A general framework for localization of classical waves: I. Inhomogeneous media and defect eigenmodes*

Abel Klein
University of California, Irvine
Department of Mathematics
Irvine, CA 92697-3875, USA
aklein@math.uci.edu

Andrew Koines
Texas A&M University
Department of Mathematics
College Station, TX 77843-3368, USA
Andrew.Koines@math.tamu.edu

February 1, 2001

Abstract

We introduce a general framework for studying the localization of classical waves in inhomogeneous media, which encompasses acoustic waves with position dependent compressibility and mass density, elastic waves with position dependent Lamé moduli and mass density, and electromagnetic waves with position dependent magnetic permeability and dielectric constant. We also allow for anisotropy. We develop mathematical methods to study wave localization in inhomogeneous media. We show localization for local perturbations (defects) of media with a spectral gap, and study midgap eigenmodes.

*This work was partially supported by NSF Grant DMS-9800883.

Contents

1 Introduction **2**

2 The mathematical framework **4**
2.1 Classical wave equations 4
2.2 Wave equations in abstract Schrödinger form 6
2.3 Wave localization 7
2.4 Classical wave operators 8
2.5 Defects and wave localization 12

3 Properties of classical wave operators **14**
3.1 A trace estimate 14
3.2 Finite volume classical wave operators 15
3.3 An interior estimate 18
3.4 Improved resolvent decay estimates in a gap 18
3.5 A Simon-Lieb-type inequality 24
3.6 The eigenfunction decay inequality 27

4 Periodic classical wave operators **30**

5 Defects and midgap eigenmodes **32**

A A useful lemma **37**

1 Introduction

We provide a general framework for studying localization of acoustic waves, elastic waves, and electromagnetic waves in inhomogeneous media, i.e., the existence of acoustic, elastic, and electromagnetic waves such that almost all of the wave's energy remains in a fixed bounded region uniformly over time. Our general framework encompasses acoustic waves with position dependent compressibility and mass density, elastic waves with position dependent Lamé moduli and mass density, and electromagnetic waves with position dependent magnetic permeability and dielectric constant. We also allow for anisotropy.

In this first article we develop mathematical methods to study wave localization in inhomogeneous media. As an application we show localization for local perturbations (defects) of media with a gap in the spectrum, and study midgap eigenmodes. In the second article [16] we use the methods developed in this article to study wave localization for random perturbations of periodic media with a gap in the spectrum.

Our results extend the work of Figotin and Klein [6, 7, 8, 9, 10, 15] in several ways: 1) We study a general class of classical waves which includes acoustic, electromagnetic and elastic waves as special cases. 2) We allow for more than one inhomogeneous coefficient (e.g., electromagnetic waves in media where both the magnetic permeability and the dielectric constant are position dependent). 3) We allow for anisotropy in our wave equations. 4) In [16] we prove strong dynamical localization in random media, using the recent results of Germinet and Klein [13] on strong dynamical localization and of Klein, Koines and Seifert [17] on a generalized eigenfunction expansion for classical wave operators.

Previous results on localization of classical waves in inhomogeneous media [6, 7, 8, 9, 10, 15, 3] considered only the case of one inhomogeneous coefficient. Acoustic and electromagnetic waves were treated separately. Elastic waves were not discussed.

Our approach to the mathematical study of localization of classical waves, as in the work of Figotin and Klein, is operator theoretic and reminiscent of quantum mechanics. It is based on the fact that many wave propagation phenomena in classical physics are governed by equations that can be recast in abstract Schrödinger form [20, 7, 15]. The corresponding self-adjoint operator, which governs the dynamics, is a first order partial differential operator, but its spectral theory may be studied through an auxiliary self-adjoint, second order partial differential operator. These second order *classical wave operators* are analogous to Schrödinger operators in quantum mechanics. The method is particularly suitable for the study of phenomena historically associated with quantum mechanical electron waves, especially Anderson localization in random media [6, 7, 10, 15] and midgap defect eigenmodes [8, 9].

Physically interesting inhomogeneous media give rise to nonsmooth coefficients in the classical wave equations, and hence in their classical wave operators. (E.g., a medium composed of two different homogeneous materials will be represented by piecewise constant coefficients.) *Thus we make no assumptions about the smoothness of the coefficients of classical wave operators.* Since we allow two inhomogeneous coefficients, we have to deal with domain questions for the quadratic forms associated with classical wave operators.

We must also take into account that many classical wave equations come with auxiliary conditions, and the corresponding classical wave operators are not elliptic (e.g., the Maxwell operator – see [20, 19, 7, 10, 15]).

This paper is organized as follows: In Section 2 we introduce our framework for studying classical waves. We discuss classical wave equations in inhomogeneous media and wave localization. We define first and second order classical wave operators, and use them to rewrite the wave equations in abstract Schrödinger form. We state our results on wave localization created by defects. In Section 3 we study classical wave operators, obtaining the technical tools that are needed for proving localization in inhomogeneous and random media. We study finite volume classical wave operators, discuss interior estimates, give an improved resolvent decay estimate in a gap, prove a Simon-Lieb-type inequality and an eigenfunction decay inequality. In Section 4 we study periodic classical wave operators, and prove a theorem that gives the spectrum of a periodic classical wave operator in terms of the spectra of its restriction to finite cubes with periodic boundary condition. In Section 5 we study the effect of defects on classical wave operators, and give the proofs and details of the results on defects and wave localization stated in Section 2.

2 The mathematical framework

2.1 Classical wave equations

Many classical wave equations in a linear, lossless, inhomogeneous medium can be written as first order equations of the form:

$$\begin{array}{rcl} \mathcal{K}(x)^{-1}\dfrac{\partial}{\partial t}\psi_t(x) & = & \mathbf{D}^*\phi_t(x) \\ \mathcal{R}(x)^{-1}\dfrac{\partial}{\partial t}\phi_t(x) & = & -\mathbf{D}\,\psi_t(x) \end{array}, \tag{2.1}$$

where $x \in \mathbb{R}^d$ (space), $t \in \mathbb{R}$ (time), $\psi_t(x) \in \mathbb{C}^n$ and $\phi_t(x) \in \mathbb{C}^m$ are physical quantities that describe the state of the medium at position x and time t, $\mathbf{D}$ is an $m \times n$ matrix whose entries are first order partial differential operators with constant coefficients (see Definition 2.1), $\mathbf{D}^*$ is the formal adjoint of $\mathbf{D}$, and, $\mathcal{K}(x)$ and $\mathcal{R}(x)$ are $n \times n$ and $m \times m$ positive, invertible matrices, uniformly bounded from above and away from 0, that describe the medium at position x (see Definition 2.3). In addition, $\mathbf{D}$ satisfies a partial ellipticity property (see Definition 2.2), and there may be auxiliary conditions to be satisfied by the quantities $\psi_t(x)$ and $\phi_t(x)$.

The physical quantities $\psi_t(x)$ and $\phi_t(x)$ then satisfy second order wave equations, with the same auxiliary conditions:

$$\frac{\partial^2}{\partial t^2}\psi_t(x) = -\mathcal{K}(x)\mathbf{D}^*\mathcal{R}(x)\mathbf{D}\psi_t(x) \quad (2.2)$$

$$\frac{\partial^2}{\partial t^2}\phi_t(x) = -\mathcal{R}(x)\mathbf{D}\mathcal{K}(x)\mathbf{D}^*\phi_t(x)\ . \quad (2.3)$$

Conversely, given (2.2) (or (2.3)), we may write this equation in the form (2.1) by introducing an appropriate quantity $\phi_t(x)$ (or $\psi_t(x)$), which will then satisfy equation (2.3) (or (2.2)).

The medium is called *homogeneous* if the coefficient matrices $\mathcal{K}(x)$ and $\mathcal{R}(x)$ are constant, i.e., they do not depend on the position x. Otherwise the medium is said to be *inhomogeneous*.

Examples:

Electromagnetic waves: Maxwell equations are given by (2.1) with $d = n = m = 3$, $\psi_t(x)$ the magnetic field, $\phi_t(x)$ the electric field, $\mathbf{D} = \mathbf{D}^*$ the curl ($\mathbf{D}\phi = \nabla \times \phi$), $\mathcal{K}(x) = \frac{1}{\mu(x)}I_3$ and $\mathcal{R}(x) = \frac{1}{\varepsilon(x)}I_3$, with $\mu(x)$ the magnetic permeability and $\varepsilon(x)$ the dielectric constant. (By I_k we denote the $k \times k$ identity matrix.) The auxiliary conditions are $\nabla \cdot \mu\psi_t = 0$ and $\nabla \cdot \varepsilon\phi_t = 0$.

Acoustic waves: The acoustic equations in d dimensions may be written as (2.1), with $n = 1$, $m = d$, $\psi_t(x)$ the pressure, $\phi_t(x)$ the velocity, $\mathbf{D}$ the gradient ($\mathbf{D}\phi = \nabla\phi$, $\mathbf{D}^*\psi = -\nabla \cdot \psi$), $\mathcal{K}(x) = \frac{1}{\kappa(x)}I_1$ and $\mathcal{R}(x) = \frac{1}{\varrho(x)}I_d$, with $\kappa(x)$ the compressibility and $\varrho(x)$ the mass density. The auxiliary condition is $\nabla \times \varrho\phi_t = 0$. The usual second order acoustic equation for the pressure is then given by (2.2).

Elastic waves: The equations of motion for linear elasticity, in an isotropic medium, can be written as the second order wave equation

$$\rho(x)\frac{\partial^2}{\partial t^2}\psi_t(x) = -\left\{\nabla[\lambda(x) + 2\mu(x)]\nabla^* + \nabla \times \mu(x)\nabla\times\right\}\psi_t(x), \quad (2.4)$$

where $x \in \mathbb{R}^3$, $\psi_t(x)$ is the medium displacement, $\rho(x)$ is the mass density, and, $\lambda(x)$ and $\mu(x)$ are the Lamé moduli. It is of the form of equation (2.2), with $n = 3$, $\mathbf{D}$ the differential operator given by $\mathbf{D}\psi = (\nabla^*\psi) \oplus (\nabla \times \psi)$ (a 4×3 matrix), $\mathcal{K}(x) = \frac{1}{\rho(x)}I_3$, and $\mathcal{R}(x) = (\lambda(x) + 2\mu(x))I_1 \oplus \mu(x)I_3$ (a 4×4 matrix).

2.2 Wave equations in abstract Schrödinger form

The wave equation (2.1) may be rewritten in abstract Schrödinger form [20, 7, 15]:

$$-i\frac{d}{dt}\Psi_t = \mathbb{W}\Psi_t \ , \tag{2.5}$$

where $\Psi_t = \begin{pmatrix} \psi_t \\ \phi_t \end{pmatrix}$ and

$$\mathbb{W} = \begin{pmatrix} 0 & -i\mathcal{K}(x)\mathbf{D}^* \\ i\mathcal{R}(x)\mathbf{D} & 0 \end{pmatrix} \ . \tag{2.6}$$

The (first order) classical wave operator $\mathbb{W}$ is formally (and can be defined as) a self-adjoint operator on the Hilbert space

$$\mathcal{H} = L^2\left(\mathbb{R}^d, \mathcal{K}(x)^{-1}dx; \mathbb{C}^n\right) \oplus L^2\left(\mathbb{R}^d, \mathcal{R}(x)^{-1}dx; \mathbb{C}^m\right), \tag{2.7}$$

where, for a $k\times k$ positive invertible matrix-valued measurable function $\mathcal{S}(x)$, we set

$$L^2\left(\mathbb{R}^d, \mathcal{S}(x)^{-1}dx; \mathbb{C}^k\right) = \left\{f\colon \mathbb{R}^d \to \mathbb{C}^k;\ \left\langle f, \mathcal{S}(x)^{-1}f\right\rangle_{L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)} < \infty\right\}.$$

The auxiliary conditions to the wave equation are imposed by requiring the solutions to equation (2.5) to also satisfy

$$\Psi_t = P_{\mathbb{W}}^{\perp}\Psi_t, \tag{2.8}$$

where $P_{\mathbb{W}}^{\perp}$ denotes the orthogonal projection onto the orthogonal complement of the kernel of $\mathbb{W}$. The solutions to the equations (2.5) and (2.8) are of the form

$$\Psi_t = \mathrm{e}^{it\mathbb{W}} P_{\mathbb{W}}^{\perp}\Phi_0 \ , \quad \Phi_0 \in \mathcal{H} \ . \tag{2.9}$$

The *energy density* at time t of a solution $\Psi \equiv \Psi_t(x) = (\psi_t(x), \phi_t(x))$ of the wave equation (2.1) is given by

$$\mathcal{E}_\Psi(t,x) = \frac{1}{2}\left\{\langle\psi(x), \mathcal{K}(x)^{-1}\psi_t(x)\rangle_{\mathbb{C}^n} + \langle\phi_t(x), \mathcal{R}(x)^{-1}\phi_t(x)\rangle_{\mathbb{C}^m}\right\}. \tag{2.10}$$

The wave *energy*, a conserved quantity, is thus given by

$$\mathcal{E}_\Psi = \tfrac{1}{2}\|\Psi_t\|^2_{\mathcal{H}} \quad \text{for any} \ \ t. \tag{2.11}$$

Note that (2.9) gives the finite energy solutions to the wave equation (2.1).

2.3 Wave localization

Let $\Psi = \Psi_t(x)$ be a finite energy solution of the wave equation (2.1). There are many criteria for wave localization, e.g.:

Simple localization: Almost all of the wave's energy remains in a fixed bounded region at all times, more precisely:

$$\lim_{R\to\infty} \inf_t \frac{1}{\mathcal{E}_\Psi} \int_{|x|\leq R} \mathcal{E}_\Psi(t,x)\, dx = 1. \tag{2.12}$$

Moment localization: For some (or we may require for all) $q > 0$, we have

$$\sup_t \int_{\mathbb{R}^d} |x|^q \mathcal{E}_\Psi(t,x)\, dx = \tfrac{1}{2} \sup_t \| \, |x|^{\frac{q}{2}} \Psi_t\|_{\mathcal{H}}^2 < \infty. \tag{2.13}$$

Exponential localization (in the L^2-sense): For some $C < \infty$ and $m > 0$, we have

$$\sup_t \|\chi_x \mathcal{E}_\Psi(t,.)\|_2 = \tfrac{1}{\sqrt{2}} \sup_t \| \, \chi_x \Psi_t\|_{\mathcal{H}} \leq C\mathrm{e}^{-m|x|} \tag{2.14}$$

for all $x \in \mathbb{R}^d$, where χ_x denotes the characteristic function of a cube of side 1 centered at x.

It is easy to see that exponential localization implies moment localization for all $q > 0$, and moment localization for some $q > 0$ implies simple localization.

