigma \in \Omega\,, \quad f : \Omega \mapsto \mathbb{R} \tag{2.3}$$

where $\nabla_{xy} = T_{xy} - \mathbb{1}$. The nonnegative real quantities $c_{xy}(\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Finite range.* $c_{xy}(\sigma)$ depends only on the spins $\sigma(z)$ with $d(\{x,y\}, z) \le r$

(2) *Detailed balance.* For all $\sigma \in \Omega$ and $[x,y] \in \mathcal{E}_{\mathbb{Z}^d}$

$$\exp\bigl[-H_{\{x,y\}}(\sigma)\bigr]\, c_{xy}(\sigma) = \exp\bigl[-H_{\{x,y\}}(\sigma^{xy})\bigr]\, c_{xy}(\sigma^{xy}) \tag{2.4}$$

(3) *Positivity and boundedness.* There exist positive real numbers $c_m(\beta)$ $c_M(\beta)$ such that

$$c_m \le c_{xy}(\sigma) \le c_M \qquad \forall x, y \in \mathbb{Z}^d\,, \sigma \in \Omega\,. \tag{2.5}$$

We denote by $L_{\Lambda,N}^\tau$ the operator L_Λ acting on $L^2(\Omega, \nu_{\Lambda,N}^\tau)$ (this amounts to fix equal to $\tau_{\Lambda_v^c}$ the configuration outside Λ_v and N as the number of particles). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L_{\Lambda,N}^\tau$, and whose semigroup we denote by $(T_t^{\Lambda,N,\tau})_{t\ge0}$. $L_{\Lambda,N}^\tau$ is a bounded operator on $L^2(\Omega, \nu_{\Lambda,N}^\tau)$ and $\nu_{\Lambda,N}^\tau$ is its unique invariant measure. Moreover $\nu^\tau_{\Lambda,N}$ is *reversible* with respect to the process, *i.e.* $L^\tau_{\Lambda,N}$ is self–adjoint on $L^2(\Omega,\nu^\tau_{\Lambda,N})$.

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\mathrm{gap}(L^\tau_{\Lambda,N}) = \inf \mathrm{spec}\,(-L^\tau_{\Lambda,N} \restriction \mathbb{1}^\perp)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega,\nu^\tau_{\Lambda,N})$ orthogonal to the constant functions. We let $\mathcal{E}$ be the Dirichlet form associated with the generator $L^\tau_{\Lambda,N}$,

$$\mathcal{E}^\tau_{\Lambda,N}(f,f) = \langle f,\, -L^\tau_{\Lambda,N} f\rangle_{L^2(\Omega,\nu^\tau_{\Lambda,N})} = \frac{1}{2} \sum_{[x,y]\in\Lambda_b} \nu^\tau_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right] \tag{2.6}$$

and $\mathrm{Var}^\tau_{\Lambda,N}$ is the variance relative to the probability measure $\nu^\tau_{\Lambda,N}$. The gap can also be characterized as

$$\mathrm{gap}(L^\tau_{\Lambda,N}) = \inf_{\substack{f\in L^2(\Omega,\nu^\tau_{\Lambda,N}),\\ \mathrm{Var}^\tau_{\Lambda,N}(f)\neq 0}} \frac{\mathcal{E}^\tau_{\Lambda,N}(f,f)}{\mathrm{Var}^\tau_{\Lambda,N}(f)}\,. \tag{2.7}$$

2.4 Main results.

We are finally in a position to formulate the main results of this paper on the spectral gap of the generator of Kawasaki dynamics in a finite volume.

Let p_c denote the critical percolation for independent bond percolation in $\mathbb{Z}^d$ (see e.g. [Gri]).

Theorem 2.1. *Assume $p < p_c$. Then there exists a set $\Theta_0 \subset \Theta$ with $P(\Theta_0) = 1$ and two positive constants c_1, c_2 such that for any $J \in \Theta_0$ and any L large enough*

$$c_1 L^{-2} \le \min_{N,\tau} \mathrm{gap}(L^\tau_{Q_L,N}) \le \max_{N,\tau} \mathrm{gap}(L^\tau_{Q_L,N}) \le c_2 L^{-2} \tag{2.8}$$

A nice consequence of the above estimate is an inverse polynomial bound on the time decay to equilibrium in $L^2\left(d\nu^\tau_{\Lambda,N}\right)$ of local observables.

Theorem 2.2. *Assume $p < p_c$. Then there exists a set $\Theta_0 \subset \Theta$ with $P(\Theta_0) = 1$ such that, for any $J \in \Theta_0$, any $\delta \in (0,1)$ and any local function f with $0 \in \Delta_f$ there exists a positive constant $C_{f,\delta}$ independent of J such that for any integer $N \in \{1,\dots,(2L)^d\}$ and provided that L and t are taken large enough*

$$\mathrm{Var}^\tau_{\Lambda,N}\left(e^{tL^\tau_{\Lambda,N}} f\right) \le C_{f,\delta}\,\frac{1}{t^{\alpha-\delta}} \tag{2.9}$$

Here $\Lambda := B_L$ and $\alpha = \frac{1}{2}$ in $d = 1$, $\alpha = 1$ for $d > 1$. Furthermore there exists a positive constant $C_{f,\delta}$ such that for all L and all t

$$\mathbb{E}\left(\mathrm{Var}^\tau_{\Lambda,N}\left(e^{tL^\tau_{\Lambda,N}} f\right)\right) \le C_{f,\delta}\,\frac{1}{t^{\alpha-\delta}} \tag{2.10}$$

Remark. An analogous result was proved in [CM2] for translation invariant interaction under a suitable mixing condition. We also refer to [JLQY] where a sharper result for the zero–range process is obtained.

3. Simple large deviations for independent bond percolation.

In this section we prove some very simple large deviations results for independent bond percolation below the percolation threshold that will allow us to prove some sort of homogenization property of the dilute Ising model at large scale. The expert reader may skip the section and just look at the final result given in Corollary 3.2.

Given an integer n, let f be a real function on the set of all finite connected subgraphs of $\mathbb{Z}^d$ which is translation invariant, that is $f(A) = f(A+x)$ for all $x \in \mathbb{Z}^d$ and for all finite connected subgraph A, and such that $|f(A)| \leq |A_v|^n$ where A_v is the set of vertices of the graph A. Let, for a fixed finite set Λ and a given disorder configuration J,

$$< f >_{\Lambda,J} = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(\hat{C}_x) \tag{3.1}$$

and let $\bar{f} := \mathbb{E}(f(C_0))$ provided that $\mathbb{E}(|C_0|^n) < \infty$.

Proposition 3.1. *Assume $p < p_c$ and let $\epsilon_0 = \frac{1}{2d(n+1)+1}$ where n is the integer governing the growth of f. Let Λ be a parallelepiped with ratio between the shortest and longest side greater than ϵ. Then for any $\epsilon \in (0, \epsilon_0)$ there exist constants $0 < \delta = \delta(n, p, \epsilon) < 1$, $m_1 = m_1(p) > 0$ and $m_2 = m_2(p, n) > 0$ such that*

$$a) \qquad \mathbb{P}\left(\sup_{x \in \Lambda} |C_x| \geq v \right) \leq |\Lambda| e^{-m_1 v}, \qquad \forall v \geq 0$$

$$b) \qquad \mathbb{P}(| < f >_{\Lambda,J} - \bar{f} \,| \geq |\Lambda|^{-\epsilon}) \leq e^{-m_2 \, |\Lambda|^\delta}$$

Proof.

Part *a)* follows immediately from the exponential tail of the cluster size distribution below p_c (see e.g. [Gri]).

To prove part *b)* we first observe that, because of part a), $\bar{f}$ is well defined. Next, given $\epsilon \in (0, \epsilon_0)$, let $l_1 = |\Lambda|^\delta$, $l_2 = |\Lambda|^{\delta(n+1)+2\epsilon}$ where $\delta = \frac{1-\epsilon[2d(n+1)+1]}{1+d(n+1)^2}$.

Let also Θ_1 be the event that $| < f >_{\Lambda,J} - \bar{f} \,| \geq |\Lambda|^{-\epsilon}$ and Θ_2 be the event that $\sup_{x \in \Lambda} |C_x| \leq l_1$. Then we write

$$\mathbb{P}(\Theta_1) \leq \mathbb{P}(\Theta_1 \cap \Theta_2) + \mathbb{P}(\Theta_2^c) \tag{3.2}$$

The term $\mathbb{P}(\Theta_2^c)$ is bounded from above by

$$\mathbb{P}(\Theta_2^c) \leq |\Lambda| e^{-m_1 l_1} \leq e^{-m_2' |\Lambda|^\delta} \tag{3.3}$$

because of *a)*.

In order to estimate the first term in the r.h.s. of (3.2), consider the maximal collection of cubes $\{Q_\alpha\}$ in Λ such that: *i)* $\text{dist}(Q_\alpha, Q_\beta) = 2l_1$ for $\alpha \neq \beta$ and *ii)* $|Q_\alpha| = l_2^d$. Let N be the number of such cubes. Clearly $|\Lambda \setminus \cup_\alpha Q_\alpha| \leq C' \frac{l_1}{l_2} |\Lambda|$ for a suitable constant C' so that $|\Lambda| \geq N l_2^d \geq |\Lambda|(1 - C' \frac{l_1}{l_2})$. Therefore, if $\sup_{x \in \Lambda} |C_x| \leq l_1$

$$| < f >_{\Lambda,J} - \bar{f} \,| \leq \frac{1}{|\Lambda|} | \sum_\alpha \xi_\alpha \,| + C'' \frac{1}{|\Lambda|^{2\epsilon}}$$

for a suitable constant C'', where $\xi_\alpha = \sum_{\substack{x\in Q_\alpha\\ C_x\subset Q_\alpha}} \left[f(\hat C_x) - \bar f\right]$. Thus, if $|\Lambda|$ is large enough, we get that

$$\mathbb{P}(\Theta_1\cap\Theta_2) \le \mathbb{P}(|\sum_\alpha \xi_\alpha| \ge \frac{|\Lambda|^{1-\epsilon}}{2}) \le e^{-t|\Lambda|^{1-\epsilon}} \left(\mathbb{E}\left(e^{2t\xi_\alpha}\right)\right)^N \tag{3.4}$$

for any positive t, because of the exponential Chebyshev inequality and the fact that the random variables $\{\xi_\alpha\}$ are i.i.d.

We now choose $t = \frac{1}{4(l_2^{d(n+1)}+|\bar f|l_2^d)}$ and observe that with this choice

$$\mathbb{E}(e^{2t\xi_\alpha}) \le e^{2\sqrt{e}\,t^2\mathbb{E}(\xi_a^2)}$$

Let us estimate the second moment of ξ_α.

We denote by $\mathcal{A}_x$ the set of all connected subgraphs of $\mathbb{Z}^d$ that contain x. Clearly $\mathbb{P}\left(\{C_x = A\}\cap\{C_y = B\}\right) = \mathbb{P}(C_x = A)\,\mathbb{P}(C_y = B)$ for all $A\in\mathcal{A}_x$, $B\in\mathcal{A}_y$ such that there is no edge with one endpoint in A and the other in B.

With these notations we write

$$\begin{aligned}
\mathbb{E}(\xi_\alpha^2) &= \sum_{x,y\in Q_\alpha}\sum_{\substack{A\in\mathcal{A}_x\\ A\subset Q_\alpha}}\sum_{\substack{B\in\mathcal{A}_y\\ B\subset Q_\alpha}} f(A)\,f(B)\left[\mathbb{P}\left(\{C_x = A\}\cap\{C_y = B\}\right) - \mathbb{P}\left(C_x = A\right)\mathbb{P}\left(C_y = B\right)\right] \\
&\le \sum_{x,y\in Q_\alpha}\sum_{\substack{A\in\mathcal{A}_x\\ A\subset Q_\alpha}}\sum_{\substack{B\in\mathcal{A}_y\\ B\subset Q_\alpha}} |f(A)|\,|f(B)|\chi(\mathrm{dist}(A,B)\le 1)[\mathbb{P}\left(\{C_x = A\}\cap\{C_y = B\}\right) \\
&\quad + \mathbb{P}\left(C_x = A\right)\mathbb{P}\left(C_y = B\right)] \\
&\le \sum_{x,y\in Q_\alpha} \mathbb{E}\left(|f(C_x)|\,|f(C_y)|\,\chi(\mathrm{dist}(C_x,C_y)\le 1)\right) \\
&\quad + \sum_{x,y\in Q_\alpha}\sum_{\substack{A\in\mathcal{A}_x\\ A\subset Q_\alpha}} |f(A)|\,\mathbb{P}\left(C_x = A\right)\mathbb{E}\left(|f(C_y)|\chi(\mathrm{dist}(A,C_y)\le 1)\right) \\
&\le k_2 l_2^d
\end{aligned}$$

for a suitable positive constant k_2, because as we are below the percolation threshold $\mathbb{E}(f(C_0)^4) < \infty$ and $\mathbb{P}(x\leftrightarrow x') \le e^{-md(x,x')}$, where $m = m(p)$ is a positive constant and $x\leftrightarrow x'$ means that x and x' are connected by a path of open bonds (see [Gri]). Thus, with the above choice of t and δ,

$$\text{r.h.s. of (3.4)} \;\le e^{-\left[t|\Lambda|^{-\epsilon} - k_3 t^2\right]|\Lambda|} \le e^{-m_2''|\Lambda|^\delta}$$

for a suitable constant m_2'' provided that $|\Lambda|$ is large enough. Putting together the above estimate and (3.3) the proof is concluded. $\square$

Here is a simple consequence of the above large deviation results.

