MAXIMUM OF FRACTIONAL BROWNIAN MOTION: PROBABILITIES OF SMALL VALUES

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Abstract

Let $b_\gamma(t)$, $b_\gamma(0) = 0$ be fractional Brownian motion, i.e., a Gaussian process with the structure function $E|b_\gamma(t)-b_\gamma(s)|^2 = |t-s|^\gamma$, $0 < \gamma < 2$. We study the logarithmic asymptotics of $P_T = P\{b_\gamma(t) < 1, \quad t \in T\Delta\}$ as $T \to \infty$, where Δ is either the interval $(0,1)$ or a bounded region that contains a vicinity of 0 for the case of multidimensional time. It is shown that $\log P_T = -D \log T(1 + o(1))$, where D is the dimension of zeroes of $b_\gamma(t)$ in the former case and the dimension of time in the latter.

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1. INTRODUCTION AND RESULTS

In connection with an analysis of the fractal nature of solutions of the Burgers equation with random initial data [13,14], Ya.Sinai posed the problem of finding the asymptotics for the probability

$$P\{b_\gamma(s) < 1, \quad s \in T \cdot \Delta\} = F_\gamma(T) \tag{1}$$

as $T \to \infty$ and Δ is the interval $(0,1)$ for the so-called fractional Brownian motion (FBM) of index $\gamma \in (0,2)$. FBM is a continuous Gaussian process with the structure function

$$E|b_\gamma(t) - b_\gamma(s)|^2 = |t-s|^\gamma,$$

$Eb_\gamma(t) = b_\gamma(0) = 0$. That family of processes naturally arises in many applications [9] in virtue of affine invariance:

$$b_\gamma(\Lambda t + a) \stackrel{\mathrm{d}}{=} b_\gamma(a) + |\Lambda|^{\gamma/2}\, b_\gamma(t) \tag{2}$$

Here, a and Λ are arbitrary parameters, $\stackrel{\mathrm{d}}{=}$ denoting equality for finite-dimensional distributions. From (2) it follows that

$$F_\gamma(t) = P\{\max_{t\in\Delta} b_\gamma(t) < x\}, \quad x = T^{-\gamma/2} \to 0 \tag{3}$$

The original problem is therefore equivalent to the asymptotics probability of small values of the maximum of $b_\gamma(t)$, $M_\Delta = \max_{t\in\Delta} b_\gamma(t)$. The theory of Gaussian processes has well-developed asymptotics for large and small deviations of Gaussian functions, i.e. $P\{\max_\Delta \xi(t) > u\}$ and $P\{\max_\Delta |\xi(t)| < 1/u\}$ as $u \to \infty$ (see [8]). In the same time the related problem (3) has been little dealt with, if at all. Exeptions are some isolated examples requiring special analytical tools. Here we note [14,5] where one can find the exact asymptotics of (3) for the integral of Brownian motion. In the case of FBM, recently Sinai [15] derived an Abel integral equation of the second kind for $F_\gamma(T)$ and gave reasons in favor of a power-law asymptotics $F_\gamma(T)$, $T \to \infty$ for γ

close to 1. Below we develop a different approach and find the logarithmic asymptotics of $F_\gamma(T)$ with any $\gamma \in (0,2)$ in the following form:

$$\log F_\gamma(T) = D \log T^{-1}(1 + O((\log T)^{-1/2})), \quad T \to \infty \tag{4}$$

We consider two types of Δ. The first is the original one, $\Delta = (0,1)$, in which 0 is an end point of Δ. In this case $D = 1 - \gamma/2$, that is, the Hausdorff dimension of the zero set of FBM. The second type of Δ is relevant to the case in which 0 is an inner point of Δ, and Δ is a bounded convex set when FBM with multidimensional time is considered. In this case the logarithmic asymptotics (4) does not depend on the correlation structure of FBM and D is the dimension of time.

It is well known from the theory of Brownian motion (see [4]) that the following random variables have the so-colled arcsine law for $\gamma = 1$:
$\tau_{\max} = \mathrm{Arg}\max_{[0,1]} b_\gamma(t)$ is the position of the maximum of $b_\gamma(t)$ on $[0,1]$;
$z_- = \sup\{t \in (0,1) : \ b_\gamma(t) = 0\}$ is the last zero of $b_\gamma(t)$ on $[0,1]$;
$s_+ = \mathrm{mes}\{t \in (0,1) : \ b_\gamma(t) > 0\}$ is the total time spent in $[0,\infty)$ by $b_\gamma(t)$ during the time interval $[0,1]$.
These variables are closely related to the problem (4), $\Delta = (0,1)$. Roughly speaking, small values of the random variables M_Δ, $\tau_{\max}$, z_- and s_+ are strongly correlated for FBM. In particular, the typical position of small global maximum of FBM on Δ belongs to a vicinity of $t = 0$ where $Eb_\gamma^2(t)$ is minimum. This contrasts with the typical area of large deviations of a Gaussian function in which its dispersion function takes largest values. We will show that, with any $\gamma \in (0,2)$, the variables $\xi : \tau_{\max}$, z_- and s_+ have the same asymptotics:

$$\log P\{\xi < x\} = D \log x(1 + o(1)), \quad x \to 0 \tag{5}$$

with $D = 1 - \gamma/2$. The invariance of $b_\gamma(t)$ with respect to the Moebius inversion

$$t^\gamma b_\gamma(t^{-1}) \overset{\mathrm{d}}{=} b_\gamma(t) \tag{6}$$

gives the relation $1/z_- \overset{\mathrm{d}}{=} z_+ = \inf\{s > 1 : \ b_\gamma(s) = 0\}$ for the duration of excursion of $b_\gamma(t)$ after time $t = 1$. For this reason (5) also gives the asymptotics of $P\{z_+ > T\}$, $T \to \infty$, which is identical with (4). A heuristic analysis of the probabilities of large distances between successive zeroes of FBM is contained in a physics work [2].