The fact that finite energy solutions are given by (2.9) suggests a method to obtain localized waves: if $\Psi_0 \in \mathcal{H}$ is an eigenfunction for the classical wave operator $\mathbb{W}$ with nonzero eigenvalue ω, i.e., $\mathbb{W}\Psi_0 = \omega\Psi_0$ with $\omega \neq 0$, then the wave $\Psi_t = \mathrm{e}^{it\omega}\Psi_0$ exhibits simple localization. If in addition $\| \, |x|^{\frac{q}{2}} \Psi_0\|_{\mathcal{H}}^2 < \infty$, we have moment localization. If Ψ_0 is exponentially decaying (in the L^2-sense), we have exponential localization.

In a homogenous medium, a classical wave operator cannot have nonzero eigenvalues. (This can be shown using the Fourier transform.) Thus an appropriate inhomogenous medium is required to produce nonzero eigenvalues, and hence localized waves. In Subsection 2.5 we will see that we can produce eigenvalues in spectral gaps of classical wave operators by introducing defects, i.e., by making local changes in the medium. Moreover, the corresponding waves will exhibit exponential localization

In the sequel [16] we show that random changes in the media can produce Anderson localization in spectral gaps of periodic classical wave operators.

2.4 Classical wave operators

We now introduce the mathematical machinery needed to make the preceding discussion mathematically rigorous.

It is convenient to work on $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)$ instead of the weighted space $L^2\left(\mathbb{R}^d, \mathcal{S}(x)^{-1}dx; \mathbb{C}^k\right)$. To do so, note that the operator $V_{\mathcal{S}}$, given by multiplication by the matrix $\mathcal{S}(x)^{-1/2}$, is a unitary map from the Hilbert space $L^2\left(\mathbb{R}^d, \mathcal{S}(x)^{-1}dx; \mathbb{C}^k\right)$ to $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)$, and if we set $\widetilde{\mathbb{W}} = (V_{\mathcal{K}} \oplus V_{\mathcal{R}})\, \mathbb{W}\, (V_{\mathcal{K}}^* \oplus V_{\mathcal{R}}^*)$, we have

$$\widetilde{\mathbb{W}} = \begin{pmatrix} 0 & -i\sqrt{\mathcal{K}(x)}\mathbf{D}^*\sqrt{\mathcal{R}(x)} \\ i\sqrt{\mathcal{R}(x)}\mathbf{D}\sqrt{\mathcal{K}(x)} & 0 \end{pmatrix} , \tag{2.15}$$

a formally self-adjoint operator on $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^n\right) \oplus L^2\left(\mathbb{R}^d, dx; \mathbb{C}^m\right)$.

In addition, if $S_- I \le \mathcal{S}(x) \le S_+ I$ with $0 < S_- \le S_+ < \infty$, as it will be the case in this article, it turns out that if $\widetilde{\varphi} = V_{\mathcal{S}}\varphi$, then the functions $\varphi(x)$ and $\widetilde{\varphi}(x)$ share the same decay and growth proporties (e.g., exponential or polynomial decay).

Thus it will suffice for us to work on $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)$, and we will do so in the remainder of this article. We set

$$\mathcal{H}^{(k)} = L^2(\mathbb{R}^d, dx; \mathbb{C}^k) . \tag{2.16}$$

Given a closed densely defined operator T on a Hilbert space $\mathcal{H}$, we will denote its kernel by $\ker T$ and its range by $\operatorname{ran} T$; note $\ker T^*T = \ker T$. If T is self-adjoint, it leaves invariant the orthogonal complement of its kernel; *the restriction of* T *to* $(\ker T)^\perp$ *will be denoted by* $T_\perp$. Note that $T_\perp$ is a self-adjoint operator on the Hilbert space $(\ker T)^\perp = P_T^\perp \mathcal{H}$, where $P_T^\perp$ denotes the orthogonal projection onto $(\ker T)^\perp$.

Definition 2.1 *A constant coefficient, first order, partial differential operator* $\mathbf{D}$ *from* $\mathcal{H}^{(n)}$ *to* $\mathcal{H}^{(m)}$ *(*$CPDO^{(1)}_{n,m}$*) is of the form* $\mathbf{D} = D(-i\nabla)$, *where, for a* d*-component vector* k, $D(k)$ *is the* $m \times n$ *matrix*

$$D(k) = [D(k)_{r,s}]_{\substack{r=1,\ldots,m \\ s=1,\ldots,n}} \ ; \ \ D(k)_{r,s} = a_{r,s} \cdot k \ , \ \ a_{r,s} \in \mathbb{C}^d \ . \tag{2.17}$$

We set

$$D_+ = \sup\{\|D(k)\|;\ \ k \in \mathbb{C}^d,\ |k| = 1\}\,, \tag{2.18}$$

so $\|D(k)\| \le D_+|k|$ for all $k \in \mathbb{C}^d$. Note that D_+ is bounded by the norm of the matrix $[|a_{r,s}|]_{\substack{r=1,\ldots,m \\ s=1,\ldots,n}}$.

Defined on

$$\mathcal{D}(\mathbf{D}) = \{\psi \in \mathcal{H}^{(n)} :\ \mathbf{D}\psi \in \mathcal{H}^{(m)} \text{ in distributional sense}\}\ , \tag{2.19}$$

a $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ is a closed, densely defined operator, and $C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$ (the space of infinitely differentiable functions with compact support) is an operator core for $\mathbf{D}$. We will denote by $\mathbf{D}^*$ the $CPDO^{(1)}_{m,n}$ given by the formal adjoint of the matrix in (2.17).

Definition 2.2 *A $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ is said to be partially elliptic if there exists a $CPDO^{(1)}_{n,q}$ $\mathbf{D}^\perp$ (for some q), satisfying the following two properties:*

$$\mathbf{D}^\perp\mathbf{D}^* \;=\; 0\ , \tag{2.20}$$

$$\mathbf{D}^*\mathbf{D} + (\mathbf{D}^\perp)^*\mathbf{D}^\perp \;\ge\; \Theta\ [(-\Delta)\otimes I_n]\ , \tag{2.21}$$

with $\Theta > 0$ being a constant. ($\Delta = \nabla\cdot\nabla$ is the Laplacian on $L^2(\mathbb{R}^d, dx)$; I_n denotes the $n\times n$ identity matrix.)

If $\mathbf{D}$ is partially elliptic, we have

$$\mathcal{H}^{(n)} = \ker \mathbf{D}^\perp \oplus \ker \mathbf{D}\ , \tag{2.22}$$

and

$$\mathbf{D}^*\mathbf{D} + (\mathbf{D}^\perp)^*\mathbf{D}^\perp = (\mathbf{D}^*\mathbf{D})_\perp \oplus ((\mathbf{D}^\perp)^*\mathbf{D}^\perp)_\perp\ . \tag{2.23}$$

Note that $\mathbf{D}$ is elliptic if and only it is partially elliptic with $\mathbf{D}^\perp = 0$. Note also that a $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ may be partially elliptic with $\mathbf{D}^*$ not being partially elliptic [17, Remark 1.1].

Definition 2.3 *A coefficient operator $\mathcal{S}$ on $\mathcal{H}^{(n)}$ (CO_n) is a bounded, invertible operator given by multiplication by a coefficient matrix: an $n\times n$ matrix -valued measurable function $\mathcal{S}(x)$ on $\mathbb{R}^d$, satisfying*

$$S_- I_n \le \mathcal{S}(x) \le S_+ I_n\ ,\ \ \textit{with}\ \ 0 < S_- \le S_+ < \infty\ . \tag{2.24}$$

Definition 2.4 *A multiplicative coefficient, first order, partial differential operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$ ($MPDO^{(1)}_{n,m}$) is of the form*

$$A = \sqrt{\mathcal{R}}\mathbf{D}\sqrt{\mathcal{K}} \quad \text{on} \quad \mathcal{D}(A) = \mathcal{K}^{-\frac{1}{2}}\mathcal{D}(\mathbf{D}) \ , \tag{2.25}$$

where $\mathbf{D}$ is a $CPDO^{(1)}_{n,m}$, $\mathcal{K}$ is a CO_n, and $\mathcal{R}$ is a CO_m. (We will write $A_{\mathcal{K},\mathcal{R}}$ for A whenever it is necessary to make explicit the dependence on the on the medium, i.e., on the coefficient operators. $\mathbf{D}$ does not depend on the medium, so it will be omitted in the notation.)

An $MPDO^{(1)}_{n,m}$ A is a closed, densely defined operator with $A^* = \sqrt{\mathcal{K}}\mathbf{D}^*\sqrt{\mathcal{R}}$ an $MPDO^{(1)}_{m,n}$. Note that $\mathcal{K}^{-\frac{1}{2}}C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$ is an operator core for A.

The following quantity will appear often in estimates:

$$\Xi_A \equiv D_+\sqrt{R_+K_+}\,. \tag{2.26}$$

Definition 2.5 *A first order classical operator ($CWO^{(1)}_{n,m}$) is an operator of the form*

$$\mathbb{W}_A = \begin{bmatrix} 0 & -iA^* \\ iA & 0 \end{bmatrix} \quad \text{on} \quad \mathcal{H}^{(n+m)} \cong \mathcal{H}^{(n)} \oplus \mathcal{H}^{(m)} \ , \tag{2.27}$$

where A is an $MPDO^{(1)}_{n,m}$. If either $\mathbf{D}$ or $\mathbf{D}^*$ is partially elliptic, $\mathbb{W}_A$ will also be called partially elliptic.

A $CWO^{(1)}_{n,m}$ is a self-adjoint $MPDO^{(1)}_{n+m,n+m}$: $\mathbb{W}_A = \sqrt{\mathcal{S}}\mathbb{W}_{\mathbf{D}}\sqrt{\mathcal{S}}$, where $\mathcal{S} = \mathcal{K}\oplus\mathcal{R}$ is a CO_{n+m} and $\mathbb{W}_{\mathbf{D}}$ is a self-adjoint $CPDO^{(1)}_{n+m,n+m}$. (Note that our definition of a first order classical wave operator is more restrictive than the one used in [17]. The definition of partial ellipticity is also different; [17] requires both $\mathbf{D}$ and $\mathbf{D}^*$ to be partially eliptic.)

The Schrödinger-like equation (2.5) for classical waves with the auxiliary condition (2.8) may be written in the form:

$$-i\frac{\partial}{\partial t}\mathbf{\Psi}_t = (\mathbb{W}_A)_\perp\mathbf{\Psi}_t \ , \quad \mathbf{\Psi}_t \in (\ker \mathbb{W}_A)^\perp = (\ker A)^\perp \oplus (\ker A^*)^\perp, \tag{2.28}$$

with $\mathbb{W}_A$ a $CWO^{(1)}_{n+m}$ as in (2.27). Its solutions are of the form

$$\Psi_t = \mathrm{e}^{it(\mathbb{W}_A)_\perp}\Psi_0 \ , \quad \Psi_0 \in (\ker \mathbb{W}_A)^\perp \ , \tag{2.29}$$

which is just another way of writing (2.9).

Since

$$(\mathbb{W}_A)^2 = \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix} , \tag{2.30}$$

if $\mathbf{\Psi}_t = (\psi_t, \phi_t) \in \mathcal{H}^{(n)} \oplus \mathcal{H}^{(m)}$ is a solution of (2.28), then its components satisfy the second order wave equations (2.2) and (2.3), plus the auxiliary conditions, which may be all written in the form

$$\frac{\partial^2}{\partial t^2}\psi_t = -(A^*A)_\perp \psi_t \ , \quad \text{with} \quad \psi_t \in (\ker A)^\perp \ , \tag{2.31}$$

$$\frac{\partial^2}{\partial t^2}\phi_t = -(AA^*)_\perp \phi_t \ , \quad \text{with} \quad \phi_t \in (\ker A^*)^\perp \ . \tag{2.32}$$

The solutions to (2.31) may be written as

$$\psi_t = \cos\left(t(A^*A)_\perp^{\frac{1}{2}}\right)\psi_0 + \sin\left(t(A^*A)_\perp^{\frac{1}{2}}\right)\eta_0 \ , \ \ \psi_0, \eta_0 \in (\ker A)^\perp, \tag{2.33}$$

with a similar expression for the solutions of (2.32).

The operators $(A^*A)_\perp$ and $(AA^*)_\perp$ are unitarily equivalent (see Lemma A.1): the operator U defined by

$$U\psi = A(A^*A)_\perp^{-\frac{1}{2}}\psi \quad \text{for} \quad \psi \in \ \operatorname{ran}(A^*A)_\perp^{\frac{1}{2}} \, , \tag{2.34}$$

extends to a unitary operator from $(\ker A)^\perp$ to $(\ker A^*)^\perp$, and

$$(AA^*)_\perp = U(A^*A)_\perp U^* \ . \tag{2.35}$$

In addition, if

$$\mathbb{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_A & I_A \\ iU & -iU \end{bmatrix} , \quad \text{with } I_A \text{ the identity on } (\ker A)^\perp \ , \tag{2.36}$$

$\mathbb{U}$ is a unitary operator from $(\ker A)^\perp \oplus (\ker A)^\perp$ to $(\ker A)^\perp \oplus (\ker A^*)^\perp$, and we have the unitary equivalence:

$$\mathbb{U}^* (\mathbb{W}_A)_\perp \mathbb{U} = (A^*A)_\perp^{\frac{1}{2}} \oplus \left[-(A^*A)_\perp^{\frac{1}{2}}\right] \ . \tag{2.37}$$

Thus the operator $(A^*A)_\perp$ contains full information about the spectral theory of the operator $(\mathbb{W}_A)_\perp$ (e.g., [7, 17]). In particular

$$\sigma((\mathbb{W}_A)_\perp) = \sigma\left((A^*A)_\perp^{\frac{1}{2}}\right) \cup \left(-\sigma\left((A^*A)_\perp^{\frac{1}{2}}\right)\right) , \tag{2.38}$$

and to find all eigenvalues and eigenfunctions for $(\mathbb{W}_A)_\perp$, it is necessary and sufficient to find all eigenvalues and eigefunctions for $(A^*A)_\perp$. For if $(A^*A)_\perp \psi_{\omega^2} = \omega^2 \psi_{\omega^2}$, with $\omega \neq 0$, $\psi_{\omega^2} \neq 0$, we have

$$(\mathbb{W}_A)_\perp \left(\psi_{\omega^2}, \pm \frac{i}{\omega} A \psi_{\omega^2} \right) = \pm \omega \left(\psi_{\omega^2}, \pm \frac{i}{\omega} A \psi_{\omega^2} \right). \tag{2.39}$$

Conversely, if $(\mathbb{W}_A)_\perp (\psi_{\pm\omega}, \phi_{\pm\omega}) = \pm\omega (\psi_{\pm\omega}, \phi_{\pm\omega})$, with $\omega \neq 0$, it follows that (see [17, Proposition 5.2])

$$(A^*A)_\perp \psi_{\pm\omega} = \omega^2 \psi_{\pm\omega} \ \textit{ and } \ \phi_{\pm\omega} = \pm \frac{i}{\omega} A \psi_{\pm\omega}. \tag{2.40}$$

Definition 2.6 *A second order classical wave operator on $\mathcal{H}^{(n)}$ ($CWO_n^{(2)}$) is an operator $W = A^*A$, with A an $MPDO_{n,m}^{(1)}$ for some m. (We write $W_{\mathcal{K},\mathcal{R}} = A^*_{\mathcal{K},\mathcal{R}} A_{\mathcal{K},\mathcal{R}}$.) If* $\mathbf{D}$ *in (2.25) is partially elliptic, the $CWO_n^{(2)}$ will also be called partially elliptic.*

Note that a first order classical wave operator $\mathbb{W}_A$ is partially elliptic if and only if one of the two second order classical wave operators A^*A and AA^* is partially elliptic.

