Families of spectral measures with mixed types

R. del Rio, S. Fuentes, and A. Poltoratski

Abstract. Consider a family of Sturm-Liouville operators H_θ on the half-axis defined as

$$H_\theta u = -u'' + q(x)u \qquad 0 \le x < \infty$$

with the boundary condition

$$u(0)\cos\theta + u'(0)\sin\theta = 0 \qquad 0 \le \theta < \pi$$

and the limit point case at infinity. We show that it is possible for all H_θ to have dense absolutely continuous and dense singular spectrum. The construction is based on integral representations of Pick functions in the upper half-plane. We also discuss applications to the Krein spectral shift.

1. Introduction

This note studies the interplay between various types of spectra under small perturbations or a change of the boundary condition.

Let A_0 be a cyclic self-adjoint operator, φ its cyclic vector and μ the corresponding spectral measure. Denote by A_λ the rank one perturbations of A_0:

$$A_\lambda = A + \lambda(\cdot, \varphi)\varphi, \quad \lambda \in \mathbb{R}.$$

Let μ_λ be the spectral measures of A_λ corresponding to φ. We study the following general problem: What can happen to μ_λ when the parameter (the coupling constant) λ is changing?

It is well known that the same problem can be formulated in terms of Sturm–Liouville operators on the half-axis. Let the operator H_θ be defined as

$$H_\theta u = -u'' + q(x)u \qquad 0 \le x < \infty$$

where q is real valued, locally integrable function defined in $[0, \infty)$). The domain is restricted by the boundary condition at zero

$$u(0)\cos\theta + u'(0)\sin\theta = 0 \qquad 0 \le \theta < \pi.$$

We assume that the limit point case occurs at infinity.

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For each θ one can define the so-called Weyl–m function $m_\theta(z)$ which is analytic and has positive imaginary part in the upper half-plane. The imaginary part of $m_\theta(z)$ has the following integral representation

$$Im\ m_\theta(z) = \int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2} d\rho_\theta(t) \qquad z = x+iy$$

The measures ρ_θ are called the Weyl spectral measures of the boundary value problem. The following relation is satisfied:

$$m_\theta(z) = \frac{-\sin(\theta-\beta) + m_\beta(z)\cos(\theta-\beta)}{\cos(\theta-\beta) + m_\beta(z)\sin(\theta-\beta)}, \tag{1}$$

see [**4**] and [**9**]. In these settings our general question transforms into: How does the spectrum of H_θ (the measure ρ_θ) depend on the parameter θ in the boundary condition? The same question can also be reformulated in terms of self-adjoint extensions of a symmetric operator with deficiency indices (1,1) or discrete Schrödinger operators. For more on these connections see [**8**] and [**19**].

In this paper we focus on the correlations between the absolutely continuous and singular parts of the spectrum under the change of the coupling constant λ (the boundary parameter θ). By the Kato-Rosenblum theorem the absolutely continuous spectrum is invariant under such small perturbations. At the same time, examples obtained by many researchers in recent years show that the singular spectrum can be extremely unstable under the change of the coupling constant λ in particular in "mixed" situations when both singular and absolutely continuous spectrum are present. In most cases, the singular spectrum of A_λ is located in the gaps of the absolutely continuous spectrum. With the change of λ the isolated eigenvalues of A_λ move inside their respective gaps and often dissappear when they hit the edge of the gap. But what happens if there are no gaps in the absolutely continuous spectrum? It seems that in this case there is no space for the singular spectrum. Even if some of A_λ's have nontrivial singular components, squeezed somewhere in the midst of the absolutely continuous spectrum, they must be easily destroyed by the change of λ. In particular, it seems unlikely that all operators can have dense absolutely continuous and dense singular spectrum. All the examples we know seem to support this intuitive argument: the examples of Naboko [**12**] and Simon [**20**] exhibit singular spectra only for a set of boundary conditions of Lebesgue measure zero (in θ), and the examples of Remling [**17**] do not have dense singular spectra. In [**5**] such a coexistence of spectra is shown for a set of positive measure in the coupling constant but not for all coupling constants. Also, in [**3**] it is shown that two other types of spectrum, the pure point and continuous, cannot coexist for all coupling constants if the spectrum of A_0 is dense. Other papers where coexistence is studied are [**1**] and [**10**].

In this paper we show that, despite all the evidence mentioned above, it is possible for *all* A_λ's to have everywhere dense absolutely continuous *and* everywhere dense singular spectrum. The corresponding example is constructed in Section 2.

