Fields, Vacuum, and the Mirror Universe

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This study builds theory of motion of charged particles with spin in four-dimensional pseudo-Riemannian space. The theory employs the mathematical apparatus of chronometric invariants by A. L. Zelmanov (physical observable values in the General Theory of Relativity). Numerous special effects in electrodynamics, physics of elementary particles, and cosmology are discussed. The effects explain anomalous annihilation of orthopositronium and link with "stop-light experiment".

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Preface

This book is addressed to readers who want to have a look at the laws of micro and macro world from a single viewpoint. This is the English translation of our *Theory of Non-Geodesic Motion of Particles*, originally published in Russian in 1999, with some recent amendements.

The background behind the book is as follows. In 1991 we initiated a study to find out what kinds of particles may theoretically inhabit four-dimensional space-time. As the instrument, we equipped ourselves with mathematical apparatus of physical observable values (chronometric invariants) developed by A. L. Zelmanov, a prominent cosmologist.

The study was completed by 1997 to reveal that aside for mass-bearing and massless (light-like) particles, those of third kind may exist. Their trajectories lay beyond regular space-time of General Relativity. For a regular observer the trajectories are of zero four-dimensional length and zero three-dimensional observable length. Besides, along these trajectories interval of observable time is also zero. Mathematically, that means such particles inhabit fully degenerated space-time with non-Riemannian geometry. We called such space “zero-space” and such particles — “zero-particles”.

For a regular observer their motion in zero-space is instant, i. e. zero-particles are carriers of long-range action. Through possible interaction with our world’s mass-bearing or massless particles zero-particles may instantly transmit signals to any point in our three-dimensional space.

Considering zero-particles in the frames of the wave-particle concept we obtained that for a regular observer they are standing waves and the whole zero-space is a system of standing light-like waves (zero-particles), i. e. a standing-light hologram. This result links with “stop-light experiment” (Cambridge, Massachusetts, January 2001).

Using methods of physical observable values we also showed that in basic four-dimensional space-time a mirror world may exist, where coordinate time has reverse flow in respect to the viewpoint of regular observer’s time.

We presented the results in 1997 in two pre-prints[1].

B. M. Levin, an expert in orthopositronium problem came across these publications. He contacted us immediately and told us about critical situation around anomalies in annihilation of orthopositronium, which had been awaiting theoretical explanation for over a decade.

Rate of annihilation of orthopositronium (the value reciprocal to its life span) is among the references set to verify the basic laws of Quantum Electrodynamics. Hence any anomalies contradict with these reliably proven laws. In 1987 *Michigan group* of researchers (Ann Arbor, Michigan, USA) using advanced precision equipment revealed that the measured rate of annihilation of orthopositronium was substantially higher compared to its theoretical value.

That implies that some atoms of orthopositronium annihilate not into three photons as required by laws of conservation, but into lesser number of photons, which breaks those laws. In the same 1987 Levin discovered what he called “isotope anomaly” in anomalous annihilation of orthopositronium (Gatchina–St.Petersburg, Russia). Any attempts to explain the anomalies by means of Quantum Electrodynamics over 10 years would fail. This made S. G. Karshenboim, a prominent expert in the field, to resume that all capacities of standard Quantum Electrodynamics to explain the anomalies were exhausted.

In our 1997 publications Levin saw a means of theoretical explanation of orthopositronium anomalies by methods of General Relativity and suggested a joint research effort in this area.

[1]Borissova L. B. and Rabounski D. D. Movement of particles in four-dimensional space-time. Lomonossov Workshop, Moscow, 1997 (*in Russian*); Rabounski D. D. Three forms of existence of matter in four-dimensional space-time. Lomonossov Workshop, Moscow, 1997 (*in Russian*).

Solving the problem we obtained that our world and the mirror Universe are separated with a space-time membrane, which is a degenerated space-time (zero-space). We also arrived to physical conditions under which exchanges may occur between our world and the mirror Universe. Thanks to this approach and using methods of General Relativity we developed a geometric concept of virtual interactions: it was mathematically proven that virtual particles are zero-particles that travel in zero-space and carry long-range action. Application of the results to annihilation of orthopositronium showed that two modes of decay are theoretically possible: (a) all three photons are emitted into our Universe; (b) one photon is emitted into our world, while two others go to the mirror Universe and become unavailable for observation.

All the above results stemmed exclusively from application of Zelmanov's mathematical apparatus of physical observable values.

When tackling the problem we had to amend the existing theory with some new techniques. In their famous *The Classical Theory of Fields*, which has already become a de-facto standard for a university reference book on General Relativity, L. D. Landau and E. M. Lifshitz give an excellent account of theory of motion of particle in gravitational and electromagnetic fields. But the monograph does not cover motion of spin-particles, which leaves no room for explanations of orthopositronium experiments (as its para and ortho states differ by mutual orientation of electron and positron spins). Besides, Landau and Lifshitz employed general covariant methods. The technique of physical observable values (chronometric invariants) has not been yet developed by that time by Zelmanov, which should be also taken into account.

Therefore we faced the necessity to introduce methods of chronometric invariants into the existing theory of motion of particles in gravitational and electromagnetic fields. Separate consideration was given to motion of particles with inner mechanical momentum (spin). We also added a chapter with account of tensor algebra and analysis. This made our book a contemporary supplement to *The Classical Theory of Fields* to be used as a reference book in university curricula.

In conclusion we would like to express our sincere gratitude to Dr. Abram Zelmanov (1913–1987) and Prof. Kyril Stanyukovich (1916–1989). Many years of acquaintance and hours of friendly conversations with them have planted seeds of fundamental ideas which by now grew up in our minds to be reflected on these pages.

We are grateful to Dr. Kyril Dombrovski whose works greatly influenced our outlooks.

We highly appreciate contribution from our colleague Dr. Boris Levin. With enthusiasm peculiar to him he stimulated our writing of this book.

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[2]http://www.tex.ac.uk/tex-archive/systems/win32/bakoma/

Chapter 1

Introduction

1.1 Geodesic motion of particles

Numerous experiments aimed at proving the conclusions of the General Theory of Relativity have also proven that its basic space-time (four-dimensional pseudo-Riemannian space) is the basis of our real world geometry. This also implies that even by progress of experimental physics and astronomy, which will discover new effects in time and space, four-dimensional pseudo-Riemannian space will remain the cornerstone for further widening of the basic geometry of General Relativity and will become one of its specific cases. Therefore, when building mathematical theory of motion of particles, we are considering their motion specifically in the four-dimensional pseudo-Riemannian space.

A terminology note should be taken at this point. Generally, the basic space-time in General Relativity is a *Riemannian space*[1] with four dimensions with sign-alternating Minkowski's label $(+---)$ or $(-+++)$. The latter implies 3+1 split of coordinate axis in Riemannian space into three spatial coordinate axis and the time axis. For convenience of calculations a Riemannian space with $(+---)$ signature is considered, where time is real while space is imaginary. Some theories, largely General Relativity, also employs $(-+++)$ label with imaginary time and real space. But Riemannian spaces may as well have non-alternating signature, e. g. $(++++)$. Therefore a Riemannian space with alternating label is commonly referred to as *pseudo-Riemannian space*, to emphasize the split of coordinate axis into two types. But even in this case all its geometric properties are still properties of Riemannian geometry and the "pseudo" notation is not absolutely proper from mathematical viewpoint. Nevertheless we are going to use this notation as a long-established and traditionally understood one.

We consider motion of a particle in four-dimensional pseudo-Riemannian space. A particle affected by gravitation only falls freely and moves along the shortest (*geodesic*) line. Such motion is referred to as *free* or *geodesic motion*. If the particle is also affected by some additional non-gravitational forces, the latter divert the particle from its geodesic trajectory and the motion becomes *non-geodesic motion*.

From geometric viewpoint motion of a particle in four-dimensional pseudo-Riemannian space is parallel transfer of some four-dimensional vector Q^α which describes motion of the particle and is therefore tangential to the trajectory at any of its points. Consequently, equations of motion of particle actually define parallel transfer of vector Q^α along its four-dimensional trajectory and are equations of absolute derivative of the vector by certain parameter ρ, which exists along all the trajectory of particle's motion and is not zero along the way,

$$\frac{DQ^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma^\alpha_{\mu\nu} Q^\mu \frac{dx^\nu}{d\rho}\,, \qquad \alpha, \mu, \nu = 0, 1, 2, 3. \tag{1.1}$$

Here $DQ^\alpha = dQ^\alpha + \Gamma^\alpha_{\mu\nu} Q^\mu dx^\nu$ is the absolute differential (the absolute increment of vector Q^α), which is different from a regular differential dQ^α by presence of Christoffel symbols of 2nd kind

[1] A metric space which geometry is defined by metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ called to as Riemann's metric. Bernhard Riemann (1826–1866), a German mathematician, the founder of Riemannian geometry (1854).

$\Gamma^{\alpha}_{\mu\nu}$ (coherence coefficients of Riemannian space), which are calculated through Christoffel symbols (coherence coefficients) of 1st kind $\Gamma_{\mu\nu,\rho}$ and are functions of first derivatives of fundamental metric tensor $g_{\alpha\beta}$[2]

$$\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\rho}\Gamma_{\mu\nu,\rho}\,, \quad \Gamma_{\mu\nu,\rho} = \frac{1}{2}\left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right). \tag{1.2}$$

When moving along a geodesic trajectory (free motion) parallel transfer occurs in the meaning of Levi-Civita[3]. Here the absolute derivative of four-dimensional vector of particle Q^{α} equals to zero

$$\frac{dQ^{\alpha}}{d\rho} + \Gamma^{\alpha}_{\mu\nu}Q^{\mu}\frac{dx^{\nu}}{d\rho} = 0\,, \tag{1.3}$$

and the square of the vector being transferred is conserved along all trajectory $Q_{\alpha}Q^{\alpha}=const$. Such equations are referred to as equations of motion of free particles.

Kinematic motion of particle is characterized by four-dimensional vector of acceleration (also referred to as *kinematic vector)*

$$Q^{\alpha} = \frac{dx^{\alpha}}{d\rho}\,, \tag{1.4}$$

in parallel transfer by Levi-Civita equations of four-dimensional trajectories of free particle are obtained (*equations of geodesic lines*)

$$\frac{d^2x^{\alpha}}{d\rho^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\rho}\frac{dx^{\nu}}{d\rho} = 0\,. \tag{1.5}$$

Necessary condition $\rho\neq 0$ along the trajectory of motion implies that derivation parameters ρ are not the same along trajectories of different kind. In pseudo-Riemannian space three kinds of trajectories are principally possible, each kind corresponds to its type of particles:

- *non-isotropic real trajectories*, that lay “within” the light hyper-cone. Along such trajectories the square of space-time interval $ds^2>0$, while the interval ds is real. These are trajectories of regular sub-light-speed particles with non-zero rest-mass and real relativistic mass;
- *non-isotropic imaginary trajectories*, which lay “outside” the light hyper-cone. Along such trajectories the square of space-time interval $ds^2<0$, while the interval ds is imaginary. These are trajectories of super-light-speed tachyon particles with imaginary relativistic mass [18, 19];
- *isotropic trajectories*, which lay on the surface of light hyper-cone and are trajectories of particles with zero rest-mass (massless light-like particles), which travel at the light speed. Along the isotropic trajectories the space-time interval is zero $ds^2=0$, but the three-dimensional interval is not zero.

As a derivation parameter to non-isotropic trajectories space-like interval ds is commonly used. But it can not be used in such capacity to trajectories of massless particles $ds=0$. Therefore as early as in 1941–1944 A. L. Zelmanov in his doctorate thesis proposed another variable that does not turn into zero along isotropic trajectories, to be used as derivation parameter to isotropic trajectories [6],

$$d\sigma^2 = \left(-g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}\right)dx^i dx^k, \tag{1.6}$$

which is a three-dimensional physical observable interval [6]. L. D. Landau and E. M. Lifshitz also arrived to the same conclusion independently (see Section 84 in their *The Classical Theory of Fields* [1]).

Substituting respective differentiation parameters into generalized equations of geodesic lines (1.5), we arrive to equations of non-isotropic geodesic lines (trajectories of mass-bearing particles)

$$\frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0\,, \tag{1.7}$$

[2]Coherence coefficients of Riemannian space (Christoffel symbols) are named after German mathematician Elvin Bruno Christoffel (1829–1900), who obtained them in 1869. In the Special Relativity space-time (Minkowski space) one can always set an inertial system of reference, where the matrix of fundamental metric tensor is a unit diagonal tensor and all Christoffel symbols become zeroes.

[3]Tullio Levi-Civita (1873–1941), an Italian mathematician, who was the first to study such parallel transfer [3].

and to equations of isotropic geodesic lines (light propagation equations)

$$\frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0\,. \tag{1.8}$$

But in order to get the whole picture of motion of particle, we have to build dynamic equations of motion, which contain physical properties of particle (mass, frequency, energy, etc.).

From geometric viewpoint dynamic equations of motion are equations of parallel transfer of four-dimensional dynamic vector of particle along its trajectory (absolute derivative of dynamic vector by parameter, not equal to zero along the trajectory, is zero).

Motion of free mass-bearing particles (non-isotropic geodesic trajectories) is characterized by four-dimensional impulse vector

$$P^\alpha = m_0\frac{dx^\alpha}{ds}\,, \tag{1.9}$$

where m_0 is rest-mass of particle. Parallel transfer in the meaning of Levi-Civita of four-dimensional impulse vector P^α gives dynamic equations of motion of free mass-bearing particles

$$\frac{dP^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}P^\mu\frac{dx^\nu}{ds} = 0\,, \qquad P_\alpha P^\alpha = m_0^2 = const. \tag{1.10}$$

Motion of massless light-like particles (isotropic geodesic lines) is characterized by four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c}\frac{dx^\alpha}{d\sigma}\,, \tag{1.11}$$

where ω is specific cyclic frequency of massless particle. Respectively, parallel transfer in the meaning of Levi-Civita of vector K^α gives dynamic equations of motion of free massless particles

$$\frac{dK^\alpha}{d\sigma} + \Gamma^\alpha_{\mu\nu}K^\mu\frac{dx^\nu}{d\sigma} = 0\,, \qquad K_\alpha K^\alpha = 0\,. \tag{1.12}$$

Therefore we have got dynamic equations of motion for free massless particles. These are presented in four-dimensional general covariant form. This form has got its own advantage as well as a substantial drawback. The advantage is invariance in all transitions from one frame of reference to another. The drawback is that in covariant form the terms of the equations do not contain actual three-dimensional values, which can be measured in experiments or observations (*physical observable values*). This implies that in general covariant form equations of motion of particle are merely an intermediate theoretical result, not applicable to practice. Therefore, in order to make results of any physical mathematical theory usable in practice, we need to formulate its equations with physical observable values. Namely, to calculate trajectories of certain particles we have to formulate general covariant dynamic equations of motion with physical observable properties of these particles as well as through observable properties of an actual physical frame of reference of the observer.

But defining physical observable values is not a trivial problem. For instance, if for a four-dimensional vector Q^α (as few as four components) we may *heuristically assume* that its three spatial components form a three-dimensional observable vector, while the temporal component is observable potential of the vector field (which generally does not prove they can be actually observed, though), a contravariant 2nd rank tensor $Q^{\alpha\beta}$ (as many as 16 components) makes the problem much more indefinite. For tensors of higher rank the problem of heuristic definition of observable components is far more complicated. Besides there is an obstacle related to definition of observable components of covariant tensors (in which indices are the lower ones) and of mixed type tensors, which have both lower and upper indices.

Therefore the most reasonable way out of the labyrinth of heuristic guesses is creating a strict mathematical theory to enable calculation of observable components for any tensor values. Such theory was created in 1941–1944 by Zelmanov and set forth in his dissertation thesis [6]. It should be noted, though, that many researchers were working on theory of observable values in 1940's. For example, Landau and Lifshitz in later editions of their well-known *The Classical Theory of Fields* [1] introduced observable time and observable three-dimensional interval, similar to those introduced by Zelmanov. But the authors limited themselves only to this part of theory and did not arrive to general mathematical methods to define physical observable values in pseudo-Riemannian space.

Over the next decades Zelmanov would improve his mathematical apparatus of physical observable values (the methods of chronometric invariants), setting forth the results in some later papers [7, 8, 9, 10]. Similar results were obtained by C. Cattaneo, an Italian mathematician, independently from Zelmanov. However Cattaneo published his first study on the theme in 1958 [11, 12, 13, 14].

A systematic description of Zelmanov's mathematical methods of chronometric invariants is given in our two pre-prints [15, 16]. Therefore in the next Section of this Chapter we will give just a brief overview of the methods of theory of physical observable values, which is necessary for understanding them and using in practice.

In Section 1.3 we will present the results of studying geodesic motion of particles using the technique of chronometric invariants [15, 16]. In Section 1.4 will focus on setting problem of building dynamic equations of particles along non-geodesic trajectories, i. e. under action of non-gravitational external forces.

1.2 Physical observable values

This Section introduces Zelmanov's mathematical apparatus of chronometric invariants.

To define mathematically which components of any four-dimensional values are physical observable values, we consider a real frame of reference of some observer, which includes *coordinate nets*, spanned over some physical body (*body of reference*), at each point of which *real clock* is installed. The body of reference being a real physical body possesses a certain gravitational potential, may rotate and deform, making the space of reference non-uniform and anisotropic. Actually, the body of reference and attributed to it space of reference may be considered as a set of real physical references, to which observer compares all results of his measurements. Therefore, physical observable values should be obtained as a result of projecting four-dimensional values on time and space of observer's real body of reference.

From geometric viewpoint three-dimensional space is *spatial section* $x^0 = ct = const$. At any point of the space-time a local spatial section (orthogonal space) can be done orthogonal to the *line of time*. If exists space-time enveloping curve to local spaces it is a spatial section everywhere orthogonal to lines of time. Such space is known as *holonomic space*. If no enveloping curve exists to such local spaces, i. e. there only exist spatial sections locally orthogonal to lines of time, such space is known as *non-holonomic*.

We assume that the observer rests in respect to his physical references (body of reference). Frame of reference of such observer in any displacements accompanies the body of reference and is called *accompanying frame of reference*. Any coordinate nets that rest in respect to the same body of reference are related through transformation

$$\tilde{x}^0 = \tilde{x}^0\left(x^0, x^1, x^2, x^3\right), \quad \tilde{x}^i = \tilde{x}^i\left(x^1, x^2, x^3\right), \quad \frac{\partial \tilde{x}^i}{\partial x^0} = 0\,, \tag{1.13}$$

where the latter equation implies independence of spatial coordinates in tilde-marked net from time of non-marked net, which is equivalent to setting coordinate net of concrete and fixed lines of time $x^i = const$ in any point of coordinate net. Transformation of coordinates is nothing but transition from one coordinate net to another within the same spatial section. Transformation of time implies changing the whole set of clocks, i. e. transition from to another spatial section (space of reference). In practice that means replacement of one body of reference along with all of its physical references with another body of references that has got its own physical references. But when using different references observer will obtain quite different results (observable values). Therefore physical observable values must be invariant in respect to transformations of time, i. e. should be *chronometrically invariant values*.

Because transformations (1.13) define a set of fixed lines of time, then chronometric invariants (physical observable values) are all values, invariant in respect to these transformations.

In practice, to obtain physical observable values in accompanying frame of reference we have to calculate chronometrically invariant projections of four-dimensional values on time and space of a real physical body of reference and formulate them with chronometrically invariant (physically observable) properties of the space of reference.

We project four-dimensional values using operators that characterize properties of real space of reference. Operator of projection on time b^α is a unit vector of four-dimensional velocity of observer's frame of reference (four-dimensional velocity of body of reference)

$$b^\alpha = \frac{dx^\alpha}{ds}\,, \tag{1.14}$$

which is tangential to four-dimensional observer's trajectory in its every point. Because any frame of reference is described by its own tangential unit vector b^α, Zelmanov called the vector *a monad*. The operator of projection on space is defined as four-dimensional symmetric tensor

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta\,, \qquad h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta, \tag{1.15}$$

which mixed components are

$$h_\alpha^\beta = -g_\alpha^\beta + b_\alpha b^\beta. \tag{1.16}$$

Previous studies show that these values possess necessary properties of projection operators [6, 10, 16]. Projection of tensor value on time is a result of its contraction with monad vector. Projection on space is contraction with tensor of projection on space.

In accompanying frame of reference three-dimensional observer's velocity in respect to the body of reference is zero b^i=0. Other components of the monad are

$$b^0 = \frac{1}{\sqrt{g_{00}}}\,, \quad b_0 = g_{0\alpha} b^\alpha = \sqrt{g_{00}}\,, \quad b_i = g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}\,. \tag{1.17}$$

Respectively, in accompanying frame of reference (b^i=0) components of tensor of projection on space are

$$\begin{array}{lll} h_{00} = 0\,, & h^{00} = -g^{00} + \dfrac{1}{g_{00}}\,, & h_0^0 = 0\,, \\ h_{0i} = 0\,, & h^{0i} = -g^{0i}, & h_0^i = \delta_0^i = 0\,, \\ h_{i0} = 0\,, & h^{i0} = -g^{i0}, & h_i^0 = \dfrac{g_{i0}}{g_{00}}\,, \\ h_{ik} = -g_{ik} + \dfrac{g_{0i} g_{0k}}{g_{00}}\,, & h^{ik} = -g^{ik}, & h_k^i = -g_k^i = \delta_k^i\,. \end{array} \tag{1.18}$$

Tensor $h_{\alpha\beta}$ in three-dimensional space of an accompanying frame of reference shows properties of fundamental metric tensor

$$h_\alpha^i h_k^\alpha = \delta_k^i - b_k b^i = \delta_k^i\,, \qquad \delta_k^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.19}$$

where δ_k^i is a unit three-dimensional tensor[4]. Therefore, in accompanying frame of reference three-dimensional tensor h_{ik} may lift or lower indices in chronometrically invariant values.

Projections on time T and space L^α of a certain vector Q^α (1st rank tensor) in accompanying frame of reference (b^i=0) are

$$T = b^\alpha Q_\alpha = b^0 Q_0 = \frac{Q_0}{\sqrt{g_{00}}}\,, \tag{1.20}$$

$$L^0 = h_\beta^0 Q^\beta = -\frac{g_{0k}}{g_{00}} Q^k, \quad L^i = h_\beta^i Q^\beta = \delta_k^i Q^k = Q^k. \tag{1.21}$$

[4]Tensor δ_k^i is the three-dimensional part of four-dimensional unit tensor δ_β^α, which can be used to replace indices in four-dimensional values.

Below are some projections of 2nd rank tensor $Q^{\alpha\beta}$ in accompanying frame of reference

$$T = b^{\alpha}b^{\beta}Q_{\alpha\beta} = b^0 b^0 Q_{00} = \frac{Q_{00}}{g_{00}}\,, \tag{1.22}$$

$$L^{00} = h^0_{\alpha}h^0_{\beta}Q^{\alpha\beta} = -\frac{g_{0i}g_{0k}}{g^2_{00}}Q^{ik}, \quad L^{ik} = h^i_{\alpha}h^k_{\beta}Q^{\alpha\beta} = Q^{ik}. \tag{1.23}$$

Experimental check of invariance of the obtained physical values in respect to transformations (1.13) says that physical observable values are projection of four-dimensional value on time and spatial components of projection on space.

Hence, projecting four-dimensional coordinates x^{α} on time and space we obtain *physical observable time*

$$\tau = \sqrt{g_{00}}\,t + \frac{g_{0i}}{c\sqrt{g_{00}}}x^i, \tag{1.24}$$

and *physical observable coordinates*, which coincide with spatial coordinates x^i. Similarly, projection of elementary interval of four-dimensional coordinates dx^{α} gives elementary interval of physical observable time

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}dx^i, \tag{1.25}$$

and elementary interval of physical observable coordinates dx^i. Respectively, *physical observable velocity* of particle for an observer is three-dimensional chronometrically invariant vector

$$\mathrm{v}^i = \frac{dx^i}{d\tau}\,, \tag{1.26}$$

which is different from the value $u^i = \frac{dx^i}{dt}$, which is the vector of its three-dimensional coordinate velocity.

Projecting fundamental metric tensor on space we obtain that in accompanying frame of reference *physical observable spatial metric tensor* consists of spatial components of tensor of projection on space

$$h^i_{\alpha}h^k_{\beta}g^{\alpha\beta} = g^{ik} = -h^{ik}, \qquad h^{\alpha}_i h^{\beta}_k g_{\alpha\beta} = g_{ik} - b_i b_k = -h_{ik}\,. \tag{1.27}$$

Therefore the square of physical observable interval $d\sigma$ is

$$d\sigma^2 = h_{ik}dx^i dx^k. \tag{1.28}$$

Four-dimensional space-time interval formulated with physical observable values can be obtained by substituting $g_{\alpha\beta}$ from (1.15)

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{1.29}$$

But aside for projections on space and time four-dimensional values of 2nd rank and above also have mixed components which have both upper and lower indices at the same time. How do we find physical observable values among them, if any? The best approach is to develop a generalized method to calculate physical observable values based solely on their property of chronometric invariance and allowing to find all observable values at the same time in any tensor. Such method was developed by Zelmanov and set forth as a theorem.

Zelmanov theorem

"We assume that $Q^{ik\ldots p}_{00\ldots 0}$ are components of four-dimensional tensor $Q^{\mu\nu\ldots\rho}_{00\ldots 0}$ of r-th rank, in which all upper indices are not zero, while all m lower indices are zeroes. Then tensor values

$$T^{ik\ldots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\ldots p}_{00\ldots 0} \tag{1.30}$$

make up chronometrically invariant three-dimensional contravariant tensor of $(r-m)$-th rank. Hence tensor $T^{ik\ldots p}$ is a result of m-fold projection on time by indices $\alpha, \beta \ldots \sigma$ and projection on space by $r-m$ indices $\mu, \nu \ldots \rho$ of the initial tensor $Q^{\mu\nu\ldots\rho}_{\alpha\beta\ldots\sigma}$".

An immediate result of the theorem is that for vector Q^{α} two values are physical observable, which were obtained earlier by projecting

$$b^{\alpha}Q_{\alpha}=\frac{Q_0}{\sqrt{g_{00}}}\,,\qquad h^{i}_{\alpha}Q^{\alpha}=Q^{i}. \tag{1.31}$$

For a symmetric 2nd rank tensor $Q^{\alpha\beta}$ the three values are physical observable ones, namely

$$b^{\alpha}b^{\beta}Q_{\alpha\beta}=\frac{Q_{00}}{g_{00}}\,,\quad h^{i\alpha}b^{\beta}Q_{\alpha\beta}=\frac{Q^{i}_{0}}{\sqrt{g_{00}}}\,,\quad h^{i}_{\alpha}h^{k}_{\beta}Q^{\alpha\beta}=Q^{ik}. \tag{1.32}$$

The calculated physical observable values (chronometric invariants) have to be compared to the references (the standards of measure) — observed properties of the space of reference which are specific for any particular body of reference. Therefore we will now consider the basic properties of the space of reference with which the final equations of our theory are to be formulated.

Physical observable properties of the space of reference are obtained with the help of chronometrically invariant operators of differentiation by time and spatial coordinates

$$\frac{^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,,\qquad \frac{^*\partial}{\partial x^{i}}=\frac{\partial}{\partial x^{i}}-\frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^{0}}\,, \tag{1.33}$$

which are not commutative, i. e. difference between 2nd derivatives with respect to time and space coordinates becomes not zero

$$\frac{^*\partial^2}{\partial x^{i}\partial t}-\frac{^*\partial^2}{\partial t\partial x^{i}}=\frac{1}{c^2}F_i\frac{^*\partial}{\partial t}\,, \tag{1.34}$$

$$\frac{^*\partial^2}{\partial x^{i}\partial x^{k}}-\frac{^*\partial^2}{\partial x^{k}\partial x^{i}}=\frac{2}{c^2}A_{ik}\frac{^*\partial}{\partial t}\,. \tag{1.35}$$

Here A_{ik} is a three-dimensional antisymmetric chronometrically invariant *tensor of angular velocities of rotation of the reference's space*

$$A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^{i}}-\frac{\partial v_i}{\partial x^{k}}\right)+\frac{1}{2c^2}\left(F_i v_k-F_k v_i\right), \tag{1.36}$$

where v_i stands for *rotation velocity of space*

$$v_i=-c\frac{g_{0i}}{\sqrt{g_{00}}}\,. \tag{1.37}$$

Tensor A_{ik} being equal to zero is the necessary and sufficient condition of holonomity of space [6, 10]. In this case g_{0i}=0 and v_i=0. In non-holonomic space $A_{ik}\neq 0$ is always not zero. Therefore, tensor A_{ik} is also a tensor of the space's non-holonomity[5].

The value F_i is a three-dimensional chronometrically invariant *vector of gravitational inertial force*

$$F_i=\frac{c^2}{c^2-w}\left(\frac{\partial w}{\partial x^{i}}-\frac{\partial v_i}{\partial t}\right),\qquad w=c^2\left(1-\sqrt{g_{00}}\,\right), \tag{1.38}$$

where w stands for *gravitational potential* of the body of reference's space[6]. In quasi-Newtonian approximation, i. e. in a weak gravitational field at speeds much lower than the speed of light and in absence of rotation of space F_i becomes a non-relativistic force

$$F_i=\frac{\partial w}{\partial x^{i}}\,. \tag{1.39}$$

[5] Special Relativity space-time (Minkowski space) in Galilean frame of reference and some cases in General Relativity are examples of holonomic spaces A_{ik}=0.

[6] Values w and v_i do not possess property of chronometric invariance of their own. Vector of gravitation inertial force and tensor of angular velocity of space's rotation, built using them, are chronometric invariants.

Because the observer's body of reference is a real physical body, coordinate nets it bears are deformed. Consequently the space of a real body of reference is deformed too. Therefore comparison of observable values with physical references of the body of reference has to take into account the field of deformation of the space of reference, i. e. that the field of tensor h_{ik} is not stationary. In practice stationary deformation of space is rather rare: field of deformation varies all the time which should be as well taken into account in measurements. This can be done by defining in equations a three-dimensional symmetric chronometrically invariant *tensor of deformation velocities*

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \qquad D = D_k^k = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \qquad h = \det\|h_{ik}\|\,. \tag{1.40}$$

Given these definitions we can generally formulate any geometric object in Riemannian space with observable parameters of the space of reference. For instance, Christoffel symbols that appear in equations of motion are not tensors [2]. Nevertheless, they can be as well formulated with physical observable values [6, 15, 16]

$$\Gamma_{00}^0 = -\frac{1}{c^3}\left[\frac{1}{1-\frac{w}{c^2}}\frac{\partial w}{\partial t} + \left(1-\frac{w}{c^2}\right)v_k F^k\right], \tag{1.41}$$

$$\Gamma_{00}^k = -\frac{1}{c^2}\left(1-\frac{w}{c^2}\right)^2 F^k, \tag{1.42}$$

$$\Gamma_{0i}^0 = \frac{1}{c^2}\left[-\frac{1}{1-\frac{w}{c^2}}\frac{\partial w}{\partial x^i} + v_k\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}v_i F^k\right)\right], \tag{1.43}$$

$$\Gamma_{0i}^k = \frac{1}{c}\left(1-\frac{w}{c^2}\right)\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}v_i F^k\right), \tag{1.44}$$

$$\begin{aligned}\Gamma_{ij}^0 = -\frac{1}{c}\frac{1}{1-\frac{w}{c^2}}\Bigg\{&-D_{ij} + \frac{1}{c^2}v_n\left[v_j\left(D_i^n + A_{i\cdot}^{\cdot n}\right) + v_i\left(D_j^n + A_{j\cdot}^{\cdot n}\right) + \frac{1}{c^2}v_i v_j F^n\right] + \\ &+\frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right) - \triangle_{ij}^n v_n\Bigg\},\end{aligned} \tag{1.45}$$

$$\Gamma_{ij}^k = \triangle_{ij}^k - \frac{1}{c^2}\left[v_i\left(D_j^k + A_{j\cdot}^{\cdot k}\right) + v_j\left(D_i^k + A_{i\cdot}^{\cdot k}\right) + \frac{1}{c^2}v_i v_j F^k\right], \tag{1.46}$$

where $\triangle_{ij}^k$ are *chronometrically invariant Christoffel symbols* which are defined similarly to regular Christoffel symbols (1.2) but through physical observable metric tensor h_{ik} and chronometrically invariant differentiation operators

$$\triangle_{jk}^i = h^{im}\triangle_{jk,m} = \frac{1}{2}h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right). \tag{1.47}$$

We have discussed the basics of mathematical apparatus of chronometric invariants. Now having any equations obtained using general covariant methods we can calculate their chronometrically invariant projections on time and on space of any particular body of reference and formulate them with its real physical observable properties. From here we arrive to equations containing only values measurable in practice.

Naturally, the first possible application of this mathematical apparatus that comes to our mind is calculation of chronometrically invariant dynamic equations of motion of free particles and studying the results. Partial solution of this problem was obtained by Zelmanov [6, 10]. We presented the general one in our previous works [15, 16]. The next Section will focus on the results of our study.

1.3 Dynamic equations of motion of free particles

Absolute derivative of vector of motion of particle to scalar parameter is actually a four-dimensional vector

$$N^{\alpha}=\frac{dQ^{\alpha}}{d\rho}+\Gamma^{\alpha}_{\mu\nu}Q^{\mu}\frac{dx^{\nu}}{d\rho}\,. \tag{1.48}$$

Therefore chronometrically invariant (physical observable) components of equation of motion are defined similarly to those of any four-dimensional vector (1.31)

$$\frac{N_0}{\sqrt{g_{00}}}=\frac{g_{0\alpha}N^{\alpha}}{\sqrt{g_{00}}}=\frac{1}{\sqrt{g_{00}}}\left(g_{00}N^0+g_{0i}N^i\right), \tag{1.49}$$

$$N^i=h^i_{\beta}N^{\beta}=h^i_0N^0+h^i_kN^k. \tag{1.50}$$

From geometric viewpoint this is a projection of vector N^{α} on time and spatial components of its projection on space in accompanying frame of reference. In a similar way we can project general covariant dynamic equations of motion of free mass-bearing particles (1.10) and of free massless particles (1.12). The technique to calculate these projections is given in details in our previous publications [15, 16]. As a result we arrive to chronometrically invariant dynamic equations of motion of free mass-bearing particles

$$\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=0\,, \tag{1.51}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau}+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k-mF^i+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=0\,, \tag{1.52}$$

and of free massless particles

$$\frac{dk}{d\tau}-\frac{k}{c^2}F_ic^i+\frac{k}{c^2}D_{ik}c^ic^k=0\,, \tag{1.53}$$

$$\frac{d\left(kc^i\right)}{d\tau}+2k\left(D^i_k+A^{\cdot i}_{k\cdot}\right)c^k-kF^i+k\triangle^i_{nk}c^nc^k=0\,, \tag{1.54}$$

where m stands for relativistic mass of mass-bearing particle, $k=\frac{\omega}{c}$ is wave number that characterizes massless particle and c^i is three-dimensional chronometrically invariant vector of light velocity. As seen, contrary to general covariant dynamic equations of motion (1.10, 1.12), chronometrically invariant equations have a single derivation parameter for both mass-bearing particles and massless particles (which is physical observable time τ).

These equations were first obtained by Zelmanov [6]. But later it was found that function of time $\frac{dt}{d\tau}$ they include is strictly positive [15, 16]. Physical time has direct flow $d\tau>0$. Flow of coordinate time dt shows change of time coordinate of particle $x^0=ct$ in respect to observer's clock. Hence the sign of the function shows where the particle travels to in time in respect to observer.

Function of time $\frac{dt}{d\tau}$ [15, 16] is obtained from the condition that the square of four-dimensional velocity of particle is constant along its four-dimensional trajectory $u_{\alpha}u^{\alpha}=g_{\alpha\beta}u^{\alpha}u^{\beta}=const$. Equations in respect to $\frac{dt}{d\tau}$ are the same for sub-light-speed mass-bearing particles, for massless particles and for super-light-speed mass-bearing particles and have two solutions, which are

$$\left(\frac{dt}{d\tau}\right)_{1,2}=\frac{v_i\mathrm{v}^i\pm c^2}{c^2\left(1-\dfrac{w}{c^2}\right)}\,. \tag{1.55}$$

As shown in [15, 16] time has direct flow if $v_i\mathrm{v}^i\pm c^2>0$, time has reverse flow if $v_i\mathrm{v}^i\pm c^2<0$, and flow of time stops if $v_i\mathrm{v}^i\pm c^2=0$. Therefore there exists a whole range of solutions for various types of particles and directions they travel in time in respect to observer. For instance, relativistic mass of particle, which is projection of its four-dimensional vector on time $\frac{P_0}{\sqrt{g_{00}}}=\pm m$ is positive if particle travels into future and negative if it travels into past. Wave number of massless particle $\frac{K_0}{\sqrt{g_{00}}}=\pm k$ is also positive for movement into future and is negative for movement into past.

In the studies [15, 16] we also showed that chronometrically invariant dynamic equations of motion of free mass-bearing particles with reverse flow of time $\frac{dt}{d\tau}<0$ that travel from future into past are

$$-\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=0\,, \tag{1.56}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau}+mF^i+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=0\,. \tag{1.57}$$

For free massless particles that travel into past we arrive to

$$-\frac{dk}{d\tau}-\frac{k}{c^2}F_ic^i+\frac{k}{c^2}D_{ik}c^ic^k=0\,, \tag{1.58}$$

$$\frac{d\left(kc^i\right)}{d\tau}+kF^i+k\triangle^i_{nk}c^nc^k=0\,. \tag{1.59}$$

For super-light-speed mass-bearing particles equations of motion are similar to those for sub-light speeds, save that relativistic mass m is multiplied by imaginary unit i.

Equations of motion of particles into future and into past are not symmetric due to different physical conditions in case of direct and reverse time flows, and some terms in equations will be missing.

Besides, in our previous studies [15, 16] we considered motion of mass-bearing and massless particles within the wave-particle concept, assuming that motion of any particles can be represented as propagation of waves in approximation of geometric optics. In this case the dynamic vector of massless particles will be [1]

$$K_\alpha=\frac{\partial\psi}{\partial x^\alpha}\,, \tag{1.60}$$

where ψ is wave phase (eikonal). In a similar way we consider dynamic vector of mass-bearing particles

$$P_\alpha=\frac{\hbar}{c}\frac{\partial\psi}{\partial x^\alpha}\,, \tag{1.61}$$

where $\hbar$ is Plank constant. Wave phase equation (eikonal equation) in approximation of geometric optics is the condition $K_\alpha K^\alpha=0$. Hence chronometrically invariant eikonal equation for massless particles will be

$$\frac{1}{c^2}\left(\frac{^*\partial\psi}{\partial t}\right)^2+h^{ik}\frac{^*\partial\psi}{\partial x^i}\frac{^*\partial\psi}{\partial x^k}=0\,, \tag{1.62}$$

and for mass-bearing particles

$$\frac{1}{c^2}\left(\frac{^*\partial\psi}{\partial t}\right)^2+h^{ik}\frac{^*\partial\psi}{\partial x^i}\frac{^*\partial\psi}{\partial x^k}=\frac{m_0^2c^2}{\hbar^2}\,. \tag{1.63}$$

Substituting wave form of dynamic vector into general covariant dynamic equations of motion (1.10, 1.12) and their projection on time and space we obtain wave form of chronometrically invariant equations of motion. For mass-bearing particles the equations are

$$\pm\frac{d}{d\tau}\left(\frac{^*\partial\psi}{\partial t}\right)+F^i\frac{^*\partial\psi}{\partial x^i}-D^i_k\mathrm{v}^k\frac{^*\partial\psi}{\partial x^i}=0\,, \tag{1.64}$$

$$\begin{gathered}\frac{d}{d\tau}\left(h^{ik}\frac{^*\partial\psi}{\partial x^k}\right)-\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\left(\pm\frac{1}{c^2}\frac{^*\partial\psi}{\partial t}\mathrm{v}^k-h^{km}\frac{^*\partial\psi}{\partial x^m}\right)\pm\\ \pm\frac{1}{c^2}\frac{^*\partial\psi}{\partial t}F^i+h^{mn}\triangle^i_{mk}\mathrm{v}^k\frac{^*\partial\psi}{\partial x^n}=0\,,\end{gathered} \tag{1.65}$$

where “plus” in alternating terms stands for motion of particles from past into future (direct flow of time), while “minus” stands for motion into past (reverse flow of time). Noteworthy, contrary to corpuscular form of equations of motion (1.51, 1.52) and (1.56, 1.57) these equations are symmetric in respect to direction of motion in time. For massless particles wave form of chronometrically invariant equations of motion shows the only difference: instead of three-dimensional observable velocity of particle v^i it includes three-dimensional chronometrically invariant vector of velocity of light c^i.

The fact that corpuscular equations of motion into past and into future are not symmetric leads to evident conclusion that in four-dimensional non-uniform space-time there exists a fundamental asymmetry of directions in time. To understand physical sense of this fundamental asymmetry in previous study we introduced *the mirror principle* or *observable effect of the mirror Universe* [16].

Imagine *a mirror* in four-dimensional space-time which coincides with spatial section and therefore separates past from future. Then particles and waves traveling from past into future (with positive relativistic mass and frequency) hit the mirror and bounce back in time, i. e. into past. And their properties take negative values. And vice versa, particles and waves traveling into past (with negative relativistic mass and frequency) bounce from the mirror to give positive values to their properties and to begin traveling into future. When bouncing from the mirror the value $\frac{{}^*\partial\psi}{dt}$ changes its sign and equations of wave propagation into future become equations of wave propagation into past (and vice versa). Noteworthy, when reflecting from the mirror equations of wave propagation transform into each other *completely* without contracting or adding new terms. In other words, wave form of matter undergoes *full reflection* from our mirror. To the contrary, corpuscular equations of motion *do not transform completely* in reflection from our mirror. Spatial components of equations for mass-bearing and massless particles, traveling from past into future, have an additional term

$$2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k, \qquad 2k\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k, \tag{1.66}$$

not found in equations of motion from future into past. Equations of motion of particle into past gain an additional term when reflecting from the mirror. And vice versa, equations of motion into future lose a term when particle hits the mirror. That implies that either in case of motion of particles (corpuscular equations) as well as in case of propagation of waves (wave equations) we come across not a simple "bouncing" from the mirror, but rather *passing* through the mirror itself into another world, i. e. into *a mirror world*.

In this *mirror world* all particles bear negative masses and frequencies and travel (from viewpoint of our world's observer) from future into past. Wave form of matter in our world does not affect events in the mirror world, while wave form of matter in the mirror world does not affect events in our world. To the contrary, corpuscular form of matter (particles) in our world may produce significant effect on events in the mirror world, while particles in the mirror world may affect events in our world. Our world is fully isolated from the mirror world (no mutual effect between particles from two worlds) under an evident condition $D^i_k\mathrm{v}^k{=}-A^{\cdot i}_{k\cdot}\mathrm{v}^k$, at which the additional term in corpuscular equations is zero. This becomes true, in particular, when $D^i_k{=}0$ and $A^{\cdot i}_{k\cdot}{=}0$, i. e. when dynamic deformation and rotation of the body of reference's space is totally absent.

So far we have only considered motion of particles along non-isotropic trajectories, where $ds^2{=}c^2dt^2{-}d\sigma^2{>}0$, and that along isotropic (light-like) trajectories, where $ds^2{=}0$ and $c^2dt^2{=}d\sigma^2{\neq}0$. Besides, in our previous studies [15, 16] we considered trajectories of third kind, which, aside for $ds^2{=}0$, meet even more strict conditions $c^2dt^2{=}d\sigma^2{=}0$

$$d\tau = \left[1 - \frac{1}{c^2}\left(w + v_i u^i\right)\right] dt = 0\,, \tag{1.67}$$

$$d\sigma^2 = h_{ik}dx^i dx^k = 0\,. \tag{1.68}$$

We will refer to such trajectories as *degenerated* or *zero trajectories*, because from viewpoint of a regular sub-light-speed observer interval of observable time and observable three-dimensional interval are zero along them. We can as well show that along zero-trajectories the determinant of fundamental metric tensor of Riemannian space is also zero $g{=}0$. In Riemannian space by definition $g{<}0$, i. e. the metric in strictly non-degenerated. We will refer to a space with fully degenerated metric as *zero-space*, while particles that move along trajectories in such space will be referred to as *zero-particles*.

Physical conditions of degeneration are obtained from (1.67, 1.68)

$$w + v_i u^i = c^2, \tag{1.69}$$

$$g_{ik} u^i u^k = c^2 \left(1 - \frac{w}{c^2}\right)^2 . \tag{1.70}$$

Respectively, mass of zero-particles M, which include physical conditions of degeneration, is different from relativistic mass m of regular particles in non-degenerated space-time and is

$$M = \frac{m}{1 - \frac{1}{c^2}\left(w + v_i u^i\right)} , \tag{1.71}$$

i. e. is a ratio between two values, each one equals to zero in case of degenerated metric, but the ratio is not zero[7].

Corpuscular and wave forms of dynamic vector of zero-particles are

$$P^\alpha = \frac{M}{c}\frac{dx^\alpha}{dt} , \qquad P_\alpha = \frac{\hbar}{c}\frac{\partial\psi}{\partial x^\alpha} . \tag{1.72}$$

Then corpuscular form of chronometrically invariant dynamic equations of motion in zero-space is

$$M D_{ik} u^i u^k = 0 , \tag{1.73}$$

$$\frac{d}{dt}\left(M u^i\right) + M \triangle^i_{nk} u^n u^k = 0 . \tag{1.74}$$

Wave form of the same equations is

$$D^m_k u^k \frac{{}^*\partial\psi}{\partial x^m} = 0 , \tag{1.75}$$

$$\frac{d}{dt}\left(h^{ik}\frac{{}^*\partial\psi}{\partial x^k}\right) + h^{mn}\triangle^i_{mk} u^k \frac{{}^*\partial\psi}{\partial x^n} = 0 . \tag{1.76}$$

Equation of eikonal for zero-particles is

$$h^{ik}\frac{{}^*\partial\psi}{\partial x^i}\frac{{}^*\partial\psi}{\partial x^k} = 0 , \tag{1.77}$$

and is a standing wave equation (information ring). Therefore, from viewpoint of a regular sub-light-speed observer all zero-space is filled with a system of standing light-like waves (zero-particles), i. e. with *a standing-light hologram*. Besides, in zero-space observable time has the same value for any two events (1.67). This implies that from viewpoint of a regular observer velocity of zero-particles is infinite, i. e. zero-particles can instantly transfer information from one point of our regular world to another, thus performing *long-range action* [15, 16].

1.4 Introducing concept of nongeodesic motion of particles. Problem statement

We obtained that free motion of particle (along geodesic lines) leaves absolute derivative of dynamic vector of particle (four-dimensional impulse vector) zero and its square is conserved along the trajectory of motion. In other words, parallel transfer is effected in the meaning of Levi-Civita.

In case of non-free (non-geodesic) motion of particle absolute derivative of its four-dimensional impulse is not zero. But equal to zero is absolute derivative of sum of four-dimensional impulse of particle P^α and impulse vector L^α, which particle gains from interaction with external fields which

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[7] This is similar to the case of massless particles, because given $\mathrm{v}^2 = c^2$ values $m_0 = 0$ and $\sqrt{1-\mathrm{v}^2/c^2} = 0$ are zero, but their ratio is $m = \frac{m_0}{\sqrt{1-\mathrm{v}^2/c^2}} \neq 0$.

deviates its motion from geodesic line. Superposition of any number of vectors can be subjected to parallel transfer [2]. Hence, building dynamic equations of non-geodesic motion of particles first of all requires definition of non-gravitational perturbation fields.

Naturally, external field will only interact with particle and deviate it from geodesic line if the particle bears physical property of the same kind as the external field does. As of today, we know of three fundamental physical properties of particles, not related to any others. These are *mass* of particle, *electric charge* and *spin*. If fundamental character of the former two was under no doubt, spin of electron over a few years after experiments by O. Stern and W. Gerlach (1921) and their interpretation by S. Gaudsmith and G. Ulenbek (1925), was considered as its specific moment of impulse caused by rotation around its own axis. But experiments done over the next decades, in particular, discovery of spin in other elementary particles, proved that views of spin particles as rotating gyroscopes were wrong. Spin proved to be a fundamental property of particles just like mass and charge, though it has dimension of moment of impulse and in interactions reveals as specific rotation moment of particle.

Gravitational field by now has received geometric interpretation. In theory of chronometric invariants gravitational force and gravitational potential (1.38) are obtained as functions of only geometric properties of the space itself. Therefore considering motion of particle in pseudo-Riemannian space we actually consider its motion in gravitational field.

But we still do not know whether electromagnetic force and potential can be expressed through geometric properties of space. Therefore electromagnetic field at the moment has no geometric interpretation and is introduced into space-time as a separate tensor field (Maxwell tensor field). By now the basic equations of electromagnetic theory have been obtained in general covariant form[8]. In this theory charged particle gains four-dimensional impulse $\frac{e}{c^2}A^\alpha$ from electromagnetic field, where A^α is four-dimensional potential of electromagnetic field and e is particle's charge [1, 4]. Adding this extra impulse to specific vector of impulse of particle and actuating parallel transfer we obtain general covariant dynamic equations of motion of charged particles.

The case of spin particles is far more complicated. To calculate impulse that particle gains due to its spin, we have to define the external field that interacts with spin as a fundamental property of particle. Initially this problem was approached using methods of quantum mechanics only (Dirac equations, 1928). Methods of General Relativity were first used by A. Papapetrou and E. Corinaldesi [20, 21] to study the problem. Their approach relied upon general view of particles as mechanical monopoles and dipoles. From this viewpoint a regular mass-bearing particle is a *mechanical monopole*. A particle that can be represented as two masses co-rotating around a common center of gravity is a *mechanical dipole*. Therefore, proceeding from representation of spin particle as a rotating gyroscope we can (to a certain extent) consider it as a mechanical dipole, which center of gravity lays over the particle's surface. Then Papapetrou and Corinaldesi considered motion of mechanic dipole in pseudo-Riemannian space with Schwarzschild metric, i. e. in every specific case when rotation of space is zero and its metric is stationary (tensor of deformation velocities is zero).

No doubt the method proposed by Papapetrou is worth attention, but it has a significant drawback. Being developed in 1940's it fully relied upon view of a spin-particle as a swiftly rotating gyroscope, which does not match experimental data of the recent decades[9].

There is another way to tackle the problem of motion of spin particles. In Riemannian space fundamental metric tensor is symmetric $g_{\alpha\beta}=g_{\beta\alpha}$. Nevertheless we can build a space in which metric tensor will have arbitrary form $g_{\alpha\beta}\neq g_{\beta\alpha}$ (such space will have non-Riemannian geometry). Then a non-zero antisymmetric part can be found in metric tensor[10]. Appropriate additions will also appear

[8] Despite this, due to complicated calculations of energy-impulse tensor of electromagnetic field in pseudo-Riemannian space, specific problems are commonly solved either for certain particular cases of General Relativity or in Galilean frame of reference in a flat Minkowski space (space-time of Special Relativity).

[9] As a matter of fact, considering electron as a ball with radius of $r_{\mathrm{e}}=2.8{\cdot}10^{-13}$ cm implies that linear speed of its rotation on the surface is $u=\frac{\hbar}{2m_0 r_{\mathrm{e}}}=2{\cdot}10^{11}$ cm/s which is about 70 times as high as the light speed. But experiments show there are no such speeds in electron.

[10] Generally, in any tensor of 2nd rank and above symmetric and antisymmetric parts can be distinguished. For instance, in 2nd rank fundamental metric tensor $g_{\alpha\beta}=\frac{1}{2}\left(g_{\alpha\beta}+g_{\beta\alpha}\right)+\frac{1}{2}\left(g_{\alpha\beta}-g_{\beta\alpha}\right)=S_{\alpha\beta}+N_{\alpha\beta}$, where $S_{\alpha\beta}$ is symmetric part and $N_{\alpha\beta}$ is antisymmetric part of tensor $g_{\alpha\beta}$. Because metric tensor of Riemannian space is symmetric $g_{\alpha\beta}=g_{\beta\alpha}$, its antisymmetric part is zero.

in Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$ (1.2) and in Riemann-Christoffel curvature tensor $R_{\alpha\beta\mu\nu}$. These additions will cause a vector transferred along a closed contour not to return into the initial point, i. e. trajectory of transfer becomes twisted like a spiral. Such space is referred to as twisted space. In such space spin rotation of particle can be considered as transfer of rotation vector along its surface contour, which generates local field of space twist.

But this method has got significant drawbacks as well. First, with $g_{\alpha\beta} \neq g_{\beta\alpha}$ functions of components with different order of indices may be varied. The functions have to be fixed somehow in to order to set a concrete field of twist, which dramatically narrows the range of possible solutions, enabling only building equations for a range of specific cases. Second, the method fully relies upon assumption of spin's physical nature as a local field of twist produced by transfer of vector of particle's rotation along a contour. This, in its turn, again implies the view of spin particle as a rotating gyroscope with a limited radius (like in Papapetrou's method), which does not match experimental data.

Nevertheless, there is little doubt in that additional impulse, which a spin-particle gains, can be represented with methods of General Relativity. Adding it to the specific dynamic vector of a particle (effect of gravitation) and accomplishing parallel transfer, we obtain general covariant dynamic equations of motion of a spin-particle.

Once we have general covariant dynamic equations of motion of spin and electric charged particles obtained, we should project them on space and time of accompanying frame of reference and express through physical observable properties of the space of reference. As a result we arrive to chronometrically invariant (physical observable) dynamic equations of nongeodesic motion of particles.

Therefore, the problem we are going to tackle in this book falls apart into a few stages. First, we should build chronometrically invariant theory of electromagnetic field in pseudo-Riemannian space and arrive to chronometrically invariant dynamic equations of motion of charged particle. This problem will be solved in Chapter 3.

Then, we have to create a theory of motion of a spin-particle. We will approach the problem in its most general form, assuming spin a fundamental property of matter (like mass or electric charge). In Chapter 4 detailed study will show that field of non-holonomity of space interacts with spin giving particle additional impulse.

In Chapter 5 we are going to discuss observable projections of Einstein equations. Proceeding from them we will study properties of physical vacuum and how they are dealt in cosmology.

In Chapter 6 we prove that fully degenerated space-time (zero-space) is an area inhabited by virtual particles, and build geometric concept of annihilation of particles using methods of General Relativity. Within the concept, we explain anomalous rate of annihilation of othopositronium.

But before turning to these studies we would like to have a look into four-dimensional tensor analysis in terms of physical observable values (chronometric invariants). Original publications by Zelmanov gave a very fragmented account of the subject, which prevented a reader not familiar with this mathematical apparatus from learning it on their own. Therefore we recommend our Chapter 2 to readers who are going to use mathematical apparatus of chronometric invariants in their theoretic studies. For general understanding of our book, though, reading this Chapter may be not necessary.

Chapter 2

Tensor algebra and the analysis

2.1 Tensors and tensor algebra

We assume a space (not necessarily a metric one) with an arbitrary frame of reference x^α. In some part of the space, there exists an object G defined by n functions f_n of coordinates x^α. We know the transformation rule to calculate these n functions in any other frame of reference $\tilde{x}^\alpha$ in this space. Given all this G is *a geometric object*, which in the frame of reference x^α has axial components $f_n(x^\alpha)$, while in any other frame of reference $\tilde{x}^\alpha$ it has components $\tilde{f}_n(\tilde{x}^\alpha)$.

We assume a tensor object (*tensor*) of zero rank is any geometric object φ, transformable according to the rule

$$\tilde{\varphi} = \varphi \frac{\partial x^\alpha}{\partial \tilde{x}^\alpha}\,, \tag{2.1}$$

where the index takes in turn numbers of all coordinate axis (such notation is referred to as *by-component notation* or *tensor notation*). Zero rank tensor has got a single component and is also known as *scalar*. Scalar in space is a point to which a certain number is attributed.

Consequently, scalar field[11] is a set of points in space, which have some common property. For instance, mass of a material point is a scalar, while distribution of mass in gas makes up a scalar field.

Contravariant tensor of 1st rank is a geometric object A^α with components transformable according to the rule

$$\tilde{A}^\alpha = A^\mu \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\,. \tag{2.2}$$

From geometric viewpoint it is a n-dimensional vector. For instance, vector of displacement dx^α is a contravariant tensor of 1st rank.

Contravariant tensor of 2nd rank $A^{\alpha\beta}$ is a geometric object with components transformable according to the rule

$$\tilde{A}^{\alpha\beta} = A^{\mu\nu} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\,. \tag{2.3}$$

From geometric viewpoint this is an area (parallelogram) constrained by two vectors. Therefore 2nd rank contravariant tensor is sometimes referred to as *bivector*.

Similarly, contravariant tensors of higher ranks are

$$\tilde{A}^{\alpha\ldots\sigma} = A^{\mu\ldots\tau} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\cdots\frac{\partial \tilde{x}^\sigma}{\partial x^\tau}\,. \tag{2.4}$$

Vector field or field of tensors of higher rank is also space distribution of these values. For instance, because mechanical stress characterizes both magnitude and direction, its distribution in a physical body can be presented as a vector field.

Covariant tensor of 1st rank A_α is a geometric object, transformable according to the rule

$$\tilde{A}_\alpha = A_\mu \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\,. \tag{2.5}$$

[11] Algebraic notations of a tensor and of a tensor field are the same: field of a tensor is represented as a tensor at a point in space, but its presence at other points in this part of the space is assumed.

In particular, gradient of scalar field of any invariant φ, i. e. the value $A_\alpha = \frac{\partial\varphi}{\partial x^\alpha}$, is a covariant tensor of 1st rank. That is, because for a regular invariant we have $\tilde{\varphi} = \varphi$, then

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{x}^\alpha} = \frac{\partial\tilde{\varphi}}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha} = \frac{\partial\varphi}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\,. \tag{2.6}$$

Covariant tensor of 2nd rank $A_{\alpha\beta}$ is a geometric object the transformation rule for which is

$$\tilde{A}_{\alpha\beta} = A_{\mu\nu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}\,. \tag{2.7}$$

Similarly, covariant tensors of higher ranks are

$$\tilde{A}_{\alpha\ldots\sigma} = A_{\mu\ldots\tau}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\cdots\frac{\partial x^\tau}{\partial\tilde{x}^\sigma}\,. \tag{2.8}$$

Mixed tensors are tensors of 2nd rank and above with both upper and lower indices. For instance, mixed symmetric tensor A^α_β is a geometric object transformable according to the rule

$$\tilde{A}^\alpha_\beta = A^\mu_\nu\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}\,. \tag{2.9}$$

Tensor objects exist both in metric and non-metric spaces, where distance between any two points can not be measured.

A tensor has a^n components, where a is dimension of the tensor and n is the rank. For instance, four-dimensional tensor of zero rank has 1 component, 1st rank tensor has 4 components, 2nd rank tensor has 16 components and so on. But indices, i. e. axial components, are found not in tensors only, but in other geometric objects as well. Therefore, if we come across a value in by-component notation, this is not necessarily a tensor value.

In practice, to know whether a given object is a tensor or not, we have to know the equation for this object in a certain frame of reference and to transform it to any other frame of reference. For instance: are coefficients of coherence of space, i. e. Christoffel symbols, tensors?

To know this, we have to calculate the values in another (tilde-marked) frame of reference

$$\widetilde{\Gamma}^\alpha_{\mu\nu} = \tilde{g}^{\alpha\sigma}\widetilde{\Gamma}_{\mu\nu,\sigma}\,, \quad \widetilde{\Gamma}_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^\nu} + \frac{\partial\tilde{g}_{\nu\sigma}}{\partial\tilde{x}^\mu} - \frac{\partial\tilde{g}_{\mu\nu}}{\partial\tilde{x}^\sigma}\right) \tag{2.10}$$

proceeding from values in non-marked frame of reference.

Now we are going to calculate the terms in brackets (1.10). Fundamental metric tensor, just like any other covariant 2nd rank tensor, is transformable to tilde-marked frame of reference according to the rule

$$\tilde{g}_{\mu\sigma} = g_{\varepsilon\tau}\frac{\partial x^\varepsilon}{\partial\tilde{x}^\mu}\frac{\partial x^\tau}{\partial\tilde{x}^\sigma}\,. \tag{2.11}$$

Because $g_{\varepsilon\tau}$ depends upon non-tilde-marked coordinates, its derivative by tilde-marked coordinates (which are also functions of non-tilde-marked ones) is calculated according to the rule

$$\frac{\partial g_{\varepsilon\tau}}{\partial\tilde{x}^\nu} = \frac{\partial g_{\varepsilon\tau}}{\partial x^\rho}\frac{\partial x^\rho}{\partial\tilde{x}^\nu}\,. \tag{2.12}$$

Then the first term in brackets (2.10) taking into account the rule of transformation of fundamental metric tensor, is

$$\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^\nu} = \frac{\partial g_{\varepsilon\tau}}{\partial x^\rho}\frac{\partial x^\rho}{\partial\tilde{x}^\nu}\frac{\partial x^\varepsilon}{\partial\tilde{x}^\mu}\frac{\partial x^\tau}{\partial\tilde{x}^\sigma} + g_{\varepsilon\tau}\left(\frac{\partial x^\tau}{\partial\tilde{x}^\sigma}\frac{\partial^2 x^\varepsilon}{\partial\tilde{x}^\nu\partial\tilde{x}^\mu} + \frac{\partial x^\varepsilon}{\partial\tilde{x}^\mu}\frac{\partial^2 x^\tau}{\partial\tilde{x}^\nu\partial\tilde{x}^\sigma}\right). \tag{2.13}$$

Similarly, calculating the rest of the terms of tilde-marked Christoffel symbols (2.10), after transposition of free indices we arrive to

$$\widetilde{\Gamma}_{\mu\nu,\sigma} = \Gamma_{\varepsilon\rho,\tau}\frac{\partial x^\varepsilon}{\partial\tilde{x}^\mu}\frac{\partial x^\rho}{\partial\tilde{x}^\nu}\frac{\partial x^\tau}{\partial\tilde{x}^\sigma} + g_{\varepsilon\tau}\frac{\partial x^\tau}{\partial\tilde{x}^\sigma}\frac{\partial^2 x^\varepsilon}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\,, \tag{2.14}$$

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\gamma}_{\varepsilon\rho}\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}} + \frac{\partial\tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial^2 x^{\gamma}}{\partial\tilde{x}^{\mu}\partial\tilde{x}^{\nu}}\,. \tag{2.15}$$

We see that coefficients of space coherence (Christoffel symbols) are transformed not in the way tensors are, hence they are not tensors.

Tensors can be represented as matrices. But in practice, such form may be illustrative for tensors of 1st and 2nd rank (single-row and flat matrices, respectively). For instance, elementary four-dimensional displacement tensor is

$$dx^{\alpha} = \left(dx^0, dx^1, dx^2, dx^3\right), \tag{2.16}$$

and four-dimensional fundamental metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{2.17}$$

Tensor of 3rd rank is a three-dimensional matrix. Representing tensors of higher ranks as matrices is even more problematic.

We now turn to tensor algebra — a part of tensor calculus that focuses on algebraic operations over tensors.

Only same-type tensors of the same rank with indices in the same position can be added or subtracted. *Adding up* two same-type n-rank tensors gives a new tensor of the same type and rank with components being sums of respective components of the tensors added up. For instance, sum of two vectors and sum of two mixed 2nd rank tensors are

$$A^{\alpha} + B^{\alpha} = D^{\alpha}, \qquad A^{\alpha}_{\beta} + B^{\alpha}_{\beta} = D^{\alpha}_{\beta}\,. \tag{2.18}$$

Multiplication is permitted not only for same-type, but for any tensors of any ranks. *External multiplication* of n-rank and m-rank tensors gives an $(n+m)$-rank tensor

$$A_{\alpha\beta}B_{\gamma} = D_{\alpha\beta\gamma}\,, \qquad A_{\alpha}B^{\beta\gamma} = D^{\beta\gamma}_{\alpha}. \tag{2.19}$$

Contraction is multiplication of same-rank tensors when indices are the same. Contraction of tensors by all indices gives a scalar value

$$A_{\alpha}B^{\alpha} = C\,, \qquad A^{\gamma}_{\alpha\beta}B^{\alpha\beta}_{\gamma} = D\,. \tag{2.20}$$

Often multiplication of tensors implies contraction by not all indices. Such multiplication is referred to as *internal multiplication* which implies contraction of some indices inside the multiplication

$$A_{\alpha\sigma}B^{\sigma} = D_{\alpha}\,, \qquad A^{\gamma}_{\alpha\sigma}B^{\beta\sigma}_{\gamma} = D^{\beta}_{\alpha}\,. \tag{2.21}$$

Using internal multiplication of geometric objects we can find whether they are tensors or not. There is a so-called theorem of fractions.

Theorem of fractions

"If $B^{\sigma\beta}$ is a tensor and its internal multiplication with a geometric object $A\left(\alpha,\sigma\right)$ is tensor $D\left(\alpha,\beta\right)$

$$A\left(\alpha,\sigma\right)B^{\sigma\beta} = D\left(\alpha,\beta\right), \tag{2.22}$$

then this object $A\left(\alpha,\sigma\right)$ is also a tensor" [10].

According to it, if internal multiplication of an object $A_{\alpha\sigma}$ with tensor $B^{\sigma\beta}$ gives tensor D^{β}_{α}

$$A_{\alpha\sigma}B^{\sigma\beta} = D^{\beta}_{\alpha}\,, \tag{2.23}$$

then object $A_{\alpha\sigma}$ is a tensor. Or, if internal multiplication of some object A^{α}_{σ} and tensor $B^{\sigma\beta}$ gives tensor $D^{\alpha\beta}$

$$A^{\alpha\cdot}_{\cdot\sigma}B^{\sigma\beta} = D^{\alpha\beta}, \tag{2.24}$$

then object $A^{\alpha\cdot}_{\cdot\sigma}$ is a tensor.

Geometric properties of metric space are defined by its fundamental metric tensor $g_{\alpha\beta}$, which may lower or lift indices in objects of metric space[12]. For example,

$$g_{\alpha\beta}A^{\beta} = A_{\alpha}\,, \qquad g^{\mu\nu}g^{\sigma\rho}A_{\mu\nu\sigma} = A^{\rho}. \tag{2.25}$$

In Riemannian space mixed fundamental metric tensor g^{β}_{α} equals to unit tensor $g^{\beta}_{\alpha}{=}g_{\alpha\sigma}g^{\sigma\beta}{=}\delta^{\beta}_{\alpha}$. Diagonal components of four-dimensional unit tensor are ones, while the rest are zeroes. Using the unit tensor we can replace indices

$$\delta^{\beta}_{\alpha}A_{\beta} = A_{\alpha}\,, \qquad \delta^{\nu}_{\mu}\delta^{\sigma}_{\rho}A^{\mu\rho} = A^{\nu\sigma}. \tag{2.26}$$

Contraction of 2nd rank tensor with fundamental metric tensor gives a scalar value known as *spur of tensor* or *trace of tensor*

$$g^{\alpha\beta}A_{\alpha\beta} = A^{\sigma}_{\sigma}\,. \tag{2.27}$$

For example, spur of fundamental metric tensor in four-dimensional Riemannian space equals to the number of coordinate axis

$$g_{\alpha\beta}g^{\alpha\beta} = g^{\sigma}_{\sigma} = g^{0}_{0} + g^{1}_{1} + g^{2}_{2} + g^{3}_{3} = 4. \tag{2.28}$$

Physical observable metric tensor h_{ik} (1.27) in three-dimensional space has properties of fundamental metric tensor. Therefore it can lower, lift or replace indices in chronometrically invariant values. Namely, we can calculate squares of four-dimensional objects. Respectively, spur of three-dimensional tensor is obtained by means of its contraction with observable metric tensor.

For instance, spur of tensor of velocities of space deformation D_{ik} (1.40) is a scalar

$$h^{ik}D_{ik} = D^{m}_{m}\,, \tag{2.29}$$

that stands for absolute value of the speed of relative expansion of elementary volume of space.

Of course our brief account can not fully cover such a vast field like tensor algebra. Moreover, there is even no need in doing that here. Detailed accounts of tensor algebra can be found in numerous mathematical books not related to General Relativity. Besides, many specific techniques of this science, which occupy substantial part of mathematical textbooks, are not used in theoretical physics. Therefore our goal was to give only a basic introduction into tensors and tensor algebra, necessary for understanding this book. For the same reasons we have not covered issues like weight of tensors or many others not used in calculations given in the below.

2.2 Scalar product of vectors

Scalar product of two vectors A^{α} and B^{α} in four-dimensional pseudo-Riemannian space is value

$$g_{\alpha\beta}A^{\alpha}B^{\beta} = A_{\alpha}B^{\alpha} = A_{0}B^{0} + A_{i}B^{i}. \tag{2.30}$$

Scalar product is contraction because multiplication of vectors at the same time contracts all indices. Therefore scalar product of two vectors (1st rank tensors) is always a scalar value (zero rank tensor).

If both vectors are the same, their scalar product

$$g_{\alpha\beta}A^{\alpha}A^{\beta} = A_{\alpha}A^{\alpha} = A_{0}A^{0} + A_{i}A^{i} \tag{2.31}$$

is the square of vector A^{α}. Consequently length of a vector A^{α} is a scalar

$$A = |A^{\alpha}| = \sqrt{g_{\alpha\beta}A^{\alpha}A^{\beta}}. \tag{2.32}$$

[12]In Riemannian space metric has quadratic form $ds^2{=}g_{\alpha\beta}dx^{\alpha}dx^{\beta}$, and respectively fundamental metric tensor is a 2nd rank tensor $g_{\alpha\beta}$.

Because four-dimensional pseudo-Riemannian space by its definition has indefinite metric (i. e. sign-alternating signature), then length of four-dimensional vector may be real, imaginary or zero. According to this, vectors with non-zero (real or imaginary) length are referred to as *non-isotropic vectors*. Vectors with zero length are referred to as *isotropic vectors*. Isotropic vectors are tangential to trajectories of propagation of light-like particles (isotropic trajectories).

In three-dimensional Euclidean space scalar product of two vectors is a scalar value with module equal to product of lengths of the two vectors multiplied by cosine of the angle between them

$$A_i B^i = \left|A^i\right| \left|B^i\right| = \cos\left(\widehat{A^i ; B^i}\right). \tag{2.33}$$

Theoretically at every point of Riemannian space a tangential flat space can be set, which basic vectors will be tangential to basic vectors of Riemannian space at the tangential points. Then metric of tangential flat space will be metric of Riemannian space at this point. Therefore this statement is also true in Riemannian space if we consider the angle between coordinate lines and replace Roman (three-dimensional) indices with Greek ones.

From here we can see that scalar product of two multiplicated vectors is zero if the vectors are orthogonal. In other words, scalar product from geometric viewpoint is projection of one vector onto another. If multiplicated vectors are the same, vector is projected onto itself and the result of projection is its length's square.

We will denote chronometrically invariant (physical observable) components of arbitrary vectors A^α and B^α as

$$a = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad a^i = A^i, \tag{2.34}$$

$$b = \frac{B_0}{\sqrt{g_{00}}}\,, \qquad b^i = B^i. \tag{2.35}$$

Then the other components are

$$A^0 = \frac{a + \frac{1}{c} v_i a^i}{1 - \frac{w}{c^2}}\,, \qquad A_i = -a_i - \frac{a}{c} v_i\,, \tag{2.36}$$

$$B^0 = \frac{b + \frac{1}{c} v_i b^i}{1 - \frac{w}{c^2}}\,, \qquad B_i = -b_i - \frac{b}{c} v_i\,. \tag{2.37}$$

Substituting values of observable components into formulas for $A_\alpha B^\alpha$ and $A_\alpha A^\alpha$ we arrive to

$$A_\alpha B^\alpha = ab - a_i b^i = ab - h_{ik} a^i b^k, \tag{2.38}$$

$$A_\alpha A^\alpha = a^2 - a_i a^i = a^2 - h_{ik} a^i a^k. \tag{2.39}$$

From here we see that the square of vector's length is difference between squares of lengths of its projections onto time and space. If both projections are equal, the vector's length is zero and it is isotropic. Hence isotropic vector equally belongs to time and space. Equality of time and space projections also implies that the vector is orthogonal to itself. If temporal projection is "longer", it becomes real. If spatial projection is "longer" the vector becomes imaginary.

Scalar product of four-dimensional vector with itself can be instanced by square of length of space-time interval

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dx_\alpha dx^\alpha = dx_0 dx^0 + dx_i dx^i. \tag{2.40}$$

In terms of physical observable values it can be represented as

$$ds^2 = c^2 d\tau^2 - dx_i dx^i = c^2 d\tau^2 - h_{ik} dx^i dx^k = c^2 d\tau^2 - d\sigma^2. \tag{2.41}$$

Length of interval $ds{=}\sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$ may be real, imaginary or zero depending upon whether ds is time-like $c^2 d\tau^2{>}d\sigma^2$ (sub-light real trajectories), space-like $c^2 d\tau^2{<}d\sigma^2$ (imaginary super-light trajectoties) or isotropic $c^2 d\tau^2{=}d\sigma^2$ (light-like trajectories).

2.3 Vector product of vectors. Antisymmetric tensors and pseudotensors

Vector product of two vectors A^α and B^α is a 2nd rank tensor $V^{\alpha\beta}$ obtained from their external multiplication according to the rule

$$V^{\alpha\beta} = \left[A^\alpha; B^\beta\right] = \frac{1}{2}\left(A^\alpha B^\beta - A^\beta B^\alpha\right) = \frac{1}{2}\begin{vmatrix} A^\alpha & A^\beta \\ B^\alpha & B^\beta \end{vmatrix}. \tag{2.42}$$

As seen, here the order in which vectors are multiplied does matter, i. e. the order in which we write down tensor indices. Therefore tensors obtained as vector products are *antisymmetric tensors.* In an antisymmetric tensor $V^{\alpha\beta}{=}-V^{\beta\alpha}$ indices being moved "reserve" their places as dots $g_{\alpha\sigma}V^{\sigma\beta}{=}V_{\alpha\cdot}^{\cdot\beta}$, thus showing from where an index was moved. In symmetric tensors there is no need of "reserving" places for moved indices, because the order in which they appear does not matter. In particular, fundamental metric tensor is symmetric tensor $g_{\alpha\beta}{=}g_{\beta\alpha}$, while tensor of space curvature $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is symmetric in respect to transposition by pair of indices and is antisymmetric inside each pair of indices. Evidently, only tensor of 2nd rank or above may be symmetric or antisymmetric.

All diagonal components of any antisymmetric tensor by its definition are zeroes. For examlpe, in antisymmetric 2nd rank tensor we have

$$V^{\alpha\alpha} = [A^\alpha; B^\alpha] = \frac{1}{2}\left(A^\alpha B^\alpha - A^\alpha B^\alpha\right) = 0. \tag{2.43}$$

In three-dimensional Euclidean space absolute value of vector product of two vectors is defined as the area of the parallelogram they make and equals to product of modules of the two vectors multiplied by sine of the angle between them

$$\left|V^{ik}\right| = \left|A^i\right|\left|B^k\right| = \sin\left(\widehat{A^i; B^k}\right). \tag{2.44}$$

This implies that vector product of two vectors (antisymmetric 2nd rank tensor) is a pad oriented in space according to directions of the forming vectors.

Contraction of an antisymmetric tensor $V_{\alpha\beta}$ with any symmetric tensor $A^{\alpha\beta}{=}A^\alpha A^\beta$ is zero due to its properties $V_{\alpha\alpha}{=}0$ and $V_{\alpha\beta}{=}-V_{\beta\alpha}$

$$V_{\alpha\beta}A^\alpha A^\beta = V_{00}A^0A^0 + V_{0i}A^0A^i + V_{i0}A^iA^0 + V_{ik}A^iA^k = 0\,. \tag{2.45}$$

According to theory of chronometric invariants physical observable components of antisymmetric 2nd rank tensor $V^{\alpha\beta}$ are values

$$\frac{V_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = -\frac{V_{\cdot 0}^{i\cdot}}{\sqrt{g_{00}}} = \frac{1}{2}\left(ab^i - ba^i\right), \tag{2.46}$$

$$V^{ik} = \frac{1}{2}\left(a^ib^k - a^kb^i\right), \tag{2.47}$$

expressed through observable components of its forming vectors A^α and B^α. Because in an antisymmetric tensor all diagonal components are zeroes, the third observable component $\frac{V_{00}}{g_{00}}$ (1.32) is also zero.

Physical observable components V^{ik} (projections of $V^{\alpha\beta}$ upon spatial section of four-dimensional space-time) are the analog of vector product in three-dimensional space, while the value $\frac{V_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}$, which is space-time (mixed) projection of tensor $V^{\alpha\beta}$, has no analogs among components of a regular three-dimensional vector product.

Square of antisymmetric 2nd rank tensor, formulated with observable components of forming vectors, is

$$V_{\alpha\beta}V^{\alpha\beta} = \frac{1}{2}\left(a_ia^ib_kb^k - a_ib^ia_kb^k\right) + aba_ib^i - \frac{1}{2}\left(a^2b_ib^i - b^2a_ia^i\right). \tag{2.48}$$

The latter two terms in the formula contain values a (2.34) and b (2.35), which are projections of multiplied vectors A^α and B^α onto time and therefore have no analogs in vector product in three-dimensional Euclidean space.