Definition 2.7 *A classical wave operator (CWO) is either a $CWO_n^{(1)}$ or a $CWO_n^{(2)}$. If the operator W is a CWO, we call $W_\perp$ a proper CWO.*

Remark 2.8 *A proper classical wave operator W has a trivial kernel by construction, so* 0 *is not an eigenvalue. However, using a dilation argument, one can show that 0 is in the spectrum of $W_\perp$ [17, Theorem A.1], so $W_\perp$ and W have the same spectrum and essential spectrum.*
.

2.5 Defects and wave localization

We now describe our results on defects and wave localization. We can produce eigenvalues in spectral gaps of classical wave operators by introducing defects. Moreover, the corresponding waves exhibit exponential localization. The proofs and details are given in Section 5.

A *defect* is a modification of a given medium in a bounded domain. Two media, described by coefficient matrices $\mathcal{K}_0(x), \mathcal{R}_0(x)$ and $\mathcal{K}(x), \mathcal{R}(x)$, are said to *differ by a defect*, if they are the same outside some bounded set Ω, i.e., $\mathcal{K}_0(x) = \mathcal{K}(x)$ and $\mathcal{R}_0(x) = \mathcal{R}(x)$ if $x \notin \Omega$. The defect is said to be supported by the bounded set Ω.

We recall that the essential spectrum $\sigma_{ess}(H)$ of an operator H consists of all the points of its spectrum, $\sigma(H)$, which are not isolated eigenvalues with finite multiplicity. Figotin and Klein [8] showed that the essential spectrum of Acoustic and Maxwell operators are not changed by defects. We extend this result to the class of classical wave operators: the essential spectrum of a partially elliptic classical wave operator (first or second order) is not changed by defects.

Theorem 2.9 *Let W_0 and W be partially elliptic classical wave operators for two media which differ by a defect. Then*

$$\sigma_{ess}(W) = \sigma_{ess}(W_0). \tag{2.41}$$

If (a,b) is a gap in the spectrum of W_0, the spectrum of W in (a,b) consists of at most isolated eigenvalues with finite multiplicity, the corresponding eigenmodes decaying exponentially fast away from the defect, with a rate depending on the distance from the eigenvalue to the edges of the gap.

In view of the unitary equivalence (2.37), Theorem 2.9 is an immediate corollary to Theorem 5.1 and Corollary 5.3. For second order classical wave operators, the exponential decay of an eigenmode is given in (5.11). For first order classical wave operators, the exponential decay of an eigenmode follows from (2.40), (5.11), and (3.19).

We now turn to the existence of midgap eigenmodes and exponentially localized waves. The next theorem shows that one can design simple defects which generate eigenvalues in a specified subinterval of a spectral gap of W_0 , extending [8, Theorem 2] to the class of classical wave operators. We insert a defect that changes the value of $\mathcal{K}_0(x)$ and $\mathcal{R}_0(x)$ inside a bounded set of "size" ℓ to given positive constants K and R. If (a,b) is a gap in the spectrum of W_0, we will show that we can deposit an eigenvalue of W inside any specified closed subinterval of (a,b), by inserting such a defect with $\frac{\ell}{\sqrt{KR}}$ large enough. We provide estimates on how large is "large enough". Note that the corresponding eigenmode is exponentially decaying by Theorem 2.9, so we construct an exponentially localized wave.

Theorem 2.10 (Existence of exponentially localized waves) *Let (a,b) be a gap in the spectrum of a partially elliptic classical wave operator $W_0 = W_{\mathcal{K}_0,\mathcal{R}_0}$, select $\mu \in (a,b)$, and pick $\delta > 0$ such that the* *interval* $[\mu-\delta,\mu+\delta]$ *is contained in the gap, i.e.,* $[\mu-\delta,\mu+\delta]\subset(a,b)$. *Given an open bounded set* Ω, $x_0\in\Omega$, $0<K,R,\ell<\infty$, *we introduce a defect that produces coefficient matrices* $\mathcal{K}(x)$ *and* $\mathcal{R}(x)$ *that are constant in the set* $\Omega_\ell = x_0+\ell(\Omega-x_0)$, *with*

$$\mathcal{K}(x)=KI_n \ \ and\ \ \mathcal{R}(x)=RI_m \ \ for\ \ x\in\Omega_\ell\,. \tag{2.42}$$

Then there is a finite constant C, *satisfying an explicit lower bound depending only on the order (first or second) of the classical wave operator, and on* D_+, μ, δ, *and the geometry of* Ω, *such that if*

$$\frac{\ell}{\sqrt{KR}}>C\,, \tag{2.43}$$

then the operator $W=W_{\mathcal{K},\mathcal{R}}$ *has at least one eigenvalue in the interval* $[\mu-\delta,\mu+\delta]$.

Theorem 2.10 follows from Theorem 5.4 and (2.37). For second order operators the explicit lower bound is given in (5.16), for first order operators it can be calculated from (5.16) and (2.37).

3 Properties of classical wave operators

In this section we discuss several important properties of classical wave operators, which provide the necessary technical tools for proving localization in inhomogeneous and random (see [16]) media.

3.1 A trace estimate

Partially elliptic second order classical wave operators satisfy a trace estimate that provides a crucial ingredient for many results.

Theorem 3.1 ([17, Theorem 1.1]) *Let* W *be a partially elliptic second order classical wave operator on* $\mathcal{H}^{(n)}$, *and let* $P_W^\perp$ *denote the orthogonal projection onto* $(\ker W)^\perp$. *Then*

$$\operatorname{tr}\left(V^*P_W^\perp(W+I)^{-2r}V\right)\le C_{d,n,K_\pm,R_\pm,D_+,D_+^\perp,\Theta}\,\|V\|_{\infty,2}^2<\infty \quad , \tag{3.1}$$

for $r \geq \nu$, where ν is the smallest integer satisfying $\nu > \frac{d}{4}$. V is the bounded operator on $\mathcal{H}^{(n)}$ given by multiplication by an $n \times n$ matrix-valued measurable function $V(x)$, with

$$\|V\|_{\infty,2}^2 = \sum_{y \in \mathbb{Z}^d} \|\chi_{y,1}(x) V^*(x) V(x)\|_\infty < \infty \ . \tag{3.2}$$

($\chi_{y,L}$ denotes the characteristic function of a cube of side length L centered at y.) The constant $C_{d,n,K_\pm,R_\pm,D_+,D_+^\perp,\Theta}$ depends only on the fixed parameters $d, n, K_\pm, R_\pm, D_+, D_+^\perp, \Theta$.

3.2 Finite volume classical wave operators

Throughout this paper we use two norms in $\mathbb{R}^d$ and $\mathbb{C}^d$:

$$|x| = \left(\sum_{i=1}^{d} |x_i|^2 \right)^{\frac{1}{2}} , \tag{3.3}$$

$$\|x\| = \max\{|x_i|, \ i = 1, \ldots, d\} . \tag{3.4}$$

We set $B_r(x)$ to be the open ball in $\mathbb{R}^d$, centered at x with radius $r > 0$:

$$B_r(x) = \{y \in \mathbb{R}^d; \ |y - x| < r\} . \tag{3.5}$$

By $\Lambda_L(x)$ we denote the open cube in $\mathbb{R}^d$, centered at x with side $L > 0$:

$$\Lambda_L(x) = \{y \in \mathbb{R}^d; \ \|y - x\| < L/2\} , \tag{3.6}$$

and by $\overline{\Lambda}_L(x)$ the closed cube. By Λ we will always denote some open cube $\Lambda_L(x)$. We will identify a closed cube $\overline{\Lambda}_L(x)$ with a torus in the usual way, and use the following distance in the torus:

$$d_L(y, y') = \min_{m \in L\mathbb{Z}^d} |y - y' + m| \leq \tfrac{\sqrt{d}}{2} L \ \text{ for } \ y, y' \in \overline{\Lambda}_L(x). \tag{3.7}$$

We set $\mathcal{H}_\Lambda^{(n)} = L^2(\Lambda, dx; \mathbb{C}^n)$. A $CPDO_{n,m}^{(1)}$ $\mathbf{D}$ defines a closed densely defined operator $\mathbf{D}_\Lambda$ from $\mathcal{H}_\Lambda^{(n)}$ to $\mathcal{H}_\Lambda^{(m)}$ with periodic boundary condition; an operator core is given by $C_{\text{per}}^\infty(\overline{\Lambda}, \mathbb{C}^n)$, the infinitely differentiable, periodic $\mathbb{C}^n$-valued functions on $\overline{\Lambda}$. The restriction of a CO_n $\mathcal{S}$ to Λ gives the bounded, invertible operator $\mathcal{S}_\Lambda$ on $\mathcal{H}_\Lambda^{(n)}$. Given an $MPDO_{n,m}^{(1)}$ A as in (2.25), we define its restriction A_Λ to the cube Λ with periodic boundary condition by

$$A_\Lambda = \sqrt{\mathcal{R}_\Lambda} \mathbf{D}_\Lambda \sqrt{\mathcal{K}_\Lambda} \ \text{ on } \ \mathcal{D}(A_\Lambda) = \mathcal{K}_\Lambda^{-\frac{1}{2}} \mathcal{D}(\mathbf{D}_\Lambda) \ , \tag{3.8}$$

a closed, densely defined operator on $\mathcal{H}_\Lambda^{(n)}$. The restriction W_Λ of the second order classical wave operator $W = A^*A$ to Λ with periodic boundary condition is now defined as $W_\Lambda = A_\Lambda^* A_\Lambda$.

If the $CPDO_{n,m}^{(1)}$ $\mathbf{D}$ is partially elliptic, then the restriction $\mathbf{D}_\Lambda$ is also partially elliptic, in the sense that equations (2.20) and (2.21) hold for $\mathbf{D}_\Lambda$, $(\mathbf{D}^\perp)_\Lambda$, and Δ_Λ. (Δ_Λ is the Laplacian on $L^2(\Lambda, dx)$ with periodic boundary condition.) This can be easily seen by using the Fourier transform; here the use of periodic boundary condition plays a crucial role. We also have (2.22) and (2.23) with $\mathcal{H}_\Lambda^{(n)}$.

If $\Lambda = \Lambda_L(x)$, we write $\mathcal{H}_{x,L}^{(n)}$, $W_{x,L}$, and so on.

Given a second order classical wave operator W on $\mathcal{H}^{(n)}$, we define its finite volume resolvent on a cube Λ by

$$R_\Lambda(z) = (W_\Lambda - z)^{-1} \quad \text{for} \quad z \notin \sigma(W_\Lambda)\,. \tag{3.9}$$

If W is partially elliptic, it turns out that $(W_\Lambda)_\perp$ has compact resolvent, i.e., $R_\Lambda(z)P_{W_\Lambda}^\perp$ is a compact operator for $z \notin \sigma(W_\Lambda)$. Note that it suffices to prove the statement for $z = -1$. We will prove a stronger statement.

In what follows, we write $F \simeq G$ if the positive self-adjoint operators F and G are unitarily equivalent, and we write $F \preceq G$ if $F \simeq J$ for some positive self-adjoint operator $J \leq G$. Note that, if $0 \leq F \preceq G$, then $\mathrm{tr} f(G) \leq \mathrm{tr} f(F)$, for any positive, decreasing function f on $[0, \infty)$.

Proposition 3.2 *Let W be a partially elliptic second order classical wave operator. Then for any finite cube Λ and $p > \frac{d}{2}$ we have*

$$\mathrm{tr}\left\{(W_\Lambda + 1)^{-p} P_{W_\Lambda}^\perp\right\} \leq n \,\mathrm{tr}\left\{(K_- R_- \Theta(-\Delta_\Lambda) + 1)^{-p}\right\} < \infty\,. \tag{3.10}$$

Proof: Using Lemma A.1 and (2.24), we get

$$\begin{aligned}(W_\Lambda)_\perp &= \left(\sqrt{\mathcal{K}_\Lambda}\mathbf{D}_\Lambda^* \mathcal{R}_\Lambda \mathbf{D}_\Lambda \sqrt{\mathcal{K}_\Lambda}\right)_\perp \geq R_- \left(\sqrt{\mathcal{K}_\Lambda}\mathbf{D}_\Lambda^* \mathbf{D}_\Lambda \sqrt{\mathcal{K}_\Lambda}\right)_\perp \\ &\simeq R_- \left(\mathbf{D}_\Lambda \mathcal{K}_\Lambda \mathbf{D}_\Lambda^*\right)_\perp \geq K_- R_- \left(\mathbf{D}_\Lambda \mathbf{D}_\Lambda^*\right)_\perp \simeq K_- R_- \left(\mathbf{D}_\Lambda^* \mathbf{D}_\Lambda\right)_\perp\,.\end{aligned} \tag{3.11}$$

It follows from (3.11), (2.23), and (2.21) that

$$\begin{aligned}\mathrm{tr}\left\{(W_\Lambda + 1)^{-p} P_{W_\Lambda}^\perp\right\} &= \mathrm{tr}\left\{((W_\Lambda)_\perp + 1)^{-p}\right\} \\ &\leq \mathrm{tr}\left\{(K_- R_- \left(\mathbf{D}_\Lambda^* \mathbf{D}_\Lambda\right)_\perp + 1)^{-p}\right\}\end{aligned} \tag{3.12}$$

$$\leq \operatorname{tr}\left\{\left(K_- R_- \left[(\mathbf{D}_\Lambda^* \mathbf{D}_\Lambda)_\perp \oplus \left((\mathbf{D}^\perp)_\Lambda^* (\mathbf{D}^\perp)_\Lambda\right)_\perp\right] + 1\right)^{-p}\right\}$$
$$\leq \operatorname{tr}\left\{(K_- R_- \Theta\left[(-\Delta_\Lambda) \otimes I_n\right] + 1)^{-p}\right\}$$
$$= n \operatorname{tr}\left\{(K_- R_- \Theta(-\Delta_\Lambda) + 1)^{-p}\right\} < \infty\,,$$

if $p > \frac{d}{2}$. ∎

Since $(W_\Lambda)_\perp \geq 0$ has compact resolvent, we may define

$$N_{W_\Lambda}(E) = \operatorname{tr} \chi_{(-\infty,E)}((W_\Lambda)_\perp)\,, \tag{3.13}$$

the number of eigenvalues of $(W_\Lambda)_\perp$ that are less than E. If $E \leq 0$, we have $N_{W_\Lambda}(E) = 0$, and if $E > 0$, $N_{W_\Lambda}(E)$ is the number of eigenvalues of W_Λ (or $(W_\Lambda)_\perp$) in the interval $(0, E)$. Notice that $N_{W_\Lambda}(E)$ is the distribution function of the measure $n_{W_\Lambda}(dE)$ given by

$$\int h(E) n_{W_\Lambda}(dE) = \operatorname{tr}\left(h((W_\Lambda)_\perp)\right)\,, \tag{3.14}$$

for positive continuous functions h of a real variable.