In Section 1 we develop our machinery. It is based on the integral representations of analytic Pick functions in the upper half-plane. Our main tool is Lemma 2, which reveals the relations between the families of Pick functions appearing in Perturbation Theory. In addition to the main example, in Section 2 we explain how one can use Lemma 2 to obtain the singular components of the spectral measures μ_λ directly from the corresponding Krein function.

2. Preliminaries

Every Pick function $F(z)$ (an analytic function which takes the upper half-plane into itself) has an integral representation of the form

$$F(z) = a + bz + \int\limits_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\mu(t) \tag{2}$$

where $a, b \in \mathbb{R}, b \geq 0$ and μ is a non negative Borel measure which satisfies

$$\mid \mu \mid := \int\limits_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty. \tag{3}$$

Conversely, any function of this form is analytic and takes the upper half-plane into itself, see [**7**] and [**18**]. The integral on the righthand side of (2) is the Cauchy integral of μ in the upper half-plane. We will denote it by $K\mu$. We will also denote by $P\mu$ the Poisson integral of μ:

$$P\mu(x+iy) = ImK\mu(x+iy) = \int\limits_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} d\mu(t).$$

The Poisson integral is a so called approximative identity: its kernel $\frac{y}{(x-\epsilon)^2+y^2}$ is positive, tends to zero uniformally outside of any neighborhood of x as $y \to 0$ and its L^1-norm is constant. This implies the following version of the Lebesgue Theorem (about the Lebesgue points of a summable function), see [**15**]:

LEMMA 1. If μ is a complex Borel measure on $\mathbb{R}$ such that $|\mu|$ satisfies (3) and $f \in L^1(\mu)$ then

$$\lim_{y\downarrow 0} \frac{Pf\mu(x+iy)}{P\mu(x+iy)} = f(x)$$

for μ-a. e. x.

COROLLARY 1. If μ and ν are complex Borel measures on $\mathbb{R}$ such that $|\mu|$ and $|\nu|$ satisfy (3) and $\mu = f\nu + \eta$, where $f \in L^1(\nu)$ and $\eta \perp \nu$ (η and ν are mutually singular), then

$$\lim_{y\downarrow 0} \frac{P\mu(x+iy)}{P\nu(x+iy)} = f(x)$$

for ν-a. e. x.

Proof: Since

$$\lim_{y\downarrow 0} \frac{P\mu(x+iy)}{P\nu(x+iy)} = \lim_{y\downarrow 0} \frac{Pf\nu(x+iy)}{P\nu(x+iy)} + \lim_{y\downarrow 0} \frac{P\eta(x+iy)}{P\nu(x+iy)}$$

by Lemma 1 it is enough to show that the last summand tends to 0 ν^s-a. e. Consider $f \in L^1(\nu+\eta)$ defined as 1 η-a.e. and as 0 ν-a.e. Then by Lemma 1

$$\lim_{y\downarrow 0} \frac{P\eta(x+iy)}{P(\nu+\eta)(x+iy)} = \lim_{y\downarrow 0} \frac{Pf(\nu+\eta)(x+iy)}{P(\nu+\eta)(x+iy)} = 0$$

ν-a.e. Therefore

$$\left[\frac{P\eta(x+iy)}{P(\nu+\eta)(x+iy)}\right]^{-1} = 1 + \frac{P\nu(x+iy)}{P(\eta)(x+iy)} \to \infty$$

ν-a.e. and we obtain our statement.

Q.E.D.

To construct our main example we will also need the following lemma. Consider the family of Pick functions

$$f_\theta(z) := \frac{\cos\theta + z\sin\theta}{\sin\theta - z\cos\theta}, \quad \theta \in \mathbb{R}. \tag{4}$$

Let $L(z)$ be another Pick function such that $0 \le ImL(z) \le \pi$. For any $\alpha \in \mathbb{R}$ denote

$$M_\alpha(z) := f_\alpha(L(z)) \quad \text{and} \quad N_\alpha(z) := f_\alpha(\exp L(z)).$$

Both $M_\alpha(z)$ and $N_\alpha(z)$ are Pick functions admitting representations similar to (2). We denote by μ_α and ν_α the measures appearing in the representations for M_α and N_α respectively. Then the singular parts of these measures, ν_α^s and μ_β^s, enjoy the following relation:

LEMMA 2. Let ν_α and μ_β be as above. Define the function $\alpha(\beta)$ as

$$\alpha(\beta) = tg^{-1}(\exp tg\beta), \quad \beta \in (-\pi/2, \pi/2)$$

then

$$\mu_\beta^s = \alpha'(\beta)\, \nu_{\alpha(\beta)}^s.$$

Proof:

Let us first show that

$$\lim_{y \downarrow 0} \frac{ImM_\beta(x+iy)}{ImN_\alpha(x+iy)} = \alpha'(\beta)$$

for μ_β^s *a.e.* x.