To the set $\{\tau_{\max},\ z_-,\ z_+,\ s_+\}$, which are all related to the problem (4), should be added local time $\mathcal{L}(t)$ [6], which serves as the measure of the time the process has been at zero on $[0,t]$. Our result for $\mathcal{L}(t)$ is incomplete. It is shown that

$$\log P\{\mathcal{L}(1) < x\} \geq \log x \cdot (1 + o(1)), \quad x \to 0$$

or, taking into account the scale invariance $\mathcal{L}$, $\mathcal{L}(t) \stackrel{\mathrm{d}}{=} t^D \mathcal{L}(1)$ for fixed t with $D = 1 - \gamma/2$,

$$\log P\{\mathcal{L}(T) < 1\} \geq D \log T^{-1} \cdot (1 + o(1)), \quad T \to \infty$$

Probably this is an exact estimate, i.e., the probabilities of events $\{M_{(0,T)} < 1\}$ and $\{\mathcal{L}(T) < 1\}$ have identical logarithmic asymptotics as $T \to \infty$.

The method of proof for (4) is based on a correlation of small values of M_Δ, $\tau_{\max}$ and z_+. To get the exact values of the exponents D we use the fact that $\tau_{\max}$ has a probability density (case $0 \notin \partial\Delta$). In the original case, $0 \in \partial\Delta$, we use, in an adapted form, one asymptotics result due to Kawazu and Tanaka [7] for an exponential functional of Brownian motion.

This paper is organized as follows. Section 2 gives some auxiliary statements. Sections 3 and 4 contain proofs of the asymptotics (4) and (5), respectively, for $\Delta = (0,1)$. Section 5 is a study of the asymptotics (1) for multiparameter fractional Brownian motion.

2. AUXILIARY STATEMENTS

Below we give some statements that are important for our proofs.

Statement 1. *Suppose that a process $x(t)$ is affine-invariant in the sense of* (2), *it has cadlag paths and*

$$E \sup_{[0,1]} |x(s)|) < \infty, \tag{7}$$

$$E\left[\int_0^1 \exp(x(u))\, du\right]^{-1} < \infty.$$

Then, as $t \to \infty$,

$$E\Big[\int_0^t \exp(x(u))\,du\Big]^{-1} = \gamma/2 \,\cdot\, t^{-(1-\gamma/2)}(E\sup_{[0,1]} x(t) + o(1)). \qquad (8)$$

This statement is implicitly contained in ([7], Section 2.2) for a Wiener process $x(t)$. The proof can be easily adapted to the general case.

Statement 2 (Fernique's estimation [3]).

Let $x(t)$ *be a Gaussian continuous function on* $T = [0,1]^d$ *with the structure function* $\mathbf{\Delta}^2(t,s) = E|x(t) - x(s)|^2$, $Ex(t) = 0$. *If*

$$c_F = \sup_{T\times T} \mathbf{\Delta}(t,s) + (2+\sqrt{2}\int_1^\infty \varphi(2^{-u^2})\,du) < \infty \qquad (9)$$

where $\varphi(h) = \sup\{\mathbf{\Delta}(t,s),\ |s-t| < h\}$, *then one has*

$$P\{\sup_T x(t) \ge c_F \cdot x\} \le 5\cdot 2^{2d-1}\int_x^\infty e^{-u^2/2}\,du \qquad (10)$$

for $x \ge \sqrt{1+4d}$. *Here* $|s-t| = \max(|s_i - t_i|,\ i = 1,\dots,d)$.

Below this statement is applied to the functions $x(t) = b_\gamma(t)$ and $x(t,s) = b_\gamma(t) - b_\gamma(s)$ for with $\varphi(h) < ch^{\gamma/2}$ and therefore $c_F < \infty$.

Statement 3 (Molchan and Golosov, [11]). *The Hilbert space* $H_-(\Delta)$ *of continuous functions* f, $f(0) = 0$ *on* $\Delta = [0,T]$ *with the reproducing kernel* [1]

$$B(t,s) = E\,b_\gamma(s)b_\gamma(t) = \frac{1}{2}(|t|^\gamma + |s^\gamma| - |t-s|^\gamma), \quad (t,s)\in\Delta^2$$

is specified by the following scalar product $(f,g)_-$ *on a dense subset of smooth functions*:

$$(f,f)_- = c_\gamma\int_\Delta |\hat{f}_\alpha(t)|^2\, t^{2\alpha}\,dt, \quad 2\alpha = \gamma - 1 \qquad (11)$$

where

$$\hat{f}_\alpha = \frac{d}{dt}\int_0^t \frac{(t-x)^{-\alpha}}{\Gamma(1-\alpha)}\, x^{-\alpha}\, f'(x)\, dx, \quad c_\gamma^{-1} = \Gamma(1+\gamma)\, \sin(\pi\,\gamma/2)$$

Elements of $H_-(\Delta)$ are admissible shifts of the measure that is relevant to the process $\{b_\gamma(t),\, t \in \Delta\}$. The formula giving the Radon-Nikodim derivative π of the shifted measure on $f \in H_-(\Delta)$ with respect to the original measure is called below the Cameron-Martin formula. It is (see [8])

$$\pi = \exp\{(f, b_\gamma)_- - (f,f)_-/2\}$$

3. FBM ON THE INTERVAL $[0, T]$

Theorem 1. *The maximum M_t of fractional Brownian motion on $[0,t]$ obeys the asymptotics*

$$\log P\{M_t < 1\} = D \log t^{-1}(1 + O((\log t)^{-1/2})), \quad t \to \infty$$

where $D = 1 - \gamma/2$.

The proof of this theorem will be divided among three lemmas. The theorem itself is an obvious corollary of lemmas 1 and 3.

Lemma 1 (lower bound of $F_t = P\{M_t < 1\}$).