We have the following "a priori" estimate:

Lemma 3.3 *Let W be a partially elliptic second order classical wave operator. Then for any finite cube Λ and $E > 0$ we have*

$$N_{W_\Lambda}(E) \leq n\, N_{-\Delta_\Lambda}\left(\tfrac{E}{K_- R_- \Theta}\right) \leq n\, C_d \left(\tfrac{E}{K_- R_- \Theta}\right)^{\frac{d}{2}} |\Lambda|\,, \tag{3.15}$$

where C_d is some finite constant depending only on the dimension d.

Proof: We have

$$N_{W_\Lambda}(E) \leq N_{K_- R_- \left(\mathbf{D}_\Lambda^* \mathbf{D}_\Lambda\right)}(E) \tag{3.16}$$
$$\leq N_{K_- R_- \left(\mathbf{D}_\Lambda^* \mathbf{D}_\Lambda + (\mathbf{D}^\perp)_\Lambda^* (\mathbf{D}^\perp)_\Lambda\right)}(E) \tag{3.17}$$
$$\leq N_{K_- R_- \Theta[(-\Delta_\Lambda)\otimes I_n]}(E) = n\, N_{-\Delta_\Lambda}\left(\tfrac{E}{K_- R_- \Theta}\right) \tag{3.18}$$

where (3.16) follows from (3.11) and the Min-max Principle, (3.17) follows from (2.23), (3.18) follows from (2.21), plus a simple computation for the equality.

The second inequality in (3.15) is given by a standard estimate. ∎

3.3 An interior estimate

The following interior estimate is an adaptation of [17, Theorem 4.1] to both finite or infinite volume.

Lemma 3.4 *Let $W = A^*A$ be a second order classical wave operator, and let Λ denote either an open cube or $\mathbb{R}^d$. Let $\rho \in C_0^1(\Lambda)$ and $\tau \in L^\infty_{loc}(\Lambda, dx)$, with $0 \le \rho(x) \le \tau(x)$ and $|\nabla \rho(x)| \le c\tau(x)$ a.e., where c is a finite constant. Then, for any $\psi \in \mathcal{D}(W_\Lambda)$ we have*

$$\|\rho A_\Lambda \psi\|^2 \le a\|\tau W_\Lambda \psi\|^2 + \left(\tfrac{1}{a} + 4c^2\Xi_A^2\right)\|\tau\psi\|^2 \tag{3.19}$$

for all $a > 0$, where Ξ_A is given in (2.26).

Proof: This is proved as [17, Theorem 4.1], keeping track of the constants. ∎

3.4 Improved resolvent decay estimates in a gap

We adapt an argument of Barbaroux, Combes and Hislop [1] to second order classical wave operators, obtaining an improvement on the rate of decay given by the usual Combes-Thomas argument (e.g., [6, Lemma 12], [7, Lemma 15]). Our proof, while based on [1, Lemma 3.1], is otherwise different from the proof for Schrödinger operators, as we use an argument based on quadratic forms avoiding the analytic continuation of the operators. This way we can accomodate the nonsmoothness of the coefficients of our classical wave operators.

We will prove the decay estimate for both infinite and finite volumes (with periodic boundary condition). We start with infinite volume. Recall that $B_r(x)$ denotes the open ball of radius r centered at x.

Theorem 3.5 *Let $W = A^*A$ be a second order classical wave operator with a spectral gap (a, b). Then for any $E \in (a, b)$ and $\ell, \ell' > 0$ we have*

$$\|\chi_{B_\ell(x)} R(E) \chi_{B_{\ell'}(y)}\| \le C_E \, \mathrm{e}^{m_E(\ell+\ell')} \, \mathrm{e}^{-m_E|x-y|} \tag{3.20}$$

for all $x, y \in \mathbb{R}^d$, with

$$m_E = \frac{1}{4\Xi_A}\sqrt{\frac{(E-a)(b-E)}{(a+b+2)\,(b+1)}} \le \frac{1}{4\Xi_A}, \tag{3.21}$$

and

$$C_E = \max\left\{\frac{a+b+2}{E-a}, \frac{4(b+1)}{b-E}\right\} . \tag{3.22}$$

In addition,

$$\begin{aligned} &\|\chi_{B_\ell(x)} A R(E) \chi_{B_{\ell'}(y)}\| \\ &\quad \le C_E \left(2E + 16\,\Xi_A^2\right)^{\frac{1}{2}} \, \mathrm{e}^{m_E(\ell+\ell'+1)} \, \mathrm{e}^{-m_E|x-y|} \end{aligned} \tag{3.23}$$

for all $x, y \in \mathbb{R}^d$ *with* $|x-y| \ge \ell + \ell' + 1$.

Proof: We start by defining the operators formally given by

$$W_\alpha = \mathrm{e}^{\alpha\cdot x} W \mathrm{e}^{-\alpha\cdot x} , \quad \alpha \in \mathbb{R}^d . \tag{3.24}$$

To do so, let us consider the bounded operator

$$G_\alpha = \sqrt{R} D(\alpha) \sqrt{K} , \ \ \|G_\alpha\| \le |\alpha| \Xi_A . \tag{3.25}$$

Then

$$A_\alpha \ = \ \mathrm{e}^{\alpha\cdot x} A \mathrm{e}^{-\alpha\cdot x} = A + iG_\alpha \ \text{ on } \ \mathcal{D}(A) . \tag{3.26}$$

$$(A^*)_\alpha \ = \ \mathrm{e}^{\alpha\cdot x} A^* \mathrm{e}^{-\alpha\cdot x} = A^* + iG_\alpha^* \ \text{ on } \ \mathcal{D}(A^*) , \tag{3.27}$$

are closed, densely defined operators. (Note $(A^*)_\alpha \ne (A_\alpha)^*$.) We define $W_\alpha = (A^*)_\alpha A_\alpha$ as a quadratric form. More precisely, for each $\alpha \in \mathbb{R}^d$,we define a quadratic form with domain $\mathcal{D}(A)$ by

$$\mathcal{W}_\alpha[\psi] = \langle (A^*)_\alpha^* \psi, A_\alpha \psi \rangle \ . \tag{3.28}$$

Note that if $\alpha = 0$, $\mathcal{W} = \mathcal{W}_0$ is the closed, nonegative quadratic form associated to the classical wave operator W.

It follows from (3.26) and (3.27) that

$$\mathcal{W}_\alpha[\psi] - \mathcal{W}[\psi] = 2i \,\mathrm{Re}\, \langle A\psi, G_\alpha \psi \rangle - \langle G_\alpha \psi, G_\alpha \psi \rangle , \tag{3.29}$$

so

$$\begin{aligned} |\mathcal{W}_\alpha[\psi] - \mathcal{W}[\psi]| &\le \|G_\alpha \psi\| \left(4\|A\psi\|^2 + \|G_\alpha\psi\|^2 \right)^{\frac{1}{2}} \\ &\le 4s\mathcal{W}[\psi] + (\tfrac{1}{s} + s)|\alpha|^2 \Xi_A^2 \|\psi\|^2 \end{aligned} \tag{3.30}$$

for any $s > 0$. It follows [14, Theorem VI.1.33] that $\mathcal{W}_\alpha$ is a closed sectorial form on the form domain of W. We define W_α as the unique m-sectorial operator associated with it [14, Theorem VI.2.1].

For each $\alpha \in \mathbb{R}^d$, we set

$$\begin{aligned} W^{(\alpha)} &= W - G_\alpha^* G_\alpha \,, && (3.31)\\ \theta^{(\alpha)} &= 1 + |\alpha|^2 \Xi_A^2 \,, && (3.32)\end{aligned}$$

note

$$W^{(\alpha)} + \theta^{(\alpha)} \geq 1 \,. \tag{3.33}$$

It follows from (3.29) that

$$\begin{aligned}\left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} (W_a - E) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} = \quad (3.34)\\ \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \left(W^{(\alpha)} - E\right) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} + iY_\alpha,\end{aligned}$$

as everywhere defined quadratic forms, where

$$\left\| \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \left(W^{(\alpha)} - E\right) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \right\| \leq 1 + \theta^{(\alpha)} + E < \infty, \tag{3.35}$$

and

$$Y_\alpha = \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} (A^* G_\alpha + G_\alpha^* A) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \tag{3.36}$$

extends to a bounded self-adjoint operator with

$$\|Y_\alpha\| \leq 2|\alpha| \Xi_A \,, \tag{3.37}$$

in view of (3.25), (3.31), (3.32), (3.33), and

$$\begin{aligned} &\left\| A \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \psi \right\|^2 && (3.38)\\ &= \left\langle \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \psi, A^* A \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \psi \right\rangle \\ &= \|\psi\|^2 + \left\langle \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \psi, \left(G_\alpha^* G_\alpha - \theta^{(\alpha)}\right) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \psi \right\rangle \\ &\leq \|\psi\|^2 - \left\| \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \psi \right\|^2 \leq \|\psi\|^2 \,. && (3.39)\end{aligned}$$

Since (a, b) is a gap in the spectrum of W and $E \in (a, b)$, we have that the interval

$$\begin{aligned} &\left(\frac{a - E}{a + \theta^{(\alpha)}} \,, \frac{b - E - |\alpha|^2 \Xi_A^2}{b + \theta^{(\alpha)} - |\alpha|^2 \Xi_A^2} \right) && (3.40)\\ &= \left(\frac{a - E}{a + 1 + |\alpha|^2 \Xi_A^2} \,, \frac{b - E - |\alpha|^2 \Xi_A^2}{b + 1} \right)\end{aligned}$$

is a gap in the spectrum of the operator

$$\left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \left(W^{(\alpha)} - E\right) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} , \tag{3.41}$$

containing 0, as long as

$$|\alpha| < \frac{\sqrt{b-E}}{\Xi_A} . \tag{3.42}$$

We now use [1, Lemma 3.1] to conclude, if in addition to (3.42) we also require

$$|\alpha| < \frac{1}{4\Xi_A}\sqrt{\frac{(E-a)\left(b-E-|\alpha|^2\Xi_A^2\right)}{\left(a+1+|\alpha|^2\Xi_A^2\right)(b+1)}} , \tag{3.43}$$

that 0 is not in the spectrum of the operator in (3.34), and

$$\begin{aligned} &\left\| \left[\left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} (W_a - E) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \right]^{-1} \right\| \\ &\leq \quad \Omega_\alpha \equiv 2 \max \left\{ \frac{a+1+|\alpha|^2\Xi_A^2}{E-a}, \frac{b+1}{b-E-|\alpha|^2\Xi_A^2} \right\} . \end{aligned} \tag{3.44}$$

Since

$$\mathcal{D}(W_\alpha) \subset \mathcal{D}(\mathcal{W}_\alpha) = \mathcal{D}(\mathcal{W}) = \mathcal{D}\left(\left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{\frac{1}{2}}\right) , \tag{3.45}$$

we may use (3.33) and (3.44) to obtain, for all $\phi \in \mathcal{D}(W_\alpha)$,

$$\begin{aligned} &\| (W_a - E)\, \phi \| \geq \\ &\quad \left\| \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} (W_a - E) \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{-\frac{1}{2}} \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{\frac{1}{2}} \phi \right\| \\ &\quad\quad \geq \Omega_\alpha^{-1} \left\| \left(W^{(\alpha)} + \theta^{(\alpha)}\right)^{\frac{1}{2}} \phi \right\| \geq \Omega_\alpha^{-1} \|\phi\| . \end{aligned} \tag{3.46}$$

Since (3.46) holds for all $\alpha \in \mathbb{R}^d$, and $W_\alpha^* = W_{-\alpha}$, $W^{(\alpha)} = W^{(-\alpha)}$, $\theta^{(\alpha)} = \theta^{(-\alpha)}$, and $\Omega^{(\alpha)} = \Omega^{(-\alpha)}$, we see that we also have (3.46) for W_α^*. We can conclude that $E \notin \sigma(W_\alpha)$ and

$$\left\| (W_\alpha - E)^{-1} \right\| \leq \Omega_\alpha , \tag{3.47}$$

for all $\alpha \in \mathbb{R}^d$ satisfying (3.42) and (3.43).

We now take $|\alpha| \leq m_E$, where m_E is given in (3.21). Then both (3.42) and (3.43) are satisfied, and we also have $|\alpha|\Xi_A \leq \sqrt{\frac{b-E}{2}}$, so $\Omega_\alpha \leq C_E$, with C_E as in (3.22), and (3.47) gives

$$\|R_\alpha(E)\| \leq C_E \quad , \quad \text{with } R_\alpha(E) = (W_\alpha - E)^{-1}. \tag{3.48}$$

We may now prove (3.20) . Let $x_0, y_0 \in \mathbb{R}^d$, $\ell > 0$, and take $\alpha = \frac{m_E}{|x_0 - y_0|}(x_0 - y_0)$. We have

$$\begin{aligned} \chi_{B_\ell(x_0)} R(E) \chi_{B_{\ell'}(y_0)} &= \chi_{B_\ell(x_0)} \mathrm{e}^{-\alpha\cdot x} R_\alpha(E) \mathrm{e}^{\alpha\cdot x} \chi_{B_{\ell'}(y_0)} \\ &= \mathrm{e}^{-m|x_0-y_0|} \chi_{B_\ell(x_0)} \mathrm{e}^{-\alpha\cdot(x-x_0)} R_\alpha(E) \mathrm{e}^{\alpha\cdot(x-y_0)} \chi_{B_{\ell'}(y_0)} , \end{aligned} \tag{3.49}$$

so

$$\begin{aligned} &\|\chi_{B_\ell(x_0)} R(E) \chi_{B_{\ell'}(y_0)}\| \\ &\quad \leq C_E \|\chi_{B_\ell(x_0)} \mathrm{e}^{-\alpha\cdot(x-x_0)}\|_\infty \|\chi_{B_{\ell'}(y_0)} \mathrm{e}^{a\cdot(x-y_0)}\|_\infty \mathrm{e}^{-m|x_0-y_0|} . \end{aligned} \tag{3.50}$$

Since

$$\|\chi_{B_\ell(x_0)} \mathrm{e}^{\pm\alpha\cdot(x-x_0)}\|_\infty \leq \mathrm{e}^{|\alpha|\ell} = \mathrm{e}^{m_E \ell} , \tag{3.51}$$

(3.20) follows from (3.50) and (3.51).