Given $\epsilon\in(0,\epsilon_0)$, an integer L and N real, translation invariant functions $\{f_i\}_{i=1}^N$ on the set of all finite connected subgraphs of of $\mathbb{Z}^d$ such that $\max_{i\le N}|f_i(A)| \le |A_v|^4$ for any A, let us consider the event $\Theta(\epsilon, M, N, L) = \cup_{R\in\mathcal{R}_L^\epsilon}\Theta_R(\epsilon, M, N, L)$ where

$$\Theta_R(\epsilon, M, N, L) =$$
$$\{\sup_{x\in R}|C_x| \geq M\log L\}\cup\{\sup_{n\leq 4}\sum_{x\in R}|C_x|^n \geq M|R|\}\cup\{\sup_{i\leq N}|\ < f_i >_{R,J} - \bar{f}_i\,| \geq |R|^{-\epsilon}\}$$

Then

Corollary 3.2. *Assume $p < p_c$. There exists M such that for any $a < \infty$ $\sum_{1=L}^{\infty}\sup_{N\leq L^a}\mathbb{P}\left(\Theta(\epsilon, M, N, L)\right) < +\infty$.*

Proof. It follows at once from proposition 3.1 applied to each f_i and to the functions $g_n(A) = |A_v|^n$, $n = 1, \ldots, 4$ and the fact that the cardinality of $\mathcal{R}_L^\epsilon$ is bounded from above by L^{2d}. $\square$

4. Preliminary results

In this section we collect several preliminary results that are essential in order to prove that, with large probability, the spectral gap of the Kawasaki dynamics for the dilute Ising model below p_c, on all scales between L^ϵ to L can be bounded from below by exactly the same methods employed in the high temperature region. More precisely we will formulate three conditions on the disorder configuration in the cube Q_L which will ensure that, if satisfied, the corresponding dilute Ising model shares all the relevant (for our purposes) features of the high temperature standard Ising model. Moreover our conditions will be meaningful in the sense that the probability of not being all verified simultaneously will be summable in L.

Let $\epsilon_0 = \frac{1}{10d+1}$ and let us fix a small positive number $\epsilon \in (0, \epsilon_0)$ and a large positive number M. For any $\Lambda \subset \mathbb{Z}^d$, any integer $N \in \{1, 2, \ldots, |\Lambda|\}$, any boundary condition $\tau \in \Omega$ and any disorder configuration J let also $\lambda = \lambda(\Lambda, N, \tau, J)$ be the (unique) constant chemical potential such that $\mu_\Lambda^{J,\tau,\lambda}(N_\Lambda) = N$ and let $\lambda_0 = \lambda_0(\Lambda, N)$ be such that $\mathbb{E}\left(|\hat{C}_0|^{-1}\mu_{\hat{C}_0}^{\beta,\lambda_0}(N_{C_0})\right) = \frac{N}{|\Lambda|}$, taht is the particle density of the cluster of the origin averaged on the disorder is equal to $N/|\Lambda|$. The existence and uniqueness of the chemical potential λ is proved in the appendix of [CM1], for λ_0 a similar reasoning can be applied. Then our conditions read as follows.

Assumption 1. *For any $R \in \mathcal{R}_L^\epsilon$*

$$\max_{x\in\Lambda}|C_x| \leq M\,\log L \qquad \text{and} \qquad \max_{n\leq 4}\sum_{x\in R}|C_x|^n \leq M\,|R|$$

Assumption 2. *Let $h_x := e^{-\nabla_x H}\sigma(x)$. Then, for any $R \in \mathcal{R}_L^\epsilon$,*

$$\sup_\tau\sup_{N\in[\epsilon|R|,(1-\epsilon)|R|]}|\,\mu_R^{\tau,\lambda_0}(N_R, \sum_{x\in R}h_x) - |R|\,\mathbb{E}\left(|C_0|^{-1}\mu_{\hat{C}_0}^{\beta,\lambda_0}(N_{C_0}, \sum_{x\in C_0}h_x)\right)\,| \leq |R|^{1-\epsilon}$$

Similarly for $\hat{h}_x := e^{-\nabla_x H}(1-\sigma(x))$ and $\tilde{h}_x = \sigma(x)$.

Assumption 3. *For any $R \in \mathcal{R}_L^\epsilon$*

$$\sup_\tau\sup_{N\in[\epsilon|R|,(1-\epsilon)|R|]}|\,\lambda(R, N, \tau, J) - \lambda_0(R, N)| \leq |R|^{-\epsilon}$$

Definition. *The set of disorder configurations J that satisfy assumptions 1, 2 and 3 will be denoted by $\Theta_{good}(L,M,\epsilon)$.*

Thanks to Corollary 3.2 we have the following result.

Proposition 4.1. *Assume $p<p_c$. Then*

(i) there exists M such that for any $\epsilon\in(0,\epsilon_0)$ $\sum_{L=1}^{\infty}\mathbb{P}\left(\Theta_{good}(L,M,\epsilon)^c\right)<\infty$.
In particular, for any large enough M and any $\epsilon\in(0,\epsilon_0)$

(ii) there exists a set $\Theta_0\subset\Theta$ such that $\mathbb{P}(\Theta_0)=1$ and for any $J\in\Theta_0$ there exists $L(J)$ such that $J\in\Theta_{good}(L,M,\epsilon)$ for any $L\geq L(J)$;

(iii) there exists $\gamma=\gamma(M)>0$, $\lim_{M\to+\infty}\gamma(M)=+\infty$, such that $\mathbb{P}(L(J)>l)\leq l^{-\gamma}$.

Proof. Once point (i) of the proposition is established point (ii) is nothing but the standard Borel Cantelli lemma.

To analyze the convergence of the series $\sum_L\mathbb{P}\left(\Theta_{good}(L,M,\epsilon)^c\right)$ we first observe that, thanks to proposition 3.1, the probability that assumption 1 is violated can be bounded from above by $c\,|\mathcal{R}_L^\epsilon|\,(L^{-mM}+e^{-m_2(\epsilon L^\epsilon)^{d\delta}})$ (where m is a positive constant depending on p and n and we used $|R|\geq(\epsilon L^\epsilon)^d$). In order to compute the probability that assumption 2 is violated in $R\in\mathcal{R}_L^\epsilon$ we define for any $A\in\hat{\mathbb{F}}$ the function $f_N(A)=|A|^{-1}\mu_A^{\beta,\lambda_0(N)}(N_A,\sum_{x\in A_v}h_x)$ for $N\in[\epsilon|R|,(1-\epsilon)|R|]$, similarly we define $\hat{f}_N(A)$ and $\tilde{f}_N(A)$ when we have $\hat{h}$ or $\tilde{h}$. With this notation and using the fact that μ_R^{τ,J,λ_0} is the product measure over the clusters in R, we can write

$$
\begin{aligned}
&|\,\mu_R^{\tau,J,\lambda_0(N)}(N_R,\sum_{x\in R}h_x)-|R|\,\mathbb{E}\left(|C_0|^{-1}\,\mu_{\hat{C}_0}^{\beta,\lambda_0}(N_{C_0},\sum_{x\in C_0}h_x)\right)|\\
&\qquad\leq|\sum_{x\in R}(f(C_x)-\bar{f})\,|+\sum_{\substack{x\in R:\\ C_x\cap R^c\neq\emptyset}}|\,\mu_{\hat{C}_x\cap\hat{R}}^{\tau,\beta,\lambda_0}(N_{C_x},\sum_{y\in C_x}h_y)\,|\\
&\qquad\leq|\sum_{x\in R}(f(C_x)-\bar{f})\,|+CL^{d-1}\sup_{x\in R}|C_x|^3
\end{aligned}
$$

for a suitable constant $C=C(d,\beta)$. We have used here the fact that any $R\in\mathcal{R}_L^\epsilon$ has surface smaller than $C''L^{d-1}$. A similar computation holds for $\hat{f}_N$ and $\tilde{f}_N$. We can at this point use proposition 3.1 and the fact that $|R|\geq(\epsilon L^\epsilon)^d$ to conclude that the probability that assumption 2 is violated can be bounded from above by $c\,|\mathcal{R}_L^\epsilon|\,|R|\,e^{-(m_2\epsilon L^\epsilon)^{d\delta}}$.

We are left with the analysis of the last assumption.

Let us fix $R\in\mathcal{R}_L^\epsilon$. Without loss of generality we can assume that $\sup_{x\in R}|C_x|\leq M\log L$ for some large, fixed M, and that, for any $N\in[\epsilon|R|,(1-\epsilon)N]$, $|\,\mu_R^{\tau,\lambda_0(N)}(N_R)-N\,|\leq|R|^{1-\epsilon}$, since the probability of the complement can be bounded from above by $c\,|R|\,e^{-m_2(\epsilon L^\epsilon)^{d\delta}}$ because of the previous reasoning.

Under this condition it is relatively simple to bound from above the difference $|\lambda(R,N,\tau,J)-$

$\lambda_0(R,N)|$. We have

$$\begin{aligned}\int_0^1 ds\, \mu_R^{\tau,\lambda_s}(N_R,N_R)\,|\,\lambda(R,N,\tau,J)-\lambda_0(R,N)\,| &= |\int_0^1 ds\,\frac{d}{ds}\mu_R^{\tau,\lambda_s}(N_R)\,| \\ &= |\,\mu_R^{\tau,\lambda_0(N)}(N_R)-N\,| \\ &\le |R|^{1-\epsilon}\end{aligned}$$

where $\lambda_s = s\lambda(R,N,\tau,J)+(1-s)\lambda_0(R,N)$. Using proposition 4.2 below, we have that $\mu_R^{\tau,\lambda_s}(N_R,N_R) \ge C\mu_R^{\tau,\lambda_s}(N_R) \ge C'|R|$ for some fixed constant $C'=C'(\epsilon)$ and any $N \in [\epsilon|R|,(1-\epsilon)|R|]$. Thus

$$|\lambda(R,N,\tau,J)-\lambda_0(R,N)| \le \frac{1}{C'}|R|^{-\epsilon}$$

Using the fact that the cardinality of $\mathcal{R}_L^{\epsilon}$ is bounded from above by L^{2d} point (i) follows provided that M is taken big enough.
We are left with the proof of point (iii). By the definition of $L(J)$

$$\mathbb{P}(L(J)>l) \le \mathbb{P}(\Theta_{good}(l,M,\epsilon)^c)$$

proceeding as for point (ii) the result follows. $\square$

4.1 Bounds on various covariances.

Here we report, for completeness, some results which follow immediately from the factorization property of the grand canonical measure over the clusters, since they enter at various levels in the analysis of the Kawasaki dynamics for the dilute Ising model.
The setting is as follows. Let $\Lambda=\cup_i\Lambda_i$ where the atoms $\Lambda_1,\cdots,\Lambda_k$ are pairwise disjoint, the chemical potential $\underline{\lambda}$ be constant on each atom of the partition and $\rho_i=\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i})/|\Lambda_i|$.
Then for each set $V\subseteq\Lambda$ and $n\in\mathbb{N}$ we define

$$V_i := V\cap\Lambda_i \quad \bar V := \cup_{x\in V}C_x \quad \text{and} \quad V^{(n)} := \sum_{x\in V}|C_x\cap V|^n \tag{4.1}$$

and say that a subset $V\subset\Lambda$ is good if $\bar V\subset\Lambda_i$ for some $i=1,\dots,k,$.
The following proposition holds for any disorder configuration J.

Proposition 4.2. *There exists a constant c depending only on β such that for any bounded local function f with support $\Delta_f\subset\Lambda$*

$$
\begin{aligned}
&a) \quad |\mu_\Lambda^{\tau,\underline{\lambda}}(N_{C_x\cap V_i}, N_{C_x\cap V_j})| \le c\,\min\{\rho_i,\rho_j\}\,|C_x\cap V_i||C_x\cap V_j|\\
&b) \quad \mu_\Lambda^{\tau,\underline{\lambda}}(\bar N^2_{C_x\cap V_i}) \ge c^{-1}\,\rho_i|C_x\cap V_i|\\
&c) \quad \mu_\Lambda^{\tau,\underline{\lambda}}(\bar N^2_{V_i}) \le c\,\rho_i\,V_i^{(1)}\\
&d) \quad \mu_\Lambda^{\tau,\underline{\lambda}}(\bar N^2_{V_i}) \ge c^{-1}\,\rho_i\,|V_i|\\
&e) \quad |\mu_\Lambda^{\tau,\underline{\lambda}}(f, N_{V_i})| \le c\,\|f\|_\infty\,\min\Big\{\,\rho_i|\bar\Delta_f\cap V_i|,\ \Big(\rho_i\big(\bar\Delta_f\cap V_i\big)^{(1)}\Big)^{\frac12}\,\Big\} \qquad (4.2)\\
&f) \quad \mu_\Lambda^{\tau,\underline{\lambda}}(\bar N^4_{V_i}) \le c\,\max\{\,(\rho_i V_i^{(1)})^2,\ (\rho_i V_i^{(3)})\,\}\\
&g) \quad |\mu_\Lambda^{\tau,\underline{\lambda}}(f, N_{V_i}, N_{V_i})| \le\ c\,\|f\|_\infty\,\rho_i|\bar\Delta_f\cap V_i|^2\\
&h) \quad |\mu_\Lambda^{\tau,\underline{\lambda}}(\bar N^3_{V_i})| \le\ c\,\rho_i V_i^{(2)}
\end{aligned}
$$

4.2 Equivalence of ensembles

Here we recall some fine results on the finite volume comparison of ensembles that will be crucial in most of our future arguments. We refer the reader to sections 6 and 7.2 of [CM1].

Let Λ be a parallelepiped in the class $\bar{\mathcal{R}}_L^\epsilon$ whose longest side is say along the d direction and is L_d. Take $L_1,\dots,L_k$ such that $\sum_{j=1}^k L_j = L_d$ and $L_j \ge \epsilon L_d$ for any $j=1,\cdots,k$. We then take $\Lambda_j = \{x\in\Lambda\ :\ L_{j-1}\le x_d\le L_j\}$ with $L_0=0$, which are elements of $\mathcal{R}_L^\epsilon$. Let also $\mathbf{N} := \{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_i}\}_{i=1}^k$ and let $\rho_i := \frac{N_i}{|\Lambda_i|}$. Given a boundary condition τ and a disorder configuration J, there exists a unique choice of the the chemical potential $\underline{\lambda}$, constant on each Λ_i, $i=1,\cdots,k$, such that $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}) = N_i,\ i=1,\cdots,k$ (see the appendix in [CM1]).
We denote by $\mu := \mu_\Lambda^{\tau,\underline{\lambda}}$ the grand canonical Gibbs measure and by $\nu := \nu_{\Lambda,\mathbf{N}}^\tau$ the multi canonical Gibbs measure $\mu_\Lambda^{\tau,\underline{\lambda}}\big(\cdot\,|\,\mathbf{N}_{\Lambda_i} = \mathbf{N}_i\big)$ and by Ω_τ the set of configurations τ' that coincide with τ in the half space $\{x\in\mathbb{Z}^d\ :\ x_d < L_d\}$, where L_d is largest among the d–coordinates of the sites in Λ.

