Local time-decay of solutions to Schrödinger equations with time-periodic potentials

A. Galtbayar[*] A. Jensen[†‡] and K. Yajima[§]

Dedicated to Professor Elliott Lieb on his seventieth birthday

Abstract

Let $H(t) = -\Delta + V(t,x)$ be a time-dependent Schrödinger operator on $L^2(\mathbf{R}^3)$. We assume that $V(t,x)$ is 2π–periodic in time and decays sufficiently rapidly in space. Let $U(t,0)$ be the associated propagator. For u_0 belonging to the continuous spectral subspace of $L^2(\mathbf{R}^3)$ for the Floquet operator $U(2\pi,0)$, we study the behavior of $U(t,0)u_0$ as $t \to \infty$ in the topology of x-weighted spaces, in the form of asymptotic expansions. Generically the leading term is $t^{-3/2}B_1u_0$. Here B_1 is a finite rank operator mapping functions of x to functions of t and x, periodic in t. If $n \in \mathbf{Z}$ is an eigenvalue, or a threshold resonance of the corresponding Floquet Hamiltonian $-i\partial_t + H(t)$, the leading behavior is $t^{-1/2}B_0u_0$. The point spectral subspace for $U(2\pi,0)$ is finite dimensional. If $U(2\pi,0)\phi_j = e^{-i2\pi\lambda_j}\phi_j$, then $U(t,0)\phi_j$ represents a quasi-periodic solution.

[*]University Street 3, School of Mathematics and Computer Science, National University of Mongolia, P.O.Box 46/145, Ulaanbaatar, Mongolia

[†]Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark. E-mail: matarne@math.auc.dk

[‡]MaPhySto, Centre for Mathematical Physics and Stochastics, funded by the Danish National Research Foundation

[§]Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. E-mail: yajima@ms.u-tokyo.ac.jp

1 Introduction

In this paper we study the large time behavior of solutions of time-dependent Schrödinger equations with potentials $V(t,x)$, which are periodic in time:

$$\begin{aligned} i\partial_t u &= (-\Delta + V(t,x))\, u, \quad (t,x) \in \mathbf{R}\times\mathbf{R}^3, \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.1}$$

We assume that $V(t,x)$ satisfies the following assumption. We write $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ for the unit circle and $\langle x\rangle = (1+x^2)^{1/2}$.

Assumption 1.1. *The function $V(t,x)$ is real-valued and is 2π-periodic with respect to t: $V(t,x) = V(t+2\pi,x)$. For $\beta > 2$ we assume that*

$$\|V\|_\beta \equiv \sum_{j=0}^{2} \sup_{x\in\mathbf{R}^3} \langle x\rangle^\beta \left(\int_0^{2\pi} |\partial_t^j V(t,x)|^2 dt\right)^{\frac12} < \infty. \tag{1.2}$$

We denote by $\mathcal{V}_\beta$ the set of all real-valued functions V on $\mathbf{T}\times\mathbf{R}^3$ which satisfy (1.2). $\mathcal{V}_\beta$ is a Banach space with the norm $\|V\|_\beta$.

Under Assumption 1.1 the operators $H(t)u = -\Delta u + V(t,x)u$ are self-adjoint in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^3)$ with the common domain $H^2(\mathbf{R}^3)$, the Sobolev space of order 2, and the equation (1.1) generates a unique propagator $\{U(t,s)\colon -\infty < t,s < \infty\}$ on $\mathcal{H}$, which satisfies the following properties (see e.g. [23]):

1. $U(t,s)$ is unitary in $\mathcal{H}$, and $(t,s) \mapsto U(t,s)$ is strongly continuous.
2. $U(t,r) = U(t,s)U(s,r)$, and $U(t,t)$ is the identity operator.
3. $U(t+2\pi, s+2\pi) = U(t,s)$ for $t,s \in \mathbf{R}$.
4. $U(t,s)H^2(\mathbf{R}^3) = H^2(\mathbf{R}^3)$. For $u_0 \in H^2(\mathbf{R}^3)$, $U(t,s)u_0$ is an $\mathcal{H}$-valued C^1-function of (t,s), and it satisfies the equations

$$i\partial_t U(t,s)u_0 = H(t)U(t,s)u_0, \quad i\partial_s U(t,s)u_0 = -U(t,s)H(s)u_0.$$

In particular, the solution to (1.1) in $\mathcal{H}$ is given by $u(t) = U(t,0)u_0$.

If V is t-independent and decays sufficiently rapidly in x, it has long been known (see e.g. [6], [16]) that for initial data $u_0(x)$, which decays sufficiently rapidly at infinity, the solution of (1.1) admits an asymptotic expansion

$$u(t,x) = \sum_{\text{finite}} a_j e^{-it\lambda_j}\phi_j(x) + t^{-\frac12}B_0u_0(x) + t^{-\frac32}B_1u_0(x) + \cdots \tag{1.3}$$

as $t \to \infty$, which is valid locally in space. Here ϕ_j are eigenfunctions of $H = -\Delta + V$ with eigenvalues λ_j, and $B_0 \equiv 0$, if 0 is neither an eigenvalue nor a resonance of H, and B_0 may be nonzero otherwise (see Remark 6.6 of [6]). The B_j, $j = 0, 1, \dots$ are finite rank operators. We show in this paper that, in spite of the possibly complex behavior in intermediate time intervals, the solution of (1.1) settles down as $t \to \infty$ to the asymptotic form

$$u(t,x) = \sum_{\text{finite}} a_j e^{-it\lambda_j} \phi_j(t,x) + t^{-\frac{1}{2}} B_0 u_0(t,x) + t^{-\frac{3}{2}} B_1 u_0(t,x) + \cdots , \tag{1.4}$$

as in the autonomous case, where $\phi_j(t,x)$ are now 2π-periodic in t and are eigenfunctions of the Floquet Hamiltonian $K = -i\partial_t - \Delta + V$, defined on the extended phase space

$$\mathcal{K} = L^2(\mathbf{T}, L^2(\mathbf{R}^3)) = L^2(\mathbf{T}) \otimes L^2(\mathbf{R}^3),$$

with eigenvalues $0 \le \lambda_j < 1$, $B_0 \equiv 0$, if 0 is neither an eigenvalue nor a resonance of K in the sense to be defined below, and B_0 may be nonzero otherwise (see Remark 1.9). Here B_j are finite rank operators from the space of functions of x to those of (t,x), 2π-periodic in t.

Recall that for the equation (1.1) the wave operators defined by the limits

$$W_\pm = \lim_{t \to \pm\infty} U(t,0)^{-1} e^{-itH_0}, \quad H_0 = -\Delta,$$

exist and are complete, viz. $\operatorname{Ran} W_\pm = \mathcal{H}_{\rm ac}(U(2\pi,0))$, the absolutely continuous subspace of $\mathcal{H}$ for $U(2\pi,0)$, and that the singular continuous spectrum is absent from $U(2\pi,0)$ (cf. [21], [5], [10]). Hence the solutions of (1.1) can be written as a superposition, with λ_j and $\phi_j(t,x)$ being as in (1.4),

$$u(t,x) = \sum a_j e^{-it\lambda_j} \phi_j(t,x) + u_{\rm scat}(t,x), \tag{1.5}$$

and $u_{\rm scat}(t,x)$ satisfies for some $\psi \in L^2(\mathbf{R}^3)$

$$\|u_{\rm scat}(t,x) - e^{-itH_0}\psi(x)\| \to 0 \tag{1.6}$$

as $t \to \infty$. Thus, our result (1.4) may be considered as a refinement of (1.6). Notice, however, the topologies defining the convergence in (1.4) and (1.6) are very different.

We remark that the expansion of the form (1.3) for autonomous systems is known also for more general equations, including higher order Schrödinger type equations (cf. [16] and references therein). For the hyperbolic equations, the asymptotic behavior of the local energy can be described by resonance poles ([13]), and such results have been extended to the time-periodic systems (cf. [2], [20]). However, to the best knowledge of the authors, an expansion formula like (1.4) has not been known for Schrödinger equations with time-periodic potentials. In particular, the threshold resonances are defined and their role in the large time behavior of the solution is made clear for the first time in this paper.

To formulate the results we introduce some terminology. The weighted L^2 spaces are defined by

$$\mathcal{H}_s \equiv L^2_s(\mathbf{R}^3) \equiv \{f \in L^2_{\mathrm{loc}}(\mathbf{R}^3) \colon \|\langle x\rangle^s f\|_{L^2} < \infty\}.$$

We use the extended phase space approach initiated by Howland ([4]) and implemented for time-periodic systems by the third author ([21], [22]). We define the one parameter family of operators $\{\mathcal{U}(\sigma) \colon \sigma \in \mathbf{R}\}$ on $\mathcal{K}$ by

$$[\mathcal{U}(\sigma)u](t) = U(t, t-\sigma)u(t-\sigma), \quad u = u(t,\cdot) \in \mathcal{K}. \tag{1.7}$$

The properties of $U(t,s)$ stated above imply that $\{\mathcal{U}(\sigma)\}$ is a strongly continuous unitary group on $\mathcal{K}$. We denote its infinitesimal generator by K:

$$\mathcal{U}(\sigma) = e^{-i\sigma K}, \quad \sigma \in \mathbf{R}.$$

K is self-adjoint in $\mathcal{K}$ and is given by

$$K = -i\partial_t - \Delta + V(t,x),$$
$$D(K) = \{u \in \mathcal{K} \colon (-i\partial_t - \Delta + V(t,x))u \in \mathcal{K}\},$$

where derivatives are in the sense of distributions. We call K the Floquet Hamiltonian for (1.1). The following properties are well known ([21], [22]):

1. $e^{-2\pi i K}$ and $I \otimes U(2\pi, 0)$ are unitarily equivalent.

2. Eigenfunctions of K are $\mathcal{H}$-valued continuous. A $\phi(t,x) \in \mathcal{K}$ is an eigenfunction of K with eigenvalue λ, if and only if $\phi(0,x)$ is an eigenfunction of the Floquet operator $U(2\pi,0)$ with eigenvalue $e^{-2\pi i\lambda}$, and

$$U(t,0)\phi(0,x) = e^{-it\lambda}\phi(t,x).$$

3. If E_n is the unitary operator defined by $E_n u(t,x) = e^{int}u(t,x)$, then

$$E_n^* K E_n = K + n, \quad \text{for all } n \in \mathbf{Z}. \tag{1.8}$$

In particular, the spectrum of K is invariant under translations by $n \in \mathbf{Z}$.

We denote by K_0 the corresponding operator for the free Schrödinger equation: $K_0 = -i\partial_t - \Delta$, $D(K_0) = \{u \in \mathcal{K} \colon (-i\partial_t - \Delta)u \in \mathcal{K}\} = D(K)$. For Banach spaces X and Y, we let $B(X,Y)$ denote the Banach space of bounded operators from X to Y. We write $B(X) = B(X,X)$. For s and $\delta \in \mathbf{R}$, we denote the $\mathcal{H}_\delta$-valued Sobolev space of order s over $\mathbf{T}$ by

$$\mathcal{K}_\delta^s = H^s(\mathbf{T}, \mathcal{H}_\delta), \quad \text{and} \quad \mathcal{Y}_\delta^s = B(\mathcal{K}_\delta^s, \mathcal{K}_{-\delta}^s).$$

If $s = 0$ or $\delta = 0$, we omit the corresponding label. We first improve the results on the properties of eigenfunctions of K. For $a \in \mathbf{R}$ we use the notation $(a)_+$ to denote any number strictly larger than a, and $(a)_-$ any number strictly smaller than a. The non-negative (positive) integers are denoted by $\mathbf{N}_0$ ($\mathbf{N}$).

Theorem 1.2. *Let $V \in \mathcal{V}_\beta$ with $\beta > 2$. Then the eigenvalues of K are discrete in $\mathbf{R}$ and are of finite multiplicities. Eigenvalues of $U(2\pi, 0)$ are finite in number and are of finite multiplicities. If $\phi(t,x)$ is an eigenfunction of K with eigenvalue λ, then $H(t)^a \partial_t^b \phi \in \mathcal{K}$ for $0 \le a + b \le 2$, $a, b \in \mathbf{N}_0$. Moreover:*
(1) If $\lambda \notin \mathbf{Z}$, then $\langle x \rangle^N H(t)^a \partial_t^b \phi \in \mathcal{K}$ for any N and $0 \le a+b \le 2$, $a, b \in \mathbf{N}_0$.
(2) If $\lambda = n \in \mathbf{Z}$, then $\langle x \rangle^{(\frac{1}{2})_-} H(t)^a \partial_t^b \phi \in \mathcal{K}$ for $0 \le a + b \le 2$. If we assume $\beta > 5/2$, then there exist constants c_1, c_2, c_3, such that

$$\psi(t,x) - e^{int} \sum_{j=1}^{3} \frac{c_j x_j}{\langle x \rangle^3} \in \mathcal{K}_{(\frac{3}{2})_-}.$$

Remark 1.3. The condition $\beta > 2$ is in general necessary for the point spectral subspace of $U(2\pi, 0)$ to be finite dimensional. If V is t-independent and $V(x) \le -C|x|^{-2}$ for a large $C > 0$, it is well known that $H = -\Delta + V$ has an infinite number of eigenvalues and the point spectral subspace of $U(2\pi, 0) = e^{-2\pi i H}$ is infinite dimensional.

Remark 1.4. It is commonly believed that the eigenvalues are absent for almost all time-periodic potentials $V(t,x)$, which are genuinely t-dependent. However, explicit classes of time-periodic potentials are known, for which K has a finite number of eigenvalues (cf. [15], [3]). In particular, it is easy to construct finite rank operators V, such that K has any finite number of eigenvalues. It is an interesting problem to characterize those potentials, for which K has no eigenvalues. It is actually known that the eigenfunctions corresponding to non-integral eigenvalues decrease exponentially as $|x| \to \infty$, see [24]. The proof below shows that the eigenfunctions ϕ satisfy $H(t)^a \partial_t^b \phi \in \mathcal{K}$ for $0 \le a + b \le m$ if $\sup \langle x \rangle^\beta \| \partial_t^j V(\cdot, x) \|_{L^2(\mathbf{T})} < \infty$ for $0 \le j \le m$.

Remark 1.5. If $V(t,x) = V_0(x) + \mu W(x)\cos t$ is a perturbation of a stationary potential $V_0(x)$, then, generically, for sufficiently small $\mu > 0$, any eigenvalue $\lambda < 0$ of $H = -\Delta + V_0$ will turn into a resonance Γ with $\operatorname{Im}\Gamma = C\mu^{2n} + O(\mu^{2n+1})$, $C < 0$, where n is the smallest integer such that $\lambda + n > 0$, and the solution $u(t,x)$ of (1.1) with $u(0,x) = \phi(x)$, ϕ being the corresponding eigenfunction of H, satisfies $(u(t,x),\phi) = e^{-it\Gamma} + O(\mu)$ uniformly in t as $\mu \to 0$ (cf. [22], see also [14], [19] and [9] for more recent works). Again, it is an interesting question to ask how the survival time $-\frac{1}{2\operatorname{Im}\Gamma}$ behaves, when μ is not small (see [3] and the references therein). These, however, are not the issues addressed in this paper.

Definition 1.6. (1) $n \in \mathbf{Z}$ *is said to be a threshold resonance of* K*, if there exists a solution* $u(t,x)$ *of*

$$-i\partial_t u - \Delta u + V(t,x)u = nu(t,x) \tag{1.9}$$

such that, with a constant $C \neq 0$*,*

$$u(t,x) = \frac{Ce^{int}}{|x|} + u_1(t,x), \quad u_1 \in \mathcal{K}. \tag{1.10}$$

Such a solution is called an n*-resonant solution.*
(2) *We say that* $V(t,x)$ *is of generic type, if* 0 *is neither an eigenvalue nor a threshold resonance of* K*. Otherwise, it is said to be of exceptional type.*

Remark 1.7. (1) Because of the identity (1.8), $n \in \mathbf{Z}$ ($\lambda + n \in \mathbf{R}$) is a threshold resonance (or an eigenvalue) of K, if and only if 0 (respectively λ) is a threshold resonance (respectively an eigenvalue) of K.
(2) The resolvent $R_0(z) = (K_0 - z)^{-1}$, considered as a $\mathcal{Y}_\delta$-valued function of $z \in \mathbf{C}^\pm$ (the upper or lower complex half plane), $\delta = \beta/2$, has continuous boundary values $R_0^\pm(\lambda) = \lim_{\varepsilon\downarrow 0} R_0(\lambda \pm i\varepsilon)$, and V is of generic type, if and only if $1 + R_0^\pm(n)V$ is invertible in $B(\mathcal{K}_{-\delta})$ for some (hence for all) $n \in \mathbf{Z}$ (see Section 2). Since $R_0^\pm(n)V$ is compact in $\mathcal{K}_{-\delta}$ and depends continuously on $V \in \mathcal{V}_\beta$, it follows that the set of generic potentials V is open and dense in $\mathcal{V}_\beta$.
(3) We do not know any explicit, genuinely time-dependent, and multiplicative example of $V(t,x)$, which is of exceptional type. For time-independent V examples are easily constructed: If $V \leq 0$, λV is of exceptional type if $1 \in \sigma(\lambda|V|^{\frac{1}{2}}(-\Delta)^{-1}|V|^{\frac{1}{2}})$, and such λ always exists, if $V \neq 0$ (cf. [6]). Here and hereafter $\sigma(T)$ denotes the spectrum of the operator T.

Now we can state the main result of the paper.

Theorem 1.8. *Let $V \in \mathcal{V}_\beta$ for $\beta > \beta_k \equiv \max\{2k+1, 4\}$, $k \in \mathbf{N}$, and let $\{\phi_j\}$ be an orthonormal basis of eigenfunctions of K corresponding to the eigenvalues $0 \le \lambda_j < 1$. Set $\delta = \beta/2$ and $\varepsilon_0 = \min\{1, \frac{\beta-\beta_k}{2}\}$. We have the following results.*
(1) *Suppose V is of generic type. Then there exist finite rank operators $B_1, \dots, B_k$ from $\mathcal{H}_\delta$ to $\mathcal{K}^1_{-\delta}$, such that $B_j = 0$, unless j is odd, and such that, for any $u_0 \in \mathcal{H}_\delta$ and for any ε, $0 < \varepsilon < \varepsilon_0$, as $t \to \infty$,*

$$U(t,0)u_0 = \sum_j c_j e^{-it\lambda_j}\phi_j(t,x) + t^{-\frac{3}{2}}B_1 u_0(t,x) + \cdots \\ \cdots + t^{-\frac{k}{2}-1}B_k u_0(t,x) + O(t^{-\frac{k+\varepsilon}{2}-1}), \quad (1.11)$$

where $c_j = 2\pi(\phi_j(0), u_0)_{\mathcal{H}}$, and $O(t^{-\frac{k+\varepsilon}{2}-1})$ stands for an $\mathcal{H}_{-\delta}$-valued function of t such that its norm in $\mathcal{H}_{-\delta}$ is bounded by $C\, t^{-\frac{k+\varepsilon}{2}-1}\|u_0\|_{\mathcal{H}_\delta}$, when $t \ge 1$.
(2) *Suppose V is of exceptional type, $\beta > \beta_k$, $k \ge 2$, and $\{\phi_{0\ell}\} \subset \{\phi_j\}$ is an orthonormal basis of eigenfunctions of K with eigenvalue 0. Then, there exist a 0-resonant solution $\psi(t,x)$, finite rank operators $B_1, \dots, B_{k-2}$ from $\mathcal{H}_\delta$ to $\mathcal{K}^1_{-\delta}$, such that $B_j = 0$, unless j is odd, and such that, for any $u_0 \in \mathcal{H}_\delta$ and for any $0 < \varepsilon < \varepsilon_0$ as $t \to \infty$,*

$$U(t,0)u_0 = \sum_j c_j e^{-it\lambda_j}\phi_j(t,x) + t^{-\frac{1}{2}}\Big(d_0\psi(t,x) + \sum_\ell d_\ell \phi_{0\ell}(t,x)\Big) \\ + t^{-\frac{3}{2}}B_1 u_0(t,x) + \cdots + t^{-\frac{k-2}{2}-1}B_{k-2}u_0(t,x) + O(t^{-\frac{k-2+\varepsilon}{2}-1}), \quad (1.12)$$

where c_j and $O(t^{-\frac{k-2+\varepsilon}{2}-1})$ are as in (1), $d_0 = 2\pi(u_0, \psi(0))_{\mathcal{H}}$, and d_ℓ are linearly independent functionals of u_0.

Remark 1.9. (1) In the statement of Theorem 1.8(2) the terms involving the resonant function, or the eigenfunctions, are to be omitted, in case n is not a threshold resonance, or not an eigenvalue. As in the autonomous case (see Remark 6.6 of [6]), we expect the linear functionals $\{d_\ell\}$ in (1.12) may be linearly independent or dependent dependening on V, however, we do not know any explicit example here (see (3) of Remark 1.7).
(2) The 2π appears in the definition of c_j because of the normalization of eigenfunctions: $\{\sqrt{2\pi}\phi_j(0,x)\}$ is the orthonormal basis of eigenfunctions of $U(2\pi, 0)$, if $\{\phi_j(t,x)\}$ is the one for K.
(3) We shall explain how the operators B_j in (1.11) (resp. (1.12)) and $F_j(0)$ in (1.16) (resp. (1.20)) below are related at the end of Introduction. In particular, B_1 in (1.11) is a rank one operator.