To prove (3.23), we use Lemma 3.4. We let $|x_0 - y_0| \geq \ell + \ell' + 1$, and pick $\rho \in C_0^1(\mathbb{R}^d)$, with $\chi_{B_\ell(x_0)} \leq \rho \leq \chi_{B_{\ell+1}(x_0)}$ and $|\nabla\rho(x)| \leq 2$. We have

$$\begin{aligned} \|\chi_{B_\ell(x_0)} A R(E) \chi_{B_{\ell'}(y_0)}\| &\leq \|\rho A R(E) \chi_{B_{\ell'}(y_0)}\| \\ &\leq \left(2E + 16\, \Xi_A^2\right)^{\frac{1}{2}} \|\chi_{B_{\ell+1}(x_0)} R(E) \chi_{B_{\ell'}(y_0)}\| , \end{aligned} \tag{3.52}$$

so (3.23) now follows from (3.20) ∎

We now turn to the torus, i.e., we prove a version of Theorem 3.5 for the restriction of a second order classical wave operator to a cube Λ with periodic boundary condition. We use the distance (3.7) in the torus.

Theorem 3.6 *W be a second order classical wave operator whose restriction with periodic boundary condition to a cube $\Lambda_L(x_0)$ has a spectral gap (a,b). Then for any $E \in (a,b)$ and $\ell > 0$, with $L > 2\ell + 8$, we have*

$$\|\chi_{B_\ell(x)} R_{x_0,L}(E) \chi_{B_{\ell'}(y)}\|_{x_0,L} \quad \leq C_E \, \mathrm{e}^{2m_{E,L,\ell}\ell} \mathrm{e}^{-m_{E,L,\ell}\, d_L(x,y)} \tag{3.53}$$

for all $x, y \in \overline{\Lambda}_L(x_0)$*, where*

$$m_{E,L,\ell} = \frac{m_E}{c_{L,\ell}}, \ \textit{with} \ c_{L,\ell} = \left(\frac{2\sqrt{d}}{1 - \frac{2(\ell+3)}{L-2}} + 1 \right) , \tag{3.54}$$

and m_E *and* C_E *are as in Theorem 3.5.*

Proof: Let us fix $x_1, y_1 \in \overline{\Lambda}_L(x_0)$, by redefining the coefficient operators we may assume $x_0 = 0 = \frac{1}{2}(x_1 + y_1)$ and $x_1, y_1 \in \overline{\Lambda}_{\frac{L}{2}}(0)$. In particular, $d_L(x_1, y_1) = |x_1 - y_1|$. Let $L > 2\ell + 8$, we pick a real valued function $\xi \in C_0^1(\mathbb{R})$ with $0 \leq \xi(t) \leq 1$ for all $t \in \mathbb{R}$, such that $\xi(t) = 1$ for $|t| \leq \frac{L}{4} + \frac{\ell}{2}$, $\xi(t) = 0$ for $|t| \geq \frac{L}{2} - 1$, and $|\xi'(t)| \leq \left(\frac{L}{4} - \frac{\ell}{2} - 2\right)^{-1}$ for all $t \in \mathbb{R}$. We set $\Phi(x) = \prod_{i=1}^{d} \xi(x_i)$ for $x \in \mathbb{R}^d$. Notice $\operatorname{supp} \Phi(x) \subset \Lambda_L(0)$.

We now proceed as in the proof of Theorem 3.5 with $\Lambda_L(0)$ substituted for $\mathbb{R}^d$ and definition (3.24) replaced by

$$(W_{0,L})_\alpha = \mathrm{e}^{\Phi(x)\alpha\cdot x} W_{0,L} \mathrm{e}^{-\Phi(x)\alpha\cdot x} \ , \quad \alpha \in \mathbb{R}^d \, , \tag{3.55}$$

and instead of (3.25), consider the bounded operators

$$(G_{0,L})_\alpha = \sqrt{R_{0,L}} \, (D(\nabla \left(\Phi(x)\alpha \cdot x\right)))_{0,L} \sqrt{K_{0,L}} \, . \tag{3.56}$$

Since

$$|\nabla \left(\Phi(x)\alpha \cdot x\right)| \quad \leq \quad \left(\frac{(\frac{L}{2}-1)\sqrt{d}}{\frac{L}{4}-\frac{\ell}{2}-2} + 1 \right) |\alpha| = c_{L,\ell}|\alpha| \tag{3.57}$$

for all $x \in \Lambda_L(0)$, with $c_{L,\ell}$ as in (3.54), we have

$$\|(G_{0,L})_\alpha\| \leq c_{L,\ell}|\alpha|\Xi_A \, . \tag{3.58}$$

We now proceed as in the proof of Theorem 3.5, except that we must now substitute $c_{L,\ell}|\alpha|$ for $|\alpha|$ in the estimates. Thus, if $|\alpha| \leq \frac{m_E}{c_{L,\ell}}$, we conclude that $E \notin \sigma((W_{0,L})_\alpha)$ and

$$\|(R_{0,L})_\alpha(E)\| \leq C_E \ , \quad \text{with } (R_{0,L})_\alpha(E) = ((W_{0,L})_\alpha - E)^{-1}. \tag{3.59}$$

To prove (3.53), we take $\alpha = \frac{m_E}{c_{L,\ell}|x_1 - y_1|}(x_1 - y_1)$, and complete the proof of as before (with x_1, y_1 substituted for x, y in (3.53)), as

$$\|\chi_{B_\ell(x_1)} \mathrm{e}^{\pm\Phi(x)\alpha\cdot x - \alpha\cdot x_1}\|_\infty = \|\chi_{B_\ell(x_1)} \mathrm{e}^{\pm\alpha\cdot(x - x_1)}\|_\infty \leq \mathrm{e}^{m_{E,L,\ell}\ell} \, . \tag{3.60}$$

∎

3.5 A Simon-Lieb-type inequality

The norm in $\mathcal{H}_\Lambda^{(r)}$ and also the corresponding operator norm will both be denoted by $\| \ \|_\Lambda$, or $\| \ \|_{x,L}$ in case $\Lambda = \Lambda_L(x)$. (We omit r from the notation.) If $\Lambda_1 \subset \Lambda_2$ are open cubes (possibly the whole space), let $J_{\Lambda_1}^{\Lambda_2}\colon \mathcal{H}_{\Lambda_1}^{(r)} \to \mathcal{H}_{\Lambda_2}^{(r)}$ be the canonical injection. If $\Lambda_i = \Lambda_{L_i}(x_i)$, $i = 1, 2$, we write $\| \ \|_{x_1,L_1}^{x_2,L_2}$ for the operator norm from $\mathcal{H}_{\Lambda_{L_1}(x_1)}^{(r)}$ to $\mathcal{H}_{\Lambda_{L_2}(x_2)}^{(r)}$, and $J_{x_1,L_1}^{x_2,L_2} = J_{\Lambda_{L_1}(x_1)}^{\Lambda_{L_2}(x_2)}$.

Given a function $\phi \in L^\infty(\mathbb{R}^d)$, with $\operatorname{supp}\phi \subset \Lambda$, we do not distinguish in the notation between ϕ as a multiplication operator on $\mathcal{H}_\Lambda^{(r)}$ and on $\mathcal{H}^{(r)}$. If $\mathbf{D}$ is a $CPDO_{n,m}^{(1)}$, and $\phi \in C_0^1(\mathbb{R}^d)$, real-valued, with $\operatorname{supp}\phi \subset \Lambda_1 \subset \Lambda_2$, we can verify that, as operators,

$$\mathbf{D}_{\Lambda_2} J_{\Lambda_1}^{\Lambda_2} \phi = J_{\Lambda_1}^{\Lambda_2} \mathbf{D}_{\Lambda_1} \phi \ \text{ on } \ \mathcal{D}(\mathbf{D}_{\Lambda_1}) \ . \tag{3.61}$$

It follows for the $MPDO_{n,m}^{(1)}$ A that

$$A_{\Lambda_2} J_{\Lambda_1}^{\Lambda_2} \phi = J_{\Lambda_1}^{\Lambda_2} A_{\Lambda_1} \phi \ \text{ on } \ \mathcal{D}(A_{\Lambda_1}) \ . \tag{3.62}$$

We set

$$A[\phi] = \sqrt{\mathcal{R}} D[\phi] \sqrt{\mathcal{K}} \tag{3.63}$$

where $D[\phi]$, given by multiplication by the matrix valued function $D(-i\nabla\phi(x))$, is a bounded operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$, with norm bounded by $D_+\|\nabla\phi\|_\infty$. Thus $A[\phi]$ is a bounded operator given by multiplication by a matrix-valued, measurable function, with (see (2.26))

$$\|A[\phi]\| \le \Xi_A \|\nabla\phi\|_\infty \, . \tag{3.64}$$

We denote by $A_\Lambda[\phi]$ its restriction to the cube Λ; it also satisfies the bound (3.64).

We will use the fact that $A_\Lambda R_\Lambda(z)$ is a bounded operator with

$$\|A_\Lambda R_\Lambda(z)\|_\Lambda^2 \le \|R_\Lambda(z)\|_\Lambda \left(|z| \|R_\Lambda(z)\|_\Lambda + 1\right) \ . \tag{3.65}$$

The basic tool to relate the finite volume resolvents in different scales is the *smooth resolvent identity* (SRI) (see [2, 6, 7, 17]).

Lemma 3.7 (SRI) *Let $W = A^*A$ be a second order classical wave operator, and let $\Lambda_1 \subset \Lambda_2$ be either open cubes or $\mathbb{R}^d$, and let $\phi \in$*

$C_0^1(\mathbb{R}^d)$ *with* $\operatorname{supp}\phi \subset \Lambda_1$*. Then, for any* $z \notin \sigma(W_{\Lambda_1}) \cup \sigma(W_{\Lambda_2})$ *we have*

$$R_{\Lambda_2}(z)\phi J_{\Lambda_1}^{\Lambda_2} = J_{\Lambda_1}^{\Lambda_2}\phi R_{\Lambda_1}(z) + R_{\Lambda_2}(z)A_{\Lambda_2}^*[\phi]J_{\Lambda_1}^{\Lambda_2}A_{\Lambda_1}R_{\Lambda_1}(z) - R_{\Lambda_2}(z)A_{\Lambda_2}^*J_{\Lambda_1}^{\Lambda_2}A_{\Lambda_1}[\phi]R_{\Lambda_1}(z) \tag{3.66}$$

as bounded operators from $\mathcal{H}^{(n}_{\Lambda_{L_1}}$ *to* $\mathcal{H}^{(n)}_{\Lambda_{L_2}}$.

Proof: This lemma can be proved as [17, Lemma 7.2]. ∎

We will now state and prove a *Simon-Lieb-type inequality* (SLI) for second order classical wave operators. This estimate is a crucial ingredient in the multiscale analysis proofs of localization for random operators, where it is used to obtain decay in a larger scale from decay in a given scale [12, 11, 5, 2, 6, 7, 13].

Let us fix $q \in \mathbb{N}$. (In [16] we will work with a periodic background medium, and we will take q to be the period.) We will take cubes $\Lambda_L(x)$ centered at sites $x \in q\mathbb{Z}^d$ with side $L \in 2q\mathbb{N}$ (so in a periodic background medium with period q the background medium will be the same in all cubes in a given scale L). For such cubes (with $L \geq 4q$), we set

$$\Upsilon_L(x) = \left\{ y \in q\mathbb{Z}^d;\ \|y - x\| = \tfrac{L}{2} - q \right\}, \tag{3.67}$$

$$\widetilde{\Upsilon}_L(x) = \overline{\Lambda}_{L-q}(x) \backslash \Lambda_{L-3q}(x) = \bigcup_{y \in \Upsilon_L(x)} \overline{\Lambda}_q(y), \tag{3.68}$$

$$\widehat{\Upsilon}_L(x) = \overline{\Lambda}_{L-\frac{3q}{2}}(x) \backslash \Lambda_{L-\frac{5q}{2}}(x), \tag{3.69}$$

$$\Gamma_{x,L} = \chi_{\widetilde{\Upsilon}_L(x)} = \sum_{y \in \Upsilon_L(x)} \chi_{y,q} \ \text{ a.e.}, \tag{3.70}$$

$$\widehat{\Gamma}_{x,L} = \chi_{\widehat{\Upsilon}_L(x)}. \tag{3.71}$$

Note

$$|\Upsilon_L(x)| \leq d(L - 2q + 1)^{d-1}. \tag{3.72}$$

In addition each cube $\Lambda_L(x)$ will be equipped with a function $\phi_{x,L}$ constructed in the following way: we fix an even function $\xi \in C_0^1(\mathbb{R})$ with $0 \leq \xi(t) \leq 1$ for all $t \in \mathbb{R}$, such that $\xi(t) = 1$ for $|t| \leq \frac{q}{4}$, $\xi(t) = 0$ for $|t| \geq \frac{3q}{4}$, and $|\xi'(t)| \leq \frac{3}{q}$ for all $t \in \mathbb{R}$. (Such a function always exists.) We define

$$\xi_L(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{L}{2} - \frac{5q}{4} \\ \xi\left(|t| - \left(\frac{L}{2} - \frac{3q}{2}\right)\right), & \text{if } |t| \geq \left(\frac{L}{2} - \frac{3q}{2}\right) \end{cases} \tag{3.73}$$

and set

$$\phi_{x,L}(y) = \phi_L(y-x) \text{ for } y \in \mathbb{R}^d, \text{ with } \phi_L(y) = \prod_{i=1}^{d} \xi_L(y_i)\,. \tag{3.74}$$

We have $\phi_{x,L} \in C_0^1(\mathbb{R}^d)$, with $\operatorname{supp}\phi_{x,L} \subset \Lambda_L(x)$ and $0 \le \phi_{x,L} \le 1$. By construction, we have

$$\chi_{x,\frac{L}{2}-\frac{5q}{4}}\phi_{x,L} = \chi_{x,\frac{L}{2}-\frac{5q}{4}}\,, \quad \chi_{x,\frac{L}{2}-\frac{3q}{4}}\phi_{x,L} = \Phi_{x,L}\,, \tag{3.75}$$

$$\widehat{\Gamma}_{x,L}\left(\nabla\phi_{x,L}\right) = \nabla\phi_{x,L}\,, \quad |\nabla\phi_{x,L}| \le \frac{3\sqrt{d}}{q}\,. \tag{3.76}$$

Similarly, we also construct a function $\rho_{x,L} \in C_0^1(\mathbb{R}^d)$, $0 \le \rho_{x,L} \le 1$, such that

$$\widehat{\Gamma}_{x,L}\,\rho_{x,L} = \widehat{\Gamma}_{x,L}\,, \quad \Gamma_{x,L}\,\rho_{x,L} = \rho_{x,L}\,, \tag{3.77}$$

$$|\nabla\rho_{x,L}| \le \frac{5\sqrt{d}}{q}\,. \tag{3.78}$$

Lemma 3.8 (SLI) *Let W be a second order classical wave operator. Let $x, y \in q\mathbb{Z}^d$, $L, \ell \in 2q\mathbb{N}$, and Ω be a set, with $\Omega \subset \Lambda_{\ell-3q}(y) \subset \Lambda_{L-3q}(x)$. Then, if $z \notin \sigma(W_{x,L}) \cup \sigma(W_{y,\ell})$, we have*

$$\|\Gamma_{x,L}R_{x,L}(z)\chi_\Omega\|_{x,L} \le \gamma_z \|\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell}\,\|\Gamma_{x,L}R_{x,L}(z)\Gamma_{y,\ell}\|_{x,L}\,, \tag{3.79}$$

with

$$\gamma_z = \tfrac{6\sqrt{d}}{q}\Xi_A\left(2|z| + \tfrac{100d}{q^2}\Xi_A^2\right)^{\frac{1}{2}}\,, \tag{3.80}$$

where Ξ_A is given in (2.26).