Proposition 4.3. *In the above setting assume $J\in\Theta_{good}(L,M,\epsilon)$. Then there exists constants $C=C(M,\epsilon)$ and $L_0=L_0(M,\epsilon)$ such that, if $L\ge L_0$*

(a) for all bounded local functions f with support $\Delta_f\subset\Lambda$ satisfying $|\Delta_f|\,(M\log L)^4 \ll |\Lambda|$ otherwise

$$
|\nu(f)-\mu(f)| \le\ C\,\|f\|_\infty \begin{cases} \frac{\bar\Delta_f^{(3)}}{|\Lambda|} & \text{if } \Delta_f \text{ is good}\\[4pt] \frac{|\Delta_f|\,(M\log L)^4}{|\Lambda|} & \text{otherwise}\end{cases}
$$

(b) for all local functions f with support $\Delta\subset\partial^-\Lambda_{n-1}\cap\partial^+\Lambda_n$, $n\le k$,

$$
\sup_{\tau'\in\Omega_\tau} |\nu^\tau(f)-\nu^{\tau'}(f)| \le C' Osc(f)\,\Big[\frac{(M\log L)^4}{L_d} + \frac{(M\log L)^{k+2-n}}{L_d^{k+1-n-(d-1)/2}}\Big]
$$

where $\bar\Delta_f^{(3)}$ has been defined in (4.1).

Proof. See theorem 6.4 and proposition 7.4 in [CM1]. $\square$

Remark. Actually the first part of proposition 4.3 holds in a much more general context (see section 6 in [CM1]).

Proposition 4.4. *In the same setting assume $J \in \Theta_{good}(L, M, \epsilon)$. Let f be such that $|\Lambda_j \setminus \Delta_f| \geq \epsilon|\Lambda_j|$ for any $j = 1, \dots k$. Then there exists a constant $A = A(M, \epsilon)$ such that*

$$\nu(|f|) \leq A\,\mu(|f|)$$

In particular

$$\nu(f, f) \leq A\,\mu(f, f)$$

Proof. The proof is identical to that of proposition 3.3 in [CM2] if we observe that, see [CM1], for any $J \in \Theta_{good}(L, M, \epsilon)$

$$\|\mu_\Lambda^{\tau,\lambda}(e^{i\sum_j \frac{t_j}{\sigma_j} N_{\Lambda_j}} \,|\, \mathcal{F}_{\Delta_f})\|_\infty \leq e^{-\alpha \sum_j t_j^2}$$

for a suitable constant $\alpha := \alpha(\epsilon, M)$, where $\sigma_j^2 := \mu_\Lambda^{\tau,\lambda}(N_{\Lambda_j}, N_{\Lambda_j})$. $\square$

4.3 A block dynamics bound

Here we give a result that is a key step in our recursive bound of the spectral gap of Kawasaki dynamics. For simplicity we discuss our estimate in two dimensions only (see however remark at the end of section 3.3 in [CM1] for its generalization to higher dimensions).

Let Λ be an element of $\bar{\mathcal{R}}_L^\epsilon$. Without loss of generality we can assume that

$$\Lambda = \left\{(x_1, x_2) \in \mathbb{Z}^2;\ 0 \leq x_1 \leq l_1 - 1,\ 0 \leq x_2 \leq l_2 - 1\right\}, \qquad l_1 \leq l_2$$

Let $\Lambda_1 = \left\{(x_1, x_2) \in \Lambda;\ 0 \leq x_2 \leq (\frac{1}{2} + 2\epsilon)\, l_2\right\}$, $\Lambda_2 = \left\{(x_1, x_2) \in \Lambda;\ (\frac{1}{2} + \epsilon)\, l_2 \leq x_2 \leq l_2\right\}$, and let $\Lambda_3 = \Lambda_1 \cap \Lambda_2$. Clearly $\Lambda_i \in \mathcal{R}_L^\epsilon$, $i = 1, 2, 3$.

Let finally N_i be possible values of the number of particles in Λ_i, $i = 1, 2, 3$, and let $\nu_{\Lambda,\mathbf{N}}^\tau$ be the multi canonical Gibbs measure $\mu_\Lambda^\tau\big(\cdot \,|\, N_{\Lambda_i} = N_i,\, i = 1, \dots, 3\big)$. Then we have

Proposition 4.5. *Assume $J \in \Theta_{good}(M, \epsilon, L)$. Then, for any $\gamma > 0$ there exist $L_0 = L_0(\epsilon, M, \gamma)$ such that, if $L \geq L_0$,*

$$\nu_{\Lambda,\mathbf{N}}^\tau(f, f) \leq (1 + \gamma)\nu_{\Lambda,\mathbf{N}}^\tau\left(\mathrm{Var}_{\Lambda_1,\mathbf{N}_1}^\eta(f) + \mathrm{Var}_{\Lambda_2,\mathbf{N}_2}^\eta(f)\right)$$

where $\mathrm{Var}_{\Lambda_i,\mathbf{N}_i}^\eta(f)$, $i = 1, 2$, denotes the variance of f w.r.t. the multicanonical measure on Λ_i with N_i particles, N_3 of which are in Λ_3, and boundary condition η on $\partial_r^+\Lambda_i$.

Proof. Thanks to proposition 4.3 the proof is identical to that of proposition 3.4 of [CM2].

4.4 On the distribution of the particle number

Here we provide some simple results on the distribution of the particle numbers in the atoms of a partition of a given set Λ. Throughout this subsection the setting will be as follows.

Let, for $i = 1, \dots, k$, $\Lambda_i \in \mathcal{R}_L^\epsilon$ be pairwise disjoint and let $\Lambda = \cup_i \Lambda_i$. Assume that also Λ belongs to $\mathcal{R}_L^\epsilon$. Let also $\mathbf{N} = \{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_i}\}_{i=1}^k$. Given a boundary condition τ, let $\underline{\lambda}$ be the chemical potential, constant on each atom, such that $\mu_\Lambda^{\tau,\underline{\lambda}}(\mathbf{N_\Lambda}) = \mathbf{N}$ (see appendix of [CM1] for the existence of $\underline{\lambda}$). Then we have (see Corollary 6.3 in [CM1])

Proposition 4.6. *Assume $J \in \Theta_{good}(M, \epsilon, L)$ and let $\sigma_i^2 := \mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_i}, N_{\Lambda_i})$. Then*

$$\frac{1}{C'}\frac{1}{\prod_i \sigma_i} \le \mu_\Lambda^{\tau,\underline{\lambda}}(\mathbf{N}_\Lambda = \mathbf{N}) \le C'\frac{1}{\prod_i \sigma_i}$$

for a suitable constant $C' = C'(M, \epsilon) > 1$.

The next result concerns the way particles distribute inside one block of the partition.

Pick $j \in [1, \dots, k]$ and divide Λ_j into two disjoint subsets V, W that we assume to be also elements of $\mathcal{R}_L^\epsilon$. Denote by N^* the average number of particles in V according to $\mu_\Lambda^{\tau,\underline{\lambda}}$ and let $\gamma(n) = \nu_{\Lambda,\mathbf{N}}^\tau(N_V = n)$.

Theorem 4.7. *Assume $J \in \Theta_{good}(M, \epsilon, L)$. Then there exists $c_0 = c_0(M, \epsilon)$ such that for all $f : \Omega_\Lambda \mapsto \mathbb{R}$ that depend only on $N_V(\sigma)$ the following Poincaré inequality holds*

$$\nu_{\Lambda,\mathbf{N}}^\tau(f, f) \le c_0 \min\{N^*, |V| - N^*\} \sum_n \gamma(n) \wedge \gamma(n+1) \left[f(n+1) - f(n)\right]^2$$

Proof. Again, thanks to the fact that $J \in \Theta_{good}(M, \epsilon, L)$ and using proposition 4.2, the proof is identical to that of theorem 1.9 in [CM2]. □

4.5 A key bound on special covariances.

The setting is the following. We fix $\epsilon \in (0, 1)$ and $l \in [L^\epsilon, L]$. We then consider a volume $\Lambda \in \mathcal{R}_L^\epsilon(l)$ such that $\Lambda = \cup_{i=1}^k \Lambda_i$, where $\Lambda_i \in \mathcal{R}_L^\epsilon(l)$ and $|\Lambda_i|/|\Lambda| \ge \epsilon$ for $i = 1, \cdots, k$. Let $\mathbf{N} := \{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_i}\}_{i=1}^k$. Let also $h_x := e^{-\nabla_x H}\sigma(x)$ and $\hat{h}_x := e^{-\nabla_x H}(1 - \sigma(x))$ and take $\epsilon' = \frac{2}{db}\epsilon$, where b is defined in proposition 5.2 and d is the dimension.

Lemma 4.8. *Assume $J \in \Theta_{good}(M, \epsilon', L)$. Then for any $\delta > 0$ there exist a positive constant C and $L_0(M, \epsilon', \delta) > 0$ such that*

$$\nu_{\Lambda,\mathbf{N}}^\tau\big(f, \frac{1}{|\Lambda_i||\Lambda_j|}\sum_{\substack{x\in\Lambda_i\\ z\in\Lambda_j}} h_x\hat{h}_z\big)^2 \le \frac{Cl^2}{|\Lambda|}\nu_{\Lambda,\mathbf{N}}^\tau\left(\sum_{[x,z]\in\mathcal{E}_\Lambda} c_{xz}(\nabla_{xz} f)^2\right) + \frac{\delta}{|\Lambda|}\operatorname{Var}_{\Lambda,\mathbf{N}}^\tau(f)$$

provided that $L \geq L_0$.

Proof. Fix $\delta > 0$. Due to lemma A.1 we can assume, without loss of generality, that $\nu_{\Lambda,\mathbf{N}}(h_x) = 0$, $\forall x \in \Lambda$. For notation simplicity define $G := \frac{1}{|\Lambda_i|}\sum_{x\in\Lambda_i} h_x$, $\mathcal{H} = \frac{1}{|\Lambda_j|}\sum_{z\in\Lambda_j} \hat{h}_z$. Let $\Delta_{\hat{h}}(z)$ be the support of $\hat{h}_z$ and write $\mathcal{H} = \mathcal{H}^{\text{in}} + \mathcal{H}^{\text{ext}}$ where $\mathcal{H}^{\text{in}}$ is the sum over those z's in Λ_j such that $\Delta_{\hat{h}}(z) \subset \Lambda_j$ and $\mathcal{H}^{\text{ext}}$ the rest. Then, using the formula relating the covariance of two functions f and g w.r.t. the measure $\nu^\tau_{\Lambda,\mathbf{N}}$ to the covariance w.r.t. the same measure conditioned to a sub σ-algebra, we get

$$
\begin{aligned}
\nu^\tau_{\Lambda,\mathbf{N}}\big(f, \frac{1}{|\Lambda_i||\Lambda_j|}\sum_{\substack{x\in\Lambda_i\\ z\in\Lambda_j}} h_x\hat{h}_z\big)^2 &= \nu^\tau_{\Lambda,\mathbf{N}}\big(f, G\mathcal{H}\big)^2 \\
&\leq 2\|\mathcal{H}^{\text{ext}}\|^2_\infty \operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(G)\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f) + 4\|\mathcal{H}^{\text{in}}\|^2_\infty \nu^\tau_{\Lambda,\mathbf{N}}\left(\left[\nu^\tau_{\Lambda,\mathbf{N}}\Big(f, G \,|\, \mathcal{F}_{\Lambda_i^c}\Big)\right]^2\right) \\
&\quad + 4\left[\nu^\tau_{\Lambda,\mathbf{N}}\left(f,\, \nu^\tau_{\Lambda,\mathbf{N}}\Big(G \,|\, \mathcal{F}_{\Lambda_i^c}\Big)\mathcal{H}^{\text{in}}\right)\right]^2 \\
&\leq \frac{1}{|\Lambda|}\left[C'\left(\frac{|\partial_r^+\Lambda_j|}{|\Lambda_j|}\right)^2 + \frac{\delta}{2}\right]\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f) + \frac{Cl^2}{|\Lambda|}\nu^\tau_{\Lambda,\mathbf{N}}\left(\sum_{[x,z]\in\mathcal{E}_\Lambda} c_{xz}(\nabla_{xz}f)^2\right) \\
&\quad + 4\left[\nu^\tau_{\Lambda,\mathbf{N}}\left(f,\, \nu^\tau_{\Lambda,\mathbf{N}}(G \,|\, \mathcal{F}_{\Lambda_i^c})\mathcal{H}^{\text{in}}\right)\right]^2
\end{aligned}
\tag{4.3}
$$

provided that L is big enough. To obtain (4.3) we used the hypothesis $\frac{|\Lambda_i|}{|\Lambda|} \geq \epsilon$ together with lemma A.1 to bound the term $\left[\nu^\tau_{\Lambda,\mathbf{N}}\Big(f, G \,|\, \mathcal{F}_{\Lambda_i^c}\Big)\right]^2$ and proposition 4.4 and the fact that $J \in \Theta_{good}(M, \epsilon', L)$ to get

$$
\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(G) \leq C\, \operatorname{Var}^{\tau,\lambda}_\Lambda(G) = \frac{1}{|\Lambda|^2}\sum_{x\in\Lambda}\frac{\operatorname{Var}^{\tau,\lambda}_\Lambda(\sum_{y\in C_x\cap\Lambda} h_y)}{|C_x\cap\Lambda|} \leq \frac{C''}{|\Lambda|}
\tag{4.4}
$$

The third term in the.r.h.s of (4.3) can be bounded from above by

$$
\|\nu^\tau_{\Lambda,\mathbf{N}}(G \,|\, \mathcal{F}_{\Lambda_i^c})\|^2_\infty \operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(\mathcal{H}^{\text{in}}) \operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f)
\tag{4.5}
$$

In turn, the second factor in the r.h.s. of (4.5), using as in (4.4) in proposition 4.4 and the hypothesis $J \in \Theta_{good}(M, \epsilon', L)$, is bounded from above by $\frac{C_1}{|\Lambda|}$. The first factor in the r.h.s. of (4.5), thanks to the hypothesis $\nu^\tau_{\Lambda,\mathbf{N}}(h_x) = 0$ $\forall x \in \Lambda$, a simple telescopic argument, part 1) of proposition 4.3 and assumption $J \in \Theta(M, \epsilon', \Lambda)$, is bounded from above by

$$
\Big[\sup_{\tau,\tau'\in\Omega_{\partial_r^+\Lambda_i}} \frac{1}{|\Lambda_i|}\sum_{x\in\Lambda_i} |\nu^\tau_{\Lambda_i,N_i}(h_x) - \nu^{\tau'}_{\Lambda_i,N_i}(h_x)|\Big]^2 \leq C'\left[\frac{|\partial_r^+\Lambda_i|}{|\Lambda_i|}\right]^2
$$

In conclusion, for any $\delta > 0$, the first and third term in the.r.h.s of (4.3) can be bounded from above by $\frac{\delta}{2|\Lambda|}\operatorname{Var}^\tau_{\Lambda,\mathbf{N}}(f)$ provided that L is large enough. The proof is complete. $\square$

5. A general lower bound on the spectral gap in a finite subgraph.

In this section we obtain a rough lower bound for the spectral gap of the dynamics in a finite subgraph Λ of $\mathbb{Z}^d$ which depends on the size of Λ and on the size of the largest cluster inside Λ. As a corollary we get that, if $J \in \Theta_{good}(M, \epsilon, L)$ and $\Lambda \in \mathcal{R}_L^\epsilon$, the spectral gap in Λ is not smaller than $|\Lambda|^{-b}$ for a suitable b independent of ϵ.