The rest of the paper is devoted to the proof of Theorem 1.8. We display the plan of the paper, explaining the main idea of the proof, when non-integral eigenvalues are absent, as the latter contribute to (1.11) or (1.12) only by eigenfunctions and by the remainder terms, and as they can be easily accommodated by a similar (but simpler) method for treating the threshold eigenvalues or threshold resonances. We write $J\colon \mathcal{H} \to \mathcal{K}$ for the identification operator $(Ju_0)(t,x) = u_0(x)$. We shall prove the theorem by studying the unitary group $e^{-i\sigma K}$ via the Fourier transform:

$$e^{-i\sigma K} Ju_0(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int e^{-i\sigma\lambda} R(\lambda + i\varepsilon) Ju_0 \, d\lambda, \tag{1.13}$$

$R(z) = (K - z)^{-1}$ being the resolvent of K. This requires a detailed study of $R(z)$ near the reals. In Section 2 we begin with the study of $R_0(z) = (K_0 - z)^{-1}$ and show that

1. $R_0(z)$ has a C^k-extension to $\overline{\mathbf{C}}^{\pm} \setminus \mathbf{Z}$ as a $\mathcal{Y}^s_\gamma$-valued function, $s \in \mathbf{N}_0$, if $k \geq 0$ and $\gamma > k + \frac{1}{2}$.

2. $R_0(z+n)$ has an asymptotic expansion in powers of $\sqrt{z}$ as $z \to 0$. We denote the boundary values on the reals by $R_0^{\pm}(\lambda) = R_0(\lambda \pm i0)$.

3. $\lambda \notin \mathbf{Z}$ is an eigenvalue of K, if and only if $-1 \in \sigma(R_0^{\pm}(\lambda)V)$, and $n \in \mathbf{Z}$ is an eigenvalue or resonance of K if and only if $-1 \in \sigma(R_0^{\pm}(n)V)$.

We then prove most of Theorem 1.2 in that section. We also show in Section 2 how the n-mode of $R(z)Ju_0$, viz. the n-th Fourier component of $R(z)Ju_0$ with respect to t, decays as $n \to \pm\infty$.

In Section 3 we study the behavior of $R(z)$ near and on the real line. The properties 1. to 3. above and Theorem 1.2 imply that $R(z)$ has boundary values $R^{\pm}(\lambda) = R(\lambda \pm i0)$ away from $\mathbf{Z} \cup \{\text{eigenvalues of } K\}$, and they are C^k functions with values in $\mathcal{Y}^1_\delta$. In Subsection 3.1 we study $R(z)$ near $\mathbf{Z}$ for generic V. In this case $G(z) = (1 + R_0(z)V)^{-1}$ exists for z near $\mathbf{Z}$, and we obtain the following theorem by a straightforward perturbation argument. We write $L^{(j)}(z)$ for the j-th derivative of $L(z)$, and $f \otimes g$ stands for the integral operator on $\mathbf{T} \times \mathbf{R}$ with the kernel $f(t,x)\overline{g(s,y)}$.

Definition 1.10. *Let X and Y be Banach spaces, and let $L(z)$ a $B(X,Y)$-valued function defined in $U = \{z \in \overline{\mathbf{C}^+}\colon 0 < |z| < \rho\}$, a punched neighborhood of the origin in $\overline{\mathbf{C}^+}$. Let $k \in \mathbf{N}_0$ and $0 \leq \varepsilon < 1$. We say $L(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$, if $L(z)$ satisfies the following properties:*
(i) $L(z) \in C^k(U)$ *and it satisfies*

$$\|(d/dz)^j L(z)\| \leq C_j |z|^{\frac{k+\varepsilon}{2} - j}, \ j = 0, 1, \ldots, k, \ z \in U. \tag{1.14}$$

(*Hence, $L(z)$ is $C^{[k/2]}(U \cup \{0\})$, if we set $L(0) = 0$.*)
(ii) *For $\ell = [(k+2)/2]$, there exist $\mu > 0$ and $\gamma > 0$ such that, for $0 < h < \gamma$,*

$$\int_{-\mu}^{\mu} \|L^{(\ell)}(z+h) - L^{(\ell)}(z)\| dz \leq \begin{cases} C|h|^{\frac{\varepsilon}{2}}, & \text{if } k \text{ is even,} \\ C|h|^{\frac{1+\varepsilon}{2}}, & \text{if } k \text{ is odd.} \end{cases} \tag{1.15}$$

For fixed (ρ, μ, γ), we write $\|L(z)\|_{\mathcal{O}((k+\varepsilon)/2)}$ for the sum of the smallest numbers $C_0, \dots, C_k$ and C, such that (1.14) and (1.15) are satisfied.

Theorem 1.11. *Let $V \in \mathcal{V}_\beta$ for $\beta > \beta_k \equiv \max\{2k+1, 4\}$, $k \in \mathbf{N}$. Let $\delta = \beta/2$ and $\varepsilon_0 = \min\{1, \frac{\beta-\beta_k}{2}\}$. Suppose that V is of generic type. Then, as a $\mathcal{Y}^s_\delta$-valued function of $z \in \overline{\mathbf{C}^+}$, $s = 0, 1$, for any $0 < \varepsilon < \varepsilon_0$, we have*

$$R(z+n) = F_0(n) + \sqrt{z} F_1(n) + z F_2(n) + \cdots + z^{k/2} F_k(n) + \mathcal{O}(z^{(k+\varepsilon)/2}) \tag{1.16}$$

in a neighborhood of $z = 0$. Here
(1) *$F_j(n) = E_n F_j(0) E_n^*$ for all $n \in \mathbf{Z}$ and $j = 0, 1, \dots$.*
(2) *If j is odd, $F_j(0)$ are operators of finite rank and may be written as a finite sum $\sum a_{j\nu} \otimes b_{j\nu}$, where $a_{j\nu}, b_{j\nu} \in \mathcal{K}^1_{-\delta}$.*
(3) *The first few terms are given as*

$$F_0(n) = G^+(n) R_0^+(n) (= R^+(n)), \tag{1.17}$$
$$F_1(n) = G^+(n) D_1(n) G^-(n)^*, \tag{1.18}$$
$$F_2(n) = G^+(n) \left[D_2(n) - D_1(n) V G^+(n) D_1(n) \right] G^-(n)^*, \tag{1.19}$$

where $G^\pm(n) = (1 + R_0^\pm(n) V)^{-1}$, and where $D_j(n)$ are the operators defined in statement (3) of Lemma 2.3.

In Subsection 3.2 we study the same problem in the case that V is of exceptional type. In this case, $-1 \in \sigma(R_0^\pm(n)V)$, and the analysis of $R(z)$ near thresholds is substantially more involved. We apply here the method developed by Murata ([16]) and prove the following theorem. We shall repeat some of the arguments of Murata here for the convenience of the readers. Note that we also could have used the approach introduced in [8]. For Borel sets I we write $E_K(I)$ for the spectral measure of K.

Theorem 1.12. *Let Assumption 1.1 be satisfied with $\beta > \beta_k \equiv 2k+1$, $k \geq 2$ an integer. Let $\delta = \beta/2$, and $0 < \varepsilon < \varepsilon_0 = \min\{1, \frac{\beta-\beta_k}{2}\}$. Suppose that V is of exceptional type. Then, as a $\mathcal{Y}^s_\delta$-valued function of $z \in \overline{\mathbf{C}^+}$, $s = 0, 1$,*

$$R(z+n) = -\frac{1}{z} F_{-2}(n) + \frac{1}{\sqrt{z}} F_{-1}(n) + F_0(n) + \cdots \cdots + z^{(k-2)/2} F_{k-2}(n) + \mathcal{O}(z^{(k-2+\varepsilon)/2}) \tag{1.20}$$

in a neighborhood of $z = 0$. *Here*
(1) $F_j(n) = E_n F_j(0) E_n^*$ *for* $n \in \mathbf{Z}$ *and* $j = -2, -1, \dots$.
(2) $F_j(n)$ *is of finite rank, when* j *is odd, and may be written as a finite sum* $\sum a_{j\nu} \otimes b_{j\nu}$, *where* $a_{j\nu}, b_{j\nu} \in \mathcal{K}^1_{-\delta}$.
(3) $F_{-2}(n) = E_K(\{n\})$.
(4) $F_{-1}(n) = E_K(\{n\}) V D_3(n) V E_K(\{n\}) - 4\pi i \bar{Q}_n$, *where* $\bar{Q}_n = \langle \cdot, \psi^{(n)} \rangle \psi^{(n)}$, *and* $\psi^{(n)}$ *is a suitably normalized n-resonant function.*

Remark 1.13. In the statement of Theorem 1.12 the terms involving the resonant function, or the eigenfunctions, are to be omitted, in case n is not a threshold resonance, or not an eigenvalue.

In Section 4, we apply (1.16) or (1.20) to the expression (1.13) for $e^{-i\sigma K} J$. Using also the properties that $\| \frac{d}{d\lambda} R_0^+(\lambda) V R_0^+(\lambda) J u_0 \|_{K^1_{-\delta}} = O(|\lambda|^{-3/2})$ and $\| R_0^+(\lambda) V R_0^+(\lambda) V R_0^+(\lambda) u_0 \|_{K^1_{-\delta}} = O(|\lambda|^{-3/2})$ as $|\lambda| \to \infty$ (see Lemma 2.5), which, in physics terminology, represents the fact that the energy spreads slowly in the resolvent, and which guarantees that the contributions to the integral of thresholds singularities at $n \in \mathbf{Z}$ are summable, we then obtain the asymptotic expansion of $e^{-i\sigma K} J$ as $\sigma \to \infty$. When V is of generic type, the result is

$$e^{-i\sigma K} J = \sigma^{-3/2} Z_1(\sigma) + \cdots + \sigma^{-(k+2)/2} Z_k(\sigma) + O(\sigma^{-(k+2+\varepsilon)/2}) \tag{1.21}$$

as a $B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})$-valued function. Here $Z_j(\sigma)$ is 2π-periodic in σ, $Z_j(\sigma) = 0$ if j is even, and if j is odd, $Z_j(\sigma)$ has the form $Z_j(\sigma) = C_j \sum_n e^{-in\sigma} F_j(n) J$ where C_j is the universal constant in (4.3). Because $F(n) = E_n F_j(0) E_n^*$ and $F_j(0) = \sum_\nu a_{j\nu} \otimes b_{j\nu}$, $a_{j\nu}, b_{j\nu} \in \mathcal{K}^1_{-\delta}$, by Theorem 1.11(2), the Fourier inversion formula implies

$$\begin{aligned} Z_j(\sigma) u_0(t,x) &= C_j \sum_\nu \sum_n e^{-in(\sigma - t)} a_{j\nu}(t,x) \int_{\mathbf{T} \times \mathbf{R}^3} b_{j\nu}(s,y) e^{-ins} u_0(y) ds dy \\ &= 2\pi C_j \sum_\nu a_{j\nu}(t,x) \int_{\mathbf{R}^3} b_{j\nu}(t - \sigma, y) u_0(y) dy. \end{aligned} \tag{1.22}$$

Since $\mathcal{K}^1_{-\delta}$ is continuously embedded in $C(\mathbf{T}, \mathcal{H}_{-\delta})$ by the Sobolev embedding theorem, (1.21) implies that, uniformly with respect to $t \in \mathbf{T}$ (hence with respect to $t \in \mathbf{R}$ by the periodicity), as $\sigma \to \infty$,

$$\begin{aligned} &\| U(t, t - \sigma) u_0 - \sigma^{-3/2} Z_1(\sigma) u_0(t) - \cdots \\ &\quad \cdots - \sigma^{-(k+2)/2} Z_k(\sigma) u_0(t) \|_{\mathcal{H}_{-\delta}} = O(\sigma^{-(k+2+\varepsilon)/2}) \| u_0 \|_{\mathcal{H}_\delta}. \end{aligned} \tag{1.23}$$

We set $t = \sigma$ in (1.23) and replace σ by t. We then obtain (1.11) with

$$B_j(t) = 2\pi C_j \sum_{\nu} a_{j\nu}(t, x) \otimes b_\nu(0, y). \tag{1.24}$$

Though the procedure will be a little more involved, as will be shown in Section 4, to settle the convergence problem at various stages, this basically proves Theorem 1.8 for generic V. The proof of Theorem 1.8 for the exceptional case can be carried out along the same lines, by applying (1.20) instead of (1.16).

In what follows the adjoints of various bounded operators between function spaces over $\mathbf{T} \times \mathbf{R}^3$ are taken with respect to the coupling

$$\langle f, g \rangle = \int_{\mathbf{T}\times\mathbf{R}^3} f(t, x)\overline{g(t, x)} dt dx.$$

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2 Limiting absorption principle

In this and next sections we study the resolvent $R(z)$, $z \in \mathbf{C}^\pm$. In this section, we begin with studying $R_0(z)$ near the boundary of $\mathbf{C}^\pm$ and, then, identify those points $\lambda \in \mathbf{R}$, where the boundary values $R^\pm(\lambda) \equiv \lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)$ do not exist, with the eigenvalues, or the threshold resonances, of K. We note that the limiting absorption principle (away from thresholds) has been proved previously in greater generality, see for example [12], [25], and the references therein.

We denote by $r_0(z) = (-\Delta - z)^{-1}$ the resolvent of the free Schrödinger operator $-\Delta$ in $L^2(\mathbf{R}^3)$, by p_n, $n \in \mathbf{Z}$, the projection in $L^2(\mathbf{T})$ onto the one dimensional subspace spanned by e^{int}, and by $P_n = p_n \otimes I$ the corresponding operator in $\mathcal{K} = L^2(\mathbf{T}) \otimes \mathcal{H}$. For $\gamma \in \mathbf{R}$, we write $\mathcal{X}_\gamma = B(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$. For the function $\sqrt{z}$, we always choose the branch such that $\operatorname{Im}\sqrt{z} \geq 0$.

2.1 The free resolvent

We write $cl(\mathbf{C})$ for the closure of $\mathbf{C} \setminus [0, \infty)$ in the Riemann surface of $\sqrt{z}$. The following is well known (cf. [1], [11], [6], [7]).

Lemma 2.1. *Consider $r_0(z)$ as an $\mathcal{X}_\gamma$-valued analytic function $\mathbf{C} \setminus [0, \infty) \ni z \to r_0(z)$, where $\gamma > k + 1/2$, $k = 0, 1, \ldots$. Then*

(1) *$r_0(z)$ has an extension to $cl(\mathbf{C})\setminus\{0\}$ as an $\mathcal{X}_\gamma$-valued $C^{\gamma-(1/2)+}$-function.*
(2) *When $\gamma > 1$, it can be extended to $cl(\mathbf{C})$ as an $\mathcal{X}_\gamma$-valued continuous function. We write $r_0^\pm(\lambda) = \lim_{\varepsilon\downarrow 0} r_0(\lambda \pm i\varepsilon)$, $\lambda \in [0,\infty)$.*
(3) *$r_0(z)\colon \mathcal{H}_\gamma \to \mathcal{H}_{-\gamma}$ is compact for any $z \in cl(\mathbf{C})$.*
(4) *For $j = 0,\dots,k$, there exist constants C_j such that*

$$\|(d/dz)^j r_0(z)\|_{\mathcal{X}_\gamma} \le C_j \langle z\rangle^{-(j+1)/2}, \quad |z| \ge 1. \tag{2.1}$$

The following is a special case of Lemma 2.2 and Lemma 2.5 of [16] where more general operators are studied. We provide an elementary proof for the convenience of readers. We use the notation $\mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ of Definition 1.10 for functions defined in $\overline{\mathbf{C}^\pm}\setminus\{0\}$. This slight abuse of notation should not cause any confusion. We let $\beta_k = \max\{2k+1, 4\}$ as above.

Lemma 2.2. *Let $\gamma > \beta_k/2$ for a $k \in \mathbf{N}$. Then:*
(1) *As an $\mathcal{X}_\gamma$-valued function on $\{z \in \overline{\mathbf{C}^\pm}\colon 0 < |z| < 1\}$, $r_0(z)$ satisfies*

$$r_0(z) = g_0 + \sqrt{z} g_1 + \cdots + z^{k/2} g_k + d_k(z), \quad d_k(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}}), \tag{2.2}$$

for any $0 \le \varepsilon < \varepsilon_0 \equiv \min\{1, \gamma - \frac{\beta_k}{2}\}$. Here g_j are the integral operators

$$g_j u(x) = \frac{i}{4\pi j!}\int (i|x-y|)^{j-1} u(y)dy, \quad j = 0,1,\dots,k, \tag{2.3}$$

and g_j are of finite rank, when j is odd.
(2) *Suppose $k \ge 2$. Let $\mathcal{H}_\gamma^\circ$ be the closed subspace of $\mathcal{H}_\gamma$ given by $\mathcal{H}_\gamma^\circ = \{u \in \mathcal{H}_\gamma\colon \int u dx = 0\}$. Then, $g_j \in B(\mathcal{H}_\gamma^\circ, \mathcal{H}_{-\gamma+1})$ for $j = 0,\dots,k$, and $r_0(z)$ satisfies (2.2) as a $B(\mathcal{H}_\gamma^\circ, \mathcal{H}_{-\gamma+1})$ valued function.*

Proof. (1) The integral kernel of $r_0(z)$ admits an expansion

$$\frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} = \sum_{j=0}^{k} \frac{1}{4\pi j!}(i\sqrt{z})^j |x-y|^{j-1} + d_k(z;x,y)$$

with the remainder given by

$$d_k(z;x,y) = \frac{(i\sqrt{z})^k |x-y|^{k-1}}{4\pi(k-1)!}\int_0^1 (1-s)^{k-1}(e^{is\sqrt{z}|x-y|}-1)ds. \tag{2.4}$$

If j is odd, $|x-y|^{j-1}$ is a sum of monomials $x^\alpha y^\beta$, $|\alpha|+|\beta| = j-1$, and g_j is of finite rank. We show that the integral operator $d_k(z)$ with the kernel $d_k(z;x,y)$ satisfies $d_k(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ as an $\mathcal{X}_\gamma$-valued function. Using

$$\frac{\partial}{\partial z}(e^{is\sqrt{z}|x-y|}-1) = \frac{s}{2z}\frac{\partial}{\partial s}\left(e^{is\sqrt{z}|x-y|}-1\right)$$

and applying integrating by parts, we have

$$\left(\frac{d}{dz}\right)^j \left\{\sqrt{z}^k \int_0^1 (1-s)^{k-1}(e^{is\sqrt{z}|x-y|}-1)ds\right\}$$
$$= \begin{cases} z^{\frac{k}{2}-j}\int_0^1 p_{jk}(s)(e^{is\sqrt{z}|x-y|}-1)ds, & j<k, \\ z^{-\frac{k}{2}}\left\{c_k(e^{i\sqrt{z}|x-y|}-1)+\int_0^1 p_{kk}(s)(e^{is\sqrt{z}|x-y|}-1)ds\right\}, & j=k, \end{cases} \tag{2.5}$$

where p_{jk}, $j=0,\dots,k$, are polynomials in s, and c_k are constants. Using the obvious estimate $|e^{i\sqrt{z}|x-y|}-1|\le C_\varepsilon |z|^{\frac{\varepsilon}{2}}|x-y|^\varepsilon$, we then obtain

$$\left|\left(\frac{d}{dz}\right)^j d_k(z;x,y)\right| \le C_{jk}|x-y|^{k-1+\varepsilon}|z|^{\frac{k+\varepsilon}{2}-j}, \quad j=0,1,\dots,k,$$

for any $0\le\varepsilon\le 1$ and $|z|\le 1$. Thus $d_k(z)$ satisfies (1.14) as an $\mathcal{X}_\gamma$-valued function. We next prove (1.15) for $d_k(z)$. If k is even, $\ell=(k+2)/2$ and

$$d_k^{(\ell)}(z,x,y)=\frac{|x-y|^{k-1}}{z}\left\{c(e^{i\sqrt{z}|x-y|}-1)+\int_0^1 p(s)(e^{is\sqrt{z}|x-y|}-1)ds\right\},$$

by virtue of (2.5), where the constant c vanishes unless $k=2$, and $p(s)$ is a polynomial. Since $|e^a-e^b|\le C_\varepsilon|a-b|^\varepsilon$ for any $0\le\varepsilon\le 1$ if $\mathrm{Re}\,a,\mathrm{Re}\,b\ge 0$, we have, uniformly with respect to $0\le s\le 1$, that

$$\begin{aligned}
&\left|\frac{1}{z+h}(e^{is\sqrt{z+h}|x-y|}-1)-\frac{1}{z}(e^{is\sqrt{z}|x-y|}-1)\right| \\
&\qquad\le \left|\frac{h}{z(z+h)}(e^{is\sqrt{z}|x-y|}-1)\right| + \left|\frac{1}{z+h}(e^{is\sqrt{z+h}|x-y|}-e^{is\sqrt{z}|x-y|})\right| \\
&\qquad\le \left(\frac{h|x-y|^\varepsilon}{|z^{1-\frac{\varepsilon}{2}}(z+h)|}+\frac{h^\varepsilon|x-y|^\varepsilon}{|(z+h)(\sqrt{z+h}+\sqrt{z})^\varepsilon|}\right)\equiv |x-y|^\varepsilon a_1^\varepsilon(z,h),
\end{aligned}$$

and, by interchanging the roles of z and $z+h$, that

$$\begin{aligned}
&\left|\frac{1}{z+h}(e^{is\sqrt{z+h}|x-y|}-1)-\frac{1}{z}(e^{is\sqrt{z}|x-y|}-1)\right| \\
&\qquad\qquad\le \left(\frac{h|x-y|^\varepsilon}{|z(z+h)^{1-\frac{\varepsilon}{2}}|}+\frac{h^\varepsilon|x-y|^\varepsilon}{|z(\sqrt{z+h}+\sqrt{z})^\varepsilon|}\right)\equiv |x-y|^\varepsilon a_2^\varepsilon(z,h).
\end{aligned}$$

It follows that for $0<\varepsilon<\varepsilon_0$

$$\|d_k^{(\ell)}(z+h)-d_k^{(\ell)}(z)\|_{\mathcal{X}_\gamma}\le \begin{cases} Ca_1^\varepsilon(z,h), & |z+h|\ge h/2, \\ Ca_2^\varepsilon(z,h), & |z+h|<h/2. \end{cases} \tag{2.6}$$

The change of variable $z \to zh$ instantly implies that

$$\int_{|z+h|\geq h/2} a_1^{\varepsilon}(z,h)dz + \int_{|z+h|<h/2} a_2^{\varepsilon}(z,h)dz = C_{\varepsilon} h^{\frac{\varepsilon}{2}}.$$

Thus, $d_k(z)$ satisfies (1.15) as an $\mathcal{X}_\gamma$-valued function when k is even. When k is odd, $d_k^{(\ell)}(z)$ has the integral kernel

$$\frac{|x-y|^{k-1}}{\sqrt{z}} \int_0^1 p(s)(e^{is\sqrt{z}|x-y|} - 1)ds,$$

and we proceed entirely similarly as above. We omit the details (see the proof of Lemma 2.4 for a similar argument).