Proof: We proceed as in [6, Lemma 26]. Using (3.75), (3.66), and $\Gamma_{x,L}\phi_{y,\ell} = 0$, we obtain

$$\begin{aligned}\Gamma_{x,L}R_{x,L}(z)J_{y,\ell}^{x,L}\chi_\Omega &= \Gamma_{x,L}R_{x,L}(z)J_{y,\ell}^{x,L}\phi_{y,\ell}\chi_\Omega \\ &= \Gamma_{x,L}R_{x,L}(z)A_{x,L}^*[\phi_{y,\ell}]J_{y,\ell}^{x,L}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega \\ &\quad - \Gamma_{x,L}R_{x,L}(z)A_{x,L}^*J_{y,\ell}^{x,L}A_{y,\ell}[\phi_{y,\ell}]R_{y,\ell}(z)\chi_\Omega\,.\end{aligned} \tag{3.81}$$

We now use (3.76) and (3.64) to get

$$\begin{aligned}&\|\Gamma_{x,L}R_{x,L}(z)A_{x,L}^*[\phi_{y,\ell}]J_{y,\ell}^{x,L}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell}^{x,L} \\ &= \|\Gamma_{x,L}R_{x,L}(z)\Gamma_{y,\ell}A_{x,L}^*[\phi_{y,\ell}]J_{y,\ell}^{x,L}\widehat{\Gamma}_{y,\ell}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell}^{x,L} \\ &\le \tfrac{3\sqrt{d}}{q}\Xi_A\|\Gamma_{x,L}R_{x,L}(z)\Gamma_{y,\ell}\|_{x,L}\|\widehat{\Gamma}_{y,\ell}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell}\,,\end{aligned} \tag{3.82}$$

and

$$\begin{aligned} &\|\Gamma_{x,L}R_{x,L}(z)A^*_{x,L}J^{x,L}_{y,\ell}A_{y,\ell}[\phi_{y,\ell}]R_{y,\ell}(z)\chi_\Omega\|_{x,L} \\ &= \|\Gamma_{x,L}R_{x,L}(z)A^*_{x,L}\widehat{\Gamma}_{y,\ell}J^{x,L}_{y,\ell}A_{y,\ell}[\phi_{y,\ell}]\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{x,L} \\ &\le \tfrac{3\sqrt{d}}{q}\Xi_A\|\Gamma_{x,L}R_{x,L}(z)A^*_{x,L}\widehat{\Gamma}_{y,\ell}\|_{x,L}\|\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell} \\ &= \tfrac{3\sqrt{d}}{q}\Xi_A\|\widehat{\Gamma}_{y,\ell}A_{x,L}R_{x,L}(\bar z)\Gamma_{x,L}\|_{x,L}\|\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell}\,. \end{aligned} \tag{3.83}$$

We now appeal to Lemma 3.4 using (3.77) and (3.78). For $\psi \in \mathcal{H}y,\ell^{(n)}$ and $a>0$ we get

$$\begin{aligned} &\|\widehat{\Gamma}_{y,\ell}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega\psi\|^2_{y,\ell} \le \|\rho_{y,\ell}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega\psi\|^2_{y,\ell} \\ &\le a\|\Gamma_{y,\ell}W_{y,\ell}R_{y,\ell}(z)\chi_\Omega\psi\|^2_{y,\ell} \\ &\qquad + \left(\tfrac{1}{a}+\tfrac{100d}{q^2}\Xi_A^2\right)\|\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\psi\|^2_{y,\ell} \\ &\le \left(a|z|^2+\tfrac{1}{a}+\tfrac{100d}{q^2}\Xi_A^2\right)\|\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\psi\|^2_{y,\ell}\,. \end{aligned} \tag{3.84}$$

Choosing $a=|z|^{-1}$, we get

$$\|\widehat{\Gamma}_{y,\ell}A_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell} \le \left(2|z|+\tfrac{100d}{q^2}\Xi_A^2\right)^{\frac{1}{2}}\|\Gamma_{y,\ell}R_{y,\ell}(z)\chi_\Omega\|_{y,\ell}. \tag{3.85}$$

Similarly, we get

$$\begin{aligned} &\|\widehat{\Gamma}_{y,\ell}A_{x,L}R_{x,L}(\bar z)\Gamma_{x,L}\|_{x,L} \\ &\le \left(2|z|+\tfrac{100d}{q^2}\Xi_A^2\right)^{\frac{1}{2}}\|\Gamma_{y,\ell}R_{x,L}(\bar z)\Gamma_{x,L}\|_{x,L} \\ &= \left(2|z|+\tfrac{100d}{q^2}\Xi_A^2\right)^{\frac{1}{2}}\|\Gamma_{x,L}R_{x,L}(z)\Gamma_{y,\ell}\|_{x,L}\,. \end{aligned} \tag{3.86}$$

Since

$$\|\Gamma_{x,L}R_{x,L}(z)\chi_\Omega\|_{x,L} = \|\Gamma_{x,L}R_{x,L}(z)J^{x,L}_{y,\ell}\chi_\Omega\|^{x,L}_{y,\ell}\,, \tag{3.87}$$

the lemma follows from (3.81)-(3.86). ∎

3.6 The eigenfunction decay inequality

The *eigenfunction decay inequality* (EDI) estimates decay for generalized eigenfunctions from decay of finite volume resolvents. resolvents.

We start by introducing generalized eigenfunctions for classical wave operators. (We refer to [17] for the details.) Given $\nu > d/4$, we define the weighted spaces (we will omit ν from the notation) $\mathcal{H}_{\pm}^{(r)}$ as follows:

$$\mathcal{H}_{\pm}^{(r)} = \mathrm{L}^2(\mathbb{R}^d, (1+|x|^2)^{\pm 2\nu}\mathrm{d}x; \mathbb{C}^r) \ .$$

$\mathcal{H}_{-}^{(r)}$ is the space of polynomially L^2-bounded functions. The sesquilinear form

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}_{+}^{(r)}, \mathcal{H}_{-}^{(r)}} = \int \overline{\phi_1(x)} \cdot \phi_2(x) \mathrm{d}x, \tag{3.88}$$

where $\phi_1 \in \mathcal{H}_{+}^{(r)}$ and $\phi_2 \in \mathcal{H}_{-}^{(r)}$, makes $\mathcal{H}_{+}^{(r)}$ and $\mathcal{H}_{-}^{(r)}$ conjugate duals to each other. By $O^{\dagger}$ we will denote the adjoint of an operator O with respect to this duality. By construction, $\mathcal{H}_{+}^{(r)} \subset \mathcal{H}^{(r)} \subset \mathcal{H}_{-}^{(r)}$, the natural injections $\imath_{+} : \mathcal{H}_{+}^{(r)} \to \mathcal{H}^{(r)}$ and $\imath_{-} : \mathcal{H}^{(r)} \to \mathcal{H}_{-}^{(r)}$ being continuous with dense range, with $\imath_{+}^{\dagger} = \imath_{-}$.

Given a second order classical wave operator $W = A^*A$, where A is a $MPDO_{n,m}^{(1)}$, we define operators $W_{\pm}$ on $\mathcal{H}^{(n)}$ as follows: A_+ is the restriction of the operator A to $\mathcal{H}_{+}^{(n)}$, i.e., A_+ is the operator from $\mathcal{H}_{+}^{(n)}$ to $\mathcal{H}_{+}^{(m)}$ with domain $\mathcal{D}(A_+) = \{\phi \in \mathcal{D}(A) \cap \mathcal{H}_{+}^{(n)};\ A\phi \in \mathcal{H}_{+}^{(m)}\}$, defined by $A_+\phi = A\phi$ for $\phi \in \mathcal{D}(A_+)$. A_+ is a closed densely defined operator, and we set $A_- = (A_+^*)^{\dagger}$, a closed densely defined operator from $\mathcal{H}_{-}^{(n)}$ to $\mathcal{H}_{-}^{(m)}$. We define $W_+ = A_+^* A_+ = A_-^{\dagger} A_+$, which is a closed densely defined operator on $\mathcal{H}_{+}^{(n)}$ with domain $\mathcal{D}(W_+) = \{\phi \in \mathcal{D}(W) \cap \mathcal{H}_{+}^{(n)};\ W\phi \in \mathcal{H}_{+}^{(n)}\}$, and $W_+\phi = W\phi$ for $\phi \in \mathcal{D}(W_+)$ [17, Theorem 4.2]. We define $W_- = W_+^{\dagger}$, a closed densely defined operator on $\mathcal{H}_{-}^{(n)}$. Note that W is the restriction of W_- to $\mathcal{H}^{(n)}$.

A measurable function $\psi : \mathbb{R}^d \to \mathbb{C}^n$ is said to be a *generalized eigenfunction* of W with generalized eigenvalue λ, if $\psi \in \mathcal{H}_{-}^{(n)}$ (for some $\nu > \frac{d}{4}$) and is an eigenfunction for W_- with eigenvalue λ, i.e., $\psi \in \mathcal{D}(W_-)$ and $W_-\psi = \lambda\psi$. In other words, $\psi \in \mathcal{H}_{-}^{(n)}$ and

$$\langle W_+\phi, \psi \rangle_{\mathcal{H}_{+}^{(n)}, \mathcal{H}_{-}^{(n)}} = \lambda \langle \phi, \psi \rangle_{\mathcal{H}_{+}^{(n)}, \mathcal{H}_{-}^{(n)}} \ \text{ for all } \phi \in \mathcal{D}(W_+). \tag{3.89}$$

Eigenfunctions of W are always generalized eigenfunctions. Conversely, if a generalized eigenfunction is in $\mathcal{H}^{(n)}$, then it is a bona fide eigenfunction.

Lemma 3.9 (EDI) *Let W be a second order classical wave operator, and let ψ be a generalized eigenfunction of W with generalized eigenvalue E. Let $x \in q\mathbb{Z}^d$ and $L \in 2q\mathbb{N}$ be such that $E \notin \sigma(W_{x,L})$. Then* *for any set $\Omega \subset \Lambda_{L-3q}(x)$ we have*

$$\|\chi_\Omega \psi\| \leq \gamma_E \|\Gamma_{x,L} R_{x,L}(E)\chi_\Omega\|_{x,L} \|\Gamma_{x,L}\psi\| \,, \tag{3.90}$$

with γ_E as in (3.80).

Proof: We fix $\nu > \frac{d}{4}$ such that $\psi \in \mathcal{D}(W_-)$ and $W_-\psi = E\psi$. We write $J_{x,L} = J_{x,L}^{\mathbb{R}^d}$.

Using [17, Lemma 4.1], we can show that, weakly in $\mathcal{H}_{x,L}^{(n)}$,

$$\begin{aligned} J_{x,L}^* \chi_\Omega \psi &= \chi_\Omega J_{x,L}^* \phi_{x,L} \psi = \chi_\Omega R_{x,L}(E)(W_{x,L} - E) J_{x,L}^* \phi_{x,L} \psi \qquad (3.91) \\ &= \chi_\Omega R_{x,L}(E) A_{x,L}^* J_{x,L}^* A[\phi_{x,L}]\psi + \chi_\Omega R_{x,L}(E) J_{x,L}^* A^*[\phi_{x,L}] A_- \psi. \end{aligned}$$

Proceeding as in the proof of Lemma 3.8, we have

$$\begin{aligned} &\left\| \chi_\Omega R_{x,L}(E) A_{x,L}^* J_{x,L}^* A[\phi_{x,L}]\psi \right\|_{x,L} \qquad (3.92) \\ &= \left\| \chi_\Omega R_{x,L}(E) A_{x,L}^* \widehat{\Gamma}_{x,L} J_{x,L}^* A[\phi_{x,L}] \Gamma_{x,L}\psi \right\|_{x,L} \\ &\leq \tfrac{3\sqrt{d}}{q} \Xi_A \|\chi_\Omega R_{x,L}(E) A_{x,L}^* \widehat{\Gamma}_{x,L}\|_{x,L} \|J_{x,L}^* \Gamma_{x,L}\psi\|_{x,L} \\ &= \tfrac{3\sqrt{d}}{q} \Xi_A \|\widehat{\Gamma}_{x,L} A_{x,L} R_{x,L}(E)\chi_\Omega\|_{x,L} \|J_{x,L}^* \Gamma_{x,L}\psi\|_{x,L} \\ &\leq \tfrac{3\sqrt{d}}{q} \Xi_A \left(2|E| + \tfrac{100d}{q^2}\Xi_A^2\right)^{\frac{1}{2}} \times \\ &\qquad\qquad \|\Gamma_{x,L} R_{x,L}(E)\chi_\Omega\|_{x,L} \|\Gamma_{x,L}\psi\| \,. \end{aligned}$$

Similarly,

$$\begin{aligned} &\left\| \chi_\Omega R_{x,L}(E) J_{x,L}^* A^*[\phi_{x,L}] A_- \psi \right\|_{x,L} \qquad (3.93) \\ &= \left\| \chi_\Omega R_{x,L}(E) \Gamma_{x,L} J_{x,L}^* A^*[\phi_{x,L}] \widehat{\Gamma}_{x,L} A_- \psi \right\|_{x,L} \\ &\leq \tfrac{3\sqrt{d}}{q} \Xi_A \|\Gamma_{x,L} R_{x,L}(E)\chi_\Omega\|_{x,L} \|\widehat{\Gamma}_{x,L} A_-\psi\| \,, \end{aligned}$$

and, using Lemma 3.4, which is also valid for the operator A_- (see [17, Theorem 4.1], we have

$$\begin{aligned} &\|\widehat{\Gamma}_{x,L} A_-\psi\|^2 \leq \|\rho_{x,L} A_-\psi\|^2 \qquad (3.94) \\ &\leq a\|\Gamma_{x,L} W_-\psi\|^2 + \left(\tfrac{1}{a} + \tfrac{100d}{q^2}\Xi_A^2\right) \|\Gamma_{x,L}\psi\|^2 \\ &= \left(a|E|^2 + \tfrac{1}{a} + \tfrac{100d}{q^2}\Xi_A^2\right) \|\Gamma_{x,L}\psi\|^2 \,, \end{aligned}$$

for any $a > 0$.