Theorem 5.1. *Let Λ be a finite subgraph of $\mathbb{Z}^d$ and let $\Gamma_J(\Lambda) := \max_{x \in \Lambda_v} |C_x \cap \Lambda|^{\frac{d-1}{d}}$. Then there exist a positive constant c, depending only on J and on d, and a numerical constant $\alpha > 8$ such that*

$$\operatorname{gap}(L^\tau_{\Lambda,N})^{-1} \leq |\Lambda_v|^\alpha \exp\left(c\,\Gamma_J(\Lambda)\right) \qquad \forall\, N,\ \tau$$

Corollary 5.2. *Let $\Lambda \in \mathcal{R}_L^\epsilon$ and assume $J \in \Theta_{good}(M, \epsilon, L)$. Then there exists a positive numerical constant $b > 8$, independent of L, such that*

$$\operatorname{gap}(L^\tau_{\Lambda,N})^{-1} \leq |\Lambda|^b \qquad \forall\, N,\ \tau$$

Proof. It follows immediately from theorem 5.1 and the fact that $\max_{x\in\Lambda} |C_x| \leq M \log L$ for any $J \in \Theta_{good}(M, \epsilon, L)$.

Proof of Theorem 5.1. The proof is divided into two distinct parts. In the first step we show that there exists a positive constant c, depending only on $\sup_{xy} |J_{xy}|$ and d, such that for any finite subgraph Λ, any τ and any N

$$\operatorname{gap}(L^\tau_{\Lambda,N})^{-1} \leq \exp(c\,|\Lambda|^{\frac{d-1}{d}}) \tag{5.1}$$

Then, using a suitable inductive procedure, we improve (5.1) to the sought bound in terms of $\Gamma_J(\Lambda)$.

Step 1. Let us therefore start by proving (5.1). We first remark that such a general bound on the spectral gap has been proved in great generality for non conservative Glauber type dynamics (see [CMM] and [M]) and for Kawasaki dynamics when the graph Λ is a cube in $\mathbb{Z}^d$ [CCM]. The general scheme of proof is the same for Glauber and Kawasaki dynamics and it goes as follows. Pick a graph $\Lambda \subset \hat{\mathcal{F}}$ and divide it into two disjoint subgraphs V and W in such a way that

i) they both have volume (*i.e.* number of vertices) of the order of $\frac{1}{2}|\Lambda_v|$;
ii) the number of edges with one endpoint in V and the other in W is not larger than $|\Lambda|^{\frac{d-1}{d}}$.

A little geometric argument shows that such a partition always exists (see proposition A1.1 in [CMM]). Then one proves that

$$\operatorname{gap}(L^\tau_{\Lambda,N})^{-1} \leq \exp(c\,|\Lambda|^{\frac{d-1}{d}}) \left[\sup_{\tau,n} \operatorname{gap}(L^\tau_{V,n})^{-1} \vee \sup_{\eta,k} \operatorname{gap}(L^\eta_{W,k})^{-1}\right] \tag{5.2}$$

Once (5.2) is available one can follow the same steps of theorem 3.8 in [M] for the non conservative case and prove (5.1).
Thus we concentrate on the proof of (5.2).

Let Λ be the disjoint union of two finite subgraphs V and W and let us consider a modified dilute Ising model in Λ in which all (if any) the interactions between V and W have been turned off. In other words we define a new interaction

$$\dot{J}_{xy} = \begin{cases} 0 & \text{if } x \in V_v \text{ and } y \in W_v \\ J_{xy} & \text{otherwise} \end{cases}$$

and the following "dotted" quantities for the "decoupled" system

$$\dot{\mu} := \mu_{\Lambda}^{\tau,\lambda,\dot{J}}\,, \quad \dot{\nu} := \nu_{\Lambda,N}^{\tau,\dot{J}}\,, \quad \dot{L} := L_{\Lambda,N}^{\tau,\dot{J}}\,, \quad \dot{\mathcal{E}}(f,f) := \mathcal{E}_{\Lambda,N}^{\tau,\dot{J}}(f,f)$$

where, as usual, λ is the chemical potential such that $\dot{\mu}(N_\Lambda) = N$. Using the formula of the conditional variance and lemma 5.3 below we have

$$\begin{aligned} \mathrm{Var}_{\dot{\nu}}(f) &= \dot{\nu}\left(\mathrm{Var}_{\dot{\nu}}(f \mid N_V)\right) + \mathrm{Var}_{\dot{\nu}}\left(\dot{\nu}(f \mid N_V)\right) \\ &\leq |\Lambda|^{\alpha} \exp\left(k\,\Gamma_J(\Lambda)\right) \left[\dot{\mathcal{E}}(f,f) + \dot{\nu}\left(\mathrm{Var}_{\dot{\nu}}(f \mid N_V)\right)\right] \end{aligned} \tag{5.3}$$

Lemma 5.3. *There exist a numerical constant $a > 8$ and a positive constant $k = k(J,d)$ such that*

$$\mathrm{Var}_{\dot{\nu}}\left(\dot{\nu}(f \mid N_V)\right) \leq |\Lambda|^{a} \exp\left(k\,\Gamma_J(\Lambda)\right) \left[\dot{\mathcal{E}}(f,f) + \dot{\nu}\left(\mathrm{Var}_{\dot{\nu}}(f \mid N_V)\right)\right]$$

We postpone the proof of lemma 5.3 and conclude the proof of (5.2). Since there is no interaction between V and W the conditional measure $\dot{\nu}(\cdot \mid N_V)$ is a product measure. Thus

$$\dot{\nu}\left(\mathrm{Var}_{\dot{\nu}}(f \mid N_V)\right) \leq \left[\sup_{\tau,n} \mathrm{gap}(L^{\tau}_{V,n})^{-1} \vee \sup_{\eta,k} \mathrm{gap}(L^{\eta}_{W,k})^{-1}\right] \dot{\mathcal{E}}(f,f)$$

so that, since $\Gamma_J(\Lambda) \leq |\Lambda|^{\frac{d-1}{d}}$,

$$\mathrm{Var}_{\dot{\nu}}(f) \leq \exp\left(c|\Lambda|^{\frac{d-1}{d}}\right) \left[\sup_{\tau,n} \mathrm{gap}(L^{\tau}_{V,n})^{-1} \vee \sup_{\eta,k} \mathrm{gap}(L^{\eta}_{W,k})^{-1}\right] \dot{\mathcal{E}}(f,f) \tag{5.4}$$

for a suitable constant c. All what is left is to restore the original interaction for all $x, y \in \Lambda$. This is straightforward since, because of property *ii)* above of the decomposition $\Lambda = V \cup W$, there exists a positive constant k_1, depending only on J and d, such that

$$e^{-k_1|\Lambda|^{\frac{d-1}{d}}} \leq \frac{\dot{\nu}(\sigma)}{\nu^{\tau}_{\Lambda,N}(\sigma)} \leq e^{k_1|\Lambda|^{\frac{d-1}{d}}}$$

Thus we can remove the "dot" in (5.4) and get (5.2) by paying a price not larger than $e^{2k_1|\Lambda|^{\frac{d-1}{d}}}$.

Proof of lemma 5.3. Let

$$\gamma(n) := \dot{\nu}(N_V = n), \qquad g(n) := \dot{\nu}(f \mid N_V = n)$$

$n_{\min} := \max\{0, N - |W|\}$, $n_{\max} := \min\{|V|, N\}$ and $\Omega := \{n \in [n_{\min}, n_{\max}] :\ n \text{ is an integer}\}$. By proposition 3.7 in [CM2] we have

$$\mathrm{Var}_{\dot\nu}\big(\dot\nu(f \mid N_V)\big) \le C_\gamma \sum_{n\in\Omega} \gamma(n) \wedge \gamma(n-1)[g(n) - g(n-1)]^2$$

where

$$C_\gamma = 4 \max\left[\left(\sup_{n\le \bar N+1} \sum_{j\le n} \frac{\gamma(j)}{\gamma(n)}\right)^2, \left(\sup_{n\ge \bar N} \sum_{j\ge n} \frac{\gamma(j)}{\gamma(n)}\right)^2\right]$$

and $\bar N$ is the largest integer such that $\sum_{n\le \bar N} \gamma(n) \le \frac{1}{2}$. Thanks to *(1)* of proposition 4.3 in [CCM]

$$\sum_{n\in\Omega} \gamma(n) \wedge \gamma(n-1)[g(n) - g(n-1)]^2 \le C\,|\Lambda|^8 \big[\dot{\mathcal{E}}(f,f) + \dot\nu\,(\mathrm{Var}_{\dot\nu}(f \mid \mathcal{F}_V))\big]$$

The result then follows if we can prove the next lemma (to be compared with proposition 4.2 in [CCM]). □

Lemma 5.4. *There exist a positive numerical constant* a *and a constant* $c = c(J,d)$ *such that*

$$C_\gamma \le |\Lambda|^a \exp\left(c\,\Gamma_J(\Lambda)\right)$$

Proof. Let $N_V^* := \dot\mu(N_V)$. We begin by showing that there exists a positive constant c' such that for any $j, n \in \Omega$ with the property that either $j \le n \le N_V^*$ or $j \ge n \ge N_V^*$

$$\frac{\gamma(j)}{\gamma(n)} \le N^2\,|V|^2\,|W|^2 \exp\{c'\,\Gamma_J(\Lambda)\} \tag{5.5}$$

For this purpose we first remark that, by the definition of the canonical measure and the fact that there is no interaction between V and W, we can write

$$\gamma(n) = \frac{\mu_V^{\mathrm{b.c.},\lambda}(N_V = n)\,\mu_W^{\mathrm{b.c.},\lambda}(N_W = N-n)}{\dot\mu(N_\Lambda = N)}$$

where the notation b.c. in $\mu_X^{\mathrm{b.c.},\lambda}$, $X = V$ or $X = W$, means τ boundary conditions on $\partial^+X \cap \partial^+\Lambda \ne \emptyset$ and free boundary conditions on $\partial^+X \cap \partial^+\Lambda = \emptyset$. Next, for $s \in [0, |X|]$, we define $\lambda_X(s)$ as the chemical potential such that $\mu_X^{\mathrm{b.c.},\lambda_X}(N_X) = s$ and $I_X(s) := s\lambda_X(s) - \log \frac{Z_X^{\mathrm{b.c},\lambda_X(s)}}{Z_X^{\mathrm{b.c.},\lambda_X(s)}}$. By construction

$$\mu_X^{\mathrm{b.c.},\lambda}(N_X = s) = e^{-I_X(s)}\mu_X^{\mathrm{b.c.},\lambda_X(s)}(N_X = s)$$

It is easy to check that

$$\frac{d^2}{ds^2} I_X(s) > 0\,, \qquad \frac{d}{ds} I_W(s)\Big|_{s=N-N_V^*} = 0\,, \qquad \frac{d}{ds} I_V(s)\Big|_{s=N_V^*} = 0$$

so that for $j \le n \le N_V^*$ or $j \ge n \ge N_V^*$

$$\frac{\gamma(j)}{\gamma(n)} \le \Big[\mu_V^{\mathrm{b.c.},\lambda_V(n)}(N_V = n)\, \mu_W^{\mathrm{b.c.},\lambda_W(N-n)}(N_W = N-n)\Big]^{-1} \tag{5.6}$$

Thanks to proposition 3.3 in [CCM], there exists a positive constant c'' depending only on $\sup_{xy} |J_{xy}|$ such that for any integer $s \in [0, |X|]$

$$\mu_X^{\lambda_X(s)}(N_X = s) \ge c''\, \Big(\frac{\mathrm{gap}(\mathcal{L}_X^{\mathrm{b.c.},\lambda_X})}{|X|} \wedge 1\Big) \frac{1}{(s+2)(|X|-s+1)} \tag{5.7}$$

where $\mathcal{L}_X^{\mathrm{b.c.},\lambda_X}$ is the generator of Glauber's dynamics on X with "heat bath" rates (see e.g. [M]). Since there is no interaction between the clusters $\{C_x\}_{x\in\Lambda}$, theorem 3.8 in [M] implies that there exists $c' = c'(J,d) > 0$ such that

$$\mathrm{gap}(\mathcal{L}_X^{\mathrm{b.c.},\lambda_X}) \ge \exp\left(-c'\, \Gamma_J(\Lambda)\right) \tag{5.8}$$

By putting together (5.6), (5.7) and (5.8) equation (5.5) follows.
Thanks to (5.5) we can conclude that there exists a positive numerical constant a such that for any $n \le N_V^*$

$$\sum_{j\le n} \frac{\gamma(j)}{\gamma(n)} \le |\Lambda|^a \exp\left(c'\, \Gamma_J(\Lambda)\right) \tag{5.9}$$

and similarly for $\sum_{j\ge n} \frac{\gamma(j)}{\gamma(n)}$, $n \ge N_V^*$.
Suppose now, for definiteness, that the median for $\gamma(\cdot)$, $\bar N$, is smaller than N_V^*. Then (5.9) implies in particular that for any $n \in [\bar N + 1, N_V^*]$

$$\frac{1}{2}\gamma(n)^{-1} \le \sum_{j\le n} \frac{\gamma(j)}{\gamma(n)} \le |\Lambda|^a \exp\left(c'\, \Gamma_J(\Lambda)\right)$$