For proving (2), we first note that $g_j \in B(\mathcal{H}_\gamma^\circ, \mathcal{H}_{-\gamma+1})$, $j = 0, \ldots, k$. This follows from the expression for $u \in \mathcal{H}_\gamma^\circ$

$$g_j u(x) = c_j \int (|x-y|^{j-1} - |x|^{j-1}) u(y) dy$$

and the obvious inequality $||x-y|^{j-1} - |x|^{j-1}| \leq C_j \langle x\rangle^{j-2} \langle y\rangle^{j-1}$, which imply $|g_j u(x)| \leq C\langle x\rangle^{j-2} \|u\|_{\mathcal{H}_\gamma}$ and, hence, $\|g_j u\|_{\mathcal{H}_{-\gamma+1}} \leq C\|u\|_{\mathcal{H}_\gamma}$. For completing the proof of (2), it then suffices to show that $r_0(z)$, considered as a $B(\mathcal{H}_\gamma^\circ, \mathcal{H}_{-\gamma+1})$-valued function, has an expansion in powers of $\sqrt{z}$ up to the order $z^{k/2}$ with the remainder $\mathcal{O}(z^{\frac{k+\varepsilon}{2}})$. We choose $\chi \in C_0^\infty(\mathbf{R}^3)$ such that $\chi(\xi) = 1$ near $\xi = 0$ and $\chi(\xi) = 0$ if $|\xi| \geq 1$, and decompose

$$r_0(z) = r_0(z)\chi(D) + r_0(z)(1 - \chi(D)).$$

Then, as the Fourier transform is an isomorphism between $L_\gamma^2(\mathbf{R}^3)$ and the Sobolev space $H^\gamma(\mathbf{R}^3)$ and the multiplication with $(\xi^2 - z)^{-1}(1 - \chi(\xi))$ is a $B(H^\gamma(\mathbf{R}^3))$-valued smooth function of z near $z = 0$, $r_0(z)(1 - \chi(D))$ is a $B(L_\gamma^2(\mathbf{R}^3))$ valued smooth function of z near $z = 0$ and has a Taylor expansion up to any order. For $u \in \mathcal{H}_\gamma^\circ$ we have $\hat{u}(0) = 0$. Choose $\tilde{\chi}$ such that $\chi\tilde{\chi} = \chi$ and define $\hat{u}_j(\xi)$ by

$$\hat{u}_j(\xi) = \tilde{\chi}(\xi) \int_0^1 \frac{\partial(\chi\hat{u})}{\partial \xi_j}(\theta\xi) d\theta = \frac{\tilde{\chi}(\xi)}{|\xi|} \int_0^{|\xi|} \frac{\partial(\chi\hat{u})}{\partial \xi_j}(\theta\hat{\xi}) d\theta.$$

We have $\chi(\xi)\hat{u}(\xi) = \sum_{j=1}^3 \xi_j \hat{u}_j(\xi)$, or $\chi(D)u = \sum_{j=1}^3 D_j u_j(x)$, and, by using Hardy's inequality and the interpolation theorem, we also have

$$\|u_j\|_{L^\infty} + \|u_j\|_{\mathcal{H}_{\gamma-1}} \leq C\|u\|_{\mathcal{H}_\gamma}.$$

Then, by integration by parts, we may write $r_0(z)\chi(D)u$ in the form

$$r_0(z)\chi(D)u(x) = \sum_{j=1}^{3} i\int \frac{e^{i\sqrt{z}|x-y|}(x_j-y_j)}{4\pi|x-y|^3}u_j(y)dy + \sum_{j=1}^{3}\sqrt{z}\int \frac{e^{i\sqrt{z}|x-y|}(x_j-y_j)}{4\pi|x-y|^2}u_j(y)dy, \tag{2.7}$$

and statement (2) follows by an argument similar to the one used for proving (1). □

The Fourier series expansion with respect to the t-variable implies

$$R_0(z) = \sum_{m\in\mathbf{Z}} \oplus(p_m \otimes r_0(z-m)) \tag{2.8}$$

on the tensor product $\mathcal{K} = L^2(\mathbf{T})\otimes L^2(\mathbf{R}^3)$, where we inserted $\oplus$ to emphasize that the summands are orthogonal to each other. Since $-i\partial/\partial t$ commutes with $R_0(z)$, it may be considered as a $\mathcal{Y}^s_\gamma$-valued function for any $s\in\mathbf{N}$ and $\gamma\geq 0$. Recall that $\mathcal{K}^s_\gamma = H^s(\mathbf{T},\mathcal{H}_\gamma)$, and $\mathcal{Y}^s_\gamma = B(\mathcal{K}^s_\gamma,\mathcal{K}^s_{-\gamma})$. Combining Lemma 2.2 with (2.8), we obtain the following lemma.

Lemma 2.3. *Let $\gamma > 1/2$ and $s\in\mathbf{N}_0$. Consider $R_0(z)$ as a $\mathcal{Y}^s_\gamma$-valued analytic function of $z\in\mathbf{C}^\pm$. Then:*
(1) $R_0(z)$ can be extended to $\overline{\mathbf{C}}^\pm\setminus\mathbf{Z}$ as a $C^{\gamma-(1/2)+}$ function and, if $\gamma>1$, to $\overline{\mathbf{C}}^\pm$ as a continuous function. We write $R_0^\pm(\lambda) = \lim_{\varepsilon\downarrow 0} R_0(\lambda\pm i\varepsilon)$ for the boundary values on the reals $\lambda\in\mathbf{R}$.
(2) For $\gamma>1$ and any $z\in\overline{\mathbf{C}}^\pm$, $R_0(z)$ is a compact operator from $\mathcal{K}^s_\gamma$ to $\mathcal{K}^s_{-\gamma}$.
(3) Let $\gamma > \beta_k/2 \equiv \max\{k+\frac{1}{2},2\}$ for an integer $k\geq 1$, $\ell = [(k+2)/2]$ and $\varepsilon_0 = \min\{1,\gamma-\frac{\beta_k}{2}\}$. Then, for any $n\in\mathbf{Z}$, in a neighborhood of 0 in $\overline{\mathbf{C}}^+$,

$$R_0(z+n) = R_0^+(n) + \sqrt{z}D_1(n) + \cdots + z^{k/2}D_k(n) + \widetilde{R}_{0k}(n,z). \tag{2.9}$$

Here
(a) $D_j(n)\colon \mathcal{K}^s_\gamma \to \mathcal{K}^s_{-\gamma}$ are compact operators, and are defined by

$$D_j(n) = \begin{cases} p_n\otimes g_j + \dfrac{1}{(j/2)!}\displaystyle\sum_{m\neq n} p_m\otimes \frac{d^{j/2}r_0^+}{dz^{j/2}}(n-m), & \text{if } j \text{ is even,}\\ p_n\otimes g_j, & \text{if } j \text{ is odd.}\end{cases} \tag{2.10}$$

In particular, $D_j(n)$ is of finite rank, if j is odd.

(b) $\widetilde{R}_{0k}(n,z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ *for any* $0 \le \varepsilon < \varepsilon_0$*, and it has the form*

$$\widetilde{R}_{0k}(n,z) = \sum_{m \in Z} p_m \otimes e_k(z, n-m), \tag{2.11}$$

where $e_k(z,0) = d_k(z)$ *and* $e_k^{(j)}(z,m) = (d/dz)^j e_k(z,m)$*,* $m \neq 0$*, satisfies*

$$\|e_k^{(j)}(z,m)\|_{\mathcal{X}_\gamma} \le C|z|^{\ell-j}\langle m\rangle^{-(\ell+1)/2}, \quad |z| < 1/2 \tag{2.12}$$

for $r = 0, \ldots, k$*, and*

$$\int_{-1/2}^{1/2} \|e_k^{(\ell)}(z+h,m) - e_k^{(\ell)}(z,m)\|_{\mathcal{X}_\gamma} dz$$

$$\le C\langle m\rangle^{-(\ell+1)/2} \begin{cases} |h|, & k \ge 3, \\ |h|^\varepsilon, & k = 2, \\ |h|^{\frac{1+\varepsilon}{2}}, & k = 1. \end{cases} \tag{2.13}$$

Proof. Since $-i\partial/\partial t$ commutes with $R_0(z)$, we have only to prove the case $s = 0$. We have $\| \sum_{n=0}^\infty \oplus A_n\|_{\mathcal{Y}_\gamma} = \sup_{-\infty<n<\infty} \|A_n\|_{\mathcal{Y}_\gamma}$. Hence the statement (1) follows from (2.8) and the properties in parts 1, 2, and 4 of Lemma 2.1. The statement (2) follows from (2.8) and the properties in parts 3 and 4 of Lemma 2.1 (cf. [21]). Note that $R_0(z+n) = E_n R_0(z) E_n^*$ by virtue of (1.8), $E_n P_m E_n^* = P_{n+m}$, and the fact that E_n is unitary in $\mathcal{K}_{\pm\delta}$. Hence it suffices to prove (3) for $n = 0$. We expand each summand of (2.8) near $z = 0$. For the term with $m = 0$, we apply (2.2). We expand those with $m \neq 0$ as

$$r_0(z-m) = \sum_{0 \le j \le k/2} \frac{z^j}{j!} \frac{d^j r_0^+}{dz^j}(-m) + e_k(z,m). \tag{2.14}$$

Estimate (2.12) and (2.13) follow from Lemma 2.1. (We assumed $\gamma > 2$ to obtain (2.13) when $k = \ell = 1$.) This implies the remainder estimate $R_{0k}(n,z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ for any $0 < \varepsilon < \varepsilon_0$. The compactness of $D_j(0)$ is obvious, as each term is a norm limits in $\mathcal{Y}_\gamma$ of difference quotients of $R(z)$ as $z \to 0$ in $\mathcal{Y}_\gamma$. This completes the proof of the Lemma. □

We remark here that that the adjoint of $R_0(z) : \mathcal{K}_\gamma \to \mathcal{K}_{-\gamma}$ is given by $R_0(z)^* = R_0(\bar{z})$ and it is bounded from $\mathcal{K}_\gamma^s$ to $\mathcal{K}_{-\gamma}^s$ for any $s \in \mathbf{N}$.

In what follows we often use the following lemma.

Lemma 2.4. *Let* X, Y, Z *be Banach spaces, Suppose that* $L_1(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ *and* $L_2(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ *for* $k \in \mathbf{N}_0$ *and* $0 \le \varepsilon < 1$ *as* $B(X,Y)$ *and* $B(Y,Z)$ *valued functions, respectively.*

(1) *If* $k \geq 1$, *then* $z^{-1/2}L_1(z) = \mathcal{O}(z^{\frac{k+\varepsilon-1}{2}})$ *and*

$$\|z^{-1/2}L_1(z)\|_{\mathcal{O}((k-1+\varepsilon)/2)} \leq C\|L_1(z)\|_{\mathcal{O}((k+\varepsilon)/2)}. \tag{2.15}$$

(2) $L_2(z)L_1(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ *as a* $B(X,Z)$*-valued function and*

$$\|L_2(z)L_1(z)\|_{\mathcal{O}((k+\varepsilon)/2)} \leq C\|L_1(z)\|_{\mathcal{O}((k+\varepsilon)/2)}\|L_2(z)\|_{\mathcal{O}((k+\varepsilon)/2)}. \tag{2.16}$$

(3) *If* $L_3(z)$ *and* $L_4(z)$ *are* $B(Y,Z)$ *and* $B(Z,Y)$*-valued smooth functions of* $\sqrt{z}$, *respectively, then,* $L_3(z)L_1(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ *and* $L_1(z)L_4(z) = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ *as* $B(X,Z)$ *and* $B(Z,X)$*-valued functions, respectively.*

$$\|L_3(z)L_1(z)\|_{\mathcal{O}((k+\varepsilon)/2)} \leq C\|L_3(z)\|_{C^{k+1}}\|L_1(z)\|_{\mathcal{O}((k+\varepsilon)/2)}, \tag{2.17}$$
$$\|L_1(z)L_4(z)\|_{\mathcal{O}((k+\varepsilon)/2)} \leq C\|L_4(z)\|_{C^{k+1}}\|L_1(z)\|_{\mathcal{O}((k+\varepsilon)/2)}, \tag{2.18}$$

where we wrote $\|u\|_{C^{k+1}} = \sup_{z^2\in U}\sum_{j=0}^{k+1}\|(d/dz)^j(u(z^2))\|$.
(4) *If* $X = Y$, *then,* $(1+L_1(z))^{-1}$ *exists in a suitable neighborhood of* 0 *and* $(1+L_1(z))^{-1} = 1 + \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$.

Proof. (1) It suffices to show that $z^{-1/2}L_1(z)$ satisfies (1.15) and (1.14) with $k-1$ in place of k. We show (1.15) only as the other is obvious. We write $\ell = [(k+2)/2]$ and $\ell^* = [(k+1)/2]$. Since $L^{(j)}(0) = 0$, $0 \leq j \leq \ell - 1$, Taylor's formula implies

$$L^{(j)}(z) = \frac{z^{\ell-j}}{(\ell-j-1)!}\int_0^1 (1-\theta)^{\ell-j-1}L^{(\ell)}(\theta z)d\theta, \quad j = 0, \dots, \ell-1. \tag{2.19}$$

If k is odd, $\ell = \ell^* = (k+1)/2$, and Leibniz formula together with (2.19) imply

$$\Big(\frac{d}{dz}\Big)^{\ell^*}\Big(\frac{L(z)}{\sqrt{z}}\Big) = \frac{L^{(\ell)}(z)}{\sqrt{z}} + \sum_{j=0}^{\ell^*}\frac{c_j}{\sqrt{z}}\int_0^1 (1-\theta)^{\ell-j-1}L^{(\ell)}(\theta z)d\theta$$

with a suitable constant c_j. We write

$$\frac{L^{(\ell)}(z+h)}{\sqrt{z+h}} - \frac{L^{(\ell)}(z)}{\sqrt{z}} = \frac{hL^{(\ell)}(z+h)}{(\sqrt{z+h}+\sqrt{z})\sqrt{z}\sqrt{z+h}} + \frac{L^{(\ell)}(z+h) - L^{(\ell)}(z)}{\sqrt{z}}.$$

Since $L^{(\ell)}(z) = L^{(\ell^*)}(z)$ satisfies (1.14), we have

$$\int_{-\mu}^{\mu}\Big\|\frac{hL^{(\ell)}(z+h)}{(\sqrt{z+h}+\sqrt{z})\sqrt{z}\sqrt{z+h}}\Big\|dz \leq C|h|^{\frac{\varepsilon}{2}},$$

and

$$\int_{|z|\le h}\Big\|\frac{L^{(\ell)}(z+h)-L^{(\ell)}(z)}{\sqrt{z}}\Big\|dz \le \int_{|z|\le h}\frac{|z+h|^{\frac{\varepsilon-1}{2}}+|z|^{\frac{\varepsilon-1}{2}}}{|\sqrt{z}|}dz = C|h|^{\frac{\varepsilon}{2}}.$$

As $L^{(\ell)}(z)=L^{(\ell^*)}(z)$ also satisfies (1.15), we have

$$\int_{h<|z|<\mu}\Big\|\frac{L^{(\ell)}(z+h)-L^{(\ell)}(z)}{\sqrt{z}}\Big\|dz \le C|h|^{-\frac{1}{2}}\cdot|h|^{\frac{1+\varepsilon}{2}} \le C|h|^{\frac{\varepsilon}{2}},$$

and, combining the last three estimates, we obtain

$$\int_{|z|\le\mu}\Big\|\frac{L^{(\ell)}(z+h)}{\sqrt{z+h}}-\frac{L^{(\ell)}(z)}{\sqrt{z}}\Big\|dz \le Ch^{\frac{\varepsilon}{2}}. \tag{2.20}$$

Applying (2.20) to $L^{(\ell)}(\theta z)$ yields

$$\int_{|z|\le\mu}\Big\|\frac{L^{(\ell)}(\theta(z+h))}{\sqrt{z+h}}-\frac{L^{(\ell)}(\theta z)}{\sqrt{z}}\Big\|dz \le C\theta^{\frac{\varepsilon-1}{2}}h^{\frac{\varepsilon}{2}}.$$

It follows that $M_j(z)=\dfrac{c_j}{\sqrt{z}}\displaystyle\int_0^1(1-\theta)^{\ell-j-1}L^{(\ell)}(\theta z)d\theta$, $j=0,\dots,\ell-1$, satisfy

$$\int_{|z|\le\mu}\|M_j(z+h)-M_j(z)\|dz \le C_jh^{\frac{\varepsilon}{2}}\int_0^1(1-\theta)^{\ell-j-1}\theta^{\frac{\varepsilon-1}{2}}d\theta \le C_jh^{\frac{\varepsilon}{2}}. \tag{2.21}$$

(2.20) and (2.21) show that $z^{-1/2}L(z)$ satisfies (1.15) with $k-1$ in place of k and $z^{-1/2}L(z)=\mathcal{O}(z^{\frac{k-1+\varepsilon}{2}})$ when k is odd.