Choosing $a = |E|^{-1}$ in (3.94), (3.90) follows from (3.91)-(3.94). ∎

4 Periodic classical wave operators

In this section we study classical wave operators in periodic media. The main theorem gives the spectrum of a periodic classical wave operator in terms of the spectra of its restriction to finite cubes with periodic boundary condition.

Definition 4.1 *A coefficient operator $\mathcal{S}$ on $\mathcal{H}^{(n)}$ is periodic with period $q > 0$ if $\mathcal{S}(x) = \mathcal{S}(x + qj)$ for all $x \in \mathbb{R}^d$ and $j \in \mathbb{Z}^d$.*

Definition 4.2 *A medium is called periodic if the coefficient operators $\mathcal{K}$ and $\mathcal{R}$ that describe the medium are periodic with the same period q. (We will always take the period $q \in \mathbb{N}$ without loss of generality.) The corresponding classical wave operators will be said to be periodic with period q (q-periodic).*

If $k, n \in \mathbb{N}$, we say that $k \preceq n$ if $n \in k\mathbb{N}$ and that $k \prec n$ if $k \preceq n$ and $k \neq n$.

Theorem 4.3 *Let W be a q-periodic second order classical wave operator. Let $\{\ell_n;\ n = 0, 1, 2, \ldots\}$ be a sequence in $\mathbb{N}$ such that $\ell_0 = q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0, 1, 2, \ldots$. Then*

$$\sigma\left(W_{0,\ell_n}\right) \subset \sigma\left(W_{0,\ell_{n+1}}\right) \subset \sigma(W) \ \text{ for all } \ n = 0, 1, 2, \ldots, \tag{4.1}$$

and

$$\sigma(W) = \overline{\bigcup_{n=0}^{\infty} \sigma\left(W_{0,\ell_n}\right)}. \tag{4.2}$$

The analogous result for periodic Schrödinger operators is well known [Ea]. Periodic acoustic and Maxwell operators are treated in [6, Theorem 14] and [7, Theorem 25], respectively. We will sketch a proof, using Floquet theory. We refer to [18, Section XIII.6] for the definitions and notations of direct integrals of Hilbert spaces.

We let $Q = \Lambda_q(0)$ be the basic period cell, $\tilde{Q} = \breve{\Lambda}_{\frac{2\pi}{q}}(0)$ the dual basic cell. ($\breve{\Lambda}_L(x) = \{y \in \mathbb{R}^d;\ x_i - \frac{L}{2} \leq y_i < x_i + \frac{L}{2}, i = 1, \ldots, d\}$; we should also take $Q = \breve{\Lambda}_q(0)$, but we will not since it will make no difference in what follows.) For any $r \in \mathbb{N}$ we define the Floquet transform

$$\mathcal{F}\colon \mathcal{H}^{(r)} \to \int_{\tilde{Q}}^{\oplus} \mathcal{H}_Q^{(r)}\, dk \equiv L^2\left(\tilde{Q}, dk; \mathcal{H}_Q^{(r)}\right) \tag{4.3}$$

by

$$(\mathcal{F}\psi)(k,x) = \left(\frac{q}{2\pi}\right)^{\frac{d}{2}} \sum_{m\in q\mathbb{Z}^d} \mathrm{e}^{ik\cdot(x-m)}\psi(x-m), \quad x \in Q,\ k \in \tilde{Q}, \tag{4.4}$$

if ψ has compact support; it extends by continuity to a unitary operator.

The q-periodic operator W is decomposable in this direct integral representation, more precisely,

$$\mathcal{F}W\mathcal{F}^* = \int_{\tilde{Q}}^{\oplus} W_Q(k)\, dk, \tag{4.5}$$

where for each $k \in \mathbb{R}^d$ we set $\mathbf{D}_Q(k)$ to be the restriction to Q with periodic boundary condition of the operator given by the matrix $D(-i\nabla + k)$ (see (2.17)), a closed, densely defined operator, and let $W_Q(k) = A_Q^*(k)A_Q(k)$ with $A_Q(k) = \sqrt{\mathcal{R}_Q}\mathbf{D}_Q(k)\sqrt{\mathcal{K}_Q}$. (If for $p \in \frac{2\pi}{q}\mathbb{Z}^d$, U_p denotes the unitary operator on $\mathcal{H}_Q^{(r)}$ given by multiplication by the function $\mathrm{e}^{-ip\cdot x}$, then for all $k \in \mathbb{R}^d$ we have $W_Q(k+p) = U_p^* W_Q(k) U_p$.)

Since

$$\|A_Q(k+h) - A_Q(k)\| \le |h|\Xi_A\,, \tag{4.6}$$

follows from the resolvent identity that the map

$$k \in \mathbb{R}^d \mapsto (W_Q(k) + I)^{-1} \in \mathcal{B}\left(\mathcal{H}_Q^{(n)}\right) \tag{4.7}$$

is operator norm continuous, so we conclude from (4.5) that

$$\sigma(W) = \overline{\bigcup_{k\in\tilde{Q}} \sigma\left(W_Q(k)\right)}. \tag{4.8}$$

If $\ell \in q\mathbb{Z}^d$, similar considerations apply to the operator $W_{0,\ell}$, which is q-periodic on the torus $\overline{\Lambda}_\ell(0)$. The Floquet transform

$$\mathcal{F}_\ell \colon \mathcal{H}_{0,\ell}^{(r)} \to \bigoplus_{k\in\frac{2\pi}{\ell}\mathbb{Z}^d\cap\tilde{Q}} \mathcal{H}_Q^{(r)} \tag{4.9}$$

is a unitary operator now defined by

$$(\mathcal{F}_\ell\psi)(k,x) = \left(\frac{q}{\ell}\right)^{\frac{d}{2}} \sum_{m\in q\mathbb{Z}^d\cap\breve{\Lambda}_\ell(0)} \mathrm{e}^{ik\cdot(x-m)}\psi(x-m), \tag{4.10}$$

where $x \in Q,\ k \in \frac{2\pi}{\ell}\mathbb{Z}^d \cap \tilde{Q}, \psi \in \mathcal{H}_{0,\ell}^{(r)},\ \psi(x-m)$ being properly interpreted in the torus $\overline{\Lambda}_\ell(0)$. We also have

$$\mathcal{F}_\ell W_{0,\ell} \mathcal{F}_\ell^* = \bigoplus_{k \in \frac{2\pi}{\ell}\mathbb{Z}^d \cap \tilde{Q}} W_Q(k)\,, \tag{4.11}$$

and

$$\sigma(W_{0,\ell}) = \bigcup_{k \in \frac{2\pi}{\ell}\mathbb{Z}^d \cap \tilde{Q}} \sigma\left(W_Q(k)\right). \tag{4.12}$$

Theorem 4.3 follows from (4.8) and (4.12).

5 Defects and midgap eigenmodes

We now prove the results in Subsection 2.5.

Theorem 5.1 (Stability of essential spectrum) *Let W_0 and W be second order partially elliptic classical wave operators for two media which differ by a defect. Then*

$$\sigma_{ess}(W) = \sigma_{ess}(W_0). \tag{5.1}$$

Proof: We will first prove the theorem when the defect only changes $\mathcal{R}$, i.e., we will show

$$\sigma_{ess}(W_{\mathcal{K}_0,\mathcal{R}}) = \sigma_{ess}(W_{\mathcal{K}_0,\mathcal{R}_0}). \tag{5.2}$$

The general case will follow, using Remark 2.8 and Lemma A.1, as then

$$\begin{aligned} \sigma_{ess}(W_{\mathcal{K}_0,\mathcal{R}_0}) &= \sigma_{ess}(W_{\mathcal{K}_0,\mathcal{R}}) = \sigma_{ess}\left((W_{\mathcal{K}_0,\mathcal{R}})_\perp\right) \\ &= \sigma_{ess}\left((W_{\mathcal{R},\mathcal{K}_0})_\perp\right) = \sigma_{ess}(W_{\mathcal{R},\mathcal{K}_0}) = \sigma_{ess}(W_{\mathcal{R},\mathcal{K}}) \\ &= \sigma_{ess}\left((W_{\mathcal{R},\mathcal{K}})_\perp\right) = \sigma_{ess}\left((W_{\mathcal{K},\mathcal{R}})_\perp\right) = \sigma_{ess}(W_{\mathcal{K},\mathcal{R}})\,. \end{aligned} \tag{5.3}$$

To prove (5.2), we proceed as in [8, Theorem 1]. Let $\mathcal{T}(x) = \mathcal{R}(x) - \mathcal{R}_0(x)$, by our hypotheses it is a bounded, measurable, self-adjoint matrix-valued function with compact support. We write $\mathcal{T}(x) = \mathcal{T}_+(x) - \mathcal{T}_-(x)$, with $\mathcal{T}_\pm(x)$ the positive/negative part of the self-adjoint matrix $\mathcal{T}(x)$. We let $\mathcal{T}_\pm$ denote the bounded operators given by the matrices $\mathcal{T}_\pm(x)$, they would be coefficients operators except for the fact that the functions $\mathcal{T}_\pm(x)$ have compact support, so they are not bounded away from zero. We may still define operators define non-negative self-adjoint operators $W_{\mathcal{K}_0,\mathcal{T}_\pm}$. We have

$$W_{\mathcal{K}_0,\mathcal{R}} = (W_{\mathcal{K}_0,\mathcal{R}_0} + W_{\mathcal{K}_0,\mathcal{T}_+}) - W_{\mathcal{K}_0,\mathcal{T}_-} \;, \tag{5.4}$$

as quadratic forms. (Note that $\mathcal{Q}(W_{\mathcal{K}_0,\mathcal{R}}) = \mathcal{Q}(W_{\mathcal{K}_0,\mathcal{R}_0}) \subset \mathcal{Q}(W_{\mathcal{K}_0,\mathcal{T}_\pm})$, where $\mathcal{Q}(W)$ denotes the form domain of the operator W.) Thus (5.2) follows from [18, Corollary 4 to Theorem XIII.14] and the following lemma.

Lemma 5.2 *Let $W_{\mathcal{K},\mathcal{R}}$ be a second order partially elliptic classical wave operator, and let $\mathcal{T}$ be like a coefficient operator, except for the fact that the function $\mathcal{T}(x)$ has compact support, so it is not bounded away from zero (i.e., $T_- = 0$). Then*

$$\operatorname{tr}\left\{(W_{\mathcal{K},\mathcal{R}} + I)^{-r} W_{\mathcal{K},\mathcal{T}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\} < \infty \tag{5.5}$$

if $r \geq \nu + 1$, where ν is the smallest integer satisfying $\nu > \frac{d}{4}$.

Proof: Let Ω denote the support of $\mathcal{T}(x)$, we pick a function $\rho \in C_0^1(\mathbb{R}^d)$ with $\chi_\Omega \leq \rho(x) \leq \chi_{\widetilde{\Omega}}$, where $\widetilde{\Omega} = \operatorname{supp}\rho$ is a compact set. We have

$$\mathcal{T} \leq T_+\rho^2 \leq T_+ R_-^{-1} \rho^2 \mathcal{R}\,. \tag{5.6}$$

Thus, using $\| \;\|_{HS}$ to denote the Hilbert-Schmidt norm, and setting $c = \|\nabla\rho\|_\infty$, we have

$$\begin{aligned}
&\operatorname{tr}\left\{W_{\mathcal{K},\mathcal{R}} + I)^{-r} W_{\mathcal{K},\mathcal{T}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\} && (5.7)\\
&\quad\leq\; T_+ R_-^{-1} \operatorname{tr}\left\{W_{\mathcal{K},\mathcal{R}} + I)^{-r} A^*_{\mathcal{K},\mathcal{R}} \rho^2 A_{\mathcal{K},\mathcal{R}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\} \\
&\quad=\; T_+ R_-^{-1} \left\|\rho A_{\mathcal{K},\mathcal{R}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\|^2_{HS} && (5.8)\\
&\quad\leq\; T_+ R_-^{-1} \left\{ \left\|\chi_{\widetilde{\Omega}} W_{\mathcal{K},\mathcal{R}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\|^2_{HS} + \right. && (5.9)\\
&\qquad\qquad \left. \left(1 + 4c^2 \Xi^2_{A_{\mathcal{K},\mathcal{R}}}\right) \left\|\chi_{\widetilde{\Omega}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\|^2_{HS} \right\} \\
&\quad\leq\; T_+ R_-^{-1} \left\{ \left(\left\|\chi_{\widetilde{\Omega}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r+1}\right\|_{HS} + \left\|\chi_{\widetilde{\Omega}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\|_{HS} \right)^2 \right. \\
&\qquad\qquad \left. + \left(1 + 4c^2 \Xi^2_{A_{\mathcal{K},\mathcal{R}}}\right) \left\|\chi_{\widetilde{\Omega}} (W_{\mathcal{K},\mathcal{R}} + I)^{-r}\right\|^2_{HS} \right\} < \infty\,, && (5.10)
\end{aligned}$$

where the final bound in (5.10) follows from Theorem 3.1 if $r - 1 \geq \nu$. To go from (5.8) to (5.9) we used Lemma 3.4. ∎

This finishes the proof of Theorem 5.1. ∎

Corollary 5.3 (Behavior of midgap eigenmodes) *Let W_0 and W be second order partially elliptic classical wave operators for two media which differ by a defect. If (a,b) is a gap in the spectrum of W_0, the spectrum of W in (a,b) consists of at most isolated eigenvalues with finite multiplicity, the corresponding eigenmodes decaying exponentially fast from the defect, with a rate depending on the distance from the eigenvalue to the edges of the gap. If the defect is supported by some ball $B_r(x_0)$, and $E \in (a,b)$ is an eigenvalue for W with a corresponding eigenmode ψ, $\|\psi\| = 1$, then*

$$\|\chi_x \psi\|^2 \leq \tag{5.11}$$
$$2C_E\, \Xi_{A_0} \left(E^{\frac{1}{2}} + \mathrm{e}^{m_E} \left(2E + 16\, \Xi_{A_0}^2 \right)^{\frac{1}{2}} \right) \mathrm{e}^{m_E(\frac{\sqrt{d}}{2}+r+2)}\, \mathrm{e}^{-|x-x_0|}$$

for all $x \in \mathbb{R}^d$ such that $|x - x_0| \geq \frac{\sqrt{d}}{2} + r + 3$, where m_E and C_E are as in Theorem 3.5.