Therefore

$$\sup_{n\le \bar N+1} \sum_{j\le n} \frac{\gamma(j)}{\gamma(n)} \le |\Lambda|^a \exp\left(c'\, \Gamma_J(\Lambda)\right)$$

and

$$\begin{aligned}\sup_{n\ge \bar N} \sum_{j\ge n} \frac{\gamma(j)}{\gamma(n)} &\le |\Lambda|^a \exp\left(c'\, \Gamma_J(\Lambda)\right) + \sup_{n\in\Omega\cap[\bar N+1, N_V^*]} \frac{1}{\gamma(n)} \\ &\le 2|\Lambda|^a \exp\left(c'\, \Gamma_J(\Lambda)\right)\end{aligned}$$

The lemma is proved. $\square$

Step 2. We can now turn to the second part of the proof of our theorem.
Let

$$\tilde{\mathcal{E}}^{\tau}_{\Lambda,N}(f,f) = \frac{1}{2} \sum_{x,y \in \Lambda_v} \nu^{\tau}_{\Lambda,N} \left[c_{xy} \, (\nabla_{xy} f)^2 \right]$$

where the rates c_{xy} satisfy all the assumption of section 2. In other words $\tilde{\mathcal{E}}$ is the Dirichlet form of a conservative Markov process on Ω_Λ, reversible w.r.t. the canonical measure $\nu^{\tau}_{\Lambda,N}$, in which the particles can jump between *any* pair of sites. Let also $\tilde{L}^{\tau}_{\Lambda}$ be the corresponding generator.
First we establish a Poincarè inequality of the form

$$\mathrm{Var}^{\tau}_{\Lambda,N}(f) \le |\Lambda|^{\alpha'} \exp\left(c\,\Gamma_J(\Lambda)\right) \tilde{\mathcal{E}}^{\tau}_{\Lambda,N}(f,f) \tag{5.10}$$

Once (5.10) is available, we get immediately the statement of the theorem with $\alpha = 3 + a$ from the rough bound (see lemma 4.3 in [Y] and proposition 3.13 in [CM2])

$$\tilde{\mathcal{E}}^{\tau}_{\Lambda,N}(f,f) \le |\Lambda_v|^3 \mathcal{E}^{\tau}_{\Lambda,N}(f,f)$$

In order to prove (5.10), let $\{C_i\}_{i=1}^k$ be the clusters inside Λ which contain more than one point and ordered in decreasing size. Define $V_0 := \Lambda$, $V_j := \Lambda \setminus \cup_{i=1}^j C_i$ and write (in order to simplify the notation in what follows we drop the subcripts Λ, N and the superscript τ)

$$\mathrm{Var}(f) = \nu\big(\mathrm{Var}(f \,|\, \mathcal{F}_1)\big) + \mathrm{Var}\big(\nu(f \,|\, \mathcal{F}_1)\big) \tag{5.11}$$

where $\mathcal{F}_1$ is the σ–algebra generated by N_{C_1}, the number of particles in the first cluster.
Consider the first term in the r.h.s. of (5.11). Since by construction there is no interaction between C_1 and V_1, the conditional measure $\nu(\cdot \,|\, N_{C_1} = n)$ is a product measure. Therefore, if we define $\alpha_j := \sup_{N,\tau} \mathrm{gap}(\tilde{L}^{\tau}_{V_j,N})^{-1}$ and we use (5.1), we have that, for some $c > 0$,

$$\nu\big(\mathrm{Var}(f \,|\, \mathcal{F}_1)\big) \le \max\{\alpha_1, \exp\left(c\,\Gamma_J(\Lambda)\right)\} \, \tilde{\mathcal{E}}(f,f) \tag{5.12}$$

Consider now the second term in the r.h.s. of (5.11). Let $\mathcal{F}_{C_1}$ be the $\sigma-$algebra generated by $\{\eta_x\}_{x \in C_1}$ and let $f_1 := \nu(f \,|\, \mathcal{F}_{C_1})$. Notice that $\mathcal{F}_1 \subseteq \mathcal{F}_{C_1}$ so that $\nu(f \,|\, \mathcal{F}_1) = \nu(f_1 \,|\, \mathcal{F}_1)$ we thus have

$$\mathrm{Var}\big(\nu(f \,|\, \mathcal{F}_1)\big) = \mathrm{Var}\big(\nu(f_1 \,|\, \mathcal{F}_1)\big) \le |\Lambda|^a \exp\left(c\,\Gamma_J(\Lambda)\right) \left[\tilde{\mathcal{E}}(f,f) + \nu\big(\mathrm{Var}(f_1 \,|\, \mathcal{F}_1)\big)\right] \tag{5.13}$$

where we used lemma 5.3 in the last inequality as there is no interaction between C_1 and V_1. we can now conclude the proof of the theorem. Since f_1 depends only on the spin variables in C_1, (5.1) gives

$$\mathrm{Var}(f_1 \,|\, N_{C_1} = n) \le \exp\left(c\,|C_1|^{\frac{d-1}{d}}\right) \mathcal{E}_{C_1,n}(f_1,f_1) \qquad \forall\, n \tag{5.14}$$

It is quite easy to check, again because there is no interaction between C_1 and its complement, that $\mathcal{E}_{C_1,n}(f_1,f_1) \le \mathcal{E}_{C_1,n}(f,f)$. Thus

$$\nu\left(\mathrm{Var}(f_1 \,|\, \mathcal{F}_1)\right) \le \exp\left(c\,\Gamma_J(\Lambda)\right) \mathcal{E}(f,f) \le \exp\left(c\,\Gamma_J(\Lambda)\right) \tilde{\mathcal{E}}(f,f) \tag{5.15}$$

Using (5.12), (5.13) and (5.15) to bound (5.11) we then obtain

$$\alpha_0 \le \max\{\alpha_1, |\Lambda|^a \exp\left(c\,\Gamma_J(\Lambda)\right)\}$$

We can now iterate the procedure and obtain

$$\alpha_j \le \max\{\alpha_{j+1}, |\Lambda|^a \exp\left(c\,\Gamma_J(\Lambda)\right) \qquad \forall\, j = 0, \cdots, k-1$$

so that

$$\alpha_0 \le \max\{\alpha_k, |\Lambda|^a \exp\left(c\,\Gamma_J(\Lambda)\right)\}$$

Since the spectral gap for the symmetric simple exclusion with long range jumps in the region V_k is greater than $|V_k|^{-1}$ (see lemma 8.1 in [Q]) we have that $\alpha_k \le |\Lambda|$ and (5.10) follows. □

6. Proof of the diffusive scaling of the spectral gap.

In this section we finally prove our main result, namely theorem 2.1, via a recursive scheme combined with theorem 5.1.

6.1 Recursive analysis from scale L to scale L^ϵ.

In this first paragraph we prove a lower bound of the spectral gap of the generator of the Kawasaki dynamics in the box Q_L in terms of the smallest spectral gap in suboxes of Q_L of side L^ϵ, $\epsilon \ll 1$, provided that the configuration of the random couplings J is "good" for all scales between $L^{\epsilon'}$ and L, namely $J \in \Theta_{good}(M, \epsilon', L)$ where $\epsilon' = \frac{\epsilon}{db}$, d is the space dimension and b is the positive numerical costant defined in corollary 5.2. The main tool is a recursive analysis of the behaviour of the spectral gap when the linear size of the volume under consideration is doubled, developed in [CM2] for the high temperature case.

For simplicity we carry out our analysis in two dimensions but the extension to higher dimension is straightforward.

Let

$$g(l, L) := g(J, l, L, \epsilon) = \min_{R \in \bar{\mathcal{R}}_L^\epsilon(l)} \min_{N,\tau} \operatorname{gap}(L_{R,N}^{\tau,J}) \tag{6.1}$$

where $\operatorname{gap}(L_{R,N}^{\tau,J})$ has been defined in (2.7). Notice that necessarily $l \ge L^\epsilon$ because of the very definition of $\bar{\mathcal{R}}_L^\epsilon$.

With the above notation we will prove the following recursive bound.

Theorem 6.1. *Assume $J \in \Theta_{good}(M, \epsilon', L)$. Then there exist $L_0(\epsilon', M)$ and $k = k(d, \beta, M, \epsilon')$ such that, if $L \ge L_0$,*

$$g(l, L)^{-1} \le \frac{3}{2}\, g(\frac{l}{2}, L)^{-1} + k\, l^2 \qquad \text{for any} \qquad l \in [2L^\epsilon, L]$$

In particular

$$\min_{N,\tau} \operatorname{gap}(L_{Q_L,N}^{\tau,J}) L^2 \ge \left[4k + L^{(1-\epsilon)\log_2 \frac{3}{8} - 2\epsilon} g(L^\epsilon, L)^{-1}\right]^{-1}$$

Proof. The fact that

$$\min_{N,\tau} \operatorname{gap}(L^{\tau,J}_{Q_L,N})L^2 \geq \left[4k + L^{(1-\epsilon)\log_2 \frac{3}{8} - 2\epsilon} g(L^\epsilon, L)^{-1}\right]^{-1}$$

is a trivial consequence of the recursive bound.

Fix now $l \in [2L^\epsilon, L]$ and let us consider a rectangle $\Lambda \in \bar{\mathcal{R}}^\epsilon_L(l)$. Without loss of generality Λ can be taken of the form

$$\Lambda = \{x = (x_1, x_2);\ 0 \leq x_1 < l_1,\ 0 \leq x_2 < l_2\}$$

with $l_1 \leq l_2$. If $l_2 \leq \frac{l}{2}$ then $\min_{N,\tau} \operatorname{gap}(L^\tau_{\Lambda,N}) \geq g(\frac{l}{2}, L)$ because of the definition of $g(l, L)$. Thus we assume $\frac{l}{2} < l_2 \leq l$. We set $d = \lfloor \epsilon l \rfloor$ and, given an integer $j \in [1, \lfloor \frac{1}{10\epsilon} \rfloor]$, we partition Λ into three atoms $\{\Lambda_i\}_{i=1}^3$ as follows (we omit the index j for simplicity)

$$\begin{aligned}
&\Lambda_1 = \{x \in \Lambda;\ 0 \leq x_2 \leq l_2/2 + jd\} \\
&\Lambda_2 = \{x \in \Lambda;\ l_2/2 + (j-1)d < x_2 \leq l_2 - 1\} \\
&\Lambda_3 = \Lambda_1 \cap \Lambda_2
\end{aligned} \tag{6.2}$$

Notice that each Λ_i belongs to $\mathcal{R}^\epsilon_L$ therefore, since $J \in \Theta_{good}(M, \epsilon', L)$, we are allowed to use any of the results of section 4 for each rectangles Λ_i, $i = 1, 2, 3$.

Fix now a boundary condition τ outside Λ and a number of particles $N \in [0, \ldots, |\Lambda|]$. We will then use twice the formula relating the variance of a function f w.r.t. the measure $\nu^\tau_{\Lambda,N}$ to the variance of f w.r.t. the measure $\nu^\tau_{\Lambda,N}$ conditioned to a sub σ–algebra $\mathcal{F}_0$

$$\nu^\tau_{\Lambda,N}(f,f) = \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \,|\, \mathcal{F}_0)\right) + \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f \,|\, \mathcal{F}_0), \nu^\tau_{\Lambda,N}(f \,|\, \mathcal{F}_0)\right) \tag{6.3}$$

to write

$$\begin{aligned}
\nu^\tau_{\Lambda,N}(f,f) &= \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \,|\, \mathcal{F}_1)\right) + \nu^\tau_{\Lambda,N}(f_1, f_1) \\
&= \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \,|\, \mathcal{F}_{1,3})\right) + \nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f_{1,3}, f_{1,3} \,|\, \mathcal{F}_1)\right) + \nu^\tau_{\Lambda,N}(f_1, f_1)
\end{aligned} \tag{6.4}$$

where $\mathcal{F}_1$ and $\mathcal{F}_{1,3}$ are the σ–algebras generated by N_{Λ_1} and $\{N_{\Lambda_1}, N_{\Lambda_3}\}$ respectively, and $f_1 := \nu^\tau_{\Lambda,N}(f \,|\, \mathcal{F}_1)$, $f_{1,3} := \nu^\tau_{\Lambda,N}(f \,|\, \mathcal{F}_{1,3})$. Formula (6.4) will represent our basic starting point. We will now examine separately each term in the r.h.s. of (6.4).

6.2 Analysis of the first term in the r.h.s. of (6.4).

For any small δ and large enough L, the first term in the r.h.s. of (6.4) can be bounded from above using proposition 4.5 by

$$\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(f,f \,|\, \mathcal{F}_{1,3})\right) \leq (1+\delta)\nu^\tau_{\Lambda,N}\left(\nu^\eta_{\Lambda_1,N_1}(f,f) + \nu^\eta_{\Lambda_2,N_2}(f,f)\right) \tag{6.5}$$

where the average is over the random variables η, N_{Λ_1} and N_{Λ_2}. As in section 5.1 of [CM2], let now examine the spectral gap of the bottom rectangle Λ_1, the reasoning being similar for the top one.

There are two cases to analyze:

a) $l_1 \leq \frac{3}{4}l$. In this case one easly verifies that $\Lambda_1 \in \bar{\mathcal{R}}^{\epsilon}_{\Lambda}(\frac{3}{4}l)$.

b) $l_1 > \frac{3}{4}l$. In this case $\Lambda_1 \in \bar{\mathcal{R}}^{\epsilon}_{L}$ but now the longest is l_1 and the shortest one is smaller than $\frac{l_2}{2} + \frac{l}{10} \leq \frac{3}{5}l$ since $l_2 \leq l$.

Therefore if

$$\hat{g}(l,L) := \min_{\substack{R\in\bar{\mathcal{R}}^{\epsilon}_{L}(l) \\ l_1<\frac{3}{5}l,\ l_2\geq\frac{3}{4}l}} \ \min_{N,\tau} \operatorname{gap}(L^{\tau,J}_{R,N})$$

the r.h.s. of (6.5) is smaller than

$$(1+\delta)\max\{\, g(\frac{3}{4}l,L)^{-1},\, \hat{g}(l,L)^{-1}\,\} \left[\mathcal{E}^{\tau}_{\Lambda,N}(f,f) + \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^{\tau}_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy}f)^2\,\right]\right] \tag{6.6}$$

uniformly in $j \in [1, \frac{1}{10\delta}]$.

Notice that the "spurious" term $\frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^{\tau}_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy}f)^2\,\right]$ comes from the fact that Λ_1 and Λ_2 overlap.

6.3 Analysis of the second and third term in the r.h.s. of (6.4).

Here we bound from above the second and third term in (6.4) (see section 5.2 in [CM2]). The necessary steps are identical in both cases and therefore, for shortness, we treat only the third term which is (notationally speaking) also the simplest.