If k is even, $\ell^*=\ell-1=k/2$. We write

$$\Big(\frac{d}{dz}\Big)^{\ell^*}\Big(\frac{L(z)}{\sqrt{z}}\Big)=\frac{L^{(\ell-1)}(z)}{\sqrt{z}}+\sum_{j=0}^{\ell^*}\frac{c_j}{\sqrt{z}}\int_0^1(1-\theta)^{\ell-j-1}L^{(\ell-1)}(\theta z)d\theta$$

and proceed as above: We use $\|L^{(\ell-1)}(z+h)\|\le C|z+h|^{\frac{\varepsilon}{2}}$ and obtain

$$\int_{-\mu}^{\mu}\Big\|\frac{hL^{(\ell-1)}(z+h)}{(\sqrt{z+h}+\sqrt{z})\sqrt{z}\sqrt{z+h}}\Big\|dz \le C|h|^{\frac{1+\varepsilon}{2}};$$

using (1.14), we estimate as

$$\|L^{(\ell-1)}(z+h)-L^{(\ell-1)}(z)\| \le h\int_0^1\|L^{(\ell)}(z+\theta h)\|d\theta \le \int_0^h\frac{d\theta}{|z+\theta|^{1-\frac{\varepsilon}{2}}},$$

from which we obtain

$$\int_{|z|\le\mu}\Big\|\frac{L^{(\ell-1)}(z+h)-L^{(\ell-1)}(z)}{\sqrt{z}}\Big\|dz \le C\int_{|z|\le\mu}\Big(\int_0^h \frac{d\theta}{|z+\theta|^{1-\frac{\varepsilon}{2}}\sqrt{z}}\Big)dz$$
$$\le \int_0^h\Big(\int_{\mathbf{R}}\frac{dz}{|z+\theta|^{1-\frac{\varepsilon}{2}}\sqrt{z}}\Big)d\theta = C\int_0^h \theta^{\frac{\varepsilon-1}{2}}d\theta = Ch^{\frac{1+\varepsilon}{2}}.$$

It follows that

$$\int_{|z|\le\mu}\Big\|\frac{L^{(\ell-1)}(z+h)}{\sqrt{z+h}}-\frac{L^{(\ell-1)}(z)}{\sqrt{z}}\Big\|dz \le Ch^{\frac{1+\varepsilon}{2}}. \tag{2.22}$$

Then, the same argument as in the case k is odd implies $z^{-1/2}L_1(z) = \mathcal{O}(z^{\frac{k-1+\varepsilon}{2}})$ also for even k. Similar and simpler proofs for other statements are left for the readers. □

We need the following lemma in the final part of the paper. Recall that J is the identification operator $(Ju)(t,x) = u(x)$. We write

$$\mathbf{Z}_c = \cup_{n\in\mathbf{Z}}\{z\in\mathbf{C}\colon |z-n|<c\}$$

for the c-neighborhood of $\mathbf{Z}$ in $\mathbf{C}$. We define $M(z) = R_0(z)VR_0(z)$ and $N(z) = R_0(z)VR_0(z)VR_0(z)$ in the following lemma.

Lemma 2.5. *Suppose $V\in\mathcal{V}_\beta$ for $\beta>\beta_k\equiv\max\{2k+1,4\}$, $k\ge 1$ being an integer. Let $\delta=\beta/2$ and $\varepsilon_0=\min\{1,\delta-\frac{\beta_k}{2}\}$. Then:*
(1) *For any $c>0$ small, $s=0,1$ and $j=0,1,\dots,k$, there exists $C>0$ such that, for all $z\notin\mathbf{Z}_c$,*

$$\|(d/dz)^j M(z)Ju_0\|_{\mathcal{K}^s_{-\delta}} \le C\langle z\rangle^{-\min\left\{\frac{j}{2}+1,\frac{3}{2}\right\}}\|u_0\|_{\mathcal{H}_\delta}, \tag{2.23}$$
$$\|(d/dz)^j N(z)Ju_0\|_{\mathcal{K}^s_{-\delta}} \le C\langle z\rangle^{-\min\left\{\frac{j}{2}+\frac{3}{2},\frac{3}{2}\right\}}\|u_0\|_{\mathcal{H}_\delta}. \tag{2.24}$$

(2) *As $B(\mathcal{K}^s_\delta,\mathcal{K}^s_{-\delta})$-valued functions of z defined in a neighborhood of 0 in $\bar{\mathbf{C}}^\pm$ and for $s=0,1$, we have the expansions*

$$M(z+n) = M_0(n)+\cdots+z^{k/2}M_k(n)+\widetilde{M}_k(n,z), \tag{2.25}$$
$$N(z+n) = N_0(n)+\cdots+z^{k/2}N_k(n)+\widetilde{N}_k(n,z). \tag{2.26}$$

Here $\widetilde{M}_k(n,z)=\mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ and $\widetilde{N}_k(n,z)=\mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ for any $0<\varepsilon<\varepsilon_0$ and, $M_j(n)$ and $N_j(n)$, $j=0,\dots,k$, satisfy the following estimates for $s=0,1$:

$$\|M_j(n)Ju_0\|_{\mathcal{K}^s_{-\delta}} \le C_j\langle n\rangle^{-\min\{1+\frac{j}{2},\frac{3}{2}\}}\|u_0\|_{\mathcal{H}_\delta}, \tag{2.27}$$

$$\|N_j(n)Ju_0\|_{\mathcal{K}^s_{-\delta}} \le C_j \langle n\rangle^{-\min\{\frac{3}{2}+\frac{j}{2},\frac{3}{2}\}}\|u_0\|_{\mathcal{H}_\delta}. \tag{2.28}$$

Moreover, as $B(\mathcal{H}_\delta, \mathcal{K}^s_{-\delta})$-valued functions of z, $s = 0, 1$,

$$\|\widetilde{M}_k(n,z)J\|_{\mathcal{O}((k+\varepsilon)/2)} \le C\langle n\rangle^{-\min\{1+\frac{k}{2},\frac{3}{2}\}}, \tag{2.29}$$

$$\|\widetilde{N}_k(n,z)J\|_{\mathcal{O}((k+\varepsilon)/2)} \le C\langle n\rangle^{-\min\{\frac{k}{2}+\frac{3}{2},\frac{3}{2}\}}. \tag{2.30}$$

Proof. We decompose V into its Fourier series with respect to t and write

$$V(t,x) = \sum_{m=-\infty}^{\infty} V_m(x)e^{imt}, \quad V_m(x) = \frac{1}{2\pi}\int_{\mathbf{T}} e^{-imt}V(t,x)dt. \tag{2.31}$$

We have that $\sup_{x\in\mathbf{R}^3}\langle x\rangle^\beta\Big(\sum_m |V_m(x)|^2\langle m\rangle^4\Big)^{1/2} < \infty$ by the Parseval formula and Assumption 1.1, a fortiori,

$$\sup_{x\in\mathbf{R}^3}\langle x\rangle^\beta |V_m(x)| \le C\langle m\rangle^{-2}. \tag{2.32}$$

Write $R_0^{(a)}(z) = (d/dz)^a R_0(z)$ etc. When $u_0 \in \mathcal{H}_\delta$, we may write

$$R_0^{(a)}(z)VR_0^{(b)}(z)Ju_0 = \sum_m e^{imt}\otimes r_0^{(a)}(z-m)V_m r_0^{(b)}(z)u_0, \tag{2.33}$$

for $z \in \overline{\mathbf{C}}^+ \setminus \mathbf{Z}_c$. It follows that for such z and $a+b=j$

$$\begin{aligned}\left\|R_0^{(a)}(z)VR_0^{(b)}(z)Ju_0\right\|^2_{\mathcal{K}_{-\delta}} &= 2\pi\sum_m \left\|r_0^{(a)}(z-m)V_m r_0^{(b)}(z)u_0\right\|^2_{\mathcal{H}_{-\delta}} \\ &\le C\sum_m \langle z-m\rangle^{-a-1}\langle m\rangle^{-4}\langle z\rangle^{-b-1}\|u_0\|^2_{\mathcal{H}_\delta} \le C\langle z\rangle^{-\min\{j+2,5\}}\|u_0\|^2_{\mathcal{H}_\delta}.\end{aligned}$$

and

$$\begin{aligned}\left\|R_0^{(a)}(z)VR_0^{(b)}(z)Ju_0\right\|^2_{\mathcal{K}^1_{-\delta}} &= 2\pi\sum_m \left\|mr_0^{(a)}(z-m)V_m r_0^{(b)}(z)u_0\right\|^2_{\mathcal{H}_{-\delta}} \\ &\le C\sum_m \langle z-m\rangle^{-a-1}\langle m\rangle^{-2}\langle z\rangle^{-b-1}\|u_0\|^2_{\mathcal{H}_\delta} \le C\langle z\rangle^{-\min\{j+2,3\}}\|u_0\|^2_{\mathcal{H}_\delta}.\end{aligned}$$

The last two estimates imply (2.23). We omit the very similar proof for (2.24). By virtue of (2.9) and (2.11), we have (2.25) and (2.26) with

$$M_j(n) = \sum_{a+b=j} D_a(n)VD_b(n),$$

$$
\begin{aligned}
N_j(n) &= \sum_{a+b+c=j} D_a(n)VD_b(n)VD_c(n),\\
\widetilde{M}_k(z,n) &= \sum_{a+b\geq k+1} \sqrt{z}^{a+b} D_a(n)VD_b(n),\\
\widetilde{N}_k(z,n) &= \sum_{a+b+c\geq k+1} \sqrt{z}^{a+b+c} D_a(n)VD_b(n)VD_c(n),
\end{aligned}
$$

where we wrote $R_0^+(n) = D_0(n)$ and $\widetilde{R}_0(n,k,z) = \sqrt{z}^{k+1}D_{k+1}(n)$, with a slight abuse of notation. We also use the shorthand notation $a+b \geq k+1$ and $a+b+c \geq k+1$ for the sum over the relevant terms involving the remainders. We prove (2.27) and (2.29) for large n. If b is odd, then $D_a(n)VD_b(n)Ju_0 = 0$ and, if $b = 2b'$ is even and a is odd

$$
D_a(n)VD_b(n)Ju_0 = (1/b'!)e^{int}(g_aV_nr_0^{(b')}(n+i0)u_0)(x),
$$

and we obviously have

$$
\|D_a(n)VD_b(n)Ju_0\|_{\mathcal{K}^s_{-\delta}} \leq C\langle n\rangle^{-\frac{5-2s}{2}-\frac{b'}{2}}\|u\|_{\mathcal{H}_\delta}. \tag{2.34}
$$

When $n \neq 0$ and $a = 2a', b = 2b'$ both are even, we have

$$
\begin{aligned}
D_a(n)VD_b(n)Ju_0 &= e^{int}g_aV_nr_0^{(b')}(n+i0)u_0\\
&\quad + \sum_{m\neq n} e^{imt}r_0^{(a')}(n-m+i0)V_mr_0^{(b')}(n+i0)u_0,
\end{aligned}
$$

and we can estimate as follows for $s = 0,1$:

$$
\begin{aligned}
&\|D_a(n)VD_b(n)Ju_0\|^2_{\mathcal{K}^s_{-\delta}}\\
&\quad = 2\pi\langle n\rangle^{2s}\|g_aV_nr_0^{(b')}(n+i0)u_0\|^2_{\mathcal{H}_{-\delta}}\\
&\qquad + 2\pi\sum_{m\neq n}\langle m\rangle^{2s}\|r_0^{(a')}(n-m+i0)V_mr_0^{(b')}(n+i0)u_0\|^2_{\mathcal{H}_{-\delta}}\\
&\quad \leq C\langle n\rangle^{2s-5}\|u_0\|_{\mathcal{H}_\delta} + C\sum_{m\neq n}\langle m\rangle^{2s-4}\langle n-m\rangle^{-1-a'}\langle n\rangle^{-1-b'}\|u_0\|_{\mathcal{H}_\delta}\\
&\quad \leq C\langle n\rangle^{-\min\{3,2+a'+b'\}}\|u_0\|_{\mathcal{H}_\delta}.
\end{aligned} \tag{2.35}
$$

The estimates (2.34) and (2.35) yield (2.27). For proving (2.29), we use the expression (2.11) for the remainder instead of (2.10) and proceed similarly, applying (2.12), remainder estimates in (2.2) and (2.13), and Lemma 2.4 in addition. We omit the details of the entirely similar proof of (2.28) and (2.30). □

2.2 Eigenvalues and resonances

In this section we assume that $V \in \mathcal{V}_\beta$ with $\beta > 2$, and set $\delta = \beta/2 > 1$. Then $R_0(z)V$ is compact in $\mathcal{K}_{-\delta}$ for all $z \in \overline{\mathbf{C}}^{\pm}$ by Lemma 2.3. Hence $-1 \notin \sigma(R_0(z)V)$ for any $z \in \mathbf{C}^{\pm}$ by the self-adjointness of K, and from the resolvent equation $R(z) = R_0(z) - R_0(z)VR(z)$ we have

$$R(z) = (1 + R_0(z)V)^{-1}R_0(z), \quad z \in \mathbf{C}^{\pm}.$$

It follows that if $-1 \notin \sigma(R_0^{\pm}(\lambda)V)$, then $R(z)$ can be extended to $\mathbf{C}^{\pm} \cup I$ as a $\mathcal{Y}_\delta$-valued continuous function, where I is a (small) neighborhood of λ on the real line. We denote the boundary values by $R^{\pm}(\lambda)$. We then have

$$R^{\pm}(\lambda) = (1 + R_0^{\pm}(\lambda)V)^{-1}R_0^{\pm}(\lambda), \quad \lambda \in I. \tag{2.36}$$

We want to identify those $\lambda \in \mathbf{R}$ with $-1 \in \sigma(R_0^{\pm}(\lambda)V)$ in $\mathcal{K}_{-\delta}$. We use the following lemma, see [1, page 157].

Lemma 2.6. (1) *Let $c > 0$ and $s \in \mathbf{R}$. Then there exists $C > 0$, such that*

$$\left\| \frac{f(\xi)}{\xi^2 + \lambda^2} \right\|_{H^s(\mathbf{R}^3)} \leq C\lambda^{-2}\|f\|_{H^s(\mathbf{R}^3)}, \quad \lambda > c. \tag{2.37}$$

(2) *Let $c > 0$ and $s > 1/2$. Then there exists $C > 0$, such that for all $\lambda > c$ and $f \in H^s(\mathbf{R}^3)$ satisfying $f(\xi)|_{|\xi|=\lambda} = 0$, we have*

$$\left\| \frac{f(\xi)}{\xi^2 - \lambda^2} \right\|_{H^{s-1}(\mathbf{R}^3)} \leq C\lambda^{-1}\|f\|_{H^s(\mathbf{R}^3)}. \tag{2.38}$$

Proof. Consider first part (1). If $s \in \mathbf{N}_0$, then (2.37) is obvious. For general $s > 0$ we use the interpolation theorem, and for negative s the duality.

In order to prove (2) we take $\phi \in C_0^\infty(\mathbf{R}^3)$ such that $\phi(\xi) = 1$ for $|\xi| < c/4$ and $\phi(\xi) = 0$ for $|\xi| \geq c/2$, and set $\tilde{\phi} = 1 - \phi$. We have as in (1)

$$\left\| \frac{\phi(\xi)f(\xi)}{\xi^2 - \lambda^2} \right\|_{H^s(\mathbf{R}^3)} \leq C\lambda^{-2}\|f\|_{H^s(\mathbf{R}^3)}, \quad \lambda > c. \tag{2.39}$$

Take a partition of unity $\sum \chi_j(\xi) = 1$ on $\xi \in \mathbf{R}^3 \setminus \{0\}$ where $\chi_j \in C^\infty(\mathbf{R}^3 \setminus \{0\})$ is homogeneous of degree zero and is supported in a cone with opening angle less than $\pi/4$, and decompose as

$$\frac{\tilde{\phi}(\xi)f(\xi)}{\xi^2 - \lambda^2} = \frac{\psi(\xi)}{|\xi| + \lambda} \sum_j \frac{\tilde{\phi}(\xi)\chi_j(\xi)f(\xi)}{|\xi| - \lambda},$$

where ψ is such that $\tilde{\phi}\psi = \tilde{\phi}$ and $\operatorname{supp}\psi \subset \{\xi\colon |\xi| > c/8\}$. Then

$$|\partial_\xi^\alpha \left(\psi(\xi)(|\xi| + \lambda)^{-1}\right)| \le C_\alpha \lambda^{-1},$$

and (2.38) follows, if we prove

$$\left\| \frac{\chi_j(\xi)\tilde{\phi}(\xi)f(\xi)}{|\xi| - \lambda} \right\|_{H^{s-1}(\mathbf{R}^3)} \le C\|f\|_{H^s(\mathbf{R}^3)}, \quad \lambda > c. \tag{2.40}$$

In order to prove (2.40), we may assume by rotating the coordinates that χ_j is supported by the set $\{\xi = (\xi_1, \xi')\colon |\xi'| < \xi_1\}$. We may then choose coordinates $(|\xi|, \xi')$ and reduce the estimate (2.40) to

$$\left\| \frac{f(\xi)}{\xi_1 - \lambda} \right\|_{H^{s-1}(\mathbf{R}^3)} \le C\|f\|_{H^s(\mathbf{R}^3)},$$

for functions f such that $f|_{\xi_1 = \lambda} = 0$, which is obvious by the Fourier transform. □

The following lemma partly improves the mapping properties of g_0 stated in Lemma 2.2.

Lemma 2.7. (1) *Let $\delta > 1/2$. Then there exists $C > 0$ such that for all $f \in H^\delta(\mathbf{R}^3)$ we have*

$$\left\| \frac{f(\xi)}{|\xi|^2} \right\|_{H^{\min\{\delta-2, (-1/2)_-\}}} \le C\|f\|_{H^\delta(\mathbf{R}^3)}.$$

(2) *Let $\delta > 3/2$. Then there exists $C > 0$ such that for any $f \in H^\delta(\mathbf{R}^3)$ with $f(0) = 0$ we have*

$$\left\| \frac{f(\xi)}{|\xi|^2} \right\|_{H^{\min\{\delta-2, (1/2)_-\}}} \le C\|f\|_{H^\delta(\mathbf{R}^3)}.$$

Here $(a)_-$ stands for any number strictly small than a, and the constants C above depend on this number and δ.

Proof. (1) We may assume $1/2 < \delta < 3/2$. We have $H^\delta(\mathbf{R}^3) \subset L^p_{\mathrm{loc}}(\mathbf{R}^3)$ for some $p > 3$ and $f(\xi)/|\xi|^2$ is integrable. Then, using the Fourier transform, we see that it suffices to show that the kernel $\langle x\rangle^{\delta-2}|x-y|^{-1}\langle y\rangle^{-\delta}$ defines a bounded operator on $L^2(\mathbf{R}^3)$. This kernel is dominated by the kernel $|x|^{\delta-2}|x-y|^{-1}|y|^{-\delta}$, which defines a bounded operator on $L^2(\mathbf{R}^3)$ by well-known results on homogeneous kernels, see for example [17], and the first

part follows.
(2) We may assume $3/2 < \delta < 5/2$. We use the condition $f(0) = 0$ to replace $|x-y|^{-1}$ by $|x-y| - (1+|x|)^{-1}$ in the kernel. We have

$$|\langle x\rangle^{\delta-2}(|x-y|^{-1} - (1+|x|)^{-1})\langle y\rangle^{-\delta}| \le C|x|^{\delta-3}|x-y|^{-1}|y|^{-\delta+1},$$

and the boundedness follows from the results on homogeneous kernels. This concludes the proof. $\square$

Lemma 2.8. *Let V satisfy Assumption* 1.1 *for some $\beta > 2$, and let $\delta = \beta/2$. Assume that $\mathcal{K}_{-\delta} \ni \psi^{\pm} \neq 0$ satisfies $(1 + R_0^{\pm}(\lambda)V)\psi^{\pm} = 0$. Then:*
(1) *If $\lambda \notin \mathbf{Z}$, then λ is an eigenvalue of K and $\psi^{\pm}$ is an associated eigenfunction. For any N and $a, b \in \mathbf{N}_0$ with $0 \le a+b \le 2$ we have $\langle x\rangle^N H(t)^a \partial_t^b \psi^{\pm} \in \mathcal{K}$. In particular, $\psi^{\pm}$ is an $H^2(\mathbf{R}^3)$-valued C^1 function. Let $0 < c < 1$. Then, for all λ,$\psi^{\pm}$ with* $\operatorname{dist}(\lambda, \mathbf{Z}) > c$ *we have*

$$\|\langle x\rangle^N H(t)^a \partial_t^b \psi^{\pm}\|_{\mathcal{K}} \le C\|\psi^{\pm}\|_{\mathcal{K}}, \quad 0 \le a+b \le 2. \tag{2.41}$$

(2) *Assume $\beta > 3$ and $\lambda \in \mathbf{Z}$. Then the following results hold.*

- (a) *If $\langle V, \psi^{\pm}\rangle_{\mathcal{K}} = 0$, then λ is an eigenvalue of K, and $\psi^{\pm}$ is an associated eigenfunction. We have $\langle x\rangle^{(1/2)-} H(t)^a \partial_t^b \psi^{\pm} \in \mathcal{K}$ for $0 \le a+b \le 2$. (This result actually holds under the assumption $\beta > 2$.) Furthermore, we have, with $C_j^{\pm} = \langle x_j V, \psi^{\pm}\rangle_{\mathcal{K}}/(8\pi^2)$,*

$$\psi^{\pm}(t,x) - e^{i\lambda t}\sum_{j=1}^{3} \frac{C_j^{\pm} x_j}{|x|^3} \in \mathcal{K}_{(\frac{3}{2})_-}. \tag{2.42}$$

- (b) *If $\langle V, \psi^{\pm}\rangle_{\mathcal{K}} \neq 0$, then λ is a threshold resonance, and $\psi^{\pm}$ associated resonant functions. We have with $C^{\pm} = \langle V, \psi^{\pm}\rangle_{\mathcal{K}}/(8\pi^2) \neq 0$,*

$$\psi^{\pm}(t,x) = e^{i\lambda t}\frac{C^{\pm}}{|x|} + u_1^{\pm}(t,x), \quad u_1^{\pm} \in \mathcal{K}^2_{(\frac{1}{2})_-}. \tag{2.43}$$

(3) $\{\lambda\colon -1 \in \sigma(R_0^{\pm}(\lambda)V)\}$ *is discrete in $\mathbf{R} \setminus \mathbf{Z}$, with possible accumulation to $\mathbf{Z}$.*