There exist constants c_1 and c_2 for large t such that

$$F_t \geq c_1\, t^{\gamma/2-1} \exp\{-c_2\sqrt{\ln t}\} \tag{12}$$

Proof. One has in virtue of (8), as $t \to \infty$,

$$c\, t^{\gamma/2-1}(1 + o(1)) = E\Big[\int_0^t e^{b_\gamma(s)}\, ds\Big]^{-1} = I_1 + I_2 \tag{13}$$

where

$$\begin{aligned} I_1 &= E\,[\cdot]^{-1}\, \mathbf{1}_{M_t < c_t} \\ I_2 &= E\,[\cdot]^{-1}\, \mathbf{1}_{M_t > c_t}, \end{aligned}$$

c_t being a constant to be specified below, as are all the others.

Estimation of I_1. One has

$$I_1 \leq E\Big[\int_0^1 e^{b_\gamma(s)}\,ds\Big]^{-1} \mathbf{1}_{M_t<c_t} = E\xi \cdot \mathbf{1}_{M_t<c_t}$$

where $\xi < \exp(c_F\sqrt{2c_0 \ln t})$ on the set of the elementary events

$$\Omega = \{\omega : \min_{[0,1]} b_\gamma(s) > -c_F\sqrt{2c_0 \ln t}\}.$$

For this reason

$$I_1 \leq P\{M_t < c_t\} e^{c_F\sqrt{2c_0 \ln t}} + E\,\xi\,\mathbf{1}_{\Omega^c}$$

where Ω^c is the complement of Ω. Using the Schwartz inequality, one has

$$(E\,\xi\,\mathbf{1}_{\Omega^c})^2 \leq P(\Omega^c)\,E\,\xi^2 \leq P\{M_1 > c_F\sqrt{2c_0 \ln t}\}\,E\,e^{2M_1}$$

If c_F is the Fernique constant (9) for $b_\gamma(t)$, then

$$E\,\xi\,\mathbf{1}_{\Omega^c} = O(t^{-c_0/2})$$

and

$$I_1 \leq P\{M_t < c_t\} e^{c_F\sqrt{2c_0 \ln t}} + c_1 t^{-c_0/2} \tag{14}$$

Estimation of I_2. The estimation of I_2 is based on the fact that, when $M_t > c_t$, the path of $b_\gamma(s)$ is above the level ρc_t, $0 < \rho < 1$ on an interval of suitable length.

We shall assume t to be integer for the sake of simplicity. Divide the interval $[0, t]$ into t equal intervals Δ_i. Let R_i be the range of $b_\gamma(t)$ in Δ_i, $0 < \rho < 1$ and $\bar\rho = 1 - \rho$. Let us estimate the probability of the event

$$A = \bigcup_i (R_i > \bar{\rho} c_t)$$

by using the Fernique inequality (10). Let

$$\bar{\rho} c_t = c_{F,2} \sqrt{2 c_2 \ln t} \tag{15}$$

where $c_{F,2}$ is the Fernique constant (9) for the field $\eta(u_1, u_2) = b_\gamma(u_1) - b_\gamma(u_2)$, $(u_1, u_2) \in I = [0,1]^2$, c_2 being an arbitrary constant. In that case one can make use of stationarity in the increments of $b_\gamma(t)$ and of the Fernique inequality (10) for η to get

$$P(A) < t\, P\{\max_I \eta(u_1, u_2) > c_{F,2} \sqrt{2 c_2 \ln t}\} = O(t^{-(c_2 - 1)}) \tag{16}$$

The event $A^c \cap \{M_t > c_t\}$ entails $B = \{\mathrm{mes}(s : \ b_\gamma(s) > \rho c_t) \geq 1\}$. Therefore,

$$\begin{aligned} I_2 &= E\Big[\int_0^t \exp(b_\gamma(s))\, ds\Big]^{-1} \mathbf{1}_{(M_t > c_t)} \\ &\leq E\Big[\int_0^t \exp(b_\gamma(s))\, ds\Big]^{-1} \mathbf{1}_A + E\Big[\int_0^t \exp(b_\gamma(s))\, ds\Big]^{-1} \mathbf{1}_B \\ &:= J_1 + J_2 \end{aligned}$$

The term J_1 is estimated in the same way as I_1 by taking (16) into account:

$$\begin{aligned} J_1 &< P(A) \exp(c_F \sqrt{2 c_0 \ln t}) + c_1 t^{-c_0/2} \\ &= O(t^{-(c_2 - 1)}) \exp(c_F \sqrt{2 c_0 \ln t}) + c_1 t^{-c_0/2} \end{aligned} \tag{17}$$

Further, $J_2 < \exp(-\rho c_t)$. The use of (15) gives

$$J_2 < e^{-\rho c_t} = \exp\Big(- \frac{\rho}{1 - \rho}\ c_{F,2} \sqrt{2 c_2 \ln t}\Big)$$

Putting $\rho = \sqrt{\ln t}/(1 + \sqrt{\ln t})$, one gets

$$J_2 \leq t^{-\sqrt{2c_2}\, c_{F,2}} \tag{18}$$

Since the constants c_0, c_2 can be chosen at will, the following bound is derived from (13), (14), (17) and (18):

$$c\, t^{\gamma/2-1}(1+o(1)) \leq P\{M_t < c_1\, \ln t\}\, e^{c_2\sqrt{\ln t}} \tag{19}$$

which is holds for large t and some new constants c_i. It follows from the scale invariance of $b_\gamma(t)$ that

$$P\{M_t < c_1\, \ln t\} = P\{M_{t'} < 1\}$$

where $t' = t/(c_1 \ln t)^{2/\gamma}$. The desired bound (12) can therefore be got by using the change of t in (19) indicated above. $\diamondsuit$