Let $\rho := \frac{N}{|\Lambda|}$, $u = \lfloor\rho\,|\Lambda_1|\rfloor$ and assume, without loss of generality, that $\rho \leq \frac{1}{2}$. Let also $N_1^* = \mu^{\tau,\lambda}_{\Lambda}(N_{\Lambda_1})$, $\mu^{\tau,\lambda}_{\Lambda}$ being the grand canonical measure with average particle number equal to N, and let $\gamma(n) := \nu^{\tau}_{\Lambda,N}(N_{\Lambda_1} = n)$. Let finally $c_n = n(|\Lambda\setminus\Lambda_1| - N + n)$, that is (number of particles in Λ_1)$\times$ (number of holes in $\Lambda\setminus\Lambda_1$), and let $c'_n = n(|\Lambda_1| - N + n)$.

Then, using theorem 4.7 and corollary 3.11 of [CM2], we can write

$$\begin{aligned}\nu^{\tau}_{\Lambda,N}(f_1,f_1) &\leq c_0 N_1^* \sum_n (\gamma(n)\wedge\gamma(n-1))\left[\nu^{\tau}_{\Lambda,N}(f\,|\,N_{\Lambda_1}=n) - \nu^{\tau}_{\Lambda,N}(f\,|\,N_{\Lambda_1}=n-1)\right]^2 \\ &\leq c_0 N_1^* \sum_n (\gamma(n)\wedge\gamma(n-1))\,[A(n)^2 + B(n)^2]\end{aligned} \tag{6.7}$$

where

$$A(n)=\begin{cases}\frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)}\sum_{\substack{x\in\Lambda_1\\ z\in\Lambda\setminus\Lambda_1}}\nu^\tau_{\Lambda,\mathbf{N}}\left[(\nabla_{zx}f)\,\mathbb{1}_{E_{zx}}e^{-\nabla_{xz}H_\Lambda}\,|\,N_{\Lambda_1}=n-1\right] & \text{if } n\le u\\[2ex] \frac{1}{c'_{N-n+1}}\frac{\gamma(n)}{\gamma(n-1)}\sum_{\substack{x\in\Lambda_1\\ z\in\Lambda\setminus\Lambda_1}}\nu^\tau_{\Lambda,\mathbf{N}}\left[(\nabla_{xz}f)\,\mathbb{1}_{E_{xz}}e^{-\nabla_{xz}H_\Lambda}\,|\,N_{\Lambda_1}=n\right] & \text{otherwise}\end{cases}$$

$$B(n)=\begin{cases}\frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)}\sum_{\substack{x\in\Lambda_1\\ z\in\Lambda\setminus\Lambda_1}}\nu^\tau_{\Lambda,\mathbf{N}}\left[\left(e^{-\nabla_{xz}H_\Lambda}-1\right)\mathbb{1}_{E_{zx}},f\,|\,N_{\Lambda_1}=n-1\right] & \text{if } n\le u\\[2ex] \frac{1}{c'_{N-n+1}}\frac{\gamma(n)}{\gamma(n-1)}\sum_{\substack{x\in\Lambda_1\\ z\in\Lambda\setminus\Lambda_1}}\nu^\tau_{\Lambda,\mathbf{N}}\left[\left(e^{-\nabla_{xz}H_\Lambda}-1\right)\mathbb{1}_{E_{xz}},f\,|\,N_{\Lambda_1}=n\right] & \text{otherwise}\end{cases} \tag{6.8}$$

and

$$E_{xz}=\{\sigma\in\Omega:\sigma(x)=1,\ \sigma(z)=0\}\,. \tag{6.9}$$

Exactly as in section 5.2 of [CM2], the first term in the r.h.s. of (6.7) is bounded from above by

$$C'l^2\mathcal{E}^\tau_{\Lambda,N}(f,f) \tag{6.10}$$

We now turn to the estimate of the second term in the r.h.s. of (6.7). Using lemma 4.8 we get that for any $\delta>0$ there exists a constant $C(\delta,M,\epsilon')$ such that the second term in the r.h.s. of (6.7) is smaller than

$$Cl^2\,\mathcal{E}^\tau_{\Lambda,N}(f,f)+\delta\,\nu^\tau_{\Lambda,N}\,(f,f) \tag{6.11}$$

In conclusion, for any $\delta>0$ there exists a constant $C''(\delta,M,\epsilon')$ such that the third term in the r.h.s. of (6.4) is smaller than

$$\nu^\tau_{\Lambda,N}\,(f_1,f_1)\le C''l^2\,\mathcal{E}^\tau_{\Lambda,N}(f,f)+\delta\,\nu^\tau_{\Lambda,N}\,(f,f) \tag{6.12}$$

A similar bound holds for the second term in the r.h.s. of (6.4).

6.4 The recursion completed.

We are finally in a position to complete the proof of theorem 6.1. If we put together (6.12) and (6.6) we get that, for any $\delta\in(0,\frac{1}{2})$

$$\begin{aligned}&\text{r.h.s. of (6.4)}\ \le\\ &\le(1+\delta)\max\{\,g(\tfrac{3}{4}l,L)^{-1},\,\hat g(l,L)^{-1}\,\}\Big[\mathcal{E}^\tau_{\Lambda,N}(f,f)+\frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}}\nu^\tau_{\Lambda,N}\left[c_{xy}\,(\nabla_{xy}f)^2\right]\Big]\\ &\quad+2C''l^2\,\mathcal{E}^\tau_{\Lambda,N}(f,f)+2\delta\,\nu^\tau_{\Lambda,N}\,(f,f)\end{aligned} \tag{6.13}$$

that is

$$
\begin{aligned}
&\nu^\tau_{\Lambda,N}(f,f) \le \\
&\le \left(\frac{1+\delta}{1-2\delta}\right) \max\Big\{ g(\frac{3}{4}l,L)^{-1},\, \hat g(l,L)^{-1} \Big\} \Big[\mathcal{E}^\tau_{\Lambda,N}(f,f) + \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_{\Lambda_3}} \nu^\tau_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy}f)^2\,\right]\Big] \\
&\quad + kl^2\, \mathcal{E}^\tau_{\Lambda,N}(f,f)
\end{aligned}
\tag{6.14}
$$

for a suitable constant $k = k(\delta, M, \epsilon)$.
Finally we average w.r.t. to the index j (see (6.2)) and use the observation that, as j varies in $[1, \frac{1}{10\epsilon}]$, the strips $\Lambda_3 := \Lambda_3^{(j)}$ are disjoint. In particular

$$
\frac{1}{2}\sum_{j\in[1,\frac{1}{10\epsilon}]}\ \sum_{[x,y]\in\mathcal{E}_{\Lambda_3^{(j)}}} \nu^\tau_{\Lambda,N}\left[\, c_{xy}\,(\nabla_{xy}f)^2\,\right] \le \mathcal{E}^\tau_{\Lambda,N}(f,f)
$$

so that

$$
\begin{aligned}
&\nu^\tau_{\Lambda,N}(f,f) \le \\
&\le \Big[(\frac{1+\delta}{1-2\delta})\,(1+\lfloor 10\epsilon\rfloor) \max\{ g(\frac{3}{4}l,L)^{-1},\, \hat g(l,L)^{-1} \} + kl^2 \Big]\, \mathcal{E}^\tau_{\Lambda,N}(f,f)
\end{aligned}
\tag{6.15}
$$

In other words

$$
\mathrm{gap}(L^\tau_{\Lambda,N})^{-1} \le \Big[(\frac{1+\delta}{1-2\delta})\,(1+\lfloor 10\epsilon\rfloor) \max\{ g(\frac{3}{4}l,L)^{-1},\, \hat g(l,L)^{-1} \} + kl^2 \Big] \tag{6.16}
$$

Notice that if the original rectangle Λ was chosen in the sub–class of $\bar{\mathcal{R}}^\epsilon_L(l)$ entering in the definition of $\hat g(l,L)$, *i.e.* $l_1 \le \frac{l_2}{2} + \frac{1}{10}l$, then we would have obtained the inequality (6.16) with the factor $\max\{ g(\frac{3}{4}l,L)^{-1},\, \hat g(l,L)^{-1} \}$ replaced by $g(\frac{3}{4}l,L)^{-1}$ simply because, for any $j \in [1, \lfloor\frac{1}{10\epsilon}\rfloor]$, both Λ_1 and Λ_2 would belong to $\bar{\mathcal{R}}^\epsilon_L(\frac{3}{4}l)$. Thus

$$
\hat g(l,L)^{-1} \le \left(\frac{1+\delta}{1-2\delta}\right)(1+\lfloor 10\epsilon\rfloor)\, g(\frac{3}{4}l,L)^{-1} + kL^2 \tag{6.17}
$$

If we combine (6.16) with (6.17) we finally get

$$
\mathrm{gap}(L^\tau_{\Lambda,N})^{-1} \le \left(\frac{1+\delta}{1-2\delta}\right)^2 (1+\lfloor 10\epsilon\rfloor)^2\ g(\frac{3}{4}L)^{-1} + k'l^2 \tag{6.18}
$$

for another constant k'. Thus

$$
g(l,L)^{-1} \le \left(\frac{1+\delta}{1-2\delta}\right)^2 (1+\lfloor 10\epsilon\rfloor)^2\ g(\frac{3}{4}l,L)^{-1} + k'l^2
$$

and two more iterations prove the recursive inequality of the theorem provided that the two parameters ϵ, δ were chosen small enough. □

6.5 Proof of theorem 2.1.

Let $\epsilon_0 = 1/(10d+1)$ and let us fix M large enough and $0 < \epsilon < \min\{\frac{\log_2 \frac{8}{3}}{\log_2 \frac{8}{3} - 2 + db}, \epsilon_0\}$, where b is the constant defined in corollary 5.2. Let also $\epsilon' = \frac{\epsilon}{db}$. Then by proposition 4.1 there exists a set $\Theta_0 \subset \Theta$ with $\mathbb{P}(\Theta_0) = 1$ such that for any $J \in \Theta_0$ there exists $L(J) < \infty$ with the property that $J \in \Theta_{good}(L, M, \epsilon')$ for any $L \geq L(J)$. Without loss of generality we can assume that $L(J)$ is larger than some fixed large constant $L_0 = L_0(M, \epsilon)$. If we now apply theorem 6.1 we obtain

$$\min_{N,\tau} \operatorname{gap}(L^{\tau,J}_{Q_L,N})L^2 \geq \left[4k + L^{(1-\epsilon)\log_2 \frac{3}{8} - 2\epsilon} g(L^\epsilon, L)^{-1}\right]^{-1} \tag{6.19}$$

provided that $L > L(J)$. Thanks to corollary 5.2

$$g(L^\epsilon, L)^{-1} \leq L^{db\epsilon} \tag{6.20}$$

so that

$$\min_{N,\tau} \operatorname{gap}(L^{\tau,J}_{Q_L,N})L^2 \geq \frac{1}{4k+1}$$

and the proof is concluded. □

6.6 Proof of theorem 2.2.

Let the constants M, ϵ, ϵ' as well as the set Θ_0 and the constant $L(J)$ be as in the proof of theorem 2.1. Fix $J \in \Theta_0$, $L > 2L(J)$ and take $\Lambda = Q_L$. Let also f be an arbitrary local function such that $0 \in \Delta_f$. Denote by E_λ the spectral projection associated to the interval $[0, \lambda]$ for the selfadjoint operator $-L^\tau_{\Lambda,N}$ on $L^2\left(\Omega_\Lambda, d\nu^\tau_{\Lambda,N}\right)$. As usual we omit the superscript J in our notation. Assume that $\nu^\tau_{\Lambda,N}(f) = 0$. Then we will prove that for any $\delta \in (0, \frac{1}{10})$ there exist $C_{f,\delta}$ independent of Λ, N, J such that

$$\|E_\lambda f\|_2^2 \leq C_{f,\delta}\, \lambda^{\alpha-\delta} \tag{6.21}$$

where $\|\cdot\|_2$ denote the $L^2\left(\Omega_\Lambda, d\nu^\tau_{\Lambda,N}\right)$–norm and $\alpha = \alpha(d)$ is as in the theorem, provided that $L \geq 2\lambda^{-1} \geq 2L(J)^{\frac{d}{\delta}}$. It is clear that once such an estimate is available then

$$\begin{aligned}
\|e^{tL^\tau_{\Lambda,N}} f\|_2^2 &\leq \sum_{j=0}^\infty e^{-j} \|E_{\frac{j+1}{t}} f - E_{\frac{j}{t}} f\|_2^2 \\
&\leq C_{f,\delta}\, \frac{1}{t^{\alpha-\delta}} \sum_{j=0}^\infty e^{-j}(j+1)^\alpha \\
&\leq C'_{f,\delta}\, \frac{1}{t^{\alpha-\delta}}
\end{aligned}$$

if $L \geq 2t \geq 2\,L(J)^{\frac{d}{\delta}}$. If instead $t \leq L(J)^{\frac{d}{\delta}}$ then we bound $\|e^{tL^\tau_{\Lambda,N}} f\|_2^2$ by $\|f\|_\infty^2$. In particular (2.9) follows at once. Moreover, because of (iii) of proposition 4.1, $\mathbb{P}(t \leq L(J)^{\frac{d}{\delta}}) \leq t^{-\frac{\gamma\delta}{d}}$ where

$\gamma = \gamma(M)$ is such that $\lim_{M\to\infty}\gamma(M) = +\infty$. Thus, if M was chosen large enough, we can safely average over the disorder J and get (2.10).