Proof. Due to the periodicity we may assume $0 \le \lambda < 1$. We consider only the $+$-case, and write ψ instead of ψ^+. If $(1 + R_0^+(\lambda)V)\psi = 0$, we have, in the sense of distributions,

$$(K_0 - \lambda)(1 + R_0^+(\lambda)V)\psi = (K_0 + V - \lambda)\psi = 0.$$

We denote the Fourier coefficient of $f(t,x)$ with respect to t by $f_n(x)$ as previously (see (2.31)) such that $f(t,x) = \sum_{n=-\infty}^{\infty} e^{int} f_n(x)$. We have from (2.8) that

$$\psi_m + r_0^+(\lambda - m)(V\psi)_m = 0, \quad m \in \mathbf{Z}. \tag{2.44}$$

To prove part (1), we fix c, $0 < c < 1/2$, and consider λ with $c \le \lambda \le 1 - c$. We prove that for any N we have

$$\|\langle x\rangle^N \partial_t^j \psi\|_{\mathcal{K}} \le C\|\psi\|_{\mathcal{K}}, \quad j = 0, \ldots, 2, \tag{2.45}$$

with C independent of λ in the interval considered. The result (2.41) will then follow from this result since the differentiation of

$$\partial_t \psi(t) = -i(H(t) + \lambda)\psi(t)$$

implies $\partial_t^2\psi(t) = -i(H(t)+\lambda)\partial_t\psi(t) - i(\partial_t V)\psi = -(H(t)+\lambda)^2\psi(t) - i(\partial_t V)\psi$ and $\partial_t V(t,x)$ is a bounded function. In particular, $\psi \in \mathcal{D}(K)$, and ψ is an eigenfunction of K with eigenvalue λ. To show (2.45) we apply Lemma 2.6 and the well-known bootstrap argument (see [1]). We have $V\psi \in \mathcal{K}_\delta$ and $(V\psi)_m \in \mathcal{H}_\delta = L^2_\delta(\mathbf{R}^3)$. It follows from (2.44) and (2.37) that for $\lambda - m < 0$ we have $\psi_m \in \mathcal{H}_\delta$ and

$$\|\psi_m\|_{\mathcal{H}_\delta} \le C\langle m\rangle^{-1}\|(V\psi)_m\|_{\mathcal{H}_\delta} \tag{2.46}$$

with a constant $C > 0$ independent of $m > \lambda$. To study the case $m < \lambda$, we note that

$$\langle V\psi, \psi\rangle = -\langle V\psi, R_0^+(\lambda)V\psi\rangle = -\sum_m \langle (V\psi)_m, r_0^+(\lambda - m)(V\psi)_m\rangle \tag{2.47}$$

is a real number, as V is real-valued. Since $\delta > 1$, the L^2-trace on the sphere $\{\xi\colon |\xi| = \sqrt{\lambda - m}\}$ of the Fourier transform $(V\psi)\widehat{\ }_m$ exists, and, as a limit of the Poisson integral, we have for $\lambda - m > 0$ that

$$\operatorname{Im}\langle (V\psi)_m, r_0^+(\lambda - m)(V\psi)_m\rangle = \frac{\pi}{2\sqrt{\lambda - m}} \int_{|\xi| = \sqrt{\lambda - m}} |(V\psi)\widehat{\ }_m(\xi)|^2 d\sigma(\xi) \ge 0,$$

where $d\sigma(\xi)$ is the surface measure on $\{\xi\colon |\xi| = \sqrt{\lambda - m}\}$. It follows that the trace vanishes:

$$(V\psi)\widehat{\ }_m(\xi)|_{|\xi| = \sqrt{\lambda - m}} = 0, \tag{2.48}$$

and, by virtue of (2.38), we obtain that, with a constant independent of m,

$$\|\psi_m\|_{\mathcal{H}_{\delta-1}} \le \|r_0^+(\lambda - m)(V\psi)_m\|_{\mathcal{H}_{\delta-1}} \le C\langle m\rangle^{-1/2}\|(V\psi)_m\|_{\mathcal{H}_\delta}. \tag{2.49}$$

It follows by combining (2.46) with (2.49) that

$$\|\psi\|^2_{\mathcal{K}^{1/2}_{\delta-1}} = \sum_m \langle m\rangle \|\psi_m\|^2_{\mathcal{H}_{\delta-1}} \le C\|V\psi\|^2_{\mathcal{K}_\delta} \le C\|\psi\|^2_{\mathcal{K}_{-\delta}}.$$

Notice this constant C does not depend on λ, as long as $c \le \lambda \le 1-c$. This result implies that $V\psi \in \mathcal{K}^{1/2}_{3\delta-1}$ because Assumption 1.2 implies that V maps $\mathcal{K}^s_\gamma$ into $\mathcal{K}^s_{\gamma+\beta}$ for any $0 \le s \le 2$, and the same argument as above yields that $\psi \in \mathcal{K}^1_{2(2\delta-1)-\delta}$ with a corresponding estimate

$$\|\psi\|_{\mathcal{K}^1_{2(2\delta-1)-\delta}} \le C\|\psi\|_{\mathcal{K}_{-\delta}}$$

with a λ-independent constant, $c \le \lambda \le 1-c$. We repeat the argument $j \ge 4$ times until $N \le j(2\delta-1)-\delta$ for a given N and $j/2 \ge 2$, to obtain (2.45).

To prove part (2) it suffices to consider $\lambda = 0$. Note that

$$\langle (V\psi)_0, r_0^+(0)(V\psi)_0\rangle = \int \frac{|(V\psi)\widehat{_0}(\xi)|^2}{|\xi|^2} d\xi \ge 0.$$

Thus the argument leading to (2.48) produces

$$(V\psi)\widehat{_m}(\xi)|_{|\xi|=\sqrt{-m}} = 0, \quad 0 > m \in \mathbf{Z}. \tag{2.50}$$

It follows that (2.49) holds for $m<0$ and, as above,

$$\sum_{m\ne 0} e^{imt}\psi_m = -\sum_{m\ne 0} e^{imt} r_0^+(-m)(V\psi)_m \in \mathcal{K}^{1/2}_{\delta-1}. \tag{2.51}$$

We have $(V\psi)_0 \in \mathcal{H}_\delta(\mathbf{R}^3)$. Suppose first that $1<\delta<3/2$ or $2<\beta<3$. Then it follows from (2.44) with $\lambda=0$ and $m=0$, and from Lemma 2.7(1) that $\hat\psi_0 = -(V\psi)\widehat{_0}/|\xi|^2 \in H^{\delta-2}(\mathbf{R}^3)$ Thus, together with (2.51), we have that $\psi \in \mathcal{K}^{1/2}_{\delta-2}$, and hence that $V\psi \in \mathcal{K}^{1/2}_{3\delta-2}$. After a few repetition of the same argument, we conclude that $\psi \in \mathcal{K}^2_{(-1/2)_-}$ and $V\psi \in \mathcal{K}^2_{\beta-(1/2)_+}$. Thus, $\sum_{m\ne 0} e^{imt}\psi_m \in \mathcal{K}^2_{\beta-(3/2)_+}$ and

$$\hat\psi_0(\xi) = -\lim_{\varepsilon\downarrow 0} \frac{(V\psi)\widehat{_0}(\xi)}{\xi^2 \pm i0} = \frac{(V\psi)\widehat{_0}(0)}{\xi^2} + \frac{(V\psi)\widehat{_0}(\xi) - (V\psi)\widehat{_0}(0)}{\xi^2} \tag{2.52}$$

where the first term can be written as

$$\frac{1}{4\pi|x|}\int_{R^3}(V\psi)_0(x)dx = \frac{\langle V,\psi\rangle_{\mathcal{K}}}{8\pi^2|x|},$$

and the second term belongs to $\mathcal{H}_{(\beta-2)-(1/2)_+}$ by Lemma 2.7(2).

Suppose now that $\langle V, \psi\rangle_{\mathcal{K}} = 0$. Then we have that $\psi \in \mathcal{K}^2_{(\beta-2)-(1/2)_+}$, and therefore $V\psi \in \mathcal{K}^2_{(2\beta-2)-(1/2)_+}$. Iteration of the argument implies that $\psi \in \mathcal{K}^2_{\min\{2(\beta-2)-(1/2)_+,(1/2)_-\}}$. After a few further iterations we find that $\psi \in \mathcal{K}^2_{(1/2)_-}$. To prove the result (2.42), we first note that $\sum_{m\neq 0} e^{imt}\psi_m(x) \in \mathcal{K}^2_{\beta-\left(\frac{1}{2}\right)_+}$ and this can be put into the remainder. By Fourier inversion formula we have from (2.52) that

$$\psi_0(x) = \frac{1}{4\pi}\int \psi(y)\left(\frac{1}{|x-y|} - \frac{1}{|x|}\right) dy$$

and the function inside the parenthesis can be expanded as $(x\cdot y)|x|^{-3} + h(x,y)$, where the remainder satisfies $|h(x,y)| \le C|y|^2|x-y^{-1}||x|^{-2}$ for $|x|$ large. Then the arguments from the proof of Lemma 2.7 prove (2.42) if $\beta > 3$. This proves part (2a).

To proceed with the case $\langle V, \psi\rangle_{\mathcal{K}} \neq 0$ we need to assume that $\beta > 3$. If actually $\beta \ge 5$, then Lemma 2.7(2) implies that the second term in (2.52) belongs to $H^{(1/2)_-}(\mathbf{R}^3)$, and we are done. Otherwise, we repeat the argument as in case (a). We omit the details here.

To prove part (3), assume that we have ψ_j satisfying $\psi_j + R_0^+(\lambda_j)V\psi_j = 0$ with $c \le \lambda_j \le 1-c$. Since ψ_j are then eigenfunctions with eigenvalues λ_j, we may assume that the set $\{\psi_j\}$ is orthonormalized. Then (2.41) implies that $\{\psi_j\}$ is a compact subset of $\mathcal{K}$, which means that it is a finite set. This argument proves the statement (3). $\square$

Remark 2.9. Let us define

$$M^{\pm,n}_{-\gamma} = \{u \in \mathcal{K}_{-\gamma} \colon (1 + R_0^\pm(n)V)u = 0\},$$
$$\widetilde{M}^{\pm,n}_{-\gamma} = \{u \in \mathcal{K}_{-\gamma} \colon (1 + R_0^\pm(n)V)u = 0, \langle V, \psi^\pm\rangle_{\mathcal{K}} = 0\}.$$

These spaces do not depend on γ for $1/2 < \gamma < \beta/2$. Neither do they depend on the signs $\pm$, since $(1+R^\mp(\lambda)V)\psi^\pm = 0$ due to (2.50). Thus we may denote them by M^n and $\widetilde{M}^n$, respectively. We obviously have $\dim(M^n/\widetilde{M}^n) \le 1$.

We prove the converse of Lemma 2.8.

Lemma 2.10. *Let V satisfy Assumption* 1.1 *with $\beta > 2$. Then we have the following results.*
(1) *Suppose that $\lambda \notin \mathbf{Z}$ is an eigenvalue of K with eigenfunction ψ. Then $\psi \in \mathcal{K}^2_N$ for any N, and it satisfies $(1 + R_0^\pm(\lambda)V)\psi = 0$.*
(2) *Suppose that $\lambda \in \mathbf{Z}$ is an eigenvalue of K with eigenfunction ψ, and that $\beta > 3$. Then ψ satisfies $\langle V, \psi\rangle_{\mathcal{K}} = 0$ and $(1 + R_0^\pm(\lambda)V)\psi = 0$. It satisfies the properties in* (2a) *of Lemma* 2.8.

(3) *Suppose that* $\lambda \in \mathbf{Z}$ *is a threshold resonance of* K*,* ψ *is a corresponding resonant solution, and* $\beta > 3$*. Then* ψ *satisfies* $(1+R_0^{\pm}(\lambda)V)\psi = 0$*. It satisfies the properties in* (2b) *of Lemma* 2.8*.*

Proof. We compare the Fourier coefficients with respect to the t variable of both sides in $K_0\psi + V\psi - \lambda\psi = 0$. We have $(n - \Delta - \lambda)\psi_n + (V\psi)_n = 0$. Hence away from the zeros of $\xi^2 + n - \lambda$ we have

$$\hat{\psi}_n(\xi) = -\frac{\widehat{(V\psi)_n}(\xi)}{\xi^2 + n - \lambda}. \tag{2.53}$$

Suppose first that $\lambda \notin \mathbf{Z}$ and $\psi \in \mathcal{K}$. Then $V\psi \in \mathcal{K}_\beta$. When $n > \lambda$, it obviously follows that

$$\psi_n(x) = -r_0(\lambda - n)(V\psi)_n. \tag{2.54}$$

Consider now $n < \lambda$. Since $\widehat{(V\psi)_n} \in H^\beta(\mathbf{R}^3)$, the L^2-trace of $\widehat{(V\psi)_n}$ on the sphere $\xi^2 = \lambda - n$ is well-defined, and by (2.53) it has to vanish. As in the proof of the previous lemma we have

$$\hat{\psi}_n(\xi) = -\frac{\widehat{(V\psi)_n}(\xi)}{\xi^2 + n - \lambda} = -\lim_{\varepsilon\downarrow 0} \frac{\widehat{(V\psi)_n}(\xi)}{\xi^2 + n - \lambda \mp i\varepsilon}$$

or

$$\psi_n = -r_0^{\pm}(\lambda - n)(V\psi)_n. \tag{2.55}$$

The results (2.54) and (2.55) imply $(1 + R_0^{\pm}(\lambda)V)\psi = 0$. The first statement of Lemma 2.8 then implies that $\psi \in \mathcal{K}_N^2$ for any N. This proves part (1) of the lemma.

To prove part (2), it suffices to consider the case $\lambda = 0$. The argument in the proof of part (1) shows that $\psi_n = -r_0^{\pm}(\lambda - n)(V\psi)_n$ for $n \neq 0$. For $n = 0$ we have $-\Delta\psi_0 + (V\psi)_0 = 0$. Since $\xi^2/(\xi^2 \pm i\varepsilon)$ is bounded by 1 in modulus and converges to 1 as $\varepsilon \to 0$, $\xi \neq 0$, we see that, in $L^2(\mathbf{R}^3)$,

$$\hat{\psi}_0(\xi) = \lim_{\varepsilon\downarrow 0} \frac{\xi^2\hat{\psi}_0(\xi)}{\xi^2 \pm i\varepsilon} = -\lim_{\varepsilon\downarrow 0} \frac{\widehat{(V\psi)_0}(\xi)}{\xi^2 \pm i\varepsilon}. \tag{2.56}$$

Here $\widehat{(V\psi)_0}(\xi)$ is of class C^1, since we assume $\beta > 3$. Hence for the right hand side to converge in $L^2(\mathbf{R}^3)$, $-\widehat{(V\psi)_0}(0)$ has to vanish and, by virtue of (2.56), $\psi_0 = -r_0(0)(V\psi)_0$. Thus we have again $(1 + R_0^{\pm}(0)V)\psi = 0$. The second statement of Lemma 2.8 then implies that ψ has the properties stated in (2a) of that lemma.

To prove part (3), it again suffices to consider $\lambda = 0$. Let ψ be a 0-resonant solution to $Ku = 0$. Then by (1.10) there exists $C \neq 0$ such that

$\psi - C|x|^{-1} \in \mathcal{K}$. Thus $\psi_n \in L^2(\mathbf{R}^3)$ for all $n \neq 0$, and $(V\psi)\widehat{}_n \in H^{\beta-(1/2)+}(\mathbf{R}^3)$ for all n. The argument in the proof of part (1) implies that the trace of $(V\psi)\widehat{}_n(\xi)$ on the sphere $\xi^2 = \lambda - n$ vanishes for all $n < \lambda$. Hence (2.55) holds for $n \neq 0$. When $n = 0$, we have that $\hat{\psi}_0(\xi) - 4\pi|\xi|^{-2} \in L^2(\mathbf{R}^3)$ by assumption. Thus $\hat{\psi}_0 \in L^1_{\rm loc}(\mathbf{R}^3)$ and (2.56) holds in $L^1_{\rm loc}(\mathbf{R}^3)$ or $\psi_0 = -r_0(0)(V\psi)_0$. Thus we have $(1 + R_0^{\pm}(0)V)\psi = 0$ and statement (2) in Lemma 2.8 completes the proof. □

3 Threshold behavior of $R(z)$

We denote by Λ the set of non-integral eigenvalues of K. We will later show that $\Lambda \cup \mathbf{Z}$ is a discrete subset of $\mathbf{R}$, and we proceed, assuming this result. Then the $\mathcal{Y}_\delta$-valued analytic function $R(z)$ of $z \in \mathbf{C}^{\pm}$ has continuous extensions to $\overline{\mathbf{C}}^{\pm} \setminus (\Lambda \cup \mathbf{Z})$, and the equation

$$R(z) = (1 + R_0(z)V)^{-1}R_0(z) \tag{3.1}$$

is satisfied for all $z \in \overline{\mathbf{C}}^{\pm} \setminus (\Lambda \cup \mathbf{Z})$. For operators A and B, we write $A \subset B$ if A is a restriction of B. Notice that the commutator relation

$$[D_t, R_0(z)V] \subset R_0(z)\,(D_tV)$$

implies that $R_0(z)V$ is also compact in $\mathcal{K}^1_{-\delta}$, and that $-1 \notin \sigma(R_0(z)V)$ in $\mathcal{K}^1_{-\delta}$, when $z \notin \Lambda \cup \mathbf{Z}$. Since (3.1) is satisfied as an identity in $\mathcal{Y}^1_\delta$ as well, we obtained the following lemma. We write $R^{\pm}(\lambda) = R(\lambda \pm i0)$, as above.

Lemma 3.1. *Let $k \geq 0$ be an integer, and let $\delta > k + 1/2$. Then for $s = 0, 1$, the analytic function $\mathbf{C}^{\pm} \ni z \to R(z) \in \mathcal{Y}^s_\delta$ can be extended to $\overline{\mathbf{C}}^{\pm} \setminus (\Lambda \cup \mathbf{Z})$ as a $C^{\delta-(1/2)+}$ function. When $z \in \overline{\mathbf{C}}^{\pm} \setminus (\Lambda \cup \mathbf{Z})$, $R^{\pm}(z)\colon \mathcal{K}^s_\delta \to \mathcal{K}^s_{-\delta}$ are compact.*

In the following two subsections, we let k, β, δ and ε_0 be as in Theorem 1.8, viz. we assume $\beta > \max\{2k+1, 4\}$ for an integer $k \geq 1$ and set $\delta = \beta/2$ and $0 < \varepsilon < \varepsilon_0 = \min\{1, \delta - k - 1/2, \delta - 2\}$. We then study the behavior of $R(z)$, when z approaches $n \in \mathbf{Z}$. We further assume $k \geq 2$ if V is of exceptional case.

3.1 The generic case

In this subsection we prove Theorem 1.11. We assume that V is of generic type. Then Lemmas 2.8 and 2.10 imply that $-1 \notin \sigma(R_0^{\pm}(n)V)$ in $\mathcal{K}^s_{-\delta}, s = 0, 1$, for any integer $n \in \mathbf{Z}$. It follows that $R(z)$ can be extended to a neighborhood I of $\mathbf{Z}$ as a $\mathcal{Y}^s_\delta$ valued continuous function, and that (3.1) holds for all $z \in \mathbf{C}^{\pm} \cup I$. In what follows we concentrate on the +-case and $n = 0$, since other cases are either reduced to this case via the identity $R(z+n) = E_n R(z) E_n^*$ or treated entirely analogously.