Antisymmetry of tensor field is defined by reference antisymmetric tensor. In Galilean frame of reference[13] such references are Levi-Civita tensors: for four-dimensional values this is *four-dimensional completely antisymmetric unit tensor* $e^{\alpha\beta\mu\nu}$ and for three-dimensional values this is *three-dimensional completely antisymmetric unit tensor* e^{ikm}. Components of these tensors, which have all indices different, are either $+1$ or -1 depending upon the number of transpositions of indices. All other components, i. e. those having at least two coinciding indices, are zeroes. Moreover, for the signature we are using $(+---)$ all non-zero components bear the sign opposite to their respective covariant components[14]. For example, in Minkowski space

$$\begin{gathered} g_{\alpha\sigma}g_{\beta\rho}g_{\mu\tau}g_{\nu\gamma}e^{\sigma\rho\tau\gamma} = g_{00}g_{11}g_{22}g_{33}e^{0123} = -e^{0123}, \\ g_{i\alpha}g_{k\beta}g_{m\gamma}e^{\alpha\beta\gamma} = g_{11}g_{22}g_{33}e^{123} = -e^{123} \end{gathered} \tag{2.49}$$

due to signature conditions $g_{00}{=}1$ and $g_{11}{=}g_{22}{=}g_{33}{=}{-1}$. Therefore, components of tensor $e^{\alpha\beta\mu\nu}$ are

$$\begin{gathered} e^{0123} = +1, \quad e^{1023} = -1, \quad e^{1203} = +1, \quad e^{1230} = -1, \\ e_{0123} = -1, \quad e_{1023} = +1, \quad e_{1203} = -1, \quad e_{1230} = +1, \end{gathered} \tag{2.50}$$

and components of tensor e^{ikm} are

$$e^{123} = +1, \quad e^{213} = -1, \quad e^{231} = +1, \quad e_{123} = -1, \quad e_{213} = +1, \quad e_{231} = -1. \tag{2.51}$$

Because the sign of the first component is arbitrary, we can assume $e^{0123}{=}{-1}$ and $e^{123}{=}{-1}$. Subsequently, other components will change too. In general, four-dimensional tensor $e^{\alpha\beta\mu\nu}$ is related to three-dimensional tensor e^{ikm} as $e^{0ikm}{=}e^{ikm}$.

Multiplying four-dimensional antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$ by itself we obtain a regular 8th rank tensor with non-zero components, which are presented in the matrix

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\rho\gamma} = -\begin{pmatrix} \delta^\alpha_\sigma & \delta^\alpha_\tau & \delta^\alpha_\rho & \delta^\alpha_\gamma \\ \delta^\beta_\sigma & \delta^\beta_\tau & \delta^\beta_\rho & \delta^\beta_\gamma \\ \delta^\mu_\sigma & \delta^\mu_\tau & \delta^\mu_\rho & \delta^\mu_\gamma \\ \delta^\nu_\sigma & \delta^\nu_\tau & \delta^\nu_\rho & \delta^\nu_\gamma \end{pmatrix}. \tag{2.52}$$

Other properties of tensor $e^{\alpha\beta\mu\nu}$ are obtained from the previous one by means of contraction of indices

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\rho\nu} = -\begin{pmatrix} \delta^\alpha_\sigma & \delta^\alpha_\tau & \delta^\alpha_\rho \\ \delta^\beta_\sigma & \delta^\beta_\tau & \delta^\beta_\rho \\ \delta^\mu_\sigma & \delta^\mu_\tau & \delta^\mu_\rho \end{pmatrix}, \tag{2.53}$$

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\mu\nu} = -2\begin{pmatrix} \delta^\alpha_\sigma & \delta^\alpha_\tau \\ \delta^\beta_\sigma & \delta^\beta_\tau \end{pmatrix} = -2\left(\delta^\alpha_\sigma\delta^\beta_\tau - \delta^\beta_\sigma\delta^\alpha_\tau\right), \tag{2.54}$$

$$e^{\alpha\beta\mu\nu}e_{\sigma\beta\mu\nu} = -6\delta^\alpha_\sigma\,, \quad e^{\alpha\beta\mu\nu}e_{\alpha\beta\mu\nu} = -6\delta^\alpha_\alpha = -24. \tag{2.55}$$

Multiplying three-dimensional antisymmetric unit tensor e^{ikm} by itself we obtain a regular 6th rank tensor

$$e^{ikm}e_{rst} = \begin{pmatrix} \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^k_r & \delta^k_s & \delta^k_t \\ \delta^m_r & \delta^m_s & \delta^m_t \end{pmatrix}. \tag{2.56}$$

Other properties of tensor e^{ikm} can be expressed as

$$e^{ikm}e_{rsm} = -\begin{pmatrix} \delta^i_r & \delta^i_s \\ \delta^k_r & \delta^k_s \end{pmatrix} = \delta^i_s\delta^k_r - \delta^i_r\delta^k_s\,, \tag{2.57}$$

[13]Galilean frame of reference is the one that does not rotate, is not subject to deformation and falls freely in a flat space-time (Minkowski space). Here lines of time are linear and so are three-dimensional coordinate axis.

[14]In case of signature $(-+++)$ this is only true for four-dimensional tensor $e^{\alpha\beta\mu\nu}$. Components of three-dimensional tensor e^{ikm} will have same signs as the respective components of e_{ikm}.

$$e^{ikm}e_{rkm} = 2\delta_r^i\,, \quad e^{ikm}e_{ikm} = 2\delta_i^i = 6. \tag{2.58}$$

Completely antisymmetric unit tensor defines for a tensor object its respective *pseudotensor* marked with asteriks.

For instance, four-dimensional scalar, vector and tensors of 2nd, 3rd, and 4th ranks have respective four-dimensional pseudotensors of the following ranks

$$\begin{gathered} V^{*\alpha\beta\mu\nu} = e^{\alpha\beta\mu\nu}V, \quad V^{*\alpha\beta\mu} = e^{\alpha\beta\mu\nu}V_\nu\,, \quad V^{*\alpha\beta} = \frac{1}{2}e^{\alpha\beta\mu\nu}V_{\mu\nu}\,, \\ V^{*\alpha} = \frac{1}{6}e^{\alpha\beta\mu\nu}V_{\beta\mu\nu}\,, \quad V^* = \frac{1}{24}e^{\alpha\beta\mu\nu}V_{\alpha\beta\mu\nu}\,, \end{gathered} \tag{2.59}$$

where 1st rank pseudotensor $V^{*\alpha}$ is sometimes called *pseudovector*, while zero-rank pseudotensor V^* is called *pseudoscalar.* Tensor and its respective pseudotensor are referred to as *dual* to each other to emphasize their common genesis. Similarly, three-dimensional tensors have respective three-dimensional pseudotensors

$$\begin{gathered} V^{*ikm} = e^{ikm}V, \qquad V^{*ik} = e^{ikm}V_m\,, \\ V^{*i} = \frac{1}{2}e^{ikm}V_{km}\,, \quad V^* = \frac{1}{6}e^{ikm}V_{ikm}\,. \end{gathered} \tag{2.60}$$

Pseudotensors are called such because contrary to regular tensors, they do not change being reflected in respect to one of the axis. For instance, being reflected in respect to abscises axis $x^1 = -\tilde{x}^1$, $x^2 = \tilde{x}^2$, $x^3 = \tilde{x}^3$. Reflected component of antisymmetric tensor V_{ik}, orthogonal to x^1 axis, is $\widetilde{V}_{23} = -V_{23}$, while its dual component of pseudovector V^{*i} is

$$\begin{gathered} V^{*1} = \frac{1}{2}e^{1km}V_{km} = \frac{1}{2}\left(e^{123}V_{23} + e^{132}V_{32}\right) = V_{23}\,, \\ \widetilde{V}^{*1} = \frac{1}{2}e^{1km}\widetilde{V}_{km} = \frac{1}{2}e^{k1m}\widetilde{V}_{km} = \frac{1}{2}\left(e^{213}\widetilde{V}_{23} + e^{312}\widetilde{V}_{32}\right) = V_{23}\,. \end{gathered} \tag{2.61}$$

Because four-dimensional antisymmetric tensor of 2nd rank and its dual pseudotensor are of the same rank, their contraction is pseudoscalar

$$V_{\alpha\beta}V^{*\alpha\beta} = V_{\alpha\beta}e^{\alpha\beta\mu\nu}V_{\mu\nu} = e^{\alpha\beta\mu\nu}B_{\alpha\beta\mu\nu} = B^*. \tag{2.62}$$

Square of pseudotensor $V^{*\alpha\beta}$ and square of pseudovector V^{*i}, expressed through their dual antisymmetric tensors of 2nd rank are

$$V_{*\alpha\beta}V^{*\alpha\beta} = e_{\alpha\beta\mu\nu}V^{\mu\nu}e^{\alpha\beta\rho\sigma}V_{\rho\sigma} = -24V_{\mu\nu}V^{\mu\nu}, \tag{2.63}$$

$$V_{*i}V^{*i} = e_{ikm}V^{km}e^{ipq}V_{pq} = 6V_{km}V^{km}. \tag{2.64}$$

In non-uniform and anisotropic pseudo-Riemannian space we can not set a Galilean frame of reference and the reference of antisymmetry of tensor field will depend upon non-uniformity and anisotropy of the space itself, which are defined by fundamental metric tensor. Here reference antisymmetric tensor is *a four-dimensional completely antisymmetric discriminant tensor*

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}\,, \qquad E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}\,. \tag{2.65}$$

Here is the proof. Transformation of a unit completely antisymmetric tensor from Galilean (non-tilde-marked) frame of reference into an arbitrary (tilde-marked) frame of reference is

$$\tilde{e}_{\alpha\beta\mu\nu} = \frac{\partial x^\sigma}{\partial\tilde{x}^\alpha}\frac{\partial x^\gamma}{\partial\tilde{x}^\beta}\frac{\partial x^\varepsilon}{\partial\tilde{x}^\mu}\frac{\partial x^\tau}{\partial\tilde{x}^\nu}e_{\sigma\gamma\varepsilon\tau} = Je_{\alpha\beta\mu\nu}\,, \tag{2.66}$$

where $J = \det\left\|\frac{\partial x^\alpha}{\partial\tilde{x}^\sigma}\right\|$ is called *the Jacobian of transformation* (the determinant of Jacobi matrix)

$$J = \det \begin{Vmatrix} \frac{\partial x^0}{\partial \tilde{x}^0} & \frac{\partial x^0}{\partial \tilde{x}^1} & \frac{\partial x^0}{\partial \tilde{x}^2} & \frac{\partial x^0}{\partial \tilde{x}^3} \\ \frac{\partial x^1}{\partial \tilde{x}^0} & \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \frac{\partial x^1}{\partial \tilde{x}^3} \\ \frac{\partial x^2}{\partial \tilde{x}^0} & \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \frac{\partial x^2}{\partial \tilde{x}^3} \\ \frac{\partial x^3}{\partial \tilde{x}^0} & \frac{\partial x^3}{\partial \tilde{x}^1} & \frac{\partial x^3}{\partial \tilde{x}^2} & \frac{\partial x^3}{\partial \tilde{x}^3} \end{Vmatrix}. \tag{2.67}$$

Because metric tensor $g_{\alpha\beta}$ is transformable according to the rule

$$\tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}\,, \tag{2.68}$$

then its determinant in tilde-marked frame of reference is

$$\tilde{g} = \det \left\| \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu} \right\| = J^2 g\,. \tag{2.69}$$

Because in Galilean (non-tilde-marked) frame of reference

$$g = \det \|g_{\alpha\beta}\| = \det \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} = -1, \tag{2.70}$$

then $J^2{=}-\tilde{g}^2$. Expressing $\tilde{e}_{\alpha\beta\mu\nu}$ in an arbitrary frame of reference as $E_{\alpha\beta\mu\nu}$ and writing down metric tensor in a regular non-tilde-marked form, we obtain $E_{\alpha\beta\mu\nu}{=}e_{\alpha\beta\mu\nu}\sqrt{-g}$ (2.65). In a similar way we obtain transformation rules for components $E^{\alpha\beta\mu\nu}$, because for them $g{=}\tilde{g}\tilde{J}^2$, where $\tilde{J} = \det \left\| \frac{\partial \tilde{x}^\alpha}{\partial x^\sigma} \right\|$.

But discriminant tensor $E^{\alpha\beta\mu\nu}$ is not a physical observable value. Physical observable reference of antisymmetry of tensor fields is *three-dimensional chronometrically invariant discriminant tensor*

$$\varepsilon^{\alpha\beta\gamma} = h^\alpha_\mu h^\beta_\nu h^\gamma_\rho b_\sigma E^{\sigma\mu\nu\rho} = b_\sigma E^{\sigma\alpha\beta\gamma}, \tag{2.71}$$

$$\varepsilon_{\alpha\beta\gamma} = h^\mu_\alpha h^\nu_\beta h^\rho_\gamma b^\sigma E_{\sigma\mu\nu\rho} = b^\sigma E_{\sigma\alpha\beta\gamma}\,, \tag{2.72}$$

which in an accompanying frame of reference ($b^i{=}0$), taking into account that $\sqrt{-g}{=}\sqrt{h}\sqrt{g_{00}}$, will take the form

$$\varepsilon^{ikm} = b_0 E^{0ikm} = \sqrt{g_{00}}\, E^{0ikm} = \frac{e^{ikm}}{\sqrt{h}}\,, \tag{2.73}$$

$$\varepsilon_{ikm} = b^0 E_{0ikm} = \frac{E_{0ikm}}{\sqrt{g_{00}}} = e_{ikm}\sqrt{h}\,. \tag{2.74}$$

With its help we can transform chronometrically invariant (physical observable) pseudotensors. For instance, from chronometrically invariant antisymmetric tensor of space's rotation A_{ik} (1.36) we obtain observable pseudovector of angular velocity of rotation of space $\Omega^{*i}{=}\frac{1}{2}\varepsilon^{ikm}A_{km}$.

2.4 Introducing absolute differential and derivative to the direction

In geometry *a differential* of a function is its variation between infinitely close points with coordinates x^α and $x^\alpha{+}dx^\alpha$. Respectively, *absolute differential* in n-dimensional space is variation of n-dimensional values between infinitely close points of n-dimensional coordinates in this space. For continuous functions, we commonly deal with in practice, variations between infinitely close points are infinitesimal. But in order to define infinitesimal variation of a tensor value we can not use simple "difference" between its values in points x^α and $x^\alpha{+}dx^\alpha$, because tensor algebra does not define the ratio between values of tensors at different points in space. This ratio can be defined only using rules of transformation of tensors from one frame of reference into another. As a consequence, differential operators and the results of their application to tensors must be tensors themselves.

For instance, absolute differential of a tensor value is a tensor of the same rank as the value itself. For a scalar φ it is a scalar

$$D\varphi = \frac{\partial \varphi}{\partial x^{\alpha}} dx^{\alpha}. \tag{2.75}$$

In accompanying frame of reference (b^i=0) it is

$$D\varphi = \frac{^*\partial \varphi}{\partial t} d\tau + \frac{^*\partial \varphi}{\partial x^i} dx^i. \tag{2.76}$$

We can see that aside for three-dimensional observable differential there is an additional term that takes into account dependence of absolute displacement $D\varphi$ from flow of physical observable time $d\tau$.

Absolute differential of contravariant vector A^{α}, formulated with operator of absolute derivation ∇ (nabla) is

$$DA^{\alpha} = \nabla_{\sigma} A^{\alpha} dx^{\sigma} = \frac{\partial A^{\alpha}}{\partial x^{\sigma}} dx^{\sigma} + \Gamma^{\alpha}_{\mu\sigma} A^{\mu} dx^{\sigma} = dA^{\alpha} + \Gamma^{\alpha}_{\mu\sigma} A^{\mu} dx^{\sigma}, \tag{2.77}$$

where $\nabla_{\sigma} A^{\alpha}$ is absolute derivative A^{α} by coordinate x^{σ} and d stands for regular differential

$$\nabla_{\sigma} A^{\alpha} = \frac{\partial A^{\alpha}}{\partial x^{\sigma}} + \Gamma^{\alpha}_{\mu\sigma} A^{\mu}, \tag{2.78}$$

$$d = \frac{\partial}{\partial x^{\alpha}} dx^{\alpha}. \tag{2.79}$$

Notation of absolute differential with physical observable values is equivalent to calculation of projection of its general covariant form onto time and space in accompanying frame of reference

$$T = b_{\alpha} DA^{\alpha} = \frac{g_{0\alpha} DA^{\alpha}}{\sqrt{g_{00}}}\,, \quad B^i = h^i_{\alpha} DA^{\alpha}. \tag{2.80}$$

Denoting observable components of vector A^{α} as

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i, \tag{2.81}$$

we arrive to its other components

$$A_0 = \varphi \left(1 - \frac{w}{c^2}\right), \quad A^0 = \frac{\varphi + \frac{1}{c} v_i q^i}{1 - \frac{w}{c^2}}\,, \quad A_i = -q_i - \frac{\varphi}{c} v_i\,. \tag{2.82}$$

Taking into account that regular differential in chronometrically invariant form is

$$d = d\tau \frac{^*\partial}{\partial t} + dx^i \frac{^*\partial}{\partial x^i}\,, \tag{2.83}$$

and substituting Christoffel symbols in accompanying frame of reference (1.41–1.46) into values T and B^i (2.80), we arrive to chronometrically invariant (observable) projections onto time and space of absolute differential of vector A^{α}

$$T = b_{\alpha} DA^{\alpha} = d\varphi + \frac{1}{c}\left(-F_i q^i d\tau + D_{ik} q^i dx^k\right), \tag{2.84}$$

$$B^i = h^i_{\sigma} DA^{\sigma} = dq^i + \left(\frac{\varphi}{c} dx^k + q^k d\tau\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \frac{\varphi}{c} F^i d\tau + \triangle^i_{mk} q^m dx^k. \tag{2.85}$$

To build equations of motion we will also need chronometrically invariant equations of absolute derivative of a vector to a direction, tangential to trajectory of motion. From geometric viewpoint *a derivative to direction* of a certain function is its change in respect to elementary displacement along a given direction. *Absolute derivative to direction* in n-dimensional space is change of n-dimensional value in respect to elementary n-dimensional interval along a given direction. For instance, absolute derivative of a scalar function φ to direction, defined by a curve x^{α}=$x^{\alpha}(\rho)$, where ρ is a parameter along this curve, shows the “rate” of change of this function

$$\frac{D\varphi}{d\rho} = \frac{d\varphi}{d\rho}\,. \tag{2.86}$$

In accompanying frame of reference it is

$$\frac{D\varphi}{d\rho} = \frac{{}^*\partial\varphi}{\partial t}\frac{d\tau}{d\rho} + \frac{{}^*\partial\varphi}{\partial x^i}\frac{dx^i}{d\rho}\,. \tag{2.87}$$

Absolute derivative of vector A^α to direction of curve $x^\alpha{=}x^\alpha\,(\rho)$ is

$$\frac{DA^\alpha}{d\rho} = \nabla_\sigma A^\alpha \frac{dx^\sigma}{d\rho} = \frac{dA^\alpha}{d\rho} + \Gamma^\alpha_{\mu\sigma}A^\mu\frac{dx^\sigma}{d\rho}\,. \tag{2.88}$$

Its physical observable projections onto time and space in accompanying frame of reference are

$$b_\alpha\frac{DA^\alpha}{d\rho} = \frac{d\varphi}{d\rho} + \frac{1}{c}\left(-F_i q^i\frac{d\tau}{d\rho} + D_{ik}q^i\frac{dx^k}{d\rho}\right), \tag{2.89}$$

$$h^i_\sigma\frac{DA^\sigma}{d\rho} = \frac{dq^i}{d\rho} + \left(\frac{\varphi}{c}\frac{dx^k}{d\rho} + q^k\frac{d\tau}{d\rho}\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \frac{\varphi}{c}F^i\frac{d\tau}{d\rho} + \triangle^i_{mk}q^m\frac{dx^k}{d\rho}. \tag{2.90}$$

Actually, these projections are “generic” chronometrically invariant equations of motion. But once we define a particular vector of motion of particle, calculate its observable components and substitute them into given equations, we immediately arrive to particle’s equations of motion formulated with physical observable values.

2.5 Divergence and rotor

Divergence of a tensor field is its “change” along coordinate axis. Respectively, *absolute divergence* of n-dimensional tensor field is its divergence in n-dimensional space. Divergence is a result of contraction of field tensor with operator of absolute derivation ∇. Divergence of vector field is a scalar value

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu}A^\mu. \tag{2.91}$$

Divergence of field of 2nd rank tensor is a vector

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu}F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu}F^{\sigma\mu}, \tag{2.92}$$

while it can be proven that $\Gamma^\sigma_{\sigma\mu}$ is

$$\Gamma^\sigma_{\sigma\mu} = \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}\,. \tag{2.93}$$

To do this we will use the definition of Christoffel symbols and write down $\Gamma^\sigma_{\sigma\mu}$ in detailed form

$$\Gamma^\sigma_{\sigma\mu} = g^{\sigma\rho}\Gamma_{\mu\sigma,\rho} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^\sigma} + \frac{\partial g_{\sigma\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\rho}\right). \tag{2.94}$$

Because σ and ρ here are free indices, they can change places. As a result, after contraction with tensor $g^{\rho\sigma}$ the first and the last terms cancel each other and $\Gamma^\sigma_{\sigma\mu}$ takes the form

$$\Gamma^\sigma_{\sigma\mu} = \frac{1}{2}g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}\,. \tag{2.95}$$

Values $g^{\rho\sigma}$ are components of a tensor reciprocal to tensor $g_{\rho\sigma}$. Therefore each component of matrix $g^{\rho\sigma}$ is

$$g^{\rho\sigma} = \frac{a^{\rho\sigma}}{g}\,, \qquad g = \det\|g_{\rho\sigma}\|\,, \tag{2.96}$$

where $a^{\rho\sigma}$ is algebraic cofactor of a matrix' element with indices $\rho\sigma$, equal to $(-1)^{\rho+\sigma}$, multiplied by determinant of matrix obtained by crossing the row and the column with numbers σ and ρ out of matrix $g_{\rho\sigma}$. As a result we obtain $a^{\rho\sigma}{=}gg^{\rho\sigma}$. Because determinant of fundamental metric tensor $g = \det\|g_{\rho\sigma}\|$ by definition is

$$g = \sum_{\alpha_0\ldots\alpha_3} (-1)^{N(\alpha_0\ldots\alpha_3)}\, g_{0(\alpha_0)} g_{1(\alpha_1)} g_{2(\alpha_2)} g_{3(\alpha_3)}\,, \tag{2.97}$$

then the value dg will be $dg{=}a^{\rho\sigma}dg_{\rho\sigma}{=}gg^{\rho\sigma}dg_{\rho\sigma}$, or

$$\frac{dg}{g} = g^{\rho\sigma}dg_{\rho\sigma}\,. \tag{2.98}$$

Integration of the left part gives $\ln(-g)$, because g is negative while logarithm is only defined for a positive function. Then $d\ln(-g) = \frac{dg}{g}$. Taking into account that $(-g)^{1/2}{=}\frac{1}{2}\ln(-g)$, we arrive to

$$d\ln\sqrt{-g} = \frac{1}{2}g^{\rho\sigma}dg_{\rho\sigma}\,, \tag{2.99}$$

and $\Gamma^{\sigma}_{\sigma\mu}$ (2.95) takes form

$$\Gamma^{\sigma}_{\sigma\mu} = \frac{1}{2}g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} = \frac{\partial\ln\sqrt{-g}}{\partial x^{\mu}}\,, \tag{2.100}$$

which had to be proven (2.93).

Now we are going to calculate physical observable components of divergence of vector field (2.91) and field of 2nd rank tensor (2.92). The formula for divergence of vector field A^{α} is a scalar, hence $\nabla_{\sigma}A^{\sigma}$ can not be projected onto time and space, while it is enough to express is through chronometrically invariant components of A^{α} and through observable properties of frame of reference. Besides, regular operators of derivation should be replaced with chronometrically invariant ones.

Assuming notations φ and q^i for observable components of vector A^{α} (2.81) we express other components of the vector through them (2.82). Then substituting regular operators of derivations, expressed through chronometrically invariant operators

$$\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t} = \frac{^*\partial}{\partial t}\,, \qquad \sqrt{g_{00}} = 1 - \frac{w}{c^2}\,, \tag{2.101}$$

$$\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}v_i\frac{^*\partial}{\partial t}\,, \tag{2.102}$$

into (2.91), and taking into account that $\sqrt{-g}{=}\sqrt{h}\sqrt{g_{00}}$ after some calculations we arrive to

$$\nabla_{\sigma}A^{\sigma} = \frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right) + \frac{^*\partial q^i}{\partial x^i} + q^i\frac{^*\partial\ln\sqrt{h}}{\partial x^i} - \frac{1}{c^2}F_iq^i. \tag{2.103}$$

In the third term the value

$$\frac{^*\partial\ln\sqrt{h}}{\partial x^i} = \triangle^j_{ji} \tag{2.104}$$

stands for chronometrically invariant Christoffel symbols $\triangle^k_{ji}$ (1.47) contracted by two symbols. Hence similarly to definition of absolute divergence of vector field (2.91), the value

$$\frac{^*\partial q^i}{\partial x^i} + q^i\frac{^*\partial\ln\sqrt{h}}{\partial x^i} = \frac{^*\partial q^i}{\partial x^i} + q^i\triangle^j_{ji} = {}^*\nabla_iq^i \tag{2.105}$$

stands for *chronometrically invariant divergence* of vector q^i. Consequently we will call *physical divergence* of vector q^i the chronometrically invariant value

$${}^*\widetilde{\nabla}_iq^i = {}^*\nabla_iq^i - \frac{1}{c^2}F_iq^i, \tag{2.106}$$

in which the additional term takes into account that the pace of time is different on opposite walls of an elementary volume [10]. As a matter of fact, in calculation of divergence we consider an elementary volume of space and calculate the difference between the amounts of "substance" which flows in and out of the volume over an elementary time interval. But presence of gravitational inertial force F^i (1.38) results in different pace of time at different points in space. Therefore, if we measure durations of time intervals at opposite walls of the volume, the beginnings and the ends of the interval will not coincide making them invalid for comparison. Synchronization of clocks at opposite walls of the volume will give the true picture, i. e. the measured durations of the intervals will be different.

The final equation for $\nabla_\sigma A^\sigma$ will be

$$\nabla_\sigma A^\sigma = \frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right) + {}^*\widetilde{\nabla}_i q^i. \tag{2.107}$$

The second term in the formula is physical observable analog of a regular divergence in three-dimensional space. The first term has no analogs and falls apart into two parts: $\frac{^*\partial\varphi}{\partial t}$ is variation in time of temporal projection φ of four-dimensional vector A^α; $D\varphi$ is variation in time of volume of three-dimensional vector field q^i, because spur of tensor of deformation velocities $D{=}D^i_i$ is rate of relative expansion of elementary volume in space of reference.

Equation $\nabla_\sigma A^\sigma{=}0$ applied to four-dimensional vector potential A^α of electromagnetic field is Lorentz condition. In chronometrically invariant form Lorentz condition is

$${}^*\widetilde{\nabla}_i q^i = -\frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right). \tag{2.108}$$

Now we are going to calculate physical observable components of divergence of an arbitrary antisymmetric tensor $F^{\alpha\beta}{=}{-}F^{\beta\alpha}$ (later we will need them to obtain chronometrically invariant Maxwell equations)

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu}F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu}F^{\sigma\mu} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}F^{\alpha\mu}, \tag{2.109}$$

where the third term $\Gamma^\alpha_{\sigma\mu}F^{\sigma\mu}$ is zero because of contraction of Christoffel symbols $\Gamma^\alpha_{\sigma\mu}$, symmetric by lower indices $\sigma\mu$, and antisymmetric tensor $F^{\sigma\mu}$ is zero (as a symmetric and an antisymmetric tensors).

The term $\nabla_\sigma F^{\sigma\alpha}$ is four-dimensional vector, so its chronometrically invariant projections are

$$T = b_\alpha \nabla_\sigma F^{\sigma\alpha}, \qquad B^i = h^i_\alpha \nabla_\sigma F^{\sigma\alpha} = \nabla_\sigma F^{i\alpha}. \tag{2.110}$$

We denote chronometrically invariant (physical observable) components of tensor $F^{\alpha\beta}$ as

$$E^i = \frac{F^{\cdot i}_{0\cdot}}{\sqrt{g_{00}}}, \qquad H^{ik} = F^{ik}. \tag{2.111}$$

Then the rest non-zero components of the tensor, being formulated with physical observable components (2.111) are

$$F^{\cdot 0}_{0\cdot} = \frac{1}{c}v_k E^k, \tag{2.112}$$

$$F^{\cdot 0}_{k\cdot} = \frac{1}{\sqrt{g_{00}}}\left(E_i - \frac{1}{c}v_n H^{\cdot n}_{k\cdot} - \frac{1}{c^2}v_k v_n E^n\right), \tag{2.113}$$

$$F^{0i} = \frac{E^i - \frac{1}{c}v_k H^{ik}}{\sqrt{g_{00}}}, \qquad F_{0i} = -\sqrt{g_{00}}\,E_i\,, \tag{2.114}$$

$$F^{\cdot k}_{i\cdot} = -H^{\cdot k}_{i\cdot} - \frac{1}{c}v_i E^k, \quad F_{ik} = H_{ik} + \frac{1}{c}\left(v_i E_k - v_k E_i\right), \tag{2.115}$$

and the square of tensor $F^{\alpha\beta}$ is

$$F_{\alpha\beta}F^{\alpha\beta} = H_{ik}H^{ik} - 2E_iE^i. \tag{2.116}$$

Substituting the components into (2.110) and replacing regular operators of derivation with chronometrically invariant operators after some algebra we arrive to

$$T = \frac{\nabla_\sigma F_{0\cdot}^{\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial E^i}{\partial x^i} + E^i \frac{{}^*\partial \ln \sqrt{h}}{\partial x^i} - \frac{1}{c} H^{ik} A_{ik}\,, \tag{2.117}$$

$$B^i = \nabla_\sigma F^{\sigma i} = \frac{{}^*\partial H^{ik}}{\partial x^k} + H^{ik} \frac{{}^*\partial \ln \sqrt{h}}{\partial x^k} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right), \tag{2.118}$$

where A_{ik} is antisymmetric chronometrically invariant tensor of non-holonomity of space. Taking into account that

$$\frac{{}^*\partial E^i}{\partial x^i} + E^i \frac{{}^*\partial \ln \sqrt{h}}{\partial x^i} = {}^*\nabla_i E^i \tag{2.119}$$

is chronometrically invariant divergence of vector E^i and that

$${}^*\nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} = {}^*\widetilde{\nabla}_k H^{ik} \tag{2.120}$$

is physical chronometrically invariant divergence of tensor H^{ik} we arrive to final equations for physical observable projections of divergence of an arbitrary antisymmetric tensor $F^{\alpha\beta}$

$$T = {}^*\nabla_i E^i - \frac{1}{c} H^{ik} A_{ik}\,, \tag{2.121}$$

$$B^i = {}^*\widetilde{\nabla}_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right). \tag{2.122}$$

We now calculate physical observable components of divergence of pseudotensor $F^{*\alpha\beta}$, dual to the given antisymmetric tensor $F^{\alpha\beta}$

$$F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}\,, \quad F_{*\alpha\beta} = \frac{1}{2} E_{\alpha\beta\mu\nu} F^{\mu\nu}. \tag{2.123}$$

We denote observable components of pseudotensor $F^{*\alpha\beta}$ as

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}\,, \qquad E^{*ik} = F^{*ik}, \tag{2.124}$$

because there are evident relations $H^{*i}{\sim}H^{ik}$ and $E^{*ik}{\sim}E^i$ between these values and observable components of antisymmetric tensor $F^{\alpha\beta}$ (2.111) because of duality of the given tensors $F^{\alpha\beta}$ and $F^{*\alpha\beta}$. Therefore, given that

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{1}{2} \varepsilon^{ipq} H_{pq}\,, \quad F^{*ik} = -\varepsilon^{ikp} E_p\,, \tag{2.125}$$

the other components of pseudotensor $F^{*\alpha\beta}$ formulated with observable components of the dual tensor $F^{\alpha\beta}$ (2.111) are

$$F_{0\cdot}^{*\cdot 0} = \frac{1}{2c} v_k \varepsilon^{kpq} \left[H_{pq} + \frac{1}{c}\left(v_p E_q - v_q E_p\right)\right], \tag{2.126}$$

$$\begin{gathered} F_{i\cdot}^{*\cdot 0} = \frac{1}{2\sqrt{g_{00}}} \left[\varepsilon_{i\cdot}^{\cdot pq} H_{pq} + \frac{1}{c} \varepsilon_{i\cdot}^{\cdot pq} \left(v_p E_q - v_q E_p\right) - \right. \\ \left. - \frac{1}{c^2} \varepsilon^{kpq} v_i v_k H_{pq} - \frac{1}{c^3} \varepsilon^{kpq} v_i v_k \left(v_p E_q - v_q E_p\right)\right], \end{gathered} \tag{2.127}$$

$$F^{*0i} = \frac{1}{2\sqrt{g_{00}}} \varepsilon^{ipq} \left[H_{pq} + \frac{1}{c}\left(v_p E_q - v_q E_p\right)\right], \tag{2.128}$$

$$F_{*0i} = \frac{1}{2}\sqrt{g_{00}}\,\varepsilon_{ipq}H^{pq}, \tag{2.129}$$

$$F_{i\cdot}^{*\cdot k} = \varepsilon_{i\cdot}^{\cdot kp}E_p - \frac{1}{2c}v_i\varepsilon^{kpq}H_{pq} - \frac{1}{c^2}v_i v_m\varepsilon^{mkp}E_p\,, \tag{2.130}$$

$$F_{*ik} = \varepsilon_{ikp}\left(E^p - \frac{1}{c}v_q H^{pq}\right), \tag{2.131}$$

while its square is

$$F_{*\alpha\beta}F^{*\alpha\beta} = \varepsilon^{ipq}\left(E_p H_{iq} - E_i H_{pq}\right), \tag{2.132}$$

where ε^{ipq} is three-dimensional chronometrically invariant discriminant tensor (2.73, 2.74). Then observable components of divergence of pseudotensor $F^{*\alpha\beta}$ are

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial H^{*i}}{\partial x^i} + H^{*i}\frac{{}^*\partial\ln\sqrt{h}}{\partial x^i} - \frac{1}{c}E^{*ik}A_{ik}\,, \tag{2.133}$$

$$\nabla_\sigma F^{*\sigma i} = \frac{{}^*\partial E^{*ik}}{\partial x^i} + E^{*ik}\frac{{}^*\partial\ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right), \tag{2.134}$$

or, outlining chronometrically invariant divergence ${}^*\nabla_i H^{*i}$ and chronometrically invariant physical divergence ${}^*\widetilde{\nabla}_k E^{*ik}$, similarly to (2.119, 2.120) we obtain

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = {}^*\nabla_i H^{*i} - \frac{1}{c}E^{*ik}A_{ik}\,, \tag{2.135}$$

$$\nabla_\sigma F^{*\sigma i} = {}^*\widetilde{\nabla}_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right). \tag{2.136}$$

Aside for vector divergence, antisymmetric tensor and 2nd rank pseudotensor we as well need to know observable projections of divergence of 2nd rank symmetric tensor (we will need them later to obtain chronometrically invariant laws of conservation). Because these formulas have been already obtained by Zelmanov, we will take them from his lectures [10].

Denoting observable components of symmetric tensor $T^{\alpha\beta}$ as

$$\frac{T_{00}}{g_{00}} = \rho\,, \qquad \frac{T_0^i}{\sqrt{g_{00}}} = K^i, \qquad T^{ik} = N^{ik}, \tag{2.137}$$

according to [10] we obtain

$$\frac{\nabla_\sigma T_0^\sigma}{\sqrt{g_{00}}} = \frac{{}^*\partial\rho}{\partial t} + \rho D + D_{ik}N^{ik} + c\,{}^*\nabla_i K^i - \frac{2}{c}F_i K^i, \tag{2.138}$$

$$\nabla_\sigma T^{\sigma i} = c\,\frac{{}^*\partial K^i}{\partial t} + cDK^i + 2c\left(D_k^i + A_{k\cdot}^{\cdot i}\right)K^k + c^2\,{}^*\nabla_k N^{ik} - F_k N^{ik} - \rho F^i. \tag{2.139}$$

Along with internal (scalar) product of tensor with operator of absolute derivation ∇, which is divergence of this tensor field, we can consider difference of covariant derivatives of tensor field. This value is referred to as *rotor* and from geometrical viewpoint is "rotation" (vortex) of tensor field. *Absolute rotor* is a rotor of n-dimensional tensor field in n-dimensional space (though this notation is rather uncommon compared to absolute divergence). Rotor of four-dimensional vector field A^α is covariant antisymmetric 2nd rank tensor defined as[15]

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\,, \tag{2.140}$$

[15]For example, see Section 98 in well-known P. K. Raschewski's book [2]. Actually, rotor is not a tensor (2.140), but its dual pseudotensor (2.142), because invariance with respect to reflection is necessary for rotation.

where $\nabla_\mu A_\nu$ is absolute derivative of A_α by coordinate x^μ

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma^\sigma_{\nu\mu} A_\sigma \,. \tag{2.141}$$

Rotor contracted with four-dimensional absolutely antisymmetric discriminant tensor $E^{\alpha\beta\mu\nu}$ (2.65) is a pseudotensor

$$F^{*\alpha\beta} = E^{\alpha\beta\mu\nu}\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) = E^{\alpha\beta\mu\nu}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right). \tag{2.142}$$

In electrodynamics $F_{\mu\nu}$ (2.140) is the tensor of electromagnetic field (Maxwell tensor) which is a rotor of four-dimensional potential A^α of electromagnetic field. Therefore later we will need formulas for observable components of four-dimensional rotor $F_{\mu\nu}$ and its dual pseudotensor $F^{*\alpha\beta}$, expressed through observable components of four-dimensional field vector A^α (2.81) that form them.

Let us calculate components of rotor $F_{\mu\nu}$ (taking into account that $F_{00}{=}F^{00}{=}0$ just like for any other antisymmetric tensor). As the result after some algebra we obtain

$$F_{0i} = \left(1 - \frac{w}{c^2}\right)\left(\frac{\varphi}{c^2}F_i - \frac{{}^*\partial\varphi}{\partial x^i} - \frac{1}{c}\frac{{}^*\partial q_i}{\partial t}\right), \tag{2.143}$$

$$\begin{aligned} F_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} + \frac{\varphi}{c}\left(\frac{{}^*\partial v_i}{\partial x^k} - \frac{{}^*\partial v_k}{\partial x^i}\right) + \\ + \frac{1}{c}\left(v_i\frac{{}^*\partial\varphi}{\partial x^k} - v_k\frac{{}^*\partial\varphi}{\partial x^i}\right) + \frac{1}{c^2}\left(v_i\frac{{}^*\partial q_k}{\partial t} - v_k\frac{{}^*\partial q_i}{\partial t}\right), \end{aligned} \tag{2.144}$$

$$F_{0\cdot}^{\cdot 0} = -\frac{\varphi}{c^3}v_k F^k + \frac{1}{c}v^k\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right), \tag{2.145}$$

$$\begin{aligned} F_{k\cdot}^{\cdot 0} = -\frac{1}{1 - \frac{w}{c^2}}\left[\frac{\varphi}{c^2}F_k - \frac{{}^*\partial\varphi}{\partial x^k} - \frac{1}{c}\frac{{}^*\partial q_k}{\partial t} + \frac{2\varphi}{c^2}v^m A_{mk} + \frac{1}{c^2}v_k v^m\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \right. \\ \left. - \frac{1}{c}v^m\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{\varphi}{c^4}v_k v_m F^m\right], \end{aligned} \tag{2.146}$$

$$F_{k\cdot}^{\cdot i} = h^{im}\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{1}{c}h^{im}v_k\frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c^2}h^{im}v_k\frac{{}^*\partial q_m}{\partial t} + \frac{\varphi}{c^3}v_k F^i + \frac{2\varphi}{c}A_{k\cdot}^{\cdot i}\,, \tag{2.147}$$

$$\begin{aligned} F^{0k} = \frac{1}{1 - \frac{w}{c^2}}\left[h^{km}\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \frac{\varphi}{c^2}F^k + \right. \\ \left. + \frac{1}{c}v^n h^{mk}\left(\frac{{}^*\partial q_n}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^n}\right) - \frac{2\varphi}{c^2}v_m A^{mk}\right], \end{aligned} \tag{2.148}$$

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha}F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik}\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right) - \frac{\varphi}{c^2}F^i, \tag{2.149}$$

$$F^{ik} = g^{i\alpha}g^{k\beta}F_{\alpha\beta} = h^{im}h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}A^{ik}, \tag{2.150}$$

where (2.149, 2.150) are physical observable projections of rotor $F_{\mu\nu}$. Respectively, observable projections of its dual pseudotensor $F^{*\alpha\beta}$ are

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{g_{0\alpha}F^{*\alpha i}}{\sqrt{g_{00}}} = \varepsilon^{ikm}\left[\frac{1}{2}\left(\frac{{}^*\partial q_k}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^k}\right) - \frac{\varphi}{c}A_{km}\right], \tag{2.151}$$

$$F^{*ik} = \varepsilon^{ikm}\left(\frac{\varphi}{c^2}F_m - \frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right), \tag{2.152}$$

where $F_{0\cdot}^{*\cdot i}{=}g_{0\alpha}F^{*\alpha i}{=}g_{0\alpha}E^{*\alpha i\mu\nu}F_{\mu\nu}$ can be calculated using components of rotor $F_{\mu\nu}$ (2.143–2.148).

2.6 Laplace and d'Alembert operators

Laplace operator is three-dimensional operator of derivation

$$\triangle = \nabla\nabla = \nabla^2 = g^{ik}\nabla_i\nabla_k\,. \tag{2.153}$$

Its generalization in four-dimensional pseudo-Riemannian space is general covariant *d'Alembert operator*

$$\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\,. \tag{2.154}$$

In Minkowski space these operators take the form

$$\triangle = \frac{\partial^2}{\partial x^1\partial x^1} + \frac{\partial^2}{\partial x^2\partial x^2} + \frac{\partial^2}{\partial x^3\partial x^3}\,, \tag{2.155}$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^1\partial x^1} - \frac{\partial^2}{\partial x^2\partial x^2} - \frac{\partial^2}{\partial x^3\partial x^3} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle\,. \tag{2.156}$$

Our goal is to apply d'Alembert operator to scalar and vector fields in pseudo-Riemannian space and to present the results in chronometrically invariant form. First, we will apply d'Alembert operator to four-dimensional field of scalar φ, because in this case the calculations will be much simpler (absolute derivative of scalar field $\nabla_\alpha\varphi$ does not contain Christoffel symbols and leads to a regular derivative)

$$\Box\varphi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\,\varphi = g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\left(\frac{\partial\varphi}{\partial x^\beta}\right) = g^{\alpha\beta}\frac{\partial^2\varphi}{\partial x^\alpha\partial x^\beta}\,. \tag{2.157}$$

Components of fundamental metric tensor will be formulated with values of theory of chronometrically invariants. For components g^{ik} from (1.18) we obtain $g^{ik}{=}{-}h^{ik}$. The values g^{oi} are obtained from formulas for vector of linear velocity of space rotation $v^i{=}{-}cg^{0i}\sqrt{g_{00}}$

$$g^{0i} = -\frac{1}{c}\frac{v^i}{\sqrt{g_{00}}}\,. \tag{2.158}$$

Component g^{00} can be obtained from the property of fundamental metric tensor $g_{\alpha\sigma}g^{\beta\sigma}{=}g^\beta_\alpha$. Expanding this equation for $\alpha{=}0$, $\beta{=}0$

$$g_{0\sigma}g^{0\sigma} = g_{00}g^{00} + g_{0i}g^{0i} = \delta^0_0 = 1, \tag{2.159}$$

and taking into account that

$$g_{00} = \left(1 - \frac{w}{c^2}\right)^2, \qquad g_{0i} = -\frac{1}{c}v_i\left(1 - \frac{w}{c^2}\right), \tag{2.160}$$

we obtain the expression for g^{00} value

$$g^{00} = \frac{1 - \frac{1}{c^2}v_iv^i}{\left(1 - \frac{w}{c^2}\right)^2}\,, \qquad v_iv^i = h_{ik}v^iv^k = v^2. \tag{2.161}$$

Substituting the obtained formulas for components of fundamental metric tensor into $\Box\varphi$ (2.157) and replacing regular operators of derivation with chronometrically invariant ones we arrive to d'Alembertian for scalar field in chronometrically invariant form

$$\Box\varphi = \frac{1}{c^2}\frac{^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{^*\partial^2\varphi}{\partial x^i\partial x^k} = {}^*\Box\varphi\,, \tag{2.162}$$

where $^*\Box$ is chronometrically invariant d'Alembert operator and $^*\triangle$ is chronometrically invariant Laplace operator

$$^*\Box = \frac{1}{c^2}\frac{^*\partial^2}{\partial t^2} - h^{ik}\frac{^*\partial^2}{\partial x^i\partial x^k}\,, \tag{2.163}$$

$$
{}^*\triangle = g^{ik}\,{}^*\nabla_i\,{}^*\nabla_k = h^{ik}\frac{{}^*\partial^2}{\partial x^i\partial x^k}\,. \tag{2.164}
$$

Now we apply d'Alembert operator to an arbitrary four-dimensional vector field A^α in pseudo-Riemannian space

$$
\Box A^\alpha = g^{\mu\nu}\nabla_\mu\nabla_\nu A^\alpha. \tag{2.165}
$$

Because $\Box A^\alpha$ is four-dimensional vector, its chronometrically invariant (physical observable) projections onto time and space are

$$
T = b_\sigma \Box A^\sigma = b_\sigma g^{\mu\nu}\nabla_\mu\nabla_\nu A^\sigma, \tag{2.166}
$$

$$
B^i = h^i_\sigma \Box A^\sigma = h^i_\sigma g^{\mu\nu}\nabla_\mu\nabla_\nu A^\sigma. \tag{2.167}
$$

In general, obtaining chronometrically invariant d'Alembertian for vector field in pseudo-Riemannian space is not a trivial task, as Christoffel symbols are not zeroes and formulas for projections of second derivatives take dozens of pages[16]. The main criterion of correct calculations is Zelmanov's rule of chronometrically invariance: "Correct calculations make all terms in final equations chronometric invariants. That is they consist of chronometrically invariant derivatives of observable components of vector field and chronometrically invariant properties of the frame of reference. If any single mistake was made during calculations, the terms of final equations will not be chronometric invariants".

Hence, after some algebra we obtain that chronometrically invariant projections of d'Alembertian for vector field in pseudo-Riemannian space (2.166, 2.167) are

$$
\begin{aligned}
T = {}^*\Box\varphi &- \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(F_k q^k\right) - \frac{1}{c^3}F_i\frac{{}^*\partial q^i}{\partial t} + \frac{1}{c^2}F^i\frac{{}^*\partial\varphi}{\partial x^i} + h^{ik}\triangle^m_{ik}\frac{{}^*\partial\varphi}{\partial x^m} - \\
&- h^{ik}\frac{1}{c}\frac{{}^*\partial}{\partial x^i}\left[\left(D_{kn}+A_{kn}\right)q^n\right] + \frac{D}{c^2}\frac{{}^*\partial\varphi}{\partial t} - \frac{1}{c}D^k_m\frac{{}^*\partial q^m}{\partial x^k} + \frac{2}{c^3}A_{ik}F^i q^k + \\
&+ \frac{\varphi}{c^4}F_iF^i - \frac{D}{c^3}F_m q^m - \frac{\varphi}{c^2}D_{mk}D^{mk} - \frac{1}{c}\triangle^m_{kn}D^k_m q^n + \frac{1}{c}h^{ik}\triangle^m_{ik}\left(D_{mn}+A_{mn}\right)q^n,
\end{aligned} \tag{2.168}
$$

$$
\begin{aligned}
B^i = {}^*\Box A^i &+ \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left[\left(D^i_k + A^{\cdot i}_{k\cdot}\right)q^k\right] + \frac{D}{c^2}\frac{{}^*\partial q^i}{\partial t} + \frac{1}{c^2}\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{{}^*\partial q^k}{\partial t} - \\
&- \frac{1}{c^3}\frac{{}^*\partial\left(\varphi F^i\right)}{\partial t} - \frac{1}{c^3}F^i\frac{{}^*\partial\varphi}{\partial t} + \frac{1}{c^2}F^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{1}{c}\left(D^{mi}+A^{mi}\right)\frac{{}^*\partial\varphi}{\partial x^m} + \\
&+ \frac{1}{c^4}q^kF_kF^i + \frac{1}{c^2}\triangle^i_{km}q^mF^k - \frac{\varphi}{c^3}DF^i + \frac{D}{c^2}\left(D^i_n + A^{\cdot i}_{n\cdot}\right)q^n - \\
&- h^{km}\left\{\frac{{}^*\partial}{\partial x^k}\left(\triangle^i_{mn}q^n\right) + \frac{1}{c}\frac{{}^*\partial}{\partial x^k}\left[\varphi\left(D^i_m + A^{\cdot i}_{m\cdot}\right)\right] + \left(\triangle^i_{kn}\triangle^n_{mp} - \triangle^n_{km}\triangle^i_{np}\right)q^p + \right. \\
&\left. + \frac{\varphi}{c}\left[\triangle^i_{kn}\left(D^n_m + A^{\cdot n}_{m\cdot}\right) - \triangle^n_{km}\left(D^i_n + A^{\cdot i}_{n\cdot}\right)\right] + \triangle^i_{kn}\frac{{}^*\partial q^n}{\partial x^m} - \triangle^n_{km}\frac{{}^*\partial q^i}{\partial x^n}\right\},
\end{aligned} \tag{2.169}
$$

where ${}^*\Box\varphi$ and ${}^*\Box q^i$ result from application of chronometrically invariant d'Alembert operator (2.163) to values $\varphi{=}A_0/\sqrt{g_{00}}$ and $q^i{=}A^i$ (physical observable components of vector A^α)

$$
{}^*\Box\varphi = \frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{{}^*\partial^2\varphi}{\partial x^i\partial x^k}\,, \quad {}^*\Box q^i = \frac{1}{c^2}\frac{{}^*\partial^2 q^i}{\partial t^2} - h^{km}\frac{{}^*\partial^2 q^i}{\partial x^k\partial x^m}\,. \tag{2.170}
$$

D'Alembert operator from tensor field, which equals to zero or not zero gives *d'Alembert equations* for the same field. From physical viewpoint these are equations of propagation of waves of the field. If

[16] This is one of the reasons why practical applications of theory of electromagnetic field and moving charge are mainly calculated in Galilean frame of reference in Minkowski space (space-time of Special Relativity), where Christoffel symbols are zeroes. As a matter of fact, general covariant notation hardly permits unambigous interpretation of calculation results, unless they are formulated with physical observable values (chronometrically invariants) or demoted to a simple specific case, like one in Minkowski space, for instance.

d'Alembertian is not zero, these are equations of propagation of waves triggered by some "external" source or by their distribution in space (d'Alembert equations with "source"). For instance, the sources in electromagnetic field are charges and currents. If d'Alembert operator for the field is zero, then these are equations of propagation of waves of the given field not related to any "sources" introduced into the space. Therefore, equality to zero of all terms not under chronometrically invariant d'Alembert operator $^*\Box$ sets physical conditions of propagation of observable waves of the studied field in four-dimensional space-time, where gravitational force F^i is zero and neither rotation A_{ik} and deformation D_{ik}, same as any additional media. In this case chronometrically invariant equations of propagation of waves of field A^α are obtained from (2.170) made equal to zero in a very simple form. If gravitational potential of space is not zero while the space itself rotates or is subject to deformation (presence of any of these geometric properties will suffice), then, as seen from (2.168, 2.169), equations of wave propagation become pretty more complicated to take account of every of the above geometric factors. If the area of space-time under consideration aside for the tensor field in question is filled with another medium, the d'Alembert equations will gain an additional term in their right parts to characterize the media, which can be obtained from the equations that define it.

2.7 Conclusions

We are now ready to outline the results of this Chapter. Aside for general knowledge of tensor and tensor algebra we obtained some convenient tools to facilitate our calculations in the next Chapters. Equality to zero of absolute derivative of dynamic vector of particle to a direction sets dynamic equations of motion of this particle. Equality to zero of divergence of vector field sets Lorentz condition and equation of continuity. Equality to zero of divergence of 2nd rank symmetric tensor sets laws of conservation, while equality to zero of antisymmetric tensor (and of pseudotensor) of 2nd rank set Maxwell equations. Rotor of vector field is tensor of electromagnetic field (Maxwell tensor). D'Alembert equations are equations of propagation of waves in generalized form, i. e. not only in approximation of geometric optics. This was a brief list of applications of mathematical techniques of which we came into possession. For instance, if we now come across an antisymmetric tensor or a differential operator, we do not have to undertake special calculations of their physical observable components, but may rather use already obtained general formulas from this Chapter.

Chapter 3

Charged particle in pseudo-Riemannian space

3.1 Problem statement

In this Chapter we will set forth the theory of electromagnetic field and moving of charged particles in four-dimensional pseudo-Riemannian space, where all properties of field will be in chronometrically invariant forms (i. e. expressed through physical observable values).

Electromagnetic field is commonly studied as vector field of four-dimensional potential A^{α} in four-dimensional space-time (pseudo-Riemannian space). Its temporal component is *scalar potential* of electromagnetic field φ, while the three-dimensional components make up so-called *vector-potential* A^i. Four-dimensional potential of electromagnetic field A^{α} in CGSE and Gaussian systems of units has the dimension

$$A^{\alpha}\ [\mathrm{g}^{1/2}\,\mathrm{cm}^{1/2}\,\mathrm{s}^{-1}]. \tag{3.1}$$

Its components φ and A^i have the same dimensions. Therefore, studying electromagnetic field is substantially different from studying gravitational field: according to the theory of chronometrically invariants gravitational inertial force F^i and gravitational potential w (1.38) are functions of geometric properties of space only, while electromagnetic field (i. e. the field of A^{α} potential) has not been "geometrically interpreted" yet and we have to study it just as an external vector field introduced into space-time.

Equations of classical electrodynamics, — Maxwell equations that define the relationship between strengths of electric and magnetic fields, — were obtained long before theoretical physics accepted the terms of distorted pseudo- Riemannian space and even flat Minkowski space. Later, when electrodynamics was set forth in Minkowski space under the name of *relativistic electrodynamics*, Maxwell equations were obtained in four-dimensional form. Then general covariant form of Maxwell equations in pseudo-Riemannian space was obtained. But having accepted general covariant form Maxwell equations became less illustrative, which used to be an advantage of classical electrodynamics. On the other hand, four-dimensional equations in Minkowski space can be simply presented as scalar (temporal) and vector (spatial) components, because in Galilean frame of reference they are observable values by definition. But when we turn to distorted, non-uniform and anisotropic pseudo-Riemannian space, the problem of comparing vector and scalar components in general covariant equations with equations of classical and relativistic electrodynamics becomes non-trivial. In other words, a question arises which values can be assumed physical observable ones.

Thus the equations of relativistic electrodynamics in pseudo-Riemannian space should be formulated in respect to physical observable components of electromagnetic field and observable properties of observer's frame of reference. We are going to tackle the problem using mathematical apparatus of chronometric invariants, i. e. projecting general covariant values onto time and space of a real body of reference, which physical and geometric properties are reference one in our measurements. The results we are going to obtain this way will help us to arrive to *observable generalization* of the basic values and laws of classical and relativistic electrodynamics in such a form that takes into account effect of physical and geometric properties of the frame of reference (the body of reference) on results of measurements.

3.2 Observable components of electromagnetic field tensor. Invariants of the field

By definition, tensor of electromagnetic field is rotor of four-dimensional electromagnetic potential A^α and is also referred to as *Maxwell tensor*

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\,. \tag{3.2}$$

As seen, the formula is a general covariant generalization of three-dimensional values in classical electrodynamics

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,, \qquad \vec{H} = \mathrm{rot}\,\vec{A}, \tag{3.3}$$

where $\vec{E}$ is vector of strength of electric component of the field. Value φ is a scalar potential, $\vec{A}$ is a three-dimensional vector-potential of electromagnetic field, and

$$\vec{\nabla} = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \tag{3.4}$$

is gradient operator of a scalar function in regular three-dimensional Euclidean space.

In this Section we are going to discuss which components of general covariant tensor of electromagnetic field $F_{\alpha\beta}$ are physical observable values and to define relationship between these values and three-dimensional vector strengths of electric field $\vec{E}$ and magnetic field $\vec{H}$ in classical electrodynamics. The latter vectors will be also obtained in pseudo-Riemannian space, which generally is distorted, non-uniform and anisotropic.

An important note should be taken. Because in a flat four-dimensional space-time (Minkowski space) in an inertial frame of reference (i. e. the one that moves linearly and with constant velocity) the metric is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2, \tag{3.5}$$

and components of fundamental metric tensor are

$$g_{00} = 1, \qquad g_{0i} = 0, \qquad g_{11} = g_{22} = g_{33} = -1, \tag{3.6}$$

no difference exists between covariant and contravariant components of four-dimensional potential A^α (partially, this is why all calculations in Minkowski space are simpler)

$$\varphi = A_0 = A^0, \qquad A_i = -A^i. \tag{3.7}$$

In pseudo-Riemannian space (and in Riemannian space in general) there is a difference, because metric has more general form. Therefore scalar potential and vector-potential of electromagnetic field should be defined as physical observable (chronometrically invariant) components of four-dimensional potential A^α

$$\varphi = b^\alpha A_\alpha = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = h^i_\sigma A^\sigma = A^i. \tag{3.8}$$

Other components of A^α, being not chronometrically invariant, are formulated with φ and q^i as

$$A^0 = \frac{\varphi + \frac{1}{c}v_i q^i}{1 - \frac{w}{c^2}}\,, \qquad A_i = -q_i - \frac{\varphi}{c}v_i\,. \tag{3.9}$$

Note that according to the theory of chronometric invariants, covariant chronometrically invariant vector q_i is obtained from contravariant vector q^i as a result of lowering the index using chronometrically invariant (observable) tensor h_{ik}, i. e. $q_i{=}h_{ik}q^k$. To the contrary, a regular covariant vector A_i, which is not a chronometric invariant, is obtained as a result of lowering the index using four-dimensional fundamental metric tensor $A_i{=}g_{i\alpha}A^\alpha$.

According to the general formula for the square of a vector (2.39), the square of four-dimensional vector of electromagnetic potential A^α in accompanying frame of reference is

$$A_\alpha A^\alpha = g_{\alpha\beta} A^\alpha A^\beta = \varphi^2 - h_{ik} q^i q^k = \varphi^2 - q^2, \tag{3.10}$$

and is: real value, if $\varphi^2 > q^2$; imaginary value, if $\varphi^2 < q^2$; zero (isotropic) value, if $\varphi^2 = q^2$.

Now using components of four-dimensional potential A^α (3.8, 3.9) in definition of electromagnetic field tensor $F_{\alpha\beta}$ (3.2), formulating regular derivatives with chronometrically invariant derivatives (1.33), and using formulas for components of rotor of an arbitrary vector field (2.143–2.150) we obtain physical observable (chronometrically invariant) components of the tensor $F_{\alpha\beta}$

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik} \left(\frac{{}^*\partial \varphi}{\partial x^k} + \frac{1}{c} \frac{{}^*\partial q_k}{\partial t} \right) - \frac{\varphi}{c^2} F^i, \tag{3.11}$$

$$F^{ik} = g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn} \left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m} \right) - \frac{2\varphi}{c} A^{ik}. \tag{3.12}$$

We denote chronometrically inveriant (physical observable) components of electromagnetic field tensor as

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}, \qquad H^{ik} = F^{ik}, \tag{3.13}$$

and covariant chronometrically invariant values formed with their help

$$E_i = h_{ik} E^k = \frac{{}^*\partial \varphi}{\partial x^i} + \frac{1}{c} \frac{{}^*\partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i \,, \tag{3.14}$$

$$H_{ik} = h_{im} h_{kn} H^{mn} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} - \frac{2\varphi}{c} A_{ik} \,, \tag{3.15}$$

while mixed components $H_{k\cdot}^{\cdot m} = -H_{\cdot k}^{m\cdot}$ are obtained from component H^{ik} using three-dimensional chronometrically invariant metric tensor h_{ik}, i. e. $H_{k\cdot}^{\cdot m} = h_{ki} H^{im}$. In this case deformation of space of reference $D_{ik} = \frac{1}{2} \frac{{}^*\partial h_{ik}}{\partial t}$ (1.40) is also present in these formulas, but in an implicit way and appears when we substitute components $q_k = h_{km} q^m$ into formulas for time derivatives.

Besides, we may as well formulate other components of electromagnetic field tensor $F_{\alpha\beta}$ with its observable components E^i and H^{ik} (3.11) using formulas for components of arbitrary antisymmetric tensor (2.112–2.115). This is possible because generalized formulas (2.112–2.115) contain variables E^i and H^{ik} in "implicit" form, irrespective of whether they are components of a rotor or of an antisymmetric tensor of any other kind.

In Minkowski space, when acceleration F^i, rotation A_{ik} and deformation D_{ik} of the space of reference are zeroes, the formula for the component E_i becomes

$$E_i = \frac{\partial \varphi}{\partial x^i} + \frac{1}{c} \frac{\partial A_i}{\partial t} \,, \tag{3.16}$$

or in three-dimensional vector form

$$\vec{E} = \vec{\nabla} \varphi + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \,, \tag{3.17}$$

which, up within the sign, matches the formula for $\vec{E}$ in classical electrodynamics.

Now to formulate strength of magnetic field in three-dimensional vector form we use components of pseudotensor $F^{*\alpha\beta}$, which is pseudo-Riemannian space is dual to Maxwell tensor of electromagnetic field $F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}$ (2.123). According to (2.124) physical observable components of this pseudotensor are values

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}, \qquad E^{*ik} = F^{*ik}. \tag{3.18}$$

Using formulas for components of pseudotensor $F^{*\alpha\beta}$, obtained in Chapter 2 (2.125–2.131) and formulas for E_i and H_{ik} (3.14, 3.15) we arrive to expanded formulas for H^{*i} and E^{*ik}

$$H^{*i} = \frac{1}{2}\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}A_{mn}\right) = \frac{1}{2}\varepsilon^{imn}H_{mn}\,, \tag{3.19}$$

$$E^{*ik} = \varepsilon^{ikn}\left(\frac{\varphi}{c^2}F_n - \frac{^*\partial\varphi}{\partial x^n} - \frac{1}{c}\frac{^*\partial q_n}{\partial t}\right) = -\varepsilon^{ikn}E_k\,. \tag{3.20}$$

We can see that the following pairs of tensors are dual conjugate: H^{*i} and H_{mn}, E^{*ik} and E_m. Chronometrically invariant (physical observable) value H^{*i} (3.19) includes the term

$$\frac{1}{2}\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m}\right) = \frac{1}{2}\varepsilon^{imn}\left(^*\nabla_n q_m - {}^*\nabla_m q_n\right), \tag{3.21}$$

which is chronometrically invariant rotor of three-dimensional vector field q_m, and the term

$$\frac{1}{2}\varepsilon^{imn}\frac{2\varphi}{c}A_{mn} = \frac{2\varphi}{c}\Omega^{*i}, \tag{3.22}$$

where $\Omega^{*i}=\frac{1}{2}\varepsilon^{imn}A_{mn}$ is chronometrically invariant pseudovector of angular rotational velocity of space of reference. In Galilean frame of reference in Minkowski space, i. e. in absence of acceleration, rotation and deformation, the formula we obtained for chronometrically invariant pseudovector of strength of magnetic field H^{*i} (3.19) takes the form

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m}\right), \tag{3.23}$$

or in three-dimensional vector form, is

$$\vec{H} = \mathrm{rot}\,\vec{A}\,. \tag{3.24}$$

Therefore, the structure of pseudo-Riemannian space affects electromagnetic field due to the fact that observable (chronometrically invariant) vectors of electric strength E_i (3.14) and magnetic strength H^{*i} (3.19) depend upon gravitational potential and rotation of the space of reference itself.

The same will be true as well in flat Minkowski space, if a non-inertial frame of reference that rotates and moves with acceleration is assumed as the observer's frame of reference. But in Minkowski space we can always find a Galilean frame of reference (which is not true in pseudo-Riemannian space), because Minkowski space itself does not accelerate the frame of reference and neither rotates nor deforms it. Therefore such effects in Minkowski space are purely relative.

In relativistic electrodynamics we introduce *invariants of electromagnetic field* (or simply *field invariants*)

$$J_1 = F_{\mu\nu}F^{\mu\nu} = 2F_{0i}F^{0i} + F_{ik}F^{ik}, \tag{3.25}$$

$$J_2 = F_{\mu\nu}F^{*\mu\nu} = 2F_{0i}F^{*0i} + F_{ik}F^{*ik}. \tag{3.26}$$

The former is a scalar, while the latter is a pseudoscalar. Formulating them with components of Maxwell tensor we obtain

$$J_1 = H_{ik}H^{ik} - 2E_iE^i, \qquad J_2 = \varepsilon^{imn}\left(E_m H_{in} - E_i H_{nm}\right), \tag{3.27}$$

and using formulas for components of dual pseudotensor $F^{*\mu\nu}$ obtained in Chapter 2 we can present the field invariants as

$$J_1 = -2\left(E_iE^i - H_{*i}H^{*i}\right), \qquad J_2 = -4E_iH^{*i}. \tag{3.28}$$

Because J_1 and J_2 are invariants we can maintain that:

- if in any frame of reference squares of lengths of vectors of strengths of electric and magnetic fields are equal $E^2{=}H^{*2}$, the equality will preserve in any other frame of reference;

- if in any frame of reference vectors of strength of electric and magnetic fields are orthogonal E_iH^{*i}=0, the orthogonality will preserve in any frame of reference.

Electromagnetic field that satisfies conditions E^2=H^{*2} and E_iH^{*i}=0, i. e. the one in which both field invariants (3.28) are zeroes is referred to in electrodynamics as *isotropic*. Here the term "isotropic" stands not for location of field in light-like area of space-time (as is assumed in space-time theory), but rather for the field's property of equal emissions in any direction in three-dimensional space.

Invariants of electromagnetic field can be also formulated with chronometrically invariant derivatives of observable scalar potential φ and vector-potential q^i (3.8) of the field

$$\begin{aligned} J_1 = 2 \Bigg[& h^{im}h^{kn}\left(\frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i}\right)\frac{{}^*\partial q_m}{\partial x^n} - h^{ik}\frac{{}^*\partial \varphi}{\partial x^i}\frac{{}^*\partial \varphi}{\partial x^k} - \frac{2}{c}h^{ik}\frac{{}^*\partial \varphi}{\partial x^i}\frac{{}^*\partial q_k}{\partial t} - \frac{1}{c^2}h^{ik}\frac{{}^*\partial q_i}{\partial t}\frac{{}^*\partial q_k}{\partial t} + \\ & + \frac{8\varphi}{c^2}\Omega_i\Omega^i - \frac{2\varphi}{c}\varepsilon^{imn}\Omega_m\frac{{}^*\partial q_i}{\partial x^n} + \frac{2\varphi}{c^2}\frac{{}^*\partial \varphi}{\partial x^i}F^i + \frac{2\varphi}{c^3}\frac{{}^*\partial q_i}{\partial t}F^i - \frac{\varphi}{c^4}F_iF^i\Bigg], \end{aligned} \tag{3.29}$$

$$J_2 = \frac{1}{2}\left[\varepsilon^{imn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{4\varphi}{c}\Omega^{*i}\right]\left(\frac{{}^*\partial \varphi}{\partial x^i} + \frac{1}{c}\frac{{}^*\partial q_i}{\partial t} - \frac{\varphi}{c^2}F_i\right). \tag{3.30}$$

Physical conditions for isotropic electromagnetic field are obtained by equaling the latter formulas (3.29, 3.30) to zero. Doing this we can see that the conditions of equality of three-dimensional lengths of vectors of strengths E^2=H^{*2} and their orthogonality E_iH^{*i}=0 in pseudo-Riemannian space at the same time depend upon not only properties of the field itself (i. e. scalar potential φ and vector-potential q^i) but also upon acceleration F^i, rotation A_{ik} and deformation D_{ik} of the space of the body of reference. In particular, vectors E_i and H^{*i} are orthogonal when the space is holonomic Ω^{*i}=0, while three-dimensional field of vector-potential q^i is rotation-free $\varepsilon^{imn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right)$=0.

3.3 Chronometrically invariant Maxwell equations. Law of conservation of electric charge. Lorentz condition

In classical electrodynamics the dependencies between strengths of electric field $\vec{E}$ [$\mathrm{g}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{s}^{-1}$] and magnetic field $\vec{H}$ [$\mathrm{g}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{s}^{-1}$] are set forth in *Maxwell equations*, which result from generalization of experimental data. In the middle 19th century J. C. Maxwell showed that if electromagnetic field is induced in vacuum by given charges and currents, the resulting field is defined by two groups of equations [4]

$$\begin{array}{ll} \text{Group I} & \text{Group II} \\ \mathrm{rot}\,\vec{H} - \dfrac{1}{c}\dfrac{\partial \vec{E}}{\partial t} = \dfrac{4\pi}{c}\vec{j}\,, & \mathrm{rot}\,\vec{E} + \dfrac{1}{c}\dfrac{\partial \vec{H}}{\partial t} = 0, \\ \mathrm{div}\,\vec{E} = 4\pi\rho\,, & \mathrm{div}\,\vec{H} = 0, \end{array} \tag{3.31}$$

where ρ [$\mathrm{g}^{1/2}\,\mathrm{cm}^{-3/2}\,\mathrm{s}^{-1}$] stands for distributed electric charge density (the amount e [$\mathrm{g}^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1}$] of charge within 1 cm^3) and $\vec{j}$ [$\mathrm{g}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{s}^{-2}$] is the vector of current density. Equations that contain field-inducing sources ρ and $\vec{j}$ are known as *the first group of Maxwell equations*, while equations that do not contain field sources are referred to as *the second group of Maxwell equations*.

The first equation in Group I is Biot-Savart law, the second is Gauss theorem, both in differential notation. The first and the second equations in Group II are differential notation of Faraday law of electromagnetic induction and the condition of absence of magnetic charges, respectively. Totally, we have 8 equations (two vector ones and two scalar ones) in 10 unknowns: three components of $\vec{E}$, three components of $\vec{H}$, three components of $\vec{j}$, and one component of ρ.

The dependence between the field sources ρ and $\vec{j}$ is set by equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0 \tag{3.32}$$

called *the law of conservation of electric charge*, which is a mathematical notation of an experimental fact that electric charge can not be destroyed, but is merely re-distributed between contacting charged bodies.

Now we have a system of 9 equations in 10 unknowns, i. e. the system that define the field and its sources is still indefinite. The 10th equation which makes the system of equations definite (the number of equations should be the same as that of unknowns) is the *Lorentz condition*, which constrains potentials of electromagnetic field and is

$$\frac{1}{c}\frac{\partial\varphi}{\partial t}+\mathrm{div}\vec{A}=0\,. \tag{3.33}$$

The Lorentz condition stems from the fact that scalar potential φ and vector-potential $\vec{A}$ of electromagnetic field related to strength values $\vec{E}$ and $\vec{H}$ with (3.3) are defined ambiguously from them: $\vec{E}$ and $\vec{H}$ in (3.3) do not change if we replace

$$\vec{A}=\vec{A}'+\vec{\nabla}\Psi\,,\qquad \varphi=\varphi'-\frac{1}{c}\frac{\partial\Psi}{\partial t}\,, \tag{3.34}$$

where ψ is an arbitrary scalar. Evidently, ambiguous definition of φ and $\vec{A}$ permits other dependencies between the values aside for Lorentz condition. Nevertheless, it is Lorentz condition that enables transformation of Maxwell equations into wave equations.

Here is how it happens. The equation $\mathrm{div}\vec{H}=0$ (3.31) is satisfied completely if we assume $\vec{H}=\mathrm{rot}\vec{A}$. In this case the first equation in Group II of Maxwell equations (3.31) takes the form

$$\mathrm{rot}\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=0\,, \tag{3.35}$$

which has the solution

$$\vec{E}=-\vec{\nabla}\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\,. \tag{3.36}$$

Substituting $\vec{H}=\mathrm{rot}\vec{A}$ and $\vec{E}$ (3.36) into Group I of Maxwell equations we arrive at

$$\triangle\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\vec{\nabla}\left(\mathrm{div}\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}\right)=-\frac{4\pi}{c}\vec{j}\,, \tag{3.37}$$

$$\triangle\varphi+\frac{1}{c}\frac{\partial}{\partial t}\left(\mathrm{div}\vec{A}\right)=-4\pi\rho\,, \tag{3.38}$$

where $\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is regular Laplace operator.

Constraining potentials φ and $\vec{A}$ with Lorentz condition (3.33) we bring equations in Group I to the form

$$\Box\varphi=-4\pi\rho\,, \tag{3.39}$$

$$\Box\vec{A}=-\frac{4\pi}{c}\vec{j}\,, \tag{3.40}$$

where $\Box=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\triangle$ is regular d'Alembert operator.

The result of applying d'Alembert operator to a field are equations of propagation of waves of the field (see Section 2.6). Therefore the obtained result implies that if Lorentz condition is true Group I equations (3.31) are a system of equations of propagation of waves of scalar and vector potentials of electromagnetic field with sources (charges and currents). The equations will be obtained in the next Sections. So far we are going to consider general covariant Maxwell equations in pseudo-Riemannian space to obtain them in chronometrically invariant form, i. e. formulated with physical observable values.

In four-dimensional pseudo-Riemannian space Lorentz condition has general covariant form

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} A^\mu = 0\,, \tag{3.41}$$

i. e. is a condition of conservation of four-dimensional potential of field A^α. Law of conservation of electric charge (*continuity equation*) is

$$\nabla_\sigma j^\sigma = 0\,, \tag{3.42}$$

where j^α is four-dimensional *current vector* (also referred to as *shift current*), observable components in which are electric charge density

$$\rho = \frac{1}{c}\frac{j_0}{\sqrt{g_{00}}}\,, \tag{3.43}$$

and three-dimensional vector of current density j^i. Using the formula for divergence of vector field in chronometrically invariant form, which we obtained in Chapter 2 (2.107), we arrive to Lorentz condition (3.41) also in chronometrically invariant form

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t} + \frac{\varphi}{c}D + {}^*\nabla_i q^i - \frac{1}{c^2}F_i q^i = 0\,, \tag{3.44}$$

and to continuity equation in chronometrically invariant notation as well

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\nabla_i\, j^i - \frac{1}{c^2}F_i\, j^i = 0\,. \tag{3.45}$$

Here $D{=}D^i_i{=}\frac{{}^*\partial \ln\sqrt{h}}{\partial t}$ stands for spur of tensor of deformation velocities of space (1.40) — the rate of relative expansion of elementary volume, while ${}^*\nabla_i$ is operator of chronometrically invariant divergence (2.105).

Because F_i (1.38) contains derivative of gravitational potential $w{=}c^2(1{-}\sqrt{g_{00}})$, the term $\frac{1}{c^2}F_i q^i$ in the obtained formulas (3.44, 3.45) takes into account the difference in time pace at opposite walls of the elementary volume. The formula for gravitational inertial force F_i (1.38) also accounts for non-stationary state of rotation velocity of space v_i. Besides, gravitational potential and velocity of space rotation appear in chronometrically invariant derivation operators (1.33)

$$\frac{{}^*\partial}{\partial t} = \frac{1}{1-\frac{w}{c^2}}\frac{{}^*\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{1}{c^2}v_i\frac{{}^*\partial}{\partial t}\,. \tag{3.46}$$

Therefore, the condition of conservation of vector field streams A^α (3.44) and j^α (3.45) directly depend upon gravitational potential and velocity of space rotation.

Chronometrically invariant values $\frac{{}^*\partial\varphi}{\partial t}$ and $\frac{{}^*\partial\rho}{\partial t}$ are changes in time of physical observable values φ and ρ. Chronometrically invariant values φD and ρD are observable changes in time of three-dimensional volumes, filled with values φ and ρ.

In absence of gravitational inertial force, rotation and deformation of space, the obtained chronometrically invariant formulas for Lorentz condition (3.44) and electric charge conservation law (3.45) become

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \frac{\partial q^i}{\partial x^i} - \frac{\partial \ln\sqrt{h}}{\partial x^i}q^i = 0\,, \tag{3.47}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial j^i}{\partial x^i} - \frac{\partial \ln\sqrt{h}}{\partial x^i}j^i = 0\,, \tag{3.48}$$

which in Galilean frame of reference in Minkowski space are

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \frac{\partial q^i}{\partial x^i} = 0\,, \qquad \frac{\partial\rho}{\partial t} + \frac{\partial j^i}{\partial x^i} = 0\,, \tag{3.49}$$

or, in a regular vector notation

$$\frac{1}{c}\frac{\partial\varphi}{\partial t}+\operatorname{div}\overrightarrow{A}=0\,,\qquad \frac{\partial\rho}{\partial t}+\operatorname{div}\overrightarrow{j}=0\,, \tag{3.50}$$

which fully matches notations of Lorentz condition (3.33) and electric charge conservation law (3.32) in classical electrodynamics.