Proof: By Theorem 5.1 W has no essential spectrum in (a,b). Thus, if $E \in \sigma(W) \cap (a,b)$, it must be an isolated eigenvalue with finite multiplicity; let ψ be a corresponding eigenvector. To estimate the decay of ψ we have to deal with the fact that the form domains of W and W_0 may be different, and ψ may not be in the form domain of W_0. Thus we pick $\rho \in C^1(\mathbb{R}^d)$ such that

$$1 - \chi_{B_{r+2}(x_0)}(x) \leq \rho(x) \leq 1 - \chi_{B_{r+1}(x_0)}(x)\ , \quad |\nabla\rho(x)| \leq 2\,. \tag{5.12}$$

Since W and W_0 differ by a defect supported by $B_r(x_0)$, it follows from (2.25) that $\mathcal{D}_\rho \equiv \rho\mathcal{D}(A) = \rho\mathcal{D}(A_0)$, and $A\varphi = A_0\varphi$ for $\varphi \in \mathcal{D}_\rho$. Thus, if $\phi \in \mathcal{D}(A_0)$, we have

$$\begin{aligned}
\langle A_0\phi, A_0\rho\psi\rangle &= \langle A_0\phi, A\rho\psi\rangle = \langle A_0\phi, \rho A\psi\rangle + \langle A_0\phi, A[\rho]\psi\rangle \\
&= \langle \rho A_0\phi, A\psi\rangle + \langle A_0\phi, A[\rho]\psi\rangle = \langle \rho A_0\phi, A\psi\rangle + \langle A_0\phi, A[\rho]\psi\rangle \\
&= \langle A_0\rho\phi, A\psi\rangle - \langle A_0[\rho]\phi, A\psi\rangle + \langle A_0\phi, A[\rho]\psi\rangle \\
&= \langle A\rho\phi, A\psi\rangle - \langle A_0[\rho]\phi, A\psi\rangle + \langle A_0\phi, A[\rho]\psi\rangle \\
&= \langle \rho\phi, W\psi\rangle - \langle A_0[\rho]\phi, A\psi\rangle + \langle A_0\phi, A_0[\rho]\psi\rangle \\
&= E\,\langle \rho\phi, \psi\rangle - \langle A_0[\rho]\phi, A\psi\rangle + \langle A_0\phi, A_0[\rho]\psi\rangle\ .
\end{aligned} \tag{5.13}$$

Taking $\phi = (W_0 - E)^{-1}\chi_x\psi$, we get

$$\begin{aligned}
\|\chi_x\psi\|^2 = &- \left\langle A_0[\rho](W_0 - E)^{-1}\chi_x\psi, A\psi \right\rangle \\
&+ \left\langle A_0(W_0 - E)^{-1}\chi_x\psi, A_0[\rho]\psi \right\rangle
\end{aligned} \tag{5.14}$$

$$= -\left\langle A_0[\rho]\chi_{B_{r+2}(x_0)}(W_0-E)^{-1}\chi_x\psi, A\psi\right\rangle$$
$$+\left\langle \chi_{B_{r+2}(x_0)}A_0(W_0-E)^{-1}\chi_x\psi, A_0[\rho]\psi\right\rangle$$
$$\leq 2\,\Xi_{A_0}\left(\sqrt{E}\,\|\chi_{B_{r+2}(x_0)}(W_0-E)^{-1}\chi_x\| + \|\chi_{B_{r+2}(x_0)}A_0(W_0-E)^{-1}\chi_x\|\right)\|\psi\|^2$$

where we used (3.64) and $\|A\psi\|^2 = \langle\psi, W\psi\rangle = E\|\psi\|^2$.

The estimate (5.11) now follows from (5.14), using (3.20) and (3.23) in Theorem 3.5. ∎

The next theorem shows that one can design simple defects which generate eigenvalues in a specified subinterval of a spectral gap of W_0, extending [8, Theorem 2] to the class of classical wave operators. Let Ω be a an open bounded subset of $\mathbb{R}^d$, $x_0 \in \Omega$. Typically, we take Ω to be the cube $\Lambda_1(x_0)$, or the ball $B_1(x_0)$. We set $\Omega_\ell = x_0 + \ell(\Omega - x_0)$ for $\ell > 0$. We insert a defect that changes the value of $\mathcal{K}_0(x)$ and $\mathcal{R}_0(x)$ inside Ω_ℓ to given positive constants K and R. If (a,b) is a gap in the spectrum of W_0, we will show that we can deposit an eigenvalue of W inside any specified closed subinterval of (a,b), by inserting such a defect with $\frac{\ell}{\sqrt{KR}}$ large enough, how large depending only on D_+, the geometry of Ω, and the specified closed subinterval.

Theorem 5.4 (Creation of midgap eigenvalues) *Let (a,b) be a gap in the spectrum of a second order partially elliptic classical wave operator $W_0 = W_{\mathcal{K}_0,\mathcal{R}_0}$, select $\mu \in (a,b)$, and pick $\delta > 0$ such that the interval $[\mu-\delta, \mu+\delta]$ is contained in the gap, i.e., $[\mu-\delta,\mu+\delta] \subset (a,b)$. Given an open bounded set Ω, $x_0 \in \Omega$, $0 < K, R, \ell < \infty$, we introduce a defect that produces coefficient matrices $\mathcal{K}(x)$ and $\mathcal{R}(x)$ that are constant in the set $\Omega_\ell = x_0 + \ell(\Omega - x_0)$, with*

$$\mathcal{K}(x) = KI_n \ \textit{and}\ \mathcal{R}(x) = RI_m \ \textit{for}\ x \in \Omega_\ell\,. \tag{5.15}$$

If

$$\frac{\ell}{\sqrt{KR}} > \frac{\sqrt{\mu}}{\delta} D_+ \inf\left\{\|\nabla\eta\|_2 + \left(\|\nabla\eta\|_2^2 + \frac{\delta}{\mu}\|\Delta\eta\|_2^2\right)^{\frac{1}{2}}\right\}, \tag{5.16}$$

where the infimum is taken over all real valued C^2-functions η on $\mathbb{R}^d$ with support in Ω and $\|\eta\|_2 = 1$, the operator $W = W_{\mathcal{K},\mathcal{R}}$ has at least one eigenvalue in the interval $[\mu-\delta, \mu+\delta]$.

Proof: We proceed as in [8, Theorem 2]. In view of Corollary 5.3, it suffices to show that

$$\sigma(W) \cap [\mu - \delta, \mu + \delta] \neq \emptyset \tag{5.17}$$

if (5.16) is satisfied. To prove (5.17), it suffices to find $\varphi \in \mathcal{D}(W)$ such that

$$\|(W - \mu)\,\varphi\| \leq \delta\,\|\varphi\|\ . \tag{5.18}$$

To do so, we will construct a function $\varphi \in \mathcal{D}(W)$, with $\|\varphi\| = 1$ and support in Ω_ℓ, such that (5.18) holds. In this case the inequality (5.18) takes the following simple form:

$$\|(KR\mathbf{D}^*\mathbf{D} - \mu)\,\varphi\| \leq \delta, \tag{5.19}$$

which is the same as

$$\left\|(\mathbf{D}^*\mathbf{D} - \mu')\,\varphi\right\| \leq \delta', \tag{5.20}$$

with $\mu' = \frac{\mu}{KR}$ and $\delta' = \frac{\delta}{KR}$.

We start by constructing generalized eigenfunctions for the non-negative operator $\mathbf{D}^*\mathbf{D}$ corresponding to μ'. In order to do this, we consider $\kappa \in \mathbb{S}^d$, pick an eigenvalue $\lambda = \lambda_\kappa > 0$ and a corresponding eigenvector $\xi = \xi_{\kappa,\lambda} \in \mathbb{C}^n$, $|\xi| = 1$, of the $n \times n$ matrix $D(\kappa)^* D(\kappa)$ (see (2.17)). We set

$$f(x) = f_{\kappa,\lambda,\xi}(x) = \mathrm{e}^{i\sqrt{\frac{\mu'}{\lambda}}\,\kappa\cdot x}\xi \in C^\infty(\mathbb{R}^n;\mathbb{C}^n)\,. \tag{5.21}$$

Note that, pointwise, we have $|f(x)| = 1$, and

$$(\mathbf{D}^*\mathbf{D}f)(x) = \mu' f(x). \tag{5.22}$$

To produce the desired φ satisfying (5.18), we will restrict f to Ω_ℓ in suitable manner, and prove (5.20). To do so, let η_ℓ be a real valued C^2 function on $\mathbb{R}^d$ with support in Ω_ℓ and $\|\eta_\ell\|_2 = 1$. We set

$$\varphi(x) = \eta_\ell(x) f(x)\ , \quad \text{note} \;\; \|\varphi\| = \|\eta_\ell\|_2 = 1\,. \tag{5.23}$$

We have $\varphi \in \mathcal{D}(\mathbf{D}^*\mathbf{D})$ with support in Ω_ℓ, and

$$\begin{aligned}(\mathbf{D}^*\mathbf{D} - \mu')\varphi& \\ = [D^*(-i\nabla)D(-i\nabla\eta_\ell)]\,f &+ \sqrt{\tfrac{\mu'}{\lambda}}D^*(-i\nabla\eta_\ell)D(\kappa)f \\ &+ \sqrt{\tfrac{\mu'}{\lambda}}D^*(\kappa)D(-i\nabla\eta_\ell)f\,.\end{aligned} \tag{5.24}$$

Thus

$$\|(\mathbf{D}^*\mathbf{D} - \mu')\varphi\| \le D_+^2 \|\Delta\eta_\ell\|_2 + 2\sqrt{\tfrac{\mu'}{\lambda}} D_+^2 \|\nabla\eta_\ell\|_2 \,. \tag{5.25}$$

We now use a scaling argument (i.e., write $\eta_\ell(x) = \eta(\ell^{-1}(x - x_0) + x_0)$) to conclude that to obtain (5.20), it suffices to find $\eta \in C^2(\mathbb{R}^d, \mathbb{R})$ with support in Ω, $\|\eta\|_2 = 1$, and a unit vector $\kappa \in \mathbb{R}^d$, such that

$$\ell^{-2} D_+^2 \|\Delta\eta\|_2 + 2\ell^{-1}\sqrt{\tfrac{\mu'}{\lambda}} D_+^2 \|\nabla\eta\|_2 \le \delta' \,, \tag{5.26}$$

which will be satisfied if

$$\ell^{-2} KRD_+^2 \|\Delta\eta\|_2 + 2\ell^{-1}\sqrt{KR} D_+ \sqrt{\mu} \|\nabla\eta\|_2 \le \delta \,. \tag{5.27}$$

where we used the fact that $\lambda \le D_+^2$. Thus (5.20) holds if (5.16) is satisfied. ∎

A A useful lemma

The following well known lemma (e.g., [4, Lemma 2]) is used throughout this paper. We recall that, given a closed densely defined operator T on a Hilbert space $\mathcal{H}$, we denote its kernel by $\ker T$ and its range by $\operatorname{ran} T$. If T is self-adjoint, it leaves invariant the orthogonal complement of its kernel; the restriction of T to $(\ker T)^\perp$ is denoted by $T_\perp$, a self-adjoint operator on the Hilbert space $(\ker T)^\perp$.

Lemma A.1 *Let B be a closed, densely defined operator from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$. Then the operators $(B^*B)_\perp$ and $(BB^*)_\perp$ are unitarily equivalent. More precisely, the operator U defined by*

$$U\psi = B(B^*B)_\perp^{-\frac{1}{2}}\psi \quad \textit{for} \quad \psi \in \operatorname{ran}(B^*B)_\perp^{\frac{1}{2}} \,, \tag{A.28}$$

extends to a unitary operator from $(\ker B)^\perp$ to $(\ker B^*)^\perp$, and

$$(BB^*)_\perp = U(B^*B)_\perp U^* \,. \tag{A.29}$$

Acknowledgement. The authors thanks Maximilian Seifert for many discussions and suggestions. A. Klein also thanks Alex Figotin, François Germinet, and Svetlana Jitomirskaya for enjoyable discussions.

References

[1] Barbaroux, J.M., Combes, J.M., Hislop, P.D.: Localization near band edges for random Schrödinger operators. *Helv. Phys. Acta* **70**, 16-43 (1997)

[2] Combes, J.M., Hislop, P.D.: Localization for some continuous, random Hamiltonian in d-dimension. *J. Funct. Anal.* **124**, 149-180 (1994)

[3] Combes, J.M., Hislop, P.D., Tip, A.: Band edge localization and the density of states for acoustic and electromagnetic waves in random media. *Ann. Inst. H. Poincare Phys. Theor.* **70** , 381-428 (1999)

[4] Deift, P.A.: Applications of a Commutation Formula. *Duke Math. J.* **45**, 267-310 (1978)

[5] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **124**, 285-299 (1989)

[Ea] Eastham, M.: *The Spectral Theory of Periodic Differential Equations.* Scottish Academic Press, 1973.

[6] Figotin, A., Klein, A.: Localization of classical waves I: Acoustic waves. *Commun. Math. Phys.* **180**, 439-482 (1996)

[7] Figotin, A., Klein, A.: Localization of classical waves II: Electromagnetic waves. *Commun. Math. Phys.* **184**, 411-441 (1997)

[8] Figotin, A., Klein, A.: Localized Classical Waves Created by Defects. *J. Stat. Phys.* **86**, 165-177 (1997)

[9] Figotin, A., Klein, A.: Midgap Defect Modes in Dielectric and Acoustic Media. *SIAM J. Appl. Math.* **58**, 1748-1773 (1998)

[10] Figotin, A., Klein, A.: Localization of Light in Lossless Inhomogeneous Dielectrics. *J. Opt. Soc. Am.* **A 15**, 1423-1435 (1998)

[11] Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **101**, 21-46 (1985)

[12] Fröhlich, J., Spencer, T.: Absence of diffusion with Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys..* **88**, 151-184 (1983)

[13] Germinet, F., Klein, A.: Bootstrap multiscale analysis and localization in random media. Preprint (2000)

[14] Kato, T.: *Perturbation Theory for Linear Operators.* Springer-Verlag, 1976

[15] Klein, A.: Localization of light in randomized periodic media. In *Diffuse Waves in Complex Media*, J.-P. Fouque, ed., pp. 73-92, Kluwer, The Netherlands, 1999

[16] Klein, A., Koines, A. : A general framework for localization of classical waves: II. Random media. In preparation

[17] Klein, A., Koines, A., Seifert, M.: Generalized eigenfunctions for waves in inhomogeneous media. *J. Funct. Anal.*, to appear

[18] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics, Vol.IV, Analysis of Operators.* Academic Press, 1978

[19] Schulenberger, J., Wilcox, C.: Coerciveness inequalities for nonelliptic systems of partial differential equations. *Arch. Rational Mech. Anal.* **88**, 229-305 (1971)

[20] Wilcox, C.: Wave operators and asymptotic solutions of wave propagation problems of classical physics. *Arch. Rational Mech. Anal.* **22**, 37-78 (1966)