Lemma 2 (upper bound of $P(z_- < h)$). *The distribution of* $z_- = \sup\{s \in (0,1) : b_\gamma(s) = 0\}$ *admits of the following bound*

$$P(z_- < h) \leq a\, h^{1-\gamma/2}\exp\{b\sqrt{|\ln h|}\} \tag{20}$$

with some positive constants a *and* b

Proof. We use (8) again and estimate the integral

$$I = E\left[\int_0^t e^{b_\gamma(u)}\, du\right]^{-1}$$

from below. To do this, we restrict the set of elementary events to the set

$$A = \{b_\gamma(u) < \varphi(u), \quad u \in (0,t)\}$$

In that case

$$I > \left[\int_0^t e^{\varphi(u)}\, du\right]^{-1} P(A) = c_\varphi P(A)$$

Take

$$\varphi(u) = \begin{cases} \sqrt{2c_1 \ln t}, & u \in (0,1) \\ c_2 \ln u^{-1}, & u \in (1,t), \quad c_2 > 1 \end{cases}$$

Then

$$c_\varphi > \left[\int_0^\infty e^{\varphi(u)} du \right]^{-1} = \left(e^{\sqrt{2c_1 \ln t}} + (c_2 - 1)^{-1} \right)^{-1}$$

Now $P(A)$ is bounded as follows:

$$P(A) = P\{b_\gamma(s) + c_2 \ln^+ s < \sqrt{2c_1 \ln t}\ \theta(1-s), \quad 0 < s < t\} \tag{21}$$

where $\ln^+ s = \ln(s \vee 1)$, $\theta(x) = \mathbf{1}_{x>0}$.

One can continue (21) by using the Cameron-Martin formula:

$$P(A) = e^{-(\phi,\phi)_-/2}\, E\, \mathbf{1}_{\tilde{A}} e^{(b_\gamma,\phi)_-}$$

where

$$\begin{aligned} \phi &= c_2 \ln^+ s, \\ \tilde{A} &= \{b_\gamma(s) < \sqrt{2c_1 \ln t}\ \theta(1-s), \quad 0 < s < t\} \end{aligned} \tag{22}$$

$(\cdot,\cdot)_-$ being a scalar product on the space $H_-(\Delta)$ with the reproducing kernel $B(x,y) = Eb_\gamma(x)b_\gamma(y)$, $(x,y) \in \Delta^2$, $\Delta = (0,t)$.

According to (11),

$$(\phi,\phi)_- = c_\gamma\, c_2^2 \int_1^t |\psi(x)|^2\, dx := \sigma_t^2$$

where

$$\begin{aligned}
\Gamma(1-\alpha)\psi(x) &= d/dx \int_1^x (1-u/x)^{-\alpha}\, u^{-\alpha-1}\, du \\
&= d/dx \left(\int_0^x - \int_0^1\right)(1-u/x)^{-\alpha}\, u^{-\alpha-1}\, du \\
&= x^{-\alpha-1}\, \Gamma^2(1-\alpha)/\Gamma(1-2\alpha) \\
&+ \alpha x^{-2} \int_0^1 u^{-\alpha}(1-u/x)^{-\alpha-1}\, du
\end{aligned}$$

and $2\alpha = \gamma - 1, \quad |\alpha| < 1/2$.
It implies $\psi(x) = O(x^{-\alpha-1})$ as $x \to \infty$ and $\psi(x) = O((x-1)^{-\alpha})$ as $x \downarrow 1$. Hence $\psi \in L^2(1,\infty)$ and $(\phi,\phi)_- = \sigma_t^2 < \sigma_\infty^2 < \infty$.

The random variable $\eta_t = (b_\gamma, \phi)_-$ is Gaussian with zero mean and variance $\sigma_t^2 < \sigma_\infty^2 := \sigma^2$. Hence

$$\begin{aligned}
I &> c_\varphi P(A) > c_\varphi e^{-\sigma^2/2}\, E\, e^{\eta_t}\, \mathbf{1}_{\tilde{A}} \\
&\geq c_\varphi e^{-\sigma^2/2}\, e^{-\sigma\sqrt{2c_2 \ln t}}\ P\{\eta_t > -\sqrt{2c_2 \ln t} \cdot \sigma_t,\ \tilde{A}\} \qquad (23)
\end{aligned}$$

However,

$$\begin{aligned}
P\{\eta_t > -\sqrt{2c_2 \ln t}\cdot\sigma_t,\ \tilde{A}\} &= P(\tilde{A}) - P\{\tilde{A}\cdot(\eta_t < -\sigma_t\sqrt{2c_2 \ln t})\} \\
&\geq P(\tilde{A}) - P\{\eta_t < -\sigma_t\sqrt{2c_2 \ln t}\} \\
&= P(\tilde{A}) - O(t^{-c_2}) \qquad (24)
\end{aligned}$$

A similar procedure can be used for estimating $P(\tilde{A})$:

$$\begin{aligned}
P(\tilde{A}) &= P\{b_\gamma(s) < \sqrt{2c_1 \ln t}\ \theta(1-s),\ 0 < s < t\} \\
&\geq P\{b_\gamma(s) < 0,\ 1 < s < t\} - P\{\max_{(0,1)} b_\gamma(s) > \sqrt{2c_1 \ln t}\} \\
&= P\{b_\gamma(s) < 0,\ 1 < s < t\} - O(t^{-c_3}) \qquad (25)
\end{aligned}$$

where $c_3 = c_1/c_F^2$, and c_F is the Fernique constant (9) for the process $b_\gamma(t)$. Note that we have

$$
\begin{aligned}
P\{b_\gamma(s) < 0, \quad 1 < s < t\} &= P\{b_\gamma(s) < 0, \quad t^{-1} < s < 1\} \\
&= 1/2 \; P(z_- < t^{-1}) \qquad (26)
\end{aligned}
$$

in virtue of (6) and $b_\gamma(t) \overset{\mathrm{d}}{=} -b_\gamma(t)$. Consequently, combining (23), (24), and (25) yields

$$
I \geq c_\varphi \, e^{-\sigma^2/2} \, e^{-\sigma\sqrt{2c_2 \ln t}} \, [P(z_- < t^{-1})/2 \, + \, O(t^{-c_2 \wedge c_3})] \qquad (27)
$$