From the definition of M_β and N_α we have $(z = x + iy)$

$$\lim_{y \downarrow 0} \frac{ImM_\beta(z)}{ImN_\alpha(z)} = \lim_{y \downarrow 0} \frac{ImL(z)}{Im \exp L(z)} \left| \frac{\sin\alpha - \exp L(z)\cos\alpha}{\sin\beta - L(z)\cos\beta} \right|^2 .$$

From the definition of α

$$\frac{\sin\alpha - \exp L(z)\cos\alpha}{\sin\beta - L(z)\cos\beta} = \frac{\cos\alpha}{\cos\beta} \cdot \frac{\exp(tg\beta) - \exp L(z)}{tg\beta - L(z)}. \tag{5}$$

It is well known that for μ_β^s-a.e. x the Cauchy integral of μ_β at $x + i\varepsilon$ tends to infinity as $\varepsilon \to 0$. Therefore (by (2)) $M_\beta(x+i\varepsilon) \xrightarrow{\varepsilon\downarrow 0} \infty$. The formula for f_β and the definition of M_β now imply that for μ_β^s- a.e. x $L(x+i\varepsilon) \xrightarrow{\varepsilon\downarrow 0} tg\beta$.

Hence the expression in the righthand side of (5) tends to

$$\frac{\cos\alpha}{\cos\beta} \quad \exp tg\beta \quad \text{when } \varepsilon \downarrow 0$$

and we get

$$\left| \frac{\sin\alpha - \exp L(z)\cos\alpha}{\sin\beta - L(z)\cos\beta} \right|^2 \xrightarrow{\varepsilon\downarrow 0} \left(\frac{\cos\alpha}{\cos\beta} \right)^2 (\exp tg\beta)^2$$

for μ_β^s *a.e.* x.

Now if $L = a(z) + ib(z)$ then

$$\frac{ImL(z)}{Im \exp L(z)} = \frac{b(z)}{e^{a(z)} \sin b(z)}.$$

For μ_β^s *a.e.* x $\quad b \xrightarrow{\varepsilon\downarrow 0} 0$ and therefore $\dfrac{b}{\sin b} \xrightarrow{\varepsilon\downarrow 0} 1$. Since $e^{a+ib} \xrightarrow{\varepsilon\downarrow 0} tg\alpha$ then $e^a \xrightarrow{\varepsilon\downarrow 0} tg\alpha$. Hence

$$\frac{ImL}{Im\exp L}\xrightarrow{\varepsilon\downarrow 0}\frac{1}{tg\alpha}\quad \text{for } \mu_\beta^s \text{ } a.e. \text{ } x.$$

Therefore we obtain

$$\lim_{y\downarrow 0}\frac{ImM_\beta(z)}{ImN_\alpha(z)}=\left(\frac{\cos\alpha}{\cos\beta}\right)^2(tg\alpha)=\alpha'(\beta)$$

for μ_β^s *a.e.* x. Since the Poisson integral $P\mu_\beta(x+i\varepsilon)$ tends to infinity at μ_β^s *a.e.* x, the last equation implies

$$\lim_{y\downarrow 0}\frac{ImM_\beta(x+iy)}{ImN_\alpha(x+iy)}=\lim_{y\downarrow 0}\frac{ImK\mu_\beta(x+iy)+C_1y}{ImK\nu_\alpha(x+iy)+C_2y}=\lim_{y\downarrow 0}\frac{P_{\mu_\beta}(z)}{P\nu_\alpha(z)}=\alpha'(\beta) \quad (6)$$

for μ_β^s *a.e.* x. If $\mu_\beta=f\nu_\alpha+\eta$, where $f\in L^1(\nu_\alpha)$ and $\eta\perp\nu_\alpha$, then Corollary 1 and (6) imply $f=\alpha'(\beta)>0$ μ_β^s-a.e. Thus $\mu_\beta^s<<\nu_\alpha^s$. In the same way one can show that

$$\lim_{y\downarrow 0}\frac{P_{\nu_\alpha}(z)}{P\mu_\beta(z)}=\frac{1}{\alpha'(\beta)}$$

for ν_α^s *a.e.* x and therefore $\nu_\alpha^s<<\mu_\beta^s$. Hence $f\nu_\alpha^s=\mu_\beta^s$. Again by (6) and Lemma 1, $f\equiv\alpha'(\beta)$.