We omit the variable $n = 0$ and write by using (2.9) in the form

$$1+R_0(z)V = 1+R_0^+V+\sqrt{z}D_1V+\cdots+z^{\frac{k}{2}}D_kV+\widetilde{L}_{0k}(z) \equiv L(z)+\widetilde{L}_{0k}(z),$$

where $\widetilde{L}_{0k}(z) = \widetilde{R}_{0k}(0,z)V = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ as a $B(\mathcal{K}^s_{-\delta})$-valued function in a neighborhood of 0. Define

$$G^+(0) = (1 + R_0^+(0)V)^{-1}$$

which exists by assumption. Then, for small z,

$$L(z)^{-1} = G^+(0)(1 + \sqrt{z}D_1VG^+(0) + \cdots + z^{\frac{k}{2}}D_kVG^+(0))^{-1} \tag{3.2}$$

also exists and is a $B(\mathcal{K}^s_{-\delta})$-valued analytic function of $\sqrt{z}$ near 0. Thus,

$$\begin{aligned}(1 + R_0(z)V)^{-1} = &\ (1 + L(z)^{-1}\widetilde{L}_{0k}(z))^{-1}L(z)^{-1} \\ = &\ L(z)^{-1} + \{(1 + L(z)^{-1}\widetilde{L}_{0k}(z))^{-1} - 1\}L(z)^{-1}\end{aligned}$$

and, by Lemma 2.4,

$$L_1(z) \equiv \{(1 + L(z)^{-1}\widetilde{L}_{0k}(z))^{-1} - 1\}L(z)^{-1} = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$$

as a $B(\mathcal{K}^s_{-\delta})$-valued function. Thus, expanding $L(z)^{-1}$ as a power series of $z^{1/2}$, we see that $(1 + R_0(z)V)^{-1}$ can be written as

$$(1 + R_0(z)V)^{-1} = Q_0 + z^{\frac{1}{2}}Q_1 + \cdots + z^{\frac{k}{2}}Q_k + \mathcal{O}(z^{\frac{k+\varepsilon}{2}}) \tag{3.3}$$

as a $B(\mathcal{K}^s_{-\delta})$-valued function. Inserting the expansion (2.9) for $R_0(z)$ and (3.3) into (3.1) and applying Lemma 2.4, we have, denoting $D_0 = R_0^+$,

$$\begin{aligned}R(z) = (Q_0 + z^{\frac{1}{2}}Q_1+ &\cdots + z^{\frac{k}{2}}Q_k) \\ &\times (D_0 + \sqrt{z}D_1 + \cdots + z^{\frac{k}{2}}D_k) + \mathcal{O}(z^{\frac{k+\varepsilon}{2}})\end{aligned} \tag{3.4}$$

as a $\mathcal{Y}^s_\delta$ valued function. Expanding the product in the right of (3.4) and putting all the terms with powers higher than $z^{k/2}$ into the remainder, we finally obtain

$$R(z) = F_0 + \sqrt{z}F_1 + zF_2 + \cdots + z^{k/2}F_k + \mathcal{O}(z^{\frac{k+\varepsilon}{2}}), \tag{3.5}$$

as a $\mathcal{Y}_\delta^s$ valued function. ¿From the explicit formula (2.9) and

$$1 - VG^+(0)R_0^+(0) = 1 - VR^+(0) = (1 + VR_0^+(0))^{-1} = G^-(0)^*,$$

we obtain the expressions in statement (3) of Theorem 1.11. Note that F_j are linear combination of the operators of the form

$$G^+(0)D_{i_1}VG^+(0)D_{i_2}V\cdots G^+(0)D_{i_{r-1}}VG^+(0)D_{i_r}, \quad i_1 + \cdots + i_r = j \tag{3.6}$$

and, if j is odd, one of $i_1, \dots, i_r$ must be odd. Suppose i_a is odd. Then, we may write the operator in (3.6) in the form $AD_{i_a}B$ with

$$A = G^+(0)D_{i_1}V\cdots D_{i_{a-1}}VG^+(0), \quad B = VG^+(0)D_{i_{a+1}}V\cdots VG^+(0)D_{i_r}$$

and $A \in B(\mathcal{K}_{-\delta}^1)$ and $B \in B(\mathcal{K}_\delta^1)$. Hence F_j is a finite rank operator from $\mathcal{K}_\delta^1$ to $\mathcal{K}_{-\delta}^1$. Moreover, the adjoint

$$B^* = D_{i_r}^* G^+(0)^*V\cdots VD_{i_{a+1}}^* G^+(0)^*V \in B(\mathcal{K}_{-\delta})$$

is bounded in $\mathcal{K}_{-\delta}^1$ because $G^+(0)^* = (1+VR_0^-(0))^{-1}$ is bounded in $\mathcal{K}_\delta^1$. Since D_{i_a} is of the form $\sum C_{\alpha\beta}x^\alpha \otimes y^\beta$, it follows that

$$AD_{i_a}B = \sum C_{\alpha\beta}(Ax^\alpha) \otimes (B^*y^\beta)$$

and $Ax^\alpha, B^*y^\beta \in \mathcal{K}_{-\delta}^1$. This completes the proof of Theorem 1.11.

3.2 The exceptional case

In this subsection we prove Theorem 1.12. Thus, we assume $n \in \mathbf{Z}$ is a threshold resonance and/or an eigenvalue and study the behavior of $R(z)$ as $z \to n$. As above, it suffices to consider the case $n = 0$ and $z \in \overline{\mathbf{C}}^+$. The following is an adaptation of Murata's argument [16] to the time periodic systems. We use (2.9) to write as an identity in $B(\mathcal{K}_{-\delta})$

$$1 + R_0(z)V = 1 + R_0^+(0)V + z^{1/2}D_1V + zD_2V + R_2(z)V \tag{3.7}$$

$$\equiv S(z) + R_2(z)V, \tag{3.8}$$

where we have simplified the notation by omitting the dependence on $n = 0$ and wrote $\tilde{R}_{02}(0, z) = R_2(z)$. We have $R_2(z) = \mathcal{O}(z^{\frac{2+\varepsilon}{2}})$ as a $\mathcal{Y}_\delta^s$-valued function. The operator $S(z)$ is compact in $\mathcal{K}_{-\delta}^1$, due to Lemma 2.3, and it is a polynomial in $\sqrt{z}$.

Lemma 3.2. *There exists $\rho > 0$, such that $S(z)$ is invertible in $B(\mathcal{K}_{-\delta})$ for $0 < |z| < \rho$ and $S(z)^{-1}$ has a Laurent expansion in $B(\mathcal{K}_{-\delta})$ of the form*

$$S(z)^{-1} = \sum_{j=-2}^{\infty} S_j z^{j/2}, \quad 0 < |z| < \rho. \tag{3.9}$$

The operators S_{-2}, S_{-1} are of finite rank, S_j are all bounded in $\mathcal{K}^1_{-\delta}$ and (3.9) is an expansion also in $B(\mathcal{K}^1_{-\delta})$. The adjoint S_j^* is bounded also in $\mathcal{K}^1_{\delta}$.

Proof. We first show that $S(z)$ is invertible in $\mathcal{K}_{-\delta}$ for some z. Suppose the contrary. Then, since $S(z) - 1$ is compact, there exists a sequence $u_m \in \mathcal{K}_{-\delta}$ such that $\|u_m\|_{\mathcal{K}_{-\delta}} = 1$ and $S(im^{-1})u_m = 0$. We have

$$1 = (1 - R(z)V)(1 + R_0(z)V) = (1 - R(z)V)(S(z) + O(z^{\frac{2+\varepsilon}{2}})), \quad z \notin \mathbf{R}.$$

We set $z = im^{-1}$ in this formula, apply it to u_m and take the norm in both sides. We have

$$1 \le Cm^{-\frac{2+\varepsilon}{2}}(1 + \|R(im^{-1})\|_{\mathcal{Y}_\delta}) \le Cm^{-\frac{2+\varepsilon}{2}}(1 + \|R(im^{-1})\|_{B(\mathcal{K})}) \le Cm^{-\frac{\varepsilon}{2}}.$$

This is a contradiction, and $S(z)$ is invertible for some $z \in \mathbf{C}$. Thus the analytic Fredholm theory implies that $S(z)^{-1}$ is meromorphic with respect to $\sqrt{z}$ with poles of finite order. Since $z = 0$ is a pole of $S(z)^{-1}$ by assumption, $S(z)^{-1}$ exists for all $0 < |z| < \rho$ for some $\rho > 0$, and it has an expansion $S(z)^{-1} = \sum_{j=-\ell}^{\infty} S_j z^{j/2}$ with finite rank operators $S_{-\ell}, \ldots, S_{-1}$. We next show that $\ell \ge -2$. We have from (3.8) the identity

$$S(z)^{-1} = (1 - R(z)V)(1 + R_2(z)VS(z)^{-1}). \tag{3.10}$$

If $\ell < -2$ and $S_\ell \neq 0$, then for some $u \in \mathcal{K}_{-\delta}$ with $\|u\|_{\mathcal{K}_{-\delta}} = 1$, $\|S(z)^{-1}u\| \ge C|z|^{\ell/2}$, and the right hand side in (3.10) is bounded by

$$\|(1 - R(z)V)(1 + R_2(z)VS(z)^{-1})u\| \le C(|z|^{-1} + |z|^{(\ell+\varepsilon)/2}), \quad |z| < 1,$$

or $C|z|^{\ell/2} \le (|z|^{-1} + |z|^{(\ell+\varepsilon)/2})$, which is a contradiction, since $\varepsilon > 0$, and we assume $\ell < -2$. Recall that $S(z) - 1$ is also compact in $\mathcal{K}^1_{-\delta}$. It follows $\sum_{j=-2}^{\infty} S_j z^{j/2}$ is also the expansion of $S(z)^{-1}$ in $B(\mathcal{K}^1_{-\delta})$ and, hence, S_j are bounded in $\mathcal{K}^1_{-\delta}$, $j = -2, -1, \ldots$. Since $(S(\bar{z})^{-1})^* = (S(\bar{z})^*)^{-1}$, we have

$$\sum_{j=-2}^{\infty} S_j^* z^{j/2} = (S(\bar{z})^{-1})^* = (1 + VR_0^-(0) + z^{1/2}VD_1^* + zVD_2)^{-1}.$$

Here $VR_0^-(0) + z^{1/2}VD_1^* + zVD_2$ is compact in $\mathcal{K}^1_\delta$ and is analytic with respect to $\sqrt{z}$. Thus, S_j^* are bounded in $\mathcal{K}^1_\delta$, $j = -2, -1, \ldots$. □

We now show that all coefficients in (3.9) are explicitly computable, and we then compute a few leading coefficients. We introduce the notation

$$L_0 = 1 + R_0^+(0)V, \quad L_1 = D_1 V, \quad L_2 = D_2 V,$$

so that $S(z) = L_0 + \sqrt{z}L_1 + zL_2$, see (3.7) and (3.8). The following lemma implies that S_j, $j \geq 1$, can be computed from S_{-j} and L_j, $j = 0, 1, 2$.

Lemma 3.3. *For $0 < |z| < \rho$, $S^{-1}(z)$ satisfies the identity*

$$\begin{aligned} S(z)^{-1} = {} & \frac{1}{z}S_{-2} + z^{-1/2}S_{-1} + S_0 \\ & - \left\{1 + z^{1/2}\left(S_0L_1 + S_{-1}L_2\right) + zS_0L_2\right\}^{-1} \times \\ & \quad \times \left\{z^{1/2}\left(S_0L_1S_0 + S_{-1}L_2S_0 + S_0L_2S_{-1}\right) + zS_0L_2S_0\right\}. \end{aligned} \tag{3.11}$$

Proof. Compare coefficients to $z^{j/2}$ on both sides of the identity $S(z)S(z)^{-1} = S(z)^{-1}S(z) = I$. We obtain, with the convention that $S_j = 0$ for $j \leq -3$, and with the notation $\delta_{j,k}$ for the Kronecker delta, the following identities

$$L_0S_j + L_1S_{j-1} + L_2S_{j-2} = \delta_{j,0}I, \tag{3.12}$$

$$S_jL_0 + S_{j-1}L_1 + S_{j-2}L_2 = \delta_{j,0}I, \tag{3.13}$$

for $j = -2, -1, 0, \ldots,$. Hence we have for $j = -2, -1, 0, \ldots$,

$$\begin{aligned} \delta_{j,0}S_0 + \delta_{j+1,0}S_{-1} + \delta_{j+2,0}S_{-2} &= \sum_{k=0}^{2} S_{-k}\left(L_0S_{j+k} + L_1S_{j+k-1} + L_2S_{j+k-2}\right) \\ &= S_j + (S_0L_1 + S_{-1}L_2)S_{j-1} + S_0L_2S_{j-2}. \end{aligned} \tag{3.14}$$

Multiply both sides by $z^{j/2}$ and sum up over $j \geq 1$ to obtain

$$\sum_{j=1}^{\infty} z^{j/2}S_j + z^{1/2}(S_0L_1 + S_{-1}L_2)\sum_{j=0}^{\infty} z^{j/2}S_j + zS_0L_2\sum_{j=-1}^{\infty} z^{j/2}S_j = 0,$$

or

$$\begin{aligned} & \left\{1 + z^{1/2}(S_0L_1 + S_{-1}L_2) + zS_0L_2\right\}\sum_{j=1}^{\infty} z^{j/2}S_j \\ & \qquad = -\left\{z^{1/2}(S_0L_1S_0 + S_{-1}L_2S_0 + S_0L_2S_{-1}) + zS_0L_2S_0\right\}, \end{aligned}$$

which implies (3.11). □

The next step is to compute S_j, $j = -2, -1, 0$, explicitly. We write $E_K(\cdot)$ for the spectral measure of K. We then have the following results.

Lemma 3.4. *We have $S_{-2} = -E_K(\{0\})V$.*

Proof. Set $z = is$ in (3.10), multiply both sides by is, and let $s \downarrow 0$. The left hand side obviously converges to S_{-2} in $B(\mathcal{K}_{-\delta})$. The right hand side converges to $-E_K(\{0\})V$ in the strong topology of $B(\mathcal{K}_{-\delta})$, as $(is)R(is) \to -E_K(\{0\})$ strongly in $\mathcal{K}$. □

Lemma 3.5. *We have the following results on the operators S_{-j} and L_j, $j = 0, 1, 2$.*

$$L_0S_0 + L_1S_{-1} + L_2S_{-2} = I, \tag{3.15}$$

$$S_0L_0 + S_{-1}L_1 + S_{-2}L_2 = I, \tag{3.16}$$

$$S_{-1}L_0 = S_{-2}L_0 = S_{-2}L_1 = S_{-1}L_1S_0L_0 = 0, \tag{3.17}$$

$$L_0S_{-1} = L_0S_{-2} = L_1S_{-2} = L_0S_0L_1S_{-1} = 0, \tag{3.18}$$

$$S_0L_2S_{-2} = S_{-2}L_0S_0 = S_{-1}L_2S_{-2} = S_{-2}L_2S_{-1} = 0. \tag{3.19}$$

Proof. The results (3.12) and (3.13) for $j = 0$ imply (3.15) and (3.16). Setting $j = -2$ in (3.12) and (3.13), we get $L_0S_{-2} = 0$ and $S_{-2}L_0 = 0$. Since

$$L_1u(t,x) = D_1Vu = \frac{i}{8\pi^2}\int_{\mathbf{T}\times\mathbf{R}^3} V(s,y)u(s,y)dsdy, \tag{3.20}$$

we obtain $L_1S_{-2} = S_{-2}L_1 = 0$ by virtue of Lemmas 2.8 and 2.10. Now set $j = -1$ in (3.12) and (3.13), and use $L_1S_{-2} = S_{-2}L_1 = 0$ to conclude that $L_0S_{-1} = S_{-1}L_0 = 0$. We then obtain by multiplying (3.12) and (3.13) by S_{-2} from the left and the right, respectively,

$$S_{-2}L_2S_j = S_jL_2S_{-2} = 0, \quad j \neq -2, \tag{3.21}$$

$$S_{-2}L_2S_{-2} = S_{-2}. \tag{3.22}$$

Setting $j = 0$ in (3.14), we have $(S_0L_1 + S_{-1}L_2)S_{-1} + S_0L_2S_{-2} = 0$. But $S_0L_2S_{-2} = 0$ as is shown above. It follows that

$$S_0L_1S_{-1} = -S_{-1}L_2S_{-1}. \tag{3.23}$$

Multiply both side of (3.23) by L_0 from the left, and use the fact $L_0S_{-1} = 0$. Thus $L_0S_0L_1S_{-1} = 0$ follows. We have $S_{-1}L_1S_0L_0 = 0$ similarly. □

We now introduce the notation

$$P_0 = L_0S_0, \quad P_1 = L_1S_{-1}, \quad P_2 = L_2S_{-2}, \tag{3.24}$$

$$Q_0 = S_0L_0, \quad Q_1 = S_{-1}L_1, \quad Q_2 = S_{-2}L_2. \tag{3.25}$$

Lemma 3.5 then implies the following Lemma. We omit the proof, which follows from the results in Lemma 3.5 and straightforward calculations.

Lemma 3.6. *The operators P_j and Q_j, $j = 0, 1, 2$ are projections in $B(\mathcal{K}_\delta)$, which satisfy*

$$P_i P_j = \delta_{i,j} P_j, \quad i, j = 0, 1, 2, \tag{3.26}$$
$$Q_i Q_j = \delta_{i,j} Q_j, \quad i, j = 0, 1, 2, \tag{3.27}$$
$$P_0 + P_1 + P_2 = I, \tag{3.28}$$
$$Q_0 + Q_1 + Q_2 = I. \tag{3.29}$$

Lemma 3.7. *We have the following results. We write $M = M^0$, see Remark* 2.9.
(1) *If* 0 *is a threshold resonance, then S_{-1} is an operator of rank one. It can be written in the form $-4\pi i\langle\cdot, V\psi\rangle\psi$, where $\psi \in M$ is the resonant function, which is uniquely determined by the conditions*

$$\langle V, \psi\rangle = 1, \quad \langle D_2 V\phi, V\psi\rangle = 0, \quad \textit{for all } \phi \in \ker_{L^2}(K).$$

(2) *If* 0 *is not a threshold resonance, then $S_{-1} = 0$.*
(3) *For odd $j \geq 1$, S_j is of finite rank. It can be written in the form*

$$(S_j u)(t, x) = \sum_{\nu=1}^{n_j} p_{j\nu}(t, x) \int_{\mathbf{T}\times\mathbf{R}^3} q_{j\nu}(s, y) u(s, y) ds dy \tag{3.30}$$

where $n_j < \infty$ and $p_{j\nu}, q_{j\nu} \in \mathcal{K}^1_{-\delta}$ for $k = 1, \ldots, n_j$.

Proof. Set $j = 0$ in (3.13), and multiply both sides by S_{-1} from the right. Then (3.18) and (3.19) imply

$$S_{-1} = S_0 L_0 S_{-1} + S_{-1} L_1 S_{-1} + S_{-2} L_2 S_{-1} = S_{-1} L_1 S_{-1}. \tag{3.31}$$

Thus $\operatorname{rank} S_{-1} \leq \operatorname{rank} L_1 = 1$. Note that we have $u = S_{-1} L_1 u + S_{-2} L_2 u$, if $u \in M(= \ker_{\mathcal{K}_{-\delta}} L_0)$. Since $\operatorname{Ran} S_{-2} \subset M$ and $\operatorname{Ran} S_{-1} \subset M$, we have

$$\operatorname{Ran} S_{-1} \dot{+} \operatorname{Ran} S_{-2} = M$$

from Lemma 3.6. Here $\dot{+}$ denotes (nonorthogonal) direct sum. It follows from Lemmas 2.8 and 3.4 that $\operatorname{rank} S_{-1}$=1, if 0 is a threshold resonance, and $S_{-1} = 0$ otherwise.

Suppose now that 0 is a threshold resonance. Set $\bar{Q}_0 = S_{-1} R_0^+(0)$. Then $S_{-1} L_0 = 0$ implies $S_{-1} = -\bar{Q}_0 V$, and hence $\operatorname{rank} S_{-1} = \operatorname{rank} \bar{Q}_0$, and furthermore $\bar{Q}_0(1 + V R_0^+(0)) = 0$. Write

$$\bar{Q}_0 = 4\pi i \psi_+ \otimes \psi_- \quad \text{so that} \quad S_{-1} = -4\pi i \psi_+ \otimes V\psi_-.$$

Then $L_0 S_{-1} = 0$ implies $L_0\psi_+ = 0$, and therefore $\psi_+ \in M$. Also $\bar{Q}_0(1 + VR_0^+(0)) = 0$ implies $(1 + VR_0^+(0))^*\psi_- = (1 + R_0^-(0)V)\psi_- = 0$, and hence $\psi_- \in M$. Moreover, the identity (3.31) implies

$$S_{-1} = S_{-1}L_1S_{-1} = \langle V, \psi_+\rangle\langle V, \psi_-\rangle S_{-1}.$$

Since $S_{-1} \neq 0$, $\psi_\pm$ are resonance solutions, and

$$\langle V, \psi_+\rangle\langle V, \psi_-\rangle = 1. \tag{3.32}$$

Moreover,

$$P_1P_2 = L_1S_{-1}L_2E_K(\{0\})V = 0, \quad \text{and} \quad Q_2Q_1 = E_K(\{0\})VL_2S_{-1}L_1 = 0,$$

respectively, imply

$$\langle L_2\phi, V\psi_-\rangle = 0, \quad \langle D_2^*V\phi, V\psi_+\rangle = 0 \tag{3.33}$$

for all $\phi \in E_K(\{0\})\mathcal{K}$. Since $\operatorname{rank} P_2 = \operatorname{rank} Q_2 = \dim M - 1$, the condition (3.33) determines $\psi_\pm \in M^\pm$ up to scalar factors. However, as the actions of $L_2 = D_2V$ and D_2^*V are identical on ϕ, since the trace of $(V\phi)\widehat{\ }_n$ on the sphere $|\xi| = \sqrt{-n}$ vanishes, as was seen in the proof of Lemma 2.8. Thus we may choose $\psi_+ = \psi_-$, and set $\langle V, \psi\rangle = 1$, so that (3.32) is satisfied.