Since the constants c_1 and c_2 can be chosen in an arbitrary manner, the exponent $c_2 \wedge c_3$ can be made to exceed $(1 - \gamma/2)$. So, taking the bound of c_φ into account, one obtains from (27) for large t:

$$
P(z_- < t^{-1}) \leq a \, t^{-(1-\gamma/2)} \, e^{b\sqrt{\ln t}}
$$

which is in agreement with (20). $\diamondsuit$

Lemma 3 (upper bound for $P(M_t < 1)$ and lower bound for $P(z_- < h)$).

The following bounds hold for large t :

$$
P(M_t < 1) \leq a \, t^{-(1-\gamma/2)} \, \exp(b\sqrt{\ln t}) \qquad (28)
$$

$$
P(z_- < t^{-1}) \geq a \, t^{-(1-\gamma/2)} \, \exp(-b\sqrt{\ln t}) \qquad (29)
$$

where a and b are positive constants.

Proof. Let $f \in C^\infty(0, \infty)$, $f = 1$, $t > 1$ and suppose f is increasing from 0 to 1 over $(0, 1)$. The function $\varphi = 1 - f$ is then finite, and

$$
P(M_t < 1) = P\{b_\gamma(s) - f(s) \leq \varphi(s), \quad 0 < s < t\}
$$

As in Lemma 2, we use the Cameron-Martin formula to estimate the right-hand side of the above equality. We have

$$P(M_t < 1) = e^{-(f,f)_-/2}\, E\, e^{-(f,b_\gamma)_-}\, \mathbf{1}_A \tag{30}$$

where $A = \{b_\gamma(s) < \varphi(s),\ 0 < s < t\} \subset A' = \{b_\gamma(s) < 0,\ 1 < s < t\}$, $(\cdot,\cdot)_-$ being the metric of the space $H_-(\Delta)$, $\Delta = (0,t)$, see (11). It is easy to verify that $(f,f)_- = \sigma_t^2 \to \sigma^2 < \infty$, $t \to \infty$. The quantity $\xi = (f,b_\gamma)_-$ is a Gaussian variable with zero mean and variance σ_t^2. For this reason we shall use the same argument as in the derivation of (14) to get from (30)

$$P(M_t < 1) = e^{-\sigma_t^2/2}\, [e^{\sigma_t\sqrt{2c_0 \ln t}}\, P(A') + ct^{-c_0/2}\, e^{2\sigma_t^2}]$$

We put $c_0/2 = 1 - \gamma/2 + \varepsilon = D + \varepsilon$ and take (26) into account. The result is

$$P(M_t < 1) \le a\, e^{b\sqrt{\ln t}} P(z_- < t^{-1}) + O(t^{-D-\varepsilon})$$

Using (20), we obtain (28) from this, while (12) yields the lower bound for $P(z_- < h)$.

4. SMALL VALUES OF $\tau_{max}, s_+, \mathcal{L}(1)$.

In Section 1 we defined random functionals ξ of $b_\gamma(s)$ on $[0,1]$: τ_{max}, z_-, s_+ and $\mathcal{L}(1)$. Below we estimate the exponents of distributions of ξ at zero, i.e.,

$$\theta = \lim_{x\to 0} \log P(\xi < x)/\log x$$

Theorem 2. (a) *The exponents of the distributions of* τ_{max}, z_- *and* s_+ *at* 0 *are identical and equal to* $D = 1 - \gamma/2$;
(b) *For local time* $\mathcal{L}(1)$ *the exponent is* $\theta \le 1$.

Proof. The exponent $\theta = \theta(z_-)$ for the random variable z_- is defined by lemmas 2 and 3. Let us estimate the distributions of τ_{max}, s_+.
The definition $\xi = \tau_{max}, s_+$ yields

$$P\{b_\gamma(s) < 0,\ s \in (h,1)\} < P(\xi < h)$$

Because of (26) the left-hand side is $1/2\, P(z_- < h)$, consequently, the lower bound (29) also holds for $\xi = \tau_{max}, s_+$.

The upper bound for $P\{\tau_{max} < h\}$. When $M_\delta = \max\{b_\gamma(s),\, 0 < s < \delta\}$, then one has

$$P(\tau_{max} < \delta) \le P(M_1 < h) + P(M_\delta > h)$$

for $\delta < 1$.

Put $h = t^{-\gamma/2}$, $t\delta = c_F^{-2/\gamma}(2c \ln t)^{-1/\gamma}$. Then one can use the scale invariance of $b_\gamma(t)$ to get

$$P(\tau_{max} < \delta) \le P(M_t < 1) + P(M_1 > c_F\sqrt{2c \ln t})$$

If c_F is the Fernique constant (9) for $b_\gamma(s)$, one gets from (28) and (10) for $c > 1 - \gamma/2 + \varepsilon$:

$$P(\tau_{max} < \delta) \le a\, t^{-D} \exp(b\sqrt{\ln t})$$

It remains to express t in terms of δ. This gives for small δ the same inequality

$$P(\tau_{max} < \delta) \le a\, \delta^D \exp(b\sqrt{|\ln \delta|})$$

with new constants a and b.