Let us prove (6.21). For any $l \in [L(J), \frac{1}{2}L]$ let $\mathcal{F}_l := \mathcal{F}_{\Lambda\setminus B_l}$ and let $f_l := \nu^\tau_{\Lambda,N}(f \mid \mathcal{F}_l) = \nu^\eta_{B_l,N_l(\eta)}(f)$. If we use proposition 4.4 and the factorization of the grand canonical measure over the clusters we have

$$\mathrm{Var}^\tau_{\Lambda,N}(f_l) \le A(M,\epsilon)\mu^{\tau,\lambda}_\Lambda(f_l,f_l) = A\mu^\lambda_{\bar B_l}(f_l,f_l) \tag{6.22}$$

where the chemical potential λ here is, as always, such that $\mu^{\tau,\lambda}_\Lambda(N_\Lambda) = N$. We can now estimate from above $\mu^{\tau,\lambda}_\Lambda(f_l,f_l)$ by applying to it the Poincarè inequality for a Glauber dynamics reversible w.r.t. the gran canonical measure $\mu^{\tau,\lambda}_\Lambda$ (e.g. Metropolis or Heat Bath). By the independence of the clusters, the fact that, by construction, $J \in \Theta_{good}(M,\epsilon',l)$ and theorem 3.8 in [M], the spectral gap of the Glauber dynamics for $\mu^{\tau,\lambda}_\Lambda$ satisfies the following bound

$$\mathrm{gap}(\mathcal{L}_{\bar B_l}) = \min_{x\in\bar B_l}\mathrm{gap}(\mathcal{L}_{C_x}) \ge e^{-c\,(\log l)^{\frac{d-1}{d}}}$$

for some constant $c = c(\beta)$, where $\mathcal{L}_V$ is the Glauber generator in $V \subset \mathbb{Z}^d$ with free boundary conditions. Thus the r.h.s of (6.22) can be bounded from above by

$$Ae^{c\,(\log l)^{\frac{d-1}{d}}}\mu^\lambda_{\bar B_l}\Big(\sum_{x:\,d(x,B_l)\le r}[\nabla_x f_l]^2\Big) \tag{6.23}$$

It is not difficult to check at this point that, using part (a) of proposition 4.3 and proceeding as in the proof of lemma A.2, the r.h.s. of (6.23), for any $0<\delta<d$ can be bounded from above by $\frac{C_f}{l^{d-\delta}}$ for some positive constant $C_f = C_f(\delta,d,\beta)$ independent of J.

Observe now that for any function g and for any $l \le \frac{1}{2}L$ the formula of the conditional covariance (see e.g. (6.3)) together with the definition of spectral gap and the result of theorem 2.1 give the following inequality

$$\begin{aligned}\nu^\tau_{\Lambda,N}(g,f)^2 &\le 2\,\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(g,f\mid\mathcal{F}_l)^2\right) + 2\ \mathrm{Var}^\tau_{\Lambda,N}(g)\ \mathrm{Var}^\tau_{\Lambda,N}(f_l)\\ &\le 2\,\nu^\tau_{\Lambda,N}\left(\nu^\tau_{\Lambda,N}(g,f\mid\mathcal{F}_l)^2\right) + 2C_f\,\frac{1}{l^{d-\delta}}\ \mathrm{Var}^\tau_{\Lambda,N}(g)\\ &\le C'_f\ \left[l^2\,\mathcal{E}^\tau_{\Lambda,N}(g,g) + \frac{1}{l^{d-\delta}}\ \mathrm{Var}^\tau_{\Lambda,N}(g)\right]\end{aligned} \tag{6.24}$$

We will use (6.24) as the starting point of a recursive procedure whose final result will be a bound like (6.24) but with the factor l^2 replaced by l^ω with $\omega = 2\delta$ if $d\ge 2$ and $\omega = 1+2\delta$ if $d=1$, provided that $l \ge L(J)^{\frac{2}{\omega}}$.

Lemma 6.2. *Let $\beta_d = 0$ if $d\ge 3$, $\beta_d = \delta$ if $d=2$ and $\beta_d = 1+\delta$ if $d=1$. In the same hypotheses of theorem 2.2 assume that for some $\beta\in[\beta_d,2)$ and $\alpha\ge 1$, some positive constant $C(f,\delta)$, all pairs $L(J)^\alpha \le l_1 \le \frac{1}{2}l_2$ and all N the following inequality holds*

$$\nu^\tau_{B_{l_2},N}(g,f)^2 \le C(f,\beta)\ \left[l_1^\beta\,\mathcal{E}^\tau_{B_{l_2},N}(g,g) + \frac{1}{l_1^{d-\delta}}\ \mathrm{Var}^\tau_{B_{l_2},N}(g)\right] \tag{6.25}$$

Then there exists a new positive constant $C'(f,\beta)$ such that the same inequality holds with β and α replaced by $\beta' = \frac{2\beta}{d-\delta+\beta}$ and $\alpha' = \alpha\frac{(\beta+d-\delta)}{2}$ respectively.

Proof. Given (6.24) the proof is identical to that of lemma 6.1 in [CM2]. $\square$

Notice that the inequality of the lemma holds for $\beta = 2$ and $\alpha = 1$ because of (6.24). Therefore, by iterating n times inequality (6.25) starting from these values for α and β, for all pairs l, L such that $\frac{1}{2}L \geq l \geq L(J)^{\alpha_n}$, we find

$$\nu^\tau_{\Lambda,N}(g,f)^2 \leq C_{f,n}\,[l^{\beta_n}\,\mathcal{E}^\tau_\Lambda(g,g) + \frac{1}{l^{d-\delta}}\,\mathrm{Var}^\tau_{\Lambda,N}(f)]$$

where $\alpha_n = \frac{2}{\beta_n}$ and the sequence β_n converges to β_d (defined in lemma 6.2) from above. Therefore, for any $\delta \in (0, \frac{1}{10})$, there exists a constant $C_{f,\delta}$ such that

$$\nu^\tau_{\Lambda,N}(g,f)^2 \leq C_{f,\delta}\,[l^{\omega}\,\mathcal{E}^\tau_\Lambda(g,g) + \frac{1}{l^{d-\delta}}\,\mathrm{Var}^\tau_{\Lambda,N}(f)\,] \tag{6.26}$$

provided that $\frac{1}{2}\,L \geq l \geq L(J)^{\frac{2}{\omega}}$, where $\omega = 1 + 2\delta$ if $d = 1$ and $\omega = 2\delta$ if $d \geq 2$. If we now take $g := E_\lambda f$, then (6.26) gives

$$\|E_\lambda f\|_2^2 \leq C'_{f,\delta}\,[l^{\omega}\,\lambda + \frac{1}{l^{d-\delta}}\,] \leq C'_{f,\delta}\begin{cases}\lambda^{\frac{1}{2}-\delta} & \text{if } d=1\\ \lambda^{1-\delta} & \text{if } d\geq 2\end{cases}$$

provided that $L^{\omega+d-\delta} \geq 2\lambda^{-1} \geq 2L(J)^{2\frac{\omega+d-\delta}{\omega}}$. Above we have fixed l equal to $\lambda^{-\frac{1}{\omega+d-\delta}}$ in order to obtain the last inequality. It is important to observe that our choice gives $l \ll L$ since $\lambda \geq CL^{-2}$ because of theorem 2.1. This end the proof of (6.21). $\square$

Appendix. Proof of Lu–Yau's two block estimate.

Here we prove Lu–Yau's two block estimate which is the key ingredient behind lemma 4.8. The proof follows essentially the same lines of the one given in [CM2] for translation invariant interaction under a mixing condition, but with some important difference due to the presence of clusters where the particle variables are strongly interacting.

The setting is the following. We fix $\epsilon \in (0,1)$ and $l \in [2L^\epsilon, L]$. We then consider a volume $\Lambda \in \mathcal{R}_L^\epsilon(l)$ such that $\Lambda = \cup_{i=1}^k \Lambda_i$, where $\Lambda_i \in \mathcal{R}_L^\epsilon(l)$ and $|\Lambda_i|/|\Lambda| \geq \epsilon$ for $i = 1, \cdots, k$. Let $\mathbf{N} := \{N_i\}_{i=1}^k$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_i}\}_{i=1}^k$. Let also $h_x := e^{-\nabla_x H}\sigma(x)$ and $\hat{h}_x := e^{-\nabla_x H}(1-\sigma(x))$ and take $\epsilon' = \frac{2}{db}\epsilon$, where b is defined in proposition 5.2 and d is the dimension. We define $G := \frac{1}{|\Lambda_i|}\sum_{x\in\Lambda_i} g_x$ where $g_x = h_x$ or $g_x = \hat{h}_x$.

Proposition A.1. *Assume $J \in \Theta_{good}(M, \epsilon', L)$. Then for any $\delta > 0$ there exist a positive constant C and $L_0(\epsilon', M, \delta) > 0$ such that*

$$\nu^\tau_{\Lambda,\mathbf{N}}\big(f, G\big)^2 \leq \frac{Cl^2}{|\Lambda|}\,\mathcal{E}^\tau_{\Lambda,\mathbf{N}}(f,f) + \frac{\delta}{|\Lambda|}\,\mathrm{Var}^\tau_{\Lambda,\mathbf{N}}(f)$$

provided that $L \geq L_0$.

Proof. To simplify the notation we prove the proposition in the case $k = 1$. When $k > 1$ the proof can be easly generalized.
Fix $\delta > 0$. If $\rho \leq \delta$ or $1-\rho \leq \delta$ the statement follows at once from the Schwartz inequality together with proposition 4.4 and the fact that using that $J \in \Theta_{good}(M, \epsilon', L)$

$$\mu^{\tau,\lambda}_\Lambda(G,G) = \frac{1}{|\Lambda|^2}\sum_{x\in\Lambda}\frac{\sum_{y,z\in C_x\cap\Lambda}\mu^{\tau,\lambda}_\Lambda(g_y, g_z)}{|C_x\cap\Lambda|} \leq C(\beta)\rho(1-\rho)\frac{1}{|\Lambda|}$$

for large values of the (constant) chemical potential λ. We will thus assume, without further notice, that $\rho \in (\delta, 1-\delta)$.

We define $\{Q_\alpha\}_{\alpha\in I}$ to be a collection of cubes of side $L^{\epsilon'}$, such that for any $\alpha \neq \beta$ $\mathrm{dist}(Q_\alpha, Q_\beta) \geq 2M\log L$, $\mathrm{dist}(Q_\alpha, \partial\Lambda) \geq 2M\log L$ and $|\Lambda \setminus \cup_\alpha Q_\alpha| \leq |\Lambda|\log L/L^{\epsilon'}$. Clearly such collection exists. Next we observe that, without loss of generality, we can replace g_x by $g_x - \gamma\sigma(x)$, γ being an arbitrary constant independent of x, because $\sum_{x\in\Lambda}\sigma(x) = N$ almost surely w.r.t. $\nu^\tau_{\Lambda,N}$. Accordingly we define $G_\gamma := G - \gamma\,\frac{N_\Lambda}{|\Lambda|}$. Our choice of γ will be made later. We then set

$$G^{\mathrm{int}}_\gamma := \frac{1}{|\Lambda|}\sum_{x\in\cup_\alpha Q^{\mathrm{int}}_\alpha}\big(g_x - \gamma\sigma(x)\big) \quad \text{and} \quad G^{\mathrm{ext}}_\gamma := G_\gamma - G^{\mathrm{in}}_\gamma$$

where $Q^{\mathrm{int}}_\alpha = \{x \in Q_\alpha \;:\; d(x, Q^c_\alpha) \geq 2M\log L\}$. Notice that

$$\mathrm{Var}^\tau_{\Lambda,N}(G^{\mathrm{ext}}_\gamma) \leq C'\mu^{\tau,\lambda}_\Lambda(G^{\mathrm{ext}}_\gamma, G^{\mathrm{ext}}_\gamma) \leq C''\frac{\sum_{x\in\Lambda\setminus\cup_\alpha Q^{\mathrm{in}}_\alpha}|C_x|}{|\Lambda|^2} \leq C''\frac{(\log L)^2}{|\Lambda|L^{\epsilon'}} \qquad (A.1)$$

because of proposition 4.4 and the definition of $\{Q_\alpha\}_{\alpha\in I}$. In particular, for any given $\delta > 0$,

$$\nu^{\tau}_{\Lambda,N}\big(f, G^{\rm ext}_{\delta}\big)^2 \le \frac{\delta}{|\Lambda|}\, \mathrm{Var}^{\tau}_{\Lambda,N}(f)$$

provided that L is big enough.

We now turn to bound the relevant part $\nu^{\tau}_{\Lambda,N}\big(f, G^{\rm int}_{\gamma}\big)^2$. Let $\mathcal{F}_0$ be the σ–algebra generated by the the random variables $\{\sigma(x)\}_{x\in\Lambda\setminus\cup_\alpha Q_\alpha}$, $\{N_\alpha\}_{\alpha\in I}$, where $N_\alpha(\sigma) := \sum_{x\in Q_\alpha}\sigma(x)$. Then we write

$$\begin{aligned}
\nu^{\tau}_{\Lambda,N}\big(f, G^{\rm int}_{\gamma}\big)^2 &\le 2\,\nu^{\tau}_{\Lambda,N}\Big(\nu^{\tau}_{\Lambda,N}\big(f, G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)\Big)^2 + 2\,\nu^{\tau}_{\Lambda,N}\Big(f, \nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)\Big)^2 \\
&\le 2\ \mathrm{Var}^{\tau}_{\Lambda,N}(G^{\rm int}_{\gamma})\nu^{\tau}_{\Lambda,N}\Big(\mathrm{Var}^{\tau}_{\Lambda,N}\big(f \,|\, \mathcal{F}_0\big)\Big) + 2\,\nu^{\tau}_{\Lambda,N}\Big(f, \nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)\Big)^2 \qquad (A.2)\\
&\le \frac{C(L)}{|\Lambda|}\mathcal{E}^{\tau}_{\Lambda,N}(f,f) + 2\ \mathrm{Var}^{\tau}_{\Lambda,N}(f)\ \mathrm{Var}^{\tau}_{\Lambda,N}\ \big(\nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)\big)
\end{aligned}$$

where we have used (A.1) to bound $\mathrm{Var}^{\tau}_{\Lambda,N}(G^{\rm int}_{\gamma})$ by $\frac{C'}{|\Lambda|}$ and the estimate

$$\nu^{\tau}_{\Lambda,N}\Big(\ \mathrm{Var}^{\tau}_{\Lambda,N}(f \,|\, \mathcal{F}_0)\Big) \le C(L)\,\mathcal{E}^{\tau}_{\Lambda,N}(f,f)$$

for some constant $C(L)$, valid since $\nu^{\tau}_{\Lambda,N}(\cdot \,|\, \mathcal{F}_0)$ is the product of canonical Gibbs measures over the cubes Q_α. Actually, using proposition 5.2, the constant $C(L)$ is not larger than $L^{bd\epsilon'} \le l^2$.

The key point is now to prove that, for any $\delta > 0$, $\mathrm{Var}^{\tau}_{\Lambda,N}\big(\nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)\big)$ is smaller than $\frac{\delta}{|\Lambda|}$ provided that L is large enough.