Q.E.D.

Remark 1. In the definition of the function N_α in the above lemma instead of $\exp(z)$ one can use any other function analytic in the neighborhood of the strip $S=\{0<Imz<\pi\}$ which takes S to the upper half-plane and $\mathbb{R}$ to $\mathbb{R}$. Such functions arise in many other problems related to Perturbation Theory. The definition of $\alpha(\beta)$ in the statement would have to be changed accordingly.

3. The Krein function and coexistence of spectra

We now reveal the meaning of Lemma 2 from the point of view of Perturbation Theory and Mathematical Physics.

First, let us notice that Lemma 2 allows one to see Cauchy integrals of spectral measures (resolvent functions) directly from the corresponding Krein function. Let us consider the following example.

Let A_0 be a self-adjoint operator, φ its cyclic vector and μ the corresponding spectral measure. Once again, denote by A_λ the rank one perturbations:

$$A_\lambda=A+\lambda(\cdot,\varphi)\varphi,\quad \lambda\in\mathbb{R}. \quad (7)$$

Then there exists a function u on $\mathbb{R}$ satisfying

$$u(x) = \arg(1 + K\mu(x + i0)) \quad \text{for } a.e.x \tag{8}$$

where arg stands for the principal branch of argument taking values in $(-\pi; \pi]$. The function u is called the Krein spectral shift for the perturbation problem (A_0, A_1).

To apply Lemma 2 notice that (8) implies that

$$1 + K\mu = \exp(Ku + c)$$

for some real c. For any $\alpha \in (-\pi/2, \pi/2]$ denote by μ_α and u_α the measures corresponding to the Pick functions $f_\alpha(1 + K\mu)$ and $f_\alpha(Ku + c)$ respectively. Lemma 2 immediately gives us the singular components of the spectral measures μ_λ:

Theorem 1.

$$u^s_\beta = \alpha'(\beta)\ \mu^s_{\alpha(\beta)} \tag{9}$$

where $tg\alpha = \exp tg\beta$

A similar result can be formulated for self-adjoint extensions, Sturm-Liouville operators on the half-axis, discrete Schrödinger operators etc.

Theorem 1 implies relations such as

$$\int_0^{\pi/2} u^s_\beta(A) d\,\beta = \int_{\pi/4}^{\pi/2} \mu^s_\alpha(A)\, d\,\alpha.$$

For more about the relation between the measures μ_λ and the Krein function see [**16**].

Next, we will show how to apply Lemma 2 to construct families of Sturm-Liouville operators (rank one perturbations , etc.) of mixed spectral types. As was announced in the introduction, in our example all the operators will have dense singular and dense absolutely continuous spectrum on an interval, regardless of the boundary condition.

Example 1. Consider a fixed interval I and take a set $E \subset I$ such that for every subinterval $J \subset I$ we have

$$0 < |E \cap J| < |J|$$

Such sets can easily be constructed, see [**16**] or [**6**, examples 4 and 5].

Let

$$u(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c \end{cases}$$

define

$$d\,\mu_{\pi/2} = u\,dx.$$

inside I and set $\mu_{\pi/2}$ outside I such that (3) holds, and the necessary decay conditions required by the Gelfand-Levitan inverse theorem see are satisfied, [**13**], the measure $\mu_{\pi/2}$ will be the spectral measure of a Sturm-Liouville operator $H_{\pi/2}$ (defined as in the introduction). Let μ_β be the spectral measures of H_β, μ_β^s and μ_β^{ac} stand for their singular and absolutely continuous components correspondingly.

We claim:

a) $\mu_\beta^s(J) > 0$, for every subinterval $J \subset I \quad \beta \in (-\pi/2, \pi/2)$
b) $\mu_\beta^{ac}(J) > 0$, for every subinterval $J \subset I \quad \beta \in (-\pi/2, \pi/2]$

Proof:
a) Let

$$K\nu_{\pi/2}(z) := \exp(K\mu_{\pi/2}(z)). \tag{10}$$

Then

$$u(x) = \frac{1}{\pi} \arg K\nu_{\pi/2}(x + i0) \quad \text{ for } a.e. x \in I$$

and using the definition of u it follows that

$$ImK\nu_{\pi/2}(x + i0) = 0 \quad \text{ for } a.e. x \in I$$

Since the support of the absolutely continuous part of ν_α is the set

$$\{x / ImK\nu_{\pi/2}(x + i0) > 0\}$$

(see [**11**]), ν_α is purely singular in I for every $\alpha \in (-\pi/2, \pi/2)$.