The upper bound for $P(s_+ < \delta)$. One has

$$P(s_+ < \delta) \le P(M_1 < h) + P(M_1 > h,\, s_+ < \delta)$$

for arbitrary $h > 0$. Set $\delta = t^{-1}$, $t^{\gamma/2}h = c_F\sqrt{2c \ln t}$ and use the scale invariance of $b_\gamma(t)$ again. The result is

$$P(s_+ < t^{-1}) \le P(M_t < c_t) + P(M_t > c_t, \quad s_+(t) < 1)$$

where $c_t = c_F\sqrt{2c\ln t}$, $s_+(t) = \int_0^t \mathbf{1}\,(b_\gamma(s) > 0)\,ds$. Let $\Delta_i = (i,\, i+1)$, $i < [t]$ and $\Delta_{[t]} = (t-1,\, t)$. Since $s_+(t) < 1$, it follows that the maximum of $b_\gamma(s)$ over $(0,t)$ occurs in one of the intervals Δ_i, $i = 0, \ldots, [t]$ in which $b_\gamma(t)$ has a zero; otherwise $s_+(t) \geq 1$, because $M_t \geq 0$. Hence we get the bound

$$\mathcal{R} = P\{M_t > c_t, \quad s_+(t) < 1\} \leq \sum_{0 \leq i \leq [t]} P_i$$

where $P_i = P\{M_{\Delta_i} > c_F\sqrt{2c\ln t},\ \exists\ z \in \Delta_i : b(z) = 0\}$.
If c_F is the Fernique constant for $b_\gamma(s)$, then $P_0 = O(t^{-c})$. One has $P_i = O(t^{-ck})$, $k = 1/25$ for P_i, $i \geq 1$. This estimate is found in a similar manner ([10], Statement 2) and so is dropped here. The result is $\mathcal{R} = O(t^{-kc+1})$. Therefore one has for a suitable c and large t

$$P(s_+ < t^{-1}) \leq 2\,P(M_t < c_1\sqrt{\ln t}) = 2\,P(M_{t'} < 1)$$

where $t' = t/(c_1^2\,\ln t)^{1/\gamma}$. The use of (28) yields

$$P(s_+ < h) \leq a\,h^D\,\exp(b\sqrt{|\ln h|}).$$

The two-sided bounds of the distributions of τ_{max}, s_+ and z_- are identical, which proves the first part of Theorem 2.

Lower bound of $P(\mathcal{L}(1) < x)$.

Suppose $\mathcal{L}(t)$ is the local time of $b_\gamma(s)$ for the interval $(0,t)$. One has for $\delta < 1$ and $h > 0$:

$$P(z_- < \delta) \leq P(\mathcal{L}(1) < h) + P(\mathcal{L}(\delta) \geq h)$$

Set $\delta = t^{-1}$, $t^D h = (c_1\ln t)^{\gamma/2}$, $D = 1-\gamma/2$. In view of the scale invariance $\mathcal{L}(t) : \mathcal{L}(\Lambda t) \overset{\mathrm{d}}{=} \Lambda^D \mathcal{L}(t)$, one has

$$\mathcal{R} = P\{\mathcal{L}(\delta) \geq h\} = P\{\mathcal{L}(1) \geq (c_1\ln t)^{\gamma/2}\}$$

According to [16],

$$P(\mathcal{L}(1) > x) \le e^{-cx^{2/\gamma}}$$

for large x. Therefore

$$\mathcal{R} \le e^{-cc_1 \ln t} = t^{-cc_1}, \quad t \gg 1.$$

For this reason the use of (29) with $cc_1 > D + \varepsilon$ yields

$$P(\mathcal{L}(1) < h) \ge a\, t^{-D} \exp(-b\sqrt{\ln t}) = a' h\, e^{-b'\sqrt{|\ln h|}}$$

or

$$\log P(\mathcal{L}(1) < h) \ge \log h (1 + O(|\log h|^{-1/2}))$$

This proves the second part of Theorem 2.

5. *FBM* WITH MULTIDIMENSIONAL TIME

Consider the problem of small values of

$$M(G) = \sup\{b_\gamma(t),\, t \in G\}$$

for fractional Brownian motion with multidimensional time $b_\gamma(t)$, $t \in \mathbb{R}^d$. Here, G is a bounded convex region on $\mathbb{R}^d$ which contains a vicinity of the point $t = 0$. The proof of the asymptotics $P(M(G) < h)$, $h \to 0$ is based on the fact that the position of the maximum

$$\tau_{\max}(G) = \text{Arg}\ \sup\{b_\gamma(t),\, t \in G\}$$

has a finite density in a vicinity of $t = 0$ (excepting $t = 0$).

Theorem 3. *If a bounded convex region* $G \subset \mathbb{R}^d$ *contains some vicinity of* 0, *then*

$$\log P\{M(TG) < 1\} = -d\,\log T \cdot (1 + O((\log T)^{-1/2})), \quad T \to \infty$$

where TG is a region that is similar to G and has similarity coefficient T. $\diamondsuit$

As in the above, the proof follows from the upper and lower bounds of the desired probability.

Lemma 4 (lower bound). *Under the conditions of Theorem 3 one has*

$$P\,(M(TG) < 1) \geq cT^{-d}(\ln T)^{-d/\gamma} \tag{31}$$

Proof. Let $S_a(r)$ be a ball of radius r centered at a. As in the proof of Theorem 2,

$$P\{|\tau_{\max}(TG)| < 1/2\} \leq P\{M(TG) < c_F\sqrt{2c\ln T}\} + \mathcal{R} \tag{32}$$

where

$$\begin{aligned}
\mathcal{R} &= P\{M(TG) > c_F\sqrt{2c\ln T}, \quad |\tau_{\max}(TG)| < 1/2\} \\
&\leq P\{M(S_0(1/2)) > c_F\sqrt{2c\ln T}\} \\
&\leq P\{M(\Pi(1)) > \sqrt{2c\ln T}\,c_F\}
\end{aligned}$$

and $\Pi(r) = [-r/2,\, r/2]^d$. If c_F is the Fernique constant (9) for $b_\gamma(s)$, $s \in \mathbb{R}^d$ then by (10) one gets $\mathcal{R} = O(T^{-c})$.