Let us now turn to Maxwell equations. In pseudo-Riemannian space each pair of equations merge into a single general covariant equation

$$\nabla_\sigma F^{\mu\sigma}=\frac{4\pi}{c}j^\mu,\qquad \nabla_\sigma F^{*\mu\sigma}=0\,, \tag{3.51}$$

where $F^{\mu\sigma}$ is contravariant form of electromagnetic field tensor, and $F^{*\mu\sigma}$ is dual pseudotensor. Using chronometrically invariant formulas for divergence of antisymmetric 2nd rank tensor (2.121, 2.122) and for its dual pseudotensor (2.135, 2.136), obtained in Chapter 2, we write down Maxwell equations in chronometrically invariant form

$$\left.\begin{aligned} &{}^*\nabla_i E^i-\frac{1}{c}H^{ik}A_{ik}=4\pi\rho\\ &{}^*\nabla_k H^{ik}-\frac{1}{c^2}F_kH^{ik}-\frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t}+DE^i\right)=\frac{4\pi}{c}j^i \end{aligned}\right\}\ \text{Group I,} \tag{3.52}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i}-\frac{1}{c}E^{*ik}A_{ik}=0\\ &{}^*\nabla_k E^{*ik}-\frac{1}{c^2}F_kE^{*ik}-\frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t}+DH^{*i}\right)=0 \end{aligned}\right\}\ \text{Group II.} \tag{3.53}$$

Chronometrically invariant Maxwell equations in this notation were first obtained by J. del Prado [22] and N. V. Pavlov [23] independently. Now we transform chronometrically invariant Maxwell equations in a way that they include E^i and H^{*i} as unknowns. Obtaining these values from definitions (2.125, 2.124, and 2.111)

$$H_{*i}=\frac{1}{2}\varepsilon_{imn}H^{mn}, \tag{3.54}$$

$$E^{*ik}=\varepsilon^{ikm}\left(\frac{\varphi}{c^2}F_m-\frac{{}^*\partial\varphi}{\partial x^m}-\frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right)=-\varepsilon^{ikm}E_m\,, \tag{3.55}$$

and multiplying the first equation by ε^{ipq} we arrive to

$$\varepsilon^{ipq}H_{*i}=\frac{1}{2}\varepsilon^{ipq}\varepsilon_{imn}H^{mn}=\frac{1}{2}\left(\delta^p_m\delta^q_n-\delta^q_m\delta^p_n\right)H^{mn}=H^{pq}. \tag{3.56}$$

Substituting the result as $H^{ik}{=}\varepsilon^{mik}H_{*m}$ into the first equation in Group I (3.52) we bring it to the form

$$^*\nabla_i E^i-\frac{2}{c}\Omega_{*m}H^{*m}=4\pi\rho\,, \tag{3.57}$$

where $\Omega^{*i}{=}\frac{1}{2}\varepsilon^{imn}A_{mn}$ is chronometrically invariant pseudovector of angular velocity of rotation of the space of reference. Substituting the second formula $E^{*ik}{=}{-}\varepsilon^{ikm}E_m$ (3.55) into the first equation of Group II (3.53), we obtain

$$^*\nabla_i H^{*i}+\frac{2}{c}\Omega_{*m}E^m=0\,. \tag{3.58}$$

Then substituting $H^{ik}{=}\varepsilon^{mik}H_{*m}$ into the second equation in Group I (3.52) we obtain

$$^*\nabla_k\left(\varepsilon^{mik}H_{*m}\right)-\frac{1}{c^2}F_k\varepsilon^{mik}H_{*m}-\frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t}+\frac{{}^*\partial\ln\sqrt{h}}{\partial t}E^i\right)=\frac{4\pi}{c}j^i, \tag{3.59}$$

and after multiplying both parts of it by $\sqrt{h}$ and taking into account that ${}^*\nabla_k \varepsilon^{mik}=0$ we bring this formula (3.59) to the form

$$\varepsilon^{ikm}\,{}^*\nabla_k\left(H_{*m}\sqrt{h}\right)-\frac{1}{c^2}\varepsilon^{ikm}F_k H_{*m}\sqrt{h}-\frac{1}{c}\frac{{}^*\partial}{\partial t}\left(E^i\sqrt{h}\right)=\frac{4\pi}{c}j^i\sqrt{h}\,, \tag{3.60}$$

or, in another notation

$$\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(H_{*m}\sqrt{h}\right)-\frac{1}{c}\frac{{}^*\partial}{\partial t}\left(E^i\sqrt{h}\right)=\frac{4\pi}{c}j^i\sqrt{h}\,, \tag{3.61}$$

where $j^i\sqrt{h}$ is volume density of current and ${}^*\widetilde{\nabla}_k={}^*\nabla_k-\frac{1}{c^2}F_k$ is chronometrically invariant physical divergence (2.106) that accounts for different time pace at opposite walls of elementary volume.

The obtained equation (3.60) is chronometrically invariant notation of Biot-Savart law in pseudo-Riemannian space.

Substituting $E^{*ik}=-\varepsilon^{ikm}E_m$ (3.55) into the second equation in Group II (3.53) after similar transformations we obtain it in the form

$$\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(E_m\sqrt{h}\right)+\frac{1}{c}\frac{{}^*\partial}{\partial t}\left(H^{*i}\sqrt{h}\right)=0\,, \tag{3.62}$$

which is chronometrically invariant notation of Faraday law of electromagnetic induction in pseudo-Riemannian space.

The final system of 10 chronometrically invariant equations in 10 unknowns (two groups of Maxwell equations, Lorentz condition and continuity equation) that define electromagnetic field and its sources in pseudo-Riemannian space becomes

$$\left.\begin{aligned}&{}^*\nabla_i E^i-\frac{2}{c}\Omega_{*m}H^{*m}=4\pi\rho\\&\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(H_{*m}\sqrt{h}\right)-\frac{1}{c}\frac{{}^*\partial}{\partial t}\left(E^i\sqrt{h}\right)=\frac{4\pi}{c}j^i\sqrt{h}\end{aligned}\right\}\text{Group I,} \tag{3.63}$$

$$\left.\begin{aligned}&{}^*\nabla_i H^{*i}+\frac{2}{c}\Omega_{*m}E^m=0\\&\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(E_m\sqrt{h}\right)+\frac{1}{c}\frac{{}^*\partial}{\partial t}\left(H^{*i}\sqrt{h}\right)=0\end{aligned}\right\}\quad\text{Group II,} \tag{3.64}$$

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t}+\frac{\varphi}{c}D+{}^*\widetilde{\nabla}_i q^i=0 \qquad \text{Lorentz condition,} \tag{3.65}$$

$$\frac{{}^*\partial\rho}{\partial t}+\rho D+{}^*\widetilde{\nabla}_i\, j^i=0 \qquad \text{equation of continuity.} \tag{3.66}$$

In Galilean frame of reference in Minkowski space the determinant of physical observable metric tensor $\sqrt{h}=1$, it is not subject to deformation $D_{ik}=0$, rotation $\Omega_{*m}=0$ or acceleration $F_i=0$. Then chronometrically invariant Maxwell equations, obtained in pseudo-Riemannian space (3.63, 3.64), bring us directly to Maxwell equations in classical electrodynamics in by-component (tensor) form

$$\left.\begin{aligned}&\frac{\partial E^i}{\partial x^i}=4\pi\rho\\&e^{ikm}\left(\frac{\partial H_{*m}}{\partial x^k}-\frac{\partial H_{*k}}{\partial x^m}\right)-\frac{1}{c}\frac{\partial E^i}{\partial t}=\frac{4\pi}{c}j^i\end{aligned}\right\}\quad\text{Group I,} \tag{3.67}$$

$$\left.\begin{aligned}&\frac{\partial H^{*i}}{\partial x^i}=0\\&e^{ikm}\left(\frac{\partial E_m}{\partial x^k}-\frac{\partial E_k}{\partial x^m}\right)-\frac{1}{c}\frac{\partial H^{*i}}{\partial t}=0\end{aligned}\right\}\quad\text{Group II.} \tag{3.68}$$

The same equations put into vector notation will be similar to classical Maxwell equations in three-dimensional Euclidean space (3.31). Besides, the obtained chronometrically invariant Maxwell equations in four-dimensional pseudo-Riemannian space (3.64) show that in absence of rotation of space chronometrically invariant mathematical divergence of magnetic field is zero ${}^*\nabla_i H^{*i}{=}0$. In other words, magnetic field conserves in holonomic space. But divergence of electric field in this case is not zero ${}^*\nabla_i E^i{=}4\pi\rho$ (3.63), i. e. electric field is linked directly to density of electric charges ρ. Hence a conclusion that "magnetic charge", if it actually exists, should be linked directly to the field of rotation of the space itself.

3.4 Four-dimensional d'Alembert equations for electromagnetic potential and their observable components

As we have already mentioned, d'Alembert operator being applied to field gives equations of propagation of waves of that field. Therefore d'Alembert equations for scalar electromagnetic potential φ are equations of propagation of waves of scalar field φ, while for three-dimensional vector-potential $\vec{A}$ these are equations of propagation of waves of three-dimensional vector field $\vec{A}$.

General covariant form of these equations for four-dimensional potential of electromagnetic field was obtained by K. P. Stanyukovich in his book [24]. Using Group I of general covariant Maxwell equations $\nabla_\sigma F^{\mu\sigma}{=}\frac{4\pi}{c}j^\mu$ (3.51) and Lorentz condition $\nabla_\sigma A^\sigma{=}0$ (3.41) he arrived to general covariant equation in respect to four-dimensional potential A^α of electromagnetic field

$$\Box A^\alpha - R^\alpha_\beta A^\beta = -\frac{4\pi}{c}j^\alpha, \tag{3.69}$$

where $R^\alpha_\beta{=}g^{\alpha\mu}R^{\sigma}_{\cdot\mu\beta\sigma}$ is Ricci tensor and $R^{\alpha}_{\cdot\mu\beta\sigma}$ is four-dimensional Riemann-Christoffel tensor of curvature. The term $R^\alpha_\beta A^\beta$ is absent in the left part if Ricci tensor is zero, i. e. metric of space satisfies Einstein equations away from masses (in vacuum). Also that term can be neglected in case space curvature is not significant. But even in flat Minkowski space the problem can be considered in presence of acceleration and rotation. Even this approximation may reveal, for instance, effect of acceleration and rotation of body of reference on observable velocity of propagation of electromagnetic waves.

The reason for the above discussion here is that obtaining chronometrically invariant projections of d'Alembert equations in full is a very labor intensive task. The resulting equations will be too bulky to make any unambiguous conclusions. Therefore we will limit the scope of our work to obtaining chronometrically invariant d'Alembert equations for electromagnetic field in non-inertial frame of reference in Minkowski space. But that does not affect other Sections in this Chapter. Calculating chronometrically invariant projections of general covariant four-dimensional d'Alembert equations

$$\Box A^\alpha = -\frac{4\pi}{c}j^\alpha \tag{3.70}$$

using general formulas (2.168, 2.169) and taking into account that $\rho{=}\frac{1}{c\sqrt{g_{00}}}g_{0\alpha}j^\alpha$ is observable charge density, in the space without dynamic deformation and in the linear approximation (without high multiplicity members — weak fields of gravitation and space's rotation) we obtain

$$\begin{gathered} {}^*\Box\varphi - \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(F_k q^k\right) - \frac{1}{c^3}F_i\frac{{}^*\partial q^i}{\partial t} + \frac{1}{c^2}F^i\frac{{}^*\partial\varphi}{\partial x^i} + h^{ik}\triangle^m_{ik}\frac{{}^*\partial\varphi}{\partial x^m} - \\ - h^{ik}\frac{1}{c}\frac{{}^*\partial}{\partial x^i}\left(A_{kn}q^n\right) + \frac{1}{c}h^{ik}\triangle^m_{ik}A_{mn}q^n = 4\pi\rho, \end{gathered} \tag{3.71}$$

$$\begin{gathered} {}^*\Box A^i + \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left(A_{k\cdot}^{\cdot i}q^k\right) + \frac{1}{c^2}A_{k\cdot}^{\cdot i}\frac{{}^*\partial q^k}{\partial t} - \frac{1}{c^3}\frac{{}^*\partial\left(\varphi F^i\right)}{\partial t} - \frac{1}{c^3}F^i\frac{{}^*\partial\varphi}{\partial t} + \frac{1}{c^2}F^k\frac{{}^*\partial q^i}{\partial x^k} - \\ - \frac{1}{c}A^{mi}\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c^2}\triangle^i_{km}q^m F^k - h^{km}\left\{\frac{{}^*\partial}{\partial x^k}\left(\triangle^i_{mn}q^n\right) + \frac{1}{c}\frac{{}^*\partial}{\partial x^k}\left(\varphi A_{m\cdot}^{\cdot i}\right) + \right. \\ \left. + \left(\triangle^i_{kn}\triangle^n_{mp} - \triangle^n_{km}\triangle^i_{np}\right)q^p + \frac{\varphi}{c}\left(\triangle^i_{kn}A_{m\cdot}^{\cdot n} - \triangle^n_{km}A_{n\cdot}^{\cdot i}\right) + \triangle^i_{kn}\frac{{}^*\partial q^n}{\partial x^m} - \triangle^n_{km}\frac{{}^*\partial q^i}{\partial x^n}\right\} = \frac{4\pi}{c}j^i. \end{gathered} \tag{3.72}$$

We see that physical properties of space of reference F^i, A_{ik}, D_{ik} and curvilinearity of three-dimensional trajectories (characterized by $\triangle^i_{km}$) make some additional "sources" that along with electromagnetic sources φ and j^i form the waves that run through electromagnetic field.

Let us now analyze the results. First we consider the equations we obtained (3.71, 3.72) in Galilean frame of reference in a flat Minkowski space. Here metric takes the form as in (3.5) and therefore chronometrically invariant d'Alembert operator $^*\Box$ (2.163) transforms into regular d'Alembert operator $^*\Box = \frac{1}{c^2}\frac{^*\partial^2}{\partial t^2} - h^{ik}\frac{^*\partial^2}{\partial x^i \partial x^k} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle = \Box$. Then the obtained equations (3.71, 3.72) will be

$$\Box\varphi = 4\pi\rho\,, \qquad \Box q^i = -\frac{4\pi}{c}j^i, \tag{3.73}$$

which fully matches equations of classical electrodynamics (3.39, 3.40).

Now we return to the obtained chronometrically invariant d'Alembert equations (3.39, 3.40). To make their analysis easier we denote all terms in left parts, which stand after chronometrically invariant d'Alembert equations $^*\Box$, as T in the scalar equation (3.39) and as B^i in the vector equation (3.40). Transpositioning the variables into the right parts of equations and expanding the formulas for $^*\Box$ operator (2.173) we obtain

$$\frac{1}{c^2}\frac{^*\partial^2\varphi}{\partial t^2} - h^{ik}\,{}^*\nabla_i\,{}^*\nabla_k\,\varphi = T + 4\pi\rho\,, \tag{3.74}$$

$$\frac{1}{c^2}\frac{^*\partial^2 q^i}{\partial t^2} - h^{mk}\,{}^*\nabla_m\,{}^*\nabla_k\, q^i = B^i + \frac{4\pi}{c}j^i, \tag{3.75}$$

where $h^{ik}\,{}^*\nabla_i\,{}^*\nabla_k = {}^*\triangle$ is chronometrically invariant Laplace operator. The structure of their parts say that if potentials φ and q^i are stationary (i. e. do not depend on time), d'Alembert wave equations become Laplace equations that characterize statical states of field

$$^*\triangle\varphi = T + 4\pi\rho\,, \tag{3.76}$$

$$^*\triangle q^i = B^i + \frac{4\pi}{c}j^i. \tag{3.77}$$

Field is uniform along a certain direction, if its regular derivative to this direction is zero. In Riemannian space field is uniform if its regular (general covariant) derivative is zero. In accompanying frame of reference projection onto time and space non-uniformity of tensor field characterizes the difference from chronometrically invariant operator $^*\nabla_i$ [8, 10]. In other words, if for a certain (e. g. scalar) value A the condition $^*\nabla_i A = 0$ is true, the field *is observed* as uniform.

Therefore, chronometrically invariant d'Alembert operator $^*\Box$ is the difference of 2nd derivatives of operator $\frac{1}{c}\frac{^*\partial}{\partial t}$, which characterizes observed non-stationary state of the field, and of operator $^*\nabla_i$, which characterizes its observable non-uniformity. If a field is stationary and uniform at the same time, left parts in d'Alembert equations (3.74, 3.75) are zeroes, and only field sources in the right parts are left. That means the field does not generate waves, i. e. it is not a wave field.

In non-uniform stationary field ($^*\nabla_i \neq 0\,,\ \frac{1}{c}\frac{^*\partial}{\partial t} = 0$), d'Alembert equations (3.74, 3.75) characterize a standing wave

$$-\,h^{ik}\,{}^*\nabla_i\,{}^*\nabla_k\,\varphi = T + 4\pi\rho\,, \tag{3.78}$$

$$-\,h^{mk}\,{}^*\nabla_m\,{}^*\nabla_k\, q^i = B^i + \frac{4\pi}{c}j^i. \tag{3.79}$$

In uniform non-stationary field ($^*\nabla_i = 0\,,\ \frac{1}{c}\frac{^*\partial}{\partial t} \neq 0$) d'Alembert equations describe changes of the field in time depending from the state of the sources of the field (charges and currents)

$$\frac{1}{c^2}\frac{^*\partial^2\varphi}{\partial t^2} = T + 4\pi\rho\,, \tag{3.80}$$

$$\frac{1}{c^2}\frac{^*\partial^2 q^i}{\partial t^2} = B^i + \frac{4\pi}{c}j^i. \tag{3.81}$$

In inertial frame of reference (Christoffel symbols are zeroes) general covariant derivative equals to the regular one ${}^*\nabla_i\varphi=\frac{{}^*\partial\varphi}{\partial t}$ and chronometrically invariant scalar d'Alembert equation (3.74) is

$$\frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2}-h^{ik}\frac{{}^*\partial^2\varphi}{\partial x^i\partial x^k}=T+4\pi\rho\,. \tag{3.82}$$

Here the left part takes the most simple form, which facilitates more detailed study of d'Alembert equation for scalar field. As known from theory of oscillations in mathematical physics, in regular (not chronometrically invariant) d'Alembert equations

$$\Box\varphi=\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2}-g^{ik}\frac{\partial^2\varphi}{\partial x^i\partial x^k} \tag{3.83}$$

the term a is absolute value of three-dimensional velocity of elastic oscillations that spread across the field φ.

Expanding chronometrically invariant derivatives by spatial coordinates (3.46) we bring scalar d'Alembert equation (3.82) to the form

$$\begin{gathered}\frac{1}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{{}^*\partial^2\varphi}{\partial t^2}-h^{ik}\frac{\partial^2\varphi}{\partial x^i\partial x^k}+\frac{2v^k}{c^2-w}\frac{\partial^2\varphi}{\partial x^k\partial t}+\\+\frac{h^{ik}}{c^2-w}\frac{\partial v_k}{\partial x^i}\frac{\partial\varphi}{\partial t}+\frac{1}{c^2}v^kF_k\frac{\partial\varphi}{\partial t}=T+4\pi\rho\,,\end{gathered} \tag{3.84}$$

where $v^2=h_{ik}v^iv^k$ and the second chronometrically invariant derivative to time formulates with regular derivatives as

$$\frac{{}^*\partial^2\varphi}{\partial t^2}=\frac{1}{\left(1-\frac{w}{c^2}\right)^2}\frac{\partial^2\varphi}{\partial t^2}+\frac{1}{c^2\left(1-\frac{w}{c^2}\right)^3}\frac{\partial w}{\partial t}\frac{\partial\varphi}{\partial t}\,. \tag{3.85}$$

We can now see that the larger is the square of space rotation velocity v^2, the lesser is effect of physical observable non-stationary state of field ($\frac{{}^*\partial\varphi}{\partial t}$ value) on propagation of waves. In the ultimate case, when $v\to c$, d'Alembert operator becomes Laplace operator, i. e. wave d'Alembert equation becomes stationary Laplace equation. At lower velocities of space rotation $v\ll c$ one can assume that observable waves of field of scalar potential propagate at the light speed.

In general case absolute value of observed velocity of waves of scalar electromagnetic potential $\mathrm{v}_{(\varphi)}$ becomes

$$\mathrm{v}_{(\varphi)}=\frac{1}{1-\frac{v^2}{c^2}}\,c^2. \tag{3.86}$$

From the formula for chronometrically invariant value (3.85), which is observable acceleration of increment of scalar potential φ in time, we see that it is the more different from "coordinate" value the higher is gravitational potential and the higher is the rate of its change in time

$$\frac{\partial^2\varphi}{\partial t^2}=\left(1-\frac{w}{c^2}\right)^2\frac{{}^*\partial^2\varphi}{\partial t^2}+\frac{1}{c^2-w}\frac{\partial w}{\partial t}\frac{\partial\varphi}{\partial t}\,. \tag{3.87}$$

In the ultimate case, when $w\to c^2$ (approaching gravitational collapse), observable acceleration of increment of scalar potential becomes infinitesimal, while coordinate rate of growth of the potential $\frac{\partial\varphi}{\partial t}$, to the contrary, becomes infinitely large. But under regular conditions gravitational potential w contributes only smaller corrections into acceleration and velocity of growth of potential φ.

All said in the above in respect to physical observable scalar value $\frac{{}^*\partial^2\varphi}{\partial t^2}$ is also true for vector observable value $\frac{{}^*\partial^2 q^i}{\partial t^2}$, because chronometrically invariant d'Alembert operator ${}^*\Box=\frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2}-h^{ik}\frac{{}^*\partial^2}{\partial x^i\partial x^k}$

shows difference from scalar and vector functions only in the second term — Laplace operator, in which chronometrically invariant derivatives of scalar and vector values are different from each other

$$^*\nabla_i\varphi = \frac{^*\partial\varphi}{\partial x^i}\,, \qquad ^*\nabla_i q^k = \frac{^*\partial q^k}{\partial x^i} + \triangle^k_{im}q^m. \tag{3.88}$$

If space rotation and gravitational potential are infinitesimal, chronometrically invariant d'Alembert operator for scalar potential becomes

$$^*\Box\varphi = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - h^{ik}\frac{\partial^2\varphi}{\partial x^i\partial x^k}\,, \tag{3.89}$$

and electromagnetic waves, produced by scalar potential φ propagate at the speed of light.

3.5 Chronometrically invariant Lorentz force. Energy-impulse tensor of electromagnetic field

In this Section we are going to formulate physical observable components of four-dimensional force with which electromagnetic field affects electric charge in pseudo-Riemannian space. The problem will be solved for two cases: (a) for a point charge; (b) for a charge distributed in space. Also, we are going to calculate physical observable components of energy-impulse tensor of electromagnetic field.

In three-dimensional Euclidean space of classical electrodynamics motion charged particle is characterized by vector equation

$$\frac{d\overrightarrow{p}}{dt} = e\overrightarrow{E} + \frac{e}{c}\,[\overrightarrow{u};\overrightarrow{H}], \tag{3.90}$$

where $\overrightarrow{p}=m\overrightarrow{u}$ is three-dimensional vector of particle's impulse and m is its relativistic mass. The formula in the right part of this equation is referred to as *Lorentz force*.

The equation that characterizes the change of kinetic (relativistic) energy of a charged particle

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{3.91}$$

due to work accomplished by electric field to displace it within a unit time takes a vector form as follows

$$\frac{dE}{dt} = e\overrightarrow{E}\,\overrightarrow{u}, \tag{3.92}$$

and is also known as *live forces theorem*.

In four-dimensional form, thanks to unification of energy and impulse, in Galilean frame of reference in Minkowski space both equations (3.90, 3.92) takes the form

$$m_0c\frac{dU^\alpha}{ds} = \frac{e}{c}F^{\alpha\cdot}_{\cdot\sigma}U^\sigma, \qquad U^\alpha = \frac{dx^\alpha}{ds}\,, \tag{3.93}$$

and are referred to as *Minkowski equations* ($F^{\alpha\cdot}_{\cdot\sigma}$ is the electromagnetic field tensor). Because the metric here is diagonal (3.5), hence

$$ds = cdt\sqrt{1-\frac{u^2}{c^2}}, \qquad u^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \tag{3.94}$$

and the components of four-dimensional velocity of particle U^α are

$$U^0 = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\,, \qquad U^i = \frac{1}{c}\frac{u^i}{\sqrt{1-\frac{u^2}{c^2}}}\,, \tag{3.95}$$

where $u^i = \frac{dx^i}{dt}$ is three-dimensional velocity of particle. Once components $\frac{e}{c}F^{\alpha\cdot}_{\cdot\sigma}U^\sigma$ in Galilean frame of reference are

$$\frac{e}{c}F^{0\cdot}_{\cdot\sigma}U^\sigma = -\frac{e}{c^2}\frac{E_i u^i}{\sqrt{1-\frac{u^2}{c^2}}}\,, \tag{3.96}$$

$$\frac{e}{c}F^{i\cdot}_{\cdot\sigma}U^\sigma = -\frac{1}{c\sqrt{1-\frac{u^2}{c^2}}}\left(eE^i + \frac{e}{c}e^{ikm}u_k H_{*m}\right), \tag{3.97}$$

then temporal (scalar) and spatial (vector) components of Minkowski equations (3.93) are (in Galilean frame of reference as well)

$$\frac{dE}{dt} = -eE_i u^i, \tag{3.98}$$

$$\frac{dp^i}{dt} = -\left(eE^i + \frac{e}{c}e^{ikm}u_k H_{*m}\right), \qquad p^i = mu^i. \tag{3.99}$$

The above relativistic equations, save for the sign in the right parts, match the live forces theorem and equations of motion of charged particle in classical electrodynamics (3.90, 3.91). Note that difference in signs in the right parts is conditioned only by choice of signatures: we use space-time signature $(+---)$, but if we accept signature $(-+++)$, the sign in the right parts of the equations will change.

We now turn to chronometrically invariant representation in pseudo-Riemannian space of four-dimensional impulse vector $\Phi^\alpha = \frac{e}{c}F^{\alpha\cdot}_{\cdot\sigma}U^\sigma$, which particle gains from interaction of its charge e with electromagnetic field. Its physical observable projections are

$$T = \frac{e}{c}\frac{F_{0\sigma}U^\sigma}{\sqrt{g_{00}}}\,, \tag{3.100}$$

$$B^i = \frac{e}{c}F^{i\cdot}_{\cdot\sigma}U^\sigma = \frac{e}{c}\left(F^{i\cdot}_{\cdot 0}U^0 + F^{i\cdot}_{\cdot k}U^k\right). \tag{3.101}$$

Given that in pseudo-Riemannian space components of U^α are

$$U^0 = \frac{\frac{1}{c^2}v_i \mathrm{v}^i \pm 1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\left(1-\frac{w}{c^2}\right)}\,, \qquad U^i = \frac{1}{c}\frac{\mathrm{v}^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{3.102}$$

then, taking into account formulas for rotor components (2.143–2.159) we obtained in Chapter 2, we arrive to

$$T = -\frac{e}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(\frac{{}^*\partial\varphi}{\partial x^i} + \frac{1}{c}\frac{{}^*\partial q_i}{\partial t} - \frac{\varphi}{c^2}F_i\right)\mathrm{v}^i, \tag{3.103}$$

$$\begin{gathered} B^i = -\frac{e}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left\{\pm\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t} - \frac{\varphi}{c^2}F_k\right)h^{ik} + \right. \\ \left. + \left[h^{im}h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}A^{ik}\right]\mathrm{v}_k\right\}. \end{gathered} \tag{3.104}$$

Scalar value T, to within multiplier $-\frac{1}{c^2}$, is a field's work to displace charge e in pseudo-Riemannian space. Vector value B^i, to within multiplier $\frac{1}{c}$, in non-relativistic case is a regular force that acts on charged particle from electromagnetic field in pseudo-Riemannian space

$$\Phi^i = cB^i = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}H_{*m}\mathrm{v}_k\right), \tag{3.105}$$

and is observable Lorentz force. Note that alternating signs stem from the fact that in pseudo-Riemannian space quadratic equation in respect to $\frac{dt}{d\tau}$ has two roots (1.55). Respectively, the "plus" sign in the Lorentz force stands for particle's motion into future (in respect to the observer), while the "minus" sign denotes motion into past. In Galilean frame of reference in Minkowski space there is no difference between physical observable time τ and coordinate time t. Hence Lorentz force (3.99) obtained from Minkowski equations will have no alternating signs.

If charge is not a point one but is distributed in space, Lorentz force $\Phi^{\alpha}=\frac{e}{c}F^{\alpha\cdot}_{\cdot\sigma}U^{\sigma}$ in Minkowski equations (3.93) will be replaced by four-dimensional vector of *Lorentz force density*

$$f^{\alpha} = -\frac{1}{c}F^{\alpha\cdot}_{\cdot\sigma}\,j^{\sigma}, \tag{3.106}$$

where four-dimensional vector of current density $j^{\sigma}=\left\{c\rho;\,j^{i}\right\}$ is defined from Group I of general covariant Maxwell equations (3.51)

$$j^{\sigma} = \frac{c}{4\pi}\nabla_{\mu}F^{\sigma\mu}. \tag{3.107}$$

Physical observable components of Lorentz force density f^{α} are

$$\frac{f_0}{\sqrt{g_{00}}} = \frac{1}{c}E_i j^i, \tag{3.108}$$

$$f^i = \rho E^i + \frac{1}{c}H^{i\cdot}_{\cdot k}j^k = \frac{1}{c}\rho E^i + \frac{1}{c}\varepsilon^{ikm}H_{*m}j_k\,. \tag{3.109}$$

In Galilean frame of reference in Minkowski space temporal and spatial projections of vector of current density (3.109) are

$$\frac{f_0}{\sqrt{g_{00}}} = \frac{q}{c} = \frac{1}{c}\vec{E}\,\vec{j}\,, \tag{3.110}$$

$$\vec{f} = \rho\vec{E} + \frac{1}{c}\,[\,\vec{j}\,;\vec{H}\,]\,, \tag{3.111}$$

where $\vec{f}$ is density of Lorentz force that acts on spread charges, while q is density of heat power released into current conductor.

Now we transform density of Lorentz force (3.106) using Maxwell equations. Substituting j^{σ} (3.107) we arrive to

$$f_{\nu} = \frac{1}{c}F_{\nu\sigma}j^{\sigma} = \frac{1}{4\pi}F_{\nu\sigma}\nabla_{\mu}F^{\mu\sigma} = \frac{1}{4\pi}\left[\nabla_{\mu}\left(F_{\nu\sigma}F^{\mu\sigma}\right) - F^{\mu\sigma}\nabla_{\mu}F_{\nu\sigma}\right]. \tag{3.112}$$

Transpositioning indices μ and σ (also known as *mute* or *free* indices), by which we add-up, and taking into account antisymmetry of Maxwell tensor $F_{\alpha\beta}$ we the second summand to the form

$$F^{\mu\sigma}\nabla_{\mu}F_{\nu\sigma} = \frac{1}{2}F^{\mu\sigma}\left(\nabla_{\mu}F_{\nu\sigma} + \nabla_{\sigma}F_{\mu\nu}\right) = -\frac{1}{2}F^{\mu\sigma}\nabla_{\nu}F_{\sigma\mu} = \frac{1}{2}F^{\mu\sigma}\nabla_{\nu}F_{\mu\sigma}\,. \tag{3.113}$$

As a result, for f_{ν} (3.112) and its contravariant form we obtain

$$f_{\nu} = -\frac{1}{4\pi}\nabla_{\mu}\left(F^{\mu\sigma}F_{\nu\sigma} + \frac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \tag{3.114}$$

$$f^{\nu} = \nabla_{\mu}\left[-\frac{1}{4\pi}\left(F^{\mu\sigma}F^{\nu\cdot}_{\cdot\sigma} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)\right]. \tag{3.115}$$

Denoting the term

$$\frac{1}{4\pi}\left(-F^{\mu\sigma}F^{\cdot\nu}_{\sigma\cdot} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) = T^{\mu\nu}, \tag{3.116}$$

we obtain the expression

$$f^{\nu} = \nabla_{\mu}T^{\mu\nu}, \tag{3.117}$$

i. e. four-dimensional vector of density of Lorentz force f^{ν} equals to absolute (general covariant) divergence of value $T^{\mu\nu}$, referred to as *tensor of energy-impulse of electromagnetic field.* Its structure shows that it is symmetric $T^{\mu\nu}=T^{\nu\mu}$ while its trace (given that trace of fundamental metric tensor $g_{\mu\nu}g^{\mu\nu}=\delta_{\nu}^{\nu}=4$) is zero

$$T_{\nu}^{\nu}=g_{\mu\nu}T^{\mu\nu}=\frac{1}{4\pi}\left(-F^{\mu\sigma}F_{\mu\sigma}+\frac{1}{4}g_{\mu\nu}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)=\frac{1}{4\pi}\left(-F^{\mu\sigma}F_{\mu\sigma}+F^{\alpha\beta}F_{\alpha\beta}\right)=0\,. \tag{3.118}$$

Physical observable components of energy-impulse tensor are values

$$q=\frac{T_{00}}{g_{00}}\,,\qquad J^{i}=\frac{cT_{0}^{i}}{\sqrt{g_{00}}}\,,\qquad U^{ik}=c^{2}T^{ik}, \tag{3.119}$$

where q is observable *density of field*, J^i is vector of observable *density of impulse*, and U^{ik} is tensor of observable *density of impulse flow.* Calculating the values for energy-impulse tensor of electromagnetic field (3.116) we obtain the expressions

$$q=\frac{E^{2}+H^{*2}}{8\pi}\,, \tag{3.120}$$

$$J^{i}=\frac{c}{4\pi}\varepsilon^{ikm}E_{k}H_{*m}\,, \tag{3.121}$$

$$U^{ik}=qc^{2}h^{ik}-\frac{c^{2}}{4\pi}\left(E^{i}E^{k}+H^{*i}H^{*k}\right), \tag{3.122}$$

where $E^2=h_{ik}E^iE^k$ and $H^{*2}=h_{ik}H^{*i}H^{*k}$. Comparing the formula for q (3.120) with that for density of energy of electromagnetic field from classical electrodynamics

$$W=\frac{E^{2}+H^{2}}{8\pi}\,, \tag{3.123}$$

where $E^2=(\vec{E};\vec{E})$ and $H^2=(\vec{H};\vec{H})$, we can see that chronometrically invariant value q we have calculated is observable density of energy of electromagnetic field in pseudo-Riemannian space. Comparing the formula for chronometrically invariant vector J^i (3.121) with that for *Poynting vector* in classical electrodynamics

$$\vec{S}=\frac{c}{4\pi}\,(\vec{E};\vec{H})\,, \tag{3.124}$$

we can see that J^i is *observable Poynting vector* in pseudo-Riemannian space. Correspondence of the third observable component U^{ik} (3.122) to values in classical electrodynamics can be established using similarities with mechanics of continuous media, where three-dimensional tensor of similar structure is *tensor of tensions* for elementary field volume. Therefore, U^{ik} is chronometrically invariant tensor of strengths of electromagnetic field in pseudo-Riemannian space.

Now we can obtain identities for chronometrically invariant components of vector of density of Lorentz force, which right parts are already formulated with charge density ρ and current density j^i (3.108, 3.109), i. e. with sources of electromagnetic field. Using equation $f^{\nu}=\nabla_{\mu}T^{\mu\nu}$ we will formulate the left parts with observable components of tensor of energy-impulse of electromagnetic field (3.120–3.122). Using ready formulas for observable components of absolute divergence of symmetric 2nd rank tensor (2.138, 2.139), we obtain

$$\frac{^*\partial q}{\partial t}+qD+\frac{1}{c^{2}}D_{ij}U^{ij}+{}^*\widetilde{\nabla}_{i}J^{i}-\frac{1}{c^{2}}F_{i}J^{i}=\frac{1}{c}E_{i}j^{i}, \tag{3.125}$$

$$\frac{^*\partial J^{k}}{\partial t}+DJ^{k}+2\left(D_{i}^{k}+A_{\cdot i}^{k\cdot}\right)J^{i}+{}^*\widetilde{\nabla}_{i}U^{ik}-qF^{k}=\rho E^{k}+\frac{1}{c}\varepsilon^{kim}H_{*i}j_{m}\,. \tag{3.126}$$

The first chronometrically invariant identity (3.125) shows that if observable vector of current density j^i is orthogonal to vector of strength of electric field E^i, the right part turns to zero. In general case, i. e. in case of arbitrary orientation of vectors j^i and E^i, observable change of electromagnetic field density in time ($\frac{^*\partial q}{\partial t}$ value) depends upon the following factors:

1. rate of change of physically observable reference volume of space, filled with electromagnetic field (qD term);
2. change of observable strength of field under action of surface forces of volume deformation ($\frac{1}{c^2}D_{ij}U^{ij}$ term);
3. effect of gravitational inertial force that decrease or increase observable density of field impulse ($\frac{1}{c^2}F_iJ^i$ term);
4. observable "spatial variation" (physical divergence) of field impulse density (${}^*\widetilde{\nabla}_i J^i$ term);
5. magnitudes and mutual orientation of current density vector j^i and electric strength vector E^i (the right part of the equation).

The second chronometrically invariant identity (3.126) shows that observable change in time of vector of density of electromagnetic field impulse ($\frac{{}^*\partial J^k}{\partial t}$ value) depends upon the following factors:

1. rate of change of observable reference volume of space, filled with electromagnetic field (DJ^k term);
2. change of deformation forces and Coriolis forces of the space itself, which is accounted by the term $2\left(D_i^k + A_{\cdot i}^{k\cdot}\right)J^i$;
3. effect of gravitational inertial force on observable density of electromagnetic field (qF^k term);
4. observable "spatial variation" of field strength ${}^*\widetilde{\nabla}_i U^{ik}$;
5. observable density of Lorentz force (the right part defined by value $f^k = \rho E^k + \frac{1}{c}\varepsilon^{kim}H_{*i}j_m$.

In conclusion we consider a specific case of isotropic electromagnetic field. Formal definition of isotropic field with the help of Maxwell tensor [4] is a set of two conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.127}$$

which implies that both field invariants $J_1 = F_{\mu\nu}F^{\mu\nu}$, $J_2 = F_{\mu\nu}F^{*\mu\nu}$ (3.25, 3.26) are zeroes. In chronometrically invariant notation, taking into account (3.28), the conditions take the form

$$E^2 = H^{*2}, \qquad E_iH^{*i} = 0\,. \tag{3.128}$$

We see that electromagnetic field in pseudo-Riemannian space is observed as isotropic one given that in observer's frame of reference three-dimensional lengths of vectors of strengths of electric and magnetic fields are equal, while Poynting vector J^i (3.121) is zero

$$J^i = \frac{c}{4\pi}\varepsilon^{ikm}E_kH_{*m} = \frac{c}{4\pi}\varepsilon^{ikm}h_{mk}E_kH^{*k} = 0\,. \tag{3.129}$$

In terms of observable components of energy-impulse tensor (3.120, 3.121) the obtained conditions (3.128) also imply that

$$J = cq\,, \tag{3.130}$$

where $J = \sqrt{J^2}$ and $J^2 = h_{ik}J^iJ^k$. In other words, length J of observable vector of density of impulse of isotropic electromagnetic field depends only upon density of field q itself.

3.6 Equations of motion of charged particle obtained using parallel transfer method

In this Section we will obtain chronometrically invariant equations of motion of test charged mass-bearing particle in four-dimensional pseudo-Riemannian space with presence of electromagnetic field.

Generally, using the method described herein we can also obtain equations of motion for a particle, which is not a test one[17].

[17] A test particle is one with charge and mass so small that they do not affect electromagnetic and gravitational fields in which it moves. Mass-bearing particles (rest-mass is non-zero) are particles which move along non-isotropic trajectories.

The equations in question are chronometrically invariant projections onto time and space of general covariant equations of parallel transfer of four-dimensional summary vector

$$Q^{\alpha}=P^{\alpha}+\frac{e}{c^{2}}A^{\alpha}, \tag{3.131}$$

where $P^{\alpha}=m_0\frac{dx^{\alpha}}{ds}$ is four-dimensional dynamic vector of particle that moves (in this case) along a non-geodesic trajectory, and $\frac{e}{c^2}A^{\alpha}$ is four-dimensional impulse that the particle gains from interaction of its charge e with electromagnetic field A^{α}, which diverts its trajectory from a geodesic line. Given this problem statement, parallel transfer of superposition on non-geodesic eigenvector of particle and the diverting vector is also a geodesic one

$$\frac{d}{ds}\left(P^{\alpha}+\frac{e}{c^{2}}A^{\alpha}\right)+\Gamma^{\alpha}_{\mu\nu}\left(P^{\mu}+\frac{e}{c^{2}}A^{\mu}\right)\frac{dx^{\nu}}{ds}=0\,. \tag{3.132}$$

By definition geodesic line is a *line of constant direction.* This means that the one for which any vector tangential to it at a given point will remain tangential along the entire line being subjected to parallel transfer [10].

Equations of motion of particle may be obtained in another way, by considering motion along line of the least (extremum) length using least action principle. Hence, extremum length lines are also lines of constant direction. But, for instance, in space with non-metric geometry length is not defined as category. Therefore lines of extremum length are neither defined and we can not use the least action method to obtain the equations. Nevertheless, even in non-metric geometry we can define lines of constant direction and derivation parameter to them. Hence one can assume that in metric spaces, to which Riemannian space belongs, lines of extremum length are merely a specific case of constant direction lines.

Hence, general formulas for chronometrically invariant projections of equations of parallel transfer, obtained in Chapter 2, take the form

$$\frac{d\tilde{\varphi}}{ds}+\frac{1}{c}\left(-F_i\tilde{q}^i\frac{d\tau}{ds}+D_{ik}\tilde{q}^i\frac{dx^k}{ds}\right)=0\,, \tag{3.133}$$

$$\frac{d\tilde{q}^i}{ds}+\left(\frac{\tilde{\varphi}}{c}\frac{dx^k}{ds}+\tilde{q}^k\frac{d\tau}{ds}\right)\left(D^i_k+A^{\cdot i}_{k\cdot}\right)-\frac{\tilde{\varphi}}{c}F^i\frac{d\tau}{ds}+\triangle^i_{mk}\tilde{q}^m\frac{dx^k}{ds}=0\,, \tag{3.134}$$

where s (space-time interval) is parameter of derivation to the trajectory, $\tilde{\varphi}$ and $\tilde{q}^i$ are observable projections of the summary vector of charged particle Q^{α} (3.131) onto time and space

$$\tilde{\varphi}=b_{\alpha}Q^{\alpha}=\frac{Q_0}{\sqrt{g_{00}}}=\frac{1}{\sqrt{g_{00}}}\left(P_0+\frac{e}{c^2}A_0\right), \tag{3.135}$$

$$\tilde{q}^i=h^i_{\alpha}Q^{\alpha}=Q^i=P^i+\frac{e}{c^2}A^i. \tag{3.136}$$

Physical observable projections of dynamic vector of a real mass-bearing particle are

$$\frac{P_0}{\sqrt{g_{00}}}=\pm m\,,\qquad P^i=\frac{1}{c}m\mathrm{v}^i=\frac{1}{c}p^i, \tag{3.137}$$

where "plus" sign stands for particle's motion into future (in respect to observer), while "minus" appears if particle moves into past, and $p^i=m\frac{dx^i}{d\tau}$ is chronometrically invariant three-dimensional impulse of particle. Observable projections of four-dimensional impulse $\frac{e}{c^2}A^{\alpha}$ are

$$\frac{e}{c^2}\frac{A_0}{\sqrt{g_{00}}}=\frac{e}{c^2}\varphi\,,\qquad\frac{e}{c^2}A^i=\frac{e}{c^2}q^i, \tag{3.138}$$

where φ is scalar potential and q^i is vector-potential of electromagnetic field — chronometrically invariant components of four-dimensional field potential A^{α} (3.8). Then values $\tilde{\varphi}$ and $\tilde{q}^i$, i. e. physical observable projections of summary vector Q^{α} (3.135, 3.136) take the form

$$\tilde{\varphi} = \pm m + \frac{e}{c^2}\varphi\,, \tag{3.139}$$

$$\tilde{q}^i = \frac{1}{c}\left(p^i + \frac{e}{c^2}q^i\right). \tag{3.140}$$

We now substitute the values $\tilde{\varphi}$ and $\tilde{q}^i$ into general formulas for chronometrically invariant equations of motion (3.133, 3.134). Moving the terms that characterize electromagnetic interaction into the right parts we arrive to chronometrically invariant equations of motion for mass-bearing charged particle in our world that moves into future in respect to a regular observer (direct flow of time)

$$\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k = -\frac{e}{c^2}\frac{d\varphi}{d\tau} + \frac{e}{c^3}\left(F_i q^i - D_{ik}q^i\mathrm{v}^k\right), \tag{3.141}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{e}{c}\frac{dq^i}{d\tau} - \frac{e}{c}\left(\frac{\varphi}{c}\mathrm{v}^k + q^k\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + \frac{e\varphi}{c^2}F^i - \frac{e}{c}\triangle^i_{nk}q^n\mathrm{v}^k,\end{gathered} \tag{3.142}$$

as well as to ones for mirror-world particle, that moves into past in respect to the observer (reverse flow of time)

$$-\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k = -\frac{e}{c^2}\frac{d\varphi}{d\tau} + \frac{e}{c^3}\left(F_i q^i - D_{ik}q^i\mathrm{v}^k\right), \tag{3.143}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + mF^i + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{e}{c}\frac{dq^i}{d\tau} - \frac{e}{c}\left(\frac{\varphi}{c}\mathrm{v}^k + q^k\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + \frac{e\varphi}{c^2}F^i - \frac{e}{c}\triangle^i_{nk}q^n\mathrm{v}^k.\end{gathered} \tag{3.144}$$

The left parts of the obtained equations fully match those of equations of motion of free mass-bearing particles. The only difference is that these equations include observable characteristics of non-free particle that characterize its non-geodesic motion (therefore the right parts here are not zeroes). These right parts account for the effect that electromagnetic field makes on particle, as well as the effect from physical and geometric properties of space (gravitational inertial force F^i, rotation A_{ik}, and deformation D_{ik} of space and curvilinearity of coordinates $\triangle^i_{nk}$). Evidently, for a non-charged particle (e=0) the right parts turn to zero and the resulting equations fully match chronometrically invariant equations of motion of free mass-bearing particle in our world (1.51, 1.52) and in the mirror world (1.56, 1.57).

We are now considering the right parts of the obtained equations of motion in details. These are absolutely symmetric ones for particles that move either into future or past and change their sign once the sign of the charge changes. We denote the right parts of temporal projections of equations of motion (3.141, 3.143) as T. Given that

$$\frac{d\varphi}{d\tau} = \frac{{}^*\partial\varphi}{\partial t} + \mathrm{v}^i\frac{{}^*\partial\varphi}{\partial x^i}\,, \tag{3.145}$$

then using the formula for chronometrically invariant strength of electric field in covariant form E_i (3.14), we can represent T as

$$T = -\frac{e}{c^2}E_i\mathrm{v}^i - \frac{e}{c^2}\frac{{}^*\partial\varphi}{\partial t} + \frac{e}{c^3}\left(\frac{{}^*\partial q_i}{\partial t} - D_{ik}q^k\right)\mathrm{v}^i + \frac{e}{c^3}\left(q^i - \frac{\varphi}{c}\mathrm{v}^i\right)F_i\,. \tag{3.146}$$

Substituting the formula into temporal projections of equations of motion (3.141, 3.143) and multiplying them by c^2, we obtain equation for relativistic energy E=$\pm mc^2$ of particles that move into future and into past, respectively

$$\begin{gathered}\frac{dE}{d\tau} - mF_i\mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -eE_i\mathrm{v}^i - e\frac{{}^*\partial\varphi}{\partial t} + \\ + \frac{e}{c}\left(\frac{{}^*\partial q_i}{\partial t} - D_{ik}q^k\right)\mathrm{v}^i + \frac{e}{c}\left(q^i - \frac{\varphi}{c}\mathrm{v}^i\right)F_i\,,\end{gathered} \tag{3.147}$$

$$
\begin{aligned}
-\frac{dE}{d\tau} - mF_i\mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -eE_i\mathrm{v}^i - e\frac{{}^*\partial\varphi}{\partial t} + \\
+\frac{e}{c}\left(\frac{{}^*\partial q_i}{\partial t} - D_{ik}q^k\right)\mathrm{v}^i + \frac{e}{c}\left(q^i - \frac{\varphi}{c}\mathrm{v}^i\right)F_i\,,
\end{aligned}
\tag{3.148}
$$

where $eE_i\mathrm{v}^i$ is work done by electric component of the field to displace the particle in unit of time.

The obtained temporal observable projections of equations of motion of charged particle (3.147, 3.148) make *theorem of live forces* in pseudo-Riemannian space, presented in chronometrically invariant notation, i. e. formulated with physical observable values. As can be easily seen, in Galilean frame of reference our equation (3.147), obtained for particles with direct flow of time, take the form of temporal component of Minkowski equation (3.98). In three-dimensional Euclidean space the obtained equation (3.147) transforms into the theorem of live forces from classical electrodynamics $\frac{dE}{dt}{=}e\vec{E}\vec{u}$ (3.92).

We now turn to the right parts of spatial projections of equations of motion (3.142, 3.144). We denote them as M^i, because these are the same for particles that move either into future or into past and change their sign once the sign of particle's charge changes. Because

$$
\frac{dq^i}{d\tau} = \frac{{}^*\partial q^i}{\partial t} + \mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k}\,,
\tag{3.149}
$$

and in it, taking into account, that $\frac{{}^*\partial h^{ik}}{\partial t} = -\,2D^{ik}$ (1.40)

$$
\frac{{}^*\partial q^i}{\partial t} = \frac{{}^*\partial}{\partial t}\left(h^{ik}q_k\right) = -2D^i_k q^k + h^{ik}\frac{{}^*\partial q_k}{\partial t}\,,
\tag{3.150}
$$

then M^i takes the form

$$
\begin{aligned}
M^i = -\frac{e}{c}h^{ik}\frac{{}^*\partial q_k}{\partial t} + \frac{e\varphi}{c^2}\left(F^i + A^{ik}\mathrm{v}_k\right) + \frac{e}{c}A^{ik}q_k + \\
+\frac{e}{c}\left(q^k - \frac{\varphi}{c}\mathrm{v}^k\right)D^i_k - \frac{e}{c}\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{c}\triangle^i_{nk}q^n\mathrm{v}^k.
\end{aligned}
\tag{3.151}
$$

Using formulas for chronometrically invariant components E^i (3.11) and H^{ik} (3.12) of Maxwell tensor $F_{\alpha\beta}$, we write down the first two terms from the formula for the value M^i (3.151) and the third term as

$$
-\frac{e}{c}h^{ik}\frac{{}^*\partial q_k}{\partial t} + \frac{e\varphi}{c^2}F^i = -eE^i + eh^{ik}\frac{{}^*\partial\varphi}{\partial x^k}\,,
\tag{3.152}
$$

$$
\frac{e\varphi}{c^2}A^{ik}\mathrm{v}_k = \frac{e}{2c}h^{im}\mathrm{v}^n\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{e}{2c}H^{ik}\mathrm{v}_k\,.
\tag{3.153}
$$

We write down value H^{ik} as $H^{ik}{=}\varepsilon^{mik}H_{*m}$ (3.56). Then we have

$$
\frac{e\varphi}{c^2}A^{ik}\mathrm{v}_k = \frac{e}{2c}h^{im}\mathrm{v}^n\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{e}{2c}\varepsilon^{ikm}H_{*m}\mathrm{v}_k\,,
\tag{3.154}
$$

$$
\begin{aligned}
M^i = -e\left(E^i + \frac{1}{2c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right) + \frac{e}{c}\left(q^k - \frac{\varphi}{c}\mathrm{v}^k\right)D^i_k + eh^{ik}\frac{{}^*\partial\varphi}{\partial x^k} + \\
+\frac{e}{c}A^{ik}q_k + \frac{e}{2c}h^{im}\mathrm{v}^k\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{e}{c}\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{c}\triangle^i_{nk}q^n\mathrm{v}^k,
\end{aligned}
\tag{3.155}
$$

and the sum of the latter three terms in M^i equals

$$
\begin{aligned}
\frac{e}{2c}h^{im}\mathrm{v}^k\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{e}{c}\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{c}\triangle^i_{nk}q^n\mathrm{v}^k = \\
= -\frac{e}{2c}h^{im}\mathrm{v}_k\frac{{}^*\partial q^k}{\partial x^m} - \frac{e}{2c}\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{2c}h^{im}q^n\mathrm{v}^k\frac{{}^*\partial h_{km}}{\partial x^n}\,.
\end{aligned}
\tag{3.156}
$$

Finally the spatial part of chronometrically invariant equations of motion of mass-bearing charged particle that move into future and into past (3.142, 3.144) take the form, respectively

$$\begin{aligned}\frac{d\left(m\mathrm{v}^i\right)}{d\tau}&-mF^i+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\&=-e\left(E^i+\frac{1}{2c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right)+\frac{e}{c}\left(q^k-\frac{\varphi}{c}\mathrm{v}^k\right)D^i_k+eh^{ik}\frac{{}^*\partial\varphi}{\partial x^k}+\\&+\frac{e}{c}A^{ik}q_k-\frac{e}{2c}h^{im}\mathrm{v}_k\frac{{}^*\partial q^k}{\partial x^m}-\frac{e}{2c}\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k}-\frac{e}{2c}h^{im}q^n\mathrm{v}^k\frac{{}^*\partial h_{km}}{\partial x^n}\,,\end{aligned}\tag{3.157a}$$

$$\begin{aligned}\frac{d\left(m\mathrm{v}^i\right)}{d\tau}&+mF^i+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\&=-e\left(E^i+\frac{1}{2c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right)+\frac{e}{c}\left(q^k-\frac{\varphi}{c}\mathrm{v}^k\right)D^i_k+eh^{ik}\frac{{}^*\partial\varphi}{\partial x^k}+\\&+\frac{e}{c}A^{ik}q_k-\frac{e}{2c}h^{im}\mathrm{v}_k\frac{{}^*\partial q^k}{\partial x^m}-\frac{e}{2c}\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k}-\frac{e}{2c}h^{im}q^n\mathrm{v}^k\frac{{}^*\partial h_{km}}{\partial x^n}\,.\end{aligned}\tag{3.157b}$$

From here we see that the first term $-e\left(E^i+\frac{1}{2c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right)$ in the right parts of the equations is different from chronometrically invariant Lorentz force $\Phi=-e\left(E^i+\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right)$ by coefficient 1/2 at the term that stands for magnetic component of the force. In Section 9 herein we are going to show the structure of electromagnetic potential A^α at which the other terms in M^i fully compensate the coefficient 1/2 so that only Lorentz force is left in the right parts of chronometrically invariant spatial equations of motion of charged particle.

3.7 Equations of motion, obtained using the least action principle as a partial case of the previous equations

In this Section we are going to obtain chronometrically invariant notation of equations of motion of mass-bearing charged particle, using the least action principle. The principle says that action S to displace a particle along the shortest trajectory is the least, i. e. variation of action is zero

$$\delta\int_a^b dS=0\,.\tag{3.158}$$

Therefore, equations of motion, obtained from the least action principle are equations of the shortest lines.

Elementary action of gravitational and electromagnetic fields to displace charged mass-bearing particle is [1]

$$dS=-m_0c\,ds-\frac{e}{c}A_\alpha dx^\alpha,\tag{3.159}$$

where ds is elementary space-time interval. We see that the value is only applicable to characterize particles that move along non-isotropic trajectories ($ds\neq0$). On the other hand, obtaining equations through method of parallel transfer (constant direction line) is equally applicable to both isotropic and non-isotropic trajectories, along which $ds=0$. Moreover, parallel transfer is as well applicable to non-metric geometries, in particular, to obtain equations of motion of particles in fully degenerated space-time (zero-space). Therefore equations of the least length lines, obtained through the least action method, are merely a narrow specific case of constant direction lines, which result from parallel transfer.

But we are returning to the least action principle (3.158). For mass-bearing charged particle the condition takes the form

$$\delta\int_a^b dS=-\delta\int_a^b m_0c\,ds-\delta\int_a^b\frac{e}{c}A_\alpha dx^\alpha=0\,,\tag{3.160}$$

where the first term can be denoted as

$$-\delta\int_a^b m_0 c ds = -\int_a^b m_0 c DU_\alpha \delta x^\alpha = \\ = \int_a^b m_0 c \left(dU_\alpha ds - \Gamma_{\alpha,\mu\nu}U^\mu dx^\nu\right)\delta x^\alpha. \tag{3.161}$$

We represent variation of the second integral from the initial formula (3.160) as the sum of the follow terms

$$-\frac{e}{c}\delta\int_a^b A_\alpha dx^\alpha = -\frac{e}{c}\left(\int_a^b \delta A_\alpha dx^\alpha + \int_a^b A_\alpha d\delta x^\alpha\right). \tag{3.162}$$

Integrating the second term part by part we obtain

$$\int_a^b A_\alpha d\delta x^\alpha = A_\alpha \delta x^\alpha|_a^b - \int_a^b dA_\alpha \delta x^\alpha. \tag{3.163}$$

Here the first term is zero, as the integral is variated with the given values of coordinates of integration limits. Taking into account that variation of covariant vector is

$$\delta A_\alpha = \frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta, \qquad dA_\alpha = \frac{\partial A_\alpha}{\partial x^\beta}dx^\beta, \tag{3.164}$$

we obtain variation of electromagnetic part of action

$$-\frac{e}{c}\delta\int_a^b A_\alpha dx^\alpha = -\frac{e}{c}\int_a^b\left(\frac{\partial A_\alpha}{\partial x^\beta}dx^\alpha\delta x^\beta - \frac{\partial A_\alpha}{\partial x^\beta}\delta x^\alpha dx^\beta\right). \tag{3.165}$$

Transpositioning free indices α and β in the first term of the formula and accounting for variation of gravitational part of the action (3.161) we arrive to variation of the total action (3.160) of gravitational and electromagnetic fields on particle as

$$\delta\int_a^b dS = \int_a^b\left[m_0 c\left(dU_\alpha - \Gamma_{\alpha,\mu\nu}U^\mu dx^\nu\right) - \frac{e}{c}F_{\alpha\beta}dx^\beta\right]\delta x^\alpha, \tag{3.166}$$

where $F_{\alpha\beta}{=}\frac{\partial A_\beta}{\partial x^\alpha}{-}\frac{\partial A_\alpha}{\partial x^\beta}$ is Maxwell tensor, and $U^\mu{=}\frac{dx^\mu}{ds}$ is four-dimensional velocity of particle. Because value δx^α is arbitrary the formula under integral is always zero. Hence, we arrive to general covariant equations of motion of mass-bearing charged particle

$$m_0 c\left(\frac{dU_\alpha}{ds} - \Gamma_{\alpha,\mu\nu}U^\mu U^\nu\right) = \frac{e}{c}F_{\alpha\beta}U^\beta, \tag{3.167}$$

or, lifting α index, we arrive to more convenient form

$$m_0 c\left(\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu\right) = \frac{e}{c}F^{\alpha\cdot}_{\cdot\beta}U^\beta. \tag{3.168}$$

Therefore, the obtained equations (3.168) are Minkowski equations in pseudo-Riemannian space. In Galilean frame of reference in a flat Minkowski space of Special Relativity the transform into regular relativistic equations (3.93).

Projecting the general covariant Minkowski equations (3.168) onto time and space in accompanying frame of reference we obtain chronometrically invariant Minkowski equations in pseudo-Riemannian space for a charged mass-bearing particle that moves in our world

$$\frac{dE}{d\tau} - mF_i \mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -eE_i\mathrm{v}^i, \tag{3.169}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau}-mF^i+2m\left(D_k^i+A_{k\cdot}^{\cdot i}\right)\mathrm{v}^k+m\triangle_{nk}^i\mathrm{v}^n\mathrm{v}^k=-e\left(E^i+\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right), \tag{3.170}$$

and for a charged mass-bearing particle in the mirror world

$$-\frac{dE}{d\tau}-mF_i\mathrm{v}^i+mD_{ik}\mathrm{v}^i\mathrm{v}^k=-eE_i\mathrm{v}^i, \tag{3.171}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau}+mF^i+m\triangle_{nk}^i\mathrm{v}^n\mathrm{v}^k=-e\left(E^i+\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right). \tag{3.172}$$

Scalar equations of motion, both in our and the mirror worlds, represent theorem of live forces. Vector equations are three-dimensional Minkowski equations, which right parts have chronometrically invariant Lorentz force (calculated in pseudo-Riemannian space). As easily seen, in Galilean frame of reference in Minkowski space these equations turn into theorem of live forces (3.92) and regular equations of motion of charged particle (3.90) from classical electrodynamics.

Evidently, the right parts of the equations (3.169–3.172), obtained through the least action method, are different from the right parts of equations of motion (3.146, 3.157) obtained by parallel transfer. The difference is in absence here (3.169–3.172) of a few terms that characterize the structure of electromagnetic field and that of the space itself. But as we have already mentioned, the least length lines, described by equations based on the least action principle, are only a specific case of constant direction lines, defined by parallel transfer. Therefore there is little surprise in that equations of parallel transfer, as more general ones, have additional terms, which account for the structures of space and electromagnetic field.

3.8 Geometric structure of electromagnetic four-dimensional potential

In this Section we are going to find a structure of four-dimensional potential of electromagnetic field A^α that conserves the length of summary vector $Q^\alpha{=}P^\alpha{+}\frac{e}{c^2}A^\alpha$ in parallel transfer.

As known, parallel transfer in the meaning of Levi-Civita conserves the length of vector Q^α being transferred, i. e. constancy condition $Q_\alpha Q^\alpha{=}const$ is true. Given that the square of vector length is invariant in pseudo-Riemannian space, this condition must be true in any frame of reference, including the case of observer accompanying his body of reference. Hence we can analyze the condition, formulating it with physical observable values in accompanying frame of reference, i. e. in chronometrically invariant form.

Components of vector Q^α in accompanying frame of reference are

$$Q_0=\left(1-\frac{w}{c^2}\right)\left(\pm m+\frac{e\varphi}{c^2}\right), \tag{3.173}$$

$$Q^0=\frac{1}{1-\frac{w}{c^2}}\left[\left(\pm m+\frac{e\varphi}{c^2}\right)+\frac{1}{c^2}v_i\left(m\mathrm{v}^i+\frac{e}{c}q^i\right)\right], \tag{3.174}$$

$$Q_i=-\frac{1}{c}\left(m\mathrm{v}_i+\frac{e}{c}q_i\right)-\frac{1}{c}\left(\pm m+\frac{e\varphi}{c^2}\right)v_i\,, \tag{3.175}$$

$$Q^i=\frac{1}{c}\left(m\mathrm{v}^i+\frac{e}{c}q^i\right), \tag{3.176}$$

and its square is

$$Q_\alpha Q^\alpha=m_0^2+\frac{e^2}{c^4}\left(\varphi^2-q_iq^i\right)+\frac{2me}{c^2}\left(\pm\varphi-\frac{1}{c}\mathrm{v}_iq^i\right). \tag{3.177}$$

From here we can see that the square of length of summary vector of mass-bearing charged particle falls apart into the following values:

- square of length of four-dimensional eigenvector of particle's impulse $P_\alpha P^\alpha{=}m_0^2$;
- square of length of four-dimensional impulse $\frac{e}{c^2}A^\alpha$ that charged particle gains from electromagnetic field (the second term);

- the term $\frac{2me}{c^2}(\pm\varphi-\frac{1}{c}\mathrm{v}_i q^i)$, that describes interaction between gravitational "charge" of particle m and its electric charge e.

In the formula for square of vector Q^α (3.177), the first term m_0^2 is conserved in any case. In other words, it is an invariant, that does not depend upon frame of reference. Our goal is to calculate the conditions, under which the whole formula for square of vector Q^α (3.177) is conserved.

We propose that vector-potential of the field has the structure

$$q^i = \frac{\varphi}{c}\mathrm{v}^i. \tag{3.178}$$

In this case[18] the second term (3.177), which is square of $\frac{e}{c^2}A^\alpha$, is

$$\frac{e^2}{c^4}A_\alpha A^\alpha = \frac{e^2\varphi^2}{c^4}\left(1-\frac{\mathrm{v}^2}{c^2}\right). \tag{3.179}$$

Transforming the third term in a similar way, we obtain formula for square of vector Q^α (3.177) in the form

$$Q_\alpha Q^\alpha = m_0^2 + \frac{e^2\varphi^2}{c^4}\left(1-\frac{\mathrm{v}^2}{c^2}\right) + \frac{2m_0 e}{c^2}\varphi\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,. \tag{3.180}$$

Then introducing notation for scalar potential of electromagnetic field

$$\varphi = \frac{\varphi_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{3.181}$$

we can represent the obtained formula (3.180) as

$$Q_\alpha Q^\alpha = m_0^2 + \frac{e^2\varphi_0^2}{c^4} + \frac{2m_0 e\varphi_0}{c^2} = const. \tag{3.182}$$

From here we see that length of summary vector Q^α of charged mass-bearing particle conserves in parallel transfer (i. e. is an invariant not depending upon frame of reference), if its observable potentials φ and q^i in the field are related to four-dimensional field potential A^α as

$$\frac{A_0}{\sqrt{g_{00}}} = \varphi = \frac{\varphi_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad A^i = q^i = \frac{\varphi}{c}\mathrm{v}^i. \tag{3.183}$$

Then for four-dimensional vector $\frac{e}{c^2}A^\alpha$, that characterizes interaction of charged particle with electromagnetic field we have

$$\frac{e}{c^2}\frac{A_0}{\sqrt{g_{00}}} = \frac{e\varphi_0}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad \frac{e}{c^2}A^i = \frac{e\varphi_0}{c^3}\frac{\mathrm{v}^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,. \tag{3.184}$$

Note that the dimensions of vectors $\frac{e}{c^2}A^\alpha$ and $P^\alpha{=}m_0\frac{dx^\alpha}{ds}$ in CGSE and Gaussian systems of units are the same and equal to a mass m [g].

Comparing physical observable components of both vectors we can easily see that the analog for relativistic mass m in particle's interaction with electromagnetic field is the value

$$\frac{e\varphi}{c^2} = \frac{e\varphi_0}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{3.185}$$

[18]A similar problem could be solved, assuming that $q^i{=}\pm\frac{\varphi}{c}\mathrm{v}^i$. But in comparative analysis of of the two groups of equations only positive values of $q^i{=}\frac{\varphi}{c}\mathrm{v}^i$ will be important, because observer's physical time τ, by definition, flows from past into future only (*reference time*), while interval of physical observable time $d\tau$ is always greater than zero.

where $e\varphi$ is potential energy of charged particle that travels at observable velocity $\mathrm{v}^i = \frac{dx^i}{d\tau}$ in electromagnetic field (which rests in respect to the observer and his body of reference). In general, scalar potential φ is potential energy of the field itself, divided by unit of charge. Then $e\varphi$ is *relativistic* potential energy of particle with charge e in electromagnetic field, while $e\varphi_0$ is its rest potential energy. When particle rests in respect to field, its rest potential energy equals to relativistic potential energy.

Comparing $E = mc^2$ and $W = e\varphi$ we can conclude that W/c^2 is electromagnetic analog for relativistic mass m. Respectively, $W_0 = \frac{e\varphi_0}{c^2}$ is electromagnetic analog for rest-mass m_0. Then physical observable value $\frac{e}{c^2} A^i = \frac{e\varphi}{c^3} \mathrm{v}^i$ is similar to $\frac{1}{c} p^i = \frac{1}{c} m \mathrm{v}^i$ (observable vector of impulse of mass-bearing particle, divided by light speed). Therefore, when particle rests in respect to field, its "electromagnetic" projection onto space (vector value) is zero, while only its temporal projection (potential rest-energy $e\varphi_0 = const$) is observable. But if a particle moves in field at velocity v^i, its observable "electromagnetic" projections will be relativistic potential energy $e\varphi$ and three-dimensional impulse $\frac{e\varphi}{c^3} \mathrm{v}^i$.

Having obtained chronometrically invariant components of vector $\frac{e}{c^2} A^\alpha$ (3.184) calculated for the given structure of components of vector A^α (3.183), we can restore it in general covariant form, formulating with scalar potential of resting charged particle φ_0 in electromagnetic field. Taking into account that component A^i is

$$A^i = q^i = \frac{\varphi}{c} \mathrm{v}^i = \frac{\varphi}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \frac{1}{c} \frac{dx^i}{d\tau} = \varphi_0 \frac{dx^i}{ds}\,, \tag{3.186}$$

we obtain the desired general covariant notation of A^α

$$A^\alpha = \varphi_0 \frac{dx^\alpha}{ds}\,, \qquad \frac{e}{c^2} A^\alpha = \frac{e\varphi_0}{c^2} \frac{dx^\alpha}{ds}\,. \tag{3.187}$$

At the same time, projecting the obtained formula for A^α (3.187) onto time and space

$$\frac{A_0}{\sqrt{g_{00}}} = \pm\varphi = \pm \frac{\varphi_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad A^i = q^i = \frac{\varphi}{c} \mathrm{v}^i, \tag{3.188}$$

we obtain alternating signs in temporal projection, which was not the case in the initial temporal projection of the value (3.183). Naturally, a question arises: how did temporal observable component of vector A^α, initially defined as φ, at the given structure of vector A^α (3.187) accept the alternating sign? The answer is that in the first case observable values φ and q^i *are defined* proceeding from general rule of building chronometrically invariant values. But without knowing the structure of the projected vector A^α itself, we can not calculate them.

Therefore in formulas for temporal and spatial projections (3.183) the symbols φ and q^i merely *denote* observable components without revealing their "inner" structure. On the other hand, in formulas (3.188) value φ and q^i are *calculated* using formulas $\varphi = \sqrt{g_{00}} A^0 + \frac{g_{0i}}{\sqrt{g_{00}}} A^i$ and $q^i = A^i$, where the structure of components A^0 and A^i is given. Hence in the second case value $\pm\varphi$ results from calculation and sets a concrete formula

$$\varphi = \pm \frac{\varphi_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,. \tag{3.189}$$

Therefore *calculated* values of chronometrically invariant components of vector $\frac{e}{c^2} A^\alpha$ have the form

$$\frac{e}{c^2} \frac{A_0}{\sqrt{g_{00}}} = \pm \frac{e\varphi}{c^2} = \pm \frac{e\varphi_0}{c^2 \sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad \frac{e}{c^2} A^i = \frac{e\varphi}{c^3} \mathrm{v}^i, \tag{3.190}$$

where "plus" sign stands for particles of our world that travel from past into future (direct flow of time), while "minus" stands for mirror-world particles that travel into past in respect to us (reverse flow of time). The square of length of this vector is

$$\frac{e^2}{c^4}A_\alpha A^\alpha = \frac{e^2\varphi^2}{c^4}\left(1-\frac{\mathrm{v}^2}{c^2}\right) = \frac{e^2\varphi_0^2}{c^4} = const \tag{3.191}$$

along the entire trajectory of particle's motion (parallel transfer lines). Vector $\frac{e}{c^2}A^\alpha$ itself has real length at $\mathrm{v}^2<c^2$, zero length at $\mathrm{v}^2=c^2$ and imaginary length at $\mathrm{v}^2>c^2$. But here we limit our study to real form of the vector (sub-light particles), because light-like or super-light charged particles are unknown to us.

Comparing formulas for vectors $P^\alpha=m_0\frac{dx^\alpha}{ds}$ and $\frac{e}{c^2}A^\alpha=\frac{e\varphi_0}{c^2}\frac{dx^\alpha}{ds}$ we can see that both vector are collinear, i. e. are tangential to the same non-isotropic trajectory, to which derivation parameter s is assumed. Hence in this case dynamic vector of particle P^α is co-directed with action of electromagnetic field on it (particle moves "along" field).

We are going to consider the general case of not co-directed vector. From the square of summary vector Q^α (3.177) we see that the third term is doubled scalar product of vectors P^α and $\frac{e}{c^2}A^\alpha$. Parallel transfer of the two vectors conserves their scalar product

$$D\left(P_\alpha A^\alpha\right) = A^\alpha DP^\alpha + P^\alpha DA^\alpha = 0\,, \tag{3.192}$$

because absolute increment of each vector is zero. Hence we obtain

$$\frac{2e}{c^2}P_\alpha A^\alpha = \frac{2me}{c^2}\left(\pm\varphi - \frac{1}{c}\mathrm{v}_i q^i\right) = const, \tag{3.193}$$

that is, scalar product of P^α and $\frac{e}{c^2}A^\alpha$ is conserved. Consequently lengths of both vectors are conserved too. In particular

$$A_\alpha A^\alpha = \varphi^2 - q_i q^i = const. \tag{3.194}$$

As known, scalar product of two vectors is product of their lengths multiplied by cosine of the angle between them. Therefore parallel transfer also conserves the angle between vectors being transferred

$$\cos\left(\widehat{P^\alpha; A^\alpha}\right) = \frac{P_\alpha A^\alpha}{m_0\sqrt{\varphi^2 - q_i q^i}} = const. \tag{3.195}$$

Taking into account the formula for relativistic mass m, we can re-write the condition (3.193) as follows

$$\frac{2e}{c^2}P_\alpha A^\alpha = \pm\frac{2m_0 e}{c^2}\frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - \frac{2m_0 e}{c^2}\frac{1}{c}\frac{\mathrm{v}_i q^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = const, \tag{3.196}$$

or as relationship between scalar and vector potentials

$$\pm\frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - \frac{1}{c}\frac{\mathrm{v}_i q^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = const. \tag{3.197}$$

For instance, we can find the relationship between potentials φ and q^i for a particle in field, when four-dimensional vector of particle P^α is orthogonal to four-dimensional impulse $\frac{e}{c^2}A^\alpha$, which particle gains from field. Because parallel transfer of two vectors conserves the angle between them, according to general formula (3.195) cosine of angle between these orthogonal vectors is zero along the entire trajectory

$$P_\alpha A^\alpha = \pm\varphi - \frac{1}{c}\mathrm{v}_i q^i = 0\,. \tag{3.198}$$

Consequently, if particle travels in electromagnetic field so that vectors P^{α} and A^{α} are orthogonal, then scalar potential of field is

$$\varphi = \pm\frac{1}{c}\mathrm{v}_i q^i, \tag{3.199}$$

i. e. is scalar product of two chronometrically invariant vectors: observable velocity of particle v^i and particle's vector-potential in field q^i.

Now we are going to obtain the formula for the square of summary particle's vector Q^{α} taking into account that structure of four-dimensional electromagnetic potential is $A^{\alpha}{=}\varphi_0\frac{dx^{\alpha}}{ds}$ (3.187), i. e. that field vector A^{α} is collinear to particle's impulse vector P^{α}. Then

$$Q_{\alpha}Q^{\alpha} = m^2 - \frac{m^2}{c^2}\mathrm{v}_i\mathrm{v}^i + \frac{e^2}{c^4}\left(\varphi^2 - q_i q^i\right) = m_0^2 + \frac{e^2}{c^4}\varphi_0^2\,. \tag{3.200}$$

Multiplying both parts of the equation by c^4 and denoting relativistic energy of particle as $E{=}mc^2$, we arrive to

$$E^2 - c^2p^2 + e^2\varphi^2 - e^2 q_i q^i = E_0^2 + e^2\varphi_0^2\,, \tag{3.201}$$

where $p^2{=}p_ip^i$, $p^i{=}m\mathrm{v}^i$ is three-dimensional chronometrically invariant potential of particle, $e\varphi$ is potential energy of charged particle in electromagnetic field, eq^i is three-dimensional impulse that particle gains from interaction between its charge and electromagnetic field.

3.9 Building Minkowski equations as partial case of the obtained equations of motion

In Section 3.6 we obtained chronometrically invariant (observable) projections of general covariant equations of motion of mass-bearing charged particle in pseudo-Riemannian space. There the initial general covariant equations of motion were obtained using method of parallel transfer of particle's summary vector.

In the same Section we showed that temporal observable projection (3.147) of these equations of motion in Galilean frame of reference takes the form of the temporal component of Minkowski equations (3.98). In three-dimensional Euclidean space our chronometrically invariant scalar equation (3.147) becomes theorem of live forces of classical electrodynamics (3.92). But the right parts of spatial observable projections, instead of chronometrically invariant Lorentz force $\Phi^i{=}-e\Big(E^i{+}\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\Big)$, have the term $-e\Big(E^i{+}\frac{1}{2c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\Big)$ and some other additional terms that depend upon observable characteristics of electromagnetic field and the space itself. Therefore for three-dimensional observable projections of equations of motion in pseudo-Riemannian space, obtained through parallel transfer, the principle of correspondence with three-dimensional components of Minkowski equations is set non-trivially.

On the other hand, equations of constant direction lines, obtained through parallel transfer in pseudo-Riemannian space, are a more general case of the least length lines, obtained with method of the least action. Note that equations of motion, obtained from principle of the least action (in Section 3.7), have the structure matching that of Minkowski equations. Hence we can suppose that temporal and spatial projections of our equations of motion of charged particle, as more general ones, in a certain special case can be transformed into spatial projections of equations of motion, obtained from the least action principle.

To find out exactly under what conditions this can be true, we are going to consider the right parts of spatial projections of equations of motion (3.157), which contain the mismatch with Lorentz force. For analyzes convenience we considered the right parts as separate formulas up denoted as M^i. Using physical observable component H^{ik} (3.12) of Maxwell tensor $F_{\alpha\beta}$, we write down the term $\frac{e\varphi}{c^2}A^{ik}\mathrm{v}_k$ from M^i as

$$\frac{e\varphi}{c^2}A^{ik}\mathrm{v}_k = \frac{e}{2c}h^{im}\mathrm{v}^n\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{e}{2c}\varepsilon^{ikm}H_{*m}\mathrm{v}_k\,, \tag{3.202}$$

where $\varepsilon^{ikm}H_{*m}=H^{ik}$. Now we substitute into (3.157) the observable components of potential of electromagnetic field A^{α} as of (3.188). With this potential, impulse vector $\frac{e}{c^2}A^{\alpha}$ that charged particle gains from electromagnetic field is tangential to trajectory. Using the first formula $q_m=\frac{\varphi}{c}\mathrm{v}_m$ we arrive to dependence of the right part under consideration from only the scalar potential of electromagnetic field

$$M^i = -\,e\left(E^i + \frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right) + e h^{ik}\left(1-\frac{\mathrm{v}^2}{c^2}\right)\frac{{}^*\partial\varphi}{\partial x^k} + \frac{e\varphi}{2}h^{ik}\frac{{}^*\partial}{\partial x^k}\left(1-\frac{\mathrm{v}^2}{c^2}\right). \tag{3.203}$$

Substituting relativistic formula for scalar potential φ (3.181) into this formula we see that the sum of the last two terms becomes zero

$$-\frac{e\varphi}{2}h^{ik}\frac{{}^*\partial}{\partial x^k}\left(1-\frac{\mathrm{v}^2}{c^2}\right) + \frac{e\varphi}{2}h^{ik}\frac{{}^*\partial}{\partial x^k}\left(1-\frac{\mathrm{v}^2}{c^2}\right) = 0\,. \tag{3.204}$$

Then M^i takes the form of chronometrically invariant Lorentz force in pseudo-Riemannian space

$$M^i = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right), \tag{3.205}$$

which is exactly what we had to prove. Therefore, spatial projections of equations of motion of charged mass-bearing particle, obtained through parallel transfer method in pseudo-Riemannian space, match spatial projections of equations of motion, obtained using the least action principle, in a specific case where four-dimensional electromagnetic potential has the structure as $A^{\alpha}=\varphi_0\frac{dx^{\alpha}}{ds}$ (3.187). Consequently, given this structure of electromagnetic potential in Galilean frame of reference in flat space-time, our chronometrically invariant equations of motion fully match Minkowski equations and in three-dimensional Euclidean space take the form of equations of motion in classical electrodynamics.

Now we are going to consider the right part c^2T of scalar equation of motion (3.147) under condition that vector A^{α} has the structure as mentioned in the above and is tangential to trajectory of particle's motion. Substituting observable components φ and q^i for vector A^{α} of the given structure into (3.146), we transform value T to bring it to the form

$$\begin{aligned} c^2T &= -\,eE_i\mathrm{v}^i - e\frac{{}^*\partial\varphi}{\partial t} + \frac{e}{c^2}\left[\frac{{}^*\partial}{\partial t}\left(\varphi h_{ik}\mathrm{v}^k\right) - \varphi D_{ik}q^k\right]\mathrm{v}^i = \\ &= -\,eE_i\mathrm{v}^i - e\frac{{}^*\partial\varphi}{\partial t}\left(1-\frac{\mathrm{v}^2}{c^2}\right) + \frac{e\varphi}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k + \frac{e\varphi}{c^2}\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t}. \end{aligned} \tag{3.206}$$

Substituting relativistic definition of potential φ (3.181) into the first derivative and after derivation returning to φ again we obtain

$$\begin{aligned} c^2T &= -\,eE_i\mathrm{v}^i - \frac{e\varphi}{2c^2}\frac{{}^*\partial}{\partial t}\left(h_{ik}\mathrm{v}^i\mathrm{v}^k\right) + \frac{e\varphi}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k + \frac{e\varphi}{c^2}\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t} = \\ &= -\,eE_i\mathrm{v}^i - \frac{e\varphi}{2c^2}\left(\frac{{}^*\partial h_{ik}}{\partial t}\mathrm{v}^i\mathrm{v}^k + 2\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t}\right) + \frac{e\varphi}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k + \frac{e\varphi}{c^2}\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t} = -\,eE_i\mathrm{v}^i, \end{aligned} \tag{3.207}$$

because we took into account that $\frac{{}^*\partial h_{ik}}{\partial t}=2D_{ik}$ by definition of the tensor of deformation velocities of the space D_{ik} (1.40).

Therefore the right part of scalar chronometrically invariant equation of motion of charged particle fully matches temporal projection of four-dimensional Minkowski equations in pseudo-Riemannian space. Consequently if four-dimensional vector-potential of electromagnetic field is tangential to four-dimensional trajectory of charged particle, then equations of motion obtained through parallel transfer fully match equations of motion obtained using the least action principle. Noteworthy, this is yet another illustration of the geometric fact that the least length lines, obtained from the least action principle, are merely a very specific case of constant direction lines, which result from parallel transfer method.

3.10 Structure of the space with stationary electromagnetic field

It is evident that setting a particular structure of electromagnetic field imposes certain limits on motion of charged particle, which, in its turn, imposes limitations on structure of pseudo-Riemannian space where the motion takes place.

We are going to find out what kind of structure pseudo-Riemannian space should have so that particle moved in stationary electromagnetic field.

Chronometrically invariant equations of motion of charged mass-bearing particle in our world have the form

$$\frac{dE}{d\tau}-mF_i\mathrm{v}^i+mD_{ik}\mathrm{v}^i\mathrm{v}^k=-e\frac{d\varphi}{d\tau}+\frac{e}{c}\left(F_iq^i-D_{ik}q^i\mathrm{v}^k\right), \tag{3.208}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau}-mF^i+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\ =-\frac{e}{c}\frac{dq^i}{d\tau}-\frac{e}{c}\left(\frac{\varphi}{c}\mathrm{v}^k+q^k\right)\left(D^i_k+A^{\cdot i}_{k\cdot}\right)+\frac{e\varphi}{c^2}F^i-\frac{e}{c}\triangle^i_{nk}q^n\mathrm{v}^k.\end{gathered} \tag{3.209}$$

Because we assume electromagnetic field to be stationary, field potentials φ and q^i depend upon spatial coordinates, but not time.

Observable components of electromagnetic field tensor (Maxwell tensor) for a stationary field are

$$E_i=\frac{{}^*\partial\varphi}{\partial x^i}-\frac{\varphi}{c^2}F_i=\frac{\partial\varphi}{\partial x^i}-\varphi\frac{\partial\ln\left(1-\frac{w}{c^2}\right)}{\partial x^i}, \tag{3.210}$$

$$H^{*i}=\frac{1}{2}\varepsilon^{imn}H_{mn}=\frac{1}{2}\varepsilon^{imn}\left(\frac{\partial q_m}{\partial x^n}-\frac{\partial q_n}{\partial x^m}-\frac{2\varphi}{c}A_{mn}\right). \tag{3.211}$$

From here we can arrive to limitations on metric of the space, imposed by stationary state of electromagnetic field.

Hence, the formulas for E_i and H^{*i}, along with chronometrically invariant derivatives of field vector and scalar potentials, also include chronometrically invariant properties of the space of reference: vector of gravitational inertial force F_i and tensor of space non-holonomity (rotation A_{ik}). Evidently, for a stationary electromagnetic field of reference physical observable properties of space

$$F_i=\frac{c^2}{c^2-w}\left(\frac{\partial w}{\partial x^i}-\frac{\partial v_i}{\partial t}\right),\quad A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{1}{2c^2}\left(F_iv_k-F_kv_i\right) \tag{3.212}$$

should be stationary as well.

From these definitions we see that: values F_i and A_{ik} are stationary (do not depend upon time) if linear velocity of space rotation is stationary too, i. e. $\frac{\partial v_i}{\partial t}=0$. Consequently, $\frac{\partial v_i}{\partial t}=0$ condition, i. e. stationary rotation of space turns chronometrically invariant derivative to spatial coordinates into a regular derivative

$$\frac{{}^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}-\frac{1}{c^2}\frac{{}^*\partial v_i}{\partial t}=\frac{\partial}{\partial x^i}. \tag{3.213}$$

Because chronometrically invariant derivative to time is different from a regular derivative only down to multiplier $\frac{\partial}{\partial t}=\left(1-\frac{w}{c^2}\right)\frac{{}^*\partial}{\partial t}$, regular derivative of stationary value is zero too.

For tensor of deformation velocities D_{ik} in case of stationary rotation of space we have the follows

$$\frac{{}^*\partial D_{ik}}{\partial t}=\frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}=\frac{1}{2}\frac{{}^*\partial}{\partial t}\left(-g_{ik}+\frac{1}{c^2}v_iv_k\right)=-\frac{1}{2}\frac{{}^*\partial g_{ik}}{\partial t}. \tag{3.214}$$

Because the right parts of the equations of motion are stationary, the left parts should be the same too. This implies, that the space is not deformed. Then according to (3.124) three-dimensional coordinate metric g_{ik} does not depend upon time and chronometrically invariant Christoffel symbols $\triangle^i_{jk}$ (1.47) are stationary as well.

Observable components of Maxwell tensor (3.210, 3.211) follow Maxwell equations (3.63, 3.64), which for a stationary field are

$$\left.\begin{aligned} & \frac{\partial E^i}{\partial x^i}+\frac{\partial \ln \sqrt{h}}{\partial x^i}E^i-\frac{2}{c}\Omega_{*m}H^{*m}=4\pi\rho \\ & \varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(H_{*m}\sqrt{h}\right)=\frac{4\pi}{c}j^i\sqrt{h} \end{aligned}\right\}\ \text{Group I,} \tag{3.215}$$

$$\left.\begin{aligned} & \frac{\partial H^{*i}}{\partial x^i}+\frac{\partial \ln \sqrt{h}}{\partial x^i}H^{*i}+\frac{2}{c}\Omega_{*m}E^{m}=0 \\ & \varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(E_{m}\sqrt{h}\right)=0 \end{aligned}\right\}\ \text{Group II.} \tag{3.216}$$

Then Lorentz condition (3.65) and equation of continuity (3.66), respectively, take the form as down

$${}^*\widetilde{\nabla}_i q^i=0\,, \qquad {}^*\widetilde{\nabla}_i\, j^i=0\,. \tag{3.217}$$

Therefore we have found the way that stationary state of electromagnetic field affects physical observable properties of pseudo-Riemannian space and the basic equations of electrodynamics.

In the next Sections we will use these results to solve equations of motion for charged particle in pseudo-Riemannian space (3.208, 3.209) in three specific cases of stationary fields: (a) in stationary electric field (magnetic component is zero); (b) in stationary magnetic field (electric component is zero); (c) in stationary electric and magnetic fields.

3.11 Motion of charged particle in stationary electric field

We are going to consider motion of charged mass-bearing particle in stationary electric field in pseudo-Riemannian space. Magnetic field is absent, i.e. does not reveal itself for the observer.

What conditions should pseudo-Riemannian space satisfy to allow existence of pure "electric type" stationary electromagnetic field? From the formula for magnetic strength of stationary field

$$H_{ik}=\frac{\partial q_i}{\partial x^k}-\frac{\partial q_k}{\partial x^i}-\frac{2\varphi}{c}A_{ik} \tag{3.218}$$

we see that $H_{ik}=0$ provided two conditions:

1. vector-potential q^i is irrotational $\frac{\partial q_i}{\partial x^k}=\frac{\partial q_k}{\partial x^i}$;
2. the space is holonomic, i.e. $A_{ik}=0$.

Strength of stationary electric field E_i (3.210) is the sum of spatial derivative of scalar potential φ and the term $\frac{\varphi}{c^2}F_i$ that characterizes interaction between field with potential φ and field of gravitational inertial force F_i. But on the Earth surface the ratio of gravitational potential and the square of light speed is

$$\frac{w}{c^2}=\frac{GM_{\oplus}}{c^2R_{\oplus}}\approx 10^{-10}. \tag{3.219}$$

Therefore in conditions of a real Earth laboratory the second term in (3.210) may be neglected so that E_i will only depend upon spatial distribution of scalar potential of field

$$E_i=\frac{\partial \varphi}{\partial x^i}\,. \tag{3.220}$$

Because the right parts of equations of motion that stand for Lorentz force are stationary, the left parts should be stationary too. Under the conditions we are considering this is trues if deformations velocity tensor is zero, i.e. space is not deformed. Therefore, if stationary electromagnetic field has non-zero electric component and zero magnetic component, then pseudo-Riemannian space should satisfy the conditions as:

1. gravitational potential is negligible $w \approx 0$;
2. space does not rotate, i. e. $A_{ik}=0$;
3. space is not deformed, i. e. $D_{ik}=0$.

Besides, to make calculations easier we assume that for our measurements the structure of three-dimensional space is close to that of Euclidean space, i. e. $\triangle^i_{nk} \approx 0$.

Then chronometrically invariant (physical observable) equations of motion of mass-bearing charged particle in our world we obtained up (3.208, 3.209) will take the form

$$\frac{dm}{d\tau} = -\frac{e}{c^2}\frac{d\varphi}{d\tau}\,, \tag{3.221}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) = -\frac{e}{c}\frac{dq^i}{d\tau}\,. \tag{3.222}$$

From scalar equation of motion (theorem of live forces) we can see that change of relativistic energy of particle $E=mc^2$ is due to work done by electric component E_i.

From vector equation of motion we can see that three-dimensional observable impulse of particle changes under action of the term $\frac{dq^i}{d\tau}$. Assuming that four-dimensional potential of field is tangential to the world line of particle, we, as shown in Section 9, will get three-dimensional Lorentz force $\Phi^i = -eE^i$ in the right part of the vector equation. That is, in the considered case three-dimensional impulse of particle also changes under action of strength of electric field.

Both groups of Maxwell equations for stationary field (3.215, 3.216) in this case take a simple form

$$\left.\begin{aligned} &\frac{\partial E^i}{\partial x^i} = 4\pi\rho \\ &j^i = 0 \end{aligned}\right\} \qquad \text{Group I,} \tag{3.223}$$

$$\left.\varepsilon^{ikm}\frac{\partial E_m}{\partial x^k} = 0\right\} \qquad \text{Group II.} \tag{3.224}$$

Integrating scalar equation of motion (called theorem of live forces) we arrive to so-called *integral of live forces*

$$m + \frac{e\varphi}{c^2} = B = const, \tag{3.225}$$

where B is integration constant.

Another consequence from Maxwell equations is that in this case scalar potential of field satisfies either:

1. Poisson equation $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=4\pi\rho$, if $\rho\neq 0$;
2. Laplace equation $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$, if density of charges $\rho=0$.

We have found out the properties of a pseudo-Riemannian space that allows motion of charged particle in constant electromagnetic field. It would be natural now to obtain exact solutions of equations of motion of particle for a certain particular case. But unless a particular structure of the field itself is set by Maxwell equations this can not be done. Hence to simplify calculations we assume electric field uniform.

We assume that covariant vector of strength of electric field E_i, which is chronometric invariant, is directed along the x axis. Following Landau and Lifshitz (see Section 20 of *The Classical Theory of Fields* [1]) we are going to consider the case of charged particle *repulsed* by field, i. e. the case of negative value of electric field strength and increasing coordinate x of particle (naturally, in case of particle *attracted* by field the strength is positive while the coordinate of particle will decrease). Then components of the vector E_i are

$$E_1 = E_x = -E = const, \qquad E_2 = E_3 = 0\,. \tag{3.226}$$

Because uniformity of electric field implies $E_i = \frac{\partial \varphi}{\partial x^i} = const$, then scalar potential φ is function of x that satisfies Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial E}{\partial x} = 0\,. \tag{3.227}$$

This result formula implies that uniform constant electric field satisfies condition of absence of density of charges $\rho=0$.

We assume that particle's motion is co-directed with vector electric field strength E_i, i. e. is directed along x. Then its equations of motion with Lorentz force in the right part will be (in by-component notation)

$$\frac{dm}{d\tau} = -\frac{e}{c^2}\frac{d\varphi}{d\tau} = -\frac{e}{c^2}\frac{d\varphi}{dx^i}\mathrm{v}^i = \frac{e}{c^2}E\frac{dx}{d\tau}\,, \tag{3.228}$$

$$\frac{d}{d\tau}\left(m\frac{dx}{d\tau}\right) = eE\,, \quad \frac{d}{d\tau}\left(m\frac{dy}{d\tau}\right) = 0\,, \quad \frac{d}{d\tau}\left(m\frac{dz}{d\tau}\right) = 0\,. \tag{3.229}$$

Integrating scalar equation of motion of charged particle (theorem of live forces), we arrive to integral of live forces in the form

$$m = \frac{eE}{c^2}x + B\,, \qquad B = const. \tag{3.230}$$

Constant B can be obtained from the initial conditions of integration $m|_{\tau=0} = m_{(0)}$ and $x|_{\tau=0} = x_{(0)}$ at the moment $\tau=0$. As a result the constant equals

$$B = m_{(0)} - \frac{eE}{c^2}x_{(0)}\,. \tag{3.231}$$

Then the solution (3.230) of scalar equation of motion takes the form

$$m = \frac{eE}{c^2}\left(x - x_{(0)}\right) + m_{(0)}\,. \tag{3.232}$$

Substituting the obtained integral of live forces into vector equations of motion (3.229) we bring them to the following form (dot stands for derivation to physical observable time τ)

$$\begin{aligned} &\frac{eE}{c^2}\dot{x}^2 + \left(B + \frac{eE}{c^2}x\right)\ddot{x} = eE\,, \\ &\frac{eE}{c^2}\dot{x}\dot{y} + \left(B + \frac{eE}{c^2}x\right)\ddot{y} = 0\,, \\ &\frac{eE}{c^2}\dot{x}\dot{z} + \left(B + \frac{eE}{c^2}x\right)\ddot{z} = 0\,. \end{aligned} \tag{3.233}$$

From here two last equations are equations with separable variables

$$\frac{\ddot{y}}{y} = \frac{-\frac{eE}{c^2}\dot{x}}{B + \frac{eE}{c^2}x}\,, \qquad \frac{\ddot{z}}{z} = \frac{-\frac{eE}{c^2}\dot{x}}{B + \frac{eE}{c^2}x}\,, \tag{3.234}$$

which can be integrated. Their solutions are

$$\dot{y} = \frac{C_1}{B + \frac{eE}{c^2}x}\,, \qquad \dot{z} = \frac{C_2}{B + \frac{eE}{c^2}x}\,, \tag{3.235}$$

where C_1 and C_2 are integration constants which can be found setting the initial conditions $\dot{y}|_{\tau=0} = \dot{y}_{(0)}$ and $\dot{x}|_{\tau=0} = \dot{x}_{(0)}$ and using the formula for B (3.121). As a result we obtain

$$C_1 = m_{(0)}\,\dot{y}_{(0)}\,, \qquad C_2 = m_{(0)}\,\dot{z}_{(0)}\,. \tag{3.236}$$

Now we are going to solve the equation of motion along x axis, i. e. the first one from (3.233). To do this we replace $\dot{x}=\frac{dx}{d\tau}=p$. Then

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt} = pp'. \tag{3.237}$$

As a result the above equation of motion along x axis transforms into an equation with separable variables

$$\frac{pdp}{1-\frac{p^2}{c^2}} = \frac{eEdx}{B+\frac{eE}{c^2}x}\,, \tag{3.238}$$

which is a table integral. After integration we arrive to the solution

$$\sqrt{1-\frac{p^2}{c^2}} = \frac{C_3}{B+\frac{eE}{c^2}x}\,, \qquad C_3 = const. \tag{3.239}$$

Assuming $p=\dot{x}|_{\tau=0}=\dot{x}_{(0)}$ and substituting B from (3.231) we find the integration constant

$$C_3 = m_{(0)}\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}\,. \tag{3.240}$$

In the case under consideration we can replace interval of physical observable time $d\tau$ with interval of coordinate time dt. Here is why. In *The Classical Theory of Fields* Landau and Lifshitz solved equations of motion of charged particle in Galilean frame of reference in a flat Minkowski space [1]. Naturally, to be able to compare our solutions with theirs we consider the same specific case — motion in stationary and uniform electric field (see Section 20 in *The Classical Theory of Fields*). But in stationary and uniform electric field, as we showed earlier in this Section, using methods of chronometric invariants, $F_i=0$ and $A_{ik}=0$, hence

$$d\tau = \left(1-\frac{w}{c^2}\right)dt - \frac{1}{c^2}v_i dx^i = dt\,. \tag{3.241}$$

In other words, in the four-dimensional area under study where particle travels in this case the metric is Galilean one.

Substituting variable $p=\frac{dx}{dt}$ into the formula (3.239) we arrive to the last equation with separable variables

$$\frac{dx}{dt} = c\frac{\sqrt{\left(B+\frac{eE}{c^2}x\right)^2 - C_3^2}}{B+\frac{eE}{c^2}x}\,, \tag{3.242}$$

which solution is

$$ct = \frac{c^2}{eE}\sqrt{\left(B+\frac{eE}{c^2}x\right)^2 - C_3^2} + C_4\,, \quad C_4 = const, \tag{3.243}$$

where integration constant C_4, taking into account the initial conditions at the moment $t=0$, is

$$C_4 = -\frac{m_{(0)}c}{eE}\dot{x}_{(0)}\,. \tag{3.244}$$

Now formulating coordinate x explicitly from (3.243) with t we obtain the final solution of equation of motion of charged mass-bearing particle along x axis

$$x = \frac{c^2}{eE}\left[\sqrt{\frac{e^2E^2}{c^4}\left(ct-C_4\right)^2 + C_3^2} - B\right], \tag{3.245}$$

or, after substituting integration constants

$$x=\sqrt{\left(ct+\frac{m_{(0)}c\dot{x}_{(0)}}{eE}\right)^2+\left(\frac{m_{(0)}c^2}{eE}\right)^2\left(1-\frac{\dot{x}_{(0)}^2}{c^2}\right)}-\frac{m_{(0)}c^2}{eE}+x_{(0)}\,.$$

If the field *attracts* charged particle (electric strength is positive $E_1{=}E_x{=}E{=}const$), we will obtain the same solution for x but bearing the opposite sign

$$x=\frac{c^2}{eE}\left[B-\sqrt{\frac{e^2E^2}{c^4}\left(ct-C_4\right)^2+C_3^2}\,\right]. \tag{3.247}$$

In *The Classical Theory of Fields* [1] a similar problem is considered, but Landau and Lifshitz solved it through integration of three-dimensional components of general covariant equations of motion of charged particle (three-dimensional Minkowski equations) without accounting for theorem of live forces. As a result their formula for x is

$$x=\frac{1}{eE}\sqrt{\left(m_0c^2\right)^2+\left(ceEt\right)^2}. \tag{3.248}$$

This formula fully matches our solution (3.245) if $x_{(0)}-\frac{m_{(0)}c^2}{eE}{=}0$ and the initial velocity of particle is zero $\dot{x}_{(0)}{=}0$. The latter stands for significant simplifications accepted in *The Classical Theory of Fields*, according to which some integration constants are assumed zeroes.

As seen, even when solving equations of motion in Galilean frame of reference in flat Minkowski space method of chronometric invariants gives certain advantages revealing hidden factors that are left unnoticed when solving regular three-dimensional components of general covariant equations of motion. That means that even when physical observable values coincide with coordinate values, it is geometrically correct to solve *a system* of chronometrically invariant equations of motion, because theorem of live forces, being scalar equation of motion, inevitably affects solution of three-dimensional vector equations of motion.

Of course in case of non-uniform non-stationary electrical field some additional terms will appear in our solution to reflect more complicated and varying in time structure of field.

Now we are going to calculate three-dimensional observable trajectory of particle that moves in stationary uniform electric field.

To do this, we integrate equations of motion along axis y and z (3.235), formulate time from there and substitute it into the solution for x we have obtained.

First, substituting the obtained solution for x (3.245) into equation for $\dot{y}$, we obtain equation with separable variables

$$\frac{dy}{dt}=\frac{C_1}{\sqrt{\frac{e^2E^2}{c^4}\left(ct-C_4\right)^2+C_3^2}}\,, \tag{3.249}$$

integrating which we have

$$y=\frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE}\,\text{arc}\sinh\frac{eEt+m_{(0)}\,\dot{x}_{(0)}}{m_{(0)}\,c\sqrt{1-\frac{\dot{x}_{(0)}^2}{c^2}}}+C_5\,, \tag{3.250}$$

where C_5 is integration constant. After substituting the initial condition $y{=}y_{(0)}$ at the moment $t{=}0$ into here we find formula for the constant C_5

$$C_5=y_{(0)}-\frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE}\,\text{arc}\sinh\frac{\dot{x}_{(0)}}{c\sqrt{1-\frac{\dot{x}_{(0)}^2}{c^2}}}\,. \tag{3.251}$$

Substituting the constant into y (3.250) we finally have

$$y = y_{(0)} + \frac{m_{(0)}\, \dot{y}_{(0)}\, c}{eE} \left\{ \operatorname{arc\,sinh} \frac{eEt + m_{(0)}\, \dot{x}_{(0)}}{m_{(0)}\, c\sqrt{1 - \frac{\dot{x}^2_{(0)}}{c^2}}} - \operatorname{arc\,sinh} \frac{\dot{x}_{(0)}}{c\sqrt{1 - \frac{\dot{x}^2_{(0)}}{c^2}}} \right\}. \tag{3.252}$$

Formulating from here t with y and $y_{(0)}$ and taking into account that a=arcsinh b if b= sinh a and substituting arcsinh b= $\ln(b+\sqrt{b^2+1})$ into the second term we have

$$t = \frac{1}{eE} \left\{ m_{(0)}\, c\sqrt{1 - \frac{\dot{x}^2_{(0)}}{c^2}}\, \sinh \left[\frac{y - y_{(0)}}{m_{(0)}\, \dot{y}_{(0)}\, c} eE + \ln \frac{\dot{x}_{(0)} + c}{c\sqrt{1 - \frac{\dot{x}^2_{(0)}}{c^2}}} \right] - m_{(0)}\, \dot{x}_{(0)} \right\}. \tag{3.253}$$

Now we substitute it into our solution for x (3.246). As a result we obtain the desired equation for three-dimensional trajectory of particle

$$x = x_{(0)} + \frac{m_{(0)}\, c^2}{eE} \sqrt{1 - \frac{\dot{x}^2_{(0)}}{c^2}}\, \cosh \left\{ \frac{y - y_{(0)}}{m_{(0)}\, \dot{y}_{(0)}\, c} eE + \ln \frac{\dot{x}_{(0)} + c}{c\sqrt{1 - \frac{\dot{x}^2_{(0)}}{c^2}}} \right\} - \frac{m_{(0)}\, c^2}{eE}. \tag{3.254}$$

The obtained formula implies that charged mass-bearing particle in our world in uniform stationary electric field travels along a curve based on *chain line*, while factors that deviate it from "pure" chain line are functions of the initial conditions.

Our formula for particle's trajectory (3.254) fully matches the result obtained in *The Classical Theory of Fields*

$$x = \frac{m_{(0)}\, c^2}{eE} \cosh \frac{eEy}{m_{(0)}\, \dot{y}_{(0)}\, c} \tag{3.255}$$

(formula 20.5 in [1]) once we assume that $x_{(0)} - \frac{m_{(0)}\, c^2}{eE} = 0$, and the initial velocity of particle $\dot{x}_{(0)}=0$ as well. The latter condition suggests that the integration constant in scalar equation of motion (theorem of live forces) is zero, which is not always true.

At low velocities of motion after equaling relativistic terms to zero and expanding hyperbolic cosine into series $\cosh b = 1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \frac{b^6}{6!} + \ldots$ our formula for three-dimensional trajectory of particle (3.254) takes the form (higher order terms withheld here)

$$x = x_{(0)} + \frac{eE\left(y - y_{(0)}\right)^2}{2m_{(0)}\, \dot{y}^2_{(0)}}, \tag{3.256}$$

i. e. particle travels along *parabola*. This conclusion, once the initial coordinates of particle are assumed zeroes, also matches the result from *The Classical Theory of Fields*

$$x = \frac{eEy^2}{2m_{(0)}\, \dot{y}^2_{(0)}}. \tag{3.257}$$

Integration of equation of motion along axis z gives the same results. This is because the only difference between equations in respect to $\dot{y}$ and $\dot{z}$ (3.235) is a fixed coefficient — integration constant (3.236), which equals to the initial impulse of particle along axis y (in the equation for $\dot{y}$) and along axis z (in equation for $\dot{z}$).

We are going to find dynamic properties of charged particle in stationary uniform electric field — its energy and impulse. Calculating the relativistic square root (accounting for the assumptions we made)

$$\sqrt{1-\frac{\mathrm{v}^2}{c^2}}=\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}=\frac{m_{(0)}\sqrt{1-\frac{\dot{x}_{(0)}^2+\dot{y}_{(0)}^2+\dot{z}_{(0)}^2}{c^2}}}{m_{(0)}+\frac{eE}{c^2}\left(x-x_{(0)}\right)}\,, \tag{3.258}$$

we obtain the energy of particle

$$E=\frac{m_{(0)}\,c^2}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}=\frac{m_{(0)}\,c^2+eE\left(x-x_{(0)}\right)}{\sqrt{1-\frac{\dot{x}_{(0)}^2+\dot{y}_{(0)}^2+\dot{z}_{(0)}^2}{c^2}}}\,, \tag{3.259}$$

which for a particle's velocity much lower than that of light speed is

$$E=m_{(0)}\,c^2+eE\left(x-x_{(0)}\right). \tag{3.260}$$

Relativistic impulse of particle is obtained in the same way, but the formula being bulky is withheld here.

We have studied motion of charged mass-bearing particle of our world in stationary uniform electric field. Now we consider motion of mass-bearing charged particle of the mirror world under the same conditions. Its chronometrically invariant equations of motion, taking into account the constraints imposed here on geometric structure of space, are

$$\frac{dm}{d\tau}=\frac{e}{c^2}\frac{d\varphi}{d\tau}\,, \tag{3.261}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right)=-\frac{e}{c}\frac{dq^i}{d\tau}\,. \tag{3.262}$$

In other words, the only difference from equations of motion for our-world particle (3.221, 3.222) is the sign in theorem of live forces.

We assume that electric strength is negative (field *repulses* the particle) and that the particle's motion is co-directed with vector of field strength along axis x. Then integrating theorem of live forces for the mirror-world particle (3.261) we obtain

$$m=-\frac{eE}{c^2}x+B\,, \tag{3.263}$$

where integration constant, calculated from the initial conditions, is

$$B=m_{(0)}+\frac{eE}{c^2}x_{(0)}\,. \tag{3.264}$$

Substituting the results into vector equations of motion (3.262) in by-component notation, we have (comp. 3.233)

$$\left.\begin{aligned}
&-\frac{eE}{c^2}\dot{x}^2+\left(B-\frac{eE}{c^2}x\right)\ddot{x}=eE\,,\\
&-\frac{eE}{c^2}\dot{x}\dot{y}+\left(B-\frac{eE}{c^2}x\right)\ddot{y}=0\,,\\
&-\frac{eE}{c^2}\dot{x}\dot{z}+\left(B-\frac{eE}{c^2}x\right)\ddot{z}=0\,.
\end{aligned}\right\} \tag{3.265}$$

After some algebra similar to that done to obtain trajectory of charged particle in our world, we arrive to

$$x=\frac{c^2}{eE}\left[B-\sqrt{C_3^2-\frac{e^2E^2}{c^4}\left(ct-C_4\right)^2}\,\right], \tag{3.266}$$

where $C_3 = m_{(0)}\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}$ and $C_4 = -\frac{cm_{(0)}x_{(0)}}{eE}$. Or,

$$x = -\sqrt{\left(\frac{m_{(0)}c^2}{eE}\right)^2\left(1+\frac{\dot{x}^2_{(0)}}{c^2}\right) - \left(ct+\frac{m_{(0)}c\dot{x}_{(0)}}{eE}\right)^2} + \frac{m_{(0)}c^2}{eE} + x_{(0)}\,. \tag{3.267}$$

The obtained formula for coordinate x of mirror-world charged particle, *repulsed* by field, is similar to that for our-world particle *attracted* by field (3.247) when electric strength is positive $E_1{=}E_x{=}E{=}const$. Hence an interesting conclusion: transition of particle from our world into the mirror world (the one with reverse flow of time) is the same as changing the sign of its charge.

Noteworthy, similar conclusion can be done in respect to particles' masses: purported transition of particle from our world into the mirror world is the same as changing the sign of its mass. Hence our-world particles and mirror-world particle are mass and charge complementary.

Let us find three-dimensional trajectory of charged mass-bearing particle of the mirror world in stationary uniform electric field. Calculating y in the same manner as for charged our-world particle, we have

$$y = y_{(0)} + \frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE}\left\{\arcsin\frac{eEt+m_{(0)}\,\dot{x}_{(0)}}{m_{(0)}\,c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}} - \arcsin\frac{\dot{x}_{(0)}}{c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}}\right\}. \tag{3.268}$$

Contrasted to the formula for our-world particle (3.252) this formula has regular arcsine and "plus" sign under the square route.

Formulating time t from here with coordinates y and $y_{(0)}$

$$t = \frac{1}{eE}\left\{m_{(0)}\,c\sqrt{1{+}\frac{\dot{x}^2_{(0)}}{c^2}}\,\sin\left[\frac{y-y_{(0)}}{m_{(0)}\,\dot{y}_{(0)}\,c}eE + \ln\frac{\dot{x}_{(0)}+c}{c\sqrt{1{+}\frac{\dot{x}^2_{(0)}}{c^2}}}\right] - m_{(0)}\,\dot{x}_{(0)}\right\}, \tag{3.269}$$

and substituting it into our formula for x (3.267) we obtain the equation of three-dimensional trajectory of charged mass-bearing particle of the mirror world that travels in uniform stationary electric field,

$$x = x_{(0)} - \frac{m_{(0)}\,c^2}{eE}\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}\,\cos\left\{\frac{y-y_{(0)}}{m_{(0)}\,\dot{y}_{(0)}\,c}eE + \arcsin\frac{\dot{x}_{(0)}}{c\sqrt{1{+}\frac{\dot{x}^2_{(0)}}{c^2}}}\right\} - \frac{m_{(0)}\,c^2}{eE}\,. \tag{3.270}$$

In other words, the motion of the particle is *harmonic oscillation*. Once we assume the initial coordinates of the particle equal to zero, as well as its initial velocity $\dot{x}_{(0)}{=}0$ and the integration constant $B{=}0$, the obtained equation of the trajectory takes a simpler form

$$x = -\frac{m_{(0)}\,c^2}{eE}\,\cos\frac{eEy}{m_{(0)}\,\dot{y}_{(0)}\,c}\,. \tag{3.271}$$

At low velocity of motion, equaling relativistic terms to zero and expanding into the cosinus series $\cos b{=}1{-}\frac{b^2}{2!}{+}\frac{b^4}{4!}{-}\frac{b^6}{6!}{+}\ldots\approx 1{-}\frac{b^2}{2!}$ (which is always possible within a smaller part of trajectory), we bring our formula for three-dimensional trajectory of the mirror-world particle (3.270) as

$$x = x_{(0)} + \frac{eE\left(y-y_{(0)}\right)^2}{2m_{(0)}\,\dot{y}^2_{(0)}}\,, \tag{3.272}$$

which is equation of *parabola*. That means charged mirror-world particle at low velocity travels along a parabola, as does our-world particle.

Therefore, relativistic charged particle from our world in uniform stationary electric field travels along a chain line, which at low velocities becomes a parabola. Relativistic charged mirror-world particle travels along a harmonic trajectory, smaller parts of which at low velocities becomes parabola (as is the case for our-world particle).

3.12 Motion of charged particle in stationary magnetic field

Let us consider motion of charged particle when electric component of field is absent while stationary magnetic field is present. In this case physical observable vectors electric and magnetic strengths are

$$E_i = \frac{{}^*\partial\varphi}{\partial x^i} - \frac{\varphi}{c^2}F_i = \frac{\partial\varphi}{\partial x^i} - \frac{\varphi}{c^2}\frac{1}{1-\dfrac{w}{c^2}}\frac{\partial w}{\partial x^i} = 0\,, \tag{3.273}$$

$$H^{*i} = \frac{1}{2}\varepsilon^{imn}H_{mn} = \frac{1}{2}\varepsilon^{imn}\left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} - \frac{2\varphi}{c}A_{mn}\right) \neq 0 \tag{3.274}$$

because in "pure" magnetic field $\varphi{=}const$ ($E_i{=}0$) then gravitational effect can be neglected. From (3.274) we can see that magnetic strength H^{*i} is not zero, if at least one of the following conditions is true:

1. field of potential q^i is rotational;
2. the space is non-holonomic, i. e. $A_{ik}{\neq}0$.

We are going to consider motion of particle in general case, when both conditions are true, because non-holonomic space we will use later as the basic space for a spin-particle. As we did in the previous Section, we assume deformation of space to be zero and three-dimensional metric to be Euclidean one $g_{ik}{=}\delta_{ik}$. But the observed metric $h_{ik}{=}{-}g_{ik}{+}\frac{1}{c^2}v_i v_k$ in this case is not Galilean, because in non-holonomic space $h_{ik}{\neq}{-}g_{ik}$.

We assume that the space of reference rotates around axis z at constant angular velocity, that is $\Omega_{12}{=}{-}\Omega_{21}{=}\Omega$. Then linear velocity of rotation of space $v_i{=}\Omega_{ik}x^k$ has two non-zero components $v_1{=}\Omega y$ and $v_2{=}{-}\Omega x$, while non-holonomity tensor has the only non-zero components $A_{12}{=}{-}A_{21}{=}{-}\Omega$. In this case the metric will take the form

$$ds^2 = c^2dt^2 - 2\Omega y dt dx + 2\Omega x dt dy - dx^2 - dy^2 - dz^2. \tag{3.275}$$

In a space with this kind of metric $F_i{=}0$ and $D_{ik}{=}0$. In the below we will use this metric to study motion of charged particle in stationary fields. In the previous Section, which focused on motion in stationary electric field, we also assumed Christoffel symbols to be zeroes. In other words, we considered motion of particle in Galilean frame of reference in Minkowski space. But in this Section three-dimensional observable metric h_{ik} is not Euclidean because of rotation of space itself and thus Christoffel symbols $\triangle^i_{jk}$ (1.47) are not zeroes.

If linear velocity of rotation of space is not infinitesimal compared to speed of light, components of observable metric tensor h_{ik} are

$$h_{11} = 1{+}\frac{\Omega^2 y^2}{c^2}\,,\quad h_{22} = 1{+}\frac{\Omega^2 x^2}{c^2}\,,\quad h_{12} = -\frac{\Omega^2 xy}{c^2}\,,\quad h_{33} = 1\,. \tag{3.276}$$

Then the determinant of this tensor and components of h^{ik} are

$$h = \det\|h_{ik}\| = h_{11}h_{22} - h_{12}^2 = 1 + \frac{\Omega^2\left(x^2+y^2\right)}{c^2}\,, \tag{3.277}$$

$$h^{11} = \frac{1}{h}\left(1+\frac{\Omega^2 x^2}{c^2}\right),\quad h^{22} = \frac{1}{h}\left(1+\frac{\Omega^2 y^2}{c^2}\right),\quad h^{12} = \frac{\Omega^2 xy}{hc^2}\,,\quad h^{33} = 1\,. \tag{3.278}$$

Respectively, from here we obtain non-zero components of chronometrically invariant Christoffel symbols $\triangle^i_{jk}$ (1.47)

$$\triangle^1_{11} = \frac{2\Omega^4 xy^2}{c^4\left(1+\dfrac{\Omega^2\left(x^2+y^2\right)}{c^2}\right)}\,, \tag{3.279}$$

$$\triangle^1_{12} = \frac{\Omega^2 y\left(1+\dfrac{2\Omega^2 x^2}{c^2}\right)}{c^2\left(1+\dfrac{\Omega^2\left(x^2+y^2\right)}{c^2}\right)}\,, \tag{3.280}$$

$$\triangle^1_{22} = -\frac{2\Omega^2 x}{c^2}\,\frac{1+\dfrac{\Omega^2 x^2}{c^2}}{1+\dfrac{\Omega^2\left(x^2+y^2\right)}{c^2}}\,, \tag{3.281}$$

$$\triangle^2_{11} = -\frac{2\Omega^2 y}{c^2}\,\frac{1+\dfrac{\Omega^2 y^2}{c^2}}{1+\dfrac{\Omega^2\left(x^2+y^2\right)}{c^2}}\,, \tag{3.282}$$

$$\triangle^2_{12} = \frac{\Omega^2 x\left(1+\dfrac{2\Omega^2 y^2}{c^2}\right)}{c^2\left(1+\dfrac{\Omega^2\left(x^2+y^2\right)}{c^2}\right)}\,, \tag{3.283}$$

$$\triangle^2_{22} = -\frac{2\Omega^4 x^2 y}{c^4\left(1+\dfrac{\Omega^2\left(x^2+y^2\right)}{c^2}\right)}\,. \tag{3.284}$$

We are going to solve equations of motion of mass-bearing charged particle in stationary magnetic field. To make calculations easier we assume that four-dimensional potential of field A^α is tangential to the world line (four-dimensional trajectory) of particle. Because electric component of field is absent, strength E_i does not perform any work, i. e. the right parts of scalar equations of motion turn into zeroes. Therefore, chronometrically invariant equations of motion of mass-bearing charged particle from our-world (3.208, 3.209) in stationary magnetic field are

$$\frac{dm}{d\tau} = 0\,, \tag{3.285}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2mA^{\cdot i}_{k\cdot}\mathrm{v}^k + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = -\frac{e}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{3.286}$$

For mirror-world charged particle that moves in stationary magnetic field, the respective equations of motion are

$$-\frac{dm}{d\tau} = 0\,, \tag{3.287}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = -\frac{e}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{3.288}$$

Integrating theorem of live forces for our-world particle and mirror-world particle we obtain, respectively

$$m=\frac{m_0}{\sqrt{1-\dfrac{\mathrm{v}^2}{c^2}}}=const=B\,, \qquad -m=\frac{m_0}{\sqrt{1-\dfrac{\mathrm{v}^2}{c^2}}}=const=\widetilde{B}\,, \tag{3.289}$$

where B and $\widetilde{B}$ are integration constants. That implies v^2=*const*, i. e. module of observable velocity of particle stays constant in absence of electric component of field. Then vector equations of motion for our-world particle (3.286) can be represented as

$$\frac{d\mathrm{v}^i}{d\tau} + 2A_{k\cdot}^{\cdot i}\mathrm{v}^k + \triangle_{nk}^i \mathrm{v}^n \mathrm{v}^k = -\frac{e}{mc}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{3.290}$$

Respectively, vector equations of motion of mirror-world particle (3.288) will have the form

$$\frac{d\mathrm{v}^i}{d\tau} + \triangle_{nk}^i \mathrm{v}^n \mathrm{v}^k = -\frac{e}{mc}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{3.291}$$

Strength of magnetic field in these equations is defined from Maxwell equations for stationary field (3.215, 3.216), which in absence of electric strength and under the constraints we assumed in this Section, are

$$\left.\begin{aligned} &\Omega_{*m}H^{*m} = -2\pi c\rho \\ &\varepsilon^{ikm}\,{}^*\nabla_k\left(H_{*m}\sqrt{h}\right) = \frac{4\pi}{c}j^i\sqrt{h} \end{aligned}\right\} \quad \text{Group I,} \tag{3.292}$$

$$\left.{}^*\nabla_i H^{*i} = \frac{\partial H^{*i}}{\partial x^i} + \frac{\partial \ln\sqrt{h}}{\partial x^i}H^{*i} = 0\right\} \quad \text{Group II.} \tag{3.293}$$

From the first equation of Group I we see that scalar product of pseudovectors of field of non-holonomity of space and of magnetic field strength is function of density of charge. Hence if density of charge ρ=0, pseudovectors Ω_{*i} and H^{*i} are orthogonal.

In the below we consider two possible orientations of magnetic field in respect to field of non-holonomity of space.

A Magnetic field is co-directed with non-holonomity field

We assume that pseudovector of strength of magnetic field H^{*i} is directed along axis z, i. e. in the same direction as pseudovector of angular velocity of space rotation $\Omega^{*i}=\frac{1}{2}\varepsilon^{ikm}A_{km}$. Then pseudovector of angular velocity has one non-zero component $\Omega^{*3}=\Omega$, while pseudovector of magnetic strength has

$$H^{*3} = \frac{1}{2}\varepsilon^{3mn}H_{mn} = \frac{1}{2}\left(\varepsilon^{312}H_{12} + \varepsilon^{321}H_{21}\right) = H_{12} = \frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_1}{\partial x} - \frac{\partial \mathrm{v}_2}{\partial y}\right) + \frac{2\varphi}{c}\Omega\,. \tag{3.294}$$

The condition φ=*const* stems from absence of electric field. Hence Maxwell equations Group I (2.392) will be

$$\left.\begin{aligned} &\Omega_{*3}H^{*3} = \frac{\Omega\varphi}{c}\left(\frac{\partial \mathrm{v}_1}{\partial x} - \frac{\partial \mathrm{v}_2}{\partial y}\right) + \frac{2\varphi\Omega^2}{c} = -2\pi c\rho \\ &\frac{\partial}{\partial y}\left(H_{*3}\sqrt{h}\right) = \frac{4\pi}{c}j^1\sqrt{h} \\ &-\frac{\partial}{\partial x}\left(H_{*3}\sqrt{h}\right) = \frac{4\pi}{c}j^2\sqrt{h} \\ &j^3 = 0 \end{aligned}\right\}. \tag{3.295}$$

Equations in Group II (3.293) will be trivial turning into simple relationship $\frac{\partial H^{*3}}{\partial z}$=0, i. e. H^{*3}=*const*. Actually this implies that the stationary magnetic field we consider is uniform along z. In the below we assume the stationary magnetic field to be fully uniform, i. e. H^{*i}=*const*. Then from the first equation from Group I (3.295) we see that magnetic field is uniform provided that

$$\left(\frac{\partial \mathrm{v}_1}{\partial x} - \frac{\partial \mathrm{v}_2}{\partial y}\right) = const, \qquad \rho = -\frac{\varphi\Omega^2}{\pi c^2} = const. \tag{3.296}$$

Hence charge density $\rho{>}0$ if scalar potential of field $\varphi{<}0$. In this case the other equations from Group I (3.295) will be

$$j^1 = \frac{c}{4\pi}\frac{\partial \ln\sqrt{h}}{\partial y}\,, \qquad j^2 = \frac{c}{4\pi}\frac{\partial \ln\sqrt{h}}{\partial x}\,. \tag{3.297}$$

Because $h{=}1{+}\frac{\Omega^2(x^2{+}y^2)}{c^2}$ (3.277) this implies: vector of current in stationary uniform magnetic field is only non-zero in *strong* field of non-holonomity, i. e. when rotation velocity is comparable to speed of light. In a weak field of non-holonomity $h{=}1$, hence $j^1{=}j^2{=}0$.

Now having obtained magnetic strength from Maxwell equations we write down by-component notation of vector equations of motion of our-world charged mass-bearing particle (3.290, 3.291)

$$\begin{gathered}
\ddot{x} + \frac{2\Omega}{h}\left[\frac{\Omega^2 xy\dot{x}}{c^2} + \left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\right] + \triangle^1_{11}\dot{x}^2 + 2\triangle^1_{12}\dot{x}\dot{y} + \\
+\triangle^1_{22}\dot{y}^2 = -\frac{eH}{mc}\left[-\frac{\Omega^2 xy\dot{x}}{c^2} + \left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\right], \\
\ddot{y} - \frac{2\Omega}{h}\left[\frac{\Omega^2 xy\dot{y}}{c^2} + \left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\right] + \triangle^2_{11}\dot{x}^2 + 2\triangle^2_{12}\dot{x}\dot{y} + \\
+\triangle^2_{22}\dot{y}^2 = \frac{eH}{mc}\left[-\frac{\Omega^2 xy\dot{y}}{c^2} + \left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\right], \\
\ddot{z} = 0\,,
\end{gathered} \tag{3.298}$$

and those of mirror-world particle

$$\begin{gathered}
\ddot{x} + \triangle^1_{11}\dot{x}^2 + 2\triangle^1_{12}\dot{x}\dot{y} + \triangle^1_{22}\dot{y}^2 = -\frac{eH}{mc}\left[-\frac{\Omega^2 xy\dot{x}}{c^2} + \left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\right], \\
\ddot{y} + \triangle^2_{11}\dot{x}^2 + 2\triangle^2_{12}\dot{x}\dot{y} + \triangle^2_{22}\dot{y}^2 = \frac{eH}{mc}\left[-\frac{\Omega^2 xy\dot{y}}{c^2} + \left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\right], \\
\ddot{z} = 0\,.
\end{gathered} \tag{3.299}$$

The terms in the right parts that contain $\frac{\Omega^2}{c^2}$ appear because in rotation of space three-dimensional observable metric h_{ik} is not Euclidean. Hence in the case under consideration there is difference between contravariant components of the observable velocity and its covariant components. The right parts include the covariant components

$$\mathrm{v}_2 = h_{21}\mathrm{v}^1 + h_{22}\mathrm{v}^2 = -\frac{\Omega^2 xy}{c^2}\dot{x} + \left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\,, \tag{3.300}$$

$$\mathrm{v}_1 = h_{11}\mathrm{v}^1 + h_{12}\mathrm{v}^2 = -\frac{\Omega^2 xy}{c^2}\dot{y} + \left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\,. \tag{3.301}$$

If rotation of space is absent, i. e. $\Omega{=}0$, the equations of motion of charged mass-bearing our-world particle (3.298) up within a sign match equations of motion in stationary uniform magnetic field as given by Landau and Lifshitz (formula 21.2 in *The Classical Theory of Fields*)

$$\ddot{x} = \frac{eH}{mc}\dot{y}\,, \qquad \ddot{y} = -\frac{eH}{mc}\dot{x}\,, \qquad \ddot{z} = 0\,, \tag{3.302}$$

while our equations (3.298) imply that

$$\ddot{x} = -\frac{eH}{mc}\dot{y}\,, \qquad \ddot{y} = \frac{eH}{mc}\dot{x}\,, \qquad \ddot{z} = 0\,. \tag{3.303}$$

The difference stems from the fact that Landau and Lifshitz assumed magnetic strength in Lorentz force to bear "plus" sign, while in our equations it bears "minus", which is not that important though, because only depends upon choice of signature.

If the space rotates (is non-holonomic), the equations of motion will include the terms that contain Ω, $\frac{\Omega^2}{c^2}$, and $\frac{\Omega^4}{c^4}$. In a strong field of non-holonomity solving equations of motion is a non-trivial task, which is likely to be tackled in future with computer-aided numerical methods. Hopefully, the results will be quite interesting.

We are going to find exact solutions in weak field of non-holonomity, i. e. truncating terms of the second order of smallness and below. Here equations of motion we obtained (3.298, 3.299) for charged mass-bearing our-world particle are

$$\ddot{x}+2\Omega\dot{y}=-\frac{eH}{mc}\dot{y}\,,\qquad \ddot{y}-2\Omega\dot{x}=\frac{eH}{mc}\dot{x}\,,\qquad \ddot{z}=0\,,\tag{3.304}$$

and for charged mass-bearing mirror-world particle are

$$\ddot{x}=-\frac{eH}{mc}\dot{y}\,,\qquad \ddot{y}=\frac{eH}{mc}\dot{x}\,,\qquad \ddot{z}=0\,.\tag{3.305}$$

First we approach equations for our-world particle. The equation along z axis can be integrated straightaway. The solution is

$$z=\dot{z}_{(0)}\tau+z_{(0)}\,.\tag{3.306}$$

From here we see that if at the initial moment particle's velocity along z is zero, the particle will be moving within xy plane only. The rest two equations we from (3.304) we re-write as

$$\frac{d\dot{x}}{d\tau}=-\left(2\Omega+\omega\right)\dot{y}\,,\qquad \frac{d\dot{y}}{d\tau}=\left(2\Omega+\omega\right)\dot{x}\,,\tag{3.307}$$

where we denote $\omega=\frac{eH}{mc}$ for convenience. The same notation was used in Section 21 of *The Classical Theory of Fields*. Then, formulating $\dot{x}$ from the second equation, we derivate it to the observable time $\dot{x}$ and substitute into the first equation. As a result we obtain

$$\frac{d^2\dot{y}}{d\tau^2}+\left(2\Omega+\omega\right)^2\dot{y}=0\,,\tag{3.308}$$

which is equation of oscillations; it solves as

$$\dot{y}=C_1\cos\left(2\Omega+\omega\right)\tau+C_2\sin\left(2\Omega+\omega\right)\tau\,,\tag{3.309}$$

where $C_1=\dot{y}_{(0)}$ and $C_2=\frac{\ddot{y}_{(0)}}{2\Omega+\omega}$ are integration constants. Substituting $\dot{y}$ (3.309) into the first equation (3.307) we obtain the expression

$$\frac{d\dot{x}}{d\tau}=-\left(2\Omega+\omega\right)\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau-\ddot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,,\tag{3.310}$$

after integration of which we have

$$\dot{x}=\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau-\frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos\left(2\Omega+\omega\right)\tau+C_3\,,\tag{3.311}$$

where integration constant $C_3=\dot{x}_{(0)}+\frac{\ddot{y}_{(0)}}{2\Omega+\omega}$.

Having all constants substituted, the obtained formulas for $\dot{x}$ (3.311) and $\dot{y}$ (3.309) finally transform into

$$\dot{x}=\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau-\frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos\left(2\Omega+\omega\right)\tau+\dot{x}_{(0)}+\frac{\ddot{y}_{(0)}}{2\Omega+\omega}\,,\tag{3.312}$$

$$\dot{y}=\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau+\frac{\ddot{y}_{(0)}}{2\Omega+\omega}\sin\left(2\Omega+\omega\right)\tau\,.\tag{3.313}$$

Hence the formulas for components of velocity of particle $\dot{x}$ and $\dot{y}$ in stationary uniform magnetic field are equations of harmonic oscillations at frequency that in a weak field of non-holonomity is $2\Omega+\omega=2\Omega+\frac{eH}{mc}$.

From integral of live forces in stationary magnetic field (3.289) we see that the square of particle's velocity is a constant value. Calculating $\mathrm{v}^2=\dot{x}^2+\dot{y}^2+\dot{z}^2$ for our-world particle we obtain that the value

$$\begin{aligned}\mathrm{v}^2 = \dot{x}^2_{(0)} + \dot{y}^2_{(0)} + \dot{z}^2_{(0)} + 2\left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\right)\times \\ \times\left[\frac{\ddot{y}_{(0)}}{2\Omega+\omega} + \dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau - \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos\left(2\Omega+\omega\right)\tau\right]\end{aligned} \qquad (3.314)$$

is constant $\mathrm{v}^2=const$, provided that $C_3=\dot{x}_{(0)}+\frac{\ddot{y}_{(0)}}{2\Omega+\omega}=0$.

Integrating $\dot{x}$ and $\dot{y}$ to τ we obtain coordinates of our-world particle in stationary uniform magnetic field

$$x = \left[\frac{\ddot{y}_{(0)}}{2\Omega+\omega}\sin\left(2\Omega+\omega\right)\tau - \dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau\right]\frac{1}{2\Omega+\omega} + \left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\right)\tau + C_4\,, \qquad (3.315)$$

$$y = \left[\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos\left(2\Omega+\omega\right)\tau\right]\frac{1}{2\Omega+\omega} + C_5\,, \qquad (3.316)$$

where integration constants are

$$C_4 = x_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega+\omega}\,, \qquad C_5 = y_{(0)} + \frac{\ddot{y}_{(0)}}{\left(2\Omega+\omega\right)^2}\,. \qquad (3.317)$$

From the formula for x we see that particle performs *harmonic oscillations* along x provided that the equation $\dot{x}_{(0)}+\frac{\ddot{y}_{(0)}}{2\Omega+\omega}=0$ is true. This is also the condition for constant square of particle's velocity (3.314), i. e. satisfies the integral of live forces. Taking this into account we arrive to equation of trajectory of particle within xy plane

$$\begin{aligned}x^2 + y^2 = \frac{1}{\left(2\Omega+\omega\right)^2}\left[\dot{y}^2_{(0)} + \frac{\ddot{y}^2_{(0)}}{\left(2\Omega+\omega\right)^2}\right] - \frac{2C_4}{2\Omega+\omega}\times \\ \times\left[\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\sin\left(2\Omega+\omega\right)\tau\right] + \\ + \left[\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos\left(2\Omega+\omega\right)\tau\right]\frac{2C_5}{2\Omega+\omega} + C_4^2 + C_5^2\,.\end{aligned} \qquad (3.318)$$

Assuming for the initial moment of time $\ddot{y}_{(0)}=0$ and integration constants C_4 and C_5 to be zeroes, we can dramatically simplify the obtained formulas for coordinates of particle (3.315, 3.316)

$$x = -\frac{1}{2\Omega+\omega}\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau\,, \qquad (3.319)$$

$$y = \frac{1}{2\Omega+\omega}\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,. \qquad (3.320)$$

Given this, our equation of trajectory (3.318) transforms into a simple equation of a circle

$$x^2 + y^2 = \frac{\dot{y}^2_{(0)}}{\left(2\Omega+\omega\right)^2}\,. \qquad (3.321)$$

Hence if the initial velocity of charged our-world particle in respect to the axis of uniform magnetic field (z axis) is zero, particle moves within xy plane along a *circle* with radius

$$r = \frac{\dot{y}_{(0)}}{2\Omega+\omega} = \frac{\dot{y}_{(0)}}{2\Omega+\dfrac{eH}{mc}}\,, \qquad (3.322)$$

which depends upon the strength of the field and the velocity of space rotation. If initial velocity of particle along the direction of magnetic field is not zero, it moves along *a spiral line* with radius r along the field. In general case particle moves along *ellipse* within xy plane (3.318), which shape deviates from that of a circle depending upon the initial conditions of the motion.

As can be easily seen, our results match those in Section 21 of *The Classical Theory of Fields*

$$x = -\frac{1}{\omega}\dot{y}_{(0)} \cos\omega\tau\,, \qquad y = \frac{1}{\omega}\dot{y}_{(0)} \sin\omega\tau\,, \tag{3.323}$$

once we assume rotation of space $\Omega{=}0$, i. e. in absence of field of non-holonomity. Given this, the radius $r{=}\frac{\dot{y}_{(0)}}{\omega}{=}\frac{mc}{eH}\dot{y}_{(0)}$ of particle's trajectory does not depend upon velocity of rotation of space. If $\Omega{\neq}0$, field of non-holonomity of space disturbs motion of particle in magnetic field adding up with field of magnetic strength, due to which correction value 2Ω to the term $\omega = \frac{eH}{mc}$ appears in equations of the theory. In a strong field of non-holonomity, when Ω can not be neglected compared to the speed of light, the disturbance is even stronger.

On the other hand, in non-holonomic space the argument of trigonometric functions in our equations contains *a sum of two terms*, one of which stems from interaction of particle's charge with strength of magnetic field, while another is a result of rotation of the space itself, which depends neither from electric charge of particle, nor from presence of magnetic field. This allows considering two special cases of motion of particle in non-holonomic space.

In first case, when particle is electrically neutral or magnetic field is absent, its motion will be the same as that under action of magnetic component of Lorentz force, save that this motion will be caused by angular velocity of rotation of space 2Ω, comparable to $\omega = \frac{eH}{mc}$.

How real this case may be? To answer this question we need at least an approximate assessment of the ratio between velocity of space rotation Ω and magnetic field H in a special case. The best example may be an atom, because on the scales of electronic orbits electromagnetic interaction is a few orders of magnitude stronger than the others and beside, orbital velocities of electrons are relatively high.

Such assessment can be made proceeding from second case of special motion of charged particle in uniform stationary magnetic field, when

$$\frac{eH}{mc} = -2\Omega\,, \tag{3.324}$$

is true and hence the argument of trigonometric functions in equations of motion becomes zero.

We consider the observer's frame of reference, whose space of reference is attributed to atomic nucleus. Then the ratio in question (in CGSE and Gaussian systems of units) for electron in atom is

$$\frac{\Omega}{H} = -\frac{e}{2m_e c} = -\frac{4.8\cdot 10^{-10}}{18.2\cdot 10^{-28}\; 3.0\cdot 10^{10}} = -\,8.8\cdot 10^{6}, \tag{3.325}$$

where their "minus" sign stems from the fact that Ω and H in (3.324) are oppositely directed. Hence field of non-holonomity if the nucleus is the basic factor that affects motion of orbiting electron and is dominant compared to magnetic component of the Lorentz force.

Now we are going to solve equations of motion of mirror-world particle in stationary uniform magnetic field we obtained in non-holonomic (self-rotating) space (3.305), which match equations of motion in absence of field of non-holonomity (rotation) of space by Landau and Lifshitz [1]

$$\ddot{x} = -\omega\dot{y}\,, \qquad \ddot{y} = \omega\dot{x}\,, \qquad \ddot{z} = 0\,. \tag{3.326}$$

The solution of the third equation of motion (along z axis) is a simpler integral $z{=}\dot{z}_{(0)}\tau{+}z_{(0)}$.

Equations of motion along x and y axis are similar to those for our-world particle, save for the fact that the argument of trigonometric functions has ω instead of $\omega{+}2\Omega$

$$\dot{x} = \dot{y}_{(0)} \sin\omega\tau - \frac{\ddot{y}_{(0)}}{\omega}\cos\omega\tau + \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}\,, \tag{3.327}$$

$$\dot{y} = \dot{y}_{(0)} \cos \omega\tau + \frac{\ddot{y}_{(0)}}{\omega} \sin \omega\tau \,. \tag{3.328}$$

Hence the formulas for components of velocity of mirror-world particle $\dot{x}$ and $\dot{y}$ are equations of harmonic oscillations at frequency $\omega = \frac{eH}{mc}$.

Consequently, their solutions, i. e. formulas for coordinates of mirror-world particle in stationary uniform magnetic field are

$$x = \frac{1}{\omega}\left(\frac{\ddot{y}_{(0)}}{\omega} \sin \omega\tau - \dot{y}_{(0)} \cos \omega\tau\right) + \left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}\right)\tau + C_4 \,, \tag{3.329}$$

$$y = \frac{1}{\omega}\left(\dot{y}_{(0)} \sin \omega\tau + \frac{\ddot{y}_{(0)}}{\omega} \cos \omega\tau\right) + C_5 \,, \tag{3.330}$$

where integration constants are

$$C_4 = x_{(0)} + \frac{\dot{y}_{(0)}}{\omega} \,, \qquad C_5 = y_{(0)} + \frac{\ddot{y}_{(0)}}{\omega^2} \,. \tag{3.331}$$

As we have already mentioned, integral of live forces in stationary magnetic field (3.289) implies constant relativistic mass of particle and hence constant square of its observable velocity. Then putting solutions for velocities of mirror-world particle $\dot{x}$, $\dot{y}$, $\dot{z}$ in the power of two and adding them up we obtain that

$$\mathrm{v}^2 = \dot{x}^2_{(0)} + \dot{y}^2_{(0)} + \dot{z}^2_{(0)} + 2\left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}\right)\left(\frac{\ddot{y}_{(0)}}{\omega} + \dot{y}_{(0)} \sin \omega\tau - \frac{\ddot{y}_{(0)}}{\omega} \cos \omega\tau\right) \tag{3.332}$$

is constant $\mathrm{v}^2 = const$ provided that

$$\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega} = 0 \,. \tag{3.333}$$

From the formula for x (3.329) we see that particle performs purely harmonic oscillations along x provided the same condition (3.333) is true. Taking this into account, putting in the power of two and adding up x (3.329) and y (3.330) for mirror-world particle in stationary uniform magnetic field, we obtain its trajectory within xy plane

$$\begin{aligned} x^2 + y^2 = \frac{1}{\omega^2}\left(\dot{y}^2_{(0)} + \frac{\ddot{y}^2_{(0)}}{\omega^2}\right) - \frac{2C_4}{\omega}\left(\dot{y}_{(0)} \cos \omega\tau + \frac{\ddot{y}_{(0)}}{\omega} \sin \omega\tau\right) + \\ + \left(\dot{y}_{(0)} \sin \omega\tau + \frac{\ddot{y}_{(0)}}{\omega} \cos \omega\tau\right)\frac{2C_5}{\omega} + C_4^2 + C_5^2 \,, \end{aligned} \tag{3.334}$$

which only differs from our-world particle trajectory (3.318) by $\omega + 2\Omega$ replaced with ω and by values of integration constants (3.331). Therefore charged mirror-world particle, with zero initial velocity along z (the direction of magnetic strength), moves along *an ellipse* within xy plane.

Once we assume $\ddot{y}_{(0)}$, as well as constants C_4 and C_5 to be zeroes, the solutions become simpler

$$x = -\frac{1}{\omega}\dot{y}_{(0)} \cos \omega\tau \,, \qquad y = \frac{1}{\omega}\dot{y}_{(0)} \sin \omega\tau \,. \tag{3.335}$$

In such simplified case mirror-world particle that rests in respect to the field direction makes *a circle* within xy plane

$$x^2 + y^2 = \frac{\dot{y}^2_{(0)}}{\omega^2} \tag{3.336}$$

with radius $r = \frac{\dot{y}_{(0)}}{\omega} = \frac{mc}{eH}\dot{y}_{(0)}$. Consequently, if the initial velocity of a mirror-world particle along the direction of magnetic field (z axis) is not zero, the particle moves along a *spiral line* around the direction of magnetic field. Hence motion of charged mirror-world particle in stationary uniform magnetic field is the same as that of our-world particle in absence of non-holonomity of space.

B Magnetic field is orthogonal to non-holonomity field

We are going to consider the case of pseudovector of magnetic field strength H^{*i} is orthogonal to pseudovector of non-holonomity field of space $\Omega^{*i}=\frac{1}{2}\varepsilon^{ikm}A_{km}$. Then the first equation from Group I of Maxwell equations for stationary magnetic field (3.292) implies that density of charges is zero $\rho=0$.

We assume that strength of magnetic field is directed along y (only component $H^{*2}=H$ is not zero), while non-holonomity field is still directed along z (only component $\Omega^{*3}=\Omega$ is not zero). We also assume that magnetic field is stationary and uniform. Hence the only non-zero component of magnetic strength is

$$H^{*2}=H_{31}=\frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_3}{\partial x}-\frac{\partial \mathrm{v}_1}{\partial z}\right)=const\,. \tag{3.337}$$

Then in weak non-holonomity field equations of motion of our-world particle will be

$$\ddot{x}+2\Omega\dot{y}=\frac{eH}{mc}\dot{z}\,,\quad \ddot{y}-2\Omega\dot{x}=0\,,\quad \ddot{z}=-\frac{eH}{mc}\dot{x}\,, \tag{3.338}$$

or, denoting $\omega=\frac{eH}{mc}$,

$$\ddot{x}+2\Omega\dot{y}=\omega\dot{z}\,,\qquad \ddot{y}-2\Omega\dot{x}=0\,,\qquad \ddot{z}=-\omega\dot{x}\,. \tag{3.339}$$

Derivating the first equation to τ and substituting $\ddot{y}$ and $\ddot{z}$ into it from the second and the third equations we have

$$\dddot{x}+\left(4\Omega^2+\omega^2\right)\dot{x}=0\,. \tag{3.340}$$

Replacing variables $\dot{x}=p$ we arrive to equation of oscillations

$$\ddot{p}+\widetilde{\omega}^2p=0\,,\qquad \widetilde{\omega}=\sqrt{4\Omega^2+\omega^2}=\sqrt{4\Omega^2+\left(\frac{eH}{mc}\right)^2}\,, \tag{3.341}$$

which solves as

$$p=C_1\cos\widetilde{\omega}\tau+C_2\sin\widetilde{\omega}\tau\,, \tag{3.342}$$

where $C_1=\dot{x}_{(0)}$ and $C_2=\frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}$ are integration constants. Integrating $\dot{x}=p$ with respect to τ we obtain the expression for x as

$$x=\frac{\dot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau-\frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\cos\widetilde{\omega}\tau+x_{(0)}+\frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\,, \tag{3.343}$$

where $x_{(0)}+\frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}=C_3$ is integration constant.

Substituting $\dot{x}=p$ (3.342) into equations of motion in respect to y and z (3.339) after integration we obtain

$$\dot{y}=\frac{2\Omega}{\widetilde{\omega}}\dot{x}_{(0)}\sin\widetilde{\omega}\tau-\frac{2\Omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\cos\widetilde{\omega}\tau+\dot{y}_{(0)}+\frac{2\Omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\,, \tag{3.344}$$

$$\dot{z}=\frac{\omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\cos\widetilde{\omega}\tau-\frac{\omega}{\widetilde{\omega}}\dot{x}_{(0)}\sin\widetilde{\omega}\tau+\dot{z}_{(0)}-\frac{\omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\,, \tag{3.345}$$

where $\dot{y}_{(0)}+\frac{2\Omega\ddot{x}_{(0)}}{\widetilde{\omega}^2}=C_4$ and $\dot{z}_{(0)}-\frac{\omega\ddot{x}_{(0)}}{\widetilde{\omega}^2}=C_5$ are new integration constants. Then integrating these equations (3.344, 3.345) with respect to τ we obtain final formulas for y and z

$$y=-\frac{2\Omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau+\frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right)+\dot{y}_{(0)}\tau+\frac{2\Omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\tau+y_{(0)}+\frac{2\Omega}{\widetilde{\omega}^2}\dot{x}_{(0)}\,, \tag{3.346}$$

$$z=\frac{\omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau+\frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right)+\dot{z}_{(0)}\tau-\frac{\omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\tau+z_{(0)}-\frac{\omega}{\widetilde{\omega}^2}\dot{x}_{(0)}\,, \tag{3.347}$$

where $y_{(0)}+\frac{2\Omega\dot{x}_{(0)}}{\widetilde{\omega}^2}=C_6$ and $z_{(0)}-\frac{\omega\dot{x}_{(0)}}{\widetilde{\omega}^2}=C_7$.

Provided that Ω=0, i. e. rotation of space is absent, and that some integration constants are zeroes, the above equations fully match well-known formulas of relativistic electrodynamics for the case of stationary magnetic field directed along z axis

$$x = \frac{\dot{x}_{(0)}}{\omega}\sin\widetilde{\omega}\tau\,, \quad y = y_{(0)} + \dot{y}_{(0)}\tau\,, \quad z = \frac{\dot{x}_{(0)}}{\omega}\cos\widetilde{\omega}\tau\,. \tag{3.348}$$

Because integral of live forces implies that square of observable velocity of particle in stationary magnetic field is constant, we can calculate $\mathrm{v}^2=\dot{x}^2+\dot{y}^2+\dot{z}^2$. Substituting into here the obtained formulas for velocity components, we obtain

$$\mathrm{v}^2 = \dot{x}_{(0)}^2 + \dot{y}_{(0)}^2 + \dot{z}_{(0)}^2 + \frac{2}{\widetilde{\omega}}\left(\ddot{x}_{(0)} + 2\Omega\dot{y}_{(0)} - \omega\dot{z}_{(0)}\right)\left(\frac{\ddot{x}_{(0)}}{\widetilde{\omega}} + \dot{x}_{(0)}\sin\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\cos\widetilde{\omega}\tau\right), \tag{3.349}$$

i. e. $\mathrm{v}^2=const$ provided that

$$\ddot{x}_{(0)} + 2\Omega\dot{y}_{(0)} - \omega\dot{z}_{(0)} = 0\,. \tag{3.350}$$

Three-dimensional (spatial) trajectory of particle in stationary uniform magnetic field, orthogonal to field of non-holonomity, can be found calculating $x^2+y^2+z^2$

$$\begin{gathered} x^2 + y^2 + z^2 = \frac{1}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}^2 + \frac{\ddot{x}_{(0)}^2}{\widetilde{\omega}^2}\right) + C_3^2 + C_6^2 + C_7^2 + \left(C_4^2 + C_5^2\right)\tau^2 + \\ + 2\left(C_4C_6 + C_5C_7\right)\tau + \left[\left(\omega C_7 - 2\Omega C_6\right) + 2\left(\omega C_5 - 2\Omega C_6\right)\tau\right]\times \\ \times\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right)\frac{1}{\widetilde{\omega}^2} + \frac{2C_3}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right), \end{gathered} \tag{3.351}$$

which includes a linear and a quadratic (to time) terms, as well as a parametric and two harmonic terms. In a specific case, if we assume integration constants to be zeroes, the obtained formula (3.351) takes the form of a regular equation of *a sphere*

$$x^2 + y^2 + z^2 = \frac{1}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}^2 + \frac{\ddot{x}_{(0)}^2}{\widetilde{\omega}^2}\right), \tag{3.352}$$

which radius is

$$r = \frac{1}{\widetilde{\omega}}\sqrt{\dot{x}_{(0)}^2 + \frac{\ddot{x}_{(0)}^2}{\widetilde{\omega}^2}}\,, \tag{3.353}$$

where $\widetilde{\omega}=\sqrt{4\Omega^2+\omega^2}=\sqrt{4\Omega^2+\left(\frac{eH}{mc}\right)^2}$. Therefore charged our-world particle in stationary uniform magnetic field, orthogonal to field of non-holonomity, moves on a surface of *a sphere* which radius depends upon magnetic strength and velocity of space rotation.

In a specific case, when field of non-holonomity is absent and the initial acceleration is zero, our equation of trajectory simplifies significantly becoming an equation of sphere

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2}\dot{x}_{(0)}^2\,, \qquad r = \frac{1}{\omega}\dot{x}_{(0)} = \frac{mc}{eH}\dot{x}_{(0)} \tag{3.354}$$

with radius that depends only upon interaction of particle's charge with magnetic field — the result well-known in electrodynamics (see Section 21 in *The Classical Theory of Fields*).

For mirror-world particle that moves in stationary uniform magnetic field, orthogonal to field of non-holonomity, the equations of motion are

$$\ddot{x} = \frac{eH}{mc}\dot{z}, \qquad \ddot{y} = 0, \qquad \ddot{z} = -\frac{eH}{mc}\dot{x}\,. \tag{3.355}$$

These are only different from equations for our-world particle (3.338) by absence of the terms that include space rotation velocity Ω. In practice that means that in the mirror world the solutions simply do not depend upon rotation of space and match the solutions in our world provided field of non-holonomity is absent.

3.13 Motion of charged particle in stationary electromagnetic field

In this Section we are going to focus on motion of charged particle under action of both magnetic and electric components of stationary electromagnetic field.

As a "background" we will consider non-holonomic space that rotates around z axis at a constant angular velocity $\Omega_{12}=-\Omega_{21}=\Omega$, i. e. the space with metric as of (3.275). In such metric space $F_i=0$ and $D_{ik}=0$. We will solve the problem assuming that the field of non-holonomity is *weak* and hence the three-dimensional observable space has Euclidean metric. Here Maxwell equations for stationary field (3.215, 3.216) are

$$\left.\begin{aligned} &\Omega_{*m}H^{*m} = -2\pi c\rho \\ &\varepsilon^{ikm}\,\nabla_k\left(H_{*m}\sqrt{h}\right) = \frac{4\pi}{c}j^i\sqrt{h} = 0 \end{aligned}\right\} \quad \text{Group I,} \tag{3.356}$$

$$\left.\begin{aligned} &\Omega_{*m}E^{m} = 0 \\ &\varepsilon^{ikm}\,\nabla_k\left(E_{m}\sqrt{h}\right) = 0 \end{aligned}\right\} \quad \text{Group II,} \tag{3.357}$$

because the condition of observable uniformity of field is equality to zero of its chronometrically invariant derivative [8, 10], while in the specific case under consideration chronometrically invariant Christoffel symbols equal to zero (metric is Galilean one) and chronometrically invariant derivative is the same as regular one. Hence Maxwell equations imply that the following conditions will be true here:

- field of non-holonomity and electric field are orthogonal to each other ($\Omega_{*m}E^m=0$);
- field of non-holonomity and magnetic field are orthogonal to each other, charge density $\rho=0$;
- current is absent ($j^i=0$).

The last condition implies that presence of current or currents $j^i\neq 0$ stems from non-uniformity of magnetic field.

Given that field of non-holonomity is orthogonal to electric field we can consider motion of particle in two cases of mutual orientation of fields: (1) $\vec{H}\perp\vec{E}$ and $\vec{H}\parallel\vec{\Omega}$; (2) $\vec{H}\parallel\vec{E}$ and $\vec{H}\perp\vec{\Omega}$.

In either case we assume that vector of electric strength is co-directed with x axis. In the background metric (3.275) pseudovector of space's rotation is co-directed with z. Hence in the first case magnetic strength is co-directed with z, while in the second case it is co-directed with x.

Equations of motion of charged particle in stationary electromagnetic field in case of vector of electromagnetic strength co-directed with x are as follows. For our-world particle

$$\frac{dm}{d\tau} = -\frac{eE_1}{c^2}\frac{dx}{d\tau}\,, \tag{3.358}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2mA_{k\cdot}^{\cdot i}\mathrm{v}^k = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right), \tag{3.359}$$

and for mirror-world particle

$$\frac{dm}{d\tau} = \frac{eE_1}{c^2}\frac{dx}{d\tau}\,, \tag{3.360}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) = -e\left(E^i + \frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right). \tag{3.361}$$

As we did before, we will consider the case of a particle *repulsed* by filed. Then components of strength of electric field E_i, co-directed with x, are (in Galilean frame of reference covariant and contravariant indices of tensor values are the same)

$$E_1 = E_x = \frac{\partial\varphi}{\partial x} = const = -E\,, \qquad E_2 = E_3 = 0\,. \tag{3.362}$$

Integration of theorem of live forces gives integral of live forces for our world and the mirror world (respectively)

$$m=\frac{eE}{c^2}x+B\,,\qquad m=-\frac{eE}{c^2}x+\widetilde{B}\,. \tag{3.363}$$

Here B is our-world integration constant and $\widetilde{B}$ is mirror-world integration constant. Calculated from the initial conditions at the moment $\tau=0$ these are

$$B=m_{(0)}-\frac{eE}{c^2}x_{(0)}\,,\qquad \widetilde{B}=m_{(0)}+\frac{eE}{c^2}x_{(0)}\,, \tag{3.364}$$

where $m_{(0)}$ is relativistic mass of particle and $x_{(0)}$ is displacement of particle at the initial moment of time $\tau=0$.

From the obtained integrals of live forces (3.363) we see that the differences between the three case under study, due to different orientation of magnetic strength $\vec{H}$, will only reveal themselves in vector equations of motion, while scalar equations (3.358, 3.360) and their solutions (3.363) will be the same.

Note that vector $\vec{E}$ can be also directed along y axis, but can not be directed along z, because in space with such metrics co-directed with z is field of non-holonomity $\vec{\Omega}$ while Group II of Maxwell equations require strength $\vec{E}$ to be orthogonal to $\vec{\Omega}$.

Now taking into account the results of integration of theorem of live forces (3.363) we will write down by-component notations of vector equations of motion of particle in stationary uniform electric and magnetic fields for all three cases under study.

1. We assume that $\vec{H}\perp\vec{E}$ and $\vec{H}\parallel\vec{\Omega}$, i.e. the vector magnetic strength $\vec{H}$ is directed along z (parallel to the field of non-holonomity of space). Then out of all components of the vector of magnetic strength the only non-zero one will be

$$H^{*3}=H_{12}=\frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_1}{\partial y}-\frac{\partial \mathrm{v}_2}{\partial x}\right)+\frac{2\varphi}{c}A_{12}=const=H\,. \tag{3.365}$$

Consequently, vector equations of motion for our-world particle in by-component notation will be

$$\begin{gathered}\frac{eE}{c^2}\dot{x}^2+\left(B+\frac{eE}{c^2}x\right)(\ddot{x}+2\Omega\dot{y})=eE-\frac{eH}{c}\dot{y}\,,\\ \frac{eE}{c^2}\dot{x}\dot{y}+\left(B+\frac{eE}{c^2}x\right)(\ddot{y}-2\Omega\dot{x})=\frac{eH}{c}\dot{x}\,,\\ \frac{eE}{c^2}\dot{x}\dot{z}+\left(B+\frac{eE}{c^2}x\right)\ddot{z}=0\,,\end{gathered} \tag{3.366}$$

and for mirror-world particle

$$\begin{gathered}\frac{eE}{c^2}\dot{x}^2+\left(\widetilde{B}-\frac{eE}{c^2}x\right)\ddot{x}=eE-\frac{eH}{c}\dot{y}\,,\\ \frac{eE}{c^2}\dot{x}\dot{y}+\left(\widetilde{B}-\frac{eE}{c^2}x\right)\ddot{y}=\frac{eH}{c}\dot{x}\,,\\ \frac{eE}{c^2}\dot{x}\dot{z}+\left(\widetilde{B}-\frac{eE}{c^2}x\right)\ddot{z}=0\,.\end{gathered} \tag{3.367}$$

Besides, Group I of Maxwell equations require that in the case under study, when magnetic field and non-holonomity field are parallel, the condition should be true

$$\Omega_{*3}H^{*3}=-2\pi c\rho\,, \tag{3.368}$$

where $\Omega_{*3}=\Omega=const$ and $H^{*3}=H=const$. Hence this mutual orientation of non-holonomity and magnetic field is only possible in case density of electric charge as a field source $\rho\neq0$.

2. $\overrightarrow{H} \parallel \overrightarrow{E}$, $\overrightarrow{H} \perp \overrightarrow{\Omega}$, and $\overrightarrow{E} \perp \overrightarrow{\Omega}$, i. e. magnetic and electric strengths are co-directed with x, while non-holonomity field is still directed along z. Here out of all components of magnetic strength only the first component will be not zero

$$H^{*1} = H_{23} = \frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_2}{\partial z} - \frac{\partial \mathrm{v}_3}{\partial y}\right) = const = H\,, \tag{3.369}$$

while vector equations of motion for our-world particle in by component notation become

$$\left.\begin{aligned} &\frac{eE}{c^2}\dot{x}^2 + \left(B + \frac{eE}{c^2}x\right)(\ddot{x} + 2\Omega\dot{y}) = eE\,,\\ &\frac{eE}{c^2}\dot{x}\dot{y} + \left(B + \frac{eE}{c^2}x\right)(\ddot{y} - 2\Omega\dot{x}) = -\frac{eH}{c}\dot{z}\,,\\ &\frac{eE}{c^2}\dot{x}\dot{z} + \left(B + \frac{eE}{c^2}x\right)\ddot{z} = \frac{eH}{c}\dot{y}\,, \end{aligned}\right. \tag{3.370}$$

and for mirror-world particle

$$\left.\begin{aligned} &\frac{eE}{c^2}\dot{x}^2 + \left(\widetilde{B} - \frac{eE}{c^2}x\right)\ddot{x} = eE\,,\\ &\frac{eE}{c^2}\dot{x}\dot{y} + \left(\widetilde{B} - \frac{eE}{c^2}x\right)\ddot{y} = -\frac{eH}{c}\dot{z}\,,\\ &\frac{eE}{c^2}\dot{x}\dot{z} + \left(\widetilde{B} - \frac{eE}{c^2}x\right)\ddot{z} = \frac{eH}{c}\dot{y}\,, \end{aligned}\right. \tag{3.371}$$

Now that we have equations of motion of charged particle for all three cases of mutual orientation of stationary fields (i. e. electric field, magnetic field and non-holonomity field) we can turn to solving them.

A Magnetic field is orthogonal to electric field and is parallel to non-holonomity field

We are going to solve vector equations of motion of charged particle (3.366, 3.367) in non-relativistic approximation, i. e. assuming absolute value of its observable velocity negligible compared to the speed of light. Hence we can also assume particle's mass at the initial moment of time equal to its rest mass

$$m_{(0)} = \frac{m_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \cong m_0\,. \tag{3.372}$$

We assume electric strength E to be negligible as well, thus the term $\frac{eEx}{c^2}$ can be truncated. Given that, vector equations of motion of charged particle will transform as follows. For our-world particle

$$m_0\,(\ddot{x}+2\Omega\dot{y}) = eE - \frac{eH}{c}\dot{y}\,,\quad m_0\,(\ddot{y}-2\Omega\dot{x}) = \frac{eH}{c}\dot{x}\,,\quad m_0\ddot{z} = 0\,, \tag{3.373}$$

and for mirror-world particle

$$m_0\ddot{x} = eE - \frac{eH}{c}\dot{y}\,,\quad m_0\ddot{y} = \frac{eH}{c}\dot{x}\,,\quad m_0\ddot{z} = 0\,. \tag{3.374}$$

These equations match those obtained in Section 22 in *The Classical Theory of Fields* [1] in case field of rotation of space Ω=0 and strength of electric field is co-directed with x.

The obtained equations for mirror-world particle are a specific case of our-world equations at Ω=0. Therefore we can only integrate our-world equations, while the mirror-world solutions are obtained automatically by assuming Ω=0. Integrating equation of motion along z we arrive to

$$z = \dot{z}_{(0)}\tau + z_{(0)}\,. \tag{3.375}$$

Integrating the second one (along y) we arrive to

$$\dot{y} = \left(2\Omega + \frac{eH}{m_0 c}\right)x + C_1\,, \tag{3.376}$$

where integration constant is $C_1 = \dot{y}_{(0)} - \left(2\Omega + \frac{eH}{m_0 c}\right) x_{(0)}$.

Substituting $\dot{y}$ into the first equation (3.373) we obtain second-order differential equation in respect to x

$$\ddot{x} + \omega^2 x = \frac{eE}{m_0} + \omega^2 x_{(0)} - \omega \dot{y}_{(0)}\,, \tag{3.377}$$

where $\omega = 2\Omega + \frac{eH}{m_0 c}$. Introducing a new variable

$$u = x - \frac{A}{\omega^2}\,, \qquad A = \frac{eE}{m_0} + \omega^2 x_{(0)} - \omega \dot{y}_{(0)}\,, \tag{3.378}$$

we obtain equation of harmonic oscillations

$$\ddot{u} + \omega^2 u = 0\,, \tag{3.379}$$

which solves as

$$u = C_2 \cos\omega\tau + C_3 \sin\omega\tau\,, \tag{3.380}$$

where integration constants are $C_2 = u_{(0)}$, $C_3 = \frac{\dot{u}_{(0)}}{\omega}$. Returning to variable x by reverse substitution of variables we finally obtain

$$x = \frac{1}{\omega}\left(\dot{y}_{(0)} - \frac{eE}{m_0\omega}\right)\cos\omega\tau + \frac{\dot{x}_{(0)}}{\omega}\sin\omega\tau + \frac{eE}{m_0\omega^2} + x_{(0)} - \frac{\dot{y}_{(0)}}{\omega}\,. \tag{3.381}$$

Substituting the formula into the obtained equation for $\dot{y}$ (3.376), after integration we arrive to formula for y

$$y = \frac{1}{\omega}\left(\dot{y}_{(0)} - \frac{eE}{m_0\omega}\right)\sin\omega\tau - \frac{\dot{x}_{(0)}}{\omega}\cos\omega\tau + \frac{eE}{m_0\omega^2} + y_{(0)} + \frac{\dot{x}_{(0)}}{\omega}\,. \tag{3.382}$$

Vector mirror-world equations have the same solutions, but because for them $\Omega=0$, the frequency equals $\omega = \frac{eH}{m_0 c}$.

Energies of our-world and mirror-world particles are $E = mc^2$ and $E = -mc^2$, respectively.

Three-dimensional impulse of our-world charged particle in stationary uniform electromagnetic field (when magnetic field is orthogonal to electric field and is parallel to non-holonomity field) takes the form

$$\begin{aligned} p^1 &= m_0\dot{x} = \left(\frac{eE}{\omega} - m_0\dot{y}_{(0)}\right)\sin\omega\tau + m_0\dot{x}_{(0)}\cos\omega\tau\,, \\ p^2 &= m_0\dot{y} = \left(\frac{2\Omega m_0}{\omega} + \frac{eH}{\omega c}\right)\left(\frac{eE}{m_0\omega} - \dot{y}_{(0)}\right) + m_0\dot{y}_{(0)} + \\ &\quad + \left(\frac{2\Omega m_0}{\omega} + \frac{eH}{\omega c}\right)\left[\left(\dot{y}_{(0)} - \frac{eE}{m_0\omega}\right)\cos\omega\tau + \dot{x}_{(0)}\sin\omega\tau\right], \\ p^3 &= m_0\dot{z} = m_0\dot{z}_{(0)}\,. \end{aligned} \tag{3.383}$$

From here we see that impulse of our-world charged particle in the given configuration performs harmonic oscillations along x and y, while along z it is a linear function of observable time τ (if initial velocity $\dot{z}\neq 0$). Within xy plane the oscillation frequency is $\omega = 2\Omega + \frac{eH}{m_0 c}$.

In the mirror world given this configuration of electric field, magnetic field and non-holonomity field we have, respectively

$$\begin{aligned} p^1 &= \left(\frac{eE}{\omega} - m_0\dot{y}_{(0)}\right)\sin\omega\tau + m_0\dot{x}_{(0)}\cos\omega\tau\,, \\ p^2 &= \frac{eE}{\omega} + m_0\left[\left(\dot{y}_{(0)} - \frac{eE}{m_0\omega}\right)\cos\omega\tau + \dot{x}_{(0)}\sin\omega\tau\right], \\ p^3 &= m_0\dot{z}_{(0)}\,, \end{aligned} \tag{3.384}$$

where contrasted to our world the frequency is $\omega = \frac{eH}{m_0 c}$.

We should note obtaining exact general solutions of equations of motion of charged particle in electric field and magnetic field at the same time is rather problematic, because elliptic integrals have to be solved in the process. Possibly in future in case of any practical necessity the general solutions will be obtained on computers, but this evidently stays beyond the goal of this book. Presumably Landau and Lifshitz faced a similar problem, because in Section 22 of *The Classical Theory of Fields* when considering a similar problem (but contrasted to this book, they used general covariant methods and did not account for non-holonomity of space) they obtained equations of motion and solved them assuming velocity to be non-relativistic and electric field weak $\frac{eEx}{c^2} \approx 0$.

B Magnetic field is parallel to electric field and is orthogonal to non-holonomity field

We are going to solve vector equations of motion of charged particle (3.370, 3.371) in the same approximation as we did in the first case. Then for our world and for the mirror world these will be, respectively

$$\ddot{x} + 2\Omega\dot{y} = \frac{eE}{m_0}\,, \quad \ddot{y} - 2\Omega\dot{x} = -\frac{eH}{m_0 c}\dot{z}\,, \quad \ddot{z} = \frac{eH}{m_0 c}\dot{y}\,, \tag{3.385}$$

$$\ddot{x} = \frac{eE}{m_0}\,, \qquad \ddot{y} = -\frac{eH}{m_0 c}\dot{z}\,, \qquad \ddot{z} = \frac{eH}{m_0 c}\dot{y}\,. \tag{3.386}$$

Integrating the first equation of motion in our world (3.385) we obtain

$$\dot{x} = \frac{eE}{m_0}\tau - 2\Omega y + C_1\,, \qquad C_1 = const = \dot{x}_{(0)} + 2\Omega y_{(0)}\,. \tag{3.387}$$

Integrating the third equation (along z) we have

$$\dot{z} = \frac{eH}{m_0 c}y + C_2\,, \qquad C_2 = const = \dot{z}_{(0)} - \frac{eH}{m_0 c}y_{(0)}\,. \tag{3.388}$$

Substituting the obtained formulas for $\dot{x}$ and $\dot{z}$ into the second equation of motion (3.385) we obtain linear differential 2nd order equation in respect to y

$$\ddot{y} + \left(4\Omega^2 + \frac{e^2H^2}{m_0^2c^2}\right)y = \frac{2\Omega eE}{m_0}\tau + 2\Omega C_1 - \frac{eH}{m_0 c}C_2\,. \tag{3.389}$$

We are going to solve it with method of replacement of variables. Introducing a new variable u

$$u = y + \frac{1}{\omega^2}\left(\frac{eH}{m_0 c}C_2 - 2\Omega C_1\right), \qquad \omega^2 = 4\Omega^2 + \frac{e^2H^2}{m_0^2c^2}\,, \tag{3.390}$$

we obtain equation of forced oscillations

$$\ddot{u} + \omega^2 u = \frac{2\Omega eE}{m_0}\tau\,, \tag{3.391}$$

which solution is sum of general solution of free oscillations equation

$$\ddot{u} + \omega^2 u = 0\,, \tag{3.392}$$

and of a partial solution of inhomogeneous equation that can be presented as

$$\tilde{u} = M\tau + N\,, \tag{3.393}$$

where $M=const$ and $N=const$. Derivating $\tilde{u}$ twice to τ and substituting the results into the inhomogeneous equation (3.391) and then equating the obtained coefficients for τ we obtain the linear coefficients

$$M = \frac{2\Omega eE}{m_0\omega^2}\,, \qquad N = 0\,. \tag{3.394}$$

Then the general solution of the initial inhomogeneous equation (3.391) becomes

$$u = C_3 \cos\omega\tau + C_4 \sin\omega\tau + \frac{2\Omega eE}{m_0\omega^2}\tau\,, \tag{3.395}$$

where integration constants can be obtained by substituting the initial conditions at τ=0 into the obtained formula. As a result we have C_3=$u_{(0)}$ and C_4=$\frac{\dot{u}_{(0)}}{\omega}$.

Returning to the old variable y (3.390) we find the final solution for this coordinate

$$\begin{aligned} y = \left[y_{(0)} + \frac{1}{\omega^2}\left(\frac{eH}{m_0 c}C_2 + 2\Omega C_1\right)\right]\cos\omega\tau + \\ + \frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau - \frac{1}{\omega^2}\left(\frac{eH}{m_0 c}C_2 + 2\Omega C_1\right) + \frac{2\Omega eE}{m_0\omega^2}\tau\,. \end{aligned} \tag{3.396}$$

Then substituting this formula into equations for $\dot{x}$ and $\dot{z}$ after integration we arrive to solutions for x and z

$$\begin{aligned} x = \frac{eE}{2m_0}\left(1 - \frac{4\Omega^2}{\omega^2}\right)\tau^2 - \frac{2\Omega}{\omega}\left(y_{(0)} + A\right)\sin\omega\tau + \\ + \frac{2\Omega\dot{y}_{(0)}}{\omega}\cos\omega\tau + \left(C_1 + 2\Omega A\right)\tau + C_5\,, \end{aligned} \tag{3.397}$$

$$z = \frac{eH}{m_0 c\omega}\left[\left(y_{(0)} + A\right)\sin\omega\tau - \frac{\dot{y}_{(0)}}{\omega}\cos\omega\tau\right] - \left(\frac{eH}{m_0 c}A - C_2\right)\tau + C_6\,, \tag{3.398}$$

where a convenience notation was introduced

$$A = \frac{1}{\omega^2}\left(\frac{eH}{m_0 c}C_2 - 2\Omega C_1\right), \tag{3.399}$$

while the new integration constants are

$$C_5 = x_0 - \frac{2\Omega\dot{y}_{(0)}}{\omega}\,, \qquad C_6 = z_{(0)} + \frac{eH\dot{y}_{(0)}}{m_0 c\omega^2}\,. \tag{3.400}$$

If we assume Ω=0, then from coordinates of our-world charged particle (3.396–3.398) we immediately obtain solutions for mirror-world charged particle

$$x = \frac{eE}{2m_0}\tau^2 + \dot{x}_{(0)}\tau + x_{(0)}\,, \tag{3.401}$$

$$y = \frac{\dot{z}_{(0)}}{\omega}\cos\omega\tau + \frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau - \frac{\dot{z}_{(0)}}{\omega} + y_{(0)}\,, \tag{3.402}$$

$$z = \frac{\dot{z}_{(0)}}{\omega}\sin\omega\tau - \frac{\dot{y}_{(0)}}{\omega}\cos\omega\tau + \frac{\dot{y}_{(0)}}{\omega} + z_{(0)}\,. \tag{3.403}$$

Consequently, components of three-dimensional impulse of our-world charged particle in stationary uniform electromagnetic field (when magnetic field is parallel to electric field and is orthogonal to non-holonomity field) take the form

$$\begin{aligned} p^1 &= m_0\dot{x}_{(0)} + eE\left(1 - \frac{4\Omega^2}{\omega^2}\right)\tau - 2m_0\Omega\left[\frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau + \left(y_{(0)} + A\right)\cos\omega\tau - \frac{\dot{y}_{(0)}}{\omega} - A\right], \\ p^2 &= m_0\left[\dot{y}_{(0)}\cos\omega\tau - \omega\left(y_{(0)} + A\right)\sin\omega\tau\right] + \frac{2\Omega eE}{\omega^2}\,, \\ p^3 &= m_0\dot{z}_{(0)} + \frac{eH}{c}\left[\left(y_{(0)} + A\right)\cos\omega\tau + \frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau - A + \frac{2\Omega eE}{m_0\omega^2}\tau - y_{(0)}\right], \end{aligned} \tag{3.404}$$

where the frequency is $\omega = \sqrt{4\Omega^2 + \frac{e^2 H^2}{m_0^2 c}}$.

In the mirror world, given this configuration of electric field, magnetic field and non-holonomity field, components of three-dimensional impulse of charged particle are

$$\begin{aligned} p^1 &= m_0 \dot{x}_{(0)} + 2eE\tau \,, \\ p^2 &= m_0 \left(\dot{y}_{(0)} \cos\omega\tau - \dot{z}_{(0)} \sin\omega\tau\right) , \\ p^3 &= m_0 \left(\dot{z}_{(0)} \cos\omega\tau - \dot{y}_{(0)} \sin\omega\tau\right) , \end{aligned} \tag{3.405}$$

where contrasted to our world the frequency is $\omega = \frac{eH}{m_{(0)}c}$.

3.14 Conclusions

In fact the theory we have built in this Chapter can be more precisely referred to as *chronometrically invariant representation of electrodynamics in pseudo-Riemannian space*. Or, because the mathematical apparatus of physical observable values initially assumes pseudo-Riemannian space, simply as *chronometrically invariant electrodynamics* (CED). Here we presented only the basics of this theory:

- chronometrically invariant components of electromagnetic field tensor (Maxwell tensor);
- chronometrically invariant Maxwell equations;
- law of electric charge conservation in chronometrically invariant form;
- chronometrically invariant Lorentz condition;
- chronometrically invariant d'Alembert equations (wave propagation equations) for scalar potential and vector-potential of electromagnetic field;
- chronometrically invariant Lorentz force;
- tensor of energy-impulse of electromagnetic field and its chronometrically invariant components;
- chronometrically invariant equations of motion of mass-bearing charged particle;
- geometric structure of four-dimensional potential of electromagnetic field.

Evidently the whole scope of chronometrically invariant electrodynamics is much wider. In addition to what has been done we could obtain chronometrically invariant equations of motion of charge distributed in space or study motion of particle that bears its own electromagnetic emission, which interacts with the field or deduce equations of motion for particle that travels at an arbitrary angle to field (either for individual particle or a distributed charge), or tackle scores of other interesting problems.

Chapter 4

Particle with spin in pseudo-Riemannian space

4.1 Problem statement

In this Chapter we are going to obtain dynamic equation of motion of particle with inner mechanical momentum (*spin*). As we mentioned in Chapter 1, these are equations of parallel transfer of four-dimensional dynamic vector of particle Q^α, which is the sum of vectors

$$Q^\alpha = P^\alpha + S^\alpha, \tag{4.1}$$

where $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ is four-dimensional vector of impulse of particle and S^α is four-dimensional impulse that particle gains from its inner momentum (spin), which makes its motion non-geodesic. Therefore we will refer to S^α as four-dimensional *spin-impulse* of particle. Because we know all components of dynamic vector of impulse P^α, to define summary dynamic vector Q^α we only need to obtain components of spin-impulse S^α, which are functions of particle's spin.

Hence our first step will be defining particle's spin as geometric value in four-dimensional pseudo-Riemannian space. Then in Section 4.2 herein we are going to deduce four-dimensional impulse S^α that particle gains from its spin. In Section 4.3 our goal will be dynamic equations of motion of spin-particle in pseudo-Riemannian space and their chronometrically invariant (physical observable) projections onto time and space. Other Sections will focus on motion of elementary particles.

So, absolute value of spin is $\pm n\hbar$, measured in fractions of Planck constant, where n is so-called *spin quantum number*. As of today it is known [5] that for various types of particles this number may be $n = 0, 1/2, 1, 3/2, 2$. Alternating sign $\pm$ stands for possible right-wise or left-wise inner rotation of particle. Besides, Planck constant $\hbar$ has dimension of impulse momentum [$\text{g}\,\text{cm}^2\,\text{s}^{-1}$]. This alone hints that spin's tensor by its geometric structure should be similar to tensor of impulse momentum, i. e. should be an antisymmetric 2nd rank tensor. We are going to check if other source prove that.

The second Bohr postulate says that the length of electron orbit should comprise *integer* number of de Broglie wavelengths $\lambda = \frac{h}{p}$, which stands for electron according to wave-particle concept. In other words, length of electron orbit $2\pi r$ comprises k de Broglie wavelengths

$$2\pi r = k\lambda = k\frac{h}{p}\,, \tag{4.2}$$

where p is orbital impulse of electron. Taking into account that Planck constant with a bar is $\hbar = \frac{h}{2\pi}$, equation (4.2) should be

$$rp = k\hbar\,. \tag{4.3}$$

Because radius-vector of electron orbit r^i is orthogonal to vector of its orbital impulse p^k, this formula in tensor notation is vector product

$$\left[r^i; p^k\right] = k\hbar^{ik}\,. \tag{4.4}$$

From here we see that Planck constant deduced from the second Bohr postulate in tensor notation is antisymmetric 2nd rank tensor.

But this representation of Planck constant is linked to orbital model of atom — of the system more complicated than electron or any other elementary particle. Nevertheless spin, also defined by this constant, is an inner property of elementary particles themselves. Therefore according to the second Bohr postulate we have to consider geometric structure of Planck constant proceeding from another experimental relationship which is related to inner structure of electron only.

We have such opportunity thanks to classical experiments by O. Stern and W. Gerlach (1921). One of their results is that electron bears inner magnetic momentum $L_{(m)}$, which is proportional to its inner mechanical momentum (spin)

$$\frac{m_{\mathrm{e}}}{e} L_{(m)} = n\hbar\,, \tag{4.5}$$

where e is charge of electron, m_{e} is its mass and n is spin quantum number (for electron $n{=}1/2$). Magnetic momentum of a contour with area $S{=}\pi r^2$, which conducts current I, is $L_{(m)}{=}IS$. Current equals to charge e divided by its period of circulation $T{=}\frac{2\pi r}{u}$ along this contour

$$I = \frac{eu}{2\pi r}\,, \tag{4.6}$$

where u is linear velocity of charge circulation. Hence, in accordance with the definition $L(m){=}IS$, inner magnetic momentum of electron is

$$L_{(m)} = \frac{1}{2} eur\,, \tag{4.7}$$

or in tensor notation[19]

$$L^{ik}_{(m)} = \frac{1}{2} e \left[r^i ; u^k \right] = \frac{1}{2} \left[r^i ; p^k_{(m)} \right]\,, \tag{4.8}$$

where r^i is radius-vector of circulation of inner current of electron and u^k is vector of circulation velocity. From here we see that Planck constant, being calculated from inner magnetic momentum of electron (4.5), is also a vector product of two vectors, i. e. antisymmetric 2nd rank tensor

$$\frac{m_{\mathrm{e}}}{2e} \left[r^i ; p^k_{(m)} \right] = n\hbar^{ik}, \tag{4.9}$$

which proves similar conclusion based on second Bohr postulate.

Subsequently, considering inter-atomic and inter-electronic quantum relationships in four-dimensional pseudo-Riemannian space, we arrive to *four-dimensional antisymmetric Planck tensor* $\hbar^{\alpha\beta}$, which spatial components are three-dimensional values $\hbar^{ik}$

$$\hbar^{\alpha\beta} = \begin{pmatrix} \hbar^{00} & \hbar^{01} & \hbar^{02} & \hbar^{03} \\ \hbar^{10} & \hbar^{11} & \hbar^{12} & \hbar^{13} \\ \hbar^{20} & \hbar^{21} & \hbar^{22} & \hbar^{23} \\ \hbar^{30} & \hbar^{31} & \hbar^{32} & \hbar^{33} \end{pmatrix}. \tag{4.10}$$

This antisymmetric tensor $\hbar^{\alpha\beta}$ corresponds to dual *Planck pseudotensor* $\hbar^{*\alpha\beta}{=}\frac{1}{2}E^{\alpha\beta\mu\nu}\hbar_{\mu\nu}$. Subsequently, spin of particle in four-dimensional pseudo-Riemannian space is characterized by antisymmetric tensor $n\hbar^{\alpha\beta}$, or by its dual pseudotensor $n\hbar^{*\alpha\beta}$. Note that physical nature of spin does not matter here, it is enough that this fundamental property of particle is characterized by a tensor (or a pseudotensor) of a certain kind. Thanks to this approach we can solve a problem of motion of spin-particle without any preliminary assumption on their inner structure, i. e. using purely formal mathematical method.

Hence from geometric viewpoint Planck constant is antisymmetric 2nd rank tensor with dimensions of impulse momentum irrespective of through what values it was obtained: mechanical or electromagnetic ones. The latter also implies that Planck tensor does not characterize rotation of masses inside

[19] Equations (4.8) and (4.9) are given for Minkowski space, which is quite acceptable for the above experiments. In Riemannian space result of integration depends upon the integration path. Therefore radius-vector of a finite length is not defined in Riemannian space, because its length depends upon constantly varying direction.

atoms or any masses inside elementary particles, but stems from some fundamental quantum rotation of space itself and sets all "elementary" rotations in space irrespective of their nature.

Specific rotation of space is characterized by three-dimensional chronometrically invariant (observable) tensor A_{ik} (1.36), which results from lowering indices $A_{ik}{=}h_{im}h_{kn}A^{mn}$ in components A^{mn} of contravariant four-dimensional tensor

$$A^{\alpha\beta} = ch^{\alpha\mu}h^{\beta\nu}a_{\mu\nu}\,, \qquad a_{\mu\nu} = \frac{1}{2}\left(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu}\right). \tag{4.11}$$

In accompanying frame of reference ($b^i{=}0$) auxiliary value $a_{\mu\nu}$ are

$$a_{00} = 0\,, \quad a_{0i} = \frac{1}{2c^2}\left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad a_{ik} = \frac{1}{2c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right), \tag{4.12}$$

and components of four-dimensional tensor of space rotation become

$$A_{00} = 0\,, \quad A_{0i} = -A_{i0} = 0\,, \quad A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right). \tag{4.13}$$

In absence of gravitational field tensor of angular velocities of space rotation formulates with linear velocity of rotation v_i only, hence we denote it as $A_{\alpha\beta}{=}\Omega_{\alpha\beta}$

$$\Omega_{00} = 0\,, \qquad \Omega_{0i} = -\Omega_{i0} = 0\,, \qquad \Omega_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right). \tag{4.14}$$

On the other hand, according to wave-particle concept any particle corresponds to a wave with energy $E{=}mc^2{=}\hbar\omega$, where m is relativistic mass of particle and ω is its specific frequency. In other words, from geometric viewpoint any particle can be considered as wave defined within infinite proximity of geometric location of particle, which specific frequency depends upon certain distribution of angular velocities $\omega_{\alpha\beta}$, also defined within this proximity. Then the above quantum relationship in tensor notation becomes $mc^2{=}\hbar^{\alpha\beta}\omega_{\alpha\beta}$.

Because Planck tensor is antisymmetric, all of its diagonal elements are zeroes. Its space-time (mixed) components in accompanying frame of reference also should be zero similar to respective components of four-dimensional tensor of angular velocities of space rotation (4.14). Values of spatial (three-dimensional) components of Planck tensor, observable in experiments, are $\pm\hbar$ depending upon direction of rotation and make three-dimensional chronometrically invariant (observable) Planck tensor $\hbar^{ik}$. In case of left-wise rotation components $\hbar^{12}$, $\hbar^{23}$, $\hbar^{31}$ are positive, while components $\hbar^{13}$, $\hbar^{32}$, $\hbar^{21}$ are negative.

Then geometric structure of four-dimensional Planck tensor, represented as matrix, becomes

$$\hbar^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hbar & -\hbar \\ 0 & -\hbar & 0 & \hbar \\ 0 & \hbar & -\hbar & 0 \end{pmatrix}. \tag{4.15}$$

In case of right-wise rotation components $\hbar^{12}$, $\hbar^{23}$, $\hbar^{31}$ change sign to become negative, while components $\hbar^{13}$, $\hbar^{32}$, $\hbar^{21}$ become positive

$$\hbar^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\hbar & \hbar \\ 0 & \hbar & 0 & -\hbar \\ 0 & -\hbar & \hbar & 0 \end{pmatrix}. \tag{4.16}$$

The square of four-dimensional Planck tensor can be calculated as follows

$$\begin{aligned} \hbar_{\alpha\beta}\,\hbar^{\alpha\beta} = 2\hbar^2 \Big[&\left(g_{11}g_{22} - g_{12}^2\right) + \left(g_{11}g_{33} - g_{13}^2\right) + \left(g_{22}g_{33} - g_{23}^2\right) + \\ &+ 2\left(g_{12}g_{23} - g_{22}g_{13} - g_{12}g_{33} + g_{13}g_{23} - g_{11}g_{23} + g_{12}g_{13}\right)\Big]\,, \end{aligned} \tag{4.17}$$

and in Minkowski space, when frame of reference is Galilean one and metric is diagonal (2.70), it equals to $\hbar_{\alpha\beta}\hbar^{\alpha\beta}=6\hbar^2$. In pseudo-Riemannian space the value $\hbar_{\alpha\beta}\hbar^{\alpha\beta}$ can be deduced by substitution of dependency of three-dimensional components of fundamental metric tensor from observable three-dimensional metric tensor $h_{ik}=-g_{ik}+\frac{1}{c^2}v_i v_k$ and space rotation velocity into (4.17). Hence, though physical observable components $\hbar^{ik}$ of Planck tensor are constant (bear opposite signs for left and right-wise rotation), its square in general case depends from angular velocity of space rotation.

Now having components of Planck tensor defined, we can approach deduction of impulse that particle gains from its spin as well as dynamic equations of motion of spin-particle in pseudo-Riemannian space. This will be the focus of the next Section.

4.2 Spin-impulse of a particle in the equations of motion

The additional impulse S^α that particle gains from its spin can be obtained from considering *action* for spin-particle.

Action S for a particle that bears inner scalar field k, with which some external scalar field A interacts and thus displaces the particle by interval ds, is

$$S = \alpha_{(kA)} \int_a^b kA ds\,, \tag{4.18}$$

where $\alpha_{(kA)}$ is a scalar constant that characterizes properties of the particle in a given interaction and equalizes dimensions [1, 4]. If inner scalar field of particle k corresponds to external filed of 1st rank tensor A_α, then action to displace particle by that field is

$$S = \alpha_{(kA_\alpha)} \int_a^b kA_\alpha dx^\alpha. \tag{4.19}$$

In interaction of particle's inner scalar field k with external field of 2nd rank tensor $A_{\alpha\beta}$ this action of field to displace the particle is

$$S = \alpha_{(kA_{\alpha\beta})} \int_a^b kA_{\alpha\beta} dx^\alpha dx^\beta. \tag{4.20}$$

And so forth. For instance, if specific vector potential of particle k^α corresponds to external vector field A_α then action of this interaction to displace the particle is

$$S = \alpha_{(k^\alpha A_\alpha)} \int_a^b k^\alpha A_\alpha ds\,. \tag{4.21}$$

Besides, action can be represented as follows irrespective of nature of inner properties of particles and external field

$$S = \int_{t_1}^{t_2} L dt\,, \tag{4.22}$$

where L is so-called *Lagrange function*. Because the dimension of action S is [erg s=g cm^2 s^{-1}], then Lagrange function has dimension of energy [erg=g cm^2 s^{-2}]. And derivative from Lagrange function to three-dimensional coordinate velocity $u^i=\frac{dx^i}{dt}$ of particle

$$\frac{\partial L}{\partial u^i} = p_i \tag{4.23}$$

is covariant notation of its three-dimensional impulse $p^i=cP^i$ which can be used to restore full notation for four-dimensional impulse vector of particle P^α. Hence having action for the particle, having Lagrange function outlined and derivated to coordinate velocity of particle, we can calculate the additional impulse that particle gains from its spin.

As known, action to displace free particle in pseudo-Riemannian space is[20]

$$S=\int_a^b m_0 c ds\,. \tag{4.24}$$

In Galilean frame of reference in Minkowski space because non-diagonal terms of metric tensor are zeroes, space-time interval is

$$ds=\sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}=cdt\sqrt{1-\frac{u^2}{c^2}}\,, \tag{4.25}$$

and hence action becomes

$$S=\int_a^b m_0 c ds=\int_{t_1}^{t_2} m_0 c^2\sqrt{1-\frac{u^2}{c^2}}\,dt\,. \tag{4.26}$$

Therefore Lagrange function of free particle in Galilean frame of reference in Minkowski space is

$$L=m_0c^2\sqrt{1-\frac{u^2}{c^2}}\,. \tag{4.27}$$

Derivating it to coordinate velocity we arrive to covariant form of its three-dimensional impulse

$$p_i=\frac{\partial L}{\partial u^i}=m_0c^2\frac{\partial\sqrt{1-\frac{u^2}{c^2}}}{\partial u^i}=-\frac{m_0u_i}{\sqrt{1-\frac{u^2}{c^2}}}\,, \tag{4.28}$$

from which, having indices lifted, we arrive to four-dimensional impulse vector of free particle as

$$P^\alpha=\frac{1}{c}\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dx^\alpha}{dt}=m_0\frac{dx^\alpha}{ds}\,. \tag{4.29}$$

Because here in the final formula both multipliers, m_0 and $\frac{dx^\alpha}{ds}$, are general covariant values, i. e. do not depend upon choice of a particular frame of reference, this formula obtained in Galilean frame of reference is also true in another arbitrary frame of reference in four-dimensional pseudo-Riemannian space.

Now we are going to consider motion of particle that possesses inner structure, which in experiments reveals itself like its *spin*. Inner rotation (spin) of particle $n\hbar^{\alpha\beta}$ in four-dimensional pseudo-Riemannian space corresponds to external field $A_{\alpha\beta}$ of rotation of space. Therefore summary action of spin-particle is

$$S=\int_a^b\left(m_0cds+\alpha_{(s)}\hbar^{\alpha\beta}A_{\alpha\beta}ds\right), \tag{4.30}$$

where $\alpha_{(s)}$ [s cm^{-1}] is a scalar constant that characterizes particle in spin-interaction. Because constant of action may include only characteristics of particle's properties or physical constants, $\alpha_{(s)}$ is evidently spin quantum number n, which is function of inner properties of particle, divided by speed of light $\alpha_{(s)}=\frac{n}{c}$. Then action to displace particle, produced by interaction of the spin with field of non-holonomity of space $A_{\alpha\beta}$ is

$$S=\alpha_{(s)}\int_a^b\hbar^{\alpha\beta}A_{\alpha\beta}ds=\frac{n}{c}\int_a^b\hbar^{\alpha\beta}A_{\alpha\beta}ds\,. \tag{4.31}$$

[20]In *The Classical Theory of Fields* [1] Landau and Lifshitz put "minus" before action, while we always have "plus" before integral of action and Lagrange function. This is because the sign of action depends upon signature of pseudo-Riemannian space. Landau and Lifshitz use signature $(-+++)$, where time is imaginary, spatial coordinates are real and three-dimensional coordinate impulse is positive (see in the below). To the contrary, we stick to Zelmanov's [10] signature $(+---)$, where time is real and spatial coordinates are imaginary, because in this case three-dimensional *observable* impulse is positive.

A note should be taken that building four-dimensional impulse vector for spin-particle using the same method as for free particle is impossible. As known, we first obtained impulse of free particle in Galilean frame of reference in Minkowski space, where formula for ds presented with interval of coordinate time dt and substituted into action had simple form (4.25). It was shown that the obtained formula (4.29) due to its property of general covariance was true in any frame of reference in pseudo-Riemannian space. But as we can see from the formula of action for spin-particle, spin affects motion of particle in non-holonomic space $A_{\alpha\beta}\neq 0$ only, i. e. when non-diagonal terms g_{0i} of fundamental metric tensor are not zeroes. In Galilean frame of reference, by definition, all non-diagonal terms in metric tensor are zeroes, hence zeroes are velocity of space rotation $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ and non-holonomity tensor $A_{\alpha\beta}$. Therefore it is pointless to deduce formula for spin-particle impulse in Galilean frame of reference in Minkowski space (where it is *a priori* zero), instead we should deduce it directly in pseudo-Riemannian space.

In arbitrary accompanying frame of reference in pseudo-Riemannian space four-dimensional interval ds can be presented in the form

$$ds = cd\tau\sqrt{1-\frac{\mathrm{v}^2}{c^2}} = cdt\left(1-\frac{w+v_iu^i}{c^2}\right)\sqrt{1-\frac{u^2}{c^2\left(1-\frac{w+v_iu^i}{c^2}\right)^2}}\,, \tag{4.32}$$

where coordinate velocity of particle $u^i = \frac{dx^i}{dt}$ can be expressed with its observable velocity $\mathrm{v}^i = \frac{dx^i}{d\tau}$ as

$$\mathrm{v}^i = \frac{u^i}{1-\frac{w+v_iu^i}{c^2}}\,, \qquad \mathrm{v}^2 = \frac{h_{ik}u^iu^k}{\left(1-\frac{w+v_iu^i}{c^2}\right)^2}\,. \tag{4.33}$$

Then the additional action (4.31), produced by interaction of spin with field of non-holonomity of space, becomes

$$S = n\int_{t_1}^{t_2}\hbar^{\alpha\beta}A_{\alpha\beta}\sqrt{\left(1-\frac{w+v_iu^i}{c^2}\right)^2-\frac{u^2}{c^2}}\,dt\,. \tag{4.34}$$

Therefore Lagrange function for action produced by particle's spin is

$$L = n\hbar^{\alpha\beta}A_{\alpha\beta}\sqrt{\left(1-\frac{w+v_iu^i}{c^2}\right)^2-\frac{u^2}{c^2}}\,. \tag{4.35}$$

Now to deduce the additional impulse produced by spin we only have to derivate the Lagrange function (4.35) to coordinate velocity of particle. Taking into account that $\hbar^{\alpha\beta}$, being a tensor of inner rotation of particle, and $A_{\alpha\beta}$ (4.13), being a tensor of specific rotation of space, are not functions of velocity of particle, after derivation we obtain

$$p_i = \frac{\partial L}{\partial u^i} = n\hbar^{mn}A_{mn}\frac{\partial}{\partial u^i}\sqrt{\left(1-\frac{w+v_iu^i}{c^2}\right)^2-\frac{u^2}{c^2}} = -\frac{1}{c^2}\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(v_i+\mathrm{v}_i\right), \tag{4.36}$$

where $\mathrm{v}_i = h_{ik}\mathrm{v}^k$. We compare (4.36) with spatial covariant component $p_i = cP_i$ of four-dimensional impulse vector $P^\alpha = m_0\frac{dx^\alpha}{ds}$ of mass-bearing particle in pseudo-Riemannian space. For mass-bearing particle in our world that travels from past into future in respect to an usual observer (direct flow of time), three-dimensional covariant impulse equals

$$p_i = cP_i = cg_{i\alpha}P^\alpha = -m\left(v_i+\mathrm{v}_i\right) = -\frac{m_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(v_i+\mathrm{v}_i\right). \tag{4.37}$$

From here we see that four-dimensional impulse vector S^α that particle gains from its spin (i.e. spin-impulse of particle) is

$$S^\alpha = \frac{1}{c^2} n\hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}\,, \tag{4.38}$$

or, introducing notation $\eta_0{=}n\hbar^{\mu\nu}A_{\mu\nu}{=}n\hbar^{mn}A_{mn}$ to make the formula simpler

$$S^\alpha = \frac{1}{c^2}\eta_0 \frac{dx^\alpha}{ds}\,. \tag{4.39}$$

Then the summary dynamic vector Q^α (4.1) that characterizes motion of spin-particle is

$$Q^\alpha = P^\alpha + S^\alpha = m_0 \frac{dx^\alpha}{ds} + \frac{1}{c^2} n\hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}\,. \tag{4.40}$$

Therefore spin-particle in non-holonomic space ($A_{\mu\nu}{\neq}0$) actually gains additional impulse that deviates its motion from geodesic (free-particle) trajectory and makes it non-geodesic. In absence of space rotation (holonomic space) values $A_{\mu\nu}{=}0$ and the spin does not affect motion of particle. But there is hardly an area in space where rotation is fully absent. Therefore spin most often affects motion of particle in the subject domain of atomic physics, where rotation is especially strong.

4.3 Equations of motion of spin-particle

Dynamic equations of motion of spin-particle are equations of parallel transfer of summary vector $Q^\alpha{=}P^\alpha{+}S^\alpha$ (4.40) along the trajectory of motion of the particle (its parallel transfer) in four-dimensional pseudo-Riemannian space

$$\frac{d}{ds}\left(P^\alpha + S^\alpha\right) + \Gamma^\alpha_{\mu\nu}\left(P^\mu + S^\mu\right)\frac{dx^\nu}{ds} = 0\,, \tag{4.41}$$

where the square of vector being transferred conserves along the entire trajectory $Q_\alpha Q^\alpha{=}const$.

Our goal is to deduce chronometrically invariant (physically observable) projections of these equations onto time and space in accompanying frame of reference. These equations in general notation, as obtained in Chapter 2, are

$$\frac{d\varphi}{ds} - \frac{1}{c}F_i q^i \frac{d\tau}{ds} + \frac{1}{c}D_{ik} q^i \frac{dx^k}{ds} = 0\,, \tag{4.42}$$

$$\frac{dq^i}{ds} + \left(\frac{\varphi}{c}\frac{dx^k}{ds} + q^k \frac{d\tau}{ds}\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \frac{\varphi}{c}F^i \frac{d\tau}{ds} + \triangle^i_{mk} q^m \frac{dx^k}{ds} = 0\,, \tag{4.43}$$

where φ is projection of summary vector Q_α on time and q^i is its projection on space

$$\varphi = b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}} = \frac{P_0}{\sqrt{g_{00}}} + \frac{S_0}{\sqrt{g_{00}}}\,, \tag{4.44}$$

$$q^i = h^i_\alpha Q^\alpha = Q^i = P^i + S^i. \tag{4.45}$$

Therefore attaining the goal requires deducing φ and q^i, substituting them into (4.42, 4.43) and canceling similar terms. Projections of impulse vector of mass-bearing particle $P^\alpha{=}m_0\frac{dx^\alpha}{ds}$ are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \qquad P^i = \frac{1}{c} m \mathrm{v}^i, \tag{4.46}$$

and now we have to deduce projections of spin-impulse S^α. Taking into account in the formula for S^α (4.39) that space-time interval, formulated with physical observable values, is $ds{=}cd\tau\sqrt{1{-}\mathrm{v}^2/c^2}$, we obtain components of spin-impulse S^α

$$S^0 = \frac{1}{c^2}\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\dfrac{\mathrm{v}^2}{c^2}}}\frac{\left(v_i \mathrm{v}^i \pm c^2\right)}{c^2\left(1-\dfrac{w}{c^2}\right)}\,, \tag{4.47}$$

$$S^i = \frac{1}{c^3}\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\mathrm{v}^i, \tag{4.48}$$

$$S_0 = \pm\frac{1}{c^2}\left(1-\frac{w}{c^2}\right)\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}, \tag{4.49}$$

$$S_i = -\frac{1}{c^3}\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(v_i \pm \mathrm{v}_i\right), \tag{4.50}$$

also formulated with physical observable values. From here we see that physical observable projections of spin-impulse of particle are

$$\frac{S_0}{\sqrt{g_{00}}} = \pm\frac{1}{c^2}\eta\,, \qquad S^i = \frac{1}{c^3}\eta\mathrm{v}^i, \tag{4.51}$$

where η is

$$\eta = \frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}, \tag{4.52}$$

while alternating signs, which results from substituting function of time $\frac{dt}{d\tau}$ (1.55) indicate motion of particle into future (the upper sign) or into past (the lower sign). Then the square of spin-impulse is

$$S_\alpha S^\alpha = g_{\alpha\beta}S^\alpha S^\beta = \frac{1}{c^4}\eta_0^2 g_{\alpha\beta}\frac{dx^\alpha dx^\beta}{ds^2} = \frac{1}{c^4}\eta_0^2\,, \tag{4.53}$$

and the square of summary dynamic vector Q^α is

$$Q_\alpha Q^\alpha = g_{\alpha\beta}Q^\alpha Q^\beta = m_0^2 + \frac{2}{c^2}m_0\eta_0 + \frac{1}{c^4}\eta_0^2\,. \tag{4.54}$$

Therefore the square of length of summary vector of spin-particle falls apart into three parts, namely:

- the square of length of specific four-dimensional vector of impulse of particle $P_\alpha P^\alpha{=}m_0^2$;
- the square of length of four-dimensional spin-impulse of particle $S_\alpha S^\alpha{=}\frac{1}{c^4}\eta_0^2$;
- the term $\frac{2}{c^2}m_0\eta_0$ that describes spin-gravitational interaction.

To effect parallel transfer (4.41) it is necessary that the square of transferred summary vector conserved along the entire path. But the obtained formula (4.54) implies that because $m_0{=}const$ the square of summary vector of spin-particle Q^α conserves only provided that $\eta_0{=}const$, i. e.

$$d\eta_0 = \frac{\partial\eta_0}{\partial x^\alpha}dx^\alpha = 0\,. \tag{4.55}$$

Dividing both parts of the equation by $d\tau$, which is always possible because elementary interval of observer's physical time is greater than zero[21], we obtain chronometrically invariant condition of conservation of the square of summary vector of spin-particle

$$\frac{d\eta_0}{d\tau} = \frac{{}^*\partial\eta_0}{\partial t} + \mathrm{v}^k\frac{{}^*\partial\eta_0}{\partial x^k} = 0\,. \tag{4.56}$$

Substituting here $\eta_0 = n\hbar^{mn}A_{mn}$ we have

$$n\hbar^{mn}\left(\frac{{}^*\partial A_{mn}}{\partial t} + \mathrm{v}^k\frac{{}^*\partial A_{mn}}{\partial x^k}\right) = 0\,. \tag{4.57}$$

[21]The condition $d\tau{=}0$ only has sense in generalized space-time, where degeneration of fundamental metric tensor $g_{\alpha\beta}$ is possible. In this case the above condition defines fully degenerated domain (zero-space) that hosts zero-particles, which are capable of instant displacement, i. e. are carriers of long-range action.

To illustrate the result we formulate three-dimensional chronometrically invariant tensor of angular velocity of space rotation A_{ik} with pseudovector of three-dimensional angular velocity of this rotation

$$\Omega^{*i} = \frac{1}{2}\varepsilon^{imn} A_{mn}\,, \tag{4.58}$$

which is also a chronometric invariant. Multiplying Ω^{*i} by ε_{ipq}

$$\Omega^{*i}\varepsilon_{ipq} = \frac{1}{2}\varepsilon^{imn}\varepsilon_{ipq}A_{mn} = \frac{1}{2}\left(\delta_p^m\delta_q^n - \delta_p^n\delta_q^m\right)A_{mn} = A_{pq}\,, \tag{4.59}$$

we obtain (4.57) as

$$\begin{gathered} n\hbar^{mn}\left[\frac{^*\partial}{\partial t}\left(\varepsilon_{imn}\Omega^{*i}\right) + \mathrm{v}^k\frac{^*\partial}{\partial x^k}\left(\varepsilon_{imn}\Omega^{*i}\right)\right] = \\ = n\hbar^{mn}\varepsilon_{imn}\left[\frac{1}{\sqrt{h}}\frac{^*\partial}{\partial t}\left(\sqrt{h}\Omega^{*i}\right) + \mathrm{v}^k\frac{1}{\sqrt{h}}\frac{^*\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)\right] = 0\,. \end{gathered} \tag{4.60}$$

Gravitational inertial force and tensor of non-holonomity are related through Zelmanov's identities, one of which (formula 13.20 in [10]) is

$$\frac{2}{\sqrt{h}}\frac{^*\partial}{\partial t}\left(\sqrt{h}\Omega^{*i}\right) + \varepsilon^{ijk}\,{}^*\nabla_j F_k = 0\,, \tag{4.61}$$

or, in a different notation

$$\frac{^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left({}^*\nabla_k F_i - {}^*\nabla_i F_k\right) = \frac{^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{^*\partial F_k}{\partial x^i} - \frac{^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{4.62}$$

where $\varepsilon^{ijk}\,{}^*\nabla_j F_k$ is chronometrically invariant (observable) rotor of field of gravitational inertial force F_k. From here we see that non-stationarity of tensor of angular velocity A_{ik} is due to *rotor character* of field of gravitational inertial force F_{ik} in the space of the body of reference. Hence taking into account equation (4.61) our formula (4.60) becomes

$$-\,n\hbar^{mn}\,{}^*\nabla_m F_n + n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\frac{1}{\sqrt{h}}\frac{^*\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right) = 0\,, \tag{4.63}$$

or in another notation

$$n\hbar^{mn}\,{}^*\nabla_m F_n = n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\left(\Omega^{*i}\frac{^*\partial\ln\sqrt{h}}{\partial x^k} + \frac{^*\partial\Omega^{*i}}{\partial x^k}\right). \tag{4.64}$$

Now we should recall that this formula is nothing but expanded chronometrically invariant notation of the condition of conservation of the square of summary vector (4.57). The left part (4.64) equals

$$\pm\,2n\hbar\left({}^*\nabla_1 F_2 - {}^*\nabla_2 F_1 + {}^*\nabla_1 F_3 - {}^*\nabla_3 F_1 + {}^*\nabla_2 F_3 - {}^*\nabla_3 F_2\right), \tag{4.65}$$

where "plus" and "minus" stand for right and left frames of reference, respectively. Therefore the left part of formula (4.64) is chronometrically invariant rotor of gravitational inertial force. The right part (4.64) depends upon spatial orientation of field of pseudovector of angular velocities of space's rotation Ω^{*i}.

Hence to conserve the square of transferred vector of spin-particle it is necessary that the right and the left parts of (4.64) are equal to each other along the entire trajectory of the particle. In general case, i. e. without any additional assumptions on geometric structure of the space of reference, this requires balance between rotor field of gravitational inertial force of space of reference and spatial distribution of pseudovector of angular velocity of its rotation.

If field of gravitational inertial force is vortless, the left part of the conservation condition (4.64) is zero and this condition becomes

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\frac{1}{\sqrt{h}}\frac{^*\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right) = 0\,. \tag{4.66}$$

After denoting chronometrically invariant derivative as $\frac{{}^*\partial}{\partial x^k}=\frac{\partial}{\partial x^k}+\frac{1}{c^2}v_k\frac{{}^*\partial}{\partial t}$ we have

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\frac{1}{\sqrt{h}}\left[\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)-\frac{1}{c^2}v_k\frac{{}^*\partial}{\partial t}\left(\sqrt{h}\Omega^{*i}\right)\right]=0\,. \tag{4.67}$$

Since the field of force F_i is vortless, because of (4.66) the second term in this formula is zero. Therefore the square of summary vector of spin particle conserves in vortless field of force F_i provided that chronometrically invariant formula (4.66) and the formula with regular derivatives are zeroes

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)=0\,. \tag{4.68}$$

For mass-bearing particles this is the case, for instance, when $\mathrm{v}^k{=}0$, i. e. when particle rests in respect to observer and his body of reference. In this case equality to zero of derivatives in (4.68) is not essential. But massless particles travel at the speed of light, hence for such in vortless field of force F_i the derivatives $\frac{\partial}{\partial x^k}(\sqrt{h}\Omega^{*i})$ must be zeroes.

Now we are going to obtain chronometrically invariant dynamic equations of motion of spin-particle in pseudo-Riemannian space. Substituting (4.46) and (4.51) into (4.44) and (4.45) we arrive to observable components of summary vector of spin-particle

$$\varphi=\pm\left(m+\frac{1}{c^2}\eta\right),\qquad q^i=\frac{1}{c}m\mathrm{v}^i+\frac{1}{c^3}\eta\mathrm{v}^i. \tag{4.69}$$

Having these values substituted for $\varphi{>}0$ into (4.42, 4.43) we obtain chronometrically invariant equations of motion of mass-bearing spin-particle that travels from past into future

$$\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=-\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}F_i\mathrm{v}^i-\frac{\eta}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{4.70}$$

$$\begin{gathered}\frac{d}{d\tau}\left(m\mathrm{v}^i\right)+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k-mF^i+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\=-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right)-\frac{2\eta}{c^2}\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k+\frac{\eta}{c^2}F^i-\frac{\eta}{c^2}\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k.\end{gathered} \tag{4.71}$$

For mass-bearing spin-particle that travels from future into past, having the values (4.69) substituted for $\varphi{<}0$ the equations become

$$-\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}F_i\mathrm{v}^i-\frac{\eta}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{4.72}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right)+mF^i+m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k==-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right)-\frac{\eta}{c^2}F^i-\frac{\eta}{c^2}\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k, \tag{4.73}$$

We wrote down the obtained equations in a way that the left parts have *geodesic part*, which describes free (geodesic) motion of particle, while the right parts have the terms produced by spin, which makes its motion non-geodesic (*non-geodesic part*). Hence for moving no-spin particle the right parts become zeroes and we obtain chronometrically invariant dynamic equations of motion of free particle. In the next Sections such form of equations will facilitate their analysis.

Within wave-particle concept massless particle is described by four-dimensional wave vector $K^\alpha{=}\frac{\omega}{c}\frac{dx^\alpha}{d\sigma}$, where $d\sigma^2{=}h_{ik}dx^idx^k$ is three-dimensional physical observable interval, not equal to zero along isotropic trajectories. Because massless particles travel along isotropic trajectories (light propagation trajectories), vector K^α is also isotropic one: its square is zero. But because the dimension of wave vector K^α is [s^{-1}], equations of motion of massless particles, obtained with its help, have the dimension different from that of equations of motion of mass-bearing particles. Besides, this fact does not permit building uniform formula of action for both massless and mass-bearing particles [10].

On the other hand, spin is a physical property, possessed by mass-bearing and massless particles (photons, for instance). Therefore deduction of equations of motion of spin-particles require using uniform dynamic vector for both types of particles. Such vector can be obtained by applying physical conditions that are true along isotropic trajectories,

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0\,, \qquad c\,d\tau = d\sigma \neq 0\,, \tag{4.74}$$

to four-dimensional impulse of mass-bearing particle $P^\alpha = m_0 \frac{dx^\alpha}{ds}$

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{m}{c}\frac{dx^\alpha}{d\tau} = m\frac{dx^\alpha}{d\sigma}\,. \tag{4.75}$$

As a result observable three-dimensional interval, not equal to zero along isotropic trajectories, becomes the derivation parameter, while the dimension of entire formula, contrasted to four-dimensional wave vector K^α [s^{-1}], coincides with that of four-dimensional impulse vector P^α [g]. Relativistic mass m, not equal to zero for massless particles, can be obtained from its energy equivalent using $E = mc^2$ formula. For instance, photon energy of $E = 1$ Mev$=1.6\cdot 10^{-6}$ erg corresponds to its relativistic mass of $m = 1.8\cdot 10^{-28}$ g.

Therefore four-dimensional impulse vector (4.75), depending upon its form, may describe motion of either mass-bearing particles (non-isotropic trajectories) or massless ones (isotropic trajectories). As a matter of fact, for massless particles $m_0 = 0$ and $ds = 0$, therefore their ratio in (4.75) is a 0/0 indeterminance. However the transition (4.75) solves the indeterminance, because relativistic mass (motion mass) of massless particles $m \neq 0$ and along their trajectory $d\sigma \neq 0$.

Evidently that in the form applicable to massless particles (i. e. along isotropic trajectories) the square of P^α (4.75) equals to zero

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = m^2 g_{\alpha\beta} \frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} = m^2 \frac{ds^2}{d\sigma^2} = 0\,. \tag{4.76}$$

Evidently also that physical observable components of four-dimensional impulse vector for massless particles $P^\alpha = m\frac{dx^\alpha}{d\sigma}$ are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \qquad P^i = \frac{1}{c} m c^i, \tag{4.77}$$

where c^i is three-dimensional chronometrically invariant vector of light velocity (its square equals $c_i c^i = h_{ik} c^i c^k = c^2$ in an accompanying frame of reference).

Along isotropic (light-like) trajectories spin-impulse of particle (4.39) is also isotropic

$$S^\alpha = \frac{1}{c^2}\eta_0 \frac{dx^\alpha}{ds} = \frac{1}{c^2}\eta\frac{dx^\alpha}{c\,d\tau} = \frac{1}{c^2}\eta\frac{dx^\alpha}{d\sigma}\,, \tag{4.78}$$

because its square is zero

$$S_\alpha S^\alpha = g_{\alpha\beta} S^\alpha S^\beta = \frac{1}{c^4}\eta^2 g_{\alpha\beta}\frac{dx^\alpha dx^\beta}{d\sigma^2} = \frac{1}{c^4}\eta^2\frac{ds^2}{d\sigma^2} = 0\,, \tag{4.79}$$

and hence the square of summary dynamic vector of massless spin-particle $Q^\alpha = P^\alpha + S^\alpha$ is also zero. Observable projections of isotropic spin-impulse (spin-impulse of massless particle) are

$$\frac{S_0}{\sqrt{g_{00}}} = \pm\frac{1}{c^2}\eta\,, \qquad S^i = \frac{1}{c^3}\eta c^i, \tag{4.80}$$

while its observable projection coincides with the one for mass-bearing particles (4.51), and the spatial projection instead of observable velocity v^i (4.51) has vector of observable light velocity c^i. Subsequently, observable components of summary vector of massless spin-particle are

$$\varphi = \pm\left(m + \frac{1}{c^2}\eta\right), \qquad q^i = \frac{1}{c} m c^i + \frac{1}{c^3}\eta c^i. \tag{4.81}$$

Having these values substituted for positive φ into the initial formulas (4.42, 4.43), we arrive to chronometrically invariant dynamic equations of motion for massless (light-like) spin-particle that travels from past into future

$$\frac{dm}{d\tau}-\frac{m}{c^2}F_ic^i+\frac{m}{c^2}D_{ik}c^ic^k=-\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}F_ic^i-\frac{\eta}{c^4}D_{ik}c^ic^k, \tag{4.82}$$

$$\begin{gathered}\frac{d}{d\tau}\left(mc^i\right)+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)c^k-mF^i+m\triangle^i_{nk}c^nc^k=\\ =-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta c^i\right)-\frac{2\eta}{c^2}\left(D^i_k+A^{\cdot i}_{k\cdot}\right)c^k+\frac{\eta}{c^2}F^i-\frac{\eta}{c^2}\triangle^i_{nk}c^nc^k.\end{gathered} \tag{4.83}$$

For massless spin-particle that travels from future into past, having the values (4.81) substituted for $\varphi<0$, the equations become

$$-\frac{dm}{d\tau}-\frac{m}{c^2}F_ic^i+\frac{m}{c^2}D_{ik}c^ic^k=\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}F_ic^i-\frac{\eta}{c^4}D_{ik}c^ic^k, \tag{4.84}$$

$$\frac{d}{d\tau}\left(mc^i\right)+mF^i+m\triangle^i_{nk}c^nc^k=-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta c^i\right)-\frac{\eta}{c^2}F^i-\frac{\eta}{c^2}\triangle^i_{nk}c^nc^k. \tag{4.85}$$

4.4 Physical conditions of spin-interaction

As we have shown, spin (inner mechanical momentum) of particle interacts with external field of space rotation — field of non-holonomity tensor $A^{\alpha\beta}=\frac{1}{2}ch^{\alpha\mu}h^{\beta\nu}\left(\frac{\partial b_\nu}{\partial x^\mu}-\frac{\partial b_\mu}{\partial x^\nu}\right)$, which is a function of rotor of four-dimensional vector of observer's velocity b^α. In electromagnetic phenomena particle also interacts with external field of 2nd rank tensor — field of Maxwell tensor $F_{\alpha\beta}=\frac{\partial A_\beta}{\partial x^\alpha}-\frac{\partial A_\alpha}{\partial x^\beta}$. Therefore it seems natural to compare physical observable components of Maxwell field $F_{\alpha\beta}$ to their analogs for non-holonomity field $A_{\alpha\beta}$.

In the previous Chapter we obtained that field of Maxwell tensor has two groups of observable values, produced by covariant tensor $F_{\alpha\beta}$ itself and by its dual pseudotensor $F^{*\alpha\beta}=\frac{1}{2}E^{\alpha\beta\mu\nu}F_{\mu\nu}$, where $E^{\alpha\beta\mu\nu}$ is four-dimensional completely antisymmetric discriminant tensor that produce pseudotensors in four-dimensional pseudo-Riemannian space

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}=E^i,\qquad F^{ik}=H^{ik},\qquad\qquad \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}=H^{*i},\qquad F^{*ik}=E^{*ik}. \tag{4.86}$$

Similar components of general covariant tensor of non-holonomity $A_{\alpha\beta}$ (4.11) and of pseudotensor $A^{*\alpha\beta}=\frac{1}{2}E^{\alpha\beta\mu\nu}A_{\mu\nu}$, deduced in accompanying frame of reference are

$$\frac{A_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}=0\,,\quad A^{ik}=h^{im}h^{kn}A_{mn}\,,\qquad\qquad \frac{A_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}=0\,,\quad A^{*ik}=0\,. \tag{4.87}$$

Comparing these formulas with those for observable components of Maxwell tensor and pseudotensor (4.86), also deduced in accompanying frame of reference, we see that spin-interaction presents only the analog for "magnetic" component $\mathcal{H}^{ik}=A^{ik}=h^{im}h^{kn}A_{mn}$ of non-holonomity field. The analog for "electric" component of non-holonomity field in spin-interaction turns to be zero $\mathcal{E}^i=\frac{A_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}=0$. Which is no surprise, because spin (inner field of rotation) of particle interacts with external field of non-holonomity of space and both fields are produced by motion.

Besides, for non-holonomity field the analog of "magnetic" component $\mathcal{H}^{ik}=A^{ik}\neq0$ can not be dual to zero value $\mathcal{H}^{*i}=\frac{A_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}=0$. Similarity with electromagnetic field turns out to be incomplete. But full matching could not even be expected, because tensor of non-holonomity and tensor of electromagnetic field have somewhat different structures: Maxwell tensor is a "pure" rotor $F_{\alpha\beta}=\frac{\partial A_\beta}{\partial x^\alpha}-\frac{\partial A_\alpha}{\partial x^\beta}$, while tensor of non-holonomity is an "add-on" rotor $A^{\alpha\beta}=\frac{1}{2}ch^{\alpha\mu}h^{\beta\nu}\left(\frac{\partial b_\nu}{\partial x^\mu}-\frac{\partial b_\mu}{\partial x^\nu}\right)$. On the other hand we have no doubts that in future comparative analysis of these fields with such similar structures will produce theory of spin interactions, similar to that of electromagnetic field.

Incomplete similarity of non-holonomity field and electromagnetic field also leads to another result. If we define force of spin-interaction in the same way we define Lorentz force $\Phi^{\alpha}=\frac{e}{c}F^{\alpha\cdot}_{\cdot\sigma}U^{\sigma}$, the obtained formula $\Phi^{\alpha}=\frac{\eta_0}{c^2}A^{\alpha\cdot}_{\cdot\sigma}U^{\sigma}$ will include *not all* the terms from the right parts of equations of motion of spin-particle. But an external force that acts on particle, by definition, must include all factors that deviate the particle from geodesic trajectory, i. e. all terms in the right parts of dynamic equations of motion. In other words, four-dimensional force of spin-interaction Φ^{α} [g s^{-1}] is defined by general covariant formula

$$\Phi^{\alpha}=\frac{DS^{\alpha}}{ds}=\frac{dS^{\alpha}}{ds}+\Gamma^{\alpha}_{\mu\nu}S^{\mu}\frac{dx^{\nu}}{ds}\,, \tag{4.88}$$

which projection onto space (after being divided by c) gives three-dimensional force of spin-interaction Φ^{i} [g cm s^{-2}]. For instance, for mass-bearing our-world particles from (4.71) we have

$$\Phi^{i}=-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^{i}\right)-\frac{2\eta}{c^2}\left(D^{i}_{k}+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^{k}+\frac{\eta}{c^2}F^{i}-\frac{\eta}{c^2}\triangle^{i}_{nk}\mathrm{v}^{n}\mathrm{v}^{k}. \tag{4.89}$$

From further comparison of electromagnetic and spin-interaction, using similarity with electromagnetic field invariants (3.25, 3.26) we deduced invariants of non-holonomity field

$$J_1=A_{\alpha\beta}A^{\alpha\beta}=A_{ik}A^{ik}=\varepsilon_{ikm}\varepsilon^{ikn}\Omega^{*m}\Omega_{*n}=2\Omega_{*i}\Omega^{*i}, \tag{4.90}$$

$$J_2=A_{\alpha\beta}A^{*\alpha\beta}=0\,. \tag{4.91}$$

Hence scalar invariant $J_1=2\Omega_{*i}\Omega^{*i}$ is always non-zero, because otherwise the space would be holonomic (not rotating) and spin-interaction will be absent.

Now we are approaching physical conditions of motion of elementary spin-particles. Using the definition of chronometrically invariant vector of gravitational inertial force

$$F_i=\frac{1}{1-\frac{w}{c^2}}\left(\frac{\partial w}{\partial x^i}-\frac{\partial v_i}{\partial t}\right)=-c^2\frac{\partial\ln\left(1-\frac{w}{c^2}\right)}{\partial x^i}-\frac{^*\partial v_i}{\partial t} \tag{4.92}$$

we formulate non-holonomity tensor A_{ik} with gravitational potential w of reference's body and linear velocity v_i of rotation of reference's space

$$A_{ik}=\frac{1}{2}\left(\frac{^*\partial v_k}{\partial x^i}-\frac{^*\partial v_i}{\partial x^k}\right)+v_i\frac{\partial\ln\sqrt{1-\frac{w}{c^2}}}{\partial x^k}-v_k\frac{\partial\ln\sqrt{1-\frac{w}{c^2}}}{\partial x^i}\,. \tag{4.93}$$

From here we see that non-holonomity tensor A_{ik} is a three-dimensional observable rotor of linear velocity of space rotation with two additional terms, produced by interaction between field of gravitational potential w and field of space rotation.

On the other hand, because of small absolute value of Planck constant, spin-interaction only affects elementary particles. And as known, on the scales of such small masses and distances gravitational interaction is a few orders of magnitude weaker than electromagnetic, weak (spin) or strong interactions.

Keeping this fact in mind, we can assume that for spin-interaction in the formula for non-holonomity tensor A_{ik} (4.93) gravitational potential $w\to 0$. Then on the microscopic scales of elementary particles A_{ik} is physical observable rotor in "pure" notation

$$A_{ik}=\frac{1}{2}\left(\frac{^*\partial v_k}{\partial x^i}-\frac{^*\partial v_i}{\partial x^k}\right), \tag{4.94}$$

while the gravitational inertial force (4.92) will have only inertial part

$$F_i=-\frac{^*\partial v_i}{\partial t}=-\frac{1}{1-\frac{w}{c^2}}\frac{\partial v_i}{\partial t}=-\frac{\partial v_i}{\partial t}\,. \tag{4.95}$$

Zelmanov's identities (see formulas 13.20 and 13.21 from [10]), that link gravitational inertial force and space rotation

$$\frac{2}{\sqrt{h}}\frac{{}^*\partial}{\partial t}\left(\sqrt{h}\Omega^{*i}\right)+\varepsilon^{ijk}\,{}^*\nabla_j F_k=0\,,\quad {}^*\nabla_k\Omega^{*k}+\frac{1}{c^2}F_k\Omega^{*k}=0 \tag{4.96}$$

for elementary particles ($w\rightarrow 0$) become

$$\begin{gathered}\frac{1}{\sqrt{h}}\frac{\partial}{\partial t}\left(\sqrt{h}\Omega^{*i}\right)+\frac{1}{2}\varepsilon^{ijk}\left(\frac{{}^*\partial^2 v_k}{\partial x^j\partial t}-\frac{{}^*\partial^2 v_j}{\partial x^k\partial t}\right)=0\,,\\ {}^*\nabla_k\Omega^{*k}-\frac{1}{c^2}\frac{{}^*\partial v_k}{\partial t}\Omega^{*k}=0\,.\end{gathered} \tag{4.97}$$

If we substitute here $\frac{{}^*\partial v_k}{\partial t}=0$, i. e. assume that the observable rotation of space is stationary, we obtain ${}^*\nabla_k\Omega^{*k}=0$, i. e. pseudovector of angular velocity of space rotation will conserve. Then the first (vector) equation will become

$$D\Omega^{*i}+\frac{{}^*\partial\Omega^{*i}}{\partial t}=0\,, \tag{4.98}$$

from which we see that $D=\det||D_j^j||=\frac{{}^*\partial\ln\sqrt{h}}{\partial t}$, i. e. rate of relative expansion of elementary volume of space is zero $D=0$.

Therefore, these equations suggest that for elementary particles ($w\rightarrow 0$) in stationary rotation of space ($\frac{{}^*\partial v_k}{\partial t}=0$) tensor of angular velocities of this rotation conserves ${}^*\nabla_k\Omega^{*k}=0$ and relative expansion (deformation) of the space is absent $D=0$.

It is possible, that stationarity of field of non-holonomity of space (as the external field in spin-interaction) is the necessary condition of stability of elementary particle. Out of this we may conclude that long-living spin-particles should possess stable inner rotation, while short-living particles must be rotors.

To study motion of short-living particles is pretty problematic as we do not have experimental data on structure of rotors that may produce them. At the same time the study for long-living ones, i. e. in vortless (stationary) field of space rotation, can give exact solutions of their equations of motion. We will focus on these issues in the next Section.

4.5 Motion of elementary spin-particles

As we have mentioned, Planck constant, being a small absolute value, only "works" on the scales of elementary particles, where gravitational interaction is a few orders of magnitude weaker than electromagnetic, weak and strong ones. Hence assuming $w\rightarrow 0$ in equations of motion of spin-particles (4.70–4.73) and (4.82–4.85), we will arrive to equations of motion of elementary particles.

Besides, as we obtained in the previous Section, in stationary rotation of space ($\frac{{}^*\partial v_k}{\partial t}=0$) on the scales of elementary particles the trace of tensor of space deformation velocities is zero $D=0$. Of course zero trace of a tensor does not necessarily imply the tensor itself is zero. On the other hand, deformation of space is a rare phenomenon and for our study of motion of elementary particles we will assume $D_{ik}=0$.

In Section 4.3 we showed that in stationary rotation of space the condition of conservation of spin-impulse of particle S^α becomes (4.68)

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*i}\right)=0\,. \tag{4.99}$$

On the other hand, with $\frac{{}^*\partial v_k}{\partial t}=0$ Zelmanov's identities we applied for elementary particles (4.97) imply that

$${}^*\nabla_k\Omega^{*k}=\frac{\partial\Omega^{*k}}{\partial x^k}+\frac{\partial\sqrt{h}}{\partial x^k}\Omega^{*k}=\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\Omega^{*k}\right)=0\,. \tag{4.100}$$

The first condition is true provided that $\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}(\sqrt{h}\Omega^{*k})=0$. This is true if pseudovector of angular velocity of space rotation is

$$\Omega^{*i} = \frac{\Omega^{*i}_{(0)}}{\sqrt{h}}\,, \qquad \Omega^{*i}_{(0)} = const\,, \tag{4.101}$$

in this case the second condition (4.100) is true too.

Taking the above into account, from the formulas (4.70, 4.71) we obtain chronometrically invariant equations of motion of our-world mass-bearing elementary particle that travels into future in respect to an regular observer (direct flow of time)

$$\frac{dm}{d\tau} = -\frac{1}{c^2}\frac{d\eta}{d\tau}\,, \tag{4.102}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2mA_{k\cdot}^{\cdot i}\mathrm{v}^k + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right) - \frac{2\eta}{c^2}A_{k\cdot}^{\cdot i}\mathrm{v}^k - \frac{\eta}{c^2}\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k, \tag{4.103}$$

and from (4.72, 4.73) we obtain the equations for mirror-world mass-bearing elementary particle that travels into past (reverse flow of time),

$$-\frac{dm}{d\tau} = \frac{1}{c^2}\frac{d\eta}{d\tau}\,, \tag{4.104}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right) - \frac{\eta}{c^2}\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k. \tag{4.105}$$

In this case scalar equations of motion (temporal projections) are the same for mass-bearing particles in our world and in the mirror world.

Integrating scalar equation of motion for direct flow of time

$$\int_{\tau_1=0}^{\tau_2}\frac{d}{d\tau}\left(m + \frac{\eta}{c^2}\right)d\tau = 0\,, \tag{4.106}$$

we obtain

$$m + \frac{\eta}{c^2} = const = B\,, \tag{4.107}$$

where B is integration constant that can be defined from the initial conditions.

To illustrate physical sense of the obtained live forces integral, we use analogy between observable components of four-dimensional impulse vector of particle $P^\alpha{=}m_0\frac{dx^\alpha}{ds}$ and those of spin-impulse $S^\alpha{=}\frac{\eta_0}{c^2}\frac{dx^\alpha}{ds}$. Both vectors are tangential to the world line of motion of particle, while their observable components are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \quad P^i = \frac{1}{c}m\mathrm{v}^i = \frac{1}{c}p^i, \qquad \frac{S_0}{\sqrt{g_{00}}} = \pm\frac{1}{c^2}\eta\,, \quad S^i = \frac{1}{c^3}\eta\mathrm{v}^i. \tag{4.108}$$

Using analogy with relativistic mass of particle $\pm m$ we will refer to the value $\pm\frac{1}{c^2}\eta$ as relativistic *spin-mass*. Then $\frac{1}{c^2}\eta_0$ is rest spin-mass of a particle. Further, live forces theorem for spin elementary particle (4.107) implies that with the assumptions we made the sum of relativistic mass of elementary particle and of its spin-mass conserves along the trajectory.

Now using live forces integral (solution of scalar equation of motion) we approach vector equations of motion of our-world mass-bearing elementary particle (4.103). Substituting (4.107) into vector equations of motion (4.103) having the constant canceled we obtain

$$\frac{d\mathrm{v}^i}{d\tau} + 2A_{k\cdot}^{\cdot i}\mathrm{v}^k + \triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \tag{4.109}$$

i. e. pure kinematic equations of motion (non-geodesic one, in this case). The term $\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k$, which is contraction of chronometrically invariant Christoffel symbols with observable velocity of particle, is relativistic in the sense that it is a quadratic function of velocity of particles. Therefore it can be neglected provided that observable metric $h_{ik}=-g_{ik}+\frac{1}{c^2}v_i v_k$ along the trajectory is close to Euclidean one. Such is possible when velocity of space rotation is much lower than the speed of light, while three-dimensional coordinate metric g_{ik} is Euclidean one. Then the diagonal components of observable metric tensor are

$$h_{11} = h_{22} = h_{33} = +1\,, \tag{4.110}$$

while the other components $h_{ik}=0$ if $i\neq k$. Noteworthy, the four-dimensional metric can not be Galilean here, because three-dimensional space rotates in respect to time. In other words, though space in this case is a flat Euclidean one, four-dimensional space-time is not a flat Minkowski space but is a pseudo-Riemannian space with metric

$$\begin{aligned} ds^2 &= g_{00}dx^0dx^0 + 2g_{0i}dx^0dx^i + g_{ik}dx^idx^k = \\ &= c^2dt^2 + 2g_{0i}cdtdx^i - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2. \end{aligned} \tag{4.111}$$

We assume that space rotates at constant angular velocity $\Omega=const$ around a single axis, axis x^3, for instance. Then linear velocity of the space of reference $v_i=\Omega_{ik}x^k$ becomes

$$v_1 = \Omega_{12}x^2 = \Omega y\,, \qquad v_2 = \Omega_{21}x^1 = -\Omega x\,, \tag{4.112}$$

where $A_{ik}=\Omega_{ik}$. Then tensor of space non-holonomity A_{ik} has only two non-zero components

$$A_{12} = -A_{21} = -\Omega\,. \tag{4.113}$$

Taking this into account vector equations of motion of our-world elementary particle (4.109) become

$$\frac{d\mathrm{v}^1}{d\tau} + 2\Omega \mathrm{v}^2 = 0\,, \qquad \frac{d\mathrm{v}^2}{d\tau} - 2\Omega \mathrm{v}^1 = 0\,, \qquad \frac{d\mathrm{v}^3}{d\tau} = 0\,. \tag{4.114}$$

The third equation solves immediately as

$$\mathrm{v}^3 = \mathrm{v}^3_{(0)} = const. \tag{4.115}$$

Taking into account that $\mathrm{v}^3=\frac{dx^3}{d\tau}$, we represent coordinate x^3 as

$$x^3 = \mathrm{v}^3_{(0)}\tau + x^3_{(0)}\,, \tag{4.116}$$

where $x^3_{(0)}$ is the value of coordinate x^3 at the initial moment of time $\tau=0$. Now we formulate v^2 from the first equation (4.114)

$$\mathrm{v}^2 = -\frac{1}{2\Omega}\frac{d\mathrm{v}^1}{d\tau}\,, \tag{4.117}$$

having this formula derivated to $d\tau$

$$\frac{d\mathrm{v}^2}{d\tau} = -\frac{1}{2\Omega}\frac{d^2\mathrm{v}^1}{d\tau^2}\,, \tag{4.118}$$

and having it substituted into the second equation (4.114) we obtain

$$\frac{d^2\mathrm{v}^1}{d\tau^2} + 4\Omega^2\mathrm{v}^1 = 0\,, \tag{4.119}$$

i. e. equation of free oscillations. It solves as

$$\mathrm{v}^1 = C_1\cos\left(2\Omega\tau\right) + C_2\sin\left(2\Omega\tau\right), \tag{4.120}$$

where C_1 and C_2 are integration constants (4.119), which can be defined from the conditions at the moment $\tau=0$

$$\mathrm{v}^1_{(0)} = C_1\,, \qquad \left.\frac{d\mathrm{v}^1}{d\tau}\right|_{\tau=0} = -2\Omega C_1\sin\left(2\Omega\tau\right)\big|_{\tau=0} + 2\Omega C_2\cos\left(2\Omega\tau\right)\big|_{\tau=0}\,. \tag{4.121}$$

Thus $C_1 = \mathrm{v}^1_{(0)}$, $C_2 = \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}$, where $\dot{\mathrm{v}}^1_{(0)} = \frac{d\mathrm{v}^1}{d\tau}|_{\tau=0}$. Then the equation for v^1 finally solves as

$$\mathrm{v}^1 = \mathrm{v}^1_{(0)} \cos(2\Omega\tau) + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega} \sin(2\Omega\tau)\,, \tag{4.122}$$

i. e. velocity of elementary particle along x^1 performs sinusoidal oscillations at frequency equal to double angular velocity of space rotation.

Taking into account that $\mathrm{v}^1 = \frac{dx^1}{d\tau}$, we integrate the obtained formula (4.122) to $d\tau$. We obtain

$$x^1 = \frac{\mathrm{v}^1_{(0)}}{2\Omega} \sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2} \cos(2\Omega\tau) + C_3\,. \tag{4.123}$$

Assuming that at the initial moment $\tau=0$ the value $x^1 = x^1_{(0)}$ we obtain the integration constant $C_3 = x^1_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2}$. Then we have

$$x^1 = \frac{\mathrm{v}^1_{(0)}}{2\Omega} \sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2} \cos(2\Omega\tau) + x^1_0 + \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2}\,, \tag{4.124}$$

i. e. coordinate x^1 of elementary particle also performs free oscillations at frequency 2Ω.

Now having the obtained v^1 (4.122) substituted into the second equation (4.114), we arrive to

$$\frac{d\mathrm{v}^2}{d\tau} = 2\Omega \mathrm{v}^1_{(0)} \cos(2\Omega\tau) + \dot{\mathrm{v}}^1_{(0)} \sin(2\Omega\tau)\,, \tag{4.125}$$

which after integration gives formula for v^2

$$\mathrm{v}^2 = \mathrm{v}^1_{(0)} \sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + C_4\,. \tag{4.126}$$

Assuming for the moment $\tau=0$ velocity $\mathrm{v}^2 = \mathrm{v}^2_{(0)}$, we obtain the constant $C_3 = \mathrm{v}^2_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}$. Then

$$\mathrm{v}^2 = \mathrm{v}^1_{(0)} \sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + \mathrm{v}^2_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\,. \tag{4.127}$$

Taking into account that $\mathrm{v}^2 = \frac{dx^2}{d\tau}$, we integrate the formula to $d\tau$. Then we obtain the formula for coordinate x^2 of elementary particle

$$x^2 = -\frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2} \sin(2\Omega\tau) - \frac{\mathrm{v}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + \mathrm{v}^2_{(0)}\tau + \frac{\mathrm{v}^1_{(0)}\tau}{2\Omega} + C_5\,. \tag{4.128}$$

Integration constant can be found from the conditions $x^2 = x^2_{(0)}$ at $\tau=0$ as $C_5 = x^2_{(0)} + \frac{\mathrm{v}^1_{(0)}}{2\Omega}$. Then coordinate x^2 finally is

$$x^2 = \mathrm{v}^2_{(0)}\tau + \frac{\mathrm{v}^1_{(0)}\tau}{2\Omega} - \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2} \sin(2\Omega\tau) - \frac{\mathrm{v}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + x^2_{(0)} + \frac{\mathrm{v}^1_{(0)}}{2\Omega}\,. \tag{4.129}$$

From this formula we see: if at the initial moment of observable time $\tau=0$ elementary particle had velocity $\mathrm{v}^2_{(0)}$ along x^2 and acceleration $\dot{\mathrm{v}}^1_{(0)}$ along x^1, then this particle, along with free oscillations of coordinate x^2 at frequency, equal to double angular velocity of space rotation Ω, is subjected to linear displacement by $\triangle x^2 = \mathrm{v}^2_{(0)}\tau + \frac{\mathrm{v}^1_{(0)}\tau}{2\Omega}$.

Refering back to live forces integral (solution of scalar equation of motion) for a spin elementary particle $m + \frac{\eta}{c^2} = B = const$ (4.107), we define integration constant B. From (4.107), presented as

$$m_0 + \frac{\eta_0}{c^2} = B\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}\,, \tag{4.130}$$

we see that the square of observable velocity of particle $\mathrm{v}^2 = const$. Because components of observable velocity of particle have been already defined, we can present the formula for its square, which, because the three-dimensional metric in question is a Euclidean one, becomes

$$\begin{gathered}\left[\mathrm{v}^1\right]^2 + \left[\mathrm{v}^2\right]^2 + \left[\mathrm{v}^3\right]^2 = \left[\mathrm{v}^1_{(0)}\right]^2 + \left[\mathrm{v}^2_{(0)}\right]^2 + \left[\mathrm{v}^3_{(0)}\right]^2 + \\ + \frac{\left[\dot{\mathrm{v}}^1_{(0)}\right]^2}{2\Omega^2} + \frac{\dot{\mathrm{v}}^1_{(0)}\mathrm{v}^2_{(0)}}{\Omega} + 2\left[\mathrm{v}^2_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\right]\left[\mathrm{v}^1_{(0)}\sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\cos(2\Omega\tau)\right].\end{gathered} \tag{4.131}$$

We see that the square of velocity conserves, if $\dot{\mathrm{v}}^2_{(0)} = 0$ and $\dot{\mathrm{v}}^1_{(0)} = 0$. The integration constant B from the live forces integral is

$$B = \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2_{(0)}}{c^2}}}\left(m_0 + \frac{\eta_0}{c^2}\right), \qquad \mathrm{v}^2_{(0)} = \left(\mathrm{v}^1_{(0)}\right)^2 + \left(\mathrm{v}^3_{(0)}\right)^2 = const, \tag{4.132}$$

while the live forces integral itself (4.170) becomes

$$m + \frac{\eta}{c^2} = \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2_{(0)}}{c^2}}}\left(m_0 + \frac{\eta_0}{c^2}\right), \tag{4.133}$$

i. e. is the condition of conservation of the sum of relativistic mass of particle m and its spin-mass $\frac{\eta}{c^2}$.

A note should be taken here concerning all we have said in the above on elementary particles. Taking into account in $\eta_0 = n\hbar^{mn}A_{mn}$ that $A_{mn} = \varepsilon_{mnk}\Omega^{*k}$, we obtain

$$\eta_0 = n\hbar^{mn}A_{mn} = 2n\hbar_{*k}\Omega^{*k}. \tag{4.134}$$

where $\hbar_{*k} = \frac{1}{2}\varepsilon_{nmk}\hbar^{mn}$. Formally, $\hbar_{*k}$ is a three-dimensional pseudovector of the inner momentum of elementary particle. Hence η_0 is a scalar product of three-dimensional pseudovectors: that of inner momentum of particle $\hbar_{*k}$ and that of angular velocity rotation of space Ω^{*k}. Hence spin interaction is absent if pseudovectors of inner rotation and external rotation of space are collinear.

Now we refer back to equations of motion of spin-particles. Taking into account integration constants, vector equations of motion solve as

$$\begin{aligned} &\mathrm{v}^1 = \mathrm{v}^1_{(0)}\cos(2\Omega\tau), && x^1 = \frac{\mathrm{v}^1_{(0)}}{2\Omega}\sin(2\Omega\tau) + x^1_{(0)}, \\ &\mathrm{v}^2 = \mathrm{v}^2_{(0)}\sin(2\Omega\tau), && x^2 = -\frac{\mathrm{v}^1_{(0)}}{2\Omega}\cos(2\Omega\tau) + \frac{\mathrm{v}^1_{(0)}}{2\Omega} + x^2_{(0)}, \\ &\mathrm{v}^3 = \mathrm{v}^3_{(0)}, && x^3 = \mathrm{v}^3_{(0)}\tau + x^3_{(0)}. \end{aligned} \tag{4.135}$$

We are going to look at the form of spatial curve along which our-world mass-bearing particle moves. We set a frame of reference so that the initial displacement of particle is zero $x^1_{(0)} = x^2_{(0)} = x^3_{(0)} = 0$. Now all its spatial coordinates at an arbitrary moment of time are

$$x^1 = x = a\sin(2\Omega\tau), \quad x^2 = y = a\left[1 - \cos(2\Omega\tau)\right], \quad x^3 = z = b\tau, \tag{4.136}$$

where $a = \frac{\mathrm{v}^1_{(0)}}{2\Omega}$, $b = \mathrm{v}^3_{(0)}$. The obtained solutions for coordinates are parametric equations of a surface, along which mass-bearing particle travels. To illustrate what kind of surface it is, we switch from parametric notation to coordinate one, removing parameter τ from the equations. Putting formulas for x and y in the power of two we obtain

$$x^2 + y^2 = 2a^2\left[1 - \cos(2\Omega\tau)\right] = 4a^2\sin^2(\Omega\tau) = 4a^2\sin^2\frac{z\Omega}{b}. \tag{4.137}$$

The obtained result reminds of a spiral line equation $x^2+y^2=a^2$, $z=b\tau$. However the similarity is not complete — the particle travels along the surface of a cylinder at a constant velocity $b=\mathrm{v}^3_{(0)}$ along its z axis, while the radius of the cylinder oscillates at frequency Ω within the range[22] from zero up to the maximum $2a=\frac{\mathrm{v}^1_{(0)}}{\Omega}$ at $z=\frac{\pi k b}{2\Omega}$.

So our-world three-dimensional trajectory of mass-bearing elementary particle reminds of a spiral line "wound" over an long oscillating cylinder. Particle's life span is the length of the cylinder divided by its speed along z axis (cylinder axis). Oscillations of the cylinder are energy "breath ins" and "breath outs" of the particle.

That means that the cylinder we obtained is *the cylinder of events* of particle from its birth in our world (act of materialization) through its death (dematerialization). But even after death (decay) of particle the cylinder of events does not disappear completely, but *splits* into a few cylinders of events of other particles, produced by the decay either in our world or in the mirror world.

Therefore analysis of births and decays of elementary particles in General Relativity implies analysis of branch points of cylinders of events taking into account possible branches that lead into the mirror world.

If we consider motion of two linked spin-particles that rotate around a common center of masses, for instance, that of positronium (dumb-bell shaped system of electron and positron), we obtain a double DNA-like spiral — a twisted "rope ladder" with a number of steps (links of particles), wound over an oscillating cylinder of events.

Now we are going to solve equations of motion of mass-bearing spin-particle in the mirror world, a world with reverse flow of time. Under physical conditions we consider (stationary rotation of space at low velocity, absence of deformation and Euclidean three-dimensional metric), these equations (4.104, 4.105) become

$$-\frac{dm}{d\tau}=\frac{1}{c^2}\frac{d\eta}{d\tau}\,, \tag{4.138}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right)=-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right). \tag{4.139}$$

Solution of the scalar equation is live forces integral in the form $m+\frac{\eta}{c^2}=B=const$, as was the case for our-world particle (4.107). Substituting it into vector equations (4.139) we solve them as

$$\frac{d\mathrm{v}^i}{d\tau}=0\,, \tag{4.140}$$

hence $\mathrm{v}^i=\mathrm{v}^i_{(0)}=const$. That implies that from viewpoint of a regular observer mirror-world mass-bearing particles travel linearly at a constant velocity, as contrasted to observable motion of our-world particles that travel along oscillating "spiral" line.

On the other hand, from viewpoint of a hypothetical observer of the mirror world, motion of our-world mass-bearing particles will be linear and even, while mirror-world particles will travel along oscillating "spiral" lines.

We could also analyze motion of massless (light-like) spin-particles in a similar way, but we don't know how adequate in such case would be our assumption that linear velocity of rotation of space of reference is much smaller compared to speed of light. And it was this assumption thanks to which we were able to obtain exact solutions of equations of motion of mass-bearing elementary spin-particles. Though in general, the methods to solve equations of motion are the same for mass-bearing and massless particles.

4.6 Spin-particle in electromagnetic field

In this Section we are going to deduce and analyze chronometrically invariant dynamic equations of motion of particle that bears electric charge and spin, and travels in external electromagnetic field in

[22] Where $k=0,1,2,3,\ldots$ If $\mathrm{v}^3_{(0)}=0$, particle simply oscillates within xy plane (plane of cylinder's section).

four-dimensional pseudo-Riemannian space. The method to be used is projection of general covariant equations of parallel transfer of summary vector on space and time

$$Q^\alpha = P^\alpha + \frac{e}{c^2}A^\alpha + S^\alpha, \tag{4.141}$$

were P^α is four-dimensional impulse vector of particle that travels, in this case, along a non-geodesic trajectory. Respectively, the rest two terms are four-dimensional impulse that particle gains from interaction of its charge with electromagnetic field and the impulse gained from interaction of the spin with field of non-holonomity of space.

Note, that because vectors P^α and S^α are tangential to four-dimensional trajectory (the world line), we will assume that the third vector A^α (four-dimensional potential of electromagnetic field) is also tangential to the world line of particle. In this case the vector is $A^\alpha{=}\varphi_0\frac{dx^\alpha}{ds}$, while the formula $q^i{=}\frac{\varphi}{c}\mathrm{v}^i$ (see Section 3.8) sets the relationship between scalar potential φ and vector potential q^i of electromagnetic field.

Then physical observable components of $\tilde{\varphi}$ and $\tilde{q}^i$ of the summary vector of charged spin-particle, which are sums of similar components of all three added-up vectors, become

$$\tilde{\varphi} = \pm\left(m + \frac{e\varphi}{c^2} + \frac{\eta}{c^2}\right), \quad \tilde{q}^i = \frac{1}{c^2}m\mathrm{v}^i + \frac{1}{c^3}\left(\eta + e\varphi\right)\mathrm{v}^i, \tag{4.142}$$

where m is relativistic mass of particle, φ is scalar potential of electromagnetic field, while η describes interaction of particle's spin with external field of non-holonomity of space

$$m = \frac{m_0}{\sqrt{1-\dfrac{\mathrm{v}^2}{c^2}}}, \quad \varphi = \frac{\varphi_0}{\sqrt{1-\dfrac{\mathrm{v}^2}{c^2}}}, \quad \eta = \frac{\eta_0}{\sqrt{1-\dfrac{\mathrm{v}^2}{c^2}}}. \tag{4.143}$$

Generally these equations can be deduced in the same way as those for charged particle and spin-particle severally, save that now we have to project absolute derivative of the sum of the three vectors. Using formulas for $\tilde{\varphi}$ and $\tilde{q}^i$ (4.142), we obtain chronometrically invariant equations of motion of charged mass-bearing spin-particle that travels in our world (from past into future)

$$\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta + e\varphi\right) + \frac{\eta + e\varphi}{c^4}F_i\mathrm{v}^i - \frac{\eta + e\varphi}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{4.144}$$

$$\begin{gathered}\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{1}{c^2}\frac{d}{d\tau}\left[\left(\eta + e\varphi\right)\mathrm{v}^i\right] - \frac{2\left(\eta + e\varphi\right)}{c^2}\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + \frac{\eta + e\varphi}{c^2}F^i - \frac{\eta + e\varphi}{c^2}\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k,\end{gathered} \tag{4.145}$$

as well as equations of motion of charged mass-bearing spin-particle that travels in the mirror world (i. e. from future into past),

$$-\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k = \frac{1}{c^2}\frac{d}{d\tau}\left(\eta + e\varphi\right) + \frac{\eta + e\varphi}{c^4}F_i\mathrm{v}^i - \frac{\eta + e\varphi}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{4.146}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + mF^i + m\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k = -\frac{1}{c^2}\frac{d}{d\tau}\left[\left(\eta + e\varphi\right)\mathrm{v}^i\right] - \frac{\eta + e\varphi}{c^2}F^i - \frac{\eta + e\varphi}{c^2}\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k. \tag{4.147}$$

Parallel transfer in Riemannian space conserves length of transferred vector. Hence its square is invariant in any frame of reference. In particular, in accompanying frame of reference it is also constant and is

$$\begin{gathered}Q_\alpha Q^\alpha = g_{\alpha\beta}\left(P^\alpha + \frac{e}{c^2}A^\alpha + S^\alpha\right)\left(P^\beta + \frac{e}{c^2}A^\beta + S^\beta\right) = \\ = g_{\alpha\beta}\left(m_0 + \frac{e\varphi_0}{c^2} + \frac{\eta_0}{c^2}\right)^2\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \left(m_0 + \frac{e\varphi_0}{c^2} + \frac{\eta_0}{c^2}\right)^2.\end{gathered} \tag{4.148}$$

In Section 3.9 we already showed that orientation of four-dimensional electromagnetic potential A^{α} along the world line substantially simplifies the right parts of chronometrically invariant equations of motion of charged particle. The right part of vector equations of motion is chronometrically invariant Lorentz force $\Phi^i{=}-e\left(E^i{+}\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right)$, while the right part of the scalar equation is scalar product of electric strength vector E_i and chronometrically invariant velocity of particle. Keeping this in mind, we present chronometrically invariant equations of motion of charged mass-bearing spin-particle (4.144–4.147) in a more specific form. For particle that travels in our world (i. e. from past into future, direct flow of time), we have

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)-\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)F_i\mathrm{v}^i+\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)D_{ik}\mathrm{v}^i\mathrm{v}^k=-\frac{e}{c^2}E_i\mathrm{v}^i, \tag{4.149}$$

$$\begin{gathered}\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right]+2\left(m+\frac{\eta}{c^2}\right)\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k- \\ -\left(m+\frac{\eta}{c^2}\right)F^i+\left(m+\frac{\eta}{c^2}\right)\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=\,-\,e\left(E^i+\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right),\end{gathered} \tag{4.150}$$

and for particle in the mirror world that travels from future into past (reverse flow of time), we have

$$-\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)-\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)F_i\mathrm{v}^i+\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)D_{ik}\mathrm{v}^i\mathrm{v}^k=-\frac{e}{c^2}E_i\mathrm{v}^i, \tag{4.151}$$

$$\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right]+\left(m+\frac{\eta}{c^2}\right)F^i+\left(m+\frac{\eta}{c^2}\right)\triangle^i_{nk}\mathrm{v}^n\mathrm{v}^k=-\,e\left(E^i+\frac{1}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right). \tag{4.152}$$

Now to make concrete conclusions on motion of charged spin-particles in pseudo-Riemannian space we have to set concrete geometric structure of the space. As we did in the previous Section, where we analyzed motion of non-charged particles, we will assume that:

- because gravitational interaction on the scales of elementary particles is infinitesimal, so we can assume $w{\to}0$;
- rotation of space is stationary, i. e. $\frac{{}^*\partial v_k}{\partial t}{=}0$;
- deformation of space is absent, i. e. $D_{ik}{=}0$;
- three-dimensional coordinate metric $g_{ik}dx^idx^k$ is Euclidean, i. e. three-dimensional metric tensor is $g_{ik}=\left|\begin{array}{l}-1,\ i{=}k\\ \ \ \,0,\ i{\neq}k\end{array}\right.$;
- space rotates at a constant angular velocity Ω around $x^3{=}z$, i. e. components of linear rotation velocity of space are $v_1{=}\Omega_{12}x^2{=}\Omega y$, $v_2{=}\Omega_{21}x^1{=}-\Omega x$.

Keeping in mind these constraints, metric of space-time on the scales of elementary particles becomes

$$ds^2=c^2dt^2-2\Omega y dt dx+2\Omega x dt dy-dx^2-dy^2-dz^2, \tag{4.153}$$

while physical observable characteristics of the reference's space in the space-time with the metric are

$$F_i=0\,,\quad D_{ik}=0\,,\quad A_{12}=-A_{21}=-\Omega\,,\quad A_{23}=A_{31}=0\,. \tag{4.154}$$

As we did in the previous Section looking at motion of elementary spin-particles, we assume that velocity of space rotation is much less than speed of light (weak field of non-holonomity of space). In such case physical observable three-dimensional metric h_{ik} is Euclidean and all Christoffel symbols $\triangle^i_{jk}$ become zeroes, which dramatically simplifies the involved algebra. Then chronometrically invariant equations of motion of charged mass-bearing spin-particle in our world (in by-component notation) become

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=-\frac{e}{c^2}E_i\frac{dx^i}{d\tau}\,, \tag{4.155}$$

$$\begin{aligned}
&\frac{d\left(m+\dfrac{\eta}{c^2}\right)\mathrm{v}^1}{d\tau}+2\left(m+\frac{\eta}{c^2}\right)\Omega\mathrm{v}^2=-e\left(E^1+\frac{1}{c}\varepsilon^{1km}\mathrm{v}_k H_{*m}\right),\\
&\frac{d\left(m+\dfrac{\eta}{c^2}\right)\mathrm{v}^2}{d\tau}-2\left(m+\frac{\eta}{c^2}\right)\Omega\mathrm{v}^1=-e\left(E^2+\frac{1}{c}\varepsilon^{2km}\mathrm{v}_k H_{*m}\right),\\
&\frac{d\left(m+\dfrac{\eta}{c^2}\right)\mathrm{v}^3}{d\tau}=-e\left(E^3+\frac{1}{c}\varepsilon^{3km}\mathrm{v}_k H_{*m}\right),
\end{aligned} \tag{4.156}$$

while for mirror-world particle these are

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=\frac{e}{c^2}E_i\frac{dx^i}{d\tau}\,, \tag{4.157}$$

$$\begin{aligned}
&\frac{d\left(m+\dfrac{\eta}{c^2}\right)\mathrm{v}^1}{d\tau}=-e\left(E^1+\frac{1}{c}\varepsilon^{1km}\mathrm{v}_k H_{*m}\right),\\
&\frac{d\left(m+\dfrac{\eta}{c^2}\right)\mathrm{v}^2}{d\tau}=-e\left(E^2+\frac{1}{c}\varepsilon^{2km}\mathrm{v}_k H_{*m}\right),\\
&\frac{d\left(m+\dfrac{\eta}{c^2}\right)\mathrm{v}^3}{d\tau}=-e\left(E^3+\frac{1}{c}\varepsilon^{3km}\mathrm{v}_k H_{*m}\right).
\end{aligned} \tag{4.158}$$

From scalar equation of motion in our world (4.155) and in the mirror world (4.157) we see that the sum of relativistic mass of elementary particle and of its spin-mass (property of spin-interaction with non-holonomity field) equals to work of electric field to displace this charged particle along dx^i interval. From vector equations of motion we see that in our world (4.156) as well as in the mirror world (4.158) the sum of spatial three-dimensional impulse vector of particle and spin-impulse of particle along $x^3{=}z$ is defined only by Lorentz force's component along the same axis.

Now our goal is to obtain trajectory of elementary charged spin-particle in a particular electromagnetic field with known properties. As we did in Chapter 3, we will assume electromagnetic field constant, i. e. not dependent from time. The strengths E_i and H^{*i} are

$$E_i=\frac{\partial\varphi}{dx^i}\,, \tag{4.159}$$

$$H^{*i}=\frac{1}{2}\varepsilon^{imn}H_{mn}=\frac{1}{2c}\varepsilon^{imn}\left[\frac{\partial\left(\varphi\mathrm{v}_m\right)}{dx^n}-\frac{\partial\left(\varphi\mathrm{v}_n\right)}{dx^m}-2\varphi A_{mn}\right]. \tag{4.160}$$

In Chapter 3 we tackled a similar problem — solving equations of motion for charged mass-bearing particles, but without taking spin into account. Evidently, in a specific case when spin is zero, solutions of equations of motion of charged spin-particle, as more general ones, should coincide with those obtained in Chapter 3 within "pure" electrodynamics.

To compare our results with those obtained in electrodynamics, it would be reasonable to analyze motion of mass-bearing spin-particle in three typical kinds of electromagnetic fields, which were under study in Chapter 3 as well as in *The Classical Theory of Fields* by Landau and Lifshitz [1]: (a) uniform electric field with magnetic component absent; (b) uniform magnetic field with electric component absent; (c) uniform electric and magnetic fields.

On the other hand, electrodynamics studies motion of regular (not elementary) particles and it is not *a priori* evident that all three cases mentioned in the above are applicable, given the metric constraints, typical for micro-world. Here is why.

First, spin of particle affects its motion only if external field of non-holonomity (rotation) of space exists, hence tensor of non-holonomity $A_{ik}{\neq}0$. But from the formulas for electric and magnetic strengths E_i and H^{*i} (4.159, 4.160) we see that non-holonomity of space only affects magnetic strength. Hence we will largely focus on motion of elementary particle in magnetic field.

Second, scalar equation of motion of mass-bearing charged spin-particle (4.155)

$$\left(m_0+\frac{\eta_0}{c^2}\right)\frac{d}{d\tau}\frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}=-\frac{e}{c^2}E_i\mathrm{v}^i \tag{4.161}$$

in non-relativistic case, when particle's velocity is much less than speed of light, becomes

$$E_i\mathrm{v}^i=0\,, \tag{4.162}$$

i. e. electric field *does not perform work* to displace charged particle under constraints on metric, typical for the world of elementary particle. Because we are looking at stationary field, the obtained condition (4.162) can be presented as

$$E_i\mathrm{v}^i=\frac{\partial\varphi}{\partial x^i}\mathrm{v}^i=\frac{\partial\varphi}{\partial x^i}\frac{dx^i}{d\tau}=\frac{d\varphi}{d\tau}=0\,, \tag{4.163}$$

which implies that scalar potential of field $\varphi=const$ and

$$H^{*i}=\frac{\varphi}{2c}\varepsilon^{imn}\left[\frac{\partial\mathrm{v}_m}{\partial x^n}-\frac{\partial\mathrm{v}_n}{\partial x^m}-2\left(\frac{\partial v_m}{\partial x^n}-\frac{\partial v_n}{\partial x^m}\right)\right]. \tag{4.164}$$

For relativistic charged elementary particle electric field reveals itself (i. e. performs work to displace it) provided that the absolute value of its velocity is not stationary

$$\frac{1}{2c^2\left(1-\frac{\mathrm{v}^2}{c^2}\right)^{\frac{3}{2}}}\left(m_0+\frac{\eta_0}{c^2}\right)\frac{d\mathrm{v}^2}{d\tau}=-\frac{e}{c^2}E_i\mathrm{v}^i\neq0\,. \tag{4.165}$$

Hence electric component of field, given the constraints on metric, typical for elementary particles, reveals itself only for relativistic particles, which velocity is not constant along the trajectory. Hence all "slow-moving" particle fall out of our consideration in electric field.

Therefore, the general case, i. e. motion of elementary particle at arbitrary velocity (either low or relativistic one) should be only studied for stationary *magnetic field* (when electric component is absent). This will be done in the next Section.

4.7 Motion in stationary magnetic field

In this Section we are going to look at motion of charged spin-particle in stationary uniform magnetic field.

As we did in the previous Section, we will assume that space-time has the metric (4.153). Then F_i=0 and D_{ik}=0. Field of non-holonomity is stationary. In rotation around z out of all components of non-holonomity tensor only the components A_{12}=$-A_{21}$=$-\Omega$=$const$ are not zeroes, i. e. space rotates within xy plane at a constant velocity Ω.

Under the considered conditions the value η_0=$n\hbar^{mn}A_{mn}$, which describes interaction between spin (inner rotation) of particle and external field of non-holonomity (rotation) of the space itself, is

$$\eta_0=n\hbar^{mn}A_{mn}=n\left(\hbar^{12}A_{12}+\hbar^{21}A_{21}\right)=-2n\hbar\Omega\,, \tag{4.166}$$

where the sign before the product $\hbar\Omega$ depends only upon mutual orientation of $\hbar$ and Ω. "Plus" stands for co-directed $\hbar$ and Ω, "minus" implies they are oppositely directed.

Equations of motion of charged spin-particle become (provided potential A^α is oriented along the world line): for our-world particle

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=0\,, \tag{4.167}$$

$$\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right]+2\left(m+\frac{\eta}{c^2}\right)A_{k\cdot}^{\cdot i}\mathrm{v}^k+\left(m+\frac{\eta}{c^2}\right)\triangle_{nk}^i\mathrm{v}^n\mathrm{v}^k=-\frac{e}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}, \tag{4.168}$$

and for mirror-world particle

$$-\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=0\,, \tag{4.169}$$

$$\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right]+\left(m+\frac{\eta}{c^2}\right)\triangle_{nk}^i\mathrm{v}^n\mathrm{v}^k=-\frac{e}{c}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{4.170}$$

Having theorem of live forces integrated (scalar equation of motion), we obtain integral of live forces for charged spin-particle in stationary uniform magnetic field. In our world and in the mirror world it is

$$m+\frac{\eta}{c^2}=B=const\,,\qquad m+\frac{\eta}{c^2}=-\widetilde{B}=const\,, \tag{4.171}$$

where B is integration constant in our world and $\widetilde{B}$ is that in the mirror world. We can obtain these constants having the initial conditions at $\tau=0$ substituted into (4.171). As a result, we obtain

$$B=m_0+\frac{\eta_0}{c^2}=m_0+\frac{n\hbar^{mn}A_{mn}}{c^2}\,, \tag{4.172}$$

$$\widetilde{B}=-m_0-\frac{\eta_0}{c^2}=-m_0-\frac{n\hbar^{mn}A_{mn}}{c^2}\,. \tag{4.173}$$

The formulas for live forces integrals (4.171) imply that in absence of electric component the square of velocity of charged spin-particle conserves $\mathrm{v}^2=h_{ik}\mathrm{v}^i\mathrm{v}^k=const$.

Having the formulas for live forces integrals substituted into (4.168, 4.170), we arrive to vector equations of motion in our world and in the mirror world, respectively

$$\frac{d\mathrm{v}^i}{d\tau}+2A_{k\cdot}^{\cdot i}\mathrm{v}^k+\triangle_{nk}^i\mathrm{v}^n\mathrm{v}^k=-\frac{e}{cB}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,, \tag{4.174}$$

$$\frac{d\mathrm{v}^i}{d\tau}+\triangle_{nk}^i\mathrm{v}^n\mathrm{v}^k=-\frac{e}{c\widetilde{B}}\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{4.175}$$

These are similar to equations of motion of non-spin charged particle in stationary magnetic field (3.290, 3.291), save that here the integration constant from live forces integral, found in the right part, is not equal to relativistic mass m, as it was in electrodynamics (3.290, 3.291), but to the formula (4.171), which accounts for interaction of spin with field of non-holonomity of space. The same is true for by-component notation of vector equations (3.298, 3.299).

For those of our readers with special interest in the method of chronometric invariants we will make a note related to by-component notation of equations of motion. When obtaining components of the term $A_{k\cdot}^{\cdot 1}\mathrm{v}^k$, found only in our-world equations, we have, for instance, for $i=1$

$$A_{k\cdot}^{\cdot 1}\mathrm{v}^k=A_{1\cdot}^{\cdot 1}\mathrm{v}^1+A_{2\cdot}^{\cdot 1}\mathrm{v}^2=h^{12}A_{12}\mathrm{v}^1+h^{11}A_{21}\mathrm{v}^2, \tag{4.176}$$

where $A_{12}=-A_{21}=-\Omega$. Then obtaining $A_{1\cdot}^{\cdot 1}$ and $A_{2\cdot}^{\cdot 1}$ we have

$$A_{1\cdot}^{\cdot 1}=h^{1m}A_{1m}=h^{11}A_{11}+h^{12}A_{12}=h^{12}A_{12}\,, \tag{4.177}$$

$$A_{2\cdot}^{\cdot 1}=h^{1m}A_{2m}=h^{11}A_{21}+h^{12}A_{22}=h^{11}A_{21}\,, \tag{4.178}$$

where h^{11} and h^{12} are elements of a matrix reciprocal to matrix h_{ik}

$$h^{11}=\frac{h_{22}}{h}\,,\qquad h^{12}=-\frac{h_{12}}{h}\,. \tag{4.179}$$

Then because determinant of three-dimensional observable metric tensor (see Section 3.12) is

$$h = \det \|h_{ik}\| = 1 + \frac{\Omega^2 \left(x^2 + y^2\right)}{c^2}\,, \tag{4.180}$$

the unknown value $A_{k\cdot}^{\cdot 1}\mathrm{v}^k$ (4.176) is

$$A_{k\cdot}^{\cdot 1}\mathrm{v}^k = \frac{\Omega}{h}\left[\frac{\Omega^2}{c^2}xy\dot{x} + \left(1 + \frac{\Omega^2 x^2}{c^2}\right)\dot{y}\right]. \tag{4.181}$$

Component $A_{k\cdot}^{\cdot 2}\mathrm{v}^k$, found in equation of motion along y, can be found in a similar way.

Now we are going to refer back to vector equations of motion of charged spin-particle in stationary uniform magnetic field. We are going to approach them in two possible cases of mutual orientation of magnetic field and non-holonomity field.

A Magnetic field is co-directed with non-holonomity field

We assume that field of non-holonomity is directed along z and is weak. Then vector equations of motion of mass-bearing charged spin-particle, in by-component notation, become: for our-world particle

$$\ddot{x} + 2\Omega\dot{y} = -\frac{eH}{cB}\dot{y}\,, \quad \ddot{y} - 2\Omega\dot{x} = -\frac{eH}{cB}\dot{x}\,, \quad \ddot{z} = 0\,, \tag{4.182}$$

and for mirror-world particle

$$\ddot{x} = -\frac{eH}{c\widetilde{B}}\dot{y}\,, \quad \ddot{y} = -\frac{eH}{c\widetilde{B}}\dot{x}\,, \quad \ddot{z} = 0\,. \tag{4.183}$$

These equations are also different from those for non-spin charged particle in stationary magnetic field, co-directed with weak non-holonomity field (3.104, 3.305) only by having in the right part the integration constant from live forces integral, which describes interaction of spin with field of non-holonomity, instead of relativistic mass of particle.

Using ready solutions from Section 3.12 we can immediately obtain the formulas for coordinates of our-world charged spin-particle

$$x = -\left[\dot{y}_{(0)}\cos\left(2\Omega + \omega\right)\tau + \dot{x}_{(0)}\sin\left(2\Omega + \omega\right)\tau\right]\frac{1}{2\Omega + \omega} + x_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega + \omega}\,, \tag{4.184}$$

$$y = \left[\dot{y}_{(0)}\sin\left(2\Omega + \omega\right)\tau - \dot{x}_{(0)}\cos\left(2\Omega + \omega\right)\tau\right]\frac{1}{2\Omega + \omega} + y_{(0)} - \frac{\dot{x}_{(0)}}{2\Omega + \omega}\,, \tag{4.185}$$

and those for mirror-world particle

$$x = -\frac{1}{\omega}\left(\dot{y}_{(0)}\cos\omega\tau + \dot{x}_{(0)}\sin\omega\tau\right) + x_{(0)} + \frac{\dot{y}_{(0)}}{\omega}\,, \tag{4.186}$$

$$y = \frac{1}{\omega}\left(\dot{y}_{(0)}\sin\omega\tau - \dot{x}_{(0)}\cos\omega\tau\right) + y_{(0)} - \frac{\dot{x}_{(0)}}{\omega}\,, \tag{4.187}$$

which are different from solutions for charged particle in electrodynamics only by the fact that frequency ω accounts for interaction of spin with field of non-holonomity.

In our world masses of particles are positive, hence frequency ω is

$$\omega = \frac{eH}{mc + \dfrac{\eta}{c}} = \frac{eH\sqrt{1 - \dfrac{\mathrm{v}_{(0)}^2}{c^2}}}{m_0 c + \dfrac{\eta_0}{c}} = \frac{eH\sqrt{1 - \dfrac{\mathrm{v}_{(0)}^2}{c^2}}}{m_0 c \mp \dfrac{2n\hbar\Omega}{c}}\,, \tag{4.188}$$

where the sign in the denominator depends upon mutual orientation of $\hbar$ and Ω: "minus" stands for co-directed $\hbar$ and Ω (their scalar product is positive), while "plus" implies they are oppositely directed, irrespective of choice of right or left-hand frame of reference.

Masses of particles that inhabit the mirror world are always negative

$$m = \frac{-m_0}{\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}} < 0\,. \tag{4.189}$$

Hence in the mirror world frequency ω is

$$\omega = \frac{eH}{mc+\frac{\eta}{c}} = \frac{eH\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}{-m_0c+\frac{\eta_0}{c}} = \frac{eH\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}{-m_0c\mp\frac{2n\hbar\Omega}{c}}\,. \tag{4.190}$$

Note that the obtained formulas for coordinates (4.184–4.187) already took account of the fact that the square of particle's velocity is constant both in our world and in the mirror world (respectively)

$$\dot{x}_{(0)} + \frac{\ddot{y}_0}{2\Omega+\omega} = 0\,, \qquad \dot{x}_{(0)} + \frac{\ddot{y}_0}{\omega} = 0\,, \tag{4.191}$$

which results from integral of live forces (Section 3.12).

The third equation of motion (along z) solves simply as

$$z = \dot{z}_{(0)}\tau + z_{(0)}\,. \tag{4.192}$$

The obtained formulas for coordinates (4.184–4.187) say that mass-bearing charged spin-particle in stationary uniform magnetic field, parallel to weak field of non-holonomity, performs *harmonic oscillations* along x and y. In our world the frequency of the oscillations is

$$\widetilde{\omega} = 2\Omega+\omega = 2\Omega + \frac{eH}{m_0c\mp\frac{2n\hbar\Omega}{c}}\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}\,. \tag{4.193}$$

In the mirror world particle performs similar oscillation at frequency ω, as obtained in (4.190).

In a weak field of non-holonomity $n\hbar\Omega$ is much less than energy m_0c^2, because for any small value α it is true that $\frac{1}{1\mp\alpha}\cong 1\pm\alpha$, for low velocities we have

$$\widetilde{\omega} \cong 2\Omega + \frac{eH}{m_0c}\left(1\pm\frac{2n\hbar\Omega}{m_0c^2}\right). \tag{4.194}$$

If at the initial moment of time the displacement and the velocity of our-world particle satisfy the conditions

$$x_{(0)} + \frac{\dot{y}_0}{2\Omega+\omega} = 0\,, \qquad y_{(0)} - \frac{\dot{x}_0}{2\Omega+\omega} = 0\,, \tag{4.195}$$

it will travel, like a charged non-spin particle, within xy plane along *a circle*[23]

$$x^2+y^2 = \frac{\dot{y}_0^2}{(2\Omega+\omega)^2}\,. \tag{4.196}$$

But in this case, its radius equal to

$$r = \frac{\dot{y}_0}{2\Omega+\omega} = \frac{\dot{y}_0}{2\Omega+\frac{eH}{m_0c\mp\frac{2n\hbar\Omega}{c}}\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}\,, \tag{4.197}$$

[23]We set axis y along the initial impulse of the particle, which is always possible. Then all formulas for coordinates will have zero initial velocity of particle along x.

will depend upon absolute value and orientation of the spin. If the initial velocity of charged particle with spin oriented along magnetic field (along z axis) is not zero, it travels along magnetic field along *spiral line* with the same radius r (4.197).

Mirror-world particle, provided its displacement and velocity at the initial moment of time satisfy the conditions

$$x_{(0)} + \frac{\dot{y}_0}{\omega} = 0 \,, \qquad y_{(0)} - \frac{\dot{x}_0}{\omega} = 0 \,, \tag{4.198}$$

will also travel along a circle

$$x^2 + y^2 = \frac{\dot{y}_0^2}{\omega^2} \,, \tag{4.199}$$

with radius

$$r = \frac{\dot{y}_0}{\omega} = \frac{\dot{y}_0}{\dfrac{eH}{-m_0 c \mp \dfrac{2n\hbar\Omega}{c}} \sqrt{1 - \dfrac{\mathrm{v}_{(0)}^2}{c^2}}} \,. \tag{4.200}$$

In general case, i. e. with no additional conditions (4.195, 4.198) imposed, the trajectory within xy plane will be not circle.

Now we are going to obtain energy and impulse of charged spin-particle in magnetic field. Using formulas for live forces integrals, we can find that the value $\eta_0 = n\hbar^{mn} A_{mn} = n(\hbar^{12} A_{12} + \hbar^{21} A_{21}) = -2n\hbar\Omega$. Then for mass-bearing our-world particle we have

$$E_{tot} = Bc^2 = \frac{m_0 c^2 \mp 2n\hbar\Omega}{\sqrt{1 - \dfrac{\mathrm{v}_{(0)}^2}{c^2}}} = const \,, \tag{4.201}$$

and for mass-bearing mirror-world particle we have

$$E_{tot} = \widetilde{B}c^2 = \frac{-m_0 c^2 \mp 2n\hbar\Omega}{\sqrt{1 - \dfrac{\mathrm{v}_{(0)}^2}{c^2}}} = const \,. \tag{4.202}$$

Because in this Section we assumed that electric component of field is absent, electromagnetic field does not contribute into the total energy of particle (as known, magnetic field does not perform work to displace electric charge).

From the obtained formulas (4.201, 4.202) we see that the total energy of spin-particle is constant, while its absolute value depends upon mutual orientation of particle's inner momentum $\hbar$ and angular velocity of space rotation Ω.

The latter statement requires some comments to be made. By definition scalar value n (absolute value of spin in $\hbar$ units) is always positive, while $\hbar$ and Ω are numerical values of components of antisymmetric tensors h^{ik} and Ω_{ik}, which take opposite signs in right or left-handed frames of reference. But because we are dealing with the product of the values, only their mutual orientation matters, which does not depend upon choice of right or left-handed frame of reference.

If $\hbar$ and Ω are co-directed, the total energy of our-world particle E_{tot} (4.201) is the sum of its relativistic energy $E = mc^2$ and "spin-energy"

$$E_s = \frac{2n\hbar\Omega}{\sqrt{1 - \dfrac{\mathrm{v}_{(0)}^2}{c^2}}} \,, \tag{4.203}$$

i. e. the total energy is greater than $E = mc^2$.

If $\hbar$ and Ω are oppositely directed, the value E_{tot} is the difference between the relativistic energy and the spin-energy. Such orientation permits a specific case, when $m_0 c^2 = 2n\hbar\Omega$ and therefore the total energy becomes zero (this case will be discussed in the next Section).

For negative masses particles, which inhabit the mirror world, the situation is different: the total energy E_{tot} (4.202) is negative and by its absolute value is greater than relativistic energy $E{=}-mc^2$, provided that $\hbar$ and Ω are oppositely oriented.

The formula for the total three-dimensional impulse of charged spin-particle in magnetic field in our world is

$$p^i_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\mathrm{v}^i = m\mathrm{v}^i \mp \frac{2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\mathrm{v}^i, \tag{4.204}$$

i. e. is an algebraic sum of relativistic observable impulse $p^i{=}m\mathrm{v}^i$ and of spin-impulse that particle gains from field of non-holonomity. The total impulse of spin-particle is greater than relativistic impulse, if $\hbar$ and Ω are co-directed and is less then relativistic impulse otherwise.

In case of opposite mutual orientation of $\hbar$ and Ω the total impulse of spin-particle becomes zero (and so does the total energy) provided the condition $m_0c^2{=}2n\hbar\Omega$ is true.

For mirror-world particle in magnetic field spatial impulse is

$$p^i_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\mathrm{v}^i = -m\mathrm{v}^i \mp \frac{2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\mathrm{v}^i, \tag{4.205}$$

i. e. the particle moves more slowly if $\hbar$ and Ω are co-directed and faster otherwise.

Components of velocity of charged spin-particle in magnetic field co-directed with non-holonomity field, taking into account conditions (4.191), in our world are

$$\dot{x} = \dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau - \dot{x}_{(0)}\cos\left(2\Omega+\omega\right)\tau\,, \tag{4.206}$$

$$\dot{y} = \dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau + \dot{x}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,, \tag{4.207}$$

and in the mirror world they are

$$\dot{x} = \dot{y}_{(0)}\sin\omega\tau - \dot{x}_{(0)}\cos\omega\tau\,, \tag{4.208}$$

$$\dot{y} = \dot{y}_{(0)}\cos\omega\tau + \dot{x}_{(0)}\sin\omega\tau\,. \tag{4.209}$$

Then components of the total impulse of particle are (the initial impulse within xy plane is directed along y): for our world

$$p^1_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,, \tag{4.210}$$

$$p^2_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau\,, \tag{4.211}$$

$$p^3_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{z}_{(0)}\,, \tag{4.212}$$

where ω is as of (4.189); and for the mirror world

$$p^1_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\sin\omega\tau\,, \tag{4.213}$$

$$p^2_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\cos\omega\tau\,, \tag{4.214}$$

$$p^3_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{z}_{(0)}\,, \tag{4.215}$$

where ω is as of (4.190). Noteworthy, though strength of magnetic field does not appear in the the total energy E_{tot}, it appears in that for the total impulse, being part of the formula for ω (4.190).

B Magnetic field is orthogonal to non-holonomity field

Now we are going to approach motion of mass-bearing charged spin-particle in magnetic field, orthogonal to field of non-holonomity of space. Of course we are still assuming magnetic field is stationary and uniform. So field of non-holonomity is directed along z and is weak, while magnetic field is directed along y. Then vector equations of motion of spin-particle will be similar to those for non-spin particle, as obtained under the above field conditions for a non-spin particle in our world (3.338)

$$\ddot{x} + 2\Omega\dot{y} = \frac{eH}{cB}\dot{z}\,, \qquad \ddot{y} - 2\Omega\dot{x} = 0\,, \qquad \ddot{z} = -\frac{eH}{cB}\dot{x}\,. \tag{4.216}$$

The difference from (3.338) is that here the denominator of the right part instead of the relativistic mass contains integration constant from the live forces integral, which accounts for interaction between the spin and the non-holonomity field. After integration the equations solve as

$$x = \frac{\dot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\cos\widetilde{\omega}\tau + x_{(0)} + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\,, \tag{4.217}$$

$$y = -\frac{2\Omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right) + \dot{y}_{(0)}\tau + \frac{2\Omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\tau + y_{(0)} + \frac{2\Omega}{\widetilde{\omega}^2}\dot{x}_{(0)}\,, \tag{4.218}$$

$$z = \frac{\omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right) + \dot{z}_{(0)}\tau - \frac{\omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\tau + z_{(0)} - \frac{\omega}{\widetilde{\omega}^2}\dot{x}_{(0)}\,, \tag{4.219}$$

which are different from the respective solutions for a non-spin particle by the fact that frequency $\widetilde{\omega}$ here depends upon spin and its mutual orientation with field of non-holonomity

$$\widetilde{\omega} = \sqrt{4\Omega^2+\omega^2} = \sqrt{4\Omega^2 + \frac{e^2H^2\left(1-\frac{\mathrm{v}^2_{(0)}}{c^2}\right)^2}{\left(m_0c^2 \mp \frac{2n\hbar\Omega}{c}\right)^2}}\,. \tag{4.220}$$

Subsequently, equation of trajectory of a spin-particle is similar to that of a non-spin particle. In a specific case, i. e. under certain initial conditions, the equation of its trajectory is that of *a sphere*

$$x^2+y^2+z^2 = \frac{1}{\widetilde{\omega}^2}\dot{x}^2_{(0)}\,, \tag{4.221}$$

which radius, as contrasted to the radius of trajectory of non-spin particle, depends upon spin of particle and its orientation in respect to the field of non-holonomity

$$r = \frac{1}{\sqrt{4\Omega^2 + \frac{e^2H^2\left(1-\frac{\mathrm{v}^2_{(0)}}{c^2}\right)^2}{\left(m_0c^2 \mp \frac{2n\hbar\Omega}{c}\right)^2}}}\,\dot{x}_{(0)}\,. \tag{4.222}$$

For mirror-world particle vector equations of motion in weak field of non-holonomity, orthogonal to magnetic field (and directed along y), are

$$\ddot{x} = \frac{eH}{c\widetilde{B}}\dot{z}\,, \qquad \ddot{y} = 0\,, \qquad \ddot{z} = -\frac{eH}{c\widetilde{B}}\dot{x}\,, \tag{4.223}$$

i. e. are different from equations of motion of our-world particle (4.216) by absence of the terms that contain space rotation velocity Ω. As a result their solutions can be obtained from the solutions for our world (4.217–4.219) if we assume $\widetilde{\omega}=\omega$. Subsequently, equation of trajectory of spin-particle in the mirror world is

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2}\dot{x}^2_{(0)}\,, \qquad r = \frac{-m_0c^2 \mp \dfrac{2n\hbar\Omega}{c}}{eH\sqrt{1-\dfrac{\mathrm{v}^2_{(0)}}{c^2}}}\dot{x}_{(0)}\,. \tag{4.224}$$

The formula for the total energy of spin-particle E_{tot} in magnetic field, orthogonal to field of non-holonomity, is the same as it was for the case of parallel orientation of fields. But the formulas for components of the total impulse (4.201, 4.205) are different, because they include velocity of particle that depends upon mutual orientation of magnetic field and non-holonomity field. In this particular case, where the fields are orthogonal to each other, components of particle's velocity (obtained by derivation of formulas for coordinates) in our world are

$$\dot{x} = \dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\,, \tag{4.225}$$

$$\dot{y} = \frac{2\Omega}{\widetilde{\omega}}\dot{x}_{(0)}\sin\widetilde{\omega}\tau - \frac{2\Omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\cos\widetilde{\omega}\tau + \dot{y}_{(0)} + \frac{2\Omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\,, \tag{4.226}$$

$$\dot{z} = \frac{\omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\cos\widetilde{\omega}\tau - \frac{\omega}{\widetilde{\omega}}\dot{x}_{(0)}\sin\widetilde{\omega}\tau + \dot{z}_{(0)} - \frac{\omega}{\widetilde{\omega}^2}\ddot{x}_{(0)}\,, \tag{4.227}$$

and in the mirror world are

$$\dot{x} = \dot{x}_{(0)}\cos\omega\tau + \frac{\ddot{x}_{(0)}}{\omega}\sin\omega\tau\,, \tag{4.228}$$

$$\dot{y} = \dot{y}_{(0)}\,, \tag{4.229}$$

$$\dot{z} = \frac{1}{\omega}\ddot{x}_{(0)}\cos\widetilde{\omega}\tau - \dot{x}_{(0)}\sin\omega\tau + \dot{z}_{(0)} - \frac{1}{\omega}\ddot{x}_{(0)}\,. \tag{4.230}$$

Now we assume that the initial acceleration of particle and the integration constants are zeroes and set axis x along the initial impulse of particle. From a frame of such consideration we obtain components of its total impulse in our world

$$p^1_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\dfrac{\mathrm{v}^2_{(0)}}{c^2}}}\dot{x}_{(0)}\cos\widetilde{\omega}\tau\,, \tag{4.231}$$

$$p^2_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\dfrac{\mathrm{v}^2_{(0)}}{c^2}}}\frac{2\Omega}{\widetilde{\omega}}\dot{x}_{(0)}\sin\widetilde{\omega}\tau\,, \tag{4.232}$$

$$p^3_{tot} = \frac{m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\dfrac{\mathrm{v}^2_{(0)}}{c^2}}}\frac{\omega}{\widetilde{\omega}}\dot{x}_{(0)}\sin\widetilde{\omega}\tau\,, \tag{4.233}$$

and in the mirror world, respectively

$$p^1_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{x}_{(0)}\cos\widetilde{\omega}\tau\,, \tag{4.234}$$

$$p^2_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)} = 0\,, \tag{4.235}$$

$$p^3_{tot} = \frac{-m_0c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{x}_{(0)}\sin\widetilde{\omega}\tau\,. \tag{4.236}$$

As easily seen, the solutions obtained here can be transformed into respective ones from electrodynamics (Section 3.12) by assuming $\hbar\to 0$.

4.8 Law of quantization of masses of elementary particles

Scalar equations of motion of charged spin-particle in electromagnetic field in our world and in the mirror world are, respectively

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right) = -\frac{e}{c^2}E_i\mathrm{v}^i, \quad -\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right) = -\frac{e}{c^2}E_i\mathrm{v}^i. \tag{4.237}$$

The equations can be easily integrated to produce live forces integrals

$$m+\frac{\eta}{c^2} = B\,, \qquad -\left(m+\frac{\eta}{c^2}\right) = \widetilde{B}\,, \tag{4.238}$$

where B is integration constant in our world and $\widetilde{B}$ is that in the mirror world. The constants depend only upon the initial conditions. Hence it is possible to choose them as to make the integration constants zeroes.

We will find out under what initial conditions integration constants in the scalar equations of motion become zeroes. For charged spin-particles in our world and in the mirror world (4.238), respectively

$$m+\frac{\eta}{c^2} = 0\,, \qquad -\left(m+\frac{\eta}{c^2}\right) = 0\,, \tag{4.239}$$

while the right parts of the vector equations of motion (4.150, 4.152), which contain three-dimensional invariant Lorentz force, also become zeroes. In other words, with integration constants in scalar equations equal to zero electromagnetic field does not affect particles.

Having relativistic square root cancelled in (4.239), which is always possible for particles that have non-zero rest-masses, we can present these formulas in a notation that does not depend upon velocity of particle. Then for our-world mass-bearing particles we have

$$m_0c^2 = -n\hbar^{mn}A_{mn}\,, \tag{4.240}$$

and for mirror-world mass-bearing particles we have

$$m_0c^2 = n\hbar^{mn}A_{mn}\,. \tag{4.241}$$

We will refer to these formulas (4.240, 4.241) as *the law of quantization of masses of elementary particles*, which reads:

> **Rest-mass of spin-particle is proportional to energy of interaction of its spin with field of non-holonomity of space, taken with the opposite sign.**

Or, in other words:

> **Rest-energy of mass-bearing elementary particle, which has a spin equals to energy of interaction of its spin with field of non-holonomity of space, taken with the opposite sign.**

Because in the mirror world energy of particle has negative value, "plus" in the right part of (4.241) stands for energy of interaction in the mirror world taken with the opposite sign. The same is true for "minus" in (4.240) for our world.

Evidently, these quantum formulas are not applicable to non-spin particles.

Let us make some quantitative estimates that stem from the obtained law. We will obtain numerical values of $\eta_0 = n\hbar^{mn}A_{mn}$, which characterize energy of interaction between spin and field of non-holonomity ("spin-energy"), as follows. We formulate tensor of angular velocities of space rotation A_{mn} with pseudovector of the rotation $\Omega^{*i} = \frac{1}{2}\varepsilon^{imn}A_{mn}$

$$\Omega^{*i}\varepsilon_{imn} = \frac{1}{2}\varepsilon^{ipq}\varepsilon_{imn}A_{pq} = \frac{1}{2}\left(\delta^p_m\delta^q_n - \delta^p_n\delta^q_m\right)A_{pq} = A_{mn}\,. \tag{4.242}$$

Hence $A_{mn} = \varepsilon_{imn}\Omega^{*i}$. Then because

$$\frac{1}{2}\varepsilon_{imn}\hbar^{mn} = \hbar_{*i} \tag{4.243}$$

is Planck pseudovector, the value $\eta_0 = n\hbar^{mn}\varepsilon_{imn}\Omega^{*i}$ is

$$\eta_0 = 2n\hbar_{*i}\Omega^{*i}, \tag{4.244}$$

i. e. is double scalar product of three-dimensional Planck pseudovector and three-dimensional pseudovector of space rotation velocity, multiplied by particle's spin quantum number (scalar).

As known, scalar product of two pseudovectors is product of their absolute values (modules) multiplied by cosine of angle between them. Then if $\hbar_{*i}$ and Ω^{*i} are co-directed then the cosinus is positive, hence

$$\eta_0 = 2n\hbar_{*i}\Omega^{*i} = 2n\hbar\Omega\cos(\widehat{\vec{\hbar}\,;\vec{\Omega}}) > 0\,, \tag{4.245}$$

while if they are oppositely directed, then

$$\eta_0 = 2n\hbar_{*i}\Omega^{*i} = 2n\hbar\Omega\cos(\widehat{\vec{\hbar}\,;\vec{\Omega}}) < 0\,. \tag{4.246}$$

Therefore for our-world mass-bearing particles integration constant from live forces integral becomes zero, provided that pseudovectors $\hbar_{*i}$ and Ω^{*i} are oppositely oriented. For mirror-world particles the constant becomes zero if pseudovectors $\hbar_{*i}$ and Ω^{*i} are co-oriented.

This implies that if energy of interaction of mass-bearing spin-particle with field of space's non-holonomity becomes equal to its rest-energy $E = m_0c^2$, impulse of particle reveals itself neither in our world nor in the mirror world.

We assume that axis z is co-directed with pseudovector of angular velocity of space rotation Ω^{*i}. Then out of all three components of Ω^{*i} the only non-zero one is

$$\Omega^{*3} = \frac{1}{2}\varepsilon^{3mn}A_{mn} = \frac{1}{2}\left(\varepsilon^{312}A_{12} + \varepsilon^{321}A_{21}\right) = \varepsilon^{312}A_{12} = \frac{e^{312}}{\sqrt{h}}A_{12}\,. \tag{4.247}$$

To simplify the algebra we assume that three-dimensional metric g_{ik} is Euclidean, while the space rotates at constant angular velocity Ω. Then components of linear velocity of rotation are $v_1 = \Omega x$, $v_2 = -\Omega y$, and $A_{12} = -\Omega$. Hence

$$\Omega^{*3} = \frac{e^{312}}{\sqrt{h}}A_{12} = \frac{A_{12}}{\sqrt{h}} = -\frac{\Omega}{\sqrt{h}}\,. \tag{4.248}$$

The square root of determinant of observable metric tensor, as defined from (4.180) is

$$\sqrt{h} = \sqrt{\det\|h_{ik}\|} = \sqrt{1 + \frac{\Omega^2\left(x^2 + y^2\right)}{c^2}}\,. \tag{4.249}$$

Because we are dealing with very small coordinate values on the scales of elementary particles, we can assume $\sqrt{h} \approx 1$ and according to (4.248) also $\Omega^{*3} = -\Omega = const$. Then *the law of quantization* *of masses of elementary particles* (4.240) becomes: for mass-bearing particles in our world and mass-bearing particles in the mirror world, respectively

$$m_0 = \frac{2n\hbar\Omega}{c^2}\,, \qquad m_0 = -\frac{2n\hbar\Omega}{c^2}\,. \tag{4.250}$$

Hence for elementary particles in our world that bear spin the following relationship between their rest-masses m_0 and angular velocity of space's rotation Ω is true

$$\Omega = \frac{m_0 c^2}{2n\hbar}\,. \tag{4.251}$$

This means that rest-mass (the true mass) of observable object, under regular conditions not dependent from properties of observer's references, on the scales of elementary particles becomes strictly dependent from such; in particular, from angular velocity of space's rotation.

Hence, proceeding from the quantization law, we can calculate frequencies of rotation of observer's spaces, corresponding to rest-masses of our-world particles. The results are given in Table 1.

Elementary particles	**Rest-mass**	**Spin**	**Ω, s^{-1}**
LEPTONS			
electron e^-, positron e^+	1	1/2	$7.782\cdot10^{20}$
electron neutrino ν_e and			
electron anti-neutrino $\tilde{\nu}_e$	$<4\cdot10^{-4}$	1/2	$<3\cdot10^{17}$
μ-meson neutrino ν_μ and			
μ-meson anti-neutrino $\tilde{\nu}_\mu$	<8	1/2	$<6\cdot10^{21}$
μ^--meson, μ^+-meson	206.766	1/2	$1.609\cdot10^{23}$
BARIONS			
nuclons			
proton p, anti-proton $\tilde{p}$	1836.09	1/2	$1.429\cdot10^{24}$
neutron n, anti-neutron $\tilde{n}$	1838.63	1/2	$1.431\cdot10^{24}$
hyperons			
Λ^0-hyperon, anti-Λ^0-hyperon	2182.75	1/2	$1.699\cdot10^{24}$
Σ^+-hyperon, anti-Σ^+-hyperon	2327.6	1/2	$1.811\cdot10^{24}$
Σ^--hyperon, anti-Σ^--hyperon	2342.6	1/2	$1.823\cdot10^{24}$
Σ^0-hyperon, anti-Σ^0-hyperon	2333.4	1/2	$1.816\cdot10^{24}$
Ξ^--hyperon, anti-Ξ^--hyperon	2584.7	1/2	$2.011\cdot10^{24}$
Ξ^0-hyperon, anti-Ξ^0-hyperon	2572	1/2	$2.00\cdot10^{24}$
Ω^--hyperon, anti-Ω^--hyperon	3278	3/2	$8.50\cdot10^{23}$

Table 1. Frequencies of rotation of observer's space of reference, which correspond to mass-bearing elementary particles

The results from Table 1 say that on the scales of elementary particles observer's space is always non-holonomic. For instance, in observation of electron r_e=2.8·10^{-13} cm linear velocity of rotation of observer's space is $v=\Omega r$=2200 km/s[24]. Because other elementary particles are even smaller this linear velocity seems to be the upper limit[25].

So, what have we got? Generally observer compares results of his measurements with the body of reference, but the body and himself are not related to the observed object and do not affect it during observations. Hence in macroworld there is no dependence of the true properties of observed bodies (e. g. rest-mass of particle) from properties of the body and space of reference — these can be arbitrary, just like for any *non-related* objects.

[24]The value v equals to velocity of electron in the first Bohr orbit, though when calculating velocity of space rotation (see Table 1) we considered *a free* electron, i. e. the one not related to an atomic nucleus and quantization of orbits in atom of hydrogen. The reason is that "genetic" quantum non-holonomity of space seems not only to define rest-masses of elementary particles, but to be the reason of rotation of electrons in atoms.

[25]Interestingly, angular velocities of rotation of spaces of barions (Table 1) up within the order of magnitude match the frequency $\sim10^{23}$ s^{-1} that characterizes elementary particles as oscillators [36].

In other words, though observed images are distorted by influence from physical properties of observer's frame of references, the observer himself and his body of reference in macroworld do not affect measured objects in any way.

But the world of elementary particles presents a big difference. In this Section we have seen that once we reach the scales of elementary particle, where spin, a quantum property of particle, significantly affects its motion, physical properties of the space of reference (the body of reference) and those of the particle become tightly linked to each other, i. e. body of reference *affects* the observed particle. In other words, observer does not just compare properties of the observed object to those of his references any longer, but instead directly *affects* the observed object. The observer shapes its properties in a tight quantum relationship with properties of the references he possesses (body and space of reference).

This means: when looking at effects in the world of elementary particles, e. g. spin effects, *there is no border* between the observer, i. e. his body of reference, and the observed elementary particle. Hence we get an opportunity to define *relationship* between field of non-holonomity of space, linked to the observer, and rest-masses of the observed particles — objects of observations, which in macroworld are not related to the body of reference. Therefore, the obtained laws of quantization of masses are only true for elementary particles.

Please note that we have obtained the result using only geometric methods of General Relativity, not methods of probabilities of quantum mechanics. In future, this result may possibly become a "bridge" between these two fields.

4.9 Compton wavelength

We have obtained that in observation of elementary particle with rest-mass of m_0 the frequency of non-holonomity of observer's space is $\Omega=\frac{m_0 c^2}{2n\hbar}$ (4.251). We are going to find the wavelength that corresponds to that frequency. Assuming that this wave, i. e. wave of non-holonomity of space, propagates at light speed $\lambda\Omega=c$, we have

$$\lambda = \frac{c}{\Omega} = 2n\frac{\hbar}{m_0 c}\,. \tag{4.252}$$

In other words, when we observe mass-bearing particle with spin $n=1/2$ the length of non-holonomity wave equals to Compton wavelength of the particle $\lambdabar_c=\frac{\hbar}{m_0 c}$.

What does that mean? Compton effect, named after A. Compton who discovered it in 1922, is "diffraction" of photon on a free electron, which results in decrease of its own frequency

$$\triangle\lambda = \lambda_2 - \lambda_1 = \frac{h}{m_{\mathrm{e}} c}\left(1-\cos\vartheta\right) = \lambda_c^{\mathrm{e}}\left(1-\cos\vartheta\right), \tag{4.253}$$

where λ_1 and λ_2 are photon wavelengths before and after the encounter, ϑ is the angle of "diffraction". The multiplier λ_c^{e}, specific to electron, at first was called *Compton wavelength* of electron. Later it was found out that other elementary particles during "diffraction" of photons also reveal their specific wavelengths $\lambda_c=\frac{h}{m_0 c}$, or, respectively, $\lambdabar_c=\frac{\hbar}{m_0 c}$. That is, every type of elementary particle (i. e. electrons, protons, neutrons etc.) have their own Compton wavelengths. The physical sense behind the value was explained later. It was obtained, within an area smaller than $\lambdabar_c$, elementary particle is no longer a point object and its interaction with other particles (and with observer) is described by quantum mechanics. Hence the $\lambdabar_c$-sized area is sometimes interpreted as "the size" of elementary particle, in a sense in which we can speak of "size" of elementary particles at all.

As for the results we obtained in the previous Section, these can be interpreted as follows: in observation of mass-bearing particle angular velocity of rotation of observer's space grows up the level that makes the wavelength, which corresponds to such velocity, equal to Compton wavelength of the observed particle, i. e. to the "size" inside which the particle is no longer a point object. In other words, it is angular velocity of space rotation (wavelength of the field of non-holonomity of space) that defines observable Compton wavelength (specific "sizes") of mass-bearing elementary particles.

4.10 Massless spin-particle

Because massless particles do not bear electric charge, their scalar equations of motion in our world and in the mirror world are, respectively,

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=0\,,\qquad -\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=0\,. \tag{4.254}$$

Their integration always gives the constant equal to zero, hence we always obtain the formulas as of (4.239). Hence for massless particles in our world and in the mirror world, respectively

$$mc^2=-\,\eta\,,\qquad mc^2=\eta\,. \tag{4.255}$$

On the other hand, the term "rest-mass" is not applicable to massless particles — they are always on the move. Their relativistic masses are defined from energy equivalent $E{=}mc^2$, measured in electron-volts. Subsequently, massless particles have no rest spin-energy $\eta_0{=}n\hbar^{mn}A_{mn}$.

Nevertheless, Planck tensor found in spin-energy η enables quantization of relativistic masses of massless particles and angular velocities of space rotation. Hence to obtain angular velocities of space rotation for massless particles we need an expanded formula of their relativistic spin-energy η, which would not contain relativistic square root.

Quantum mechanics speaks of "spirality" of massless particles — projection of spin onto direction of impulse. The reason for introducing such term is the fact that massless particle can not rest in respect to a regular observer, as it always travels at light speed in respect to such. Hence we can assume that spin of massless particle is tangential to light-like trajectory (either co-directed or oppositely directed to it).

Keeping in mind that spin quantum number n of massless particles is 1, we assume for them that

$$\eta=\hbar^{mn}\tilde{A}_{mn}\,, \tag{4.256}$$

where $\tilde{A}_{mn}$ is three-dimensional tensor of angular velocities of rotation of massless particles' space (light-like space).

Hence to obtain relativistic spin-energy of massless particle (4.256) we need to find components of tensor of angular velocities of rotation of light-like space. We are going to build the tensor similar to four-dimensional tensor of angular velocity $A^{\alpha\beta}$ (4.11), which describes rotation of space of a frame of reference that travels in respect to observer and his body of reference at an arbitrary velocity (non-accompanying frame of reference). As a result we obtain

$$\tilde{A}^{\alpha\beta}=\frac{1}{2}c\tilde{h}^{\alpha\mu}\tilde{h}^{\beta\mu}\tilde{a}_{\mu\nu}\,,\qquad \tilde{a}_{\mu\nu}=\frac{\partial\tilde{b}_\nu}{\partial x^\mu}-\frac{\partial\tilde{b}_\mu}{\partial x^\nu}\,, \tag{4.257}$$

where $\tilde{b}^\alpha$ is four-dimensional velocity of light-like frame of reference in respect to observer and

$$\tilde{h}^{\alpha\mu}=-g^{\alpha\mu}+\tilde{b}^\alpha\tilde{b}^\mu \tag{4.258}$$

is four-dimensional generalization of "observable" metric tensor of space of light-like frame of reference.

The space inhabited by massless particles is an area of space-time, which corresponds with four-dimensional light-like (isotropic) cone set by equation $g_{\alpha\beta}dx^\alpha dx^\beta{=}0$. This cone exists at any point of Riemannian space with alternating signature $(+\,-\,-\,-)$, i. e. at any point of four-dimensional pseudo-Riemannian space.

Four-dimensional vector of velocity of light-like frame of reference of massless particles is

$$\tilde{b}^\alpha=\frac{dx^\alpha}{d\sigma}=\frac{1}{c}\frac{dx^\alpha}{d\tau}\,,\qquad \tilde{b}_\alpha\tilde{b}^\alpha=0\,, \tag{4.259}$$

its physical observable components in frame of reference of a regular "sub-light-speed" observer are

$$\frac{\tilde{b}_0}{\sqrt{g_{00}}}=\pm1,\qquad \tilde{b}^i=\frac{1}{c}\frac{dx^i}{d\tau}=\frac{1}{c}c^i, \tag{4.260}$$

while the other components of isotropic vector (4.259) are

$$\tilde{b}^0 = \frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}v_i c^i \pm 1\right), \qquad \tilde{b}_i = -\frac{1}{c}\left(c_i \pm v_i\right), \tag{4.261}$$

where c^i is chronometrically invariant vector of light velocity.

Now we are going to consider properties of massless particles' space in details. The condition of isotropy of four-dimensional velocity of massless particle $b_\alpha b^\alpha{=}0$ in chronometrically invariant form becomes

$$h_{ik}c^i c^k = c^2 = const, \tag{4.262}$$

where h_{ik} is observable metric tensor of space of reference (of a regular "sub-light-speed" observer). Components of four-dimensional tensor $\tilde{h}^{\alpha\beta}$ (4.258), which three-dimensional components make up observable metric tensor of space of massless particle $\tilde{h}^{ik}$, are

$$\tilde{h}^{00} = \frac{v_k v^k \pm 2v_k c^k + \frac{1}{c^2}v_k v_n c^k c^n}{c^2\left(1-\frac{w}{c^2}\right)^2}, \quad \tilde{h}^{0i} = \frac{v^i \pm c^i + \frac{1}{c^2}v_k c^k c^i}{c\left(1-\frac{w}{c^2}\right)}, \quad \tilde{h}^{ik} = h^{ik} + \frac{1}{c^2}c^i c^k, \tag{4.263}$$

where "plus" stands for space with direct flow of time (our world) and "minus" stands for reverse-time (mirror) world.

Now we have to deduce components of rotor of four-dimensional velocity of massless particle, found in the formula for tensor of rotation of massless particle's space (4.257). After some algebra we obtain

$$\tilde{a}_{00} = 0\,, \quad \tilde{a}_{0i} = \frac{1-\frac{w}{c^2}}{2c^2}\left(\pm F_i - \frac{{}^*\partial c_i}{\partial t}\right), \quad \tilde{a}_{ik} = \frac{1}{2c}\left(\frac{\partial c_i}{\partial x^k} - \frac{\partial c_k}{\partial x^i}\right) \pm \frac{1}{2c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right). \tag{4.264}$$

Generally, to define spin-energy of massless particle (4.256) we need covariant spatial components of tensor of rotation of its space, i. e. components with lower indices $\tilde{A}_{ik}$. To deduce them we take the formula for contravariant components $\tilde{A}^{ik}$ and lower their indices, as for any chronometrically invariant value using three-dimensional observable metric tensor of observer's space of reference.

Substituting into

$$\tilde{A}^{ik} = c\left(\tilde{h}^{i0}\tilde{h}^{k0}\tilde{a}_{00} + \tilde{h}^{i0}\tilde{h}^{km}\tilde{a}_{0m} + \tilde{h}^{im}\tilde{h}^{k0}\tilde{a}_{m0} + \tilde{h}^{im}\tilde{h}^{kn}\tilde{a}_{mn}\right) \tag{4.265}$$

the obtained components $\tilde{h}^{\alpha\beta}$ and $\tilde{a}_{\alpha\beta}$, we arrive to

$$\begin{aligned} \tilde{A}^{ik} = {} & h^{im}h^{kn}\left[\frac{1}{2}\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n c_m - F_m c_n\right)\right] \pm \\ & \pm h^{im}h^{kn}\left[\frac{1}{2}\left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n v_m - F_m v_n\right)\right] + \\ & + \left(\frac{1}{c^2}v_n c^n \pm 1\right)\left(c^k h^{im} - c^i h^{km}\right)\frac{{}^*\partial c_m}{\partial t} - \left(v^k h^{im} - v^i h^{km}\right)\frac{{}^*\partial c_m}{\partial t} + \\ & + \frac{1}{2c^2}c^m\left(c^i h^{kn} - c^k h^{in}\right)\left[\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right) \pm \left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right)\right]. \end{aligned} \tag{4.266}$$

The value $\frac{1}{2}\left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n v_m - F_m v_n\right)$, by definition, is chronometrically invariant (observable) tensor of angular velocities of rotation of observer's space of reference A_{mn}, i. e. tensor of non-holonomity of *non-isotropic space*[26].

[26] We will refer as *non-isotropic space* to an area of four-dimensional space-time where particles with non-zero rest-masses exist. This is the area of world trajectories along which $ds{\neq}0$. Subsequently, if interval ds is real, the particles travel at sub-light speeds (regular particles); if it is imaginary, the particles travel at super-light speeds (tachyons). Space of both types of particles is non-isotropic by definition.

By its structure the value $\frac{1}{2}\left(\frac{\partial c_m}{\partial x^n}-\frac{\partial c_n}{\partial x^m}\right)+\frac{1}{2c^2}\left(F_n c_m-F_m c_n\right)$ is similar to tensor A_{mn}, but instead of velocity of rotation v_i of non-isotropic observer's space it has components of covariant light velocity $c_m=h_{mn}c^n$. We denote that tensor as $\breve{A}_{mn}$, where the inward curved cap means the value belongs to *isotropic space*[27] with direct flow of time — the "upper" part of light cone, which in twisted space-time gets "round" shape. Then

$$\breve{A}_{mn}=\frac{1}{2}\left(\frac{\partial c_m}{\partial x^n}-\frac{\partial c_n}{\partial x^m}\right)+\frac{1}{2c^2}\left(F_n c_m-F_m c_n\right). \tag{4.267}$$

In a specific case, when gravitational potential is negligible ($w\approx 0$) the tensor becomes

$$\breve{A}_{mn}=\frac{1}{2}\left(\frac{\partial c_m}{\partial x^n}-\frac{\partial c_n}{\partial x^m}\right), \tag{4.268}$$

i. e. is chronometrically invariant rotor of light velocity. Therefore we will refer to $\breve{A}_{mn}$ as *rotor of isotropic space*.

The following example gives geometric illustration of rotor of isotropic space. As known, the necessary and sufficient condition of equality $A_{mn}=0$ (condition of space holonomity) is equality to zero of all components $v_i=-c\frac{g_{0i}}{\sqrt{g_{00}}}$, i. e. absence of space rotation. Tensor $\breve{A}_{mn}$ is defined only in isotropic space, inhabited by massless particles. Outside isotropic space it is senseless, because the "interior" of the light cone is inhabited by sub-light-speed particles, while tachyons inhabit its "exterior".

Our subject are spin massless particles, i. e. photons. From (4.268) it is seen that presence of field of non-holonomity of isotropic space is linked to rotor character of velocity of motion of massless particles c_m. Hence photons are rotors of isotropic space, while photon spin results from interaction between the inner field of the rotor with external field of tensor $\breve{A}_{mn}$.

To make the explanations even more illustrative, we depict areas of existence of different types of particles. Light cone exists in every point of space. Equation of light cone $g_{\alpha\beta}dx^\alpha dx^\beta=0$ in chronometrically invariant notation is

$$c^2\tau^2-h_{ik}x^i x^k=0\,,\qquad h_{ik}x^i x^k=\sigma^2. \tag{4.269}$$

On Minkowski diagram the "interior" of light cone is filled with non-isotropic space, where sub-light-speed particles exist. Outside there is also an area of non-isotropic space, inhabited by super-light-speed particles (tachyons). The specific space of massless particles is space-time *membrane* between these two non-isotropic areas. The picture is mirror-symmetric: in the upper part of cone there is sub-light-speed space with direct flow of time (our world), separated with spatial section from the lower part — a sub-light-speed space with reverse flow of time (mirror world). In other words, the upper part is inhabited by real particles with positive mass and energy, while the lower part is inhabited by their mirror "counterparts", whose mass and energy are negative (from our viewpoint).

Therefore, rotation of sub-light-speed non-isotropic space "inside" the cone involves the surrounding light membrane (isotropic space). As a result, the light cone begins rotation described by tensor $\breve{A}_{mn}$ — rotor of isotropic space. Of course we can assume a reverse order of events, where rotation of the light cone involves "the content" of its inner part. But because particles "inside" the cone bear non-zero rest-mass they are "heavier" that massless particles on the light membrane. Hence the inner "content" of the light cone is too inertial media.

Now we return to the formula for relativistic spin-energy of massless particle $\eta=\hbar^{mn}\tilde{A}_{mn}$ (4.256). By lowering indices in contravariant tensor of non-holonomity of isotropic space $\tilde{A}^{ik}$ (4.266) we obtain

$$\begin{gathered}\tilde{A}_{ik}=\pm A_{ik}+\breve{A}_{ik}+\frac{1}{2c^2}c^m\left\{c_i\left[\frac{\partial\left(c_m\pm v_m\right)}{\partial x^k}-\frac{\partial\left(c_k\pm v_k\right)}{\partial x^m}\right]-c_k\left[\frac{\partial\left(c_m\pm v_m\right)}{\partial x^i}-\frac{\partial\left(c_i\pm v_i\right)}{\partial x^m}\right]\right\}+\\+\left(v_i\frac{{}^*\partial c_k}{\partial t}-v_k\frac{{}^*\partial c_i}{\partial t}\right)+\left(\frac{1}{c^2}v_n v^n\pm 1\right)\left(c_k\frac{{}^*\partial c_i}{\partial t}-c_i\frac{{}^*\partial c_k}{\partial t}\right).\end{gathered} \tag{4.270}$$

[27] We will refer as *isotropic space* to an area of four-dimensional space-time, inhabited by massless (light-like) particles. This area can be also called *light membrane*. From geometric viewpoint light membrane is the surface of isotropic cone, i. e. the set of its four-dimensional elements (world lines of light propagation).

Having $\tilde{A}_{ik}$ contracted with Planck tensor $\hbar^{ik}$ we have

$$\eta=\eta_0+n\hbar^{ik}\breve{A}_{ik}+\left[\left(\frac{1}{c^2}v_n v^n\pm 1\right)\left(c_k\frac{{}^*\partial c_i}{\partial t}-c_i\frac{{}^*\partial c_k}{\partial t}\right)+\left(v_i\frac{{}^*\partial c_k}{\partial t}-v_k\frac{{}^*\partial c_i}{\partial t}\right)\right]n\hbar^{ik}+ \\ +\frac{1}{2c^2}n\hbar^{ik}c^m\left\{c_i\left[\frac{\partial\left(c_m\pm v_m\right)}{\partial x^k}- -\frac{\partial\left(c_k\pm v_k\right)}{\partial x^m}\right]-c_k\left[\frac{\partial\left(c_m\pm v_m\right)}{\partial x^i}-\frac{\partial\left(c_i\pm v_i\right)}{\partial x^m}\right]\right\}, \tag{4.271}$$

where "plus" stands for our world and "minus" — for the mirror world.

The value $\eta_0=\eta\sqrt{1-\mathrm{v}^2/c^2}$ for massless particles is zero, because they travel at speed of light. Hence keeping in mind that $\eta_0=n\hbar^{mn}A_{mn}$ we obtain an additional condition imposed on tensor of non-holonomity of isotropic space $\tilde{A}_{ik}$: at any point of trajectory of massless particle the following condition must be true

$$\hbar^{mn}A_{mn}=2\hbar\left(A_{12}+A_{23}+A_{31}\right)=0\,, \tag{4.272}$$

or, in another notation, $\Omega^1+\Omega^2+\Omega^3=0$.

Therefore, in an area, where observer "sees" massless particle, angular velocity of rotation of non-isotropic observer's space equals to zero. Other terms in the formula for relativistic spin-energy of massless particle (4.271) are due to possible non-stationarity of light velocity $\frac{{}^*\partial c_i}{\partial t}$ and other dependencies which include squares of light velocity.

We analyze the obtained formula (4.271) to make two simplification assumptions:

1. gravitational potential is negligible ($w\approx 0$);
2. three-dimensional chronometrically invariant velocity of light is stationary.

In this case the formulas for A_{ik} and $\breve{A}_{ik}$, i. e. for tensor of space non-holonomity and for rotor of isotropic space, become

$$A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right),\qquad \breve{A}_{ik}=\frac{1}{2}\left(\frac{\partial c_k}{\partial x^i}-\frac{\partial c_i}{\partial x^k}\right), \tag{4.273}$$

and relativistic spin-energy of massless particle (4.271) becomes

$$\eta=n\left(\hbar^{ik}\breve{A}_{ik}+\frac{1}{c^2}c_i c^m\hbar^{ik}\breve{A}_{km}\right). \tag{4.274}$$

Therefore the value η (4.274) that describes action of spin of massless particle, is defined (aside for spin) only by rotor of isotropic space and in no way depends upon non-holonomity (rotation) of observer's space of reference.

To make further deductions simpler we transform η (4.274) as follows. Similar to pseudovector of angular velocity of space rotation $\Omega^{*i}=\frac{1}{2}\varepsilon^{ikm}A_{km}$ we introduce pseudovector

$$\breve{\Omega}^{*i}=\frac{1}{2}\varepsilon^{ikm}\breve{A}_{km}\,, \tag{4.275}$$

which can be formally interpreted as pseudovector of angular velocity of rotation of isotropic space, i. e. of space where only isotropic curves exist — trajectories of massless (light-like) particle travelling at light speed.

Subsequently, $\breve{A}_{km}=\varepsilon_{kmn}\breve{\Omega}^{*n}$. Then the formula for η (4.274) can be presented as

$$\eta=n\left(\hbar_{*i}\breve{\Omega}^{*i}+\frac{1}{c^2}c_i c^m\hbar^{ik}\varepsilon_{kmn}\breve{\Omega}^{*n}\right). \tag{4.276}$$

That means that inner rotor (spin) of massless particle only reveals itself in interaction with rotor of isotropic space. The result of the interaction is scalar product $\hbar_{*i}\breve{\Omega}^{*i}$, to which spin of massless particle is attributed. Hence massless particles are elementary light-like rotors of isotropic space itself.

Now we are going to estimate rotations of isotropic space for massless particles with different energies. At present we know for sure that among the massless particles are photons — quanta of electromagnetic field.

Spin quantum number of photon is 1. Besides, its energy $E=\hbar\omega$ is positive in our world. Hence taking into account integral of live forces (4.255), for observable our-world photons we have

$$\hbar\omega = \hbar_{*i}\breve{\Omega}^{*i} + \frac{1}{c^2}c_i c^m \hbar^{ik}\varepsilon_{kmn}\breve{\Omega}^{*n}. \tag{4.277}$$

We assume that pseudovector of rotation of isotropic space Ω^{*i} is directed along z, while light velocity is directed along y. Then the relationship (4.277) obtained for photons becomes $\hbar\omega=2\hbar\breve{\Omega}$, or, after having Planck constant cancelled,

$$\breve{\Omega} = \frac{\omega}{2} = \frac{2\pi\nu}{2} = \pi\nu\,, \tag{4.278}$$

i. e. frequency $\breve{\Omega}$ of rotor of isotropic space, each interacts with photon spin, up within a constant coincides with its own frequency ν. Thanks to this formula, which results from the law of quantization of relativistic masses of light-like particles, we can estimate angular velocities of rotation of isotropic space, which correspond to photons with different energy levels. Table 2 gives the results.

Kind of photons	**Frequency $\breve{\Omega}$, s^{-1}**
Radiowaves	$10^3 - 10^{11}$
Infra-red rays	$10^{11} - 1.2\cdot 10^{15}$
Visible light	$1.2\cdot 10^{15} - 2.4\cdot 10^{15}$
Ultraviolet rays	$2.4\cdot 10^{15} - 10^{17}$
X-rays	$10^{17} - 10^{19}$
Gamma rays	$10^{19} - 10^{23}$ and above

Table 2. Frequencies of rotation of isotropic space, which correspond to photons

From Table 2 we see that angular velocities of rotation of isotropic space of photons in gamma range match rotation frequencies of regular (non-isotropic) space of electrons and other elementary particles (see Table 1).

4.11 Conclusions

Here is what we have discussed in this Chapter.

Spin of particle is characterized by four-dimensional antisymmetric 2nd rank Planck tensor, which diagonal and space-time components are zeroes, while non-diagonal spatial components are $\pm\hbar$ depending upon orientation of spin and choice of right or left-handed frame of reference.

Spin (inner vortex field of particle) interacts with external field of non-holonomity of space; as a result, particle gains additional impulse that deviates its trajectory from geodesic line. Energy of the interaction is found in scalar equation of motion (live forces theorem), which must be taken into account when solving vector (spatial) equations of motion.

Partial solution of scalar equation is law of quantization of masses of elementary spin-particles, which unambiguously links rest-masses of mass-bearing elementary particles with angular velocities of rotation of observer's space, as well as between relativistic masses of photons and angular velocities of rotation of their isotropic (light-like[28]) space.

[28]Because the area of existence of light-like particles is the area of four-dimensional isotropic trajectories, the terms "isotropic space" and "light-like space" can be used as synonyms.

Chapter 5

Physical vacuum and the mirror Universe

5.1 Introduction

According to the recent data the average density of matter in our Universe is $5\text{–}10{\cdot}10^{-30}$ g/cm^3. That of substance concentrated in galaxies is even lower at $\approx 3{\cdot}10^{-31}$ g/cm^3, which seems to be due to so-called "hidden masses" in galaxies. Besides, astronomical observations show that most part of the cosmic mass is accumulated in compact objects, e. g. in stars, which total volume is incomparable to that of the whole Universe ("island" distribution of substance). We can therefore assume that our Universe is predominantly empty.

For a long time the words "emptiness" and "vacuum" have been considered synonyms. But since 1920's geometric methods of General Relativity have showed that those are different states of matter.

Distribution of matter in space is characterized by energy-impulse tensor, which is linked to geometric structure of space-time (fundamental metric tensor) with *equations of gravitational field*. In Einstein's theory of gravitation, which is an application of the geometrical methods of General Relativity, the equations referred to as *Einstein equations* are[29]

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-æ\,T_{\alpha\beta}+\lambda g_{\alpha\beta}\,, \tag{5.1}$$

which aside for energy-impulse tensor and fundamental metric tensor, include other values, namely:

- $R_{\alpha\sigma}{=}R_{\alpha\beta\sigma\cdot}^{\cdots\beta}$ is Ricci tensor, which is a result of contraction of curvature Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ by two indices;
- $R{=}g^{\alpha\beta}R_{\alpha\beta}$ is scalar curvature;
- $æ=\frac{8\pi G}{c^2}{=}1.862{\cdot}10^{-27}$ [cm g^{-1}] is Einstein gravitational constant, where $G{=}6.672{\cdot}10^{-8}$ is gravitational constant [cm^3 g^{-1}s^{-2}]. Note that some researchers prefer to use not $æ=\frac{8\pi G}{c^2}$ [6, 8, 10], but $æ=\frac{8\pi G}{c^4}$ [1]. To understand the reason we have to look at chronometrically invariant components of energy-impulse tensor $T_{\alpha\beta}$: $\frac{T_{00}}{g_{00}}{=}\rho$ is observable density of mass, $\frac{cT_0^i}{\sqrt{g_{00}}}{=}J^i$ is vector of observable density of impulse, and $c^2T^{ik}{=}U^{ik}$ is tensor of observable impulse flux density [8, 10]. Scalar observable component of Einstein equations is $\frac{G_{00}}{g_{00}}{=}-\frac{æT_{00}}{g_{00}}{+}\lambda$. As known, Ricci tensor has dimension [cm^{-2}], hence Einstein tensor $G_{\alpha\beta}$ and the value $\frac{æT_{00}}{g_{00}}{=}\frac{8\pi G\rho}{c^2}$ has the same one. Consequently, it is evident that the dimension of energy-impulse tensor $T_{\alpha\beta}$ is that of mass density [g cm^{-3}]. That implies that when we use $\frac{8\pi G}{c^4}$ in the right part of Einstein equations, we actually use not energy-impulse tensor itself, but $c^2T_{\alpha\beta}$, which scalar and vector observable components are density of energy $\frac{c^2T_{00}}{g_{00}}{=}\rho c^2$ and energy flux $\frac{c^3T_0^i}{\sqrt{g_{00}}}{=}c^2J^i$;

[29] The left part of Einstein field equations (5.1) is often referred to as *Einstein tensor* $G_{\alpha\beta}{=}R_{\alpha\beta}{-}\frac{1}{2}g_{\alpha\beta}R$, i. e. in brief notation $G_{\alpha\beta}{=}{-}æ\,T_{\alpha\beta}{+}\lambda g_{\alpha\beta}$.

- λ [cm^{-2}] is a cosmological term that describes non-Newtonian forces of attraction or repulsion, depending upon sign before λ ($\lambda>0$ stands for repulsion, $\lambda<0$ stands for attraction). The term is referred to as cosmological one, because it is assumed that forces described by λ grow up proportional to distance and therefore reveal themselves on a full scale at "cosmological" distances comparable to size of the Universe. Because non-Newtonian gravitational fields (λ-fields) have never been observed, for our Universe in general the cosmological term is $|\lambda|<10^{-56}$ (as of today's measurement accuracy).

From Einstein equations (5.1) we see that energy-impulse tensor (which describes distribution of matter) is genetically linked to metric tensor and Ricci tensor, and hence to Riemann-Christoffel curvature tensor. Equality of Riemann-Christoffel tensor to zero is the necessary and sufficient condition for the given space-time to be flat. Riemann-Christoffel tensor is not zero for curved space only. It reveals itself as increment of vector V^{α} in its parallel transfer along a closed contour

$$\triangle V^{\mu} = -\frac{1}{2} R_{\alpha\beta\gamma\cdot}^{\cdot\cdot\cdot\mu} V^{\alpha} \triangle\sigma^{\beta\gamma}, \tag{5.2}$$

where $\triangle\sigma^{\beta\gamma}$ is the area of this contour. As a result, the initial vector V^{α} and vector $V^{\alpha}+\triangle V^{\alpha}$ have different directions. From quantitative viewpoint the difference is described by K, referred to as *four-dimensional curvature* of pseudo-Riemannian space along the given parallel transfer (for detailed account to Chapter 9 in Zelmanov's lectures [10])

$$K = \lim_{\triangle\sigma\to 0} \frac{\tan\varphi}{\triangle\sigma}\,, \tag{5.3}$$

where $\tan\varphi$ is the tangent of angle between vector V^{α} and the projection of vector $V^{\alpha}+\triangle V^{\alpha}$ on the area constrained by the transfer contour. For instance, we consider a surface and "a geodesic" triangle on it, produced by crossing of three geodesic lines. We transfer a vector, defined in any arbitrary point of that triangle, parallel to itself along the sides of the triangle. The summary angle of rotation φ after the vector returns to the initial point will be $\varphi=\Sigma-\pi$ (where Σ is the sum of the inner angles of the triangle). We assume curvature of the surface K equal in all its points, then

$$K = \lim_{\triangle\sigma\to 0} \frac{\tan\varphi}{\triangle\sigma} = \frac{\varphi}{\sigma} = const, \tag{5.4}$$

where σ is the triangle's area and $\varphi=K\sigma$ is called *spherical excess*. If $\varphi=0$, then the curvature $K=0$, i. e. the surface is flat. In this case the sum of all inner angles of the geodesic triangle is π (flat space). If $\Sigma>\pi$ (the transferred vector is rotated towards the circuit), then there is positive spherical excess and the curvature $K>0$. An example of such space is surface of a sphere: a triangle on surface of a sphere is convex. If $\Sigma<\pi$ (the transferred vector is rotated counter the circuit), spherical excess is negative and the curvature $K<0$.

Einstein postulated that gravitation is curvature of space-time. He understood curvature as not equality to zero of Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}\neq 0$ (the same is assumed in Riemannian geometry). This concept fully includes Newtonian gravitational concept, i. e. Einstein's four-dimensional gravitation-curvature for a regular physical observer can reveal itself as: (a) Newtonian gravitation; (b) rotation of three-dimensional space; (c) deformation of three-dimensional space; (d) three-dimensional curvature, i. e. when Christoffel symbols are not zeroes (see Section 13.5 in Zelmanov's lectures [10]). According to Mach Principle, on which Einstein theory of gravitation rests, "...the property of inertia is fully determined by interaction of matter" [25], i. e. curvature of space-time is produced by matter that fills it (in a certain form). Proceeding from that and from Einstein equations (5.1) we can give mathematical definitions of emptiness and vacuum:

- **emptiness** is the state of space-time for which Ricci tensor $R_{\alpha\beta}=0$, i. e. absence of substance $T_{\alpha\beta}=0$ and non-Newtonian gravitational fields $\lambda=0$. Field equations (5.1) in emptiness are as simple as $R_{\alpha\beta}=0$[30];

[30]If we put down Einstein equations for empty space $R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=0$ in a mixed form $R_{\alpha}^{\beta}-\frac{1}{2}g_{\alpha}^{\beta}R=0$, after contraction ($R_{\alpha}^{\alpha}-\frac{1}{2}g_{\alpha}^{\alpha}R=0$) we obtain $R-\frac{1}{2}4R=0$, i. e. scalar curvature in emptiness $R=0$. Hence field equations (Einstein equations) in empty space are $R_{\alpha\beta}=0$.

- **vacuum** is the state in which substance is absent $T_{\alpha\beta}=0$, but $\lambda\neq 0$ and hence $R_{\alpha\beta}\neq 0$. Emptiness is a specific case of vacuum in absence of λ-fields. Equations of field in vacuum are

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=\lambda g_{\alpha\beta}\,. \tag{5.5}$$

Einstein equations are applicable to the most varied cases of distribution of matter, aside for the cases when the density is close to that of substance in atomic nuclei. It is hard to give accurate mathematical description to all cases of distribution of matter because such problem is too general one and can't be approached *per se.* On the other hand, average density of substance in our Universe is so small $5\text{–}10\cdot 10^{-30}\,\text{g/cm}^3$, that we can assume it is nearly vacuum. Einstein equations say that energy-impulse tensor is functionally dependent from metric tensor and Ricci tensor (i. e. from curvature tensor contracted by two indices). At such small values of density we can assume energy-impulse tensor proportional to metric tensor $T_{\alpha\beta}\sim g_{\alpha\beta}$ and hence proportional to Ricci tensor. Therefore aside for field equations in vacuum (5.5) we can consider field equations as

$$R_{\alpha\beta}=kg_{\alpha\beta}\,,\qquad k=const, \tag{5.6}$$

i. e. when energy-impulse tensor is different from metric tensor only by a constant. This case, including absence of masses (vacuum) and some conditions close to it and related to our Universe, where studied in details by A. Z. Petrov [3]. Spaces for which energy-impulse tensor is proportional to metric tensor (and to Ricci tensor) he called *Einstein spaces.*

Spaces with $R_{\alpha\beta}=kg_{\alpha\beta}$ (Einstein spaces) are uniform in every their point, have no mass fluxes, while the density of matter that fills them (including any substances, if any) is every where constant. In this case

$$R=g^{\alpha\beta}R_{\alpha\beta}=kg_{\alpha\beta}g^{\alpha\beta}=4k\,, \tag{5.7}$$

while Einstein tensor takes the form

$$G_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-kg_{\alpha\beta}\,, \tag{5.8}$$

where $kg_{\alpha\beta}$ is the analog of energy-impulse tensor for matter that fills Einstein spaces.

To find out what types of matter fill Einstein spaces, Petrov studied algebraic structure of energy-impulse tensor. This is what he did: tensor $T_{\alpha\beta}$ was compared to metric tensor in an arbitrary point; for this point the difference $T_{\alpha\beta}-\xi g_{\alpha\beta}$ is calculated, where ξ are so-called eigenvalues of matrix $T_{\alpha\beta}$; the difference is equaled to zero to find ξ values which make the equality true. This problem is also referred to as the problem of matrix eigenvalues[31]. The set of matrix eigenvalues allows to define the matrix's algebraic type. For sign-constant metric the problem had been already solved, but Petrov proposed a method to bring a matrix to canonical form for indefinite (sign-alternating) metric, which allowed using is in pseudo-Riemannian space, in particular, to study algebraic structure of energy-impulse tensor. This can be illustrated as follows. Eigenvalues of matrix $T_{\alpha\beta}$ are similar to basic vectors of metric tensor matrix, i. e. are a sort of "skeleton" of $T_{\alpha\beta}$ (skeleton of matter); but even if we know what is the skeleton like, we may not know exactly what are the muscles. Nevertheless, the structure of such skeleton (length and mutual orientation of vectors) we can judge on the properties of matter, such as uniformity or isotropy, and their relation to curvature of space.

As a result, Petrov obtained that Einstein spaces have three basic algebraic types of energy-impulse tensor and a few subtypes. According to algebraic classification of energy-impulse tensor and curvature tensor, all Einstein spaces are sub-divided into three basic types (so-called *Petrov classification*)[32].

Type I spaces are best intuitively comprehensible, because field of gravitation there is produced by a massive island ("island" distribution of substance), while the space itself may be empty or filled with vacuum. Curvature of such space is created by island mass and by vacuum. At the infinite distance from the island mass, in absence of vacuum, space remains flat. Devoid of island mass but filled with

[31] Generally, the problem of matrix eigenvalues should be solved in a given point, but the obtained result is applicable to any point of the space.

[32] Chronometrically invariant interpretation of algebraic classification of Einstein spaces (or, in other words, of Petrov fields of gravitation) was obtained in 1970 by a co-author of this book (L. B. Borissova, née Grigoreva [26]).

vacuum, Type I space also bears curvature (e. g. de Sitter space). Empty Type I space, i. e. the one devoid of island masses or vacuum, is flat.

Types II and III spaces are more exotic because are curved by themselves. Their curvature is not related to island distribution of masses or presence of vacuum. Types II and III are generally attributed to radiation fields, for instance, to gravitational waves.

A few years later E. B. Gliner [27, 28, 29] in his study of algebraic structure of energy-impulse tensor of vacuum-like states of matter ($T_{\alpha\beta}{\sim}g_{\alpha\beta}$, $R_{\alpha\beta}{=}kg_{\alpha\beta}$) outlined its special type for which all four eigenvalues are the same, i. e. three space vectors and one temporal vector of "ortho-reference" of tensor $T_{\alpha\beta}$ are equal to each other[33]. The matter that corresponds to energy-impulse tensor of such structure has constant density $\mu{=}const$, equal to the value of coinciding eigenvalues of energy-impulse tensor $\mu{=}\xi$ (the dimension of $[\mu]{=}[T_{\alpha\beta}]{=}[\text{g cm}^{-3}]$). Energy-impulse tensor itself in this case is[34]

$$T_{\alpha\beta} = \mu g_{\alpha\beta}\,. \tag{5.9}$$

Equations of field at $\lambda{=}0$ are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -\text{æ}\mu g_{\alpha\beta}\,, \tag{5.10}$$

and with cosmological term $\lambda{\neq}0$

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -\text{æ}\mu g_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{5.11}$$

Gliner called such state of matter *μ-vacuum* [27, 28, 29], because it is related to vacuum-like states of substance ($T_{\alpha\beta}{\sim}g_{\alpha\beta}$, $R_{\alpha\beta}{=}kg_{\alpha\beta}$), but is not exactly vacuum (in vacuum $T_{\alpha\beta}{=}0$). At the same time Gliner showed that spaces filled with μ-vacuum are Einstein spaces and three basic types of μ-vacuum exist, which correspond to three basic algebraic types of energy-impulse tensor (and curvature tensor). In other words, Einstein space of each type (I, II, and III), if matter is present in them, is filled with μ-vacuum of corresponding type (I, II, or III).

Actually, because for "ortho-reference" of energy-impulse tensor of μ-vacuum all three space vectors and one temporal vector are the same (all four directions are equal), μ-vacuum is the highest degree of isotropy of matter. Besides, because Einstein spaces are uniform and density of matter in their every point is everywhere equal [3], then μ-vacuum that fills them does not only have constant density, but is uniform as well.

As we have seen, Einstein spaces can be filled with μ-vacuum, with regular vacuum $T_{\alpha\beta}{=}0$ or with emptiness. Besides, there may exist isolated "islands" of mass, which also produce curvature. Therefore Type I Einstein spaces are the best illustration of our knowledge of our Universe as a whole. And thus to study geometry of our Universe and physical states of matter that fills it is to study Type I Einstein spaces.

Petrov has proposed and proven a theorem: "Any space with constant curvature is Einstein space" (see Section 13 in [3]). And also that "... Types II and III Einstein spaces can not be constant curvature spaces". Hence constant curvature spaces are type I spaces according to Petrov classification (Einstein spaces). If $K{=}0$ Type I Einstein space is flat. This makes the study of vacuum and vacuum-like states of matter in our Universe even simpler, because by today we have well studied constant curvature spaces. These are *de Sitter spaces*, or, in other words, spaces with de Sitter metric.

In de Sitter space $T_{\alpha\beta}{=}0$, and $\lambda{\neq}0$, it is spherically symmetric, filled with regular vacuum and does not contain "islands" of substance. On the other hand we know that the average density of matter in our Universe is rather low. Looking at it in general, we can neglect presence of occasional "islands" and inhomogeneities, which locally distort spherical symmetry. Hence our space can be generally assumed as de Sitter space with radius equal to that of the Universe.

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[33] If we introduce a local flat space, tangential to Riemannian space in a given point, then eigenvalues ξ of tensor $T_{\alpha\beta}$ are values in oath-reference, corresponded to this tensor, as contrasted to eigenvalues of metric tensor $g_{\alpha\beta}$ in ortho-reference, defined in this tangential space.

[34] Gliner used signature $({-}{+}{+}{+})$, hence he had $T_{\alpha\beta}{=}{-}\mu g_{\alpha\beta}$ and because observable density is positive $\rho{=}\frac{T_{00}}{g_{00}}{=}{-}\mu{>}0$, Gliner had negative μ values. In our book we use signature $({+}{-}{-}{-})$, because in this case three-dimensional observable interval is positive. Hence we have $\mu{>}0$ and $T_{\alpha\beta}{=}\mu g_{\alpha\beta}$.

Theoretically de Sitter space may bear either positive ($K>0$) or negative ($K<0$) curvature. Analysis (J. L. Synge) shows that in de Sitter world with $K<0$ time-like geodesic lines are closed: test particle repeats its motion again and again along the same trajectory. This hints some ideas, which seem to be too "revolutionary" from viewpoint of today's physics [30]. Consequently, most physicists (Synge, Gliner, Petrov, *et al.*) have left negative curvature de Sitter space beyond the scope of their consideration.

As known, positive curvature Riemannian spaces are generalization of a regular sphere, while the negative curvature ones are generalization of Lobachewski-Bolyai space, an imaginary-radius sphere. In Poincaré interpretation spaces with negative curvature reflect onto the inner surface of sphere. Using methods of chronometric invariants, Zelmanov showed that in pseudo-Riemannian space (which metric is indefinite) three-dimensional observable curvature is negative to four-dimensional curvature. Because we percept our planet as a sphere, the observable three-dimensional curvature of our world is positive. If any hypothetical beings inhabited the "inner" surface of Earth, they would percept it as concave and their world will be negative curvature one.

Such illustration inspired some researchers for the idea of possible existence of our mirror twin, *the mirror Universe* inhabited by antipodes. Initially it was assumed that once our world has positive curvature, the mirror Universe must be negative curvature space. But Synge showed ([30], Chapter VII) that in de Sitter positive curvature space space-like geodesic trajectories are open, while in negative curvature de Sitter space they are closed. In other words, negative curvature de Sitter space is not a mirror reflection of its positive curvature counterpart.

On the other hand, in our previous studies [15, 16] (see also Section 1.3 herein) we found another approach to concept of the mirror Universe. Study of motion of free particles with time flow reversed in respect to that of observer, showed that observable scalar component of their four-dimensional impulse vector is negative relativistic mass. Noteworthy, particles with "mirror" masses were obtained as a formal result of projecting four-dimensional impulse on time and was not related to changing sign of space curvature: particles with either direct or reverse flow of time may either exist positive or negative curvature spaces.

These results obtained by geometric methods of General Relativity inevitably affect our view of matter and cosmology of our Universe.

In Section 5.2 we are going to obtain formula for energy-impulse tensor of vacuum and at the same time the formula for its observable density. We will also introduce classification of matter according to form of energy-impulse tensor (*T-classification*). In Section 5.3 we are going to look at physical properties of vacuum in type I Einstein spaces; in particular, we will discuss properties of vacuum in de Sitter space and make conclusions on global structure of our Universe. Following this approach in Section 5.4 we will set forth the concept of origination and development of the Universe as a result of *Inversion Explosion* from the pra-particle that possessed some specific properties. In Section 5.5 we will obtain the formula for non-Newtonian gravitational inertial force that is proportional to distance. Sections 5.6 and 5.7 will focus on collapse in Schwarzschild space (gravitational collapse, *black hole*) and in de Sitter space (inflational collapse, *inflanton*). Section 5.8 will show that our Universe and the mirror Universe are worlds with mirror time that co-exist in de Sitter space with four-dimensional negative curvature. Also we will set forth physical conditions, which allow transition through the membrane that separates our world and the mirror Universe.

5.2 Observable density of vacuum. T-classification of matter

Einstein equations (field equations in Einstein gravitation theory) are functions that link curvature of space to distribution of matter. Generally they are $R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-æ\,T_{\alpha\beta}+\lambda g_{\alpha\beta}$. The left part, as known, describe geometry of space, while the right one describes matter. The sign of the second term depends upon that of λ. As we are going to see, the sign of λ, i. e. behavior of Newtonian gravitation (attraction or repulsion) is directly linked to the sign of vacuum density.

Einstein space are defined by condition $T_{\alpha\beta}\sim g_{\alpha\beta}$, field equations for them are $R_{\alpha\beta}=kg_{\alpha\beta}$. Such field equations can exist in two cases: (a) when $T_{\alpha\beta}\neq 0$ (substance); (b) when $T_{\alpha\beta}=0$ (vacuum). But because in Einstein spaces, filled with vacuum, energy-impulse tensor equals to zero, it can not be proportional to metric tensor, it contradicts with the definition of Einstein spaces ($T_{\alpha\beta}\sim g_{\alpha\beta}$). So what is the problem here? In absence of any substance (i. e. in vacuum) field equations become $R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=\lambda g_{\alpha\beta}$, i. e. curvature is produced by λ-fields (non-Newtonian fields of gravitation) rather then by substance. In absence of both substance and λ-fields $R_{\alpha\beta}=0$, i. e. space empty but generally is not flat.

As a result we can see that λ-fields and vacuum are practically the same thing, i. e. *vacuum is non-Newtonian field of gravitation* (we will call this *physical definition of vacuum*). Hence λ-fields are action of own potential of vacuum.

This means that the term $\lambda g_{\alpha\beta}$ can not be omitted in field equations in vacuum, whatever small it is, because it describes vacuum, which is among the reasons that make space curved. Then field equations $R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-æ\,T_{\alpha\beta}+\lambda g_{\alpha\beta}$ can be put down as

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-æ\,\widetilde{T}_{\alpha\beta}\,, \tag{5.12}$$

in the right part of which the 2nd rank tensor

$$\widetilde{T}_{\alpha\beta}=T_{\alpha\beta}+\breve{T}_{\alpha\beta}=T_{\alpha\beta}-\frac{\lambda}{æ}g_{\alpha\beta} \tag{5.13}$$

is energy-impulse tensor that describes matter in general (both substance and vacuum). The first term here is energy-impulse tensor of substance. The second term

$$\breve{T}_{\alpha\beta}=-\frac{\lambda}{æ}g_{\alpha\beta} \tag{5.14}$$

is analog to energy-impulse tensor for vacuum.

Therefore because Einstein spaces may be filled with vacuum, their mathematical definition is better to be set forth in a more general form to take account for presence of both substance and vacuum (λ-fields): $\widetilde{T}_{\alpha\beta}\sim g_{\alpha\beta}$. In particular, doing this helps to avoid contradictions when considering Einstein empty spaces.

Noteworthy, the obtained formula for energy-impulse tensor of vacuum (5.14) is a direct consequence of field equations in general form.

If $\lambda>0$, then non-Newtonian forces of gravitation repel and the physical observable density of vacuum is negative

$$\breve{\rho}=\frac{\breve{T}_{00}}{g_{00}}=-\frac{\lambda}{æ}=-\frac{|\lambda|}{æ}<0\,, \tag{5.15}$$

while if $\lambda<0$ (non-Newtonian forces of gravitation attract) the observable density of vacuum is, to the contrary, positive

$$\breve{\rho}=\frac{\breve{T}_{00}}{g_{00}}=-\frac{\lambda}{æ}=\frac{|\lambda|}{æ}>0\,. \tag{5.16}$$

The latter fact, as we will see in the next Section, is of great importance, because de Sitter space with $\lambda<0$, which is constantly negative (four-dimensional) curvature space filled with vacuum only (no substance present), best fits our observation data on our Universe in general.

Therefore proceeding from studies by Petrov and Gliner and taking into account our note on existence of own energy-impulse tensor (and hence physical properties) in vacuum (λ-fields), we can set forth "geometric" classification of states of matter according to energy-impulse tensor. We will call this *T-classification of matter*

I. **emptiness:** $T_{\alpha\beta}=0$, $\lambda=0$ (space-time without matter), field equations are $R_{\alpha\beta}=0$;

II. **vacuum:** $T_{\alpha\beta}=0$, $\lambda\neq0$ (produced by λ-fields), field equations are $G_{\alpha\beta}=-\lambda g_{\alpha\beta}$;

III. **μ-vacuum:** $T_{\alpha\beta}=\mu g_{\alpha\beta}$, $\mu=const$ (vacuum-like state of substance), in this case field equations are $G_{\alpha\beta}=-æ\mu g_{\alpha\beta}$;

IV. **substance:** $T_{\alpha\beta}\neq0$, $T_{\alpha\beta}\nsim g_{\alpha\beta}$ (this state comprises both regular substance and electromagnetic field).

Generally energy-impulse tensor of substance (Type IV in T-classification) is not proportional to metric tensor. On the other hand, there are states of substance in which energy-impulse tensor contains a term proportional to metric tensor, but because it also contains other terms it is not μ-vacuum. Such are, for instance, ideal fluid

$$T_{\alpha\beta} = \left(\rho - \frac{p}{c^2}\right) U_\alpha U_\beta - \frac{p}{c^2} g_{\alpha\beta}\,, \tag{5.17}$$

and electromagnetic field

$$T_{\alpha\beta} = F_{\rho\sigma} F^{\rho\sigma} g_{\alpha\beta} - F_{\alpha\sigma} F_{\beta\cdot}^{\cdot\sigma}\,, \tag{5.18}$$

where $F_{\rho\sigma}F^{\rho\sigma}$ is the first invariant of electromagnetic field (3.27), $F_{\alpha\beta}$ is Maxwell tensor, and p is fluid pressure. If $p{=}\rho c^2$ (substance inside atomic nuclei) and $p = const$, energy-impulse tensor of ideal fluid seems to be proportional to metric tensor.

But in the next Section we will show that equation of state of μ-vacuum has fully different form $p{=}-\rho c^2$ (state of inflation, expansion in case of positive density). Hence pressure and density in atomic nuclei should not be constant as to prevent transition of their inner substance into vacuum-like state.

Noteworthy, this T-classification, just like equations of field, is only about *distribution of matter* that affects space curvature, but not about test particles — material points which own masses and sizes are so small that their effect on curvature of space can be neglected. Therefore energy-impulse tensor is not defined for particles, and they should be considered beyond this T-classification.

5.3 Physical properties of vacuum. Cosmology

Einstein spaces are defined by field equations like $R_{\alpha\beta}{=}kg_{\alpha\beta}$, where $k{=}const$. With $\lambda{\neq}0$ and $T_{\alpha\beta}{=}\mu g_{\alpha\beta}$ space is filled with matter, which energy-impulse tensor is proportional to fundamental metric tensor, i. e. with μ-vacuum. As we saw in the previous Section, for vacuum energy-impulse tensor is also proportional to metric tensor. This means that physical properties of vacuum and those of μ-vacuum are mostly the same, save for a scalar coefficient that defines composition of matter (λ-fields or substance) and absolute values of the acting forces. Therefore we are going to consider Einstein space filled with vacuum and μ-vacuum. In this case field equations become

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\left(\text{æ}\mu - \lambda\right) g_{\alpha\beta}\,. \tag{5.19}$$

Putting them down in a mixed form and then contracting we arrive to scalar curvature

$$R = 4\left(\text{æ}\mu - \lambda\right), \tag{5.20}$$

substituting which into the initial equations (5.19) we obtain field equations in final form

$$R_{\alpha\beta} = \left(\text{æ}\mu - \lambda\right) g_{\alpha\beta}\,, \tag{5.21}$$

where the term $\text{æ}\mu{-}\lambda{=}const{=}k$.

Now we are going to look at physical properties of vacuum and μ-vacuum. We deduce chronometrically invariant components of energy-impulse tensor: observable density of matter $\rho{=}\dfrac{T_{00}}{g_{00}}$, observable density of impulse $J^i{=}\dfrac{cT_0^i}{\sqrt{g_{00}}}$, and observable tensor of strengths $U^{ik}{=}c^2T^{ik}$.

For energy-impulse tensor of μ-vacuum $T_{\alpha\beta}{=}\mu g_{\alpha\beta}$ physical observable components are

$$\rho = \frac{T_{00}}{g_{00}} = \mu\,, \tag{5.22}$$

$$J^i = \frac{cT_0^i}{\sqrt{g_{00}}} = 0\,, \tag{5.23}$$

$$U^{ik} = c^2 T^{ik} = -\mu c^2 h^{ik} = -\rho c^2 h^{ik}. \tag{5.24}$$

For energy-impulse tensor $\breve{T}_{\alpha\beta}=-\frac{\lambda}{æ}g_{\alpha\beta}$ (5.14) that describes vacuum, observable values are

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = -\frac{\lambda}{æ}\,, \tag{5.25}$$

$$\breve{J}^i = \frac{c\breve{T}_0^i}{\sqrt{g_{00}}} = 0\,, \tag{5.26}$$

$$\breve{U}^{ik} = c^2\breve{T}^{ik} = \frac{\lambda}{æ}c^2h^{ik} = -\breve{\rho}c^2h^{ik}. \tag{5.27}$$

From here we see that vacuum (λ-fields) and μ-vacuum have constant density, i. e. are *uniformly distributed matter* and are also *non-emitting media*, because energy flux c^2J^i in them is zero

$$c^2\breve{J}^i = \frac{c^3\breve{T}_0^i}{\sqrt{g_{00}}} = 0\,, \qquad c^2J^i = \frac{c^3T_0^i}{\sqrt{g_{00}}} = 0\,. \tag{5.28}$$

In the frame of reference that accompanies the medium, tensor of strengths equals (according to Zelmanov's works [6, 8])

$$U_{ik} = p_0h_{ik} - \alpha_{ik} = ph_{ik} - \beta_{ik}\,, \tag{5.29}$$

where p_0 is equilibrium pressure, defined from the equation of state, p is the true pressure, α_{ik} is *2nd type viscosity* (viscous strengths tensor) $\beta_{ik}=\alpha_{ik}-\frac{1}{3}\alpha h_{ik}$ is its anisotropic part (*1st type viscosity*, which reveal itself in anisotropic deformation), where $\alpha=\alpha_i^i$ is trace of 2nd type viscosity tensor.

Formulating tensor of strengths for μ-vacuum (5.24) in the frame of reference that accompanies μ-vacuum itself, we arrive to

$$U_{ik} = ph_{ik} = -\rho c^2h_{ik}\,, \tag{5.30}$$

and similarly to tensor of strengths for vacuum (5.27)

$$\breve{U}_{ik} = \breve{p}h_{ik} = -\breve{\rho}c^2h_{ik}\,. \tag{5.31}$$

This implies that vacuum and μ-vacuum are non-viscous media (α_{ik}=0, β_{ik}=0) which equations of state[35]

$$\breve{p} = -\breve{\rho}c^2, \qquad p = -\rho c^2. \tag{5.32}$$

Such state of matter is referred to as *inflation* because at positive density of matter pressure becomes negative and the media expands.

These are the basic physical properties of vacuum and μ-vacuum: uniform (ρ=*const*), non-viscous (α_{ik}=0, β_{ik}=0), and non-emitting (c^2J^i=0) media in the state of inflation.

From general physical properties we are going to turn now to analysis of vacuum that fills constant curvature spaces, in particular, de Sitter space, which is the closest approximation of our Universe as a whole.

In constant curvature spaces Riemann-Christoffel tensor is (see Chapter VII in Synge's book [30])

$$R_{\alpha\beta\gamma\delta} = K\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right), \qquad K = const. \tag{5.33}$$

Having the tensor contracted by two indices, we obtain the formula for Ricci tensor, which subsequent contraction allows to deduce a scalar value. As a result we have

$$R_{\alpha\beta} = -3Kg_{\alpha\beta}\,, \qquad R = -12K\,. \tag{5.34}$$

Assuming our Universe a constant curvature space, we obtain field equations formulated with curvature

$$3Kg_{\alpha\beta} = -æ\,T_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{5.35}$$

[35] Equation of state of distributed matter is dependence of its pressure from density. For example, p=0 is equation of state of dust media, $p=\rho c^2$ is equation of state of matter in atomic nuclei, $p=\frac{1}{3}\rho c^2$ is equation of state of ultra-relativistic gas.

We put them down in Synge's notation as $(\lambda-3K)\,g_{\alpha\beta}=æ\,T_{\alpha\beta}$. Then energy-impulse tensor of substance in constant curvature spaces is

$$T_{\alpha\beta}=\frac{\lambda-3K}{æ}\,g_{\alpha\beta}\,. \tag{5.36}$$

From here we see that in constant curvature space the problem of geometrization of matter solves by itself: energy-impulse tensor (5.36) contains metric tensor and constants only.

De Sitter space is a constant curvature space where $T_{\alpha\beta}=0$ and $\lambda\neq0$, i. e. the one filled with vacuum (substance is absent). Then having energy-impulse tensor of substance (5.36) equaled to zero we obtain the same result as did Synge: in de Sitter space $\lambda=3K$.

Taking into account this relation, the formula for observable density of vacuum in de Sitter world becomes

$$\breve{\rho}=-\frac{\lambda}{æ}=-\frac{3K}{æ}=-\frac{3Kc^2}{8\pi G}\,. \tag{5.37}$$

Now we are approaching the key question: what is the sign of four-dimensional curvature in our Universe? The reason to ask is not pure curiosity. Depending from the answer the de Sitter world cosmology we have built may fit the available data of observations or may lead to results totally alien to commonly accepted astronomical facts.

As a matter of fact, given that four-dimensional curvature is positive ($K>0$) density of vacuum will be negative and hence inflational pressure will be greater than zero — vacuum contracts. Then because $\lambda>0$ non-Newtonian forces of gravitation are those of repulsion. We will then witness struggle of two actions: positive inflational pressure of vacuum, which tend to compress the space, and repulsion forces of non-Newtonian gravitation. The result will be as follows: firstly, because λ-forces are proportional to distance, their expanding effect would grow along with growth of radius of the Universe and the expansion would accelerate. Secondly, if the Universe has ever been of size less than the distance, at which contracting pressure of vacuum is equal to expanding action of λ-forces, the expansion would become impossible.

If to the contrary the four-dimensional curvature is negative (i. e. $K<0$), the inflational pressure will be less than zero — vacuum expands. Besides, because in this case $\lambda<0$, non-Newtonian forces of gravitation are those of attraction. Then the Universe can keep expanding from nearly a point until density of vacuum becomes so low that its expanding action becomes equal to non-Newtonian λ-forces of attraction.

As seen, the question of curvature sign is the most crucial one for cosmology of our Universe.

But human perception is three-dimensional and a regular observer can not judge anything on sign of four-dimensional curvature by means of direct observations. What can be done then? The way out of the situation is in theory of chronometric invariants — a method to define physical observable values.

Among the goals that Zelmanov set for himself was to build tensor of curvature of three-dimensional non-holonomic space, which would possess properties of Riemann-Christoffel tensor and, at the same time, would be chronometrically invariant. Zelmanov decided to build such tensor using similarity with Riemann-Christoffel tensor, which results from non-commutativity of the second derivatives from an arbitrary vector in a given space. Deducing the difference of the second chronometrically invariant derivatives from an arbitrary vector, he arrived to

$${}^*\nabla_i\,{}^*\nabla_k Q_l-{}^*\nabla_k\,{}^*\nabla_i Q_l=\frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t}+H_{lki}^{\cdots j}Q_j\,, \tag{5.38}$$

which contains chronometrically invariant tensor

$$H_{lki}^{\cdots j}=\frac{{}^*\partial\triangle_{il}^{j}}{\partial x^k}-\frac{{}^*\partial\triangle_{kl}^{j}}{\partial x^i}+\triangle_{il}^{m}\triangle_{km}^{j}-\triangle_{kl}^{m}\triangle_{im}^{j}\,, \tag{5.39}$$

which is similar to Schouten tensor from theory of non-holonomic manifolds[36]. But in general case in presence of space rotation ($A_{ik}\neq0$), tensor $H_{lki}^{\cdots j}$ is algebraically different from Riemann-Christoffel

[36] J. A. Schouten built the theory of non-holonomic manifolds for an arbitrary number of measurements, considering m-dimensional sub-space in n-dimensional space, where $m<n$ [31]. In theory of chronometric invariants we actually consider an ($m=3$)-dimensional sub-space in ($n=4$)-dimensional space.

tensor. Therefore Zelmanov introduced a new tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} + H_{iljk}\right), \tag{5.40}$$

which was not only chronometrically invariant, but also possessed all algebraic properties of Riemann-Christoffel tensor. Therefore C_{lkij} is tensor of curvature of three-dimensional space of reference of observer, who accompanies his body of reference. Having it contracted, we obtain chronometrically invariant values

$$C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im} C_{kimj}\,, \qquad C = C_j^j = h^{lj} C_{lj}\,, \tag{5.41}$$

which also describe curvature of three-dimensional space. Because C_{lkij}, C_{kj}, and C are chronometrically invariant, they are physical observable values. In particular C is *three-dimensional observable curvature* [8, 10].

Concerning our analysis of vacuum properties and cosmology, we need to know how observable three-dimensional curvature C is linked to four-dimensional curvature K in general and in de Sitter space in particular. We are going to tackle this problem step-by-step.

Four-dimensional Riemann-Christoffel curvature tensor is a 4th-rank tensor, hence it has $n^4{=}256$ components, out of which only 20 are significant. Other components are either zeroes or contain each other, because Riemann-Christoffel tensor is:

- symmetric by each pair of indices $R_{\alpha\beta\gamma\delta}{=}R_{\gamma\delta\alpha\beta}$;
- antisymmetric in respect to transposition inside each pair of indices $R_{\alpha\beta\gamma\delta}{=}{-}R_{\beta\alpha\gamma\delta}$, $R_{\alpha\beta\gamma\delta}{=}{-}R_{\alpha\beta\delta\gamma}$;
- its components are constrained with the relationship $R_{\alpha(\beta\gamma\delta)}{=}0$, where round brackets stand for transpositions by indices β, γ, δ.

Significant components of Riemann-Christoffel tensor produce three chronometrically invariant (physical observable) tensors

$$X^{ik} = -c^2\frac{R_{0\cdot 0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \quad Y^{ijk} = -c\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \quad Z^{ijkl} = -c^2 R^{ijkl}. \tag{5.42}$$

Tensor X^{ik} has 6 components, tensor Y^{ijk} has 9 components, while tensor Z^{ijkl} has only 9 due to its symmetry. Components of the second tensor are constrained by $Y_{(ijk)}{=}Y_{ijk}{+}Y_{jki}{+}Y_{kij}{=}0$. Formulating the values with chronometrically invariant properties of space of reference and having indices lowered we obtain

$$X_{ij} = \frac{^*\partial D_{ij}}{\partial t} - \left(D_i^l + A_{i\cdot}^{\cdot l}\right)\left(D_{jl} + A_{jl}\right) + \frac{1}{2}\left(^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}F_i F_j\,, \tag{5.43}$$

$$Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2}A_{ij}F_k\,, \tag{5.44}$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2 C_{iklj}\,. \tag{5.45}$$

From these Zelmanov formulas we see that spatial observable components of Riemann-Christoffel curvature tensor (5.45) are directly linked to chronometrically invariant tensor of three-dimensional observable curvature C_{iklj}.

Now we are going to deduce the formula for three-dimensional observable curvature in a constant curvature space. In this case Riemann-Christoffel tensor is as of (5.33), then

$$R_{0i0k} = -K h_{ik} g_{00}\,, \tag{5.46}$$

$$R_{0ijk} = \frac{K}{c}\sqrt{g_{00}}\left(v_j h_{ik} - v_k h_{ij}\right), \tag{5.47}$$

$$R_{ijkl} = K\left[h_{ik}h_{jl} - h_{il}h_{kj} + \frac{1}{c^2}v_i\left(v_l h_{kj} - v_k h_{jl}\right) + \frac{1}{c^2}v_j\left(v_k h_{il} - v_l h_{ik}\right)\right]. \tag{5.48}$$

Having deduced its physical observable components (5.42), we obtain Having deduced its chronometrically invariant (physical observable) components (5.42), we obtain

$$X^{ik} = c^2 K h^{ik}, \quad Y^{ijk} = 0\,, \quad Z^{ijkl} = c^2 K \left(h^{ik}h^{jl} - h^{il}h^{jk}\right), \tag{5.49}$$

hence spatial observable components with lower indices will be

$$Z_{ijkl} = c^2 K \left(h_{ik}h_{jl} - h_{il}h_{jk}\right). \tag{5.50}$$

Contracting this value step-by-step we obtain

$$Z_{jl} = Z^{i\cdots}_{\cdot jil} = 2c^2 K h_{jl}\,, \qquad Z = Z^j_j = 6c^2 K\,. \tag{5.51}$$

On the other hand, we know the formula for Z_{ijkl} in an arbitrary curvature space (5.45), which explicitly contains tensor of three-dimensional observable curvature. Evidently it is true for $K = const$ as well. Then having the general formula (5.45) contracted we have

$$Z_{il} = D_{ik}D^k_l - D_{il}D + A_{ik}A^{\cdot k}_{l\cdot} + 2A_{ik}A^{k\cdot}_{\cdot l} - c^2 C_{il}\,, \tag{5.52}$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2 C\,. \tag{5.53}$$

In a constant curvature space $Z = 6c^2 K$ (5.51), hence in such space the relationship between four-dimensional curvature K and physical observable three-dimensional curvature C is

$$6c^2 K = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2 C\,. \tag{5.54}$$

From here we see that in absence of rotation and deformation of space four-dimensional curvature has the opposite sign in respect to three-dimensional observable curvature. In de Sitter space (because there rotation and deformation are absent) we have

$$K = -\frac{1}{6}C\,, \tag{5.55}$$

i. e. three-dimensional observable curvature equals $C = -6K$.

Now we are able to build a model for development of our Universe relying upon two experimental facts: (a) the sign of observable density of matter, and (b) the sign of observable three-dimensional curvature.

Firstly, our everyday experience shows that density of matter in our Universe is positive however sparse it may be. Then to ensure that density of vacuum (5.37) is positive, the cosmological term should be negative $\lambda < 0$ (non-Newtonian forces attract) and hence four-dimensional curvature should be negative $K < 0$.

Secondly, as D. Ivanenko wrote in his preface to J. Weber's book [25] "Though the data of cosmological observations are evidently not exact, but, for instance, McWittie [32] maintains that the best results of observation of Hubble red shift $H \approx 75$–100 km/s $\cdot$Mpc and of average density of matter $\rho \approx 10^{-31}$ g/sm^3 support the idea of non-disappearing cosmological term $\lambda < 0$".

As a result we can assume that density of vacuum in our Universe is positive and three-dimensional observable curvature $C > 0$. Hence four-dimensional curvature $K < 0$ and hence cosmological term $\lambda < 0$. Then from (5.37) we obtain observable density of vacuum in our Universe, formulated with observable three-dimensional curvature

$$\breve{\rho} = -\frac{\lambda}{æ} = -\frac{3K}{æ} = \frac{C}{2æ} > 0\,, \tag{5.56}$$

i. e. inflational pressure of vacuum is negative $\breve{p} = -\breve{\rho}c^2$ (vacuum expands). And because uniform distribution in space is among the physical properties of vacuum, negative inflational pressure also implies expansion of the Universe as a whole.

Therefore observable three-dimensional space of our Universe ($C > 0$) is a three-dimensional expanding sphere, which is a sub-space of four-dimensional space-time ($K < 0$), a space with generalized Lobachewski-Bolyai geometry.

Of course de Sitter space is merely an approximation of our Universe. Astronomical data say that though “islands” of masses are occasional and hardly affect the global curvature, their effect on space curvature in their vicinities is significant (deviation of light rays within the gravitational field and similar effects). But in study of the Universe as a whole we can neglect occasional “islands” of substance and local non-uniformities in curvature. In such cases de Sitter space with negative four-dimensional curvature (observable three-dimensional curvature is positive) can be assumed the background space of our Universe.

5.4 Concept of Inversion Explosion of the Universe

From the previous Section we know that in a de Sitter space $\lambda=3K$, i. e. that according to its physical sense λ-term is the same as curvature. For three-dimensional spherical sub-space observable curvature $C=-6K$ is

$$C = \frac{1}{R^2}, \tag{5.57}$$

where R is observable radius of curvature (sphere radius). Then four-dimensional curvature of space-time equals

$$K = -\frac{1}{6R^2}\,, \tag{5.58}$$

i. e. the larger is the radius of sphere, the less is curvature K. According to astronomical estimates, our Universe emerged 10–20 billion years ago. Hence the distance covered by a photon since it was born at the dawn of the Universe is $R_H \approx 10^{27}$–10^{28} cm. This distance is referred to as *radius of the horizon of events*. Assuming our Universe as whole to be a de Sitter space with $K<0$ for four-dimensional curvature and hence for λ-term $\lambda=3K$ we have the estimate

$$K = -\frac{1}{6R_H^2} \approx -10^{-56}\,\text{cm}^{-2}. \tag{5.59}$$

On the other hand, similar figures for the horizon of events, curvature and λ-term are available from Roberto di Bartini [29, 30], who studied relationships between physical constants from topological viewpoint. In his works *the space radius of the Universe* is interpreted as *the longest distance*, defined from topological context. According to *di Bartini's inversion relationship*

$$\frac{R\rho}{r^2} = 1, \tag{5.60}$$

the space radius R (the longest distance) is an inversion image of gravitational radius of electron $\rho = 1.347\cdot 10^{-55}$ cm in respect to radius of spherical inversion $r = 2.818\cdot 10^{-13}$ cm, which equals to classical radius of electron (according to di Bartini — radius of spherical inversion). The space radius (the largest radius of the horizon of events) equals

$$R = 5.895 \cdot 10^{29}\,\text{cm}\,. \tag{5.61}$$

From topological context di Bartini also defined *the space mass* (the mass within the space radius) and *the space density*, which are

$$M = 3.986 \cdot 10^{57}\,\text{g}\,, \qquad \rho = 9.87 \cdot 10^{-34}\,\text{g/cm}^3. \tag{5.62}$$

As a matter of fact, studies done by di Bartini say that the space of the Universe (from classical radius of electron up to the horizon of events) is an external inversion image of the inner space of a certain particle with size of electron (its radius can be estimated within the range from the classical radius of electron up to its gravitational radius). From other viewpoints the particle is different from electron: its mass equals to space mass $M=3.986\cdot 10^{57}$ g, while that of electron is $m=9.11\cdot 10^{-28}$ g.

The space within that particle can not be represented as a de Sitter space. As a matter of fact, the density of vacuum in de Sitter space with $K<0$ and observable radius of curvature $r = 2.818\cdot 10^{-13}$ cm is

$$\breve{\rho} = -\frac{3K}{æ} = -\frac{1}{2ær^2} = 3.39 \cdot 10^{51}\,\text{g/cm}^3, \tag{5.63}$$

while that inside di Bartini's particle is

$$\rho = \frac{M}{2\pi^2 r^3} = 9.03 \cdot 10^{93}\ \text{g/cm}^3. \tag{5.64}$$

On the other hand, an outer space, being the inversion image of the inner one, according to its properties can be assumed as a de Sitter space. Let us assume that a space with radius of curvature, equal to di Bartini radius R=5.895 $\cdot 10^{29}$ cm, is a de Sitter space with K<0. Then four-dimensional curvature and λ-term are

$$K = -\frac{1}{6R^2} = -4.8 \cdot 10^{-61}\ \text{cm}^{-2}, \tag{5.65}$$

$$\lambda = 3K = -\frac{1}{2R^2} = -14.4 \cdot 10^{-61}\ \text{cm}^{-2}, \tag{5.66}$$

i.e. are five orders of magnitude less than the observed estimate, which equals $|\lambda|<10^{-56}$. This can be explained because the Universe keeps on expanding and in a distant future absolute values of its curvature and the cosmological term will grow down to approach the figures in (5.65, 5.66), calculated for the longest distance (the space radius). Estimated density of vacuum in de Sitter space within the space radius is

$$\breve{\rho} = -\frac{3K}{æ} = -\frac{3Kc^2}{8\pi G} \approx 7.7 \cdot 10^{-34}\ \text{g/cm}^3 \tag{5.67}$$

is also less than observed average density of matter in the Universe (5–10$\cdot 10^{-30}$ g/cm^3) and is close to the density of matter within the space radius according to di Bartini 9.87$\cdot 10^{-34}$ g/cm^3.

To find how long will our Universe keep expanding we have to define the gap between the observed radius of the horizon of events R_H and the radius of curvature R. Assuming the maximum radius of the horizon of events in the Universe $R_{H(\max)}$ equal to the space radius (the outer inversion distance), which according to di Bartini is R=5.895$\cdot 10^{29}$ cm (5.61), and comparing it with the observed radius of the horizon of events ($R_H \approx 10^{27}$–10^{28} cm), we obtain $\triangle R = R_{H(\max)} - R_H \approx 5.8 \cdot 10^{29}$ cm, i.e. the time left for expansion is

$$t = \frac{\triangle R}{c} \approx 600 \text{ billion years}. \tag{5.68}$$

These calculations of the density of vacuum and of other properties of de Sitter space pave the way for conclusions on the origin and evolution of our Universe and allow the only interpretation of di Bartini's inversion relationship. We will call it cosmological concept of *Inversion Explosion*. The concept based upon our analysis of properties of de Sitter space using geometric methods of General Relativity, and di Bartini's inversion relationship as a result of contemporary knowledge of physical constants. We can set forth the concept as follows:

In the beginning there existed a single pra-particle with radius equal to classical radius of electron and with mass equal to mass of the entire Universe.

Then the inversion explosion occurred: a topological transition inverted matter in the pra-particle in respect to its surface into the outer world, which gave birth to our expanding Universe. At present, 10–20 billion years since the explosion, the Universe is in the early stage of its evolution. The expansion will continue for almost 600 billion years.

At the end of this period the expanding Universe will reach its radius of curvature, at which non-Newtonian forces of gravitation, proportional to distance, will equalize inflational expanding pressure of vacuum. The expansion will discontinue and stability will be reached, which will last until the next inversion topological transition occurs.

Parameters of matter at stages of evolution are calculated in Table 3 — pra-particle before the inversion explosion, the stage of inversion expansion at the present time, and the stage after the expansion.

The reasons for topological transition, which led to spherical inversion of matter from pra-particle (Inversion Explosion), remain unknown... but so do the reasons for the "emerge" of the Universe in some other contemporary cosmological concepts, for instance, in the concept of the Big Bang from a singular point.

Evolution stage	**Age, years**	**Space radius, cm**	**Density, g/cm^3**	**λ-term, cm^{-2}**
Pra-particle	0	$2.82\cdot 10^{-13}$	$9.03\cdot 10^{93}$	?
Present time	10–$20\cdot 10^{9}$	10^{27}–10^{28}	5–$10\cdot 10^{-30}$	$<10^{-56}$
After expansion	$623\cdot 10^{9}$	$5.89\cdot 10^{29}$	$9.87\cdot 10^{-34}$	$1.44\cdot 10^{-60}$

Table 3. Parameters of matter at stages of evolution of the Universe

5.5 Non-Newtonian gravitational forces

Type I Einstein spaces, including constant curvature spaces, aside for having occasional "islands of matter" may be either empty or filled with uniform matter. But empty Type I Einstein space (curvature $K=0$) is dramatically different from not empty one ($K=const\neq 0$).

To make our discourse more concrete, we are going to look at the most typical examples of empty and not-empty Type I Einstein space.

If an island of mass is a ball (spherically symmetric distribution of mass in the island) placed into emptiness, then curvature of such space is produced by Newtonian field of gravitation of island and such is not a constant curvature space. At an infinite distance from the island space becomes flat again, i. e. constant curvature space with $K=0$. A typical example of field of gravitation produced by spherically symmetric island of mass in emptiness is a field described by Schwarzschild metric

$$ds^2=\left(1-\frac{r_g}{r}\right)c^2dt^2-\frac{dr^2}{1-\dfrac{r_g}{r}}-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{5.69}$$

where r is distance from the island and r_g is its gravitational radius.

In Schwarzschild metric space rotation and deformation are absent. Components of vector of gravitational inertial force (1.38) can be deduced as follows. According to the metric (5.69), g_{00} is

$$g_{00}=1-\frac{r_g}{r}\,, \tag{5.70}$$

then derivative from potential $w=c^2(1-\sqrt{g_{00}})$ is

$$\frac{\partial w}{\partial x^i}=-\frac{c^2}{2\sqrt{g_{00}}}\frac{\partial g_{00}}{\partial x^i}\,. \tag{5.71}$$

Having this derivative substituted into the formula for gravitational inertial force (1.38) in absence of rotation we have

$$F_1=-\frac{c^2r_g}{2r^2}\frac{1}{1-\dfrac{r_g}{r}}\,,\qquad F^1=-\frac{c^2r_g}{2r^2}\,. \tag{5.72}$$

Therefore, vector F^i in Schwarzschild metric space describes Newtonian gravitational force, which is reciprocal to square of distance r from the mass (the source of field).

If space is *filled* with spherically symmetric distribution of vacuum and does not include an island of mass, its curvature will be everywhere the same. An example of such field is that described by de Sitter metric

$$ds^2=\left(1-\frac{\lambda r^2}{3}\right)c^2dt^2-\frac{dr^2}{1-\dfrac{\lambda r^2}{3}}-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right). \tag{5.73}$$

Note that though de Sitter space has not islands of mass that produce Newtonian fields of gravitation. So, in de Sitter space we can consider motion of small (test) particles, which own Newtonian fields are so weak that can be neglected.

De Sitter metric space is a constant curvature one, which becomes flat space only in absence of λ-fields. Rotation or deformation are also absent here, while components of gravitational inertial force vector are

$$F_1 = \frac{\lambda c^2}{3} \frac{r}{1-\dfrac{\lambda r^2}{3}}\,, \qquad F^1 = \frac{\lambda c^2}{3} r\,, \tag{5.74}$$

i. e. vector F^i in de Sitter space describes non-Newtonian gravitational forces, proportional to r: if $\lambda<0$, those are attraction forces, if $\lambda>0$ those are repulsion forces. Therefore forces of non-Newtonian gravitation (λ-forces) grow along with distance at which they act.

Therefore we can see the principal difference between empty and non-empty Type I Einstein space: in empty one with an island of mass only Newtonian forces exist, while in the one filled with vacuum without islands of mass there are non-Newtonian gravitation forces only. An example of "mixed" Type I space is that with Kottler metric [35]

$$\begin{gathered} ds^2 = \left(1+\frac{ar^2}{3}+\frac{b}{r}\right)c^2dt^2 - \frac{dr^2}{1+\dfrac{ar^2}{3}+\dfrac{b}{r}} - r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \\ F_1 = -c^2\frac{\dfrac{ar}{3}-\dfrac{b}{2r^2}}{1+\dfrac{ar^2}{3}+\dfrac{b}{r}}\,, \qquad F^1 = -c^2\left(\frac{ar}{3}-\frac{b}{2r^2}\right), \end{gathered} \tag{5.75}$$

where both Newtonian and λ-forces exist: it is filled with vacuum and includes islands of mass, which produce Newtonian forces of gravitation. On the other hand, F. Kottler proposed his metric with two unknown constants a and b to define which some additional constraints are required. Hence despite some of attractive features of Kottler metric, only two its "ultimate" cases are of practical interest for us — Schwarzschild metric (Newtonian forces) and de Sitter metric (λ-forces).

5.6 Gravitational collapse

Evidently, representing our Universe as either a de Sitter space (filled with vacuum without islands of mass) or a Schwarzschild space (islands of mass in emptiness) is a certain approximating assumption. The real metric of our world in "something in the between". Nevertheless, in some problems dealt with non-Newtonian gravitation (produced by vacuum), where influence of concentrated masses can be neglected, de Sitter metric is optimal. And vice versa, in problems with field of concentrated masses Schwarzschild metric is more reasonable. An illustrative example of such "split" of models is collapse — a state of space-time in which $g_{00}=0$.

Gravitational potential w for an arbitrary metric is (1.38). Then

$$g_{00} = \left(1-\frac{w}{c^2}\right)^2 = 1-\frac{2w}{c^2}+\frac{w^2}{c^4}\,, \tag{5.76}$$

i. e. collapse ($g_{00}=0$) occurs at $w=c^2$.

Commonly, *gravitational collapse* is considered — compression of an island of mass under action of Newtonian gravitation until it reaches its gravitational radius. Hence "pure" gravitational collapse occurs in Schwarzschild metric space (5.69), where only Newtonian field of spherically symmetric island of mass in emptiness is present.

At larger distances from concentrated mass gravitational field becomes weak and Newtonian law of gravitation becomes true. Hence in a weak field of Newtonian gravitation potential is

$$w = \frac{GM}{r}\,, \tag{5.77}$$

where G is Newton-Gauss gravitational constant, M is mass of the body that produced that field of gravitation. In a weak field the third term in (5.76) is small and can be neglected; hence the formula for g_{00} becomes

$$g_{00} = 1 - \frac{2GM}{c^2 r}\,, \tag{5.78}$$

i. e. gravitational collapse in Schwarzschild space occurs if

$$\frac{2GM}{c^2 r} = 1\,, \tag{5.79}$$

where the value

$$r_g = \frac{2GM}{c^2}\,, \tag{5.80}$$

which has the dimension of length, is referred to as *gravitational radius*. Then g_{00} can be presented as

$$g_{00} = 1 - \frac{r_g}{r}\,. \tag{5.81}$$

From here we see that at $r{=}r_g$ in Schwarzschild space collapse occurs. In such case all mass of spherically symmetric body (the source of Newtonian field) becomes concentrated within its gravitational radius. Therefore the surface of a spherical body, which radius equals to its gravitational radius, is referred to as *Schwarzschild sphere*. Such objects are also called *black holes* because within the gravitational radius escape velocity is above that of light and hence light can not be emitted from such objects outside.

As seen from metric formula (5.69), in Schwarzschild field of gravitation three-dimensional space does not rotate ($g_{0i}{=}0$) and hence interval of observable space (1.25) is

$$d\tau = \sqrt{g_{00}}\,dt = \sqrt{1 - \frac{r_g}{r}}\,dt\,, \tag{5.82}$$

i. e. at the distance $r{=}r_g$ interval of observable time equals zero $d\tau{=}0$: from viewpoint of an external observer the time on the surface of Schwarzschild sphere stops[37]. Inside Schwarzschild sphere interval of observable time becomes imaginary. We can also be sure that a regular observer who lives on the surface of the Earth, apparently stays outside its Schwarzschild sphere with radius of 0.443 cm and can only look at process of gravitational collapse from "outside".

If $r{=}r_g$ then the value

$$g_{11} = -\frac{1}{1 - \dfrac{r_g}{r}} \tag{5.83}$$

grows up to infinity. But the determinant of metric tensor $g_{\alpha\beta}$ is

$$g = -r^4 \sin^2\theta < 0\,, \tag{5.84}$$

and hence space-time inside gravitational collapser is generally not degenerated, though collapse is also possible in zero-space.

At this point a note concerning photometric distance and metric physically observable distance should be taken.

The value r is not a metric distance along axis $x^1{=}r$, because the formula for metric (5.69) contains dr^2 with coefficient $\left(1{-}\frac{r_g}{r}\right)^{-1}$. Value r is *photometric distance* defined as function of illumination produced by a stable source of light and reciprocal to square of distance. In other words, r is radius of non-Euclidean sphere with surface are $4\pi r^2$ [10].

[37] At $g_{00}{=}0$ (collapse) interval of observable time (1.25) equals to $d\tau{=}{-}\frac{1}{c^2}v_i dx^i$, where $v_i{=}{-}c\frac{g_{0i}}{\sqrt{g_{00}}}$ is velocity of space rotation (1.37). Only assuming $g_{oi}{=}0$ and $v_i{=}0$ the condition of collapse can be defined correctly: for an external observer time on the surface of collapser stops $d\tau{=}0$, while four-dimensional interval equals to $ds^2{=}{-}d\sigma^2{=}g_{ik}dx^i dx^k$. From here a single conclusion can be made: on the surface of collapser space is holonomic (collapser does not rotate). In our previous studies [15, 16] we showed that zero-space collapses if it does not rotate. Here we proved a more general theorem: if $g_{00}{=}0$ space is holonomic irrespective of whether it is degenerated ($g{=}0$, zero space) or for it $g{<}0$ (space-time of General Relativity).

According to theory of chronometric invariants, metric elementary observable distance between two points in Schwarzschild space is

$$d\sigma = \sqrt{\frac{dr^2}{1-\dfrac{r_g}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)}\,. \tag{5.85}$$

At $\theta{=}const$ and $\varphi{=}const$ it is

$$\sigma = \int_{r_1}^{r_2}\sqrt{h_{11}}\,dr = \int_{r_1}^{r_2}\frac{dr}{\sqrt{1-\dfrac{r_g}{r}}} \tag{5.86}$$

and is not the same as photometric distance r.

Now we are going to define metric of space-time inside Schwarzschild sphere. To do this we formulate external metric (5.69) for radius $r<r_g$. As a result we have

$$ds^2 = -\left(\frac{r_g}{r}-1\right)c^2dt^2 + \frac{dr^2}{\dfrac{r_g}{r}-1} - r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right). \tag{5.87}$$

Introducing notations $r{=}c\tilde{t}$ and $ct{=}\tilde{r}$ we obtain

$$ds^2 = \frac{c^2d\tilde{t}^2}{\dfrac{r_g}{c\tilde{t}}-1} - \left(\frac{r_g}{c\tilde{t}}-1\right)d\tilde{r}^2 - c^2d\tilde{t}^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{5.88}$$

i. e. metric of space-time inside Schwarzschild sphere is similar to the external metric provided that the temporal coordinate and the spatial coordinate r swap their roles: photometric distance r outside black hole is coordinate time $c\tilde{t}$ inside, while coordinate time outside black hole ct is photometric distance $\tilde{r}$ inside.

From the first term of Schwarzschild inner metric (5.88) we see that it is not stationary and exists within a limited period of time

$$\tilde{t} = \frac{r_g}{c}\,. \tag{5.89}$$

For the Sun, which gravitational radius is 3 km, life span of such space would be approximately $<10^{-5}$ s. For the Earth, which gravitational radius is a small as 0.443 cm, life span of inner Schwarzschild metric would be even less at $1.5{\cdot}10^{-11}$ s.

Comparison of metrics inside gravitational collapser (5.88) and outside of the collapsed body (5.69) implies that:

1. space of reference of both metrics is holonomic, i. e. does not rotate ($A_{ik}{=}0$);
2. external metric is stationary, vector of gravitational inertial force is $F^1{=}-\dfrac{GM}{r^2}$;
3. internal metric is non-stationary, vector of gravitational inertial force is zero.

Now we are going to give external and internal metrics more detailed analysis; to make it simpler we assume $\theta{=}const$ and $\varphi{=}const$, i. e. out of all possible spatial directions we limit our study to radial directions only. Then the external metric will be

$$ds^2 = -\left(\frac{r_g}{r}-1\right)c^2dt^2 + \frac{dr^2}{\dfrac{r_g}{r}-1}\,, \tag{5.90}$$

while the internal metric is, respectively

$$ds^2 = \frac{c^2d\tilde{t}^2}{\dfrac{r_g}{c\tilde{t}}-1} - \left(\frac{r_g}{c\tilde{t}}-1\right)d\tilde{r}^2. \tag{5.91}$$

Now we will define physical observable distance (5.86) along radial direction to the attracting mass (gravitational collapser)

$$\sigma = \int \frac{dr}{\sqrt{1-\frac{r_g}{r}}} = \sqrt{r\,(r-r_g)} + r_g \ln\left(\sqrt{r} + \sqrt{r-r_g}\right) + const. \tag{5.92}$$

From here we see: at $r{-}r_g$ observable distance

$$\sigma_g = r_g \ln \sqrt{r_g} + const, \tag{5.93}$$

and is a constant value. This means that Schwarzschild sphere, defined by photometric radius r_g, for an external observer is a sphere with observable radius $\sigma_g{=}r_g \ln \sqrt{r_g}{+}const$ (5.93). Therefore for an external observer gravitational collapser (black hole) is a sphere with constant observable radius, on which surface time stops.

Now we are going to analyze gravitational collapser's interiors. Interval of observable time (5.82) inside Schwarzschild sphere is imaginary for an external observer

$$d\tau = i\sqrt{\frac{r_g}{r}-1}\;dt\,, \tag{5.94}$$

or, in "interior" coordinates $r{=}c\tilde{t}$ and $ct{=}\tilde{r}$ (from viewpoint of an "inner" observer),

$$d\tilde{\tau} = \frac{1}{\sqrt{\frac{r_g}{c\tilde{t}}-1}}\,d\tilde{t}\,. \tag{5.95}$$

Hence for an external observer the internal "imaginary" time of collapser (5.94) stops at its surface, while the "inner" observer sees the pace of observable time on the surface grow infinitely.

From external viewpoint three-dimensional metric distance inside collapser according to (5.87) is

$$\sigma = \int \frac{dr}{\sqrt{\frac{r_g}{r}-1}} = -\sqrt{r\,(r-r_g)} + r_g \arctan \sqrt{\frac{r_g}{r}-1} + const, \tag{5.96}$$

or, from viewpoint of the "inner" observer

$$\tilde{\sigma} = \int \sqrt{\frac{r_g}{c\tilde{t}}-1}\;dr\,. \tag{5.97}$$

From here we see: at $r{=}c\tilde{t}{=}r_g$ for an external observer observable distance between any two points converges to a constant, while for the "inner" observer observable spatial interval grows down to zero.

In conclusion we will address the question of what happens to particles, which fall from "outside" on Schwarzschild sphere along a radial direction. External metric can be presented as

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \quad d\tau = \left(1-\frac{r_g}{r}\right) dt\,, \quad d\sigma = \frac{dr}{1-\frac{r_g}{r}}\,. \tag{5.98}$$

For real-mass particles $ds^2{>}0$, for light-like particles $ds^2{=}0$, for super-light-speed tachyons $ds^2{<}0$ (their mass is imaginary). In radial motion towards black hole these conditions can be represented as:

1. mass-bearing real particles $\left(\frac{d\tau}{dt}\right)^2 < c^2\left(1-\frac{r_g}{r}\right)^2$;
2. light-like particles $\left(\frac{d\tau}{dt}\right)^2 = c^2\left(1-\frac{r_g}{r}\right)^2$;
3. imaginary particles-tachyons $\left(\frac{d\tau}{dt}\right)^2 > c^2\left(1-\frac{r_g}{r}\right)^2$.

On Schwarzschild sphere $r{=}r_g$. Hence $\frac{d\tau}{dt}{=}0$, i. e. any particle, including light-like one, will stop there. Four-dimensional interval on Schwarzschild sphere is space-like

$$ds^2 = -d\sigma^2, \tag{5.99}$$

i. e. $ds^2{<}0$. This implies that Schwarzschild sphere is filled with particles with imaginary rest-mass.

5.7 Inflational collapse

There are no islands of mass in de Sitter space, hence field of Newtonian gravitation is absent too — gravitational collapse is impossible. Nevertheless, condition $g_{00}=0$ is a purely geometric definition of collapse, not necessarily related to Newtonian fields. Subsequently, we can consider it in any arbitrary space.

We are going to look at a de Sitter metric space (5.73), which describes non-Newtonian field of gravitation in a constant curvature space without islands of mass. In this case collapse may occur due to non-Newtonian gravitational forces. From de Sitter metric (5.73) we see that

$$g_{00} = 1 - \frac{\lambda r^2}{3}\,, \tag{5.100}$$

i. e. gravitational potential $w=c^2(1-\sqrt{g_{00}})$ in de Sitter space is

$$w = c^2\left(1 - \sqrt{1 - \frac{\lambda r^2}{3}}\right). \tag{5.101}$$

Because it is a potential of non-Newtonian gravitation, produced by vacuum, we will call it λ-potential. From this formula we see that λ-potential equals to zero if de Sitter space is flat (in this case $\lambda=3$ and $K=0$).

Because in de Sitter space $\lambda=3K$, hence

- $g_{00}=1-Kr^2>0$ at distances $r<\dfrac{1}{\sqrt{K}}$;
- $g_{00}=1-Kr^2<0$ at distances $r>\dfrac{1}{\sqrt{K}}$;
- $g_{00}=1-Kr^2=0$ (collapse) at distances $r=\dfrac{1}{\sqrt{K}}$.

At curvature $K<0$ the value $g_{00}=1-Kr^2$ is always greater than zero. Hence collapse is only possible in a de Sitter space with $K>0$.

In Section 5.3 we showed that space in our Universe as a whole has $K<0$. But we can assume presence of local non-uniformities with $K>0$, which do not affect curvature of the space in general. In particular, on such non-uniformities collapse may occur. Therefore it is reasonable to consider de Sitter space with $K>0$ as a local space in the vicinities of some compact objects.

Three-dimensional physical observable curvature C is linked to four-dimensional curvature with relationship $C=-6K$ (5.55). Then assuming three-dimensional space is a sphere, we obtain $C=\dfrac{1}{R^2}$ (5.57) and hence $K=-\dfrac{1}{6R^2}$ (5.58), where R is three-dimensional observable radius of curvature. In case $K<0$ value R is real, at $K>0$ it becomes imaginary.

Collapse in de Sitter space is only possible at $K>0$. In this case observable radius of curvature is imaginary. We denote $R=iR^*$, where R^* is its absolute value. Then in de Sitter space with $K>0$

$$K = \frac{1}{6R^{*2}}\,, \tag{5.102}$$

and the collapse condition $g_{00}=1-Kr^2$ can be represented as

$$r = R^*\sqrt{6}\,, \tag{5.103}$$

i. e. at the distance $r=R^*\sqrt{6}$ in de Sitter space with $K>0$ the value $g_{00}=0$ and hence observable time stops and collapse occurs.

In other words, an area of de Sitter space within radius $r=R^*\sqrt{6}$ stays in collapse. Taking into account that vacuum that fills de Sitter space stays in inflation, we will refer to such collapsed area as *inflational collapse*, while the value $r=R^*\sqrt{6}$ (5.77) will be referred to as *inflational radius* $r_{\rm inf}$. Then collapsed area of de Sitter space within inflational radius will be referred to as *inflational collapser* (or as *inflanton*).

Inside inflanton $K>0$ (observable three-dimensional curvature $C<0$). In this case density of vacuum is negative (inflational pressure is positive, vacuum compresses) and $\lambda>0$, i.e. non-Newtonian forces repulse. This means that inflational collapser (inflanton) is filled with vacuum with negative density and is in the state of fragile balance between compacting pressure of vacuum and expanding forces of non-Newtonian gravitation.

Interval of observable time in de Sitter space with $K>0$ is

$$d\tau = \sqrt{g_{00}}\, dt = \sqrt{1 - Kr^2}\, dt = \sqrt{1 - \frac{r^2}{r_{\text{inf}}^2}}\, dt\,, \tag{5.104}$$

i.e. on the surface of inflational sphere observable time stops $d\tau=0$. The signature we have accepted $(+---)$, i.e. the condition $g_{00}>0$ is true at $r<r_{\text{inf}}$.

Using inflational radius we represent de Sitter metric with $K>0$ as

$$ds^2 = \left(1 - \frac{r^2}{r_{\text{inf}}^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \dfrac{r^2}{r_{\text{inf}}^2}} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5.105}$$

Components of gravitational inertial force (5.74) in this case are

$$F_1 = \frac{c^2}{1 - \dfrac{r^2}{r_{\text{inf}}^2}} \frac{r}{r_{\text{inf}}^2}\,, \qquad F^1 = c^2 \frac{r}{r_{\text{inf}}^2}\,. \tag{5.106}$$

Now we are going to deduce observable distance and observable inflational radius. To make calculations simpler we assume $\theta=const$ and $\varphi=const$, i.e. out of all spatial directions only radial one will be considered. Then observable three-dimensional interval is

$$\sigma = \int \sqrt{h_{11}}\, dr = \int \frac{dr}{\sqrt{1 - Kr^2}} = r_{\text{inf}} \arcsin \frac{r}{r_{\text{inf}}} + const, \tag{5.107}$$

and hence observable inflational radius is constant

$$\sigma_{\text{inf}} = \int_0^{r_{\text{inf}}} \frac{dr}{\sqrt{1 - Kr^2}} = \frac{\pi}{2} r_{\text{inf}}\,. \tag{5.108}$$

In Schwarzschild metric space, which we looked at in the previous Section, collapser is a collapsed compact mass, which produces curvature of space as a whole: regular observer stays outside gravitational collapser.

In de Sitter metric space collapser is vacuum, which fills the whole space. Collapse area in de Sitter space is comparable to surface, which radius equals to radius of curvature of space, hence regular observer stays under the surface of inflational collapser and "watches" it from inside.

To look beyond inflational collapser we present de Sitter metric with $K>0$ (5.105) for $r>r_{\text{inf}}$. Considering radial directions, in coordinates of a regular observer ("inner" coordinates of the collapser) we obtain

$$ds^2 = -\left(\frac{r^2}{r_{\text{inf}}^2} - 1\right) c^2 dt^2 + \frac{dr^2}{\dfrac{r^2}{r_{\text{inf}}^2} - 1}\,, \tag{5.109}$$

or, from viewpoint of an observer, who styes outside collapser (in "external" coordinates of this collapser $r=c\tilde{t}$ and $ct=\tilde{r}$)

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\dfrac{c^2\tilde{t}^2}{r_{\text{inf}}^2} - 1} - \left(\frac{c^2\tilde{t}^2}{r_{\text{inf}}^2} - 1\right) d\tilde{r}^2. \tag{5.110}$$

5.8 Concept of the mirror Universe. Conditions of transition through membrane from our world into the mirror Universe

As we mentioned in Section 5.1, attempts to represent our world and the mirror Universe as two spaces with positive and negative curvature failed: even within de Sitter metric, which is among the simplest space-time metrics, trajectories in positive curvature spaces are substantially different from those in negative curvature spaces (see Chapter VII in J. L. Synge's book [30]).

On the other hand, numerous researchers, beginning from P. Dirac, intuitively predicted that the mirror Universe (as the antipode to our Universe) must be sought not in a space with opposite curvature sign, but rather in a space with different sign of mass and energy. That is, because masses of particles in our Universe are positive, then those of the mirror Universe particles must be evidently negative.

Joseph Weber [25] wrote that neither law of universal gravitation nor relativistic theory of gravitation ruled out existence of negative masses; rather, our empirical experience says they have never been observed. Both Newtonian theory of gravitation and General Relativity predicted behavior of negative masses, totally different from what electrodynamics prescribes for negative charges. For two bodies, one of which bears positive mass and another bears negative one, but equal to the first one in absolute value, it would be expected that positive mass will attract the negative one, while the negative mass will repulse the positive one, so that one will chase the other! If motion occurs along line that links the centers of the two bodies, such system will move with constant acceleration. This problem was studied by H. Bondi [37]. Assuming gravitational mass of positron to be negative (observations say its inertial mass is positive) and using methods of quantum electrodynamics, L. I. Schiff obtained the difference between inertial and gravitational masses of positron. The difference proved to be much greater than the error margin in the experiment by Eötvös, who showed equality of gravitational and inertial masses [38]. As a result, Schiff concluded that negative gravitational mass in positron can not exist (see Chapter 1 in J. Weber's book [25]).

Besides, "co-habitation" of positive and negative masses particles in the same space-time area would cause ongoing annihilation. Possible consequences of "mixed" particles with positive and negative masses were also studied by Ya. P. Terletskii [39, 40].

Therefore the idea of the mirror Universe as a world of negative masses and energies faced two obstacles: (a) *experimentum crucis*, which would point directly at exchange interactions between our world and the mirror Universe, and (b) absence of theory that would clearly explain separation of worlds with positive and negative masses in space-time of General Relativity.

In this Section we are going to tackle the second (theoretical) part of the problem. In the next Chapter we will show that the *experimentum crucis* has been actually accomplished in the recent decade by various researchers (anomalous rate of orthopositronium annihilation), but in absence of proper theoretical "back-up" its results have not been interpreted as proof of existence of the mirror Universe and are still open for discussion.

We are going to look at the term "mirror properties" as applied to space-time metric. To solve the problem we present the square of space-time interval in chronometrically invariant form

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{5.111}$$

where

$$d\sigma^2 = h_{ik}dx^i dx^k, \tag{5.112}$$

$$d\tau = \left(1 - \frac{w}{c^2}\right)dt - \frac{1}{c^2}v_i dx^i = \left(1 - \frac{w + v_i u^i}{c^2}\right)dt\,. \tag{5.113}$$

From here we see that elementary spatial interval (5.112) is a quadratic function of elementary spatial increments dx^i. Spatial coordinates x^i are all equal, i. e. there is no principal difference between translational movement to the right or to the left, up or down. Therefore we will no longer consider mirror reflections in respect to spatial coordinates.

Time is a different thing. Physical observable (observer's own) time τ of regular observer always flows from past into future, hence $d\tau > 0$. But there are two cases when time stops. Firstly, it is possible in a regular space-time in a state of collapse. Secondly, this happens in zero-space — degenerated four-dimensional space-time. Therefore the state of an observer, whose own time stops, may be regarded transitional one, i. e. unavailable under regular conditions.

We will consider the problem of the mirror world for both $d\tau>0$ and $d\tau=0$. In the latter case the analysis will be done separately for collapsed areas of regular space-time and for zero-space. We begin the analysis from a regular case of $d\tau>0$. From the formula for physical observable time (5.113) it is evident, that this condition is true when

$$w+v_iu^i<c^2. \tag{5.114}$$

In absence of space rotation ($v_i=0$) it becomes $w<c^2$, which corresponds to space-time structure in state of collapse.

The square of four-dimensional interval (5.111) can be expanded as

$$ds^2=\left(1-\frac{w}{c^2}\right)^2c^2dt^2-2\left(1-\frac{w}{c^2}\right)v_idx^idt-h_{ik}dx^idx^k+\frac{1}{c^2}v_iv_kdx^idx^k, \tag{5.115}$$

on the other hand

$$ds^2=c^2d\tau^2-d\sigma^2=c^2d\tau^2\left(1-\frac{\mathrm{v}^2}{c^2}\right),\qquad \mathrm{v}^2=h_{ik}\mathrm{v}^i\mathrm{v}^k. \tag{5.116}$$

Let us divide both parts of the formula for space-time interval ds^2 (5.115) by different values in accordance with the kind of space-time trajectory of particle (real non-isotropic, zero isotropic, or imaginary non-isotropic):

1. $c^2d\tau^2\left(1-\frac{\mathrm{v}^2}{c^2}\right)$ if space-time interval is real $ds^2>0$;
2. $c^2d\tau^2$ if space-time interval equals zero $ds^2=0$;
3. $-c^2d\tau^2\left(\frac{\mathrm{v}^2}{c^2}-1\right)$ if space-time interval is imaginary $ds^2<0$.

As a result in all cases we obtain the same quadratic equation in respect to function of coordinate time of the object dt from the observer's own time $d\tau$

$$\left(\frac{dt}{d\tau}\right)^2-\frac{2v_i\mathrm{v}^i}{c^2\left(1-\frac{w}{c^2}\right)}\frac{dt}{d\tau}+\frac{1}{\left(1-\frac{w}{c^2}\right)^2}\left(\frac{1}{c^4}v_iv_k\mathrm{v}^i\mathrm{v}^k-1\right)=0\,, \tag{5.117}$$

which has two solutions

$$\left(\frac{dt}{d\tau}\right)_1=\frac{1}{1-\frac{w}{c^2}}\left(\frac{1}{c^2}v_i\mathrm{v}^i+1\right), \tag{5.118}$$

$$\left(\frac{dt}{d\tau}\right)_2=\frac{1}{1-\frac{w}{c^2}}\left(\frac{1}{c^2}v_i\mathrm{v}^i-1\right). \tag{5.119}$$

Having coordinate time of the object t integrated to τ we obtain

$$t=\frac{1}{c^2}\int\frac{v_idx^i}{1-\frac{w}{c^2}}\pm\int\frac{d\tau}{1-\frac{w}{c^2}}+const. \tag{5.120}$$

It can be easily integrated if space does not rotate and gravitational potential $w=0$. Then the integral is $t=\pm\tau+const$. Proper choice of the initial conditions can make integration constant zero. In this case the formula for coordinate time t becomes

$$t=\pm\tau\,,\qquad \tau>0\,, \tag{5.121}$$

which graphically represents two beams, which are mirror reflections of each other in respect to $\tau>0$. We can say that observer's own time serves here as the mirror (membrane), while the mirror itself separates two worlds: one with coordinate time (observable change of temporal coordinate) that flows from past into future $t=\tau$, and the other, mirror one, where coordinate time flows from future into past $t=-\tau$.

Noteworthy, world with reverse flow of time is not like a videotape being rewound. Both worlds are quite equal, but for a regular observer values of temporal coordinate in the mirror world a negative. The mirror (membrane) in this case only reflects flow of time, but does not affect it.

Now we assume that space does not rotate ($v_i=0$), but gravitational potential is not zero ($w\neq 0$). Then coordinate time equals

$$t = \pm \int \frac{d\tau}{1-\frac{w}{c^2}} + const. \tag{5.122}$$

If gravitational potential is weak ($w \ll c^2$), the integral is

$$t = \pm \left(\tau + \frac{1}{c^2}\int w d\tau\right) = \pm\left(\tau + \triangle t\right), \tag{5.123}$$

where $\triangle t$ is a correction to take account for presence of field w, which produces acceleration. Value w may define any scalar field — either field of Newtonian potential or field of non-Newtonian gravitation.

If gravitational field produced by potential w is strong, integral will become as of (5.122) and will depend upon potential w: the stronger is field w, faster flows coordinate time t (5.122). In the ultimate case, when $w=c^2$, $t\rightarrow\infty$. On the other hand, at $w=c^2$ collapse occurs ($d\tau=0$). We will look at that case in the below, but now we are still assuming $w<c^2$.

Now we are going to look at coordinate time in Schwarzschild and de Sitter spaces. If potential w describes Newtonian gravitational field (Schwarzschild metric space), then

$$t = \pm \int \frac{d\tau}{1-\frac{GM}{c^2 r}} = \pm \int \frac{d\tau}{1-\frac{r_g}{r}}\,, \tag{5.124}$$

which implies that the closer we approach the gravitational radius of the mass, the bigger is the difference between coordinate time and observer's own time. If w is potential of non-Newtonian field of gravitation (de Sitter metric space), then

$$t = \pm \int \frac{d\tau}{\sqrt{1-\frac{\lambda r^2}{3}}} = \pm \int \frac{d\tau}{\sqrt{1-\frac{r^2}{r_{\rm inf}^2}}}\,, \tag{5.125}$$

which implies that the closer is photometric distance r to inflational radius of collapser, the faster (in its absolute value) flows coordinate time t. In the ultimate case at $r\rightarrow r_{\rm inf}$ coordinate time $t\rightarrow\infty$.

Therefore in absence of rotation of space but in presence of gravitation coordinate time t flows the faster the stronger is potential of field.

Now we turn to a more general case, when both rotation and gravitational field are present. Then integral for t takes the form (5.120), i. e. coordinate time in non-holonomic (rotating) space includes:

1. "rotational" time determined by presence of the term $v_i dx^i$, which has dimension of rotational moment divided by unit mass;
2. regular coordinate time, linked to pace of observer's own time.

From integral for t (5.120) we see that rotational coordinate time, produced by rotation of space, exist independently from observer (because does not depend from τ). For an observer who rests on Earth's surface (anywhere aside for the poles) it can be interpreted as time flow determined by rotation of the planet. It always exists irrespectively of whether observer records it in this particular location or not. Regular coordinate time is linked to presence of observer (depends from his own time τ) and to the field that exists at the point of observation; in particular, to field of Newtonian potential.

Noteworthy, at $v_i \neq 0$ temporal coordinate t at the initial time of observation (when observer's own time $\tau_0=0$) is not zero.

Presenting integral for t (5.120) as

$$t = \int \frac{\frac{1}{c^2} v_i dx^i \pm d\tau}{1 - \frac{w}{c^2}} \,, \tag{5.126}$$

we obtain that the formula under the integral sign is

- positive, if $\frac{1}{c^2} v_i dx^i > \mp d\tau$;
- zero, if $\frac{1}{c^2} v_i dx^i = \pm d\tau$;
- negative, if $\frac{1}{c^2} v_i dx^i < \mp d\tau$.

Hence coordinate time t for real observer stops if scalar product of rotation velocity of space by physical observable velocity of the object is $v_i \mathrm{v}^i = \pm c^2$. This happens when absolute values of both velocities equal to that of light, and are either co-directed or oppositely directed.

An area of space-time which satisfies condition $v_i \mathrm{v}^i = \pm c^2$, at which coordinate time stops for a real observer, is the *membrane* (the mirror) that separates areas of space with positive and negative time coordinate — areas with direct and reverse flow of time.

It is also evident that no real (material) observer can accompany such space of reference (or body of reference).

We will refer as *the mirror space* to an area of space-time where coordinate time takes negative values. We are going to analyze properties of particles in the mirror space in respect to those of particles in regular world, where temporal coordinate is positive.

Physical observable components of four-dimensional impulse vector of mass-bearing particle

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} \,, \tag{5.127}$$

i. e. of a particle with non-zero rest mass are [15, 16]

$$\frac{P_0}{\sqrt{g_{00}}} = m \frac{dt}{d\tau} = \pm m \,, \qquad P^i = \frac{m}{c} \mathrm{v}^i, \tag{5.128}$$

where "plus" stands for direct flow of coordinate time, while "minus" stands for reverse flow of coordinate time in respect to observer's own time. Square of four-dimensional impulse of mass-bearing particle is

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = m_0^2 \,, \tag{5.129}$$

while its length is

$$\left| \sqrt{P_\alpha P^\alpha} \right| = m_0 \,. \tag{5.130}$$

Therefore any particle with non-zero rest-mass, being a space-time (four-dimensional) structure, is projected onto time as dipole, which consists of positive mass $+m$ and negative mass $-m$. But in projection of P^α onto three-dimensional space both projections merge into a single one — three-dimensional observable impulse $p^i = m\mathrm{v}^i$. In other words, each observable particle with positive relativistic mass has its own *mirror twin* with negative mass: particle and its mirror twin are only different by the sign of mass, while three-dimensional observable impulses of both particles are positive.

Similarly, for four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma} = k \frac{dx^\alpha}{d\sigma} \,, \tag{5.131}$$

that describes massless (light-like) particle, physical observable projections are [15, 16]

$$\frac{K_0}{\sqrt{g_{00}}} = \pm k \,, \qquad K^i = \frac{k}{c} c^i. \tag{5.132}$$

This implies that any massless particle, as a four-dimensional object, also exists in two states: in our world with direct flow of time it is a massless particle with positive frequency, while in the world with reverse flow of time it is a massless particle with negative frequency.

We define *material Universe* as four-dimensional space-time, filled with substance and fields. Then because any particle is a space-time dipole object, we can say that the material Universe as a combination of basic space-time and particles is also a four-dimensional dipole object, which exists in two states: as *our Universe*, where masses of particles and the temporal coordinate are positive, and as its *mirror twin* (*the mirror Universe*), where masses of particles and the temporal coordinate are negative, while three-dimensional observable impulse is positive. On the other hand, our Universe and the mirror world have the same background four-dimensional space-time.

For instance, analyzing properties of the Universe as a whole, we neglect action of Newtonian fields of occasional islands of substance and hence assume the space of our Universe to be a de Sitter space with negative four-dimensional curvature (three-dimensional observable curvature is positive, see Section 5 in this Chapter). Hence we can assume that our Universe as a whole is an area in de Sitter space with negative four-dimensional curvature, where the temporal coordinate and masses of particles are positive, and vice versa, the mirror Universe is an area of the same de Sitter space, where the temporal coordinate and masses of particles are negative.

The membrane that separates our Universe and the mirror Universe in the basic space-time and does not allow them to "mix", thus preventing total annihilation, will be discussed at the end of this Section.

Now we are turning to dipole structure of the Universe for $d\tau=0$, i. e. for collapsed areas of regular space-time (collapsers) and for degenerated space-time (zero-space).

As we have shown, condition $d\tau=0$ is true in a regular (non-degenerated) space-time when collapse occurs and the space is holonomic (does not rotate). Then

$$d\tau = \left(1 - \frac{w}{c^2}\right) dt = 0\,. \tag{5.133}$$

This condition is true for collapse of any type, i. e. for any type of gravitational potential w, including non-Newtonian potential. At $d\tau=0$ (5.133) four-dimensional metric becomes

$$ds^2 = -d\sigma^2 = -h_{ik}dx^i dx^k = g_{ik}dx^i dx^k = g_{ik}u^i u^k dt^2, \tag{5.134}$$

hence in this case absolute value of the interval ds equals

$$|ds| = id\sigma = i\sqrt{h_{ik}u^i u^k}\, dt = iudt\,, \qquad u^2 = h_{ik}u^i u^k, \tag{5.135}$$

and four-dimensional impulse vector on the surface of collapser is

$$P^\alpha = m_0 \frac{dx^\alpha}{d\sigma}\,, \qquad d\sigma = udt\,. \tag{5.136}$$

Its square is

$$P_\alpha P^\alpha = g_{\alpha\beta}P^\alpha P^\beta = -m_0^2\,, \tag{5.137}$$

hence length of vector P^α (5.136) is imaginary

$$\left|\sqrt{P_\alpha P^\alpha}\right| = im_0\,. \tag{5.138}$$

The latter, in particular, implies that surface of collapser is inhabited by particles with imaginary masses. But, at the same time, this does not imply that super-light-speed particles (tachyons) should be found there, because their masses are imaginary too. On surface of collapser the term "observable velocity" is void, because observable time stops there ($d\tau=0$).

Components of four-dimensional impulse vector of particles found on surface of collapser (5.136), can be formally presented as

$$P^0 = \frac{m_0 c}{u}\,, \qquad P^i = \frac{m_0}{u}u^i. \tag{5.139}$$

But as a matter of fact we can not observe them because at the surface of collapser own time of a regular (real) observer stops. On the other hand, velocity $u^i = \frac{dx^i}{dt}$, found in this formula, is coordinate one and does not depend from observer's own time. Hence we can interpret spatial vector $P^i = \frac{m_0}{u} u^i$ as coordinate impulse of particle and $\frac{m_0 c^3}{u}$ as its energy on the surface of collapser. Here energy of particle has only one sign and thus the surface of collapser as four-dimensional area of space-time (material Universe) is not a dipole four-dimensional object that exists as two mirror twins; surface of collapser (irrespective of its nature) exists in a single state.

On the other hand, the surface of collapser $g_{00}=0$ can be regarded as a membrane that separates four-dimensional areas of space-time before the collapse and after the collapse. Before collapse $g_{00}>0$ and observer's own time τ is real. After collapse $g_{00}<0$ and thus τ becomes imaginary. When observer crosses the surface of collapser his own time subjects to 90° "rotation", swapping roles with spatial coordinates.

The term "light-like particle" has no sense at the surface of collapser, as for light-like particles $d\sigma = c d\tau$ and on the surface ($d\tau=0$) for them

$$u = \sqrt{h_{ik} u^i u^k} = \sqrt{\frac{h_{ik} dx^i dx^k}{dt^2}} = \frac{d\sigma}{dt} = \frac{c d\tau}{dt} = 0 \,. \tag{5.140}$$

Observer's own time also stops ($d\tau=0$) in a fully degenerated space-time (zero-space): there, by definition, $d\tau=0$ and $d\sigma=0$ [15, 16]. These conditions (conditions of degeneration) can be presented as

$$w + v_i u^i = c^2, \qquad g_{ik} u^i u^k = c^2 \left(1 - \frac{w}{c^2}\right)^2 . \tag{5.141}$$

Particles found in degenerated space-time (zero-particles) bear zero regular relativistic mass $m=0$, but non-zero mass M (1.71) and non-zero constant-sign impulse

$$M = \frac{m}{1 - \frac{1}{c^2}\left(w + v_i u^i\right)} \,, \qquad p^i = M u^i . \tag{5.142}$$

Therefore, mirror twins are only found in regular matter — massless and mass-bearing particles not in state of collapse. Collapsed objects in regular space-time (collapsers, including black holes), which do not possess the property of mirror dipoles, are common objects for our Universe and the mirror Universe. Zero-space objects, which neither possess the property of mirror dipoles, lay beyond the basic space-time due to full degeneration of their metric. It is possible to enter "neutral zones" on surfaces of collapsed objects of regular space and in zero-space from either our Universe (where coordinate time is positive) or the mirror Universe (where coordinate time is negative).

Now we need to discuss the question of the membrane that separates our world and the mirror Universe in the basic space-time thus preventing total annihilation of all particles with negative and positive masses.

In our world $dt>0$, in the mirror Universe $dt<0$. Hence the membrane is an area of space-time where $dt=0$ (coordinate time stops); i. e. it is an area where

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i \mathrm{v}^i \pm 1\right) = 0 \,, \tag{5.143}$$

which can be also presented as the physical condition

$$dt = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i dx^i \pm d\tau\right) = 0 \,. \tag{5.144}$$

The latter notation is more versatile, because of being applicable not only in General Relativity space, but also in generalized space-time that permits degeneration of metric.

Physical conditions inside the membrane $t{=}const$ (i. e. $dt{=}0$) according to (5.144) are defined by the formula

$$v_i dx^i \pm c^2 d\tau = 0\,, \tag{5.145}$$

which can be also presented as

$$v_i \mathrm{v}^i = \pm c^2. \tag{5.146}$$

This condition is scalar product of velocity of space rotation and observable velocity of the space of reference (body of reference) of the observer. It can be presented as

$$v_i \mathrm{v}^i = |v_i|\left|\mathrm{v}^i\right| \cos\left(\widehat{v_i;\mathrm{v}^i}\right) = \pm c^2. \tag{5.147}$$

From here we see that it is true when absolute values of velocities v_i and v^i equal to that of light and are either co-directed ("plus") or oppositely directed ("minus").

Thus the membrane from physical viewpoint is a space which experiences translational motion at light speed and at the same time rotates also at light speed, i. e. travels along right or left-hand light-like spiral. In the world of elementary particles such space may be attributed to particles that possess property of spirality (e. g. photons).

Having $dt{=}0$ substituted into the formula for ds^2 we obtain metric inside the membrane

$$ds^2 = g_{ik}dx^i dx^k, \tag{5.148}$$

which is the same as on the surface of collapser. Because it is a specific case of space-time metric with signature $(+---)$, then ds^2 is always positive. This implies that in the area of space-time, which serves the membrane between our world and the mirror world, four-dimensional interval is space-like. The difference from space-like metric on the surface of collapser (5.134) is that during collapse rotation of space is absent ($g_{ik}{=}{-}h_{ik}$), while in this case $g_{ik}{=}{-}h_{ik}{+}\frac{1}{c^2}v_i v_k$ (1.18). Or

$$ds^2 = g_{ik}dx^i dx^k = -h_{ik}dx^i dx^k + \frac{1}{c^2}v_i v_k dx^i dx^k, \tag{5.149}$$

i. e. four-dimensional metric in the membrane becomes space-like due to rotation of space which makes the condition $v_i dx^i{=}{\pm}c^2 d\tau$ true.

As a result regular mass-bearing particle (irrespective of the sign of its mass) can not in its "natural" form pass through the membrane: this area of space-time is inhabited by light-like particles that move along right or left-handed light-like spirals.

On the other hand the ultimate case of particles with $m{>}0$ or $m{<}0$ are particles with relativistic mass $m{=}0$. From geometric viewpoint the area where such particles are found is tangential to areas inhabited by particles with either $m{>}0$ or $m{<}0$. This implies that zero-mass particles may have exchange interactions with either our-world particles ($m{>}0$) or mirror-world particles ($m{<}0$).

Particles with zero relativistic mass, by definition, exist in an area of space-time where $ds^2{=}0$ and $c^2d\tau^2{=}d\sigma^2{=}0$. Equalling ds^2 to zero inside the membrane (5.148) we obtain

$$ds^2 = g_{ik}dx^i dx^k = 0\,. \tag{5.150}$$

Analysis shows that this condition may be true in two cases: (1) when all $dx^i{=}0$; (2) three-dimensional metric is degenerated $\tilde{g}{=}\,\det||g_{ik}||{=}0$.

The first case may occur in regular space-time at the ultimate conditions on surface of collapser: when all the surface shrinks into a point, all $dx^i{=}0$ and the metric on the surface according to $ds^2{=}{-}h_{ik}dx^i dx^k{=}g_{ik}dx^i dx^k$ (5.134) becomes zero.

The second case occurs on surface of collapser in zero-space: because in zero-space we have the condition $g_{ik}dx^i dx^k{=}\left(1{-}\frac{w}{c^2}\right)^2 c^2 dt^2$, then at $w{=}c^2$ then $g_{ik}dx^i dx^k{=}0$ always.

The first case is asymptotic because never occurs in reality. Hence we can expect that "middlemen" in exchanges between our world and the mirror Universe are particles with zero relativistic mass (zero-particles) on surfaces of collapsers in degenerated space-time (zero-space).

5.9 Conclusions

We have shown that our Universe is observable area of basic space-time where temporal coordinate is positive and all particles bear positive masses (energies). The mirror Universe is an area of the basic space-time, where from viewpoint of regular observer temporal coordinate is negative and all particles bear negative masses. Also, from viewpoint of our-world observer the mirror Universe is a world with reverse flow of time, where particles travel from future into past in respect to us.

The two worlds are separated with the membrane — an area of space-time inhabited by light-like particles that travel along light-like right or left-handed (isotropic) spirals. On the scales of elementary particles such space can be attributed to particles that possess spirality (e. g. photons). The membrane prevents mixing of positive and negative-mass particles and thus their total annihilation. Exchange interactions between the two worlds can be effected through particles with zero relativistic masses (zero-particles) under physical conditions that exist on surfaces of collapsers in degenerated space-time (zero-space).

Chapter 6

Annihilation and the mirror Universe

6.1 Isotope anomaly and λ_{T}-anomaly of orthopositronium. The history and problem statement

Recently our colleague B. M. Levin in connection with his experimental studies of anomalies of orthopositronium decay suggested that the results of precision measurements of these anomalies (Ann Arbor, Michigan, USA, 1982–1990 and Moscow–Gatchina, Russia, 1984–1987) may be explained by exchange interactions between our world and the mirror Universe [44], which presumably occurs during the experiments.

In a nutshell, the problem that has been a subject of discussions for over a decade is as follows.

Positronium is an atom-like orbital system that includes electron and its anti-particle, positron, coupled by electrostatic forces. There are two kinds of positronium: parapositronium $^{\mathrm{S}}\mathrm{Ps}$, in which spins of electron and positron are oppositely directed and the summary spin is zero, and orthopositronium $^{\mathrm{T}}\mathrm{Ps}$, in which the spins are co-directed and the summary spin is one. Because a particle-antiparticle system is unstable, life span of positronium is rather small. In vacuum parapositronium decays in $\tau \simeq 1.25{\cdot}10^{-10}$ s, while orthopositronium is $\tau \simeq 1.4{\cdot}10^{-7}$ s after birth. In a medium the life span is even shorter because positronium tends to annihilate with electrons of the media. Due to law of conservation of charge parity parapositronium decays into even number of γ-quanta $(2, 4, 6, \dots)$ while orthopositronium annihilates into odd number of γ-quanta $(3, 5, 7, \dots)$. The older modes of annihilation are less probable and their contributions are very small. For instance, the rate of five-photons annihilation of $^{\mathrm{T}}\mathrm{Ps}$ compared to that of three-photons annihilation is as small as $\lambda_5 \approx 10^{-6}\lambda_3$. Hence parapositronium actually decays into two γ-quanta $^{\mathrm{S}}\mathrm{Ps}\rightarrow 2\gamma$, while orthopositronium decays into three γ-quanta $^{\mathrm{T}}\mathrm{Ps}\rightarrow 3\gamma$.

In laboratory environment positronium can be obtained by placing a source of free positrons into a matter, for instance, one-atom gas. The source of positrons is β^{+}-decay, self-triggered decays of protons in neutron-deficient atoms[38]

$$\mathrm{p} \rightarrow \mathrm{n} + \mathrm{e}^{+} + \nu_{\mathrm{e}}\,. \tag{6.1}$$

Some of free positrons released from β^{+}-decay source into gas quite soon annihilate with free electrons and electrons in the container's walls. Other positrons capture electrons from gas atoms thus producing orthopositronium and parapositronium (in 3:1 statistical ratio).

Temporal spectrum of positrons (number of positrons vs. life span) is the basic characteristics of their annihilation in matter. In particular, in such spectrum one can see parts corresponding to annihilation with free electrons and annihilation of $^{\mathrm{S}}\mathrm{Ps}$ and $^{\mathrm{T}}\mathrm{Ps}$.

In inert gases temporal spectrum of annihilation of free positrons generally reminds of exponential curve with a plateau in its central part, known as "shoulder" [41, 42].

In 1965 P. E. Osmon published [41] pictures of observed temporal spectra of annihilation of positrons in inert gases (He, Ne, Ar, Kr, Xe). In his experiments he used $^{22}\mathrm{NaCl}$ as a source of β^{+}-decay positrons. Analyzing the results of the experiments, Levin noted that the spectrum in neon

[38] It is also known as positron β-decay. During β^{-}-decay in nucleus neutron decays $\mathrm{n}\rightarrow\mathrm{p}+\mathrm{e}^{-}+\widetilde{\nu}_{\mathrm{e}}$.

was peculiar compared to those in other one-atom gases: in neon points in the curve were so widely scattered, that presence of a "shoulder" was unsure. Repeated measurements of temporal spectra of annihilation of positrons in He, Ne, and Ar, later accomplished by Levin [43, 44], have proven existence of anomaly in neon. Specific feature of the experiments done by Osmon, Levin and some other researchers in the UK, Canada, and Japan is that the source of positrons was ^{22}Na, while the moment of birth of positron was registered according to $\gamma_{\rm n}$-quantum of decay of excited ^{22*}Ne

$$^{22*}\mathrm{Ne} \rightarrow {}^{22}\mathrm{Ne} + \gamma_{\rm n}\,, \tag{6.2}$$

from one of products of β^+-decay of ^{22}Na. This method is quite justified and is commonly used, because life span of excited ^{22*}Ne is as small as $\tau \simeq 4{\cdot}10^{-12}$ s, which is a few orders of magnitude less than those of positron and parapositronium.

In his further experiments [45, 46] Levin discovered that the peculiarity of annihilation spectrum in neon (abnormally wide scattered points) is linked to presence in natural neon of substantial quantity of its isotope ^{22}Ne (around 9%). Levin named this effect *isotope anomaly*. Temporal spectra were measured in neon environments of two isotopic compositions: (1) natural neon (90.88% of ^{20}Ne, 0.26% of ^{21}Ne, and 8.86% of ^{22}Ne); (2) neon with reduced content of ^{22}Ne (94.83% of ^{20}Ne, 0.22% of ^{21}Ne, and 4.91% of ^{22}Ne). Comparison of temporal spectra of positron decay revealed: in natural neon (composition 1) the shoulder is fuzzy, while in neon poor with ^{22}Ne (composition 2) the shoulder is always clearly pronounced. In the part of spectrum, to which $^{\rm T}$Ps-decay mostly contributes, the ratio between intensity of decay in poor neon and that in natural neon (with much isotope ^{22}Ne) is 1.85±0.1 [46].

The relationship between anomaly of positron annihilation in neon and presence of ^{22}Ne admixture, as shown in [45, 46], hints on existence in gas neon of *collective nuclear excitation* of ^{22}Ne isotopes. In the terminal stage of β^+-decay nuclear excitation of ^{22*}Ne (life expectancy $\tau \simeq 4{\cdot}10^{-12}$ s) is somehow passed to a set of ^{22}Ne nuclea around the source of positrons and is taken away by nuclear $\gamma_{\rm n}$-quantum after a long delay at the moment of self-annihilation of orthopositronium (free positrons and parapositronium live much longer).

Hence collective excitation of ^{22}Ne atoms seems to be the reason of isotope anomaly (phenomenology). On the other hand it is still unclear what is the material carrier that passes excitation of nuclear ^{22*}Ne to the surrounding ^{22}Ne atoms and what links orthopositronium with this collective excitation: by far collective nuclear excitation is only known in crystals (Mössbauer effect, 1958).

In 1990 Levin [47] suggested that as a result of relationship between orthopositronium and collective nuclear excitation, 1-photon mode of its annihilation should be observed. But decay of $^{\rm T}$Ps into one γ-quantum would break laws of conservation of Quantum Electrodynamics (QED). To justify this phenomenological conclusion without breaking QED laws, Levin in his generalization study [48] suggested, that in the specific experimental environment annihilation of some orthopositronium atoms releases one γ-quantum into our world and two γ-quanta into the mirror Universe, which makes them unavailable for observation. However before any experiments are accomplished to prove or disprove existence of such "1-photon" mode or any theory is developed to explain the observed effect, the problem still welcomes discussion.

Another anomaly is substantially higher measured rate of annihilation of orthopositronium (the value reciprocal to its life span) compared to that predicted by QED.

Measurement of orthopositronium annihilation rate is among the main tests aimed to experimental verification of QED laws of conservation. Before the middle 1980's no difference between theory and practice was observed, as the measurement precision stayed at the same low level.

In 1987 thanks to new precision technology a group of researchers based in the University of Michigan (Ann Arbor) made a breakthrough in this area. The obtained results showed substantial gap between experiment and theory. The anomaly that the Michigan group revealed was that measured rates of annihilation at $\lambda_{\rm T(exp)} = 7.0514 \pm 0.0014\,\mu\mathrm{s}^{-1}$ and $\lambda_{\rm T(exp)} = 7.0482 \pm 0.0016\,\mu\mathrm{s}^{-1}$ (with unseen-before precision of 0.02% and 0.023% using vacuum and gas methods [49, 50, 51, 52]) were much higher compared to $\lambda_{\rm T(theor)} = 7.00383 \pm 0.00005\,\mu\mathrm{s}^{-1}$ as predicted by QED [53, 54, 55, 56]. As a result the measured anomalous effect was 0.2% from theoretically predicted value, i. e. the effect was 10 times higher than the measurement precision 0.02%! The effect was later called $\lambda_{\rm T}$-anomaly [48].

Theorists foresaw possible annihilation rate anomaly not long before the first experiments were accomplished in Michigan. In 1986 Robert Holdom [57] suggested that "mixed type" particles may exist, which being in the state of oscillation stay for some time in our world and for some time in the mirror Universe, possessing negative masses and energies. In the same year S. Glashow [58] gave further development to the idea and showed that in case of 3-photon annihilation ${}^{\mathrm{T}}\mathrm{Ps}$ will "mix up" with its mirror twin thus producing two effects: (1) higher annihilation rate due to additional mode of decay ${}^{\mathrm{T}}\mathrm{Ps}\to$nothing, because products of decay passed into the mirror Universe can not be detected; (2) the ratio between orthopositronium and parapositronium numbers will decrease from ${}^{\mathrm{T}}\mathrm{Ps} : {}^{\mathrm{S}}\mathrm{Ps}=3:1$ to 1.5:1. But because at that time (in 1986) no such effects were reported, Glashow concluded that no interaction is possible between our-world and mirror-world particles.

On the other hand, by the early 1990's these theoretic studies encouraged many researchers worldwide for experimental search of various "exotic" (i.e. not explained in QED) modes of ${}^{\mathrm{T}}\mathrm{Ps}$ decay, which could lit some light on abnormally high rate of decay. These were, to name just a few, search for ${}^{\mathrm{T}}\mathrm{Ps}\to$nothing mode [55], check of possible contribution from 2-photon mode [60, 61, 62] or from other exotic modes [63, 64, 65]. As a result it has been shown that no exotic modes can contribute to the anomaly, while contribution of ${}^{\mathrm{T}}\mathrm{Ps}\to$nothing mode is limited to

$$\left({}^{\mathrm{T}}\mathrm{Ps}\to\mathrm{nothing}\right) < 5.8\cdot 10^{-4}\left({}^{\mathrm{T}}\mathrm{Ps}\to 3\gamma\right). \tag{6.3}$$

In a generalization study in 1995 Levin pointed out [48] that the program of critical experiments was limited to search of 1-photon mode ${}^{\mathrm{T}}\mathrm{Ps}\to\gamma\backslash 2\gamma'$ involving the mirror Universe and to search of the mode ${}^{\mathrm{T}}\mathrm{Ps}\to$nothing. The situation has not changed significantly over the past five years. The most recent (as of time of writing of this book) publication on this subject in May 2000 [66] still focused on Holdom-Glashow suggestion of possible explanation of λ_{T}-anomaly by interaction of orthopositronium with its mirror-world twin, as well as on search of ${}^{\mathrm{T}}\mathrm{Ps}\to$nothing mode. But no theory has been yet suggested to prove possibility of such interaction and to describe its mechanism.

The absence of theoretical explanation of λ_{T}-anomaly encouraged G. S. Adkins *et al.* [67] to suggest experiments made in Japan [68] in 1995 as an alternative to the basic Michigan experiments. No doubt, high statistical accuracy of Japanese measurements [68] puts them on the same level with the basic experiments [49, 50, 51, 52]. But all Michigan measurements possessed the property of a "full experiment", which in this particular case means no external influence could affect wave function of positronium. Such influence is inevitable due to electrodynamic nature of positronium and can be avoided only using special technique. In Japanese measurements [68] this was not taken into account and thus they do not possess property of "full experiment".

As early as in 1993 S. G. Karshenboim [69] showed that QED had actually run out of any of its theoretical capabilities to explain orthopositronium anomaly. Given that we assume our goal as to study the annihilation anomaly in another domain, using methods of General Relativity. To do this, we are going first to study process of annihilation in general with methods of chronometric invariants and then to apply the results to parapositronium and orthopositronium. As a result we should be able to answer the question of what channels of annihilation of orthopositronium are physical observable in our world and whether "drain" of some energy into the mirror Universe is possible.

6.2 Zero-space as home space for virtual particles. Interpretation of Feynman diagrams in General Relativity

Feynman diagrams are graphical description of interactions between elementary particles. The diagrams clearly show that the actual carriers of interactions are virtual particles. In other words, almost all physical processes rely upon emission and absorption of virtual particles [70].

Another notable property of Feynman diagrams is that they are capable of describing particles (e.g. electrons) and antiparticles (e.g. positrons) at the same time. In this example positron is represented as electron that moves back in time [70].

According to QED, interaction of particles at branching points of Feynman diagram conserves four-dimensional impulse. This suggests interpretation of Feynman diagrams in General Relativity.

As a matter of fact, in four-dimensional pseudo-Riemannian space, which is the basic space-time of General Relativity, the following objects can get correct formal definitions:

1. free particle as particle that moves along geodesic trajectory;
2. non-free (dependent) particle as particle that under action of external non-gravitational fields moves along non-geodesic trajectory;
3. antiparticle (either free or dependent) as particle that travels back in time in respect to regular observer [15, 16].

Hence to translate Feynman diagrams into "geometrese" we only need formal definition of virtual particles in General Relativity.

In QED virtual particles are particles for which contrary to regular ones the relationship between energy and impulse is not true

$$E^2 - c^2p^2 = E_0^2\,, \tag{6.4}$$

where $E{=}mc^2$, $p^2{=}m^2\mathrm{v}^2$, $E_0{=}m_0c^2$. In other words, for virtual particles $E^2{-}c^2p^2{\neq}E_0^2$.

In pseudo-Riemannian space this relationship in chronometrically invariant form is similar [15, 16] but $p^2{=}h_{ik}p^ip^k$, where $p^i{=}m\mathrm{v}^i$ stands for (physical observable) vector of particle's impulse.

Dividing both parts of the equation by c^4, we obtain

$$m^2 - \frac{p^2}{c^2} = m_0^2\,, \tag{6.5}$$

which is chronometrically invariant notation of the requirement of constancy of four-dimensional impulse vector of mass-bearing real particle

$$P_\alpha P^\alpha = g_{\alpha\beta}P^\alpha P^\beta = m_0^2 g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = m_0^2 \tag{6.6}$$

in parallel transfer along trajectory, where $ds^2{>}0$, i. e. along sub-light-speed trajectory. For super-light-speed particles (tachyon), which four-dimensional impulse vector is

$$P^\alpha = m_0\frac{dx^\alpha}{|ds|}\,, \tag{6.7}$$

the relationship between mass and impulse (6.5) becomes

$$\frac{p^2}{c^2} - m^2 = (im_0)^2\,, \tag{6.8}$$

therefore rest-mass of tachyons is imaginary. For photons, i. e. particles that move along isotropic (light-like) trajectories, rest-mass is zero and the relationship between mass and impulse transforms as

$$m^2 = \frac{p^2}{c^2}\,, \tag{6.9}$$

where relativistic mass m is defined from energy equivalent $E{=}mc^2$, while observable impulse $p^i{=}mc^i$ is expressed through chronometrically invariant vector of light velocity.

So equations (6.5, 6.8, and 6.9) characterize relationship between mass and impulse for regular particles that inhabit pseudo-Riemannian space. Interactions between them are carried by virtual particles. Given that, to geometrically interpret Feynman diagrams we need geometric description of virtual particles.

By definition the relationship between mass and impulse (6.5) is not true for virtual particles. From geometric viewpoint that implies that the square of four-dimensional impulse of virtual particles does not conserve in parallel transfer. In Riemannian space, in particular in four-dimensional pseudo-Riemannian space (the basic space of General Relativity) the square of vector conserves in parallel transfer by definition. That implies that trajectories of virtual particles lay in a space with non-Riemannian geometry, i. e. outside four-dimensional pseudo-Riemannian space — the basic space-time of General Relativity.

In our studies [15, 16], not related to virtual particles, we showed that trajectories along which the square of vector being transferred does not conserve lay in fully degenerated space-time ($g = \det||g_{\alpha\beta}||=0$), also known as zero-space. In pseudo-Riemannian space $g<0$ is always true by definition. Hence zero-space lays beyond four-dimensional pseudo-Riemannian space and its geometry is not Riemannian. Besides, as was shown, relativistic mass of particles that zero-space hosts (zero-particles) is zero and from viewpoint of our-world observer their motion is perceived as instant displacement (long-range action).

Analysis of the above facts brings us to the conclusion that zero-particles can be equaled to virtual particles in *generalized space-time* ($g\leq 0$), which we also considered in our previous studies. This space permits degeneration of metric and considering not only motion of regular massless or mass-bearing particles, but also their interaction by means of exchange with virtual particles (zero-particles) in zero-space. In fact, this is the way of geometric interpretation of Feynman diagrams in General Relativity.

We are going to show why the square of a vector being transferred does not conserve. Zero-space is defined by the following conditions

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0\,, \qquad c^2 d\tau^2 = d\sigma^2 = 0\,, \tag{6.10}$$

i. e. physical observable time and physical observable three-dimensional interval are degenerated

$$d\tau = \left(1 - \frac{w}{c^2}\right) dt - \frac{1}{c^2} v_i dx^i = 0\,, \qquad d\sigma^2 = h_{ik} dx^i dx^k = 0\,. \tag{6.11}$$

Substituting into the second condition the formula for physical observable metric tensor

$$h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k\,, \tag{6.12}$$

dividing the equation by dt^2 and substituting $v_i u^i = c^2 - w$ from the first condition, we arrive to inner coordinate metric of zero-space

$$d\mu^2 = g_{ik} dx^i dx^k = \left(1 - \frac{w}{c^2}\right)^2 c^2 dt^2 \neq inv, \tag{6.13}$$

which is not physical observable value and not zero. Hence coordinate metric of zero-space can be deduced from the condition of degeneration of three-dimensional observable metric and is not invariant by definition.

In particular, because metric of zero-space $d\mu^2$ is not invariant, the square of four-dimensional vector in zero-space does not conserve

$$U^\alpha = \frac{dx^\alpha}{dt}\,, \qquad U_\alpha U^\alpha = g_{ik} u^i u^k = \left(1 - \frac{w}{c^2}\right)^2 c^2 \neq const. \tag{6.14}$$

But applying theory of observable values to this situation again shows us the way out. Because within that theory we consider all values from viewpoint of a regular observer in pseudo-Riemannian space, then all values, including those in zero-space, can be expressed through parameters of his space of reference. Therefore zero-particles from viewpoint of a regular observer possess four-dimensional impulse (1.72) which square is zero and conserves

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{M}{c} \frac{dx^\alpha}{dt}\,, \qquad P_\alpha P^\alpha = \frac{M^2}{c^2} \frac{ds^2}{dt^2} = 0\,, \tag{6.15}$$

because in zero-space, by definition, $ds^2 = 0$. But once we turn to the frame of reference of a hypothetical observer in zero-space, i. e. to the space with metric $d\mu^2$ (6.13), the square of transferred vector does not conserve any longer.

Now we are going to see what kinds of particles zero-space hosts. First we look at degeneration conditions (1.69, 1.70) in absence of gravitational potential ($w=0$). These are

$$v_i u^i = c^2, \qquad g_{ik} u^i u^k = c^2, \tag{6.16}$$

i. e. in absence of gravitational potential zero-particles travel at coordinate velocities, that equal to light speed

$$u = \sqrt{g_{ik}u^i u^k} = c\,. \tag{6.17}$$

The first condition of degeneration is scalar product of the velocity of space's rotation and three-dimensional coordinate velocity of particle

$$v_i u^i = vu\cos\left(\widehat{v_i; u^i}\right) = c^2. \tag{6.18}$$

Because $u{=}c$, this condition is true if vectors v_i and u^i are co-directed (or coincide like in this case). Hence in absence of gravitational potential zero-particles move at forward velocity equal to speed of light at the same time rotating at light speed as well. We will refer to such particles as *virtual photons*. Zero-space metric along their trajectories is

$$d\mu^2 = g_{ik}dx^i dx^k = c^2dt^2 \neq 0\,, \tag{6.19}$$

similar to metric $d\sigma^2{=}c^2d\tau^2{\neq}0$ along trajectories of regular photons in pseudo-Riemannian space.

Now we will look at degeneration conditions (1.69, 1.70) when gravitational potential $w{\neq}0$. Here

$$v_i u^i = c^2 - w\,, \qquad u^2 = g_{ik}dx^i dx^k = \left(1-\frac{w}{c^2}\right)^2 c^2. \tag{6.20}$$

Then scalar product $v_i u^i{=}c^2{-}w$ can be represented as

$$v_i u^i = vu\cos\left(\widehat{v_i; u^i}\right) = vc\left(1-\frac{w}{c^2}\right)\cos\left(\widehat{v_i; u^i}\right) = \left(1-\frac{w}{c^2}\right)c^2. \tag{6.21}$$

This equation is true given that vectors v_i and u^i are co-directed and $v{=}c$, i. e. when particle travels in zero-space at velocity which magnitude equals to

$$v = c\left(1-\frac{w}{c^2}\right), \tag{6.22}$$

and rotates along with space at speed of light $v{=}c$.

But when we turn to metric along zero-trajectories in presence of gravitational potential

$$d\mu^2 = g_{ik}dx^i dx^k = \left(1-\frac{w}{c^2}\right)^2 c^2dt^2, \tag{6.23}$$

we see that the "temporal" parameter here is the following variable ("gravitational" time)

$$t_* = \left(1-\frac{w}{c^2}\right)t\,, \tag{6.24}$$

i. e. coordinate velocity of zero-particles along such trajectories depends upon gravitational potential

$$u^i_* = \frac{dx^i}{dt_*} = \frac{u^i}{1-\dfrac{w}{c^2}}\,. \tag{6.25}$$

Because of the second degeneration condition (6.20) the square of coordinate velocity of these zero-particles equals to square of light speed

$$u^2_* = g_{ik}u^i_* u^k_* = \frac{g_{ik}dx^i dx^k}{\left(1-\dfrac{w}{c^2}\right)^2} = c^2, \tag{6.26}$$

i. e. they are *virtual photons* as well. Basing on the first degeneration condition we can as well show that in presence of gravitational potential they also rotate at speed of light

$$v_i u^i_* = c^2. \tag{6.27}$$

Noteworthy, considering virtual mass-bearing particles is senseless, because all particles in zero-space by definition possess zero rest-mass and therefore are not mass-bearing particles. Therefore only virtual photons and their varieties are virtual particles.

Now we are going to define virtual particles in the state of collapse, i. e. when $w=c^2$. We will refer to them as *virtual collapsers*. For them the degeneration conditions (1.69, 1.70) become

$$v_i u^i = 0\,, \qquad g_{ik}dx^i dx^k = 0\,, \tag{6.28}$$

i. e. zero-collapsers either rest in respect to the space of reference or for an observer in zero-space the world around him compacts into a point (all $dx^i=0$), or three-dimensional metric is degenerated $\tilde{g} = \det\|g_{ik}\|=0$.

Metric of zero-space along trajectories of virtual collapsers is

$$d\mu^2 = g_{ik}dx^i dx^k = 0\,. \tag{6.29}$$

Therefore two kinds of virtual particles can exist in zero-space, which is four-dimensional degenerated space-time:

1. virtual photons with forward motion and rotation at light speed;
2. virtual collapsers that rest in respect to the space.

We can assume that all interactions between regular mass-bearing and massless particles in four-dimensional pseudo-Riemannian space, i. e. in the basic space-time of General Relativity, are effected through *an exchange buffer*, in which capacity zero-space acts. Material carriers of interactions within such buffer are virtual particles of the two aforementioned kinds.

In our previous studies [15, 16] from considering motion of particles in the frames of particle-wave concept we obtained that eikonal equation (wave phase equation) for particles in zero-space is standing wave equation (1.77). Hence virtual particles are actually *standing waves* and hence interaction between regular particles in our space-time is transmitted through a system of standing light-like waves (*a standing-light hologram*), that fills the exchange buffer (zero-space).

6.3 Building mathematical concept of annihilation. Parapositronium and orthopositronium

In this Section we are going to focus on process of annihilation using the same methods which are employed in General Relativity to study motion of particles.

From geometric viewpoint positronium is a system of two charged particles with spin, linked together with electromagnetic force. The only difference between parapositronium and orthopositronium is that the summary spin of $^{\mathrm{T}}$Ps is one, while that of $^{\mathrm{S}}$Ps is zero.

As we showed in Chapter 4, charged particle with spin is characterized by four-dimensional summary impulse vector

$$Q^\alpha = P^\alpha + S^\alpha + \frac{e}{c^2}A^\alpha, \qquad P^\alpha = m_0\frac{dx^\alpha}{ds}\,. \tag{6.30}$$

If spin-impulse of particle is directed along its four-dimensional trajectory, i. e. is co-directed with its impulse vector P^α, then four-dimensional spin-impulse of particle is

$$S^\alpha = \frac{1}{c^2}\eta_0\frac{dx^\alpha}{ds}\,, \qquad \eta_0 = n\hbar^{\mu\nu}A_{\mu\nu}\,, \tag{6.31}$$

and hence the summary impulse vector is

$$Q^\alpha = \left(m_0 + \frac{\eta_0}{c^2}\right)U^\alpha + \frac{e}{c^2}A^\alpha. \tag{6.32}$$

As a matter of fact this summary vector characterizes charged elementary particle with spin that bears rest-mass

$$\mu_0 = m_0 + \frac{\eta_0}{c^2}\,, \tag{6.33}$$

where m_0 is rest-mass of particle. The second term stands for additional "spin-mass" which spin particle gains from interaction with external field of non-holonomity of space. In other words, ${}^{\mathrm{T}}$Ps is different from ${}^{\mathrm{S}}$Ps by bearing additional "spin-mass".

Now we will estimate how strong may be effect of "spin-mass" on motion of electron and positron in orthopositronium. Value η_0 can be calculated as

$$\eta_0 = n\hbar^{\mu\nu}A_{\mu\nu} = 2n\hbar\Omega\,, \tag{6.34}$$

where Ω is angular velocity of space rotation, which for positronium can be obtained from the 2nd Bohr postulate $m\Omega r^2{=}k\hbar$.

Substituting the values of electron-positron rest-mass $m_{\mathrm{e}}{=}9.1{\cdot}10^{-28}$ g and of orthopositronium radius $r{=}10.6{\cdot}10^{-9}$ cm, which equals to double radius of Bohr orbit (we re considering the first level $k{=}1$), we obtain

$$\frac{\eta_0}{c^2} = 10^{-32}\,\mathrm{g}\,, \tag{6.35}$$

i. e. "spin-mass" is $10^{-5}\,m_{\mathrm{e}}$ for electron and positron. At higher orbital levels in orthopositronium "spin-mass" will be k times greater.

Evidently, before the moment of annihilation the relative distance between electron and positron

$$\xi = \sqrt{g_{\alpha\beta}\xi^{\alpha}\xi^{\beta}}\,, \qquad \xi^{\alpha} = \frac{\partial x^{\alpha}}{\partial v}dv\,, \tag{6.36}$$

is real. Here ξ^{α} is an infinitesimal vector that connects points of two neighbor trajectories, v is a parameter, constant along each of two trajectories, but different for a neighbor one by dv.

In chronometrically invariant form this distance looks like

$$\xi = \sqrt{\chi^2 - r^2}\,, \qquad r^2 = h_{ik}r^i r^k, \tag{6.37}$$

where values

$$\chi = \frac{\xi_0}{\sqrt{g_{00}}}\,, \qquad r^i = \xi^i, \tag{6.38}$$

are physical observable projections of vector ξ^{α}.

At the moment of annihilation electron and positron (as space-time objects) merge into the same event, i. e. four-dimensional interval ξ between them becomes zero. Evidently, in this case three-dimensional observable distance r between them also becomes zero. And from (6.37) we have $\chi{=}0$. In other words, at the moment of annihilation physical observable time interval χ and spatial distance r between electron and positron are zeroes. This actually implies that the process of annihilation takes place in zero-space.

Hence we will consider annihilation of electron and positron as a process of exchange with virtual photons in zero-space. We can assume that positron emits virtual photons while electron absorbs them (or vice versa). From mathematical viewpoint such description is quite correct provided that generalized four-dimensional space-time, which metric can be fully degenerated $g = \det\|g_{\alpha\beta}\| \leqslant 0$, is considered the basic space-time. Such generalized space-time can be represented as a combination of four-dimensional pseudo-Riemannian space (basic space-time of General Relativity) and zero-space, which metric is degenerated.

For a regular observer the distance between electron and positron in zero-space is nil. But this does not imply they coincide in zero-space itself, because the "externally" observed equation

$$r^2 = h_{ik}r^i r^k = 0 \tag{6.39}$$

for an "internal" observer is

$$g_{ik}r^i r^k = \left(1 - \frac{w}{c^2}\right)c^2 t^2, \tag{6.40}$$

i. e. three-dimensional coordinate interval between electron and positron in zero-space turn into zero only provided that $w{=}c^2$ (collapse).

In electromagnetic interactions, to which annihilation belongs, Newtonian gravitation is infinitesimal. But non-Newtonian gravitational forces may exist as well.

For instance, a space without rotation or deformation, filled with field of gravitation of a spheric island of mass, is characterized by Schwartzshild metric (5.69). In this case vector of gravitational-inertial force F^i takes the form of (5.72) and characterizes Newtonian gravitational force, which is usually proportional to the square of distance from the field source (the mass of the island). In other words, Schwartzshild's metric space is filled with Newtonian gravitational field.

An example of space without rotation or deformation, but filled with spherical symmetric distribution of vacuum without islands of mass, is a space with de Sitter metric (5.73). As was shown in Chapter 5, vector F^i in such space takes the form of (5.74) and characterizes non-Newtonian gravitational force, which is proportional to distance and is conditioned by presence of "cosmological" λ-term in equations of field. If $\lambda<0$, this is an attraction force, otherwise this is repelling force.

For us that suggests that three-dimensional interval between electron and positron in zero-space (6.39) becomes zero only when collapse is effected by action of non-Newtonian gravitational forces (λ-forces produced by vacuum).

So at the moment of annihilation electron and positron exchange with virtual photons in zero-space. In the previous Section we showed that virtual photons, aside for forward motion at the speed of light, feature rotation at the same speed as well. Hence they plot light-like spirals on a cylinder (*cylinder of annihilation events*), that connects electron and positron through zero-space.

Now using our geometric method we can formulate the difference between decay of parapositronium and that of orthopositronium.

${}^{\mathrm{T}}$Ps and ${}^{\mathrm{S}}$Ps themselves differ only by orientation of spins of electron and positron. Spin-impulse of each particle results from interaction of its internal field of non-holonomity with external field of non-holonomity (rotation), which for positron is field of orbital rotation of electron (and vice versa). By definition zero-space is non-holonomic, i. e. is a space of rotation. Hence at the moment of annihilation when physical observable distance r between electron and positron becomes zero (6.39), the summary spin of orthopositronium ${}^{\mathrm{T}}$Ps interacts with field of non-holonomity of zero-space in the cylinder of annihilation events.

Without going into details of mechanism of exchange interactions of virtual particles in zero-space, we can still maintain that additional spin-mass (energy) of orthopositronium after its annihilation creates third virtual photon in zero-space, which in turn creates third annihilation γ-quantum in our world.

Evidently, this third virtual photon is absent in decay of parapositronium, because spins of electron and positron are oppositely directed and hence parapositronium does not possess inner field of non-holonomity, which could interact with field of non-holonomity of zero-space in the cylinder of annihilation events.

6.4 Annihilation of orthopositronium: 2+1 split of 3-photon annihilation

Now we are going to consider decay of orthopositronium in details using geometric methods.

In Section 6.2 we showed that two kinds of virtual particles, which carry interactions between regular particles, including interactions during annihilation, can exist:

1. virtual photons that combine forward and rotation motion at the speed of light;
2. virtual collapsers that rest in respect to the space.

In the previous Chapter we showed that zero-particles, which inhabit the surface of collapsed objects in zero-space, i. e. virtual collapsers, may perform exchange interactions between our Universe and the mirror Universe (where time flows backward in respect to ours). Hence if aside for regular virtual photons virtual collapsers are involved into exchange interaction of particles, then part of the summary energy they carry is emitted into the mirror Universe. In our world this could be perceived as *observable* break of law of conservation of energy-impulse, though in reality full energy and impulse are conserved with the mirror Universe taken into account.

We are going to find out whether virtual collapsers may be involved into exchange interactions between electron and positron in decay of parapositronium or orthopositronium. This, for instance, can be found out by applying the relationship of physical conditions in zero-space to parapositronium and orthopositronium.

The relationship of physical conditions in zero-space in absence of gravitational potential (regular virtual photons) is $v_i u^i = c^2$. Because we are looking at events from viewpoint of an observer in a regular space-time, we can multiply both parts of this relationship by generalized mass of particle

$$\mu_0 = m_0 + \frac{\eta_0}{c^2}\,, \tag{6.41}$$

which accounts for additional energy that particle gains from spin interaction. As a result

$$\mu_0 v_i u^i = E\,, \tag{6.42}$$

where the left part has energy of virtual particles of this interaction, formulated with properties of interacting material particles. The right part has energy of decay products $E = \mu_0 c^2$.

For parapositronium, which spin is zero, this relationships becomes

$$m_0 v_i u^i = m_0 c^2 \qquad m_0 = m_{\mathrm{e}^-} + m_{\mathrm{e}^+}\,, \tag{6.43}$$

which left part (summary energy of virtual particles that carry action)

$$m_0 v_i u^i = m_0 v u \cos\,(\widehat{v_i; u^i}) \tag{6.44}$$

is not zero, because the summary energy of annihilation γ-quanta $E_{2\gamma}$ (the right part) is not zero.

For virtual collapsers, which carry interactions between our world and the mirror Universe, the collapse condition $w = c^2$ is true. In this case the conditions in zero-space $w + v_i u^i = c^2$ become $v_i u^i = 0$. Multiplying both parts by rest mass m_0 of parapositronium, we obtain

$$m_0 v_i u^i = 0\,, \tag{6.45}$$

which *would be true* if annihilation of orthopositronium was effected not through exchange of regular virtual photons, but rather through exchange of virtual collapsers.

This formula (6.45) does not match (6.43) which is actually true for parapositronium. Hence interactions between electron and positron in virtual cylinder of events during annihilation are effected by *regular virtual photons*, not by virtual collapsers. This, in its turn, implies that annihilation of parapositronium does not involve the mirror Universe, because there is only one channel of decay: both photons are emitted into our Universe.

Now we look at decay of orthopositronium. Spin-energy of orthopositronium is not zero, hence (6.42) becomes

$$v_i p^i + \frac{\eta_0}{c^2} v_i u^i = E\,, \tag{6.46}$$

i. e. to energy of two "basic" virtual photons, that carry interactions between electron and positron during annihilation, energy of *spinino* is either added of subtracted; a virtual particle produced by transformation of spin-energy (the second term in the equation) in zero-space. Spinino may be either regular virtual photon or virtual collapser (if spin-energy transforms into virtual particle in a collapsed area of zero-space).

As a result, two space-time relationships may be true for annihilation of orthopositronium

$$1.\;\; v_i p^i + \frac{\eta_0}{c^2} v_i u^i \neq 0\,, \qquad\qquad 2.\;\; v_i p^i + \frac{\eta_0}{c^2} v_i u^i = 0\,. \tag{6.47}$$

In other words, contrary to decay of parapositronium, that of orthopositronium has two possible channels of exchange with virtual particles in zero space.

As was already mentioned, the conditions in zero-space $w + v_i u^i = c^2$ in case of collapse ($w = c^2$) become $v_i u^i = 0$. Multiplying both parts by generalized mass of orthopositronium (6.41), we obtain the relationship for energies of virtual collapsers

$$v_i p^i + \frac{\eta_0}{c^2} v_i u^i = 0\,, \tag{6.48}$$

which would be true, if they carried interactions in virtual cylinder of events (in zero-space) in annihilation of orthopositronium. Comparing it with formula for second possible channel of annihilation (6.47) shows they are the same.

That means virtual collapsers, which link our world with the mirror Universe, *can be involved* into exchange interactions between electron and positron in annihilation of orthopositronium.

As a matter of fact, the relationship (6.48), obtained for orthopositronium, is an equation

$$E_{2\gamma} + E_{\gamma s} = 0\,, \tag{6.49}$$

that suggests: if decay of electron and positron in orthopositronium is effected by exchanging not regular virtual photons but rather virtual collapsers, then energy of two "basic" annihilation photons $E_{2\gamma}$ is negative to energy of third photon $E_{\gamma s}$, produced by virtual spinino. Because virtual collapsers link our world and the mirror Universe, this has two consequences. First, both "basic" virtual photons being in the state of collapse, like any other virtual collapsers in interactions of material particles in our world, cause emission of particles (here two "basic" annihilation photons) into the mirror Universe. Second, third annihilation photon, produced by virtual spinino in the state of collapse, which energy is negative, is emitted into our world. In this case law of conservation of energy-impulse is observed.

We will refer to this phenomenon as *2+1 split of 3-photon annihilation of orthopositronium.*

Hence in decay of orthopositronium by means of virtual collapsers our-world observer instead of regular 3-photon mode will observe 1-photon mode in which two "basic" photons are emitted into the mirror Universe, while third "additional" photon produced by virtual spinino, is emitted into our world thus becoming observable.

Comparing these theoretical statements with experimental data, we can conclude the following.

Because most observable effects of orthopositronium are explained by its 3-photon decay, when all 3 photons are emitted into our world, we can assume that decay of majority (over 99.8%) of atoms of orthopositronium is effected by regular virtual photons.

On the other hand, observed λ_{T}-anomaly and isotope anomaly suggest that for a very small number (less than 0.2%) of atoms of orthopositronium interaction in cylinder of events is carried by virtual collapsers. It is because of exchange with virtual collapsers "anomalous" 1-photon mode becomes possible, with 1 photon emitted into our world and 2 photons are emitted into the mirror Universe, i. e. 2+1 split of 3-photon mode occurs.

6.5 Isotope anomaly of orthopositronium

Phenomenology of isotope anomaly and λ_{T}-anomaly, set forth by Levin [48, 71], relies upon view of positron β-decay as topological quantum transition. As a result of this process at the final stage of β^+-decay that occurs in ^{22}Ne, some space-like resonance structure of limited volume is formed, against which background non-perturbating processes of orthopositronium annihilation occur. "This space-like structure carries long-range action for barion charge, which is concentrated in its nodes, where ^{22}Ne nuclea are initiated in resonance conditions" [71].

But experiments have not answered the question of what is "the building material" of such space-like structure, which is also a material "agent" to carry long-range action for barion charge.

Terminology of General Relativity is built around space-time views of objects and phenomena and is dramatically different from terminology used in phenomenology of orthopositronium annihilation. Therefore to study annihilation of orthopositronium with geometric methods of General Relativity we can not employ terms like "collective nuclear resonance state" or "topological quantum transition".

On the other hand, possible existence of space-like structure, which properties are similar to those of fundamental space-like structure that reveal itself in the terminal stage of β^+-decay, was theoretically justified in our study of motion of test particles in General Relativity (Chapter 1). Two conclusions lay in the cornerstone of this theoretical justification. First, because eikonal equation for zero-particles is standing-wave equation, all zero-space is filled with a system of standing waves (hologram). Second, because for a regular observer the observable three-dimensional interval and the interval of observable time in zero-space are zeroes, we percept motion of zero-particles as instant displacement in space.

In other words, zero-particles are carriers of *long range-action*, i. e. should be perceived by a regular observer as *space-like structures*.

As was shown in Sections 6.2 and 6.3, virtual particles, which are material carriers of interaction between regular particles of our world, can be unambiguously represented as zero-particles, that travel along their degenerated trajectories in zero-space. This helped us to study annihilation of orthopositronium and to show that its 2+1 split is possible, in which exchange with virtual collapsers rather than regular photons results in 1 photon being emitted into our world and 2 photons being emitted into the mirror Universe.

The latter means that virtual collapsers, which carry interaction between electron and positron in annihilation, from viewpoint of a regular observer must be a space-like hologram, linked to existence of orthopositronium and being a material agent to carry interaction between our world and the mirror Universe.

As was shown in Section 1.3 (1.77) for particles in zero-space, i. e. for virtual photons, eikonal equation is standing wave equation

$$h^{ik}\frac{{}^*\partial\psi}{\partial x^i}\frac{{}^*\partial\psi}{\partial x^k}=0\,, \tag{6.50}$$

This implies that for virtual photons

$$\frac{{}^*\partial\psi}{\partial t}=0\,. \tag{6.51}$$

Because chronometrically invariant derivatives of wave phase to spatial three-dimensional coordinates and to time are

$$\frac{{}^*\partial\psi}{\partial x^i}=\frac{\partial\psi}{\partial x^i}+\frac{1}{c^2}v_i\frac{{}^*\partial\psi}{\partial t}\,,\qquad \frac{{}^*\partial\psi}{\partial t}=\frac{1}{1-\dfrac{w}{c^2}}\frac{\partial\psi}{\partial t} \tag{6.52}$$

then eikonal equation for particles in zero-space can be presented as

$$h^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k}=-g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k}=0\,. \tag{6.53}$$

Theoretically it should be true for any type of zero-particles: both for regular virtual photons or virtual photons in collapse (virtual collapsers). But for virtual collapsers, due to collapse condition $w{=}c^2$, chronometrically invariant derivative of wave phase to time is not zero, as is the case for virtual photons in general (6.51), but instead is 0/0 indefiniteness. This puts certain obstacles on the way of particular calculations of characteristics of space-like hologram, which appears in 2+1 split of 3-photon annihilation, because it results from action of virtual collapsers only. Evidently, such calculations rely upon advancement of method of chronometric invariants. In particular, theory of observable values should be developed for hypothetical observers in the mirror world (mirror observer) and in zero-space (virtual observer). This should be the subject of future studies.

6.6 Conclusions

Finally we can build the whole picture of positronium annihilation in its para and ortho states, thanks to the results we have obtained using geometric methods of General Relativity.

Annihilation of electron and positron in parapositronium is effected by means of exchange of regular virtual photons, not subjected to collapse. All products of annihilation (two annihilation γ-quanta) can be emitted into our Universe only.

Annihilation of orthopositronium permits exchange of virtual photons in zero-space through two channels. In first channel electron and positron exchange virtual photons through regular (not collapsed) zero-space. In this case regular 3-photon annihilation occurs, i. e. all three γ-quanta are emitted into our Universe. Second channel is implemented through collapsed areas of zero-space, which are "gateways" to the mirror Universe: annihilation of orthopositronium is effected by means of exchange of virtual collapsers, two of which cause emission of two γ-quanta into the mirror Universe. Third virtual collapser produced by transformation of spin-energy of orthopositronium in zero-space, bears energy of opposite sign and causes emission of one γ-quantum into our Universe. Therefore in

3-photon annihilation through the second channel 2+1 split occurs: 2 photons emitted into the mirror Universe are unavailable for observation, while 1 photon is observable.

Noteworthy, in Section 6.4 we mentioned that in electromagnetic interaction of particles, to which annihilation belongs, Newtonian gravitational forces are negligible. That means virtual collapsers responsible for 2+1 split of 3-photon mode of annihilation, inhabit surfaces of zero-space objects, collapsed under action of non-Newtonian gravitational forces. For instance, in a space with de Sitter metric (5.73) gravitational force are non-Newtonian (5.74). Therefore if we consider zero-space as a case of full degeneration of de Sitter space, collapse in such zero-space will occur under action of non-Newtonian gravitational force only.

This is in parallel with a statement from phenomenology of anomalous annihilation of orthopositronium, according to which space-like resonance structure, that brings ^{22}Ne nuclea in gas into collective excitation, seems to result from combination of de Sitter spaces with positive and negative curvature.

Appendix A

Notation

Theory of chronometric invariants

b^α	four-dimensional monad vector
h_{ik}	three-dimensional chronometrically invariant metric tensor
τ	physical observable time
$d\sigma$	spatial physical observable interval
v^i	three-dimensional chronometrically invariant velocity
A_{ik}	three-dimensional antisymmetric chronometrically invariant tensor of space's rotation (non-holonomity tensor)
v_i	three-dimensional linear velocity of space's rotation
F^i	three-dimensional chronometrically invariant vector of gravitational inertial force
w	gravitational potential
c^i	three-dimensional chronometrically invariant light velocity
D_{ik}	three-dimensional chronometrically invariant tensor of velocities of space's deformation
$\triangle^i_{jk}$	chronometrically invariant Christoffel symbols of 2nd rank

Motion of particles

u^α	four-dimensional velocity
u^i	three-dimensional coordinate velocity
P^α	four-dimensional vector of impulse
p^i	three-dimensional vector of impulse
K^α	four-dimensional wave vector
k^i	three-dimensional wave vector
ψ	wave phase (eikonal)
S	action
L	Lagrange function (Lagrangian)
$\hbar^{\alpha\beta}$	four-dimensional antisymmetric Planck tensor
$\hbar^{*\alpha\beta}$	dual four-dimensional Planck pseudotensor

Electromagnetic field

A^α	four-dimensional potential of electromagnetic field
φ	chronometrically invariant temporal component of A^α (physical observable scalar potential of electromagnetic field)
A^i	chronometrically invariant spatial components of A^α (physical observable vector-potential of electromagnetic field)
$F^{\alpha\beta}$	four-dimensional Maxwell tensor of electromagnetic field
E_i, E^{*ik}	three-dimensional chronometrically invariant strength of electric field
H_{ik}, H^{*i}	three-dimensional chronometrically invariant strength of magnetic field

Riemannian space

x^α	four-dimensional coordinates
x^i	three-dimensional coordinates
t	coordinate time
ds	space-time interval
$g_{\alpha\beta}$	four-dimensional fundamental metric tensor
δ_β^α	unit four-dimensional tensor
J	determinant of Jacobi matrix (Jacobian)
$e^{\alpha\beta\mu\nu}$	four-dimensional completely antisymmetric unit tensor
e^{ikm}	three-dimensional completely antisymmetric unit tensor
$E^{\alpha\beta\mu\nu}$	four-dimensional completely antisymmetric tensor
ε^{ikm}	physical observable completely antisymmetric tensor
$\Gamma^\alpha_{\mu\nu}, \Gamma_{\mu\nu,\rho}$	Christoffel symbols of 2nd and 1st rank
$R_{\alpha\beta\mu\nu}$	Riemann-Christoffel four-dimensional tensor of curvature
$T_{\alpha\beta}$	energy-impulse tensor
J^i	vector of physical observable density of impulse
U^{ik}	tensor of observable density of impulse stream (tension tensor)
$R_{\alpha\beta}$	Ricci tensor
K	four-dimensional curvature
C	three-dimensional physical observable curvature
λ	cosmological term (λ-term)

Appendix B

Special expressions

$dA^\alpha = \dfrac{\partial A^\alpha}{\partial x^\sigma}\, dx^\sigma$	ordinary differential of vector
$DA^\alpha = dA^\alpha + \Gamma^\alpha_{\mu\nu} A^\mu dx^\nu$	absolute differential of vector
$\nabla_\alpha A^\beta = \dfrac{\partial A^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\sigma} A^\sigma$	absolute derivative of contravariant vector
$\nabla_\alpha A_\beta = \dfrac{\partial A_\beta}{\partial x^\alpha} - \Gamma^\sigma_{\beta\alpha} A_\sigma$	absolute derivative of covariant vector
$\nabla_\alpha A^\alpha = \dfrac{\partial A^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\sigma} A^\sigma$	absolute divergence of vector
${}^*\nabla_i q^i = \dfrac{{}^*\partial q^i}{\partial x^i} + q^i \dfrac{{}^*\partial \ln\sqrt{h}}{\partial x^i}$	chronometrically invariant divergence of the vector q^i
${}^*\widetilde{\nabla}_i q^i = {}^*\nabla_i q^i - \dfrac{1}{c^2} F_i q^i$	chronometrically invariant physical divergence of the vector q^i
$\Box = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$	general covariant d'Alembert operator
$\triangle = g^{ik} \nabla_i \nabla_k$	ordinary three-dimensional Laplace operator
${}^*\triangle = h^{ik}\, {}^*\nabla_i\, {}^*\nabla_k$	chronometrically invariant Laplace operator
$\dfrac{{}^*\partial}{\partial t} = \dfrac{1}{\sqrt{g_{00}}} \dfrac{\partial}{\partial t}$	chronometrically invariant derivative with respect to temporal coordinate t
$\dfrac{{}^*\partial}{\partial x^i} = \dfrac{\partial}{\partial x^i} + \dfrac{1}{c^2} v_i \dfrac{{}^*\partial}{\partial t}$	chronometrically invariant derivative with respect to x^i
$\mathrm{v}^2 = \mathrm{v}_i \mathrm{v}^i = h_{ik} \mathrm{v}^{ik}$	square of physical observable velocity
$v^i = -c g^{0i} \sqrt{g_{00}}, \quad v_i = h_{ik} v^k$	components of the velocity of space's rotation
$v^2 = h_{ik} v^i v^k$	square of v_i (because $g_{\alpha\sigma} g^{\sigma\beta} = g^\beta_\alpha$, then with $\alpha=\beta=0$ we obtain $g_{0\sigma} g^{\sigma 0} = \delta^0_0 = 1$, hence $v^2 = c^2(1 - g_{00} g^{00})$
$\sqrt{-g} = \sqrt{h}\sqrt{g_{00}}$	relation between determinants of physical observable metric tensor and fundamental metric tensor
$\dfrac{d}{d\tau} = \dfrac{{}^*\partial}{\partial t} + \mathrm{v}^k \dfrac{{}^*\partial}{\partial x^k}$	derivative with respect to physical observable time τ

$$\frac{d}{ds}=\frac{1}{c\sqrt{1-\mathrm{v}^2/c^2}}\frac{d}{d\tau}$$

1st derivative with respect to space-time interval

$$\frac{d^2}{ds^2}=\frac{1}{c^2-\mathrm{v}^2}\frac{d^2}{d\tau^2}+\frac{1}{\left(c^2-\mathrm{v}^2\right)^2}\times$$
$$\times\left(D_{ik}\mathrm{v}^i\mathrm{v}^k+\mathrm{v}_i\frac{d\mathrm{v}^i}{d\tau}+\frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial x^m}\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m\right)\frac{d}{d\tau}$$

2nd derivative with respect to space-time interval

$$h_{ik}=-g_{ik}+\frac{1}{c^2}v_iv_k,\quad h^{ik}=-g^{ik}\quad h_i^k=\delta_i^k$$

components of physical observable metric tensor

$$\frac{{}^*\partial A_{ik}}{\partial t}+\frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i}-\frac{{}^*\partial F_i}{\partial x^k}\right)=0$$

Zelmanov's identity

$$\frac{{}^*\partial A_{km}}{\partial x^i}+\frac{{}^*\partial A_{mi}}{\partial x^k}+\frac{{}^*\partial A_{ik}}{\partial x^m}+$$
$$+\frac{1}{2}\left(F_iA_{km}+F_kA_{mi}+F_mA_{ik}\right)=0$$

Zelmanov's identity

$$g^{i\alpha}g^{k\beta}\Gamma^m_{\alpha\beta}=h^{iq}h^{ks}\triangle^m_{qs}\,,$$
$$D_k^i+A_{k\cdot}^{\cdot i}=\frac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k}-\frac{g_{0k}\Gamma^i_{00}}{g_{00}}\right),$$
$$F^k=-\frac{c^2\Gamma^k_{00}}{g_{00}}$$

Zelmanov's relations between regular Christoffel symbols
and chronometrically invariant characteristics
of observer's space of reference

$$\frac{d}{d\tau}\mathrm{v}^2=\frac{d}{d\tau}\left(h_{ik}\mathrm{v}^i\mathrm{v}^k\right)=2D_{ik}\mathrm{v}^i\mathrm{v}^k+$$
$$+\frac{{}^*\partial h_{ik}}{\partial x^m}\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m+2\mathrm{v}_k\frac{d\mathrm{v}^k}{d\tau}$$

derivative from v^2 with respect to observable time

$$\varepsilon^{ikm}=\sqrt{g_{00}}E^{0ikm}=\frac{e^{0ikm}}{\sqrt{h}}\,,$$
$$\varepsilon_{ikm}=\frac{E_{0ikm}}{\sqrt{g_{00}}}=e_{0ikm}\sqrt{h}$$

physical observable completely antisymmetric tensor

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