If we write $T_1 = S_0L_1 + S_{-1}L_2$, $T_2 = S_0L_2$, $\tilde{T}_1 = S_0L_1S_0 + S_{-1}L_2S_0 + S_0L_2S_{-1}$ and $\tilde{T}_2 = S_0L_2S_0$. Then (3.11) implies that S_j, $j \geq 1$ is a linear combination of

$$T_{i_1}\cdots T_{i_m}\tilde{T}_r, \quad i_1 + \cdots + i_m + r = j$$

Since $\operatorname{rank} T_1 \leq 2$ and $\operatorname{rank}\tilde{T}_1 \leq 3$, this shows that $\operatorname{rank} S_j$ is finite, if j is odd. Moreover, by using the concrete expression $L_1u = c\langle V, u\rangle$ and $S_1u = \langle u, V\psi\rangle\psi$ and the facts that L_i and L_i^*, $i = 1, 2$ and S_i and S_i^*, $i = -2, -1, 0$ are bounded in $\mathcal{K}^1_{-\delta}$, we see that S_j is of the form (3.30), if j is odd, as in the last part of Subsection 3.1. □

We have now determined S_{-1} and S_{-2} explicitly, and we want to show how S_0 is determined from (3.12). Write $X_j = P_j\mathcal{K}_{-\delta}$, $j = 0, 1, 2$. Then Lemma 3.6 implies the direct sum decomposition

$$\mathcal{K}_{-\delta} = X_0 \dot{+} X_1 \dot{+} X_2.$$

As $S_0P_2 = S_0L_2S_{-2} = 0$ by (3.24) and (3.19), S_0 acts on X_2 trivially. Recall (3.23): $S_0P_1 = -S_{-1}L_2S_{-1}$. Thus on X_1 we define $S_0u = -S_{-1}L_2S_{-1}v$, if $u = P_1v$. On X_0, we define S_0 as follows. Multiplying (3.16) by L_0 from the left, we have $L_0S_0L_0 = L_0$. Hence $X_0 = \mathrm{Ran}\, L_0$. Moreover, $\ker S_0 \cap \mathrm{Ran}\, L_0 = \{0\}$ and $L_0S_0 = I$ on $\mathrm{Ran}\, L_0 = X_0$, and S_0 is the right inverse of L_0.

We now show that $R(z)$ has the expansion as in (1.20). We write

$$R(z) = S(z)^{-1}\left(1 + R_2(z)VS(z)^{-1}\right)^{-1} R_0(z). \tag{3.34}$$

Since $\|R_2(z)VS(z)^{-1}\|_{B(\mathcal{K}^s_{-\delta})} = O(|z|^{\varepsilon/2})$, we may expand the second factor on the right by Neumann series and obtain

$$\begin{aligned} R(z) &= \sum_{j=0}^{\infty} S(z)^{-1}(-R_2(z)VS(z)^{-1})^j R_0(z) \\ &= \left(\sum_{j=0}^{N} + \sum_{j=N+1}^{\infty}\right)(-S(z)^{-1}R_2(z)V)^j S(z)^{-1}R_0(z). \end{aligned} \tag{3.35}$$

Here, because $R_2(z)VS(z)^{-1}$ is $C^{(k)+}$ outside $z = 0$ and it satisfies the estimates $(d/dz)^j R_2(z)VS(z)^{-1} = O(z^{\frac{\varepsilon}{2}-j})$, $j = 0, \ldots, k$, the second sum on the right will become, if N is taken sufficiently large, a C^k function in a neighborhood of $z = 0$ (including $z = 0$) with vanishing derivatives at $z = 0$ up to the order $\le k$. Thus, we may ignore the second sum from our consideration.

We first show that the summand with $j = 0$, $S(z)^{-1}R_0(z)$, may be expanded in the powers of $\sqrt{z}$ starting from z^{-1} up to the order $z^{\frac{k-1}{2}}$ as a $\mathcal{Y}^s_\delta$-valued function, $s = 0, 1$, as follows:

$$S(z)^{-1}R_0(z) = z^{-1}E_K(\{0\}) + \cdots + z^{\frac{k-1}{2}}W_{k-1} + \mathcal{O}(z^{\frac{k-1+\varepsilon}{2}}) \tag{3.36}$$

To see this, we replace $S(z)^{-1}$ by its expansion (3.9). By virtue of Lemma 2.4 and (2.9), the part $(z^{-1/2}S_{-1} + S_0 + \cdots)R_0(z)$ has an expansion of the desired form starting from a term with $z^{-1/2}$. For the part $z^{-1}S_{-2}R_0(z)$, $S_{-2} = -E_K(\{0\})V$, we write $E_K(\{0\}) = \sum \phi_j \otimes \phi_j$ by using the orthonormal system of eigenfunctions. We have $V\phi_j \in \mathcal{K}^2_{\beta+(1/2)_-}$ and $\sum_m \langle m\rangle \|(V\phi_j)_m\|_{\mathcal{H}_{\beta+(1/2)_-}} < \infty$ by virtue of Theorem 1.2, and, by virtue of Lemma 2.10 (2), the zero mode of $V\phi_j$ satisfies $\int (V\phi_j)_0(x)dx = \int V\phi_j dxdt = 0$. It follows, by applying Lemma 2.2 (2) for the zero mode and Lemma 2.1 for $m \neq 0$ modes, that

$$\begin{aligned} z^{-1}S_{-2}R_0(z) &= -z^{-1}\sum_j \phi_j \otimes (R_0(z)^* V\phi_j) \\ &= -z^{-1}\sum_j \phi_j \otimes (\sum_m e^{imt} r_0(z-m)^*(V\phi_j)_m) \end{aligned}$$

can be expanded as in (3.36) with $-z^{-1}E_K(\{0\})VR_0^{\pm}(0) = z^{-1}E_K(\{0\})$ as the leading term.

The same argument shows that $S(z)^{-1}R_2(z)V$ and $z^{-1}R_2(z)VE_K(\{0\})$, as $B(\mathcal{K}^s_{-\delta})$-valued functions, can be expanded in the forms

$$S(z)^{-1}R_2(z)V = z^{\frac{1}{2}}\tilde{W}_1 + \cdots + z^{\frac{k-1}{2}}\tilde{W}_{k-1} + \mathcal{O}(z^{\frac{k-1+\varepsilon}{2}}). \tag{3.37}$$

$$z^{-1}R_2(z)E_K(\{0\}) = z^{\frac{1}{2}}\tilde{Y}_1 + \cdots + z^{\frac{k-1}{2}}\tilde{Y}_{k-1} + \mathcal{O}(z^{\frac{k-1+\varepsilon}{2}}). \tag{3.38}$$

We next show that the summand with $j = 1$, $S(z)^{-1}R_2(z)V \cdot S(z)^{-1}R_0(z)$, has an expansion of the following form as a $\mathcal{Y}^s_\delta$-valued function, $s = 0, 1$:

$$S(z)^{-1}R_2(z)V \cdot S(z)^{-1}R_0(z) = z^{-\frac{1}{2}}Y_{-1} + \cdots + z^{\frac{k-2}{2}}Y_{k-2} + \mathcal{O}(z^{\frac{k-2+\varepsilon}{2}}) \tag{3.39}$$

By virtue of (3.36) and (3.37), it suffices to show that $z^{-1}S(z)^{-1}R_2(z)V \cdot E_K(\{0\})$ has desired expansion. We again replace $S(z)^{-1}$ by (3.9). Then, by virtue of Lemma 2.4 and (3.38), the part $z^{-1}(z^{-1/2}S_{-1} + S_0 + \cdots)R_2(z)V \cdot E_K(\{0\})$ has the expansion of the form (3.39) and we have only to examine $z^{-2}S_{-2}R_2(z)VE_K(\{0\})$, which may be written as

$$z^{-2}E_K(\{0\})V(R_0(z) - R_0^+(0) - \sqrt{z}D_1(0) - zD_2(0))VE_K(\{0\}).$$

Because eigenfunctions ϕ_j satisfy the properties mentioned above, Lemma 2.1 and Lemma 2.2 imply that the right hand side may be expanded in the form

$$z^{-1/2}X_{-1} + \cdots + z^{\frac{2k-3}{2}}X_{2k+1} + \mathcal{O}(z^{\frac{2k-3+\varepsilon}{2}}).$$

The expansion (3.39) follows since $2k - 3 \geq k - 2$ when $k \geq 1$.

Lemma 2.4 together with (3.37) and (3.39) implies that for any $j \geq 2$

$$(S(z)^{-1}R_2(z)V)^j S^{-1}(z)R_0(z) = z^{\frac{j-2}{2}}Y_{-1,j} + \cdots + z^{\frac{k-2}{2}}Y_{k-2,j} + \mathcal{O}(z^{\frac{k-2+\varepsilon}{2}}). \tag{3.40}$$

Combination of (3.36), (3.39) and (3.40) implies that, as a $\mathcal{Y}^s_\delta$-valued function, $s = 0, 1$, $R(z)$ has the expansion of the desired form

$$R(z) = F_{-2}z^{-1} + F_{-1}z^{-1/2} + F_0 + \cdots + z^{(k-2)/2}F_{k-2} + \mathcal{O}(z^{(k-2)/2+\varepsilon}). \tag{3.41}$$

Here, as the computations above show, F_j are linear combinations of

$$S_{i_0}D_{i_1}VS_{i_1}D_{i_2}VS_{i_2}\cdots D_{i_m}VS_{i_m}D_{i_{m+1}}, \quad i_0 + \cdots + i_{m+1} = j, \tag{3.42}$$

and if j is odd, one of $i_r, 0 \leq r \leq m+1$ is odd. Since S_j and D_j are of finite rank if j is odd, F_j is also finite rank if j is odd. Moreover, exactly the same argument used for proving (3.30) shows that F_j has the expression as in statement (2), when j is odd.

For reference, we compute the first three terms of (3.41) of the expansions in $\mathcal{Y}^j_\delta$, $\delta > 5/2$ are given by

$$F_{-2} = -E_K(\{0\}), \tag{3.43}$$

$$F_{-1} = E_K(\{0\})VD_3VE_K(\{0\}) - 4\pi i(\psi \otimes \psi) \tag{3.44}$$

$$\begin{aligned} F_0 &= S_{-2}D_2 + S_{-1}D_1 + [S_0 - S_{-2}\{D_3VS_{-1} \\ &\quad +D_4VS_{-2} - (D_3VS_{-2})^2\} - S_{-1}D_3VS_{-2}]\, R_0^+(0). \end{aligned} \tag{3.45}$$

Here we have used the fact $S_{-2}D_1 = E_K(\{0\})VD_1 = 0$ to eliminate a few terms, together with the results $S_{-2}R_0^+(0) = E_K(\{0\})VR_0^+(0) = -E_K(\{0\})$ and $S_{-1}R_0^+(0) = -4\pi i(\psi \otimes \psi)$. This completes the proof of Theorem 1.12.

Completion of the proof of Theorem 1.2 The argument above shows, in particular, that if n is an eigenvalue or threshold resonance of K, then $1 + R_0(z)V$ is invertible, if z is sufficiently close to n in the closed upper plane. Since this is true including $z = n$, if $1 + R_0^+(n)V$ is invertible, we see in all cases that there are no eigenvalues of K in a neighborhood of n, except possibly n itself. As the eigenvalues of K are discrete outside $\mathbf{Z}$, we conclude that they are discrete in $\mathbf{R}$. This completes the proof of Theorem 1.2.

3.3 $R(z)$ near non-integral eigenvalues

On the behavior of $R(z)$ at non-integral eigenvalue λ, we have the following lemma. Parameters satisfy $\delta = \beta/2$, $\beta > \beta_k \equiv \max\{2k+1, 4\}$ for $k \in \mathbf{N}$, $s = 0, 1$ and $\varepsilon_0 = \min\{1, \frac{\beta-\beta_k}{2}\}$ as previously and we assume $V \in \mathcal{V}_\beta$.

Lemma 3.8. *Let $\lambda \in \mathbf{R} \setminus \mathbf{Z}$ be an eigenvalue of K. Then, as a $\mathcal{Y}^s_\delta$-valued function of z in a neighborhood of λ in $\overline{\mathbf{C}}^\pm \setminus \{0\}$, $R(z+\lambda)$ has the following expansion as $z \to 0$ for any $0 < \varepsilon < \varepsilon_0$:*

$$R(z+\lambda) = \frac{P_K(\{\lambda\})}{-z} + \tilde{R}^\pm(\lambda) + zR_1^\pm(\lambda) + \cdots + z^kR_k^\pm(\lambda) + O(z^{k+\varepsilon}), \tag{3.46}$$

where $O(z^{k+\varepsilon})$ is $C^{k+\varepsilon}$ and has vanishing derivatives up to the order k at $z = 0$, and $\tilde{R}(\lambda) = \lim_{z\to\lambda}(z-\lambda)R(z)$ is the so-called reduced resolvent.

Proof. We follow the argument in the proof of Theorem 1.12 and we shall be sketchy here. We set $\tilde{S}(z) = 1 + R_0^\pm(\lambda)V + zR_0^{\pm'}(\lambda)V$ where $R_0^{\pm'}(\lambda)$ is the derivative of $R_0^\pm(\lambda)$ with respect to λ. $\tilde{S}(z) - 1$ is a compact operator in $\mathcal{K}^s_{-\delta}$,

$s = 0, 1$ and the argument as in the proof of Lemma 3.2 shows that $\tilde{S}(z)^{-1}$ has the Laurent expansion of the following form in $B(\mathcal{K}^s_{-\delta})$:

$$\tilde{S}(z)^{-1} = z^{-1}\tilde{S}_{-1} + \tilde{S}_0 + z\tilde{S}_1 + \cdots, \quad \tilde{S}_{-1} = PV$$

where $P = E_K(\{\lambda\})$ is the eigenprojection. We define

$$R_3^{\pm}(z, \lambda) = R_0^{\pm}(z + \lambda) - R_0^{\pm}(\lambda) - zR_0^{\pm'}(\lambda).$$

Since $\|R_3^{\pm}(z, \lambda)V\tilde{S}(z)^{-1}\|_{B(\mathcal{K}^s_{-\delta})} \le C|z|$ for small $|z|$, we may expand $(1 + R_3^{\pm}(z, \lambda)V\tilde{S}(z)^{-1})^{-1}$ by Neumann series and obtain the following expression for $R(z + \lambda)$ near $z = 0$, $z \in \overline{\mathbf{C}}\pm$:

$$\begin{aligned}(1+&R_0^{\pm}(z + \lambda)V)^{-1}R_0^{\pm}(\lambda + z)\\ &= \tilde{S}(z)^{-1}(1 + R_3^{\pm}(z, \lambda)V\tilde{S}(z)^{-1})^{-1}R_0^{\pm}(\lambda + z)\\ &= \left(\sum_{j=0}^{N} + \sum_{j=N+1}^{\infty}\right)(-\tilde{S}(z)^{-1}R_3^{\pm}(z, \lambda)V)^j\tilde{S}(z)^{-1}R_0^{\pm}(\lambda + z)\end{aligned} \tag{3.47}$$

If N is taken sufficiently large, the sum $\sum_{N+1}^{\infty}(\cdots)$ becomes a $\mathcal{Y}^s_\delta$-valued $C^{k+\varepsilon}$ (including $z = 0$) function of z as previously. We have $(1 + R_0^{\pm}(\lambda)V)P = P(1 + VR_0^{\pm}(\lambda)) = 0$. This and the resolvent equation yield

$$PV(R_0^{\pm}(\lambda + z) - R_0^{\pm}(\lambda)) = -zPR_0^{\pm}(\lambda + z). \tag{3.48}$$

Recall that eigenfunctions decays rapidly at infinity. Differentiating (3.48) by z and setting $z = 0$, we have $PVR_0^{\pm'}(\lambda) = -PR_0^{\pm}(\lambda)$. It follows that

$$z^{-1}\tilde{S}_{-1}R_3^{\pm}(z, \lambda)V = -P(R_0^{\pm}(z + \lambda) - R_0^{\pm}(\lambda))V, \tag{3.49}$$

$$z^{-1}\tilde{S}_{-1}R_0^{\pm}(z + \lambda) = -z^{-1}P - PR_0^{\pm}(z + \lambda). \tag{3.50}$$

Thus the summand with $j = 0$ in (3.47) has the expansion as in the desired form (3.46). We next show that all terms in (3.47) with $j \ge 1$ have expansions of the form

$$Y_0 + zY_1 + \cdots + z^kY_k + O(z^{k+\varepsilon}) \tag{3.51}$$

with the same meaning for $O(z^{k+\varepsilon})$ as in (3.46). We define $T(z) = \tilde{S}(z)^{-1} - z^{-1}\tilde{S}_{-1}$. Then, $T(z)$ is a $B(\mathcal{K}^s_{-\delta})$-valued analytic function and (3.49) implies that $\tilde{S}(z)^{-1}R_3^{\pm}(z, \lambda)V = (z^{-1}\tilde{S}_{-1} + T(z))R_3^{\pm}(z, \lambda)V$ has the expansion in the form (3.51) as a $B(\mathcal{K}^s_{-\delta})$-valued function (with $Y_0 = 0$). Thus, if we show that the summand with $j = 1$ has an expansion of the form (3.51), we are done. To see that this is indeed the case, we write

$$\tilde{S}(z)^{-1}R_3^{\pm}(z, \lambda)V\tilde{S}(z)^{-1}R_0^{\pm}(z + \lambda)$$

$$= z^{-2}\tilde{S}_{-1}R_3^{\pm}(z,\lambda)V\tilde{S}_{-1}R_0^{\pm}(z+\lambda) + z^{-1}\tilde{S}_{-1}R_3^{\pm}(z,\lambda)VT(z)R_0^{\pm}(z+\lambda)$$
$$+ z^{-1}T(z)R_3^{\pm}(z,\lambda)V\tilde{S}_{-1}R_0^{\pm}(z+\lambda) + T(z)R_3^{\pm}(z,\lambda)VT(z)R_0^{\pm}(z+\lambda).$$

Then, by virtue of (3.49) and the analyticity of $T(z)$, all terms on the right except the first may be expanded as in (3.46). We may write the first term on the right in the following form by using (3.48):

$$-Pz^{-1}(R_0^{\pm}(z+\lambda) - R_0^{\pm}(\lambda))VPVR_0^{\pm}(z+\lambda) = PR_0^{\pm}(z+\lambda)PVR_0^{\pm}(z+\lambda)$$

and this has the desired expansion by virtue of Lemma 2.3. This proved the Lemma. □

4 Proof of the main theorems

In this section we prove the main Theorem 1.8 for $t > 0$. The case $t < 0$ can be treated similarly. We write $\mathcal{Y}_\delta^s = B(\mathcal{K}_\delta^s, \mathcal{K}_{-\delta}^s)$ as above, $s = 0, 1$. By the spectral theorem $e^{-i\sigma K}$, $\sigma > 0$, can be written in terms of the upper boundary value of the resolvent:

$$e^{-i\sigma K}u = \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\frac{1}{2\pi i}\int_{-N}^{N} e^{-i\sigma\lambda}R(\lambda+i\varepsilon)u\,d\lambda, \tag{4.1}$$

where the right hand side should be understood as a weak integral.

We let $u = Ju_0$, $u_0 \in L_\delta^2(\mathbf{R}^3)$. Via the second resolvent equation, we get

$$R(z) = R_0(z) - M(z) + (1 + R_0(z)V)^{-1}N(z).$$

Here we wrote $M(z) = R_0(z)VR_0(z)$ and $N(z) = R_0(z)VR_0(z)VR_0(z)$ as in Lemma 2.5. Insert this for $R(\lambda+i\varepsilon)$ in the right hand side of (4.1) and write $e^{-i\sigma K}Ju_0$ as $I_0(\sigma)u_0 + I_1(\sigma)u_0 + I_2(\sigma)u_0$, where

$$I_0(\sigma)u_0 = \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\frac{1}{2\pi i}\int_{-N}^{N} e^{-i\sigma\lambda}R_0(\lambda+i\varepsilon)Ju_0\,d\lambda,$$
$$I_1(\sigma)u_0 = -\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\frac{1}{2\pi i}\int_{-N}^{N} e^{-i\sigma\lambda}M(\lambda+i\varepsilon)Ju_0\,d\lambda,$$
$$I_2(\sigma)u_0 = \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\frac{1}{2\pi i}\int_{-N}^{N} e^{-i\sigma\lambda}(1 + R_0(\lambda+i\varepsilon)V)^{-1}N(\lambda+i\varepsilon)Ju_0\,d\lambda.$$

We study $I_0(\sigma)$, $I_1(\sigma)$, and $I_2(\sigma)$ separately, as they converge for different reasons. Throughout the proofs always assume at least $\delta = \beta/2$, $\beta > \beta_k = \max\{2k+1, 4\}$ and $k \geq 1$, and we assume $V \in \mathcal{V}_\beta$.

We use the following two well known results.