Notice that $\nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)(\eta)$ is the sum of local functions

$$\nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)(\eta) = \frac{1}{|\Lambda|}\sum_{\alpha\in I}\nu^{\eta}_{Q_\alpha,N_\alpha(\eta)}\Big(\sum_{x\in Q^{\rm int}_\alpha} g_x - \gamma\sigma(x)\Big) := \frac{1}{|\Lambda|}\sum_{\alpha\in I} G^{\gamma}_{\alpha}(\eta)$$

Thus, if we order in an arbitrary way the set I, we can split the above sum into the sum of even and odd α's and apply proposition 4.4 to each term and get

$$\mathrm{Var}^{\tau}_{\Lambda,N}\ \big(\nu^{\tau}_{\Lambda,N}\big(G^{\rm int}_{\gamma} \,|\, \mathcal{F}_0\big)\big) \le C'\frac{1}{|\Lambda|^2}\,\mu^{\tau,\lambda}_{\Lambda}\left(\sum_\alpha G^{\gamma}_{\alpha}, \sum_\alpha G^{\gamma}_{\alpha}\right)$$

for some constant C' independent of Λ and L_0.

Let now $\xi^{\gamma}_{\alpha}(\eta) := \mu^{\eta,\lambda(\eta)}_{Q_\alpha}\Big(\sum_{x\in Q^{\rm int}_\alpha}[g_x - \gamma\sigma(x)]\Big)$, where the chemical potential $\lambda(\eta)$ is such that $\mu^{\eta,\lambda(\eta)}_{Q_\alpha}(N_\alpha) = N_\alpha(\eta)$. In the (rare) case in which $N_\alpha(\eta) = 0$ ($N_\alpha(\eta) = |Q_\alpha|$) the measure $\mu^{\eta,\lambda(\eta)}_{Q_\alpha}$ will simply be the Dirac measure on the constant configuration identically equal to 0 (1). Thanks to *1)* of proposition 4.3 and the hypothesis $J \in \Theta_{good}(M,\epsilon',L)$ we have $\sup_\eta |G^{\gamma}_{\alpha}(\eta) - \xi^{\gamma}_{\alpha}(\eta)| \le C'$. In particular, using the fact that $\mathrm{dist}(Q_\alpha, Q_\beta) \ge 2M\log L$ for any $\alpha \ne \beta$, we get

$$\begin{aligned}
\frac{1}{|\Lambda|^2}\mu^{\tau,\lambda}_{\Lambda}\Big(\sum_\alpha G^{\gamma}_{\alpha} - \xi^{\gamma}_{\alpha}, \sum_\alpha G^{\gamma}_{\alpha} - \xi^{\gamma}_{\alpha}\Big) &\le \frac{C}{|\Lambda|^2}\sum_\alpha \mu^{\tau,\lambda}_{\Lambda}\left(G^{\gamma}_{\alpha} - \xi^{\gamma}_{\alpha}, G^{\gamma}_{\alpha} - \xi^{\gamma}_{\alpha}\right) \\
&\le C'\frac{1}{|\Lambda| L^{d\epsilon'}} \le \frac{\delta}{|\Lambda|}
\end{aligned}$$

for L large enough. It is therefore enough to bound $\mu_\Lambda^{\tau,\lambda}\Big(\sum_\alpha \xi_\alpha^\gamma, \sum_\alpha \xi_\alpha^\gamma\Big)$.

We can now apply the Poincarè inequality $\mu_\Lambda^{\tau,\lambda}(f,f) \leq C'(L)\mu_\Lambda^{\tau,\lambda}\left(\sum_{x\in\Lambda}(\nabla_x f)^2\right)$ where $C'(L) = \mathrm{gap}(\mathcal{L}_\Lambda^\tau)^{-1}$ and $\mathcal{L}_\Lambda^\tau$ is Glauber generator. By the independence of the clusters, the fact that $J \in \Theta_{good}(M,\epsilon',L)$ and theorem 3.8 in [M] we have

$$\mathrm{gap}(\mathcal{L}_\Lambda^\tau) = \min_{x\in\Lambda} \mathrm{gap}(\mathcal{L}_{C_x}) \geq \exp\{-c(\log L)^{\frac{d-1}{d}}\}$$

so that

$$\frac{1}{|\Lambda|^2}\,\mu_\Lambda^{\tau,\lambda}\Big(\sum_\alpha \xi_\alpha^\gamma, \sum_\alpha \xi_\alpha^\gamma\Big) \leq \frac{1}{|\Lambda|^2}\, e^{c(\log L)^{\frac{d-1}{d}}}\, \mu_\Lambda^{\tau,\lambda}\Big(\sum_{y\in\Lambda}\big[\nabla_y \sum_{\alpha\in I}\xi_\alpha^\gamma\big]^2\Big) \tag{A.3}$$

Observe now that, by construction, $\nabla_y\, \xi_\alpha^\gamma = 0$ unless $\mathrm{dist}(y, Q_\alpha) \leq r$. Thus

$$\mu_\Lambda^{\tau,\lambda}\Big(\sum_{y\in\Lambda}\big[\nabla_y \sum_{\alpha\in I}\xi_\alpha^\gamma\big]^2\Big) = \sum_{\alpha\in I}\ \sum_{\substack{y\in\Lambda\\ \mathrm{dist}(y,Q_\alpha)\leq r}} \mu_\Lambda^{\tau,\lambda}\Big(\big[\nabla_y\, \xi_\alpha^\gamma\big]^2\Big) \tag{A.4}$$

Let us estimate a generic term $\mu_\Lambda^{\tau,\lambda}\left(\big[\nabla_y\, \xi_\alpha^\gamma\big]^2\right)$. It is at this stage that the subtraction with the free parameter γ made at the beginning becomes important. Let $\tilde\nabla_y f(\sigma) := (1-\sigma(y))\nabla_y\, f - \sigma(y)\nabla_y f$ and notice that

$$\big[\nabla_y\, \xi_\alpha^\gamma\big]^2 = \big[\tilde\nabla_y\, \xi_\alpha^\gamma\big]^2 = \big[\tilde\nabla_y\, \mu_{Q_\alpha}^{\eta,\lambda(\eta)}\big(\sum_{x\in Q_\alpha^{\mathrm{int}}} g_x\big) - \gamma\big]^2$$

Let $\lambda_0 = \lambda_0(\Lambda, N)$ be the chemical potential such that $\mathbb{E}\big(|C_0|^{-1}\,\mu_{C_0}^{\beta,\lambda_0}(N_{C_0})\big) = N/|\Lambda|$ where C_0 is the cluster of the center of Λ (see section 3). The following lemma concludes the proof of the proposition. Equation (A.5) below and the fact that $J \in \Theta_{good}(M,\epsilon',L)$ imply that (A.4) can be bounded from above by $c|\Lambda|(\frac{\log L}{L^{\epsilon'}} + \frac{1}{|Q_\alpha|^{2\epsilon'}})$. Thus the right hand side of (A.3) is smaller than $\frac{\delta}{|\Lambda|}$ provided that L is large enough.

Lemma A.2. *In the same setting of proposition A.1 define*

$$\gamma := \frac{\mathbb{E}\Big[|C_0|^{-1}\,\mu_{C_0}^{\lambda_0}\left(\sum_{x\in C_0} g_x, N_{C_0}\right)\Big]}{\mathbb{E}\Big[|C_0|^{-1}\,\mu_{C_0}^{\lambda_0}\left(N_{C_0}, N_{C_0}\right)\Big]}$$

Then there exist two positive constants k_1, k_2 independent of L and $L_0 = L_0(\epsilon', M, \delta)$ such that $\gamma \leq k_1$ and

$$\mu_\Lambda^{\tau,\lambda}\left(\big[\tilde\nabla_y\, \xi_\alpha^\gamma\big]^2\right) \leq \begin{cases} k_2|C_y| & \text{if } y \in \partial_r^+ Q_\alpha \\ \frac{k_2}{|Q_\alpha|^{2\epsilon'}} & \text{if } y \in Q_\alpha \end{cases} \tag{A.5}$$

provided that $L \geq L_0$

Proof. The fact that γ is bounded from above uniformly in L follows immediately from its definition and the fact that $\mathbb{E}(|C_0|^n) \leq k$ for $p < p_c$ (see e.g. [Gri]).

We first consider the case $y \in \partial_r^+ Q_\alpha$ then

$$
\begin{aligned}
|\tilde\nabla_y \xi_\alpha^\gamma| &= |\tilde\nabla_y \mu_{Q_\alpha}^{\eta,\lambda(\eta)}(\sum_{x\in Q_\alpha^{in}} g_x) - \gamma| \\
&\le \sum_{x\in Q_\alpha^{in}} |\mu_{C_x}^{\lambda(\eta^y)}(g_x) - \mu_{C_x}^{\lambda(\eta)}(g_x)| + \gamma \\
&\le c \sum_{x\in Q_\alpha^{in}} \frac{|C_x|\,|C_y|}{|Q_\alpha|} + \gamma \le k_2\,|C_y|
\end{aligned}
$$

where we used that $J \in \Theta_{good}(M, \epsilon', L)$ in the last inequality.

Let us now consider $y \in Q_\alpha$.

In this case, under the flip of the variable $\eta(y)$, the value of ξ_α^γ changes only because the number of particles of η varies by ± 1. Define $G_\alpha := \sum_{x\in Q_\alpha^{in}} g_x$, call $N_\alpha(\eta) = n$ and let $\lambda(s) = \lambda(\eta, s)$ be the chemical potential such that $\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}\big(N_\alpha\big) = s$ with $s \in [0, |Q_\alpha|]$. Then

$$
\frac{d\lambda(s)}{ds} = \frac{1}{\mu_{Q_\alpha}^{\eta,\lambda(s)}(N_\alpha, N_\alpha)} \tag{A.6}
$$

so that

$$
\tilde\nabla_y \mu_{Q_\alpha}^{\eta,\lambda(\eta,n)}(G_\alpha) = (1-\eta(y)) \int_n^{n+1} \frac{\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(G_\alpha, N_\alpha)}{\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(N_\alpha, N_\alpha)} ds \; - \eta(y) \int_n^{n-1} \frac{\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(G_\alpha, N_\alpha)}{\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(N_\alpha, N_\alpha)} ds
$$

By adding and subtracting inside the integrals the term $\mu_{Q_\alpha}^{\eta,\lambda}\big(G_\alpha, N_\alpha\big)/\mu_{Q_\alpha}^{\eta,\lambda}\big(N_\alpha, N_\alpha\big)$ (λ here is the chemical potential of the grand canonical measure on the volume Λ), we have

$$
\begin{aligned}
&\mu_\Lambda^{\tau,\lambda}\left(\left[\tilde\nabla_y \mu_{Q_\alpha}^{\eta,\lambda(s)}\big(G_\alpha\big) - \gamma\right]^2\right) \le \\
&2\,\mu_\Lambda^{\tau,\lambda}\Big(\left|\frac{\mu_{Q_\alpha}^{\eta,\lambda}\big(G_\alpha, N_\alpha\big)}{\mu_{Q_\alpha}^{\eta,\lambda}\big(N_\alpha, N_\alpha\big)} - \gamma\right|^2\Big) + 2\,\mu_\Lambda^{\tau,\lambda}\Big(\left|\int_{n-1}^{n+1} R(s)ds\right|^2\Big)
\end{aligned} \tag{A.7}
$$

where

$$
R(s) = \frac{\mu_{Q_\alpha}^{\eta,\lambda(s)}\big(G_\alpha, N_\alpha\big)}{\mu_{Q_\alpha}^{\eta,\lambda(s)}\big(N_\alpha, N_\alpha\big)} - \frac{\mu_{Q_\alpha}^{\eta,\lambda}\big(G_\alpha, N_\alpha\big)}{\mu_{Q_\alpha}^{\tau,\lambda}\big(N_\alpha, N_\alpha\big)}
$$

Notice that, using the fact that $J \in \Theta_{good}(M, \epsilon', L)$ it is easy to prove that

$$
\left|\frac{\mu_{Q_\alpha}^{\eta,\lambda}(G_\alpha, N_\alpha)}{\mu_{Q_\alpha}^{\eta,\lambda}(N_\alpha, N_\alpha)} - \gamma\right| \le \frac{c}{\rho_0 |Q_\alpha|^{\epsilon'}} \tag{A.8}
$$

where $\rho_0 := \mu_\Lambda^{\tau,\lambda_0}(N_\Lambda)/|\Lambda|$.
Let $\tilde n = \mu_{Q_\alpha}^{\eta,\lambda}(N_\alpha)$. Then using (A.6) we can write

$$R(s) = \frac{1}{\mu_{Q_\alpha}^{\eta,\lambda}(N_\alpha,N_\alpha)}\int_{\tilde n}^{s} ds' \frac{\mu_{Q_\alpha}^{\eta,\lambda(s')}(G_\alpha,N_\alpha,N_\alpha)}{\mu_{Q_\alpha}^{\eta,\lambda(s')}(N_\alpha,N_\alpha)}$$
$$+ \frac{\mu_{Q_\alpha}^{\eta,\lambda(s)}(G_\alpha,N_\alpha)}{\mu_{Q_\alpha}^{\eta,\lambda(s)}(N_\alpha,N_\alpha)\mu_{Q_\alpha}^{\eta,\lambda}(N_\alpha,N_\alpha)}\int_{s}^{\tilde n} ds' \frac{\mu_{Q_\alpha}^{\eta,\lambda(s')}(N_\alpha,N_\alpha,N_\alpha)}{\mu_{Q_\alpha}^{\eta,\lambda(s')}(N_\alpha,N_\alpha)}$$

so that by proposition 4.2, (A.6) and the fact that $J \in \Theta_{good}(M,\epsilon',L)$, it is easy to see that

$$\begin{aligned}\mu_\Lambda^{\tau,\lambda}\left(\left(\int_{n-1}^{n+1} R(s)ds\right)^2\right) \leq & c\|g\|_\infty^2\, \mu_\Lambda^{\tau,\lambda}\left(\frac{(n-\tilde n)^2}{\tilde n^2}\right) = \\ & c\|g\|_\infty^2\, \mu_\Lambda^{\tau,\lambda}\left(\frac{\mu_{Q_\alpha}^{\eta,\lambda}(N_\alpha,N_a)}{\mu_{Q_\alpha}^{\eta,\lambda}(N_\alpha)^2}\right) \leq \frac{c}{\rho|Q_\alpha|}\end{aligned} \tag{A.9}$$

Putting together (A.8), (A.9), the fact that $|\rho_0-\rho|\leq|\Lambda|^{-\epsilon}$ and $\rho>\gamma$ the lemma follows for L big enough. □

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