Given an interval $J \subset I$ assume that $\nu_{\pi/2}(J) = 0$. Then $K\nu_{\pi/2}(z)$ can be extended analytically across J and from (10) the same follows for $K\mu_{\pi/2}(z)$. Since $\mu_{\pi/2} > 0$, this implies $\mu_{\pi/2}(J) = 0$, which contradicts the construction of $\mu_{\pi/2}$. Hence $\nu_{\pi/2}(J) > 0$ for every $J \subset I$. From this we obtain $\nu_\alpha^s(J) > 0$ for every $\alpha \in (-\pi/2, \pi/2)$ (see, for instance, [**9**, p. 38, theorem 2.52]).

Now to obtain a) we just recall that from Theorem 1 we have

$$\mu_\beta^s(F) = \alpha'(\beta)\, \nu_\alpha^s(F)$$

when $\alpha(\beta) = tg^{-1}(\exp tg\beta)$ for every Borel set F. $\beta \in (-\pi/2, \pi/2), \alpha \in (0, \pi/2)$

b) Follows from the well known stability of the absolutely continuous part of the spectrum (see [**19**, theorem 2.1]) since $\mu_{\pi/2}$ is a.c. by construction.

Q.E.D.

REMARK 2. Note that in our construction the absolutely continuous spectra is recurrent (see [**2**]).

REMARK 3. In [**6**] five examples are given of families of measures $\{\mu_\beta\}$ where $d\mu = \chi_B(x)dx$, B is Lebesgue measurable set and

$$\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

The occurrence of the singular spectrum embedded in the a.c. spectrum is only shown for a set of $\beta's$ of positive Lebesgue measure. The construction above proves in [**6**, examples 4 and 5], coexistence for all $\beta's$ with the exception of one $(\pi/2)$.

To construct a family of measures such that a) holds for all $\beta \in (-\pi/2, \pi/2]$ observe that

$$f_\theta(z) = i\frac{-e^{2i\theta} + \varphi(z)}{-e^{2i\theta} - \varphi(z)} \text{ where } \varphi(z) = \frac{z-i}{z+i}.$$

Also,

$$f_\theta(z) = f_{\theta+\pi}(z)$$

and

$$i\frac{-e^{2i\theta} + \varphi^2(z)}{-e^{2i\theta} - \varphi^2(z)} = \frac{1}{2}\, f_{\frac{\theta}{2}+\frac{\pi}{4}}(z) + \frac{1}{2}\, f_{\frac{\theta}{2}-\frac{\pi}{4}}(z)$$

Denote by $\tilde{\mu}_\beta$ the measure corresponding to the Pick function

$$i\frac{-e^{2i\beta} + \varphi^2(K\mu_{\pi/2}(z))}{-e^{2i\beta} + \varphi^2(K\mu_{\pi/2}(z))}.$$

Then

$$\tilde{\mu}_\beta = \frac{1}{2}\, \mu_{\frac{\beta}{2}+\frac{\pi}{4}} + \frac{1}{2}\, \mu_{\frac{\beta}{2}-\frac{\pi}{4}}. \tag{11}$$

The measures $\tilde{\mu}_\beta$ satisfy the decay condition and can be realized as spectral measures for a family of Sturm-Liouville operators. Also, from (11) it follows that $\tilde{\mu}_\beta$ have dense singular and absolutely continuous parts on I for every $\beta \in (-\pi/2, \pi/2]$.

REMARK 4. If we multiply the measure $\mu_{\pi/2}$ used in Example 1 by a constant less than 1, then we get singular components only for some of the coupling constants β. More precisely, let $H_{\pi/2}$ be the Sturm-Liouville operator whose spectral measure is defined as $\gamma_{\pi/2} = \alpha\mu_{\pi/2} = \alpha u(x)dx$ where u is as in Example 1 and $0 < \alpha < 1$. Let γ_α be the spectral measures of H_α's. Then using the same methods as in Example 1 one can prove the following claim:

For the family of measures γ_α we have $\gamma_\alpha^s(J) > 0$ for every subinterval $J \subset I$ if $\alpha \in (0, \pi/2)$. If $\alpha \in (-\pi/2, 0]$ then γ_α is absolutely continuous.

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IIMAS-UNAM, Circuito Escolar, Ciudad Universitaria, 04510, México, D.F., México.
E-mail address: delrio@servidor.unam.mx

Department of Mathematics, College Station, Texas 77843–3368, U.S.A.
E-mail address: alexei.poltoratski@math.tamu.edu