Lemma 4.1. *Let $\chi \in C_0^\infty(\mathbf{R})$ be even, and assume $\chi(\lambda) = 1$ near $\lambda = 0$. Then for $n = -1, 0, 1, \dots,$ and for all N, we have*

$$h_n(\sigma) = \frac{1}{2\pi i}\int_{\mathbf{R}} e^{-i\sigma\lambda}\chi(\lambda)\lambda^{n/2}d\lambda = C_n\sigma^{-\frac{n+2}{2}} + O(\sigma^{-N}) \tag{4.2}$$

as $\sigma \to \infty$, where

$$C_n = \begin{cases} 0, & \text{for } n = 0, 2, 4, \dots, \\ \dfrac{e^{-3\pi i/4} n!!}{(2i)^{\frac{n+1}{2}}\sqrt{\pi}} & \text{for } n = -1, 1, 3, \dots. \end{cases} \tag{4.3}$$

Here $n!! = n(n-2)\cdots 1$ for $n \geq 1$ and odd, and $(-1)!! = 1$.

Proof. When n is even, integration by parts implies $h_n(\sigma) = O(\sigma^{-N})$. When n is odd, we write

$$h_n(\sigma) = \frac{1}{2\pi i}\int_0^\infty e^{-i\sigma\lambda}\chi(\lambda)\lambda^{n/2}d\lambda + \frac{e^{i\pi n/2}}{2\pi i}\int_0^\infty e^{i\sigma\lambda}\chi(\lambda)\lambda^{n/2}d\lambda,$$

make a change of variable $\lambda \to \lambda^2$, and rewrite in the form

$$h_n(\sigma) = \frac{1}{2\pi i}\int_{\mathbf{R}} \left(e^{-i\sigma\lambda^2} + e^{i\sigma\lambda^2 + i\pi n/2}\right)\chi(\lambda^2)\lambda^{n+1}d\lambda.$$

We first apply integration by parts $j = (n+1)/2$ times by using

$$\frac{1}{\pm 2i\sigma\lambda}\frac{d}{d\lambda}e^{\pm i\sigma\lambda^2} = e^{\pm i\sigma\lambda^2} \tag{4.4}$$

to see that

$$h_n(\sigma) = \frac{n!!}{2\pi i(2i\sigma)^j}\int_{\mathbf{R}} e^{-i\sigma\lambda^2}\chi(\lambda^2)d\lambda + \frac{n!!}{2\pi i(-2i\sigma\lambda)^j}\int_{\mathbf{R}} e^{i\sigma\lambda^2 + i\pi n/2}\chi(\lambda^2)d\lambda + O(\sigma^{-N}).$$

We then use well known results for the Gauss integral to complete the proof. □

Lemma 4.2. (1) *Let X be a Banach space and let $f \in L^1(\mathbf{R}, X)$ satisfy*

$$\int_{\mathbf{R}} \|f(x+h) - f(x)\|dx \leq Ch^\varepsilon, \quad 0 < h < 1$$

for some $0 < \varepsilon \leq 1$. Then, $\|\hat{f}(\lambda)\| \leq C\lambda^{-\varepsilon}$ for $\lambda > 1$.
(2) *Let $f = \mathcal{O}(z^{\frac{k+\varepsilon}{2}})$ has compact support. Then, $\|\hat{f}(\lambda)\| \leq C\lambda^{-\frac{k+2+\varepsilon}{2}}$ for $\lambda > 1$.*

Proof. We have for $0 < h < 1$

$$\|(e^{ih\lambda} - 1)\hat{f}(\lambda)\| \leq \frac{1}{2\pi}\Big\| \int_{\mathbf{R}} e^{-i\lambda x}(f(x+h) - f(x))dx\Big\| \leq Ch^{\varepsilon}.$$

When $\lambda > 1$, set $h = \lambda^{-1}$. It follows that $\|\hat{f}(\lambda)\| \leq Ch^{\varepsilon}/\sin(1/2)$. For proving (2), we first perform integration by parts $\ell = [(k+2)/2]$ times

$$\hat{f}(\lambda) = \frac{1}{2\pi(i\lambda)^{\ell}} \int_{\mathbf{R}} e^{-i\lambda x} f^{(\ell)}(x)dx$$

and then apply part (1). □

The term $I_0(\sigma)$. As is well known we have

$$I_0(\sigma)Ju_0(x) = \frac{1}{(2\pi i\sigma)^{3/2}} \int e^{\frac{i(x-y)^2}{2\sigma}} u_0(y)dy$$

and we immediately obtain by expanding the exponential into power series

$$I_0(\sigma)Ju_0 = \sigma^{-3/2}C_1Jg_1u_0 + \cdots + \sigma^{-(k+2)/2}\varepsilon_k C_k J g_k u_0 + E_k^0(\sigma)u_0, \quad (4.5)$$

where $\varepsilon_j = 0$, when j is even, and $\varepsilon_j = 1$, when j is odd and

$$\|E_k^0(\sigma)\|_{B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})} \leq C\langle\sigma\rangle^{-\frac{k+2+\varepsilon}{2}}. \quad (4.6)$$

The term $I_1(\sigma)$. For this term we use Lemma 2.5. Choose a partition of unity of the following form: $\chi \in C_0^\infty(\mathbf{R})$, χ even, and

$$\sum_{n=-\infty}^{\infty} \chi(\lambda - n) = 1, \quad \chi(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq 1/4, \\ 0 & \text{if } |\lambda| \geq 3/4. \end{cases} \quad (4.7)$$

Since $R_0^+(\lambda)VR_0^+(\lambda)Ju_0$ and its derivative satisfy estimates (2.23), (2.27) and (2.29) of Lemma 2.5, $\frac{d}{d\lambda}R_0^+(\lambda)VR_0^+(\lambda)Ju_0$ is absolutely integrable in $\mathcal{K}^1_{-\delta}$ and $I_1(\sigma)u_0$ can be written in the form

$$\begin{aligned} I_1(\sigma)u_0 &= \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-i\sigma\lambda}\frac{d}{d\lambda}(R_0^+(\lambda)VR_0^+(\lambda))Ju_0 d\lambda \\ &= \frac{1}{2\pi\sigma}\sum_{n=-\infty}^{\infty} e^{-in\sigma}\int_{\mathbf{R}} e^{-i\sigma\lambda}\chi(\lambda)\frac{d}{d\lambda}(R_0^+(\lambda+n)VR_0^+(\lambda+n))Ju_0 d\lambda. \quad (4.8)\end{aligned}$$

We then insert (2.25) for $R_0^+(\lambda+n)VR_0^+(\lambda+n)$ and apply Lemma 4.1. We obtain

$$I_1(\sigma)u_0 = \sum_{n\in\mathbf{Z}}\left(\sum_{j=1}^{k}\sigma^{-(j+2)/2}e^{-in\sigma}\varepsilon_j C_j M_j(n)Ju_0 + E_k^1(\sigma,n)\right). \tag{4.9}$$

Since $\widetilde{M}_k(n,z)$ satisfies (2.29), Lemma 4.2 implies that the remainder

$$E_k^1(\sigma,n) = \frac{1}{2\pi\sigma}e^{-in\sigma}\int_{\mathbf{R}} e^{-i\sigma\lambda}\chi(\lambda)\frac{d}{d\lambda}\widetilde{M}_k(n,\lambda)d\lambda$$

satisfies $\|E_k^1(\sigma,n)\|_{B(\mathcal{H}_\delta,\mathcal{K}^1_{-\delta})} \le C\langle n\rangle^{-\frac{3}{2}}\langle\sigma\rangle^{-\frac{k+2+\varepsilon}{2}}$, $n\in\mathbf{Z}$. Thus, for $E_k^1(\sigma) = \sum_n E_k^1(\sigma,n)$, we have

$$\|E_k^1(\sigma)\|_{B(\mathcal{H}_\delta,\mathcal{K}^1_{-\delta})} \le C\langle\sigma\rangle^{-\frac{k+2+\varepsilon}{2}}. \tag{4.10}$$

Note also $\sum_{n=-\infty}^{\infty}\|M_j(n)Ju_0\|_{\mathcal{K}^1_{-\delta}} \le C\|u_0\|_{\mathcal{H}_d}$ by (2.27).

We treat $I_2(\sigma)$ separately for the generic case and for the exceptional case. We need the following lemma.

Lemma 4.3. *Suppose that $B = \sum_{j=1}^N f_j\otimes g_j \in \mathcal{Y}^1_\delta$ is of finite rank and $f_j, g_j \in \mathcal{K}^1_{-\delta}$, $j = 1,\dots,n$. Let $Z(\sigma)u_0 = \sum_{n=-\infty}^{\infty} e^{-in\sigma}E_nBE_n^*Ju_0$, $u_0\in\mathcal{H}_\delta$. Then, $Z(\sigma)$ is an integral operator with the kernel $2\pi\sum_{j=1}^N f_j(t,x)g_j(t-\sigma,y)$.*

Proof. By the Fourier inversion formula

$$\begin{aligned} Z(\sigma)u_0 =& \sum_{j=1}^{n} f_j(t,x)\sum_{n=-\infty}^{\infty} e^{in(t-\sigma)}\int_{\mathbf{T}} e^{-ins}\left(\int_{\mathbf{R}^3} g_j(s,y)u_0(y)dy\right)ds \\ =& 2\pi\sum_{j=1}^{n} f_j(t,x)\int_{\mathbf{R}^3} g_j(t-\sigma,y)u_0(y)dy \end{aligned} \tag{4.11}$$

and the lemma follows. □

Completion of the proof, generic case. Assume V is generic and that non-integral eigenvalues are absent for K. We will comment on the necessary modifications to accommodate non-integral eigenvalues at the end of the proof. We write $R_1(z) = (1+R_0(z)V)^{-1}N(z)$. The integral

$$I_2(\sigma)Ju_0 = \frac{1}{2\pi i}\int e^{-i\sigma\lambda}R_1^+(\lambda)Ju_0 d\lambda \tag{4.12}$$

is absolutely convergent in $\mathcal{K}^1_{-\delta}$ by virtue of Lemma 2.5, and, using the partition of unity (4.7), we may write as above

$$I_2(\sigma)Ju_0 = \frac{1}{2\pi i}\sum_{n\in\mathbf{Z}} e^{-i\sigma n}\int e^{-i\sigma\lambda}\chi(\lambda)R_1^+(\lambda+n)Ju_0 d\lambda. \tag{4.13}$$

We then expand $R_1(z+n)$ as $z \to 0$ as in the proof of Theorem 1.11 by using (2.26) and (3.3). Then, (2.28) and (2.30) implies that $R_1(z+n)J$ may be written as

$$R_1(z+n)J = W_0(n) + z^{1/2}W_1(n) + \cdots + z^{k/2}W_k(n) + \tilde{W}_k(z,n),$$

and, as $B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})$-valued functions, we have

$$\|W_j(n)\| \le C\langle n\rangle^{-\frac{3}{2}}, \quad j = 0,\ldots,k, \quad \|\tilde{W}_k(z,n)\|_{\mathcal{O}((k+\varepsilon)/2)} \le \langle n\rangle^{-\frac{3}{2}}. \tag{4.14}$$

We insert this expansion into (4.13), and apply Lemma 4.1 and Lemma 4.2. The same argument as for $I_1(\sigma)u_0$ implies that

$$I_2(\sigma)J = \sum_{j=1}^{k}\sum_{n\in\mathbf{Z}} e^{-i\sigma n}\sigma^{-(j+2)/2}\varepsilon_j C_j W_j(n) + E_k^2(\sigma), \tag{4.15}$$

as $\sigma \to \infty$, where $E_k^2(\sigma)$ satisfies the same estimate as in (4.10) and the sum converges in $B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})$ by virtue of (4.14). We combine (4.5) and (4.9) with (4.15). Since $Jg_ju_0 = D_j(0)Ju_0$, when j is odd, and $D_j(0)+M_j(0)+W_j(0) = F_j(0)$ for $j = 0,\ldots,k$, we thus obtain

$$e^{-i\sigma K}Ju_0 = \sum_{j=1}^{k}\sigma^{-(j+2)/2}\Big(\sum_{n\in\mathbf{Z}} e^{-i\sigma n}\varepsilon_j F_j(n)Ju_0\Big) + O(\sigma^{-\frac{k+2+\varepsilon}{2}}). \tag{4.16}$$

Here, for odd j, $F_j(0) = \sum_\nu a_{j\nu}\otimes b_{j\nu}$ with $a_{j\nu}, b_{j\nu} \in \mathcal{K}^1_{-\delta}$ by Theorem 1.11 and $F_j(n) = E_nF_j(0)E_n^*$, and, therefore, Lemma 4.3 implies that

$$Z_j(\sigma) = \sum_{n\in\mathbf{Z}} e^{-i\sigma n}E_nF_j(0)E_n^*J \tag{4.17}$$

is the integral operator with kernel $2\pi\sum_\nu a_{j\nu}(t,x)b_{j\nu}(t-\sigma,y)$. The Sobolev embedding theorem implies $\sup_{t\in\mathbf{T}}\|u(t)\|_{\mathcal{H}_{-\delta}} \le C\|u\|_{\mathcal{K}^1_{-\delta}}$. Hence, we deduce from (4.16) that

$$\sup_{t\in\mathbf{T}}\|U(t,t-\sigma)u_0 - \sum_{j=1}^{k}\varepsilon_j\sigma^{-(j+2)/2}Z_j(\sigma)Ju_0(t)\|_{\mathcal{H}_{-\delta}} \le C\sigma^{-\frac{k+2+\varepsilon}{2}}\|u_0\|_{\mathcal{H}_\delta}$$

and, setting $t = \sigma$ and replacing σ by t,

$$\|U(t,0)u_0 - \sum_{j=1}^{k} \varepsilon_j t^{-(j+2)/2} B_j(t)u_0\|_{\mathcal{H}_{-\delta}} \le C t^{-\frac{k+2+\varepsilon}{2}} \|u_0\|_{\mathcal{H}_\delta}.$$

Here $B_j(t)$ is the integral operator with kernel $2\pi \sum_\nu a_{j\nu}(t,x) b_{j\nu}(0,y)$. This completes the proof of Theorem 1.8 for generic V if no non-integral eigenvalues exist for K.

Completion of the proof, exceptional case. For treating $I_2(\sigma)J_0$ when V is of exceptional case, we further decompose

$$R_1(z) = (1 + R_0(z)V)^{-1} N(z) = N(z) - R(z)VN(z)$$

and $I_2(\sigma) = I_{21}(\sigma) + I_{22}(\sigma)$ accordingly. For studying

$$I_{21}(\sigma)Ju_0 = \frac{1}{2\pi i} \sum_{n \in \mathbf{Z}} e^{-i\sigma n} \int e^{-i\sigma\lambda} \chi(\lambda) N(\lambda + n) J u_0 d\lambda \tag{4.18}$$

we insert (2.26) for $N(z+n)$, apply Lemma 4.1 and Lemma 4.2 to the resulting expression, and argue as in the case for $I_1(\sigma)J_0$. We obtain

$$I_{21}(\sigma)u_0 = \sum_{n \in \mathbf{Z}} \left(\sum_{j=1}^{k} \sigma^{-(j+2)/2} e^{-in\sigma} \varepsilon_j C_j N_j(n) J u_0 \right) + O(\sigma^{-\frac{k+2+\varepsilon}{2}}) \tag{4.19}$$

where $O(\sigma^{-\frac{k+2+\varepsilon}{2}})$ satisfies the same estimate as in (4.10). We have

$$I_{22}(\sigma)Ju_0 = \lim_{\varepsilon \downarrow 0} \frac{-1}{2\pi i} \sum_{n \in \mathbf{Z}} e^{-i\sigma n} E_n \int e^{-i\sigma\lambda} \chi(\lambda) R(\lambda + i\varepsilon) V N(\lambda + i\varepsilon) E_n^* J u_0 d\lambda. \tag{4.20}$$

If we use (1.20) and (2.26), then, omitting the variable 0, we have

$$\begin{aligned} R(z)VN(z) = (-z^{-1}F_{-2} + z^{-\frac{1}{2}}F_{-1} + \cdots + z^{\frac{k-2}{2}}F_{k-2} + \mathcal{O}(z^{\frac{k-2+\varepsilon}{2}})) \\ \times V(N_0 + \cdots + z^{\frac{k}{2}}N_k + \mathcal{O}(z^{\frac{k+\varepsilon}{2}})). \end{aligned} \tag{4.21}$$

Since $F_{-2} = E_K(\{0\}) \equiv P$ and $F_{-1}V = PVD_3VPV + S_{-1}$, we have

$$F_{-2}VN_0 = F_{-2}, \quad F_{-2}VN_1 = 0, \quad F_{-1}VN_0 = F_{-1}$$

by virtue of (3.17) and (3.18) and Lemma 2.4 implies the expansion

$$R(z)VN(z) = -z^{-1}F_{-2} + z^{-1/2}F_{-1} + T_0 + \cdots + z^{(k-2)/2}T_{k-2} + O(z^{\frac{k-2+\varepsilon}{2}}). \tag{4.22}$$

Note that we may change the order of $\lim_{\varepsilon\downarrow 0}$ and $\sum_{n\in\mathbf{Z}}$ by virtue of (2.28) and (2.30). Since

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int e^{-i\sigma\lambda}\frac{\chi(\lambda)}{\lambda+i\varepsilon}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\sigma}\hat{\chi}(x)dx=1+(\sigma^{-N}),$$

the first term of (4.22) contributes to $I_{22}(\sigma)Ju_0$ by

$$\Big(\sum_{n\in\mathbf{Z}}e^{-i\sigma n}E_nE_K(\{0\})E_n^*Ju_0\Big)(1+O(\sigma^{-N}))$$
$$=2\pi\sum_j\phi_j(t,x)\otimes\phi_j(t-\sigma,y)+O(\sigma^{-N}).$$

The contributions of the other terms in (4.22) may be computed and estimated by using Lemma 4.1 and Lemma 4.2 and the rest of the argument is exactly same as in the generic case. We omit the repetitive details.

Non-integral eigenvalues. We now show how to modify the argument, when non-integral eigenvalues $\{\lambda_1,\lambda_2,\dots,\lambda_N\}\subset(0,1)$ are present for K. We proceed as in the exceptional case. We treat I_{21} as in the previous section, however, for $I_{22}(\sigma)$, we use a different partition of unity: We take $\chi_j(\lambda)\in C_0^\infty(\mathbf{R})$, $j=0,\dots,N$ such that

$$\sum_{n\in\mathbf{Z}}\sum_{j=0}^{N}\chi_j(\lambda+n)=1$$

and such that $\chi_j(\lambda)=1$ near $\lambda=\lambda_j$ and $\chi_j(\lambda)=0$ near $\lambda=\lambda_k, k\neq j$, where we defined $\lambda_0=0$. We then further decompose $I_{22}=I_{22}^{(0)}+\cdots+I_{22}^{(N)}$ where $I_{22}^{(j)}(\sigma)J$ is given by (4.20) with $\chi_j(\lambda)$ in place of $\chi(\lambda)$. $I_{22}^{(0)}(\sigma)J$ can be treated exactly in the same fashion as above and $(I_0(\sigma)+I_1(\sigma)+I_{21}(\sigma)+I_{22}^{(0)}(\sigma))J$ gives the desired formula (1.12) except for the terms coming from non-integral eigenvalues. To see that $I_{22}^{(j)}(\sigma)Ju_0$, $j\neq 0$, contributes only to the eigenfunctions and to the remainder, we insert (3.46) for $R(z)$ in $R(z)VN(z)$. Then, with $P=E_K(\{\lambda_j\})$

$$R(z)VN(z)J=\frac{PVN(z)J}{\lambda_j-z}+$$
$$+\Big(\tilde{R}(\lambda_j)+\cdots+(z-\lambda_j)^kR_k(\lambda_j)+O((z-\lambda_j)^{k+\varepsilon})\Big)N(z)J$$

Here the second term on the right is $B(\mathcal{H}_\delta,\mathcal{K}^1_{-\delta})$-valued $C^{k+\varepsilon}$ on the support of χ_j and its norm decays like $O(\langle n\rangle^{-3/2})$ with its derivatives when translated by n by virtue of (2.28) and (2.30). Thus, its contribution to $I_{22}^{(j)}(\sigma)J$ is $O(\sigma^{-k-\varepsilon})$ as a $B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})$-valued function and it may be included in the remainder. If we use the identity (3.50) repeatedly, we see that

$$(\lambda - z)^{-1}PVN(z) = (\lambda - z)^{-1}P + PR_0(z) - PM(z) + PN(z).$$

Since eigenfunctions ϕ_ν are two times differentiable with respect to t and hence $\|\langle x\rangle^\ell P(p_n \otimes 1)\langle x\rangle^\ell\| \le \langle n\rangle^{-2}$ the last three terms contributes to $I_{22}^{(j)}(\sigma)J$ by $O(\sigma^{-k-\varepsilon})$ as a $B(\mathcal{H}_\delta, \mathcal{K}^1_{-\delta})$-valued function of σ again. The first term contributes by $2\pi e^{-i\lambda_j\sigma}\sum_\nu \phi_\nu(t,x) \otimes \phi_\nu(t-\sigma, y)$ as previously. The proof of Theorem 1.8 is completed.

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