We now are going to estimate the left-hand side of (32). Suppose $\delta = T^{-1}$ and let ε be small, so that $S_0(\delta/2)$ and $S_0(\varepsilon/2)$ belong to G. One has

$$\begin{aligned}
&P\{|\tau_{\max}(TG)| < 1/2\} = P\{|\tau_{\max}(G)| < \delta/2\} \\
&= P\{\sup(b_\gamma(t),\ t \in S_0(\delta/2)) > b_\gamma(s), \quad s \in G\}
\end{aligned} \tag{33}$$

Let $a \in S_0(\varepsilon/2)$ and R_0 be such that $S_0(R_0)$ contains the ε-vicinity of region G. Since $b_\gamma(t) \stackrel{\mathrm{d}}{=} b_\gamma(t+a) - b_\gamma(a)$,

$$
\begin{aligned}
(33) \;\; &= \;\; P\{\sup(b_\gamma(t+a)-b_\gamma(a),\\
&\qquad t\in S_0(\delta/2)) \; > \; b_\gamma(s+a)-b_\gamma(a), \quad s\in G\}\\
&\geq \;\; P\{\sup(b_\gamma(t'), \quad t'\in S_a(\delta/2)) > b_\gamma(s'), \quad s'\in S_0(R_0)\}\\
&= \;\; P\{\tau_{\max}(S_0(R_0))\in S_a(\delta/2)\} \qquad (34)
\end{aligned}
$$

It will be shown in Lemma 6 (see below) that

$$
P\{\tau_{\max}(S_0(R_0))\in S_a(\delta/2)\} = c\,\delta^d(1+o(1)) \qquad (35)
$$

as $\delta = T^{-1}\to 0$.
The statement of Lemma 4 follows from (32), (35) and the estimation of $\mathcal{R}$ after the change of variables $T : T' = cT(\ln T)^{-1/\gamma}$.

Lemma 5 (upper bound). *In the conditions of Theorem 3 one has*

$$
P(M(TG)<1) \leq a\,T^{-d}\exp(b\sqrt{\ln t})
$$

Proof. In a similar manner to the proof of Lemma 3, we use the decomposition $1 = f(t)+\varphi(|t|)$, $t\in TG$ where $\varphi\geq 0$ is a finite smooth function with support in the ε-vicinity of 0, $\varphi\leq 1$ and $\varphi = 1$ in the $\varepsilon/2$-vicinity of 0. The next step is to use the Cameron-Martin formula to find the probability

$$
\begin{aligned}
P\{M(TG)<1\} &= P\{b_\gamma(t)-f(t)<\varphi(t), \quad t\in TG\}\\
&= e^{-\sigma_T^2/2}\,E\,e^{-\xi}\,\mathbf{1}_A
\end{aligned}
$$

where $\sigma_T^2 = (f,f)_-$ and $(\cdot,\cdot)_-$ is the scalar product of continuous functions f, $f(0)=0$ on the Hilbert space H_T, with reproducing kernel $B(t,s) = E\,b_\gamma(t)\,b_\gamma(s)$, $(t,s)\in TG\times TG$; $\xi = (b_\gamma, f)_-$ is a Gaussian variable with zero mean and variance σ_T^2;

$$
A = \{b_\gamma(t)<\varphi(t), \quad t\in TG\}
$$

If $\sigma_T^2 < \sigma^2 < \infty$ as $T \to \infty$, then the same argument that was employed above to derive (14) can be applied again to get

$$
\begin{aligned}
P\{M(TG) < 1\} &< c_1\, e^{\sqrt{2c\ln T}}\, P(A) + O(T^{-c}) \\
&< c_1\, P\{|\tau_{\max}(TG)| < \varepsilon/2\}\; e^{\sqrt{2c\ln T}} + O(T^{-c}) \quad (36)
\end{aligned}
$$

where $c > 0$ is arbitrary.

Let $-a \in G$ and r be such that $0 \in S_{-a}(r) \subset G$. Setting $\delta = T^{-1}\,\varepsilon/2$ one gets

$$
\begin{aligned}
&P(|\tau_{\max}(TG)| < \varepsilon/2) = P(|\tau_{\max}(G)| < \delta) \\
&< P(|\tau_{\max}(S_{-a}(r))| < \delta) \\
&= P\{\tau_{\max}(S_0(r)) \in S_a(\delta)\},\ T \gg 1
\end{aligned}
$$

The last manipulations are similar to those in (34), so are here given in shorter form. From Lemma 6 it follows that the last probability is of order $O(T^{-d})$ for a suitably small a and $\delta \to 0$. Consequently, (36) proves Lemma 5.

It remains to show that $\sigma_T^2 < \sigma^2 < \infty$, $T \to \infty$.

Spaces H_T are naturally embedded in $H := H_\infty$, namely,

$$
\| g \|_T = \inf \| \tilde{g} \| \tag{37}
$$

where $\|\cdot\|_T$, $\|\cdot\|$ are the norms on H_T and H, $\inf$ being taken over all elements in H that are identical with g on TG.

It follows from the spectral representation of the reproducing kernel

$$
B_\gamma(t,s) = c_\gamma \int (e^{i(t,\lambda)} - 1)(e^{-i(s,\lambda)} - 1)\, |\lambda|^{-d-\gamma}\, d\lambda
$$

that the norm on H is defined by the relation

$$
\| g \|^2 = \tilde{c}_\gamma \int_{\mathbb{R}^d} |\hat{g}(\lambda)|^2\, |\lambda|^{d+\gamma}\, d\lambda
$$

where $\hat{g}$ is the Fourier transform of g, H being obtained by closure of the finite smooth functions g, $g(0) = 0$ in the metric indicated.

Let $\tilde{f}(t) = \varphi(|t|\,T^{-1}k^{-1}) - \varphi(|t|)$, where $k = 2\,\varepsilon^{-1}|G|$ and $|G|$ is the diameter of G. It follows from the definition of φ that $\tilde{f}(t) = 1 - \varphi(|t|) = f(t)$ on TG and $\tilde{f} \in H$. Therefore,

$$\| f \|_T \; < \; \| \tilde{f} \| \; \leq \; \| \varphi(|t|T^{-1}k^{-1}) \| + \| \varphi \| = \; ((Tk)^{-\gamma/2} + 1) \| \varphi \|$$

But the region G is convex, hence $TG \subset T'G$ for $T < T'$. So $\sigma_T^2 = \| f \|_T^2$ is increasing with T in virtue of (37) and $\sigma_T^2 < 2 \; \| \varphi \|$ for large T. The proof is now complete.

Lemma 6. *The position of the maximum of $b_\gamma(t)$ in the ball $S_0(R_0) \subset \mathbb{R}^d$ has a density of type $\psi(|t|)$ with a continuous and strictly positive function ψ in $(0, R_0)$*

Proof. The field $b_\gamma(t)$, $t \in \mathbb{R}^d$ is rotation-invariant. Consequently, the same is true for the distribution of $\xi = \tau_{\max}(S_0(R_0))$, i.e., $P(\xi \in \delta a) = d\,F(|a|)\,d\omega$, where $d\omega$ is a uniform distribution on the unit sphere. We shall prove that the distribution $F(r) = P(|\xi| < r)$ has a density. The function $F(r)$ can be differentiated almost everywhere on $(0, R_0)$. Assume this to be true at r_0, i.e., $dF(r_0) = \psi(r_0)\,dr_0$. Take $r_1 \in (0, r_0)$. Using the scale invariance of $b_\gamma(t)$, one has for $\lambda = r_0/r_1 > 1$:

$$\begin{aligned} &P\{|\tau_{\max}(S_0(R_0))| \in dr_1\} = P\{|\tau_{\max}(\lambda S_0(R_0))| \in \lambda dr_1\} \\ &\quad \leq P\{|\tau_{\max}(S_0(R_0))| \in \lambda dr_1\} = \psi(r_0)\lambda dr_1 \end{aligned} \tag{38}$$

i.e., the measure $dF(r)$ is absolutely continuous on $(0, r_0)$. Since the points of type r_0 are dense on $(0, R_0)$, $dF(r)$ is absolutely continuous on $(0, R_0)$, $dF(r) = \psi(r)dr$. Note that $r\,\psi(r)$ is nondecreasing, because (38) gives

$$\psi(r_1) \leq \psi(r_0)\; r_0/r_1 \tag{39}$$

It follows that $\psi(r)$ can be modified on a set of zero measure in such a manner as to make ψ continuous except at a countable number of points where the limits on the right and on the left are finite.

Now we deduce from the relation

$$b_\gamma(t+a) - b_\gamma(a) \stackrel{\mathrm{d}}{=} b_\gamma(t), \quad a = \varepsilon \cdot e_0, \quad |e_0| = 1$$

that $\psi(r)$ is continuous, since

$$\begin{aligned} &P\{\tau_{\max}(S_0(R_0)) \in dv\} = P\{\tau_{\max}(S_a(R_0) \in dv + a\} \\ &\le P\{\tau_{\max}(S_0(R_0 - \varepsilon)) \in dv + a\} \\ &= P\{(1 - \varepsilon/R_0)\, \tau_{\max}(S_0(R_0)) \in dv + a\} \end{aligned}$$

The last equality is a consequence of the scale invariance of $b_\gamma(t)$.

Put $dv = (r_0, r_0 + dr) \times d\omega$, $r_0 < R_0$; $e_0 = -\omega$. Then one has under $\varepsilon \to 0$

$$\begin{aligned} \psi(r_0 + 0) &\le \lim_{\varepsilon \to 0} \psi\left(\frac{r_0 - \varepsilon}{1 - \varepsilon/R_0}\right)\left(1 - \frac{\varepsilon}{R_0}\right)^{-d} \\ &= \lim_{\varepsilon \to 0} \psi(r_0 - \varepsilon(1 - r_0/R_0)) = \psi(r_0 - 0) \end{aligned}$$

One the other hand it follows from (39) that $\psi(r_0+0) \ge \psi(r_0-0)$. Therefore $\psi(r)$ is continuous.

Now we are going to show that $\psi(r) > 0$ on $(0, R_0)$. Let $\psi(r_0) = 0$. Since $r\,\psi(r)$ is nondecreasing, it then follows that $\psi(r) \equiv 0$ on $(0, r_0)$, but in that case

$$P\{\max_{|s| \le r_0} b_\gamma(s) \;\le \max_{r_0 \le |s| \le R_0} b_\gamma(s)\} = 1 \tag{40}$$

Let us decompose $b_\gamma(s)$ into the sum $b_\gamma^\perp(s) + \hat{b}_\gamma(s)$, where

$$\hat{b}_\gamma(s) = E\,\{b_\gamma(s) \,|\, b_\gamma(t), \quad |t| \in [r_0,\, R_0]\}$$

Then (40) entails

$$\max_{|s| < r_0} b_\gamma^\perp(s) \le \max_{|s| \in [r_0,\, R_0]} b_\gamma(s) + \max_{|s| < r_0} \hat{b}_\gamma(s), \quad \text{a.s.}$$

i.e., the process $b_\gamma^\perp(s)$, $|s| < r_0$ which is independent of $\{b_\gamma(s),\ r_0 < |s| < R_0\}$ and $\{\hat{b}(s),\, |s| < r_0\}$ is a.s. bounded. However, $b_\gamma^\perp(s)$ is locally unpredictable (see [12]), i.e., $b_\gamma^\perp(s)$ is a nondegenerate Gaussian process. The contradiction proves Lemma 6.

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