Singularities of Transition Processes In Dynamical Systems *

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Abstract

Abstract. The paper gives the systematic analysis of singularities of transition processes in general dynamical systems. Dynamical systems depending on parameter are studied. A system of relaxation times is constructed. Each relaxation time depends on three variables: initial conditions, parameters k of the system and accuracy ε of relaxation. This system of times contains: the time before the first entering of the motion into ε-neighbourhood of the limit set, the time of final entering in this neighbourhood and the time of stay of the motion outside the ε-neighbourhood of the limit set. The singularities of relaxation times as functions of (x_0, k) under fixed ε are studied. A classification of different bifurcations (explosions) of limit sets is performed. The bifurcations fall into those with appearance of new limit points and bifurcations with appearance of new limit sets at finite distance from the existing ones. The relations between the singularities of relaxation times and bifurcations of limit sets are studied. The peculiarities of dynamics which entail singularities of transition processes without bifurcations are described as well. The peculiarities of transition processes under perturbations are studied. It is shown that the perturbations simplify the situation: the interrelations between the singularities of relaxation times and other peculiarities of dynamics for general dynamical system under small perturbations are the same as for smooth two-dimensional structural stable systems.

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Introduction

Are there "white spots" in topological dynamics? Undoubtedly, they exist: the transition processes in dynamical systems are investigated not very well. Because of this, it is difficult to interpret the experiments which reveal singularities of transition processes, in particular, anomalously slow relaxation. "Anomalously slow" means here "unexpectedly slow"; but what on the whole one can expect from a dynamical system?

In this paper the transition processes in general dynamical systems are studied. The approach based on topological dynamics possess wide generality, but one should pay for this: the wider notions, the harder calculations. Nevertheless, the phase of general consideration is necessary.

Limit behaviour (at $t \to \infty$) of dynamical systems have been studied very intensively in XX century [1–6]. New types of limit sets ("strange attractors") were discovered [7,8]. Fundamental results concerning the structure of limit sets were obtained, such as Pugh lemma [9] and Kolmogorov-Arnold-Moser theory [10,11]. The theory of limit behaviour "on the average" – ergodic theory [12] – was considerably furthered. Theoretical and applied achievements of the bifurcation theory are evident [13–15]. In this theory the dependencies of limit sets on parameters are studied for many important cases.

Achievements for transition processes are much more modest. Perhaps, only relaxations in linear and linearized systems are well studied. Applications of this elementary theory got the name "relaxation spectroscopy". Construction of this applied discipline was marked by Nobel Prize (M.Eigen [16]).

There are not any general theory of transition processes for essentially non-linear systems. We encountered this when studying transition processes in catalytic reactions. It was necessary to give an interpretation on anomalously long transition processes. In this connection a discussion arose and even some papers were written. The main question of the discussion was: whether slow relaxations comes from slow "strange processes" (diffusion, phase transitions and so on) or they may be of purely kinetic (that is dynamic) nature?

Since there was not any general theory of relaxation times and their singularities, we had to construct it on our's own [17–22]. In the present paper the first – topological – part of this theory is given. It is elementary enough, but lengthy $\varepsilon - \delta$ reasonings may demand for time and efforts from the reader. Chemical examples, theoretical and numerical analysis of slow relaxations and also a more elementary introduction into the theory one can find in the monograph [23].

Two simplest mechanisms of slow relaxations appearing can be easily indicated: retardation of motion near unstable fixed point and (for parameter depending systems) retardation of motion in that region where under small change of parameters appears a fixed point. Let us give simplest examples for motion over the segment $[-1, 1]$.

Retardation near unstable fixed point exists in the system $\dot{x} = x^2 - 1$. There are two fixed points $x = \pm 1$ on the segment $[-1, 1]$, $x = 1$ is unstable and $x = -1$ is stable one. The equation is integrable in explicit form:

$$x = [(1 + x_0)e^{-t} - (1 - x_0)e^{t}]/[(1 + x_0)e^{-t} + (1 - x_0)e^{t}],$$

where $x_0 = x(0)$ is a value of x at initial time moment $t = 0$. If $x_0 \neq 1$, then after some time the motion will come to be in ε-neighborhood of the point $x = -1$, whatever would be $\varepsilon > 0$. This process requires the time

$$\tau(\varepsilon, x_0) = -\frac{1}{2}\ln\frac{\varepsilon}{2-\varepsilon} - \frac{1}{2}\ln\frac{1-x_0}{1+x_0}.$$

Naturally, it is supposed that $x_0 > \varepsilon$. If ε is fixed, then τ tends to $+\infty$ as $x_0 \to 1$ like $-\frac{1}{2}\ln(1-x_0)$. The motion that begins near the point $x = 1$ remains for a long time (within $-\frac{1}{2}\ln(1-x_0)$) near this point and then goes to the point $x = -1$. In order to show it clearer, let us compute the time τ' of being over the segment $[-1+\varepsilon,\ 1-\varepsilon]$ of the motion, beginning near the point $x = 1$, i.e. the time of its stay outside ε-neighborhoods of fixed points $x = \pm 1$. Supposing $1 - x_0 < \varepsilon$, we obtain

$$\tau'(\varepsilon, x_0) = \tau(\varepsilon, x_0) - \tau(2-\varepsilon, x_0) = -\ln\frac{\varepsilon}{2-\varepsilon}.$$

One can see that $\tau'(\varepsilon, x_0)$ does not depend on x_0 if $1 - x_0 < \varepsilon$. This is evident: the time τ' is the time of travel from the point $1-\varepsilon$ to the point $-1+\varepsilon$.

We obtain the example of retardation of motion in the area where under small change of parameter appears a fixed point, considering the system $\dot{x} = (k + x^2)(x^2 - 1)$ over $[-1, 1]$. If $k > 0$, there are again only two fixed points $x = \pm 1$, $x = -1$ is a stable point and $x = 1$ is an unstable one. If $k = 0$, there appears the third point $x = 0$. It is not stable, but "semistable“ in following sense. If the initial position is $x_0 > 0$, then the motion goes from x_0 to $x = 0$. If $x_0 < 0$, then the motion goes from x_0 to $x = -1$. If $k < 0$, then besides $x = \pm 1$ there are two fixed points $x = \pm\sqrt{|k|}$ more. The positive point is stable, and the negative point is unstable. Let us consider the case $k > 0$. The time of motion from the point x_0 to the point x_1 can be found in explicit form ($x_{0,1} \neq \pm 1$):

$$t = \frac{1}{2}\ln\frac{1-x_1}{1+x_1} - \frac{1}{2}\ln\frac{1-x_0}{1+x_0} - \frac{1}{\sqrt{k}}\left(\arctan\frac{x_1}{\sqrt{k}} - \arctan\frac{x_0}{\sqrt{k}}\right).$$

If $x_0 > 0,\ x_1 < 0,\ k > 0,\ k \to 0$, then $t \to \infty$ like $\pi/\sqrt{k}$.

The listed examples do not exhaust all the possibilities: they illustrate only two important mechanisms of slow relaxations appearance. Further there are studied slow relaxations of dynamical systems.

There are studied parameter depending dynamical systems. The point of view of topological dynamics is accepted (see [1–3,6,24,25]). In the first instance it means that for us, as a rule, the properties associated with the smoothness, analyticness and so on will be of no importance. Therefore the phase space X and parameter space K are further compact metric spaces: for any points x_1, x_2 from X (k_1, k_2 from K) is defined the quantity $\rho(x_1, x_2)$ $(\rho_K(k_1, k_2))$ – a distance, possessing following properties:

$$\rho(x_1, x_2) = \rho(x_2, x_1), \quad \rho(x_1, x_2) + \rho(x_2, x_3) \geq \rho(x_1, x_3),$$
$$\rho(x_1, x_2) = 0 \text{ if and only if } x_1 = x_2 \text{ (similarly for } \rho_K).$$

The sequence x_i converges to x^* ($x_i \to x^*$) if $\rho(x_i, x^*) \to 0$. The compactness means that from any sequence a convergent subsequence can be chosen.

The states of the system are represented by the points of the phase space X. The reader can suppose that X and K are closed, restricted subsets of finite-dimensional Euclidean spaces, for example polyhedrons, and ρ and ρ_K are the ordinary Euclidean distances.

Instead of giving the dynamical system by differential equations, let us from the very beginning define the transformation *"shift over the time t"* – function f of three arguments: $x \in X$ (of the initial condition), $k \in K$ (parameter values) and $t \geq 0$, with values in X: $f(t, x, k) \in X$. This function is supposed to be continuous by totality of arguments and satisfying the following conditions:
$f(0, x, k) = x$ (shift over zero time leaves any point on its place);
$f(t, f(t', x, k), k) = f(t + t', x, k)$ (the result of sequentially performed shifts over t and t' is the shift over $t + t'$);
if $x \neq x'$, then $f(t, x, k) \neq f(t, x', k)$.

For given value of parameter $k \in K$ every initial state $x \in X$ can be associated with *ω-limit set* $\omega(x, k)$ – the set of all limit points of $f(t, x, k)$ for $t \to \infty$:

$$y \in \omega(x, k) \text{ if and only if there is such sequence } t_i \geq 0 \\ \text{that } t_i \to \infty \text{ and } f(t_i, x, k) \to y.$$

The examples of ω-limit points are stationary (fixed) points, points of limit cycles and so on.

The relaxation of a system can be represented as its motion to ω-limit set corresponding to given initial state and value of parameter. *The relaxation time* can be defined as the time of this motion. However, there are several possibilities here.

Let $\varepsilon > 0$. Denote by $\tau_1(x, k, \varepsilon)$ the time during which the system under given value of parameter k will come from the initial state x into ε-neighbourhood of $\omega(x, k)$ (for the first time). Entering ε-neighborhood of ω-limit set, the system then can get out of it, then again enter it, and so several times. After all, the motion will finally enter this neighbourhood, but this may take more time than the first entry. Therefore, let us introduce for the system the time of being outside ε-neighborhood of $\omega(x, k)$ (τ_2) and the time of final entry in it (τ_3). So,

$$\begin{aligned}
\tau_1(x, k, \varepsilon) &= \inf\{t > 0 \mid \rho^*(f(t, x, k),\ \omega(x, k)) < \varepsilon\}; \\
\tau_2(x, k, \varepsilon) &= \operatorname{mes}\{t > 0 \mid \rho^*(f(t, x, k),\ \omega(x, k)) \geq \varepsilon\}; \\
\tau_3(x, k, \varepsilon) &= \inf\{t > 0 \mid \rho^*(f(t', x, k),\ \omega(x, k)) < \varepsilon \text{ for } t' > t\}.
\end{aligned}$$

Here mes is Lebesgue measure (on the real line it is length), ρ^* is the distance from the point to the set: $\rho^*(x, P) = \inf_{y \in P} \rho(x, y)$.

To different initial states can correspond different ω-limit sets (even under fixed value of parameter k). The limit behavior of the system can be characterized by *total limit set*

$$\omega(k) = \bigcup_{x \in X} \omega(x, k).$$

The set $\omega(k)$ is the union of all $\omega(x, k)$ under given k. Whatever would be the initial state, the system after some time will be in the ε-neighborhood of $\omega(k)$. The relaxation can be considered as motion towards $\omega(k)$. Introduce the corresponding relaxation times:

$$\begin{aligned}
\eta_1(x, k, \varepsilon) &= \inf\{t > 0 \mid \rho^*(f(t, x, k),\ \omega(k)) < \varepsilon\}; \\
\eta_2(x, k, \varepsilon) &= \operatorname{mes}\{t > 0 \mid \rho^*(f(t, x, k),\ \omega(k)) \geq \varepsilon\}; \\
\eta_3(x, k, \varepsilon) &= \inf\{t > 0 \mid \rho^*(f(t', x, k),\ \omega(k)) < \varepsilon \text{ for } t' > t\}.
\end{aligned}$$

Now we are to define what is to be considered as a *slow transition process*. There is not some marked out scale of time, which could be compared with relaxation times. Except that, by decrease of the relaxation times can become of any large amount even in the simplest situations – of motion to unique stable fixed point. For every initial state and given k and ε all relaxation times are finite. But they can be unrestricted in total. Just in this case we speak about slow relaxations.

Give the simplest example. Let us consider over the segment $[-1, 1]$ the differential equation $\dot{x} = x^2 - 1$. The point $x = -1$ is stable, the point $x = 1$ is unstable. For any fixed $\varepsilon > 0$, $\varepsilon < \frac{1}{2}$ – the relaxation times $\tau_{1,2,3}, \eta_3(x, k, \varepsilon) \to \infty$ as $x \to 1$, $x < 1$. The times η_1, η_2 remain limited in this case.

Let us say that the system has τ_i- (η_i)-*slow relaxations*, if for some $\varepsilon > 0$ the function $\tau_i(x, k, \varepsilon)$ ($\eta_i(x, k, \varepsilon)$) is unbounded above in $X \times K$, i.e. for any $t > 0$ there are such $x \in X$, $k \in K$, that $\tau_i(x, k, \varepsilon) > t$ ($\eta_i(x, k, \varepsilon) > t$).

One of possible causes of slow relaxations is a jump change of ω-limit set $\omega(x, k)$ by change of x, k (as well as jump change of $\omega(k)$ by change of k). These "explosions" (or bifurcations) of ω-limit sets are studied in the section 1. In the next section 2 there are listed theorems, giving necessary and sufficient conditions of slow relaxations. Set forth two of them.

Theorem 2.1′. The system has τ_1-slow relaxations if and only if there is a singularity of the dependence $\omega(x, k)$ of the following kind: there are such points $x^* \in X$, $k^* \in K$, sequences $x_i \to x^*$, $k_i \to k^*$, and number $\delta > 0$ that for any i, $y \in \omega(x^*, k^*)$, $z \in \omega(x_i, k_i)$ the distance $\rho(y, z) > \delta$.

The singularity of $\omega(x, k)$ described in the statement of the theorem is that the whole ω-limit set changes by a jump: the distance from any point of $\omega(x_i, k_i)$ to any point of $\omega(x^*, k^*)$ is greater than δ.

In the following theorem necessary and sufficient conditions of τ_3-slow relaxations are given. Since $\tau_3 \geq \tau_1$, the conditions of τ_3-slow relaxations are weaker than the conditions of the theorem 2.1′ – τ_3-slow relaxations are "more often" than τ_1-slow relaxation (the interrelations between different kinds of slow relaxations with corresponding examples you can see below in the section 3.2). This is why the discontinuities of ω-limit sets in the following theorem are weaker.

Theorem 2.7. τ_3-slow relaxations exist if and only if at least one of the following conditions is accomplished:

1) there are points $x^* \in X$, $k^* \in K$, $y^* \in \omega(x^*, k^*)$, sequences $x_i \to x^*$, $k_i \to k^*$ and number $\delta > 0$ such that for any i and $z \in \omega(x_i, k_i)$ the inequality $\rho(y^*, z) > \delta$ is true (note that the existence of one such y is sufficient – compare with the theorem 2.1′);

2) there are $x \in X$, $k \in K$ such that $x \notin \omega(x, k)$, for any $t > 0$ can be found $y(t) \in X$, for which $f(t, y(t), k) = x$ ($y(t)$ is a shift of x over $-t$), and for some $z \in \omega(x, k)$ can be found such a sequence $t_i \to \infty$ that $y(t_i) \to z$.

As an example of the point satisfying the condition 2 can be taken any point lying on the loop – trajectory, starting from the singular point and returning to the same point.

Other theorems of the section 2 also establish connections between slow relaxations and peculiarities of the limit behaviour under different initial conditions and parameter values. In general, in topological and differential dynamics the principal attention is paid to the limit behavior of dynamical systems [1–6,24–28]. In applications, however, it is often important how rapidly the motion approaches the limit one. In chemistry, long-time retardations of reactions far from equilibrium (induction periods) are studied

since Vant-Goff ([29,30], from the latest works note [31]). When minimizing functions by relaxation methods, difficulties arise bound with analogous retardations. The paper [32], for example, deals with their elimination. In simplest cases the slow relaxations are bound with delays near unstable fixed points. In general case there is a complicated system of interrelations between different types of slow relaxations and other dynamical peculiarities, as well as of different types of slow relaxations between themselves. These relations are the subject of the sections 2, 3. The investigation is performed generally in the way of classic topological dynamics [1–3]. There are, however, some differences:
a) from the very beginning is considered not one system, but practically more important case of parameter dependent systems;
b) the motion in these systems is defined, generally speaking, only for positive times.

The last circumstance is bound with the fact that for applications (in particular, chemical) makes sense the motion only in a set, being only positive invariant (in balance polyhedron). Some results can be accepted for the case of general semidynamical systems [33–37], however for majority of applications the considered degree of generality is more than sufficient.

The investigation of smooth systems permits to obtain results having no analogy in topological dynamics. So, in the section 2 is shown that "almost always" η_2-slow relaxations are absent in one separately taken C^1-smooth dynamical system (system, given by differential equations with C^1-smooth right parts). Let us explain what "almost always" means in this case. A set Q of C^1-smooth dynamical systems with common phase space is called nowhere-dense in C^1-topology, if for any system from Q can be chosen an infinitesimal perturbation of right parts (perturbation of right parts and its first derivatives are smaller than previously given $\varepsilon > 0$) such that the perturbed system should not belong to Q and should exist $\varepsilon_1 > 0$ ($\varepsilon_1 < \varepsilon$) such that under ε_1-small variations of right parts (and of first derivatives) the perturbed system could not return in Q. Union of finite number of nowhere-dense sets is also nowhere-dense. It is not the case for countable union: for example, one point on a line makes nowhere-dense set, but the countable set of rational numbers is dense on the real line: a rational number is on any segment. However, both on line and in many other cases countable union of nowhere-dense sets can be considered as very "meagre". In particular, for C^1-smooth dynamical systems the union of countable number of nowhere-dense sets has the following property: any system, belonging to this union, can be eliminated from it by infinitesimal perturbation. The above words "almost always" meant: except for union of countable number of nowhere-dense sets.

Note, by the way, that η_1-slow relaxations in separate system are quite impossible. In two-dimensional case (two variables), "almost any" C^1-smooth dynamical system is rough, i.e. its phase portrait under small perturbations is only slightly deformed, qualitatively remaining the same. For rough two-dimensional systems ω-limit sets consist of fixed points and limit cycles, and the stability of these points and cycles can be verified by linear approximation. The correlation of six different kinds of slow relaxations between themselves for rough two-dimensional systems becomes considerably more simple.

Theorem 3.5. Let M be a smooth compact two-dimensional manifold, F be C^1-smooth rough dynamical system on M, X be a positive invariant subset of M (the motion at positive times does not go out of X), $F|_X$ is a restriction of F on X. Then :

1) the availability of τ_3-slow relaxations for $F|_X$ is equivalent to the availability for that of $\tau_{1,2}$ -and η_3-slow relaxations;

2) $F|_X$ does not possess τ_3-slow relaxations if and only if in X is one fixed point (and none limit cycle), or one limit cycle (and none fixed point);

3) $\eta_{1,2}$-slow relaxations are impossible for $F|_X$.

For smooth rough two-dimensional systems it is easy to estimate measure (area) of the region of durable retardations $\mu_i(t) = \text{mes}\{x \in X \mid \tau_i(x, \varepsilon) > t\}$ under fixed sufficiently small ε and large t (the parameter k is absent – a separate system is studied). Asymptotical behaviour of $\mu_i(t)$ as $t \to \infty$ does not depend on i and

$$\lim_{t\to\infty} \frac{\ln \mu_i(t)}{t} = -\min\{\varkappa_1, \dots, \varkappa_n\},$$

where n is a number of unstable limit motions (of fixed points and cycles) in X, and the numbers are determined as follows. Denote by $B_i, \dots, B_n$ unstable limit motions lying in X.

1. Let B_i be an unstable node or focus. Then $\varkappa_1$ is the trace of matrix of linear approximation in the point b_i.

2. Let b_i be a saddle. Then $\varkappa_1$ is positive proper value of the matrix of linear approximation in this point.

3. Let b_i be an unstable limit cycle. Then $\varkappa_i$ is characteristic indicator of cycle (see [38], p. 111).

Thus, the area of the region of initial conditions, which result in durable retardation of the motion, in the case of smooth rough two-dimensional systems behaves at large retardation times as $\exp(-\varkappa t)$, where t is a retardation time, $\varkappa$ is the smallest number of $\varkappa_i, \dots, \varkappa_n$. If $\varkappa$ is close to zero (the system is close to bifurcation [5,38]), then this area decreases slowly enough at large t. One can find here analogy with linear time of relaxation to fixed point

$$\tau_l = -1/\max \text{Re}\lambda$$

where λ runs through all the proper values of the matrix of linear approximation of right parts in this point,
$\max \text{Re}\lambda$ is the largest (the smallest by value) real part of proper value,
$\tau_l \to \infty$ as $\text{Re}\lambda \to 0$.

However, there are essential differences. In particular, τ_l comprises the proper values of linear approximation matrix in that (stable) point, to which the motion is going, and the asymptotical estimate μ_i comprises the proper values of the matrix in that (unstable) point or cycle, near which the motion is retarded.

In typical situations for two-dimensional parameter depending systems the singularity of τ_l entails existence of singularities of relaxation times τ_i (to this statement can be given an exact meaning and it can be proved as a theorem). The inverse is not true. As an example should be noted the retardations of motions near unstable fixed points. Besides, for systems of higher dimensions the situation becomes more complicated, the rough systems cease to be "typical" (this was shown by S.Smeil [39], the discussion see in [5]), and the limit behaviour even of rough systems does not come to tending of motion to fixed point or limit cycle. Therefore the area of reasonable application of estimate of properties of transitional processes by means of τ_l becomes in this case even more restricted.

Any real system exists under the permanent perturbing influence of its neighbourhood. It is hardly possible to construct a model taking into account all such perturbations. Besides that, the model usually only approximately considers also the internal properties of the system. The discrepancy between the real system and the model, arising from these two circumstances, is different for different models. So, for the systems of celestial mechanics it can be done very small. Quite the contrary, for chemical kinetics, especially for kinetics of heterogeneous catalysis, this discrepancy can be if not too large but, however, not such small to be neglected. Strange as it may seem, the presence of such an unpredicted divergence of the model and reality can simplify the situation – the perturbations "conceal" some fine details of dynamics.

The section 4 is devoted to the problems of slow relaxations in presence of small perturbations. As a model of perturbed motion here are taken *ε-motions*: the function of time $\varphi(t)$ with values in X, defined at $t \geq 0$, is called ε-motion ($\varepsilon > 0$) under given value of $k \in K$, if for any $t \geq 0,\ \tau \in [0, T]$ the inequality $\rho(\varphi(t+\tau),\ f(\tau, \varphi(t), k)) < \varepsilon$ holds. In other words, if for an arbitrary point $\varphi(t)$ one considers its motion on the force of dynamical system, this motion will diverge $\varphi(t+\tau)$ from no more than at ε for $\tau \in [0, T]$. Here $[0, T]$ is a certain interval of time – its length is not very important (it is important that it is fixed), because later we'll consider the case $\varepsilon \to 0$.

There are two traditional approaches to the consideration of perturbed motions. One of them is to investigate the motion in the presence of small constantly acting perturbations [40–46], the other is the study of fluctuations under the influence of small stochastic perturbations [47–52]. The stated results join the first direction, there to are used some ideas bound with the second one. The ε-motions were studied earlier in differential dynamics, in general in connection with the theory of Anosov about ε-trajectories and its applications [27,53–56], see also [57].

When studying perturbed motions, each point is compared with not one trajectory, but with "a bundle" of ε-motions, going out from this point ($\varphi(0) = x$) under given value of parameter k. The totality of all ω-limit points of these ε-motions (of limit points of $\varphi(t)$ as $t \to \infty$) is denoted by $\omega^{\varepsilon}(x, k)$. Firstly, it is necessary to notice that $\omega^{\varepsilon}(x, k)$ does not always tend to $\omega(x, k)$ as $\varepsilon \to 0$: the set $\omega^{0}(x, k) = \bigcap_{\varepsilon>0} \omega^{\varepsilon}(x, k)$ may not coincide with $\omega(x, k)$. In the section 4 there are studied relaxation times of ε-motions and corresponding slow relaxations. In contrast to the case of nonperturbed motion, all natural kinds of slow relaxations are not considered – they are too numerous (eighteen), and the principal attention is paid to two of them, which are analyzed in more details than in the section 2.

Further the structure of limit sets of one perturbed system is studied. The analogy of general perturbed systems and smooth rough two-dimensional systems is revealed. Let us quote in this connection the review by Professor A.M.Molchanow of the thesis of A.N.Gorban (1981): *"After classic works of Andronov, devoted to the rough systems on the plane, for a long time it seemed that division of plane into finite number of cells with source and drain is an example of structure of multidimensional systems too... The most interesting (in the opinion of opponent) is the fourth chapter "Slow relaxations of the perturbed systems". Its principal result is approximately as follows. If a complicated dynamical system is made rough (by means of ε-motions), then some its important properties are similar to the properties of rough systems on the plane. This is quite positive result, showing in what sense the approach of Andronov can be generalized for arbitrary systems".*

To study limit sets of perturbed system, two relations are introduced: of preorder $\succsim$ and of equivalence $\sim$:
$x_1 \succsim x_2$ if for any $\varepsilon > 0$ there is such ε-motion $\varphi(t)$ that $\varphi(0) = x_1$ and $\varphi(\tau) = x_2$ for

some $\tau > 0$;
$x_1 \sim x_2$ if $x_1 \gtrsim x_2$ and $x_2 \gtrsim x_1$.

For smooth dynamical systems similar relation of equivalence had been introduced with the help of action functionals in studies on stochastic perturbations of dynamical systems ([52] p. 222 and further). Similar concepts can be found in [57]. Let $\omega^0 = \bigcup_{x \in X} \omega^0(x)$ (k is omitted, because only one system is studied). Let us identify equivalent points in ω^0. The obtained factor-space is quite disconnected (each point possessing a fundamental system of neighborhoods open and closed simultaneously). Just this space $\omega^0/\sim$ with the order over it can be considered as a system of sources and drains analogous to the system of limit cycles and fixed points of smooth rough two-dimensional dynamical system. The sets $\omega^0(x)$ can change by jump only on the boundaries of the region of attraction of corresponding "drains" (theorem 4.4). The interrelation of six principal kinds of slow relaxations in perturbed system is analogous to their interrelation in smooth rough two-dimensional system described in the theorem 3.5.

Let us enumerate the most important results of the investigations being stated.

1. It is not always necessary to search for "foreign" reasons of slow relaxations – in the first place one should determine if there are slow relaxations of dynamical origin in the system.

2. One of possible causes of slow relaxations is bifurcations (explosions) of ω-limit sets. Usually, the dependence of ω-limit set on parameter is studied. Here, it is necessary to study the dependence $\omega(x,k)$ of limit set both on parameters and initial data. It is violation of the joint continuity with respect to x and k that leads to slow relaxations.

3. The perturbances make the system rough – the interrelation of slow relaxations in perturbed system is the same as in smooth rough two-dimensional systems.

4. Owing to a large quantity of different unreducible to each other slow relaxations, it is important, observing them in experiment, to try to understand which namely of relaxation times is large.

5. Slow relaxations in real systems often are "bounded slow" – the relaxation time is large (essentially greater than could be expected proceeding from the coefficients of equations and notions about the characteristic times), but nevertheless limited. When studying such singularities, appears to be useful the following method, ascending to the works of A.A.Andronov: the considered system is included in appropriate family for which slow relaxations are to be studied in the sense accepted in the present work. This study together with the mention of degree of proximity of particular systems to the initial one can give an important information.

1 Bifurcations (Explosions) of ω-limit Sets

Let X be a compact metric space with the metrics ρ, and K be a compact metric space (the space of parameters) with the metrics ρ_K,

$$f : [0, \infty) \times X \times K \to X \tag{1}$$

be a continuous mapping for any $t \geq 0$, $k \in K$; let mapping $f(t,\cdot,k) : X \to X$ be homeomorphism of X into subset of X and under every $k \in K$ let these homeomorphisms form monoparametric semigroup:

$$f(0,\cdot,k) = id, \ f(t, f(t',x,k),k) = f(t+t',x,k) \tag{2}$$

for any $t, t' \geq 0, x \in X$.

Below we will call the semigroup of mappings $f(t,\cdot,k)$ under fixed k a semiflow of homeomorphisms (or, for short, semiflow), and the mapping (1) a family of semiflows or simply a system (1). It is obvious that all results, concerning the system (1), are valid also in the case when X is a phase space of dynamical system, i.e. when every semiflow can be prolonged along t to the left onto the whole axis $(-\infty,\infty)$ up to flow (to monoparametric group of homeomorphisms of X onto X).

1.1 Extension of Semiflows to the Left

It is clear that under fixed x and k the mapping $f(\cdot,x,k)$: $t \to f(t,x,k)$ can be, generally speaking, defined also for certain negative t, preserving semigroup property (2). Really, consider under fixed x and k the set of all non-negative t for which there is point $q_i \in X$ such that $f(t,q_i,k) = x$. Denote the upper bound of this set by $T(x,k)$:

$$T(x,k) = \sup\{t \mid \exists q_t \in X, \ f(t,q_t,k) = x\}. \tag{3}$$

Under given t,x,k the point q_t, if it exists, has a single value, since the mapping $f(t,\cdot,k) : X \to X$ is homeomorphism. Introduce the denotation $f(-t,x,k) = q_t$. If $f(-t,x,k)$ is determined, then for any τ within $0 \leq \tau \leq t$ is determined $f(-\tau,x,k) = f(t-\tau, f(-t,x,k),k)$. Let $T(x,k) < \infty$, $\quad T(x,k) > t_n > 0 \quad (n = 1,2,\ldots)$, $\quad t_n \to T$. Let us choose from the sequence $f(-t_n,x,k)$ a subsequence converging to some $q^* \in X$ and denote it by $\{q_j\}$, and the corresponding times denote by $-t_j$ ($q_j = f(-t_j,x,k)$). Owing to the continuity of f we obtain: $f(t_j,q_j,k) \to f(T(x,k),q^*,k)$, therefore $f(T(x,k),q^*,k) = x$. Thus, $f(-T(x,k),x,k) = q^*$.

So, under fixed x,k the mapping f was determined by us in interval $[-T(x,k),\infty)$, if $T(x,k)$ is finite, and in $(-\infty,\infty)$ in the opposite case. Let us denote by S the set of all triplets (t,x,k), in which f is now determined. For enlarged mapping f the semigroup property in following form is valid:

Proposition 1.1. (Enlarged semigroup property).

A) If (τ,x,k) and $(t, f(\tau,x,k),k) \in S$, then $(t+\tau,x,k) \in S$ and the equality

$$f(t, f(t,x,k),k) = f(t+\tau,x,k) \tag{4}$$

is true.

B) Inversely, if $(t+\tau,x,k)$ and $(\tau,x,k) \in S$, then $(t, f(\tau,x,k),k) \in S$ and (4) is true.

Thus, if the left part of the equality (4) makes sense, then its right part is determined too and the equation is valid. If there are determined both the right part and $f(\tau,x,k)$ in the left part, then the whole left part makes sense and (4) is true.

Proof. The proof consists in consideration of several variants. Since the parameter k is assumed to be fixed, for the purpose of shortening the record it is absent in following formulas.

1. $f(t, f(-\tau, x)) = f(t-\tau, x)$ $(t, \tau > 0)$ $a)$ $t > \tau > 0$.
Let the left part make sense: $f(-\tau, x)$ is determined. Then, taking into account that $t - \tau > 0$, we have $f(t, f(-\tau, x)) = f(t-\tau+\tau, f(-\tau, x)) = f(t-\tau, f(\tau, f(-\tau, x))) = f(t-\tau, x)$, since $f(\tau, f(-\tau, x)) = x$ by definition.

Therefore the equality 1 is true (the right part makes sense since $t > \tau$)- the part for the case 1a is proved. Inversely, if $f(-\tau, x)$ is determined, then the whole left part of 1 $(t > 0)$ makes sense, and then according to the proved the equality is true.

The other variants are considered in analogous way.

Proposition 1.2. The set S is closed in $(-\infty, \infty) \times X \times K$ and the mapping $f : S \to X$ is continuous.

Proof. Denote by $\langle -T(x, k), \infty)$ the interval $[-T(x, k), \infty)$, if $T(x, k)$ is finite, and the whole axis $(-\infty, \infty)$ in opposite case. Let $t_n \to t^*$, $x_n \to x^*$, $k_n \to k^*$, and $t_n \in \langle -T(x_n, k_n), \infty)$. To prove the proposition, it should be made certain that $t^* \in \langle -T(x^*, k^*), \infty)$ and $f(t_n, x_n, k_n) \to f(t^*, x^*, k^*)$. If $t^* > 0$, this follows from the continuity of f in $[0, \infty) \times X \times K$. Let $t^* \leq 0$. Then it can be supposed that $t_n < 0$. Let us redenote, changing the signs, t_n by $-t_n$ and t^* by $-t^*$. Let us choose from the sequence $f(-t_n, x_n, k_n)$ using the compactness of X a subsequence converging to some $q^* \in X$. Denote it by q_j, and the sequences of corresponding t_n, x_n and k_n denote by t_j, x_j and k_j. The sequence $f(t_j, q_j, k_j)$ converges to $f(t^*, q^*, k^*)$ $(t_j > 0,\ t^* > 0)$. But $f(t_j, q_j, k_j) = x_j \to x^*$. That is why $f(t^*, q^*, k^*) = x^*$ and $f(-t^*, x^*, k^*) = q^*$ is determined. Since q^* is an arbitrary limit point of $\{q_n\}$, and the point $f(-t^*, x^*, k^*)$, if it exists, is determined by given t^*, x^*, k^* and has a single value, the sequence q_n converges to q^*. The proposition is proved.

Later on we'll denominate the mapping $f(\cdot, x, k) :\ \langle -T(x, k), \omega) \to X$ *k-motion* of the point x ((k, x)-motion), the image of (k, x)-motion – *k-trajectory* of the point x ((k, x)-trajectory), the image of the interval $\langle -T(x, k), 0)$ a negative, and the image of $0, \infty)$ a positive *k-semitrajectory* of the point x ((k, x)-semitrajectory). If $T(x, k) = \infty$, then let us call k-motion of the point x *whole* k-motion, and the corresponding k-trajectory – whole k-trajectory.

Let $(x_n, k_n) \to (x^*, k^*)$, $t_n \to t^*$, $t_n, t^* > 0$ and for any n the (k_n, x_n)-motion be determined in the interval $[-t_n, \infty)$, i.e. $[-t_n, \infty) \subset \langle -T(x_n, k_n), \infty)$. Then (k^*, x^*)-motion is determined in $[-t^*, \infty]$. In particular, if all (k_n, x_n)-motions are determined in $[-\bar{t}, \infty)$ $(\bar{t} > 0)$, then (k^*, x^*)-motion is determined in too. If $t_n \to \infty$ and (k_n, x_n)-motion is determined in $[-t_n, \infty)$, then (k^*, x^*)-motion is determined in $(-\infty, \infty)$ and is a whole motion. In particular, if all the (k_n, x_n)-motions are whole, then (k^*, x^*)-motion is whole too. All this is a direct consequence of the closure of the set S – of the domain of definition of extended mapping f. It should be noted that from $(x_n, k_n) \to (x^*, k^*)$ and $[-t^*, \infty) \subset \langle -T(x^*, k^*), \infty)$ does not follow that for any $\varepsilon > 0$ $[-t^* + \varepsilon, \infty) \subset \langle -T(x_n, k_n), \infty)$ for n large enough.

Let us note an important property of uniform convergence in compact intervals. Let $(x_n, k_n) \to (x^*, k^*)$ and all (k_n, x_n)-motions and correspondingly (k^*, x^*)-motion be determined in compact interval $[a, b]$. Then (k_n, x_n)-motions converge uniformly in $[a, b]$ to (k^*, x^*)-motion: $f(t, x_n, k_n) \rightrightarrows f(t, x^*, k^*)$. This is a direct consequence of continuity of the mapping $f : S \to X$

1.2 Limit Sets

Definition 1.1. Point $p \in X$ is called ω- (α-)-*limit point* of the (k, x)-motion (correspondingly of the whole (k,x)-motion), if there is such sequence $t_n \to \infty$ ($t_n \to -\infty$) that $f(t_n, x, k) \to p$ as $n \to \infty$. The totality of all ω- (α-)-limit points of (k,x)-motion is called its ω- (α-)-*limit set* and is denoted by $\omega(x,k)$ ($\alpha(x,k)$).

Definition 1.2. Set $W \subset X$ is called k-*invariant*, if from $x \in W$ follows that (k,x)-motion is whole and the whole (k,x)-trajectory lies in W. In similar way, let us call the set $V \subset X$ $(k,+)$-*invariant* ((k,*positive*)-*invariant*), if for any $x \in V,\ t > 0\ f(t,x,k)\ \in\ V$.

Proposition 1.3. The sets $\omega(x,k)$ and $\alpha(x,k)$ are k-invariant.

Proof. Let $p \in \omega(x,k),\ t_n \to \infty,\ x_n = f(t_n,x,k) \to p$. Note that (k,x_n)-motion is determined at least in $[-t_n,\infty)$. Therefore, as it was noted above, (k,p)-motion is determined in $(-\infty,\infty)$, i.e. it is whole. Let us show that the whole (k,p)-trajectory consists of ω-limit points of (k,x)-motion. Let $f(\bar{t},p,k)$ be an arbitrary point of (k,p)-trajectory. Since $t \to \infty$, from some $n$is determined a sequence $f(\bar{t}+t_n,x,k))$. It converges to $f(\bar{t},p,k)$, since $f(\bar{t}+t_n,x,k) = f(\bar{t}, f(t_n,x,k),k)$ (according to the proposition 1.1.), $f(t_n,x,k) \to p$ and $f : S \to X$ is continuous (proposition 1.2).

Now, let $q \in \alpha(x,k),\ t_n \to -\infty$ and $x_n = f(t_n,x,k) \to q$. Since (according to the definition of α-limit points) (k,x)-motion is whole, then all (k,x_n)-motions are whole too. Therefore, as it was noted, (k,q)-motion is whole. Let us show that every point $f(\bar{t},q,k)$ of (k,q)-trajectory is α-limit for (k,x)-motion. Since (k,x)-motion is whole, then the semigroup property and continuity of f in S give

$$f(\bar{t}+t_n,x,k) = f(\bar{t}, f(t_n,x,k),k) \to f(\bar{t},q,k),$$

and since $\bar{t}+t_n \to -\infty$, then $f(\bar{t},q,k)$ is α-limit point of (k,x)-motion. The proposition 1.3 is proved.

Further we need also the complete ω-limit set $\omega(k) : \omega(k) = \bigcup_{x\in X}\omega(x,k)$. The set $\omega(k)$ is k-invariant, since it is the union of k-invariant sets.

Proposition 1.4. The sets $\omega(x,k),\ \alpha(x,k)$ (the last in the case when (k,x)-motion is whole) are nonempty, closed and connected.

The proof practically literally coincides with the proof of analogous statements [6, p.356-362]. The set $\omega(k)$ can be unclosed already.

Example 1.1. (Unclosure of $\omega(k)$). Let us consider the system given by the equations $\dot{x} = y(x-1),\ \dot{y} = -x(x-1)$ in the circle $x^2+y^2 \le 1$ on the plane.

The complete ω-limit set is $\omega = \{(1,0)\}\bigcup\{(x,y)\mid x^2+y^2<1\}$. It is unclosed. The closure of coincides with the whole circle, the boundary of ω consists of two trajectories: of the fixed point $(1,0) \in \omega$ and of the loop $\{(x,y)\mid x^2+y^2=1,\ x\neq 1\} \notin \omega$

Proposition 1.5. The sets $\partial\omega(k),\ \partial\omega(k)\backslash\omega(k)$ and $\partial\omega(k)\bigcap\omega(k)$ are $(k,+)$-invariant. Thereto, if $\partial\omega(k)\setminus\omega(k) \neq \varnothing$, then $\partial\omega(k)\bigcap\omega(k) \neq \varnothing$ ($\partial\omega(k) = \overline{\omega(k)}\setminus \mathrm{int}\,\omega(k)$ is the boundary of the set $\omega(k)$).

Let us note that for the propositions 1.4 and 1.5 to be true, the compactness of X is important – for non-compact spaces analogous propositions are incorrect, generally speaking.

To study slow relaxations, we need also sets composed of ω-limit sets $\omega(x,k)$:

$$\Omega(x,k) = \{\omega(x',k)\mid \omega(x',k)\subset\omega(x,k),\ x'\in X\};$$
$$\Omega(k) = \{\omega(x,k)\mid x\in X\}, \qquad (5)$$

$\Omega(x,k)$ is a set of all ω-limit sets, lying in $\omega(x,k)$, $\Omega(k)$ is a set of ω-limit sets of all k-motions.

1.3 Convergences in the Spaces of Sets

Further we consider the connection between slow relaxations and violations of continuity of the dependencies $\omega(x,k)$, $\omega(k)$, $\Omega(x,k)$, $\Omega(k)$. Let us introduce convergences in spaces of sets and investigate the mappings continuous with respect to them. One notion of continuity, used by us, is well known (see [58] sec.18 and [59] sec.43, lower semicontinuity). Two other ones are some more "exotic". In order to reveal the resemblance and distinctions between these convergences, let us consider them simultaneously (all the statements, concerning lower semicontinuity, are variations of known ones – see [58,59]).

Let us denote the set of all nonempty subsets of X by $B(X)$, and the set of all nonempty subsets of $B(X)$ by $B(B(X))$.

Let us introduce in $B(X)$ the following proximity measures: let $p, q \in B(X)$, then

$$d(p,q) = \sup_{x\in p} \inf_{y\in q} \rho(x,y); \tag{6}$$

$$r(p,q) = \inf_{x\in p, y\in q} \rho(x,y). \tag{7}$$

The "distance" $d(p,q)$ represents "a half" of known Hausdorff metrics ([59], p.223):

$$\mathrm{dist}(p,q) = \max\{d(p,q),\ d(q,p)\}. \tag{8}$$

It should be noted that, in general, $d(p,q) \neq d(q,p)$. Let us determine in $B(X)$ converges using the introduced proximity measures. Let q_n be a sequence of points of $B(X)$. We say that q_n d-converges to $p \in B(X)$, if $d(p,q_n) \to 0$. Analogously, q_n r-converges to $p \in B(X)$, if $r(p,q_n) \to 0$. Let us notice that d-convergence defines topology in $B(X)$ with a countable base in every point and the continuity with respect to this topology is equivalent to d-continuity (λ-topology [58], p.183). As a basis of neighborhoods of the point $p \in B(X)$ in this topology can be taken, for example, the family of sets $\{q \in B(X) \mid d(p,q) < 1/n\ (n = 1,2,\ldots)\}$. The topology conditions can be easily verified, since the triangle inequality

$$d(p,s) \leq d(p,q) + d(q,s) \tag{9}$$

is true (in regard to these conditions see, for example, [60], p.19-20), r-convergence does not determine topology in $B(X)$. To prove this, let us use the following evident property of convergence in topological spaces: if $p_i \equiv p,\ q_i \equiv q$ and $s_i \equiv s$ are constant sequences of the points of topological space and $p_i \to q,\ q_i \to s$, then $p_i \to s$. This property is not valid for r-convergence. To construct an example, it is enough to take two points $x, y \in X$ ($x \neq y$) and to make $p = \{x\},\ q = \{x,y\},\ s = \{y\}$. Then $r(p,q) = r(q,s) = 0,\ r(p,s) = \rho(x,y) > 0$. Therefore $p_i \to q,\ q_i \to s,\ p_i \not\to s$, and r-convergence does not determine topology for any metric space $X \neq \{x\}$.

Introduce also a proximity measure in $B(B(X))$ – in the set of nonempty subsets of $B(X)$: let $P, Q \in B(B(X))$, then

$$D(P,Q) = \sup_{p\in P} \inf_{q\in Q} r(p,q). \tag{10}$$

Note that the formula (10) is similar to the formula (6), but instead of $\rho(x,y)$ in (10) appears $r(p,q)$. The expression (10) can be somewhat simplified by introducing the following denotations. Let $Q \in B(B(X))$. Let us define $SQ = \bigcup_{q\in Q} q, SQ \in B(X)$; then

$$D(P,Q) = \sup_{p\in P} r(p, SQ). \tag{11}$$

Let us introduce convergence in $B(B(X)$ (D-convergence): $Q_n \to P$, if $D(P, Q_n) \to 0$. D-convergence, as well as r-convergence, does not determine topology. This can be illustrated in the way similar to that used for r-convergence. Let $x, y \in x, x \neq y, P = \{\{x\}\}, Q = \{\{x, y\}\}, R = \{\{y\}\}, P_i = P, Q_i = Q$. Then $D(Q, P) = D(R, Q) = 0,\ P_i \to Q,\ Q_i \to R,\ D(R, P) = \rho(x, y) > 0, P_i \not\to R$.

Later we'll need the following criteria of convergence of sequences in $B(X)$ and in $B(B(X))$.

Proposition 1.6. (see [58]). The sequence of sets $q_n \in B(X)$ d-converges to $p \in B(X)$ if and only if $\inf_{y \in q_n} \rho(x, y) \to 0$ as $n \to \infty$ for any $x \in p$.

Proposition 1.7. The sequence of sets $q_n \in B(X)$ r-converges to $p \in B(X)$ if and only if there are such $x_n \in p$ and $y_n \in q_n$ that $\rho(x_n, y_n) \to 0$ as $n \to \infty$.

This immediately follows from the definition of r-proximity.

Before treating the criterion of D-convergence, let us prove the following topological lemma.

Lemma 1.1. Let p_n, q_n $(n = 1, 2, \ldots)$ be subsets of compact metric space X and for any n $r(p_n, q_n) > \varepsilon > 0$. Then there are such $\gamma > 0$ and an infinite set of indices J that for some number N $r(p_N, q_n) > \gamma$ for $n \in J$.

Proof. Choose in X $\varepsilon/5$-network M; let to each $q \subset X$ correspond $q^M \subset M$:

$$q^M = \left\{ m \in M \mid \inf_{x \in q} \rho(x, m) \leq \varepsilon/5 \right\}. \tag{12}$$

For any two sets $p, q \subset X$ $r(p^M, q^M) + \frac{2}{5}\varepsilon \geq r(p, q)$. Therefore $r(p_n^M, q_n^M) > 3\varepsilon/5$. Since the number of different pairs p^M, q^M is finite (M is finite), there exists an infinite set J of indices n, for which the pairs p_n^M, q_n^M coincide: $p_n^M = p^M,\ q_n^M = q^M$ as $n \in J$. For any two indices $n, l \in J$ $r(p_n^M, q_l^M) = r(p^M, q^M) > 3\varepsilon/5$, therefore $r(p_n, q_l) > \varepsilon/5$, and this fact completes the proof of the lemma. It was proved more important statement really: there exists such infinite set J of indices that for any $n, l \in J$ $r(p_n, q_l) > \gamma$ (and not only for one N).

Proposition 1.8. The sequence of sets $Q_n \in B(B(X))$ D-converges to $p \in B(X))$ if and only if for any $p \in P$ $\inf_{q \in Q} r(p, q) \to 0$.

Proof. In one direction this is evident: if $Q_n \to P$, then according to definition $D(P, Q_n) \to 0$, i.e. the upper bound by $p \in P$ of the value $\inf_{q \in Q_n} r(p, q)$ tends to zero and all the more for any $p \in P$ $\inf_{q \in Q} r(p, q) \to 0$. Now, suppose that for any $p \in P$ $\inf_{q \in Q_n} r(p, q) \to 0$. If $D(P, Q_n) \not\to 0$, then one can consider that $D(P, Q_n) > \varepsilon > 0$. Therefore (because of (11)) there are such $p_n \in P$ for which $r(p_n, SQ_n) > \varepsilon$ $\left(SQ_n = \bigcup_{q \in Q_n} q\right)$. Using the lemma 1.1, we conclude that for some N $r(p_N, SQ_n) > \gamma > 0$, i.e. $\inf_{q \in Q_n} r(p_N, q) \not\to 0$. The obtained contradiction proves the second part of the proposition 1.8.

Everywhere further, if there are no another mentions, the convergence in $B(X)$ implies d-convergence, and the convergence in $B(B(X))$ implies D-convergence, and as continuous are considered the functions with respect to these convergences.

1.4 Bifurcations of ω-limit Sets

Definition 1.3. Let us say that the system (1) possesses:
A) $\omega(x, k)$-*bifurcations*, if $\omega(x, k)$ is not continuous function in $X \times K$;

B) $\omega(k)$-*bifurcations*, if $\omega(k)$ is not continuous function in K;
C) $\Omega(x,k)$-*bifurcations*, if $\Omega(x,k)$ is not continuous function in $X \times K$;
D) $\Omega(k)$-*bifurcations*, if $\Omega(k)$ is not continuous function in K.

The points of $X \times K$ or K, in which the functions $\omega(x,k)$, $\omega(k)$, $\Omega(x,k)$, $\Omega(k)$ are not d- or not D-continuous, we call *the points of bifurcation*. The considered discontinuities in the dependencies $\omega(x,k)$, $\omega(k)$, $\Omega(x,k)$, $\Omega(k)$ could be also called *"explosions"* of ω-limit sets (compare with the explosion of the set of non-wandering points in differential dynamics [26, sec. 6.3., p.185-192], which, however, is a violation of semidiscontinuity from above).

Proposition 1.9. A). If the system (1) possesses $\Omega(k)$-bifurcations, then it possesses $\Omega(x,k)$-, $\omega(x,k)$- and $\omega(x,k)$-bifurcations.
B) If the system (1) possesses $\Omega(x,k)$-bifurcations, then it possesses $\omega(x,k)$-bifurcations too.
C) If the system (1) possesses $\omega(k)$-bifurcations, then it possesses $\omega(x,k)$-bifurcations.

It is convenient to illustrate the proposition 1.9 by the scheme (the word "bifurcation" is omitted on the scheme):

$$\begin{array}{ccccc} \swarrow & & \Omega(k) & & \searrow \\ \Omega(x,k) & & & & \omega(k) \\ & \searrow & \omega(x,k) & \swarrow & \end{array} \qquad (13)$$

Proof. Let us begin from the point C. Let the system (1) (family of semiflows) possess $\omega(k)$-bifurcations. This means that there are such $k^* \in K$ (point of bifurcation), $\varepsilon > 0$, $x^* \in \omega(k^*)$ and sequence $k_n \in K$, $k_n \to k^*$, for which $\inf_{y \in \omega(x_0,k_n)} \rho(x^*, y) > \varepsilon$ for any n (according to the proposition 1.6). The point x^* belongs to some $\omega(x_0, k^*)$ ($x_0 \in X$). Note that $\omega(x_0, k_n) \subset \omega(k_n)$, consequently, $\inf_{y \in \omega(k_n)} \rho(x^*, y) > \varepsilon$, therefore the sequence $\omega(x_0, k_n)$ does not converge to $\omega(x_0, k^*)$ – there exist $\omega(x,k)$-bifurcations, and the point of bifurcation is (x_0, k^*).

Prove the statement of the point B. Let the system (1) possess $\Omega(x,k)$-bifurcations. Then, (according to the proposition 1.8) there are such $(x^*, k^*) \in X \times K$ (the point of bifurcation), $\omega(x_0, k^*) \subset \omega(x^*, k^*)$ and sequence $(x_n, k_n) \to (x^*, k^*)$ that

$$r(\omega(x_0, k^*), S\,\Omega(x_n, k_n)) > \varepsilon > 0 \text{ for any } n.$$

But the last means that $r(\omega(s_0, k^*), \omega(x_n, k_n)) > \varepsilon > 0$ and, consequently,

$$\inf_{y \in \omega(x_n, k_n)} \rho(\xi, y) > \varepsilon \text{ for any } \xi \in \omega(x^0, k^*).$$

Since $\xi \in \omega(x^*, k^*)$, from this follows the existence of $\omega(x,k)$-bifurcations ((x^*, k^*) is the point of bifurcation).

Prove the statement of the point A. Let the system (1) possess $\Omega(k)$-bifurcations. Then there are $k^* \in K$ (the point of bifurcation), $\varepsilon > 0$ and sequence of points $k_n, k_n \to k^*$, for which $D(\Omega(k^*), \Omega(k_n)) > \varepsilon$ for any n, that is for any n there is such $x_n \in X$ that $r(\omega(x_n, k^*), \omega(k_n)) > \varepsilon$ (according to (11)). But by the lemma 1.1 there are such $\gamma > 0$ and natural N that for infinite set J of indices $r(\omega(x_N, k^*), \omega(k_n)) > \gamma$ for $n \in J$. All the more $r(\omega(x_N, k^*), \omega(x_N, k_n)) > \gamma$ ($n \in J$), consequently, there are $\Omega(x,k)$-bifurcations:

$$\begin{gathered} (x_N, k_n) \to (x_N, k^*) \text{ as } n \to \infty,\ n \in J; \\ D(\Omega(x_N, k^*),\ \Omega(x_N, k_n)) = \sup\nolimits_{\omega(x,k^*) \subset \Omega(x_N,k^*)} r(\omega(x, k^*), \omega(x_N, k_n)) \geq \\ \geq r(\omega(x_N, k^*),\ \omega(x_N, k_n)) > \gamma. \end{gathered}$$

The point of bifurcation is (x_N, k^*).

We are only to show that if there are $\Omega(k)$-bifurcations, then $\omega(k)$-bifurcations exist. Let us prove this. Let the system (1) possess $\Omega(k)$-bifurcations. Then, as it was shown just above, there are such $k^* \in K, x^* \in X,\ \gamma > 0$ $(x^* = x_N)$ and a sequence of points $k_n \in K$ that $k_n \to k^*$ and $r(\omega(x^*, k^*),\ \omega(k_n)) > \gamma$. All the more, for any $\xi \in \omega(x^*, k^*)$ $\inf_{y\in\omega(k_n)} \rho(\xi, y) > \gamma$, therefore $d(\omega(k^*), \omega(k_n)) > \gamma$ and there are $\omega(k)$-bifurcations (k^* is the point of bifurcation). The proposition 1.9 is proved.

Proposition 1.10. The system (1) possesses $\Omega(x, k)$-bifurcations if and only if $\omega(x, k)$ is not r-continuous function in $X \times K$.

Proof. Let the system (1) possess $\Omega(x, k)$-bifurcations, then there are $(x^*, k^*) \in X \times K$, the sequence $(x_n, k_n) \in X \times K, (x_n, k_n) \to (x^*, k^*)$ for which for any n

$$D(\Omega(x^*, k^*), \Omega(x_n, k_n)) > \varepsilon > 0.$$

The last means that for any n there is $x_n^* \in X$ for which $\omega(x_n^*, k^*) \subset \omega(x^*, k^*)$, and $r(\omega(x_n^*, k^*), \omega(x_n, k_n)) > \varepsilon$. From the lemma 1.1 follows the existence of such $\gamma > 0$ and natural N that for infinite set J of indices $r(\omega(x_N^*, k^*), \omega(x_n, k_n)) > \gamma$ as $n \in J$. Let x_0^* be an arbitrary point of $\omega(x_N^*, k^*)$. As it was noted already, (k^*, x_0^*)-trajectory lies in $\omega(x_N^* < k^*)$ and because of the closure of the last $\omega(x_0^*, k^*) \subset \omega(x_N^*, k^*)$. Therefore $r(\omega(x_n, k_n)) > \gamma$ as $n \in J$. As $x_0^* \in \omega(x^*, k^*)$, there is such sequence $t_i \to \infty, t_i > 0$, that $f(t_i, x^*, k^*) \to x_0^*$ as $i \to \infty$. Using the continuity of f, choose for every i such $n(i) \in J$ that $\rho(f(t_i, x^*, k^*), f(t_i, x_{n(i)}, k_{n(i)})) < 1/i$. Denote $f(t_i, x_{n(i)}, k_{n(i)}) = x_i',\ k_{n(i)} = k_i'$. Note that $\omega(x_i', k_i') = \omega(x_{n(i)}, k_{n(i)})$. Therefore for any i $r(\omega(x_0^*, k^*), \omega(x_i', k_i')) > \gamma$. Since $(x_i', k_i') \to (x_0^*, k^*)$, we conclude that $\omega(x, k)$ is not r-continuous function in $X \times K$.

Let us emphasize that the point of $\Omega(x, k)$-bifurcations can be not the point of r-discontinuity.

Now, suppose that $\omega(x, k)$ is not r-continuous in $X \times K$. Then there exist $(x^*, k^*) \in X \times K$, sequence of points $(x_n, k_n) \in X \times K$, $(x_n, k_n) \to (x^*, k^*)$, and $\varepsilon > 0$, for which $r(\omega(x^*, k^*), \omega(x_n, k_n)) > \varepsilon$ for any n. But, according to (11), from this follows that $D(\Omega(x^*, k^*), \Omega(x_n, k_n)) > \varepsilon$ for any n. Therefore (x^*, k^*) is the point of $\Omega(x, k)$-bifurcation. The proposition 1.10 is proved.

The $\omega(k)$- and $\omega(x, k)$-bifurcations can be called bifurcations with appearance of new ω-limit points, and $\Omega(k)$- and $\Omega(x, k)$-bifurcations with appearance of ω-limit sets. In the first case there is such sequence of points k_n (or (x_n, k_n)), converging to the point of bifurcation k^* (or (x^*, k^*)) that there is such point $x_0 \in \omega(k^*)$ (or $x_0 \in \omega(x^*, k^*)$) which is removed away from all $\omega(k_n)$ $(\omega(x_n, k_n))$ more than at some $\varepsilon > 0$. It could be called "new" ω-limit point. In the second case, as it was shown, the existence of bifurcations is equivalent to existence of a sequence of the points k_n (or $(x_n, k_n) \in X \times K$), converging to the point of bifurcation k^* (or (x^*, k^*)), together with existence of some set $\omega(x_0, k^*) \subset \omega(k^*)$ $(\omega(x_0, k^*) \subset \omega(x^*, k^*))$, being r-removed from all $\omega(k_n)$ $(\omega(x_n, k_n))$ more than at $\gamma > 0$: $\rho(x, y) > \gamma$ for any $x \in \omega(x_0, k^*)$ and $y \in \omega(k_n)$. It is natural to call the set $\omega(x_0, k^*)$ "new" ω-limit set. A question arises: are there bifurcations with appearance of new ω-limit points, but without appearance of new ω-limit sets? The following example gives positive answer to this question.

Example 1.2. ($\omega(x, k)$-, but not $\Omega(x, k)$-bifurcations). Consider at first the system, given in the cone $x^2 + y^2 \leq z^2$, $0 \leq z \leq 1$ by differential equations (in cylindrical coordinates)

$$\dot{r} = r(2z - r - 1)^2 - 2r(1 - r)(1 - z);$$

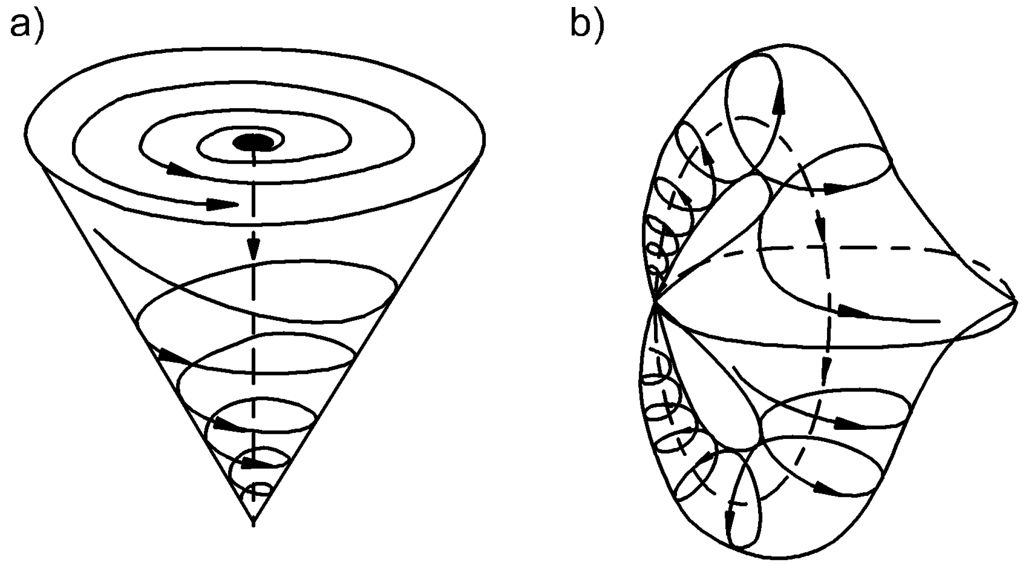

Fig.1. $\omega(x,k)$-, but not $\Omega(x,k)$-bifurcations:
a - phase portrait of the system (14);
b - the same portrait after gluing all fixed points.

$$\begin{aligned} \dot{\varphi} &= r\cos\varphi + 1; \\ \dot{z} &= -z(1-z)^2. \end{aligned} \tag{14}$$

The solutions of (14) under initial conditions $0 \le z(0) \le 1,\ 0 \le r(0) \le z(0)$ and arbitrary φ tend as $t \to \infty$ to their unique ω-limit point – to the equilibrium position $z = r = 0$. If $0 < r(0) < 1$, then as $t \to \infty$ the solution tends to the circumference $z = r = 1$. If $z(0) = 1,\ r(0) = 0$, then ω-limit point is unique: $z = 1,\ r = 0$. If $z(0) = r(0) = 1$, then ω-limit point is also unique: $z = r = 1,\ \varphi = \pi$ (see fig. 1). Thus,

$$\omega(r_0, \varphi_0, z_0) = \begin{cases} (z = r = 0), & \text{if } z_0 < 1; \\ \{(r, \varphi, z) \mid r = z = 1\}, & \text{if } z_0 = 1,\ r_0 \ne 0, 1; \\ (z = r = 1),\ \varphi = \pi, & \text{if } z_0 = r_0 = 1; \\ (r = 0, z = 1), & \text{if } z_0 = 1,\ r_0 = 0. \end{cases}$$

Consider, the sequence of points of the cone $(r_n, \varphi_n, z_n) \to (r^*, \varphi^*, 1),\ r^* \ne 0, 1$ and $z_n < 1$ for all n. For all points of the sequence ω-limit set includes one point, and for $(r^*, \varphi, 1)$ the set includes circumference. If all the positions of equilibrium were identified, then there would be $\omega(x,k)$-, but not $\omega(x,k)$-bifurcations.

The correctness of the identification procedure should be grounded. Let the studied semiflow f have fixed points $x_i, \ldots, x_n$. Define a new semiflow $\tilde{f}$ as follows:

$$\tilde{X} = X \setminus \{x_i, \ldots, x_n\} \bigcup \{x^*\}$$

is a space obtained from X when the points $x_i, \ldots, x_n$ are deleted and a new point x^* is added. Let us give metrics over $\tilde{X}$ as follows: let $x, y \in \tilde{X},\ x \ne x^*$,

$$\tilde{\rho}(x,y) = \begin{cases} \min\left\{\rho(x,y), \min_{1\le j\le n} \rho(x,x_j) + \min_{1\le j\le n} \rho(y,x_j)\right\}, & \text{if } y \ne x^*; \\ \min_{1\le j\le n} \rho(x,x_j), & \text{if } y = x^*. \end{cases}$$

Let $\tilde{f}(t,x) = f(t,x)$ if $x \in X \bigcap \tilde{X}$, $\tilde{f}(t,x^*) = x^*$.

Lemma 1.2. The mapping $\tilde{f}$ determines semiflow in $\tilde{X}$.

Proof. Injectivity and semigroup property are evident from the corresponding properties of f. If $x \in X \bigcap \tilde{X}$, $t \geq 0$ then the continuity of $\tilde{f}$ in the point (t,x) follows from the fact that $\tilde{f}$ coincides with f in some neighbourhood of this point. The continuity of $\tilde{f}$ in the point (t,x^*) follows from the continuity of f and the fact that any sequence converging in $\tilde{X}$ to x^* can be divided into finite number of sequences, each of them being either (a) a sequence of points $X \bigcap \tilde{X}$, converging to one of x_j or (b) a constant sequence, all elements of which are x^* and some more, maybe, a finite set. Mapping $\tilde{f}$ is a homeomorphism, since it is continuous and injective, and $\tilde{X}$ is compact.

Proposition 1.11. Let each trajectory lying in $\omega(k)$ be recurrent for any k. Then the existence of $\omega(x,k)$- ($\omega(k)$-)-bifurcations is equivalent to the existence of $\Omega(x,k)$- ($\Omega(k)$-)-bifurcations. More exact,

A) if $(x_n,k_n) \to (x^*,k^*)$ and $\omega(x_n,k_n) \not\to \omega(x^*,k^*)$, then $\Omega(x_n,k_n) \not\to \Omega(x^*,k^*)$[1],

B) if $k_n \to k^*$ and $\omega(k_n) \not\to \omega(k^*)$, then $\Omega(k_n) \not\to \Omega(k^*)$.

Proof. A) Let $(x_n,k_n) \to (x^*,k^*)$, $\omega(x_n,k_n) \not\to \omega(x^*,k^*)$. Then, according to the proposition 1.6, there is such $\tilde{x} \in (x^*,k^*)$ that $\inf_{y\in\omega(x_n,k_n)} \rho(\tilde{x},y) \not\to 0$. Therefore from $\{(x_n,k_n)\}$ we can choose a subsequence (denote it as $\{(x_m,k_m)\}$) for which there exists such $\varepsilon > 0$ that $\inf_{y\in\omega(x_m,k_m)} \rho(\tilde{x},y) > \varepsilon$ for any $m = 1,2 < \dots$. Denote by L the set of all limit points of sequences of the kind $\{y_m\}$, $y_m \in \omega(x_m,k_m)$. The set L is closed and k^*-invariant. Note that $\rho^*(\tilde{x},L) \geq \varepsilon$. Therefore $\omega(\tilde{x},k^*) \bigcap L = \varnothing$ as $\omega(\tilde{x},k^*)$ is a minimal set (Birkhoff's theorem, see [6], p.404). From this follows the existence of such $\delta > 0$ that $r(\omega(\tilde{x},k^*),L) > \delta$ and from some M $r(\omega(\tilde{x},k^*),\ (x_m,k_m)) > \delta/2$ (when $m > M$). Therefore (proposition 1.8) $\Omega(x_m,k_m) \not\to \Omega(x^*,k^*)$.

B) The proof practically literally (it should be substituted $\omega(k)$ for $\omega(x,k)$) coincides with that for the part A.

Corollary 1.1. Let for every pair $(x,k) \in X \times K$ the ω-limit set be minimal: $\Omega(x,k) = \{\omega(x,k)\}$. Then the statements A, B of the proposition 1.11 are true.

Proof. According to one of Birkhoff's theorems (see [6], p.402), each trajectory lying in minimal set is recurrent. Therefore the proposition 1.11 is applicable.

2 Slow Relaxations

2.1 Relaxation Times

The principal object of our consideration is *the relaxation time.*

Proposition 2.1. For any $x \in X$, $k \in K$ and $\varepsilon > 0$ the numbers $\tau_i(x,k,\varepsilon)$ and $\eta_i(x,k,\varepsilon)$ $(i = 1,2,3)$ are defined. The inequalities $\tau_i \geq \eta_i$, $\tau_1 \leq \tau_2 \leq \tau_3$, $\eta_1 \leq \eta_2 \leq \eta_3$ are true.

Proof. If τ_i, η_i are defined, then the validity of inequalities is evident ($\omega(x,k) \subset \omega(k)$, the time of the first entry in ε-neighbourhood of the set of limit points is included into the time of being outside of this neighbourhood, and the last is not larger than the time of final entry in it). The numbers τ_i, η_i are definite (bounded): there are

[1]Let us recall that everywhere further, if there are no another mentions, the convergence in $B(X)$ implies d-convergence, and the convergence in $B(B(X))$ implies D-convergence, and continuity is considered as continuity with respect to these convergences

$t_n \in [0, \infty)$, $t_n \to \infty$ and $y \in \omega(x,k)$, for which $f(t_n,x,k) \to y$ and from some n $\rho(f(t_n,x,k),y) < \varepsilon$, therefore the sets $\{t > 0 \mid \rho^*(f(t,x,k),\omega(x,k)) < \varepsilon\}$ and $\{t > 0 \mid \rho^*(f(t,x,k),\omega(k)) < \varepsilon\}$ are nonempty. Since X is compact, there is such $t(\varepsilon) > 0$ that for $t > t(\varepsilon)$ $\rho^*(f(t,x,k),\omega(x,k)) < \varepsilon$. Really, let us suppose the contrary: there are such $t_n > 0$ that $t_n \to \infty$ and $\rho^*(f(t_n,x,k),\omega(x,k)) > \varepsilon$. Let us choose from the sequence $f(t_n,x,k)$ a convergent subsequence and denote its limit x^*; x^* satisfies the definition of ω-limit point of (k,x)-motion, but it lies outside of $\omega(x,k)$. The obtained contradiction proves the required, consequently, τ_3 and η_3 are defined. According to the proved, the sets

$$\{t > 0 \mid \rho^*(f(t,x,k),\omega(x,k)) \geq \varepsilon\},$$
$$\{t > 0 \mid \rho^*(f(t,x,k),\omega(k)) \geq \varepsilon\}$$

are bounded. They are measurable because of the continuity with respect to t of the functions $\rho^*(f(t,x,k),\omega(x,k))$ and $\rho^*(f(t,x,k),\omega(k))$. The proposition is proved. Note that the existence (finiteness) of $\tau_{2,3}$ and $\eta_{2,3}$ is associated with the compactness of X.

Definition 2.1. We say that the system (1) possesses τ_i- (η_i-)-*slow relaxations*, if for some $\varepsilon > 0$ the function $\tau_i(x,k,\varepsilon)$ (correspondingly $\eta_i(x,k,\varepsilon)$) is not bounded above in $X \times K$.

Proposition 2.2. For any semiflow (k is fixed) the function $\eta_1(x,\varepsilon)$ is bounded in X for every $\varepsilon > 0$.

Proof. Suppose the contrary. Then there is such sequence of points $x_n \in X$ that for some $\varepsilon > 0$ $\eta_1(x_n,\varepsilon) \to \infty$. Using the compactness of X and, if it is needed, choosing a subsequence, assume that $x_n \to x^*$. Let us show that for any $t > 0$ $\rho^*(f(t,x^*),\omega(k)) > \varepsilon/2$. Because of the property of uniform continuity on limited segments there is such $\delta = \delta(\tau) > 0$ that $\rho(f(t,x^*),f(t,x)) < \varepsilon/2$ if $0 \leq t \leq \tau$ and $\rho(x,x^*) < \delta$. Since $\eta_1(x_n,\varepsilon) \to \infty$ and $x_n \to x^*$, there is such N that $\rho(x_N,x^*) < \delta$ and $\eta_1(x_N,\varepsilon) > \tau$, i.e. $\rho^*(f(t,x_N),\omega(k)) \geq \varepsilon$ under $0 \leq t \leq \tau$. From this we obtain the required: for $0 \leq t \leq \tau$ $\rho^*(f(t,x^*),\omega(k)) > \varepsilon/2$ or $\rho^*(f(t,x^*),\omega(k)) > \varepsilon/2$ for any $t > 0$, since τ was chosen arbitrarily. This contradicts to the finiteness of $\eta_1(x^*,\varepsilon/2)$ (proposition 2.1). The proposition 2.2 is proved.

For $\eta_{2,3}$ and $\tau_{1,2,3}$ does not exist proposition analogous to the proposition 2.2 – slow relaxations are possible for one semiflow too.

Example 2.1. (η_2-slow relaxations for one semiflow). Let us consider on the plane in the circle $x^2 + y^2 \leq 1$ a system given in the polar coordinates by the equations

$$\begin{aligned} \dot{r} &= -r(1-r)(r \ \cos\varphi + 1); \\ \dot{\varphi} &= r \ \cos\varphi + 1. \end{aligned} \tag{15}$$

Total ω-limit set consists of two fixed points $r = 0$ and $r = 1$, $\varphi = \pi$ (fig. 2,a).

The following series of simple examples is given to demonstrate the existence of slow relaxations of some kinds without some other kinds.

Example 2.2. (η_3- but not η_2-slow relaxations). Let us rather modify the previous example, substituting unstable limit cycle for the boundary loop:

$$\begin{aligned} \dot{r} &= -r(1-r); \\ \dot{\varphi} &= 1. \end{aligned} \tag{16}$$

Now the total ω-limit set includes the whole boundary circumference and the point $r = 0$ (fig .2,b), the time of the system being outside of its ε-neighborhood is limited for any $\varepsilon > 0$. Nevertheless, $\eta_3((r,\varphi),1/2) \to \infty$ as $r \to 1$, $r \neq 1$

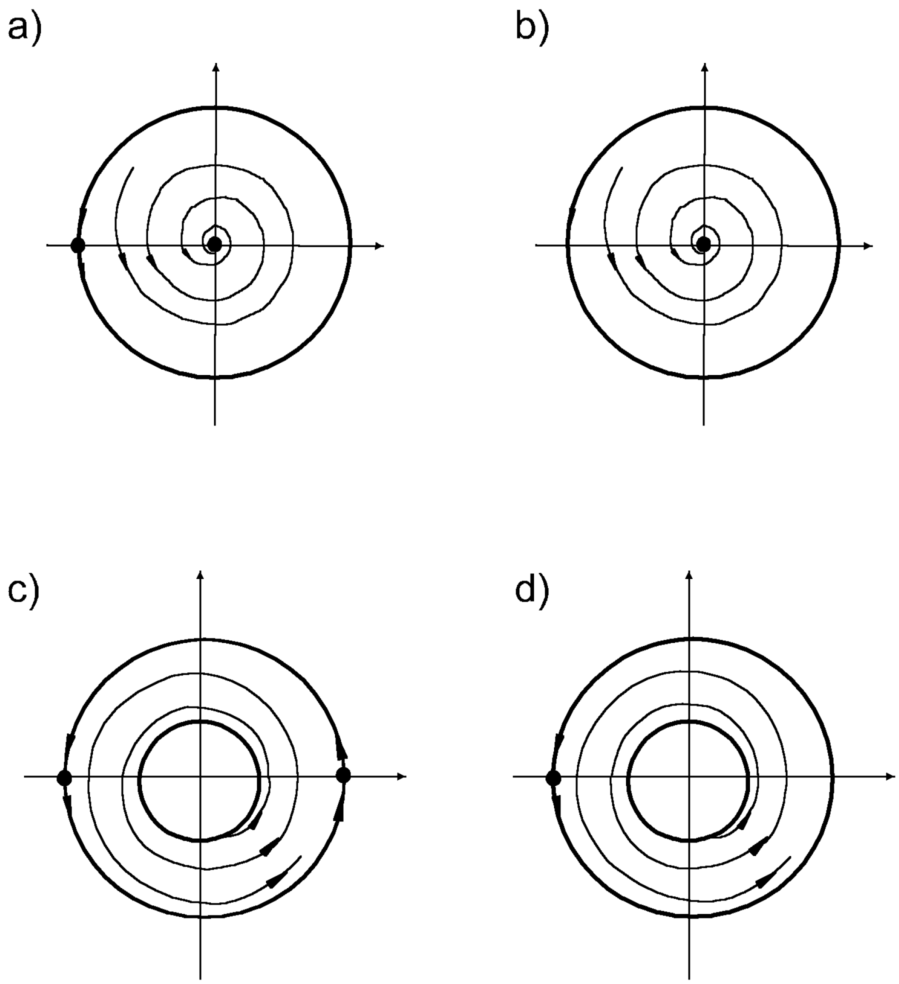

Fig.2. Phase portraits of the systems:
a - (15); b - (16); c - (17); d - (18).

Example 2.3. (τ_1 – but not $\eta_{2,3}$-slow relaxations). Let us analyze in the ring $\frac{1}{2} \leq x^2 + y^2 \leq 1$ a system given by differential equations in polar coordinates

$$\dot{r} = (1 - r)(r\cos\varphi + 1)(1 - r\cos\varphi);$$
$$\dot{\varphi} = (r\cos\varphi + 1)(1 - r\cos\varphi). \qquad (17)$$

In this case the total ω-limit set is the whole boundary circumference $r = 1$ (fig. 2,c). Under $r = 1,\ \varphi \to \pi,\ \varphi > \pi\ \tau_1(r, \varphi, 1/2) \to \infty$ since for these points $\omega(r, \varphi) = \{(r = 1,\ \varphi = 0)\}$.

Example 2.4. (τ_3 – but not $\tau_{1,2}$ – and not η_3-slow relaxations). Let us modify the preceding example of the system in the ring, leaving only one equilibrium point on the boundary circumference $r = 1$:

$$\dot{r} = (1 - r)(r\cos\varphi + 1);$$
$$\dot{\varphi} = r\cos\varphi + 1. \qquad (18)$$

In this case under $r = 1,\ \varphi \to \pi,\ \varphi \to \pi\ \tau_3((r, \varphi), 1/2) \to \infty$ and $\tau_{1,2}$ remain limited for any fixed $\varepsilon > 0$, because for these points $\omega(r, \varphi) = \{(r = 1,\ \varphi = \pi)\}$ (fig. 2,d). $\eta_{2,3}$ are limited, since the total ω-limit set is the circumference $r = 1$.

Example 2.5. (τ_2 – but not τ_1 – and not η_2-slow relaxations). We could not find a simple example on the plane without using the lemma 1.2. Consider at first a semiflow in the circle $x^2 + y^2 \leq 2$ given by the equations

$$\dot{r} = -r(1 - r)^2[(r\cos\varphi + 1)^2 + r^2\sin\varphi];$$
$$\dot{\varphi} = (r\cos\varphi + 1)^2 + r^2\sin^2\varphi. \qquad (19)$$

ω-limit sets of this system are as follows (fig. 3,a):

$$\omega(r_0, \varphi_0) = \begin{cases} \text{circumference } r = 1, & \text{if } r_0 > 1; \\ \text{point } (r = 1,\ \varphi = \pi), & \text{if } r_0 = 1; \\ \text{point } (r = 0), & \text{if } r_0 < 1. \end{cases}$$

Let us identify the fixed points $(r = 1, \varphi = \pi)$ and $(r = 0)$ (fig. 3,b). We obtain that under $r \to 1,\ r < 1\ \tau_2(r, \varphi, 1/2) \to \infty$, although τ_1 remains bounded as well as η_2. However, η_3 is unbounded.

The majority of the above examples is represented by nonrough systems, and there are serious reasons for this nonroughness. In rough systems on a plane $\tau_{1,2,3}$- and η_3-slow relaxations can occur only simultaneously (see subsection 3.3).

2.2 Slow Relaxations and Bifurcations of ω-limit Sets

In the simplest situations the connection between slow relaxations and bifurcations of ω-limit sets is evident. We should mention the case when the motion tending to its ω-limit set is retarded near unstable equilibrium position. In general case the situation becomes more complicated at least because there are several relaxation times (and consequently several corresponding kinds of slow relaxations). Except that, as it will be shown below, bifurcations are not a single possible reason of slow relaxation appearance. Nevertheless, for the time of the first hit (both for the proper τ_1 and for the non-proper η_1) the connection between bifurcations and slow relaxations is manifest.

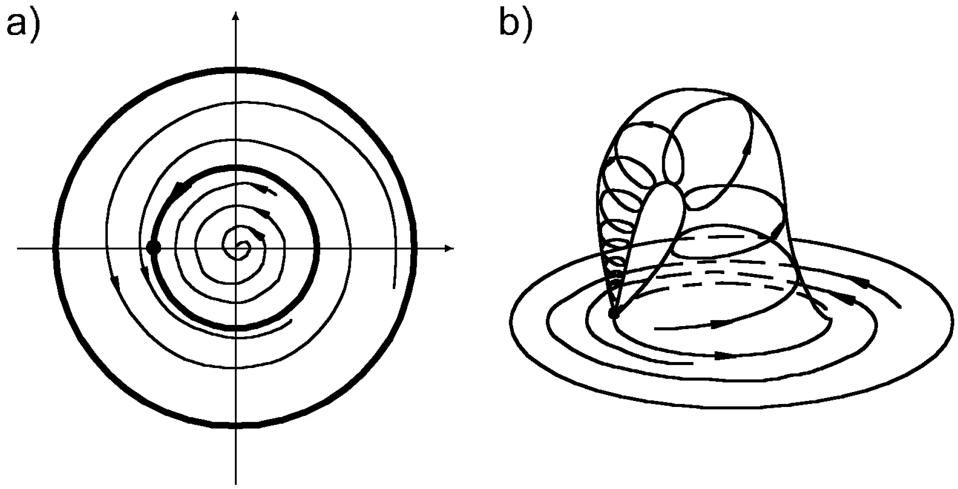

Fig.3. Phase portrait of the system (19):
a - without gluing fixed points; b - after gluing.

Theorem 2.1. The system (1) possesses τ_1-slow relaxations if and only if it possesses $\Omega(x,k)$-bifurcations.

Proof. Let the system possess $\Omega(x,k)$-bifurcations, (x^*,k^*) be the point of bifurcation. This means that there are such $x' \in X, \varepsilon > 0$ and sequence of points $(x_n,k_n) \in X \times K$, for which $\omega(x',k^*) \subset \omega(x^*,k^*)$, $(x_n,k_n) \to (x^*,k^*)$, and $r(\omega(x',k^*),\omega(x_n,k_n)) > \varepsilon$ for any n. Let $x_0 \in \omega(x',k^*)$. Then $\omega(x_0,k^*) \subset \omega(x',k^*)$ and $r(\omega(x_0,k^*),\omega(x_n,k_n)) > \varepsilon$ for any n. Since $x_0 \in \omega(x^*,k^*)$, there is such sequence $t_i > 0$, $t \to \infty$, for which $f(t_i,x^*,\ k^*) \to x_0$. As for every i $f(t_i,x_n,k_n) \to f(t_i,x^*,k^*)$, then there is such sequence $n(i)$ that $f(t_i,x_{n(i)},k_{n(i)}) \to x_0$ as $i \to \infty$. Denote $k_{n(i)}$ as k'_i and $f(t_i,x_{n(i)},k_{n(i)})$ as y_i. It is evident that $\omega(y,k'_i) = \omega(x_{n(i)},k_{n(i)})$. Therefore $r(\omega(x',k^*),\omega(y_i,k'_i)) > \varepsilon$.

Let us show that for any $\tau > 0$ there is such i that $\tau_1(y_i,k'_i,\varepsilon/2) > \tau$. To do that, let us use the property of uniform continuity of f on compact segments and choose such $\delta > 0$ that $\rho(f(t,x_0,k^*),\ f(t,y_i,k'_i)) < \varepsilon/2$ if $0 \le t \le \tau,\ \rho(x_0,y_i)+\rho_K(k^*,k'_i) < \delta$. The last inequality is true from some i_0 (when $i > i_0$), since $y_i \to x_0,\ k'_i \to k^*$. For any $t \in (-\infty,\infty)$ $f(t,x_0,k^*) \in \omega(x',k^*)$, consequently, $\rho^*(f(t,y_i,k'_i),\omega(y_i,k'_i)) > \varepsilon/2$ for $i > i_0,\ 0 \le t \le \tau$, therefore for these i $\tau_1(y_i,k'_i,\varepsilon/2) > \tau$. The existence of τ_1-slow relaxations is proved.

Now, let us suppose that there are τ_1-slow relaxations: there can be found such a sequence $(x_n,k_n) \in X \times K$ that for some $\varepsilon > 0$ $\tau_1(x_n,k_n,\varepsilon) \to \infty$. Using the compactness of $X \times K$, let us choose from this sequence a convergent one, preserving the denotations: $(x_n,k_n) \to (x^*,k^*)$. For any $y \in \omega(x^*,k^*)$ there is such $n = n(y)$ that when $n > n(y)$ $\rho^*(y,\omega(x_n,k_n)) > \varepsilon/2$. Really, as $y \in \omega(x^*,k^*)$, there is such $t > 0$ that $\rho(f(t,x^*,k^*),y) < \varepsilon/4$. Since $(x_n,k_n) \to (x^*,k^*)$, $\tau_1(x_n,k_n,\varepsilon) \to \infty$, there is such n (we will denote it by $n(y)$) that for $n > n(y)$ $\rho^*(\bar{t},x_n,k_n)) < \varepsilon/4$, $\tau_1(x_n,k_n,\varepsilon) > t$. Thereby, since $\rho^*(f(\bar{t},x_n,k_n),\omega(x_n,k_n)) > \varepsilon$, then $\rho^*(f(\bar{t},x^*,k^*),\omega(x_n,k_n)) > 3\varepsilon/4$, and, consequently, $\rho^*(y,\omega(x_n,k_n)) > \varepsilon/2$. Let $y_i,\dots,y_m$ be $\varepsilon/4$-network in $\omega(x^*,k^*)$. Let $N = \max\ n(y_i)$. Then for $n > N$ and for any i $(1 \le i \le m)$ $\rho^*(y_i,\omega(x_n,k_n)) > \varepsilon/2$. Consequently for any $y \in \omega(x^*,k^*)$ for $n > N$ $\rho^*(y,\omega(x_n,k_n) > \varepsilon/4$, i.e. for $n > N$ $r(\omega(x^*,k^*),\omega(x_n,k_n)) > \varepsilon/4$. The existence of $\Omega(x,k)$-bifurcations is proved (according to the proposition 1.8.). Using the theorem 2.1 and the proposition 1.10 we obtain the following theorem.

Theorem 2.1′. The system (1) possesses τ_1-slow relaxations if and only if $\omega(x,k)$ is not r-continuous function in $X \times K$.

Theorem 2.2. The system (1) possesses η_1-slow relaxations if and only if it possesses $\Omega(k)$-bifurcations.

Proof. Let the system possess $\Omega(k)$-bifurcations. Then (according to the proposition 1.8) there is such sequence of parameters $k_n \to k^*$ that for some $\omega(x^*,k^*) \in \Omega(k^*)$ and $\varepsilon > 0$ for any n $r(\omega(x^*,k^*),\omega(k_n)) > \varepsilon$. Let $x_0 \in \omega(x^*,k^*)$. Then for any n and $t \in (-\infty,\infty)$ $\rho^*(f(t,x_0,k^*),\omega(k_n)) > \varepsilon$ because $f(t,x_0,k^*) \in \omega(x^*,k^*)$. Let us prove that $\eta_1(x_0,k_n,\varepsilon/2) \to \infty$ as $n \to \infty$. To do this, use the uniform continuity of f on compact segments and for any $\tau > 0$ find such $\delta = \delta(\tau) > 0$ that $\rho(f(t,x_0,k^*),f(t,x_0,k_n)) < \varepsilon/2$ if $0 \le t \le \tau$ and $\rho_K(k^*,k_n) < \delta$. Since $k_n \to k^*$, there is such $N = N(\tau)$ that for $n > N$ $\rho_K(k_n,k) < \delta$. Therefore for $n > N$, $0 \le t \le \tau$ $\rho^*(f(t,x_0,k_n),\omega(k_n)) > \varepsilon/2$. The existence of η_1-slow relaxations is proved.

Now, suppose that there exist η_1-slow relaxations: there are such $\varepsilon > 0$ and sequence $(x_n,k_n) \in X \times K$ that $\eta_1(x_n,k_n,\varepsilon) \to \infty$. Use the compactness of $X \times K$ and turn to converging subsequence (retaining the same denotations): $(x_n,k_n) \to (x^*,k^*)$. Using the way similar to the proof of the theorem 2.1, let us show that for any $y \in \omega(x^*,k^*)$ there is such $n = n(y)$ that if $n > n(y)$, then $\rho^*(y,\omega(k_n)) > \varepsilon/2$. Really, there is such $\tilde{t} > 0$ that $\rho(f(\tilde{t},x^*,k^*),y) < \varepsilon/4$. As $\eta_1(x_n,k_n,\varepsilon) \to \infty$ and $(x_n,k_n) \to (x^*,k^*)$, there is such $n = n(y)$ that for $n > n(y)$ $\rho(f(\tilde{t},x^*,k^*),f(\tilde{t},x_n,k_n)) < \varepsilon/4$ and $\eta_1(x_n,k_n,\varepsilon) > \tilde{t}$. Thereafter we obtain

$$\rho^*(y,\omega(k_n)) \ge \\ \ge \rho^*(f(t,x_n,k_n),\omega(k_n)) - \rho(y,f(\tilde{t},x^*,k^*)) - \rho(f(\tilde{t},x^*,k^*),f(\tilde{t},x_n,k_n)) > \varepsilon/2.$$

Further the reasonings about $\varepsilon/4$-network of the set $\omega(x^*,k^*)$ (as in the proof of the theorem 2.1) lead to the inequality $r(\omega(x^*,k^*),\omega(k_n)) > \varepsilon/4$ for n large enough. On account of the proposition 1.8 the existence of $\Omega(k)$-bifurcations is proved, therefore is proved the theorem 2.2.

Theorem 2.3. If the system (1) possesses $\omega(x,k)$-bifurcations then it possesses τ_2-slow relaxations.

Proof. Let the system (1) possess $\omega(x,k)$-bifurcations: there is such sequence $(x_n,k_n) \in X \times K$ and such $\varepsilon > 0$ that $(x_n,k_n) \to (x^*,k^*)$ and

$$\rho^*(x',\omega(x_n,k_n)) > \varepsilon \text{ for any } n \text{ and some } x' \in \omega(x^*,k^*).$$

Let $t > 0$. Define the following auxiliary function:

$$\Theta(x^*,x',t,\varepsilon) = \mathrm{mes}\{t' \ge 0 \mid t' \le t,\ \rho(f(t',x^*,k^*),x') < \varepsilon/4\}, \tag{20}$$

$\Theta(x^*,x',t,\varepsilon)$ is "the time of dwelling" of (k^*,x^*)-motion in $\varepsilon/4$-neighbourhood of x over the time segment $[0,t]$. Let us prove that $\Theta(x^*,x',t,\varepsilon) \to \infty$ as $t \to \infty$. We will need the following corollary of continuity of f and compactness of X

Lemma 2.1. Let $x_0 \in X,\ k \in K,\ \delta > \varepsilon > 0$. Then there is such $t_0 > 0$ that for any $x \in X$ the inequalities $\rho(x,x_0) < \varepsilon$ and $0 \le t' < t_0$ lead to $\rho(x_0,f(t',x,k)) < \delta$.

Proof. Let us suppose the contrary: there are such sequences x_n and t_n that $\rho(x_0,x_n) < \varepsilon,\ t'_n \to 0$, and $\rho(x_0,f(t'_n,x_n,k)) \ge \delta$. Due to the compactness of X one can choose from the sequence x_n a convergent one. Let it converge to $\bar{x}$. The function $\rho(x_0, f(t,x,k))$ is continuous. Therefore $\rho(x_0, f(t'_n, x_n, k)) \to \rho(x_0, f(0,x,k)) = \rho(x_0, \bar{x})$. Since $\rho(x_0, x_n) < \varepsilon$, then $\rho(x_0, \bar{x}) \leq \varepsilon$. This contradicts to the initial supposition ($\rho(x_0, f(t'_n, x_n, k)) \geq \delta \geq \varepsilon$).

Let us return to the proof of the theorem 2.3. Since $x' \in \omega(x^*, k^*)$, then there is such monotonic sequence $t_j \to \infty$ that for any j $\rho(f(t_j, x^*, k^*), x') < \varepsilon/8$. According to the lemma 2.1 there is $t_0 > 0$ for which $\rho(f(t_j + \tau, x^*, k^*), x') < \varepsilon/4$ as $0 \leq \tau \leq t_0$. Suppose (turning to subsequence, if it is necessary) that $t_{j+1} - t_j > t_0$. $\Theta(x^*, x', t, \varepsilon) > jt_0$ if $t > t_j + t_0$. For any $j = 1, 2, \dots$ there is such $N(j)$ that $\rho(f(t, x_n, k_n), f(t, x^*, k^*)) < \varepsilon/4$ under the conditions $n > N(j)$, $0 \leq t \leq t_j + t_0$. If $n > N(j)$, then $\rho(f(t, x_n, k_n), x') < \varepsilon/2$ for $t_j \leq t \leq t_j + t_0$ $(i \leq j)$. Consequently, $\tau_2(x_n, k_n, \varepsilon/2) > jt_0$ if $n > N(j)$. The existence of τ_2 slow relaxations is proved.

Theorem 2.4. If the system (1) possesses $\omega(k)$-bifurcations, then it possesses η_2-slow relaxations too.

Proof. Let the system (1) possess $\omega(k)$-bifurcations: there are such sequence $k_n \in K$ and such $\varepsilon > 0$ that $k_n > k^*$ and $\rho^*(x', \omega(k_n)) > \varepsilon$ for some $x' \in \omega(k^*)$ and any n. The point x' lies in ω-limit set of some motion: $x' \in \omega(x^*, k^*)$. Let $\tau > 0$ and t^* be such that $\Theta(x^*, x', t^*, \varepsilon) > \tau$ (the existence of such t^* is shown when proving the theorem 2.3). Due to the uniform continuity of f on compact intervals there is such N that $\rho(f(x^*, k^*), f(t, x^*, k_n)) < \varepsilon/4$ for $0 \leq t \leq t^*$, $n > N$. But from this fact follows that $\eta_2(x^*, k_n, \varepsilon/2) \geq \Theta(x^*, x', t^*, \varepsilon) > \tau$ $(n > N)$. Because of the arbitrary choice of τ the theorem 2.4 is proved.

The two following theorems provide supplementary sufficient conditions of τ_2 - and η_2 -slow relaxations.

Theorem 2.5. If for the system (1) there are such $x \in X, k \in K$ that (k,x)-motion is whole and $\alpha(x,k) \not\subset \omega(x,k)$, then the system (1.1) possesses τ_2-slow relaxations.

Proof. Let be such x and k that (k,x)-motion is whole and $\alpha(x,k) \not\subset \omega(x,k)$. Denote by x^* an arbitrary α-, but not ω-limit point of (k,x)-motion. Since $\omega(x,k)$ is closed, $\rho^*(x^*, \omega(x,k)) > \varepsilon > 0$. Define an auxiliary function

$$\varphi(x, x^*, t, \varepsilon) = \mathrm{mes}\{t' \mid -t \leq t' \leq 0,\ \rho(f(t', x, k), x^*) < \varepsilon/2\}.$$

Let us prove that $\varphi(x, x^*, \varepsilon) \to \infty$ as $t \to \infty$. According to the lemma 2.1 there is such $t_0 > 0$ that $\rho(f(t, y, k), x^*) < \varepsilon/2$ if $0 \leq t \leq t_0$ and $\rho(x^*, y) < \varepsilon/4$. Since x^* is α-limit point of (k,x)-motion, there is such sequence $t_j < 0$, $t_{j+1} - t_j < -t_0$, for which $\rho(f(t_j, x, k), x^*) < \varepsilon/4$. Therefore, by the way used in the proof of the theorem 2.3 we obtain: $\varphi(x, x^*, t_j, \varepsilon) > jt_0$. This proves the theorem 2.5, because $\tau_2(f(-t, x, k), k, \varepsilon/2) \geq \varphi(x, x^*, t, \varepsilon)$.

Theorem 2.6. If for the system (1) exist such $x \in X, k \in K$ that (k,x)-motion is whole and $\alpha(x,k) \not\subset \overline{\omega(k)}$, then the system (1.1) possesses η_2-slow relaxations.

Proof. Let (k,x)-motion be whole and

$$\alpha(x,k) \not\subset \overline{\omega(k)},\ x^* \in \alpha(x,k) \setminus \overline{\omega(k)},\ \rho^*(x^*, \overline{\omega(k)}) = \varepsilon > 0.$$

As in the proof of the previous theorem, let us define the function $\varphi(x, x^*, t, \varepsilon)$. Since $\varphi(x, x^*, t, \varepsilon) \to \infty$ as $t \to \infty$ (proved above) and $\eta_2(f(-t, x, k), k, \varepsilon/2) \geq \varphi(x, x^*, t, \varepsilon)$, the theorem is proved.

Note that the conditions of the theorems 2.5, 2.6 do not imply bifurcations.

Example 2.6. (τ_2-, η_2-slow relaxations without bifurcations). Examine the system given by the set of equations (15) in the circle $x^2 + y^2 \leq 1$ (see fig. 2, a, example

2.1). Identify the fixed points $r = 0$ and $r = 1$, $\varphi = \pi$ (fig. 4). The complete ω-limit set of the system obtained consists of one fixed point. For initial data $r_0 \to 1, r_0 < 1$ (φ_0 is arbitrary) the relaxation time $\eta_2(r_0, \varphi_0, 1/2) \to \infty$ (hence, $\tau_2(r_0, \varphi_0, 1/2) \to \infty$).

Before analyzing τ_3, η_3-slow relaxations, let us define *Poisson's stability* according to [6, p.363]: (k, x)-motion is it Poisson's positively stable (P^+-*stable*), if $x \in \omega(x, k)$.

Note that any P^+-stable motion is whole.

Lemma 2.2. If for the system (1) exist such $x \in X,\ k \in K$ that (k, x)-motion is whole but not P^+-stable, then the system (1) possesses τ_3-slow relaxations.

Proof. Let $\rho^*(x, \omega(x, k)) = \varepsilon > 0$ and (k, x)-motion be whole. Then

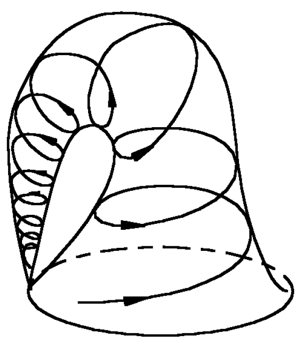

Fig.4. Phase portrait of the system (15) after gluing fixed points.

$$\tau_3(f(-t, x, k), k, \varepsilon) \geq t,$$
$$\text{since } f(t, f(-t, x, k), k) = x \text{ and } \rho^*(x, \omega(f(-t, x, k), k)) = \varepsilon$$

(because $\omega(f(-t, x, k), k) = \omega(x, k)$). Therefore τ_3-slow relaxations exist.

Lemma 2.3. If for the system (1) exist such $x \in X,\ k \in K$ that (k, x)-motion is whole and $x \notin \omega(k)$, then this system possesses η_3 -slow relaxations.

Proof. Let $\rho^*(x, \omega(k)) = \varepsilon > 0$ and (k, x)-motion be whole. Then

$$\eta_3(f(-t, x, k), k, \varepsilon)) \geq t,$$
$$\text{since } f(t, f(-t, x, k), k) = x \text{ and } \rho^*(x, \omega(k)) = \rho^*(x, \overline{\omega(k)}) = \varepsilon.$$

Consequently, η_3-slow relaxations exist.

Lemma 2.4. Let for the system (1) be such $x_0 \in X,\ k_) \in K$ that (k_0, x_0)-motion is whole. If $\omega(x, k)$ is d-continuous function in $X \times K$ (there are no $\omega(x, k)$-bifurcations), then:

1) $\omega(x^*, k_0) \subset \omega(x_0, k_0)$ for any $x^* \in \alpha(x_0, k_0)$, i.e. $\omega(\alpha(x_0, k_0), k_0) \subset \omega(x_0, k_0)$;

2) in particular, $\omega(x_0, k_0) \bigcap \alpha(x_0, k_0) \neq \varnothing$.

Proof. Let $x^* \in \alpha(x_0, k_0)$. Then there are such $t_n > 0$ that $t_n \to \infty$ and $x_n = f(-t_n, x_0, k_0) \to x^*$. Note that $\omega(x_n, k_0) = \omega(x_0, k_0)$. If $\omega(x^*, k_0) \not\subset \omega(x_0, k_0)$, then, taking into account closure of $\omega(x_0, k_0)$, we would obtain inequality $d(\omega(x^*, k_0), \omega(x_0, k_0)) > 0$. In this case $x_n \to x^*$, but $\omega(x_n, k_0) - / \to \omega(x^*, k_0)$, i.e. there is $\omega(x, k)$-bifurcation. But according to the assumption there are no $\omega(x, k)$-bifurcations. The obtained contradiction proves the first statement of the lemma. The second statement follows from the facts that $\alpha(x_0, k_0)$ is closed, k_0-invariant and nonempty. Really, let $x^* \in \alpha(x_0, k_0)$. Then $\overline{f((-\infty, \infty), x^*, k_o)} \subset \alpha(x_0, k_0)$ and, in particular, $\omega(x^*, k_0) \subset \alpha(x_0, k_0)$. But it has been proved that $\omega(x^*, k_0) \subset \omega(x_0, k_0)$. Therefore $\omega(x_0, k_0) \bigcap \alpha(x_0, k_0) \supset \omega(x^*, k_0) \neq \varnothing$.

Theorem 2.7. The system (1) possesses τ_3-slow relaxations if and only if at least one of the following conditions is satisfied:

1) there are $\omega(x, k)$-bifurcations;

2) there are such $x \in X,\ k \in K$ that (k, x)-motion is whole but not P^+-stable.

Proof. If there exist $\omega(x, k)$-bifurcations, then the existence of τ_3-slow relaxations follows from the theorem 2.3 and the inequality $\tau_2(x, k, \varepsilon) \leq \tau_3(x, k, \varepsilon)$. If the condition 2 is satisfied, then the existence of τ_3-slow relaxations follows from the lemma 2.2. To finish the proof, it must be ascertained that if the system (1) possesses τ_3-slow relaxations and does not possess $\omega(x,k)$-bifurcations, then there exist such $x \in X,\ k \in K$ that (k,x)-motion is whole and not P^+-stable. Let there be τ_3-slow relaxations and $\omega(x,k)$-bifurcations be absent. There can be chosen such convergent (because of the compactness of $X \times K$) sequence $(x_n,k_n) \to (x^*,k^*)$ that $\tau_3(x_n,k_n,\varepsilon) \to \infty$ for some $\varepsilon > 0$. Consider the sequence $y_n = f(\tau_3(x_n,k_n,\varepsilon),x_n,k_n)$. Note that $\rho^*(y_n,\omega(x_n,k_n)) = \varepsilon$. This follows from the definition of relaxation time and continuity of the function $\rho^*(f(t,x,k),s)$ of t at any $(x,k) \in X \times K,\ s \subset X$. Let us choose from the sequence y_n a convergent one (preserving the denotations y_n,x_n,k_n). Let us denote its limit: $y_n \to x_0$. It is clear that (k^*,x_0)-motion is whole. This follows from the results of the section 1.1 and the fact that (k_n,y_n)-motion is defined in the time interval $[-\tau_3(x_n,k_n,\varepsilon),\infty)$, and $\tau_3(x_n,k_n,\varepsilon) \to \infty$ as $n \to \infty$. Let us prove that (k^*,x_0)-motion is not P^+-stable, i.e. $x_0 \notin \omega(x_0,k^*)$. Suppose the contrary: $x_0 \in \omega(x_0,k^*)$. Since $y_n \to x_0$, then there is such N that $\rho(x_0,y_n) < \varepsilon/2$ for any $n \geq N$. For the same $n \geq N$ $\rho^*(x_0,\omega(y_n,k_n)) > \varepsilon/2$, since $\rho^*(y_n,\omega(y_n,k_n)) = \varepsilon$. But from this fact and from the assumption $x_0 \in \omega(x_0,k^*)$ follows that for $n \geq N$ $d(\omega(x_0,k^*),\omega(y_n,k_n)) > \varepsilon/2$, and that means that there are $\omega(x,k)$-bifurcations. So far as it was supposed d-continuity of $\omega(x,k)$, it was proved that (k^*,x_0)-motion is not P^+-stable, and this completes the proof of the theorem.

Using the lemma 2.4, the theorem 2.7 can be formulated as follows.

Theorem 2.7$'$. The system (1) possesses τ_3-slow relaxations if and only if at least one of the following conditions is satisfied:

1) there are $\omega(x,k)$-bifurcations;

2) there are such $x \in X,\ k \in K$ that (k,x)-motion is whole but not P^+-stable and possesses the following property: $\omega(\alpha(x,k),k) \subset \omega(x,k)$.

As an example of motion satisfying the condition 2 can be considered a trajectory going from a fixed point to the same point (for example, the loop of a separatrice), or a homoclinical trajectory of a periodical motion.

Theorem 2.8. The system (1) possesses η_3-slow relaxations if and only if at least one of the following conditions is satisfied:

1) there are $\omega(k)$-bifurcations;

2) there are such $x \in X,\ k \in K$ that (k,x)-motion is whole and $x \notin \overline{\omega(k)}$.

Proof. If there are $\omega(k)$-bifurcations, then, according to the theorem 2.4, there are η_2- and all the more η_3-slow relaxations. If there is accomplished the condition 2, then the existence of η_3-slow relaxations follows from the lemma 2.3. To complete the proof, it must be established that if the system (1) possesses η_3-slow relaxations and does not possess $\omega(k)$-bifurcations then the condition 2 of the theorem is accomplished: there are such $x \in X,\ k \in K$ (k,x)-motion is whole and $x \notin \overline{\omega(k)}$. Let there be η_3 -slow relaxation and $\omega(k)$-bifurcations be absent. Then we can choose such convergent (because of the compactness of $X \times K$) sequence $(x_n,k_n) \to (x^*,k^*)$ that $\eta_3(x_n,k_n,\varepsilon) \to \infty$ for some $\varepsilon > 0$. Consider the sequence $y_n = f(\eta_3(x_n,k_n,\varepsilon),x_n,k_n)$. Note that $\rho^*(y_n,\omega(k_n)) = \varepsilon$. Choose from the sequence y_n a convergent one (preserving the denotations y_n,x_n,k_n). Let us denote its limit as $x_0 : y_n \to x_0$. From the results of the section 1.1 and the fact that (k_n,y_n)-motion is defined at least on the segment $[-\eta_3(x_n,k_n,\varepsilon),\infty)$ and $\eta_3(x_n,k_n,\varepsilon) \to \infty$ we obtain that (k^*,x_0)-motion is whole. Let us prove that $x_0 \notin \overline{\omega(k)}$. Really, $y_n \to x_0$, hence there is such N that for any $n \geq N$ the inequality $\rho(x_0,y_n) < \varepsilon/2$ is true. But $\rho^*(y_n,\overline{\omega(k_n)}) = \varepsilon$, consequently for $n > N$ $\rho^*(x_0,\overline{\omega(k_n)}) > \varepsilon/2$. If x_0 belonged to $\overline{\omega(k^*)}$, then for $n > N$ the inequality $d(\overline{\omega(k^*)},\omega(k)n)) > \varepsilon/2$ would be true and there would exist $\omega(k)$-bifurcations. But according to the assumption they do not exist. Therefore is proved that $x_0 \notin \overline{\omega(k^*)}$.

Formulate now some corollaries from $\omega(k)$ the proved theorems.

Corollary 2.1. Let any trajectory from $\omega(k)$ be recurrent for any $k \in K$ and there be not such $(x,k) \in X \times K$ that (k,x)-motion is whole, not P^+-stable and $\omega(\alpha(x,k),k) \subset \omega(x,k)$ (or weaker, $\omega(x,k) \bigcap \alpha(x,k) \neq \varnothing$). Then the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$-slow relaxations.

Evidently, this follows from the theorem 2.7 and the proposition 1.11.

Corollary 2.2. Let the set $\omega(x,k)$ be minimal ($\Omega(x,k) = \{\omega(x,k)\}$) for any $(x,k) \in X \times K$ and there be not such $(x,k) \in X \times K$ that (k,x)-motion is whole, not P^+-stable and $\omega(\alpha(x,k),k) \subset \omega(x,k)$ (or weaker, $\alpha(x,k) \bigcap \omega(x,k) \neq \varnothing$).. Then the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$ -slow relaxations.

This follows from the theorem 2.7 and the corollary 1.1 of the proposition 1.11.

3 Slow Relaxations of One Semiflow

3.1 η_2-slow Relaxations

As it was shown (proposition 2.2), η_1-slow relaxations of one semiflow are impossible. Also was given an example of η_2-slow relaxations in one system (example 2.1). It is be proved below that a set of smooth systems possessing η_2-slow relaxations on a compact variety is a set of first category in C^1-topology. As for general dynamical systems, for them is true the following theorem.

Theorem 3.1. Let a semiflow f possess η_2-slow relaxations. Then one can find a non-wandering point $x^* \in X$ which does not belong to ω_f (here and further we designate complete ω-limit sets of one semiflow f as ω_f and Ω_f instead of $\omega(k)$ and $\Omega(k)$).

Proof. Let for some $\varepsilon > 0$ the function $\eta_2(x,\varepsilon)$ be unlimited in X. Consider a sequence $x_n \in X$ for which $\eta_2(x_n,\varepsilon) \to \infty$. Let V be a close subset of the set $\{x \in X \mid \rho^*\ (x,\omega_f) \geq \varepsilon\}$. Define an auxiliary function: the dwelling time of x-motion in the intersection of δ-neighbourhood of the point $y \in V$ with V:

$$\psi(x,y,\delta,V) = \mathrm{mes}\{t > 0 \mid \rho(f(t,x),y) \leq \delta,\ f(t,x) \in V\}. \tag{21}$$

From the inequality $\psi(x,y,\delta,V) \leq \eta_2(x,\varepsilon)$ and the fact that finite $\eta_2(x,\varepsilon)$ exists for each $x \in X$ (see proposition 2.1) follows that the function ψ is defined for any $x, y, \delta > 0$ and V with indicated properties (V is closed, $r(V,f) \geq \varepsilon$). Let us fix some $\delta > 0$. Suppose that $V_0 = \{x \in X \mid \rho^*(x,\omega_f) \geq \varepsilon\}$. Examine a finite overlapping of V_0 with closed spheres centered in V_0 : $V_0 \subset \bigcup_{j=1}^k \bar{U}_\delta(y_i)$ (here $\bar{U}_\delta(y_j)$ is a closed sphere of radius δ centered in $y_j \in V_0$). The inequality

$$\sum_{j=1}^{k} \psi(x,y_i,\delta,V_0) \geq \eta_2(x,\varepsilon) \tag{22}$$

is true (it is evident: being in V_0, x-motion is always in some $\bar{U}_\delta(y_i)$). From (22) follows that $\sum_{j=1}^k \psi(x,y_i,\delta,V_0) \to \infty$ as $n \to \infty$. Therefore there is j_0 ($1 \leq j_0 \leq k$) for which there is such subsequence $\{x_{m(i)}\} \subset \{x_n\}$ that $\psi(x_{m(i)},y_{j_0},\delta,V_0) \to \infty$. Let $y_0^* = y_{j_0}$.

Note that if $\rho(x,y_0^*) < \delta$ then for any $I > 0$ there is $t > T$ for which

$$f(t,\bar{U}_{2\delta}(x)) \bigcap \bar{U}_{2\delta}(x) \neq \varnothing.$$

Designate $V_1 = \bar{U}_\delta(y_0^*) \bigcap V_0$. Consider the finite overlapping of V_1 with closed spheres of radius $\delta/2$ with centers $V_1: \ v_1 = \bigcup_{j=1}^{k_1} \bar{U}_{\delta/2}(y_j^1);\ y_j^1 \in V_1$. The following inequality is true:

$$\sum_{j=1}^{k_1} \psi(x, y_j^1, \delta/2, V_1) \geq \psi(x, y_0^*, \delta, V_0). \tag{23}$$

Therefore exists j_0' $(1 \geq j_0' \geq k_1)$ for which there is such sequence $\{x_{l(i)}\} \subset \{x_{m(i)}\} \subset \{x_n\}$ that $\psi(x_{l(i)}, y^1_{j_0'}, \delta/2, V_1) \to \infty$ as $i \to \infty$. Designate $y_1^* = y^1_{j_0'}$.

Note that if $\rho(x, y_1^*) \geq \delta/2$ then for any $T > 0$ there is such $t > T$ that

$$f(t, \bar{U}_\delta(x)) \bigcap \bar{U}_\delta(x) \neq \varnothing.$$

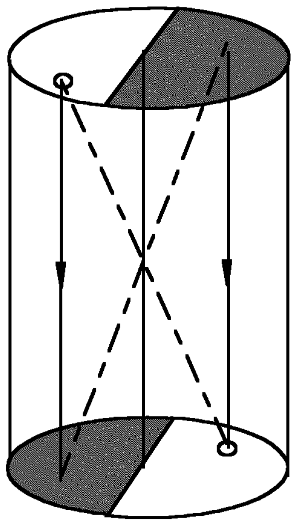

Fig.5. Phase space of the system (example 3.1). All the points of the axis are non-wandering; ◯ is the dwelling near fixed points.

Designate $V_2 = \bar{U}_{\delta/2}(y_1^*) \bigcap V_1$ and repeat the construction, substituting $\delta/2$ for $\delta, \delta/4$ for $\delta/2$, $V_{1,2}$ for $V_{0,1}$.

Repeating this constructing further, we obtain the fundamental sequence $y_0^*, y_1^*, \ldots$. Designate its limit x^*. The point x^* is non-wandering: for any its neighbourhood U and for any $T > 0$ there is such $t > T$ that $f(t, u) \bigcap U \neq \varnothing$. The theorem 3.1 is proved.

The inverse is not true in general case.

Example 3.1. (The existence of non-wandering point $x^* \notin \overline{\omega_f}$ without η_2-slow relaxations). Consider a cylinder in $R^3: \ x^2 + y^2 \leq 1, \ -1 \leq z \leq 1$. Define in it a motion by the equations $\dot{x} = \dot{y} = 0, \ \dot{z} = (x, y, z)$, where is a smooth function, ≥ 0, and it is equal to zero only at (all) points of the sets $(z = -1, x \leq 0)$ and $(z = 1, x \geq 0)$. Since the sets are closed, such function exists (even infinitely smooth). Identify the opposite bases of the cylinder, preliminary turning them at angle π. In the obtained dynamical system the closures of trajectories, consisting of more than one point, form up Zeifert foliation (fig.5) (see, for example, [5, p.158]).

Trajectory of the point $(0, 0, 0)$ is a loop, tending at $t \to \pm\infty$ to one point which is the identified centers of cylinder bases. The trajectories of all other nonfixed points are also loops, but before to close they make two turns near the trajectory $(0, 0, 0)$ The nearer is the initial point of motion to $(0, 0, 0)$, the larger is the time interval between it and the point of following hit of this motion in small neighborhood of $(0, 0, 0)$ (see fig.5).

3.2 Slow Relaxations and Stability

Let us recall the definition of Lyapunov stability of closed invariant set given by Lyapunov (see [25, p.31-32]), more general approach is given in [61].

Definition 3.1. A closed invariant set $W \subset X$ is *Lyapunov stable* if and only if for any $\varepsilon > 0$ there is such $\delta = \delta(\varepsilon) > 0$ that if $\rho^*(x, W) < \delta$ then the inequality $\rho^*(f(t, x), W) < \varepsilon$ is true for all $t \geq 0$.

The following lemma follows directly from the definition.

Lemma 3.1. A closed invariant set W is Lyapunov stable if and only if it has a fundamental system of positive-invariant closed neighborhoods: for any ε there are such

$\delta > 0$ and closed positive-invariant set $V \subset X$ that

$$\{x \in X \mid \rho^*(x, W) < \delta\} \subset V \subset \{x \in X \mid \rho^*(x, W) < \varepsilon\}. \tag{24}$$

To get the set V, one can take for example the closure of following (evidently positive-invariant) set: $\{f(t, x) \mid \rho^*(x, W) \leq \delta = \delta(\varepsilon/2),\ t \in [0, \infty)\}$, i.e. of the complete image (for all $t \leq 0$) of δ-neighbourhood of W, where $\delta(\varepsilon)$ is that spoken about in the definition 3.1.

The following lemma can be deduced from the description of Lyapunov stable sets [25, sec.11, p.40-49].

Lemma 3.2. Let a closed invariant set $W \subset X$ be not Lyapunov stable. Then for any $\lambda > 0$ there is such $y_0 \in X$ that y_0-motion is whole, $\rho^*(y_0, W) < \lambda$, $d(\alpha(y_0), W) < \lambda$ (i.e. α-limit set of y_0-motion lies in λ-neighbourhood of W), and $y_0 \notin W$.

Theorem 3.2. Let for semiflow f exist closed invariant set $W \subset X$ possessing the following property (isolation): there is such $\lambda > 0$ that for any $y \in \omega_f$ from the condition $\rho^*(y, W) < \lambda$ follows that $y \in W$. If this isolated set is not Lyapunov stable, then this semiflow possesses η_3-slow relaxations.

Proof. Let W be a closed invariant isolated Lyapunov unstable set. Let $\lambda > 0$ be the value from the definition of isolation. Then the lemma 3.2 guarantees the existence of such $y_0 \in X$ that y_0-motion is whole, $\rho^*(y_0, W) < \lambda$ and $y_0 \notin W$. It gives (due to closure of W) $\rho^*(y_0, W) = d > 0$. Let $\delta = \min\{d/2, (\lambda - d)/2\}$. Then δ-neighbourhood of the point y_0 lies outside of the set W, but in its λ-neighbourhood, and the last is free from the points of the set $\omega_f \setminus W$ (isolation of W). Thus, δ-neighbourhood of the point y_0 is free from the points of the set $\omega_f \subseteq W \bigcup (\omega_f \setminus W)$, consequently $y_0 \notin \overline{\omega_f}$. Since y_0-motion is whole, the theorem 2.8 guarantees the presence of η_3-slow relaxations. The theorem 3.2 is proved.

Lemma 3.3. Let X be connected and $\overline{\omega_f}$ be disconnected, then $\overline{\omega_f}$ is not Lyapunov stable.

Proof. Since $\overline{\omega_f}$ is disconnected, there are such nonempty closed W_1, W_2 that $\overline{\omega_f} = W_1 \bigcup W_2$ and $W_1 \bigcap W_2 = \varnothing$. Since any x-trajectory is connected and $\overline{\omega_f}$ is invariant, then and W_1 and W_2 are invariant too. The sets $\omega(x)$ are connected (see proposition 1.4), therefore for any $x \in X$ $\omega(x) \subset W_1$ or $\omega(x) \subset W_2$. Let us prove that at least one of the sets W_i $(i = 1, 2)$ is not stable. Suppose the contrary: W_1 and W_2 are stable. Define for each of them *attraction domain*:

$$At(W_i) = \{x \in X \mid \omega(x) \subset W_i\}. \tag{25}$$

It is evident that $W_i \subset At(W_i)$ owing to closure and invariance of W_i. The sets $At(W_i)$ are open due to the stability of W_i. Really, there are non-intersecting closed positive-invariant neighborhoods V_i of the sets W_i, since the last do not intersect and are closed and stable (see lemma 3.1). Let $x \in At(W_i)$. Then there is such $t \geq$ that $f(t, x) \in \text{int}V_i$. But because of the continuity of f there is such neighbourhood of x in X that for each its point x' $f(t, x) \in \text{int}V_i$. Now positive-invariance and closure of V_i ensure $\omega(x') \subset W_i$, i.e. $x' \in At(W_i)$. Consequently, x lies in $At(W_i)$ together with its neighbourhood and the sets $At(W_i)$ are open in X. Since $At(W_i) \bigcup At(W_2) = X,\ At(W_1) \bigcap At(W_2) = \varnothing$, the obtained result contradicts to the connectivity of X. Therefore at least one of the sets W is not Lyapunov stable. Prove that from this follows unstability of $\overline{\omega_f}$. Note that if a closed positive-invariant set V is union of two non-intersecting closed sets, $V = V_1 \bigcup V_2,\ V_1 \bigcap V_2 = \varnothing$, then V_1 and V_2 are also positive-invariant because of the connectivity of positive semitrajectories. If $\overline{\omega_f}$ is stable, then it possess fundamental system of closed positive-invariant neighborhoods $V_1 \supset V_2 \supset \ldots V_n \supset \ldots$. Since $\overline{\omega_f} = W_1 \bigcup W_2,\ W_1 \bigcap W_2 = \varnothing$ and W_i are nonempty and closed, then from some N $V_n = V_n' \bigcup V_n'' = \varnothing$ for $n \geq N$, and the families of the sets $V_n' \supset V_{n+1}' \supset \ldots,\ V_n'' \supset V_{n+1}'' \ldots$ form fundamental systems of neighborhoods of W_1 and W_2 correspondingly. So long as V_n', V_n'' are closed positive-invariant neighborhoods, from this follows stability of both W_1 and W_2, but it was already proved that it is impossible. This contradiction shows that $\overline{\omega_f}$ is not Lyapunov stable and completes the proof of the lemma.

Theorem 3.3. Let X be connected and $\overline{\omega_f}$ be disconnected. Then the semiflow f possesses η_3- and $\tau_{1,2,3}$-slow relaxations.

Proof. The first part (the existence of η_3-slow relaxations) follows from the lemma 3.3 and the theorem 3.2. (in the last as a closed invariant set one should take $\overline{\omega_f}$). Let us prove the existence of τ_1-slow relaxations. Let ω_f be disconnected: $\overline{\omega_f} = W_1 \bigcup W_2,\ W_1 \bigcap W_2 = \varnothing,\ W_i\ (i = 1, 2)$ are closed and, consequently, invariant due to the connectivity of trajectories. Consider the sets $At(W_i)$ (25). Note that at least one of these sets $At(W_i)$ does not include any neighbourhood of W_i. Really, suppose the contrary: $At(W_i)\ (i = 1, 2)$ includes ε-neighbourhood of W_i. Let $x \in At(W_i),\ \tau = \tau_1(x, \varepsilon/3)$ be the time of the first hit of the x-motion into $\varepsilon/3$-neighbourhood of the set $\omega(x) \subset W_i$. The point x possesses such neighbourhood $U \subset X$ that for any $y \in U$ $\rho(f(t, x), f(t, y)) < \varepsilon/3$ as $y \in U,\ 0 \leq t \leq \tau$. Therefore $d(f(\tau, U), W_i) \leq 2\varepsilon/3,\ U \subset At(W_i)$. Thus, x lies in $At(W_i)$ together with its neighbourhood: the sets $At(W_i)$ are open. This contradicts to the connectivity of X, since $X = At(W_1) \bigcup At(W_2)$ and $At(W_1) \bigcap At(W_2) = \varnothing$. To be certain, let $At(W_1)$ contain none neighbourhood of W_1. Then (owing to the compactness of X and the closure of W_1) there is a sequence $x_i \in At(W_2),\ x_i \to y \in W_1,\ \omega(x_i) \subset W_2,\ \omega(y) \subset W_1$. Note that $r((x_i), \omega(y)) \geq r(W_1, W_2) > 0$, therefore there are $\Omega(x)$-bifurcations (y is the bifurcation point) and, consequently, (the theorem 2.1) there are τ_1-slow relaxations. This yields the existence of $\tau_{2,3}$-slow relaxations ($\tau_1 \leq \tau_2 \leq \tau_3$).

3.3 Slow Relaxations in Smooth Systems

Consider in this item the application of the above developed approach to the semiflows associated with smooth dynamical systems. Let M be a smooth (of class C^∞) finite-dimensional manifold, $F: (-\infty, \infty) \times M \to M$ be a smooth dynamical system over M, generated by vector field of class C^1, X be a compact set positive-invariant with respect to the system F (in particular, $X = M$ if M is compact). The restriction of F to the set we call semiflow over X, associated with F, and designate it as $F|_X$.

We often will use the following condition: the semiflow $F|_X$ has not non-wandering points at the boundary of X (∂X); if X is positive-invariant submanifold of M with smooth boundary, $\text{int}X \neq \varnothing$, then this follows, for example, from the requirement of transversality of the vector field corresponding to the system F and the boundary of X. All the below results are valid, in particular, in the case when X is the whole manifold M and M is compact (the boundary is empty).

Theorem 3.4. The supplement of the set of smooth dynamical systems on compact manifold M possessing the following attribute 1, is the set of first category (in C^1-topology in the space of smooth vector fields).

Attribute 1. Every semiflow $F|_X$ associated with a system F on any compact positive-invariant set $X \subset M$ without non-wandering points on ∂X has not η_2-slow relaxations.

This theorem is a direct consequence of the closing lemma of Pugh, the density theorem [9,26], and the theorem 3.1 of the present work.

Note that if X is positive-invariant submanifold with smooth boundary in M, int$X \neq \varnothing$, then by infinitesimal (in C^1-topology) perturbation of F preserving positive-invariance of X one can obtain that semiflow over X, associated with the perturbed system, would not have non-wandering points on ∂X. This can be easily proved by standard in differential topology reasonings about transversality. In the present case the transversality of vector field of "velocities" F to the boundary of X is meant.

The structural stable systems over compact two-dimensional manifolds are studied much better than in general case [62,63]. They possess a number of characteristics which do not remain in higher dimensions. In particular, for them the set of non-wandering points consists of a finite number of limit cycles and fixed points, and the "loops" (trajectories whose α- and ω-limit sets intersect, but do not contain points of the trajectory itself) are absent. Slow relaxations in these systems also are different from the relaxations in the case of higher dimensions.

Theorem 3.5. Let M be C^∞-smooth compact manifold, $\dim M = 2$, F be a structural stable smooth dynamical system over M, $F|_X$ be an associated with M semiflow over connected compact positive-invariant subset $X \subset M$. Then:

1) for $F|_X$ the existence of τ_3-slow relaxations is equivalent to the existence of $\tau_{1,2}$- and η_3-slow relaxations;

2) $F|_X$ does not possess τ_3-slow relaxations if and only if $\omega_F \bigcap X$ consists of one fixed point or of points of one limit cycle;

3) $\eta_{1,2}$-slow relaxations are impossible for $F|_X$.

Proof. To prove the part 3, it is sufficient to refer to the theorem 3.1 and the proposition 2.2. Let us prove the first and the second parts. Note that $\omega_{F|_X} = \omega_F \bigcap X$. Let $\omega_F \bigcap X$ consist of one fixed point or of points of one limit cycle. Then $\omega(x) = X \bigcap \omega_F$ for any $x \in X$. Also there are not such $x \in X$ that x-motion would be whole but not P^+-stable and $\alpha(x) \bigcap \omega(x) \neq \varnothing$ (owing to the structural stability). Therefore (theorem 2.7) τ_3-slow relaxations are impossible. Suppose now that $\omega_F \bigcap X$ includes at least two limit cycles or a cycle and a fixed point or two fixed points. Then $\omega_{F|_X}$ is disconnected, and using the theorem 3.3 we obtain that $F|_X$ possesses η_3-slow relaxations. Consequently, exist τ_3-slow relaxations. From the corollary 2.1, i.e. the fact that every trajectory from ω_F is a fixed point or a limit cycle and also from the fact that rough two-dimensional systems have no loops we conclude that τ_1-slow relaxations do exist. Thus, if $\omega_F \bigcap X$ is connected, then $F|_X$ has not even τ_3-slow relaxations, and if $\omega_F \bigcap X$ is disconnected, then there are η_3 and $\tau_{1,2,3}$-slow relaxations. The theorem 3.5 is proved.

In general case (for structural stable systems with dim $M > 2$) the statement 1 of the theorem 3.5 is not true. Really, let us consider topologically transitive U-flow F over the manifold M [64]. $\omega_F = M$, therefore $\eta_3(x, \varepsilon) = 0$ for any $X \in M$, $\varepsilon > 0$. The set of limit cycles is dense in M. Let us choose two different cycles P_1 and P_2, whose stable (P_1) and unstable (P_2) manifolds intersect (such cycles exist, see for example [4,28]). For the point x of their intersection $\omega(x) = P_1$, $\alpha(x) = P_2$, therefore x-motion is whole and not Poisson's positive stable, and (lemma 2.2) τ_3-slow relaxations exist. And what is more, there exist τ_1-slow relaxations too. These appears because the motion beginning at point near P_2 of x-trajectory delays near P_2 before to enter small neighbourhood of P_1. It is easy to prove the existence of $\Omega(x)$-bifurcations too. Really, consider a sequence $t_1 \to \infty$, from the corresponding sequence $F(t_i, x)$ choose convergent subsequence: $F(t_j, x) \to y \in P_2$, $\omega(y) = P_2$, $\omega(F(t_j, x)) = P_1$, i.e. there are both τ_1-slow relaxations and $\Pi(x)$-bifurcations. For A-flows a weaker version of the statement 1 of the theorem 3.5 is valid (A-flow is called a flow satisfying S.Smeil A-axiom [4], in regard to A-flows see also [28, p.106-143]).

Theorem 3.6. Let F be A-flow over compact manifold M. Then for any compact connected positive-invariant $X \subset M$ which does not possess non-wandering points of $F|_X$ on the boundary the existence of τ_3-slow relaxations involves the existence of $\tau_{1,2}$-slow relaxations for $F|_X$.

Proof. Note that $\omega_{F|_X} = \omega_f \bigcap \mathrm{int}X$. If $\omega_F \bigcap \mathrm{int}X$ is disconnected, then, according to the theorem 3.3, $F|_X$ possesses η_3- and $\tau_{1,2,3}$-slow relaxations. Let $\omega_F \bigcap \mathrm{int}X$ be connected. The case when it consists of one fixed point or of points of one limit cycle is trivial: there are no any slow relaxations. Let $\omega_F \bigcap \mathrm{int}X$ consist of one non-trivial (being neither point nor cycle) basic set (in regard to these basic sets see [4,28]): $\omega_F \bigcap \mathrm{int}X = \Omega_0$. Since there are no non-wandering points over ∂X, then every cycle which has point in X lies entirely in int$X$. And due to positive-invariance of X, unstable manifold of such cycle lies in X. Let P_1 be some limit cycle from X. Its unstable manifold intersects with stable manifold of some other cycle $P_2 \subset X$ [4]. This follows from the existence of hyperbolic structure on Ω_0 (see also [28], p.110). Therefore there is such $x \in X$ that $\omega(x) = P_2$, $\alpha(x) = P_1$. From this follows the existence of τ_1- (and $\tau_{2,3}$-)-slow relaxations. The theorem is proved.

Remark. We have used only very weak consequence of the hyperbolicity of the set of non-wandering points: the existence in any non-trivial (being neither point nor limit cycle) isolated connected invariant set of two closed trajectories, stable manifold of one of which intersects with unstable manifold of another one. It seems very likely that the systems for which the statement of the theorem 3.6 is true are typical, i.e. the supplement of their set in the space of flows is a set of first category (in C^1- topology).

4 Slow Relaxation of Perturbed Systems

4.1 Limit Sets of ε-motions

As models of perturbed motions let us take ε-motions – mappings $f^\varepsilon : [0,\infty) \to X$, which during some fixed time T depart from the real motions at most at ε.

Definition 4.1. Let $x \in X$, $k \in K$, $\varepsilon > 0$, $T > 0$. The mapping $f^\varepsilon : [0,\infty) \to X$ is called (k, ε, T)-*motion* of the point x for the system (1) if $f^\varepsilon(0) = x$ and for any $t \geq 0$, $\tau \in [0, T]$

$$\rho(f^\varepsilon(t+\tau),\ f(\tau, f^\varepsilon(t), k)) < \varepsilon. \tag{26}$$

We call (k, ε, T)-motion of the point x (k, x, ε, T)-motion and use the denotation $f^\varepsilon(t|x, k, T)$. It is obvious that if $y = f^\varepsilon(\tau|x, k, T)$ then the function $f^*(t) = f^\varepsilon(t + \tau|x, k, T)$ is (k, y, ε)-motion.

The condition (26) is fundamental in study of motion with constantly functioning perturbations. Different restrictions on the value of perturbations of the right parts of differential equations (uniform restriction, restriction at the average etc. – see [65], p.184 and further) are used as a rule to obtain analogous to (26) estimations, on the base of which the further study is performed.

Let us introduce two auxiliary functions:

$$\varepsilon(\delta, t_0) = \sup\{\rho(f(t,x,k), f(t,x',k')) \mid 0 \le t \le t_0,\ \rho(x,x') < \delta,\ \rho_K(k,k') < \delta\}; \quad (27)$$

$$\delta(\varepsilon, t_0) = \sup\{\delta \ge 0 \mid \varepsilon(\delta, t_0) \le \varepsilon\}. \quad (28)$$

Due to the compactness of X and K the following statement is true.

Proposition 4.1. A) For any $\delta > 0$ and $t_0 > 0$ is defined (is finite) $\varepsilon(\delta, t_0)$; as $\delta \to 0$, $\varepsilon(\delta, t_0) \to 0$ uniformly over any compact segment $t_0 \in [t_1, t_2]$.
B) For any $\varepsilon > 0$ and $t_0 > 0$ is defined $\delta(\varepsilon, t_0) > 0$.

Proof. A) Let $\delta > 0$, $t_0 > 0$. Finiteness of $\varepsilon(\delta, t_0)$ ensues immediately from the compactness of x. Let $\delta_i > 0$, $\delta_i \to 0$. Let us prove that $\varepsilon(\delta_i, t_0) \to 0$. Suppose the contrary. In this case one can choose in $\{\delta_i\}$ such subsequence that corresponding $\varepsilon(\delta_i, t_0)$ are separated from zero by common constant: $\varepsilon(\delta_i, t_0) > \alpha > 0$. Let us turn to this subsequence, preserving the same denotations. For every i there are such $t_i, x_i, x'_i, k_i, k'_i$ that $0 \le t_i \le t_0$, $\rho(x_i, x'_i) < \delta_i$, $\rho_K(k_i, k'_i) < \delta_i$ and $\rho(f(t_i, x_i, k_i), f(t_i, x'_i, k'_i)) > \alpha > 0$. The product $[0, t_0] \times X \times X \times K \times K$ is compact. Therefore from the sequence $(t_i, x_i, x'_i, k_i, k'_i)$ one can choose a convergent subsequence. Let us turn to it preserving the denotations: $(t_i, x_i, x'_i, k_i, k'_i) \to (\tilde{t}, x_0, x'_0, k_0, k'_0)$. It is evident that $\rho(x_0, x'_0) = \rho_K(k_0, k'_0) = \rho_K(k_0, k'_0) = 0$, therefore $x_0 = x'_0$, $k_0 = k'_0$. Consequently, $f(\tilde{t}, x_0, k_0) = f(t, x'_0, k'_0)$. On the other hand, $\rho(f(t_i, x_i, k_i), f(t_i, x'_i, k'_i)) > \alpha > 0$, therefore $\rho(f(\tilde{t}, x_0, k_0), f(\tilde{t}, x'_0, k'_0)) \ge \alpha > 0$ and $f(\tilde{t}, x_0, k_0) \ne f(\tilde{t}, x'_0, k'_0)$. The obtained contradiction proves that $\varepsilon(\delta_i, t_0) \to 0$.

The uniformity of tending to 0 follows from the fact that for any $t_1, t_2 > 0$, $t_1 < t_2$ the inequality $\varepsilon(\delta, t_1) \le \varepsilon(\delta, t_2)$ is true, $\varepsilon(\delta, t)$ is a monotone function.

The statement of the point B) follows from the point A).

The following estimations of divergence of the trajectories are true. Let $f^{\varepsilon}(t|x,k,T)$ be (k,x,ε,T)-motion. Then[2]

$$\rho(f^{\varepsilon}(t|x,k,T), f(t,x,k)) \le \chi(\varepsilon, t, T), \quad (29)$$

where $\chi(\varepsilon, t, T) = \sum_{i=0}^{[t/T]} \varkappa_i$, $\varkappa_0 = \varepsilon, \varkappa_i = \varepsilon(\varkappa_{i-1}, T) + \varepsilon$.

Let $f^{\varepsilon_1}(t|x_1, k_1, T)$, $f^{\varepsilon_2}(t|x_2, k_2, T)$ be correspondingly $(k_1, x_1, \varepsilon_1, T)$- and $(k_2, x_2, \varepsilon_2, T)$-motion. Then

$$\rho(f^{\varepsilon_1}(t|x_1, k_1, T), f^{\varepsilon_2}(t|x_2, k_2, T)) \le$$
$$\le \varepsilon(\max\{\rho(x_1, x_2, , \rho_K(k_1, k_2)\}, T) + \chi(\varepsilon_1, t, T) + \chi(\varepsilon_2, t, T). \quad (30)$$

From the proposition 4.1 follows that $\chi(\varepsilon, t, T) \to 0$ as $\varepsilon \to 0$ uniformly over any compact segment $t \in [t_1, t_2]$.

Let $T_2 > T_1 > 0$, $\varepsilon > 0$. Then any (k, x, ε, T_2)-motion is (k, x, ε, T_1)-motion, and any (k, x, ε, T_1)-motion is $(k, x, \chi(\varepsilon, T_2, T_1), T_2)$-motion. Since we are interested in perturbed motions behavior at $\varepsilon \to 0$, and $\chi(\varepsilon, T_2, T_1) \to 0$ as $\varepsilon \to 0$, then the choice of T is unimportant. Therefore let us fix some $T > 0$ and omit references to it in formulas ((k, x, ε)-motion instead of (k, x, ε, T)-motion and $f^{\varepsilon}(t|x,k)$ instead of $f^{\varepsilon}(t|x,k,T)$).

The following propositions allow "to glue together" ε-motions.

Proposition 4.2. Let $\varepsilon_1, \varepsilon_2 > 0$, $f^{\varepsilon_1}(t|x,k)$ be ε_1-motion, $\tau > 0$, $f^{\varepsilon_2}(t|f^{\varepsilon_1}(\tau|x,k), k)$ be ε_2-motion. Then the mapping

$$f^*(t) = \begin{cases} f^{\varepsilon_1}(t|x,k), & \text{if } 0 \le t \le \tau; \\ f^{\varepsilon_2}(t - \tau | f^{\varepsilon_1}(\tau|x,k), k), & \text{if } t \ge \tau, \end{cases}$$

[2]Here and further are omitted trivial verifications, representing applications of the triangle inequality.

is $(k, x, 2\varepsilon_1, +\varepsilon_2)$-motion.

Proposition 4.3. Let $\delta, \varepsilon_1, \varepsilon_2 > 0$, $f^{\varepsilon_1}(t|x,k)$ be ε_1-motion, $\tau > 0$, $f^{\varepsilon_2}(t|y,k')$ be ε_2-motion, $\rho_K(k,k') < \delta$, $\rho(y, f^{\varepsilon_1}(\tau|x,k)) < \delta$. Then the mapping

$$f^*(t) = \begin{cases} f^{\varepsilon_1}(t|x,k), & \text{if } 0 \le t < \tau; \\ f^{\varepsilon_2}(t-\tau|y,k), & \text{if } t \ge \tau, \end{cases}$$

is $(k, x, 2\varepsilon_1 + \varepsilon_2 + \varepsilon(\delta, T))$-motion.

Proposition 4.4. Let $\delta_j, \varepsilon_j > 0$, $x_j \in X$, $k_j \in K$, $k^* \in K$, $\tau_0 > T$, $j = 0,1,2,\dots$, $i = 1,2,\dots$, $f^{\varepsilon_j}(t|x_j,k_j)$ be ε_j-motions, $\rho(f^{\varepsilon_j}(\tau_j|x_j,k_j), x_{j+1}) < \delta_j$, $\rho_K(k_j,k^*) < \delta_j/2$. Then the mapping

$$f^*(t) = \begin{cases} f^{\varepsilon_0}(t|x_0,k_0), & \text{if } 0 \le t < \tau_0; \\ f^{\varepsilon_j}\left(t - \sum_{j=0}^{i-1}\tau_j|x_j,k_j\right), & \text{if } \sum_{j=0}^{i-1}\tau_j \le t < \sum_{j=0}^{i}\tau_j, \end{cases}$$

is (k^*, x_0, β)-motion, if the numbers ε_j, δ_j are bounded above,

$$\beta = \sup_{0 \le j < \infty}\{\varepsilon_{j+1} + \varepsilon(\varepsilon_j + \delta_j + \delta_{j+1} + \varepsilon(\delta_j, T), T)\}.$$

The proof of the propositions 4.2-4.4 follows directly from the definitions.

Proposition 4.5. Let $x_i \in X$, $k_i \in K$, $k_i \to k^*$, $\varepsilon_i > 0$, $\varepsilon_i \to 0$, $f^{\varepsilon_i}(t|x_i,k_i)$ be $(k_i, x_i, \varepsilon_i)$-motions, $t_i > 0$, $t_i > t_0$, $f^{\varepsilon_i}(t_i|x_i,k_i) \to x^*$. Then (k^*, x^*)-motion is defined over the segment $[-t_0, \infty)$ and $f^{\varepsilon_i}(t_0 + t|x_i,k_i)$ tends to $f(t, x^*, k^*)$ uniformly over any compact segment from $[-t_0, \infty)$.

Proof. Let us choose from the sequence $\{x_i\}$ a convergent subsequence (preserving the denotations): $x_i \to x_0$. Note that $f^{\varepsilon_i}(t|x_i,k_i) \to f(t, x_0, k^*)$ uniformly over any compact segment $t \in [t_1,\ t_2] \subset [0, \infty)$; this follows from the estimations (30) and the proposition 4.1. Particularly, $f(t_0, x_0, k^*) = x^*$. Using the injectivity of f, we obtain that x_0 is a unique limit point of the sequence $\{x_i\}$, therefore $f^{\varepsilon_i}(t_0 + t|y_i,k_i)$ tends to $f(t, x^*, k^*) = f(t_0 + t, x_0, k^*)$ uniformly over any compact segment $t \in [t_1, t_2] \subset [-t_0, \infty)$.

Proposition 4.6. Let $x_i \in X$, $k_i \in K, k_i \to K^*$, $\varepsilon_i > 0$, $f^{\varepsilon_i}(t|x_i,k_i)$ be $(k_i, x_i, \varepsilon_i)$-motions, $t_i > 0$, $t_i \to \infty$, $f^{\varepsilon_i}(t_i|x_i,k_i) \to x^*$. Then (k^*, x^*)-motion is whole and the sequence $f^{\varepsilon_i}(t + t_i|x_i,k_i)$ defined for $t > t_0$ for any t_0, from some $i(t_0)$ (for $i \ge i(t_0)$) tends to $f(t, x^*, k^*)$ uniformly over any compact segment.

Proof. Let $t_0 \in (-\infty, \infty)$. From some i_0 $t_i > -t_0$. Let us consider the sequence of $(k_i, f^{\varepsilon_i}(t_i + t_0|x_i,k_i), \varepsilon_i)$-motions: $f^{\varepsilon_i}(t|f^{\varepsilon_i}(t_i + t_0|x_i,k_i),k_i) \stackrel{\text{def}}{=} f^{\varepsilon_i}(t + t_i + t_0|x_i,k_i)$.

Applying to the sequence the precedent proposition, we obtain the required statement (due to the arbitrariness of t_0).

Definition 4.2. Let $x \in X$, $k \in K$, $\varepsilon > 0$, $f^{\varepsilon}(t|x,k)$ be (k,x,ε)-motion. Let us call $y \in X$ *ω-limit point* of this ε-motion, if there is such a sequence $t_i \to \infty$ that $f^{\varepsilon}(t_i|x,k) \to y$. Denote the set of all ω-limit points of $f^{\varepsilon}(t|x,k)$ by $\omega(f^{\varepsilon}(t|x,k))$, the set of all ω-limit points of all (k,x,ε)-motions under fixed k,x,ε by $\omega^{\varepsilon}(x,k)$, and

$$\omega^0(x,k) \stackrel{\text{def}}{=} \bigcap_{\varepsilon > 0} \omega^{\varepsilon}(x,k).$$

Proposition 4.7. For any $\varepsilon > 0$, $\gamma > 0$, $x \in X$, $k \in K$

$$\overline{\omega^{\varepsilon}(x,k)} \subset \omega^{\varepsilon+\gamma}(x,k).$$

Proof. Let $y \in \overline{\omega^\varepsilon(x,k)}$. For any $\delta > 0$ there are (k,x,ε)-motion $f^\varepsilon(t|x,k)$ and subsequence $t_i \to \infty$, for which $\rho(f^\varepsilon(t_i|x,k), y) < \delta$. Let $\delta = \frac{1}{2}\delta(\gamma, T)$. As $(k,x,\varepsilon+\gamma)$-motion let us choose

$$f^*(t) = \begin{cases} f^*(t|x,k), & \text{if } t \neq t_i; \\ y, & \text{if } t = t_i (i = 1,2,\ldots, t_{i+1} - t_i > T). \end{cases}$$

It has y as its ω-limit point, consequently, $y \in \omega^{\varepsilon+\gamma}(x,k)$.

Proposition 4.8. For any $x \in X,\ k \in K$ the set $\omega_0(x,k)$ is closed and k-invariant.

Proof. From the proposition 4.7 follows

$$\omega^0(x,k) = \bigcap_{\varepsilon>0} \overline{\omega^\varepsilon(x,k)}. \tag{31}$$

Therefore $\omega^0(x,k)$ is closed. Let us prove that it is k-invariant. Let $y \in \omega^0(x,k)$. Then there are such sequences $\varepsilon_j > 0,\ \varepsilon_j \to 0,\ t_i^j \to \infty$ as $i \to \infty$ $(j = 1,2,\ldots)$ and such family of (k,x,ε_j)-motions $f^{\varepsilon_j}(t|x,k)$ that $f^{\varepsilon_j}(t_i^j|x,k) \to y$ as $i \to \infty$ for any $j = 1,2,\ldots$. From the proposition 4.6 follows that (k,y)-motion is whole. Let $z = f(t_0, y, k)$. Let us show that $z \in \omega^0(x,k)$. Let $\gamma > 0$. Construct (k,x,γ)-motion which has z as its ω-limit point.

Let $t_0 > 0$. Find such $\delta_0 > 0,\ \varepsilon_0 > 0$ that $\chi(\varepsilon_0, t_0 + T, T) + \varepsilon(\delta_0, t_0 + T) < \gamma/2$ (this is possible according to the proposition 4.1). Let us take $\varepsilon_j < \varepsilon_0$ and choose from the sequence t_i^j $(i = 1,2,\ldots)$ such monotone subsequence t_l $(l = 1,2,\ldots)$ for which $t_{l+1} - t_l > t_0 + T$ and $\rho(f^{\varepsilon_j}(t_l|x,k), y) < \delta_0$. Let

$$f^*(t) = \begin{cases} f^{\varepsilon_j}(t|x,k), & \text{if } t \notin [t_l, t_l + t_0] \text{ for any } l = 1,2,\ldots; \\ f(t - t_l, y, k), & \text{if } t \in [t_l, t_l + t_0]\ (l = 1,2,\ldots). \end{cases}$$

We have constructed (k,x,γ)-motion with z as its ω-limit point.

If $t_0 < 0$, then at first it is necessary to estimate the divergence of the trajectories for "backward motion". Let $\delta > 0$. Denote

$$\tilde{\varepsilon}(\delta, t_0, k) = \sup\left\{\rho(x,x') \,\middle|\, \inf_{0 \le t \le -t_0} \{\rho(f(t,x,k), f(t,x',k)\} \le \delta\right\}. \tag{32}$$

Lemma 4.1. For any $\delta > 0,\ t_0 < 0$ and $k \in K$ $\tilde{\varepsilon}(\delta, t_0, k)$ is defined (finite). $\tilde{\varepsilon}(\delta, t_0, k) \to 0$ as $\delta \to 0$ uniformly by $k \in K$ and by t_0 from any compact segment $[t_1, t_2] \subset (-\infty, 0]$.

The proof can be easily obtained from the injectivity of $f(t,\cdot,k)$ and compactness of X, K (similarly to the proposition 4.1).

Let us return to the proof of the proposition 4.8. Let $t_0 < 0$. Find such $\varepsilon_0 > 0$ and $\delta_0 > 0$ that $\tilde{\varepsilon}(\chi(\varepsilon_0, T - t_0, T), t_0, K) + \tilde{\varepsilon}(\delta_0, t_0 - T, k) < \gamma/2$. According to the proposition 4.1 and the lemma 4.1 this is possible. Let us take $\varepsilon_j < \varepsilon_0$ and choose from the sequence t_i^j $(i = 1,2,\ldots)$ such monotone subsequence t_l $(l = 1,2,\ldots)$ that $t_l > -t_0,\ \rho(f^{\varepsilon_i}(t_l|x,k), y) < \delta_0$ and $t_{l+1} - t_l > T - t_0$. Suppose

$$f^*(t) = \begin{cases} f^{\varepsilon_j}(t|x,k), & \text{if } t \notin [t_l + t_0, t_l] \text{ for any } l = 1,2,\ldots; \\ f(t - t_l, y, k), & \text{if } t \in [t_l + t_0, t_l]\ (l = 1,2,\ldots). \end{cases}$$

where $f^*(t)$ is (k,x,γ)-motion with z as its ω-limit point.

Thus, $z \in \omega^\gamma(x,k)$ for any $\gamma > 0$. The proposition is proved.

Proposition 4.9. Let $x \in \omega^0(x,k)$. Then for any $\varepsilon > 0$ there exists periodical (k,x,ε)-motion.

Proof. Let $x \in \omega^0(x,k),\ \varepsilon > 0,\ \delta = \frac{1}{2}\delta(\frac{\varepsilon}{2},T)$. There is (x,k,δ)-motion with x as its ω-limit point: $x \in (f^\delta(t),x,k)$. There is such $t_0 > T$ that $\rho(f^\delta(t_0|x,k),x) < \delta$. Suppose

$$f^*(t) = \begin{cases} x, & \text{if } t = nt_0,\ n = 0,1,2,\ldots; \\ f^\delta(t - nt_0|x,k), & \text{if } nt < t < (n+1)t_0. \end{cases}$$

Here $f^*(t)$ is a periodical (k,x,ε)-motion with the period t_0.

Thus, if $x \in \omega^0(x,k)$, then (k,x)-motion possesses the property of chain recurrence [57]. The inverse statement is also true: if for any $\varepsilon > 0$ there is a periodical (k,x,ε)-motion, then $x \in \omega^0(x,k)$ (this is evident).

Proposition 4.10. Let $x_i \in X,\ k_i \in K,\ k_i \to k^*,\ \varepsilon_i > 0,\ \varepsilon_i \to 0,\ f^{\varepsilon_i}(t|x_i,k_i)$ be (k_i,x_i,ε_i)-motion, $y_i \in \omega(f^{\varepsilon_i}(t|x_i,k_i)),\ y_i \to y^*$. Then $y^* \in \omega^0(y,k)$. If simultaneously $x_i \to x^*$ then $y^* \in \omega^0(x,k)$.

Proof. Let $\varepsilon > 0$ and $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2},T)$. It is possible to find such i that $\varepsilon_i < \delta/2,\ \rho_K(k_i,k^*) < \delta$, and $\rho(f^{\varepsilon_i}(t_j|x_i,k_i),y^*) < \delta$ for some monotone sequence $t_j \to \infty,\ t_{j+1} - t_j > T$. Suppose

$$f^*(t) = \begin{cases} y^*, & \text{if } t = t_j - t_1\ (j = 1,2,\ldots); \\ f^{\varepsilon_i}(t + t_1|x_i,k_i), & \text{otherwise,} \end{cases}$$

where $f^*(t)$ is (k^*,y^*,ε)-motion, $y^* \in \omega(f^*)$. Since $\varepsilon > 0$ was chosen arbitrarily, $y^* \in \omega^0(y^*,k^*)$. Suppose now that $x_i \to x^*$ and let us show that $y^* \in \omega^0(x^*,k^*)$. Let $\varepsilon > 0,\ \delta = \frac{1}{2}\delta(\frac{\varepsilon}{2},T)$. Find such i for which $\varepsilon_\tau < \delta/2,\ \rho(x_i,x^*) < \delta,\ \rho_K(k_i,k^*) < \delta$ and there is such monotone subsequence $t_j \to \infty$ that $t_1 > T,\ t_{j+1} - t_j > T;\ \rho(f^{\varepsilon_i}(t_j|x_i,k_i),y^*) < \delta$. Suppose

$$f^*(t) = \begin{cases} x^*, & \text{if } t = 0; \\ y^*, & \text{if } t = t_j (j = 1,2,\ldots); \\ f^{\varepsilon_i}(t|x_i,k_i), & \text{otherwise,} \end{cases}$$

where $f^*(t)$ is (k^*,x^*,ε)-motion and $y^* \in \omega(f^*)$. Consequently, $y \in \omega^0(x^*,k^*)$.

Corollary 4.1. If $x \in X,\ k \in K,\ y^* \in \omega^0(x,k)$ then $y^* \in \omega^0(y^*,k)$.

Corollary 4.2. Function $\omega^0(x,k)$ is upper semicontinuous in $X \times K$.

Corollary 4.3. For any $k \in K$

$$\omega^0(k) \stackrel{\text{def}}{=} \bigcup_{x\in X} \omega^0(x,k) = \bigcup_{x\in X} \bigcap_{\varepsilon>0} \omega^\varepsilon(x,k) = \bigcap_{\varepsilon>0} \bigcup_{x\in X} \omega^\varepsilon(x,k). \tag{33}$$

Proof. Inclusion $\bigcup_{x\in X} \bigcap_{\varepsilon>0} \omega^\varepsilon(x,k) \subset \bigcap_{\varepsilon>0} \bigcup_{x\in X} \omega^\varepsilon(x,k)$ is evident. To prove the equality, let us take arbitrary element y of the right part of this inclusion. For any natural n there is such $x_n \in X$ that $y \in \omega^{1/n}(x_n,k)$. Using the proposition 4.10, we obtain $y \in \omega^0(y,k) \subset \bigcup_{x\in X} \bigcap_{\varepsilon>0} \omega^\varepsilon(x,k)$, and this proves the corollary.

Corollary 4.4. For any $k \in K$ the set $\omega^0(k)$ is closed and k-invariant, and the function $\omega^0(k)$ is upper semicontinuous in K.

Proof. k-invariance of $\omega^0(k)$ follows from the k-invariance of $\omega^0(x,k)$ for any $x \in X,\ k \in K$ (proposition 4.8), closure and semicontinuity follow from the proposition 4.10.

Note that the statements analogous to the corollaries 4.2. and 4.4 are incorrect for the true limit sets $\omega(x,k)$ and $\omega(k)$.

Proposition 4.11. Let $k \in K$, $Q \subset \omega^0(k)$ and Q be connected. Then $Q \subset \omega^0(y,k)$ for any $y \in Q$.

Proof. Let $y_1, y_2 \in Q$, $\varepsilon > 0$. Construct nonperiodical ε-motion which passes through the points y_1, y_2. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. With Q being connected, there is such finite set $\{x_1, \ldots, x_n\} \subset Q$ that $x_1 = y_1$, $x_n = y_2$ and $\rho(x_i, x_{i+1}) < \frac{1}{2}\delta$ $(i = 1, \ldots, n-1)$ and for every $i = 1, \ldots, n$ there is a periodical $(k, x_i, \delta/2)$-motion $f^{\delta/2}(t|x_i, k)$ (see the proposition 4.9 and the corollary 4.1). Let us choose for every $i = 1, \ldots, n$ such $T_i > T$ that $f^{\delta/2}(T_i|x_i, k) = x_i$. Construct a periodical (k, y_1, ε)-motion passing through the points $x_1, \ldots, x_n$ with the period $T_0 = 2\sum_{t=1}^{n} T_i - T_1 - T_n$: let $0 \le t \le T_0$, suppose

$$f^*(t) = \begin{cases} f^{\delta/2}(tx_1, k), & \text{if } 0 \le t < T; \\ f^{\delta/2}\left(t - \sum_{i=1}^{j-1} T_i | x_i, k\right), & \text{if } \sum_{i=1}^{j-1} T_i \le t < \sum_{i=1}^{j} T_i \ (j = 2, \ldots, n); \\ f^{\delta/2}\left(t - \sum_{i=1}^{n-1} T_i + \sum_{i=j+1}^{n} T_i | x_j, k\right), & \text{if } \sum_{i=1}^{n-1} T_i + \sum_{i=j+1}^{n} T_i \le t < \\ & < \sum_{i=1}^{n-1} T_i + \sum_{i=j}^{n} T_i. \end{cases}$$

If $mT_0 \le t < (m+1)T_0$, then $f^*(t) = f^*(t - mT_0)$. $f^*(t)$ is periodical (k, y_1, ε)-motion passing through y_2. Consequently (due to the arbitrary choice of $\varepsilon > 0$), $y_2 \in \omega^0(y,k)$ and (due to the arbitrary choice of $y_2 \in Q$) $Q \subset \omega^0(y_1, k)$. The proposition is proved.

Definition 4.3. Let us say that the system (1) possesses $\omega^0(x,k)$- $(\omega^0(k)$-)-*bifurcations*, if the function $\omega^0(x,k)$ $(\omega^0(k))$ is not lower semicontinuous (i.e. d-continuous) in $X \times K$. The point in which the lower semicontinuity gets broken is called the point of (corresponding) bifurcation.

Proposition 4.12. If the system (1) possesses ω^0 -bifurcations, then it possesses $\omega^0(x,k)$-bifurcations.

Proof. Assume that $\omega^0(k)$-bifurcations exist. Then there are such $k^* \in K$ (the point of bifurcation), $x^* \in \omega^0(k^*)$, $\varepsilon > 0$, and sequence $k_i \to k$, that $\rho^*(x^*, \omega^0(k_i)) > \varepsilon$ for any $i = 1, 2, \ldots$. Note that $\omega^0(x^*, k_i) \subset \omega^0(k_i)$, consequently, $\rho^*(x^*, \omega^0(x^*, k_i)) > \varepsilon$ for any i. However $x^* \in \omega^0(x^*, k^*)$ (corollary 4.1). Therefore $d(\omega^0(x^*, k^*),\ \omega^0(x^*, k_i)) > \varepsilon$, (x^*, k^*) is the point of $\omega^0(x,k)$-bifurcation.

Proposition 4.13. The sets of all points of discontinuity of the functions $\omega^0(x,k)$ and $\omega^0(k)$ are subsets of first category in $X \times K$ and K correspondingly. For each $k \in K$ the set of such $x \in X$ that (x,k) is the point of $\omega^0(x,k)$-bifurcation is $(k,+)$-invariant.

Proof. The statement that the sets of points of $\omega^0(x,k)$- and $\omega^0(k)$-bifurcations are the sets of first category follows from the upper semicontinuity of the functions $\omega^0(x,k)$ and $\omega^0(k)$ and from known theorems about semicontinuous functions [59, p.78-81]. Let us prove $(k,+)$-invariance. Note that for any $t > 0$ $\omega^0(f(t,x,k),k) = \omega^0(x,k)$. If $(x_i, k_i) \to (x,k)$, then $(f(t, x_i, k_i), k_i) \to (f(t,x,k),k)$. Therefore, if $(x,k)$is the point of $\omega^0(x,k)$-bifurcation, then $(f(t,x,k),k)$ is also the point of $\omega^0(x,k)$-bifurcation for any $t > 0$.

Let (x_0, k_0) be the point of $\omega^0(x,k)$-bifurcation, Γ be a set of such $\gamma > 0$ for which there exist $x^* \in \omega^0(x_0, k_0)$ and such sequence $(x_i, k_i) \to (x_0, k_0)$ that $\rho^*(x^*, \omega^0(x_i, k_i)) \ge \gamma$ for all $i = 1, 2, \ldots$. Let us call the number $\tilde{\gamma} = \sup \Gamma$ *the value of discontinuity* of $\omega^0(x,k)$ in the point (x_0, k_0).

Proposition 4.14. Let $\gamma > 0$. The set of those $(x,k) \in X \times K$, in which the function $\omega^0(x,k)$ is not continuous and the value of discontinuity $\tilde{\gamma} \ge \gamma$, is nowhere dense in $X \times K$.

The proof can be easily obtained from the upper semicontinuity of the functions $\omega^0(x,k)$ and from known results about semicontinuous functions [59, p.78-81].

Proposition 4.15. If there is such $\gamma > 0$ that for any $\varepsilon > 0$ there are $(x,k) \in X \times K$ for which $d(\omega^\varepsilon(x,k), \omega^0(x,k)) > \gamma$ then the system (1) possesses $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma$.

Proof. Let the statement of the proposition be true for some $\gamma > 0$. Then there are sequences $\varepsilon_i > 0,\ \varepsilon_i \to 0$ and $(x_i, k_i) \in X \times K$, for which $d(\omega^{\varepsilon_i}(x_i,k_i), \omega^0(x_i,k_i)) > \gamma$. For every $i = 1, 2, \ldots$ choose such point $y_i \in \omega^{\varepsilon_i}(x_i,k_i)$ that $\rho^*(y_i, \omega^0(x_i,k_i)) > \gamma$. Using the compactness of X and K, choose subsequence (preserving the denotations) in such a way that the new subsequences y_i and (x_i,k_i) would be convergent: $y_i \to y_0,\ (x_i,k_i) \to (x_0,k_0)$. According to the proposition 4.10 $y_0 \in \omega^0(x_0,k_0)$. For any $\varkappa > 0$ $\rho^*(y_0, \omega^0(x_i,k_i)) > \gamma - \varkappa$ from some $i = i(\varkappa)$. Therefore (x_0,k_0) is the point of $\omega^0(x,k)$-bifurcation with the discontinuity $\tilde{\gamma} \geq \gamma$.

4.2 Slow Relaxations of ε-motions

Let $\varepsilon > 0,\ f^\varepsilon(t|x,k)$ be (k,x,ε)-motion, $\gamma > 0$. Let us define the following relaxation times:

$$
\begin{aligned}
&\text{(a)}\ \ \tau_1^\varepsilon(t|x,k),\gamma) = \inf\{t \geq 0 \mid \rho^*(f^\varepsilon(t|x,k), \omega^\varepsilon(x,k)) < \gamma\};\\
&\text{(b)}\ \ \tau_2^\varepsilon(f^\varepsilon(t|x,k),\gamma) = \overline{\mathrm{mes}}\{t \geq 0 \mid \rho^*(f^\varepsilon(t|x,k), \omega^\varepsilon(x,k)) \geq \gamma\};\\
&\text{(c)}\ \ \tau_3^\varepsilon(f^\varepsilon(t|x,k),\gamma) = \inf\{t \geq 0 \mid \rho^*(f^\varepsilon(t'|x,k), \omega^\varepsilon(x,k)) < \gamma \text{ for } t' > t\};\\
&\text{(d)}\ \ \eta_1^\varepsilon(f^\varepsilon(t|x,k),\gamma) = \inf\{t \geq 0 \mid \rho^*(t|x,k), \omega^\varepsilon(k)) < \gamma\};\\
&\text{(e)}\ \ \eta_2^\varepsilon(f^\varepsilon(t|x,k),\gamma) = \overline{\mathrm{mes}}\{t \geq 0 \mid \rho^*(f^\varepsilon(t|x,k), \omega^\varepsilon(k)) \geq \gamma\};\\
&\text{(f)}\ \ \eta_3^\varepsilon(f^\varepsilon(t|x,k), = \inf\{t \geq 0 \mid \rho^*(f^\varepsilon(t'|x,k), \omega^\varepsilon(k)) < \gamma \text{ for } t' > t\}.
\end{aligned}
\tag{34}
$$

Here $\overline{\mathrm{mes}}\{\quad\}$ is external measure, $\omega^\varepsilon(k) = \bigcup_{x\in X} \omega^\varepsilon(x,k)$.

There are another three important relaxation times. They are bound up with the hit of ε-motion in its ω-limit set. We do not consider them in this work.

Proposition 4.16. For any $x \in X,\ k \in K,\ \varepsilon > 0,\ \gamma > 0$ and (k,x,ε)-motion $f^\varepsilon(t|x,k)$ the relaxation times (34a-f) are defined (finite) and the inequalities $\tau_1^\varepsilon \leq \tau_2^\varepsilon \leq \tau_3^\varepsilon,\ \eta_1^\varepsilon \leq \eta_2^\varepsilon \leq \eta_3^\varepsilon,\ \tau_i^\varepsilon > \eta_i^\varepsilon\ (i = 1,2,3)$ are true.

Proof. The validity of the inequalities is evident due to the corresponding inclusions relations between the sets or their complements from the right parts of (34). For the same reason it is sufficient to prove definiteness (finiteness) of $\tau_3^\varepsilon(f^\varepsilon(t|x,k),\gamma)$. Suppose the contrary: the set from the right part of (34c) is empty for some $x \in X,\ k \in K,\ \gamma > 0$ and (k,x,ε)-motion $f^\varepsilon(t|x,k)$. Then there is such sequence $t_i \to \infty$ that $\rho^*(f^\varepsilon(t_i|x,k), \omega^\varepsilon(x,k)) \geq \gamma$. Owing to the compactness of X, from the sequence $f^\varepsilon(t_i|x,k)$ can be chosen a convergent one. Denote its limit as y. Then y satisfies the definition of ω-limit point of (k,x,ε)-motion but does not lie in $\omega^\varepsilon(x,k)$. The obtained contradiction proves the existence (finiteness) of $\tau_3(f^\varepsilon(t|x,k),\gamma)$.

In connection with the introduced relaxation times (34a-f) it is possible to study many different kinds of slow relaxations: infiniteness of the relaxation time for given ε, infiniteness for any ε small enough e.c. We will confine ourselves to one variant only. The most attention will be paid to the times τ_1^ε and τ_3^ε.

Definition 4.4. We say that the system (1) possesses τ_i^0- (η_i^0-)-*slow relaxations*, if there are such $\gamma > 0$, sequences of numbers $\varepsilon_j > 0,\ \varepsilon_j \to 0$, of points $(x_j,k_j) \in X \times K$, and of (k_j,x_j,ε_j)-motions $f^{\varepsilon_j}(t|x_j,k_j)$ that $\tau_i^{\varepsilon_j}(f^{\varepsilon_j}(t|x_j,k_j),\gamma) \to \infty$ ($\eta_i^{\varepsilon\times}(f^{\varepsilon_j}(t|x_j,k_j),\gamma) \to \infty$) as $j \to \infty$.

Theorem 4.1. The system (1) possesses τ_3^0-slow relaxations if and only if it possesses $\omega^0(x,k)$-bifurcations.

Proof. Suppose that the system (1) possesses τ_0^3-slow relaxations: there are such $\gamma > 0$, sequences of numbers $\varepsilon_j > 0$, $\varepsilon_j \to 0$, of points $(x_j,k_j) \in X \times K$ and of (k_j,x_j,ε_j)-motions $f^{\varepsilon_j}(t|x_j,k_j)$ that

$$\tau_3^{\varepsilon_j}(f^{\varepsilon_j}(t|x,k),\gamma) \to \infty \tag{35}$$

as $j \to \infty$.

Using the compactness of $X \times K$, choose from the sequence (x_j,k_j) a convergent one (preserving the denotations): $(x_j,k_j) \to (x^*,k^*)$. According to the definition of the relaxation time τ_3^ε there is such sequence $t_j \to \infty$ that

$$\rho^*(f^{\varepsilon_j}(t_j|x_j,k_j),\omega^{\varepsilon_j}(x_j,k_j)) \geq \gamma. \tag{36}$$

Choose again from (x_j,k_j) a sequence (preserving the denotations) in such a manner, that the sequence $y_j = f^{\varepsilon_j}(t_j|x_j,k_j)$ would be convergent: $y_j \to y^* \in X$. According to the proposition 4.6 (k^*,y^*)-motion is whole and $f^{\varepsilon_j}(t_j+t|x_j,k_j) \to f(t,y^*,k^*)$ uniformly over any compact segment $t \in [t_1,t_2]$. Two cases are possible: $\omega^0(y^*,k^*)\bigcap\alpha(y^*,k^*) \neq \varnothing$ or $\omega^0(y^*,k^*)\bigcap\alpha(y^*,k^*) = \varnothing$. We will show that in the first case there are $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$ ((y^*,k^*) is the point of bifurcation), in the second case there are $\omega^0(x,k)$-bifurcations too ((p,k^*) is the point of bifurcation, where p is any element from $\alpha(y^*,k^*)$), but the value of discontinuity can be less than $\gamma/2$. We need four lemmas.

Lemma 4.2. Let $x \in X$, $k \in K$, $\varepsilon > 0$, $f^\varepsilon(t|x,k)$ be (k,x,k)-motion, $t > 0, y = f^\varepsilon(t|x,k)$. Then $\omega^0(y,k) \subset \omega^{2\varepsilon+\sigma}(x,k)$ for any $\sigma > 0$.

The proof is an evident consequence of the definitions and the proposition 4.2.

Lemma 4.3. Let $x \in X$, $k \in K$, $t_0 > 0$, $y = f(t_0,x,k)$, $,\delta > 0$, $\varepsilon = \varepsilon(\chi(\delta,t_0,T),T)+\delta$. Then $\omega^\delta(x,k) \subset \omega^\varepsilon(y,k)$.

Proof. Let $f^\delta(t|x,k)$ be (k,x,δ)-motion. Then

$$f^*(t) = \begin{cases} y, & \text{if } t = 0; \\ f^\delta(t+t_0|x,k), & \text{if } t > 0, \end{cases} \tag{37}$$

is (k,y,ε)-motion, $\omega(f^*) \subset \omega^\varepsilon(y,k)$, and $\omega(f^*) = \omega(f^\delta(t|x,k))$.

Since $\varepsilon(\chi(\delta,t_0,T),T) \to 0$, for $\delta \to 0$ we obtain

Corollary 4.5. Let $x \in X$, $k \in K$, $t_0 > 0, y = f(t_0,x,k)$. Then $\omega^0(x,k) = \omega^0(y,k)$.

Lemma 4.4. Let (k,y)-motion be whole and $\omega^0(y,k)\bigcap\alpha(y,k) \neq \varnothing$. Then $y \in \omega^0(y,k)$.

Proof. Let $\varepsilon > 0$, $p \in \omega^0(y,k)\bigcap\alpha(y,k)$. Let us construct a periodical (k,y,ε)-motion. Suppose that $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{2},T)$. There is such $t_1 > T$ that for some (k,y,δ)-motion $f^\delta(t|y,k)$ $\rho(f^\delta(t_1|y,k),p) < \delta$. There is also such $t_2 < 0$ that $\rho(f(t_2,y,k),p) < \delta$. Then it is possible to construct a periodical (k,y,ε)-motion, due to the arbitrariness of $\varepsilon > 0$ and $y \in \omega^0(y,k)$.

Lemma 4.5. Let $y \in X$, $k \in K$, (k,y)-motion be whole. Then for any $p \in \alpha(y,k)$ $\omega^0(p,k) \supset \alpha(y,k)$.

Proof. Let $p \in \alpha(y,k)$, $\varepsilon > 0$. Let us construct a periodical (k,p,ε)-motion. Suppose that $\delta = \frac{1}{2}\delta(\varepsilon,T)$. There are two such $t_1,t_2 < 0$ that $t_1 - t_2 > T$ and $\rho(f(t_{1,2},y,k),p) < \delta$.

Suppose

$$f^*(t) = \begin{cases} p, & \text{if } t = 0 \text{ or } t = t_1 - t_2; \\ f(t+t_2|y,k), & \text{if } 0 < t < t_1 - t_2, \end{cases}$$

where $f^*(t+n(t_1-t_2)) = f^*(t)$. Periodical (k,p,ε)-motion is constructed. Since $\varepsilon > 0$ is arbitrary, $p \in \omega^0(p,k)$. Using the proposition 4.11 and the connectivity of $\alpha(y,k)$, we obtain the required: $\alpha(y,k) \subset \omega^0(p,k)$.

Let us return to the proof of the theorem 4.1. Note that according to the proposition 4.15 if there are not $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$, then from some $\varepsilon_0 > 0$ (for $0 < \varepsilon \leq \varepsilon_0$) $d(\omega^\varepsilon(x,k), \omega^0(x,k)) \leq \gamma/2$ for any $x \in X,\ k \in K$. Suppose that the system has τ_3^0-slow relaxations and does not possess $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$. Then from (36) follows that for $0 < \varepsilon \leq \varepsilon_0$

$$p^*(f^{\varepsilon_j}(t_j|x_j,k_j), \omega^\varepsilon(x_j,k_j)) \geq \gamma/2. \tag{38}$$

According to the lemma 4.2 $\omega^0(y_j,k_j) \subset \omega^{3\varepsilon_j}(x_i,k_j)$. Let $0 < \varkappa < \gamma/2$. From some j_0 (for $j > j_0$) $3\varepsilon_j < \varepsilon_0$ and $\rho(f^{\varepsilon_j}(t_j|x_j,k_j), y^*) < \gamma/2 - \varkappa$. For $j > j_0$ from (38) we obtain

$$\rho^*(y^*, \omega^0(y_j,k_j)) > \varkappa. \tag{39}$$

If $\omega^0(y^*,k^*) \bigcap \alpha(y^*,k^*) \neq \varnothing$, then from (39) and the lemma 4.4 follows the existence of $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$. The obtained contradiction (if $\omega^0(y^*,k^*) \bigcap \alpha(y^*,k^*) \neq \varnothing$ and there are not $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$, then they are) proves in this case the existence of $\omega^0(x,k)$-bifurcations with the discontinuity $\tilde{\gamma} \geq \gamma/2$. If $\omega^0(y^*,k^*) \bigcap \alpha(y^*,k^*) = \varnothing$, then there also exist $\omega^0(x,k)$-bifurcations. Really, let $p \in \alpha(y^*,k^*)$. Consider such a sequence $t_i \to -\infty$ that $f(t_i,y^*,k^*) \to p$. According to the corollary 4.5 $\omega^0(f(t_i,y^*,k^*),k^*) = \omega^0(y^*,k^*)$, consequently, according to the lemma 4.5, $d(\omega^0(p,k^*), \omega^0(f(t_i,y^*,k^*)) \geq d(\alpha(y^*,k^*), \omega^0(y^*,k^*)) > 0$ – there are $\omega^0(x,k)$-bifurcations. The theorem is proved.

Note that inverse to the theorem 4.1 is not true: for unconnected X from the existence of $\omega^0(x,k)$-bifurcations does not follow the existence of τ_3^0-slow relaxations.

Example 4.1. ($\omega^0(x,k)$-bifurcations without τ_3^0 -slow relaxations). Let X be a subset of plane, consisting of points with coordinates $(\frac{1}{n},0)$ and vertical segment $J = \{(x,y)|x = 0,\ y \in [-1,1]\}$. Let us consider on X a trivial dynamical system $f(t,x) \equiv x$. In this case $\omega_f^0((\frac{1}{n},0)) = \left\{(\frac{1}{n},0)\right\}$, $\omega_f^0((0,y)) = J$. There are $\omega^0(x,k)$ bifurcations: $(\frac{1}{n},0) \to (0,0)$ as $n \to \infty$, $\omega_f^0((\frac{1}{n},0)) = \left\{(\frac{1}{n},0)\right\}$, $\omega_f^0((0,0)) = J$. But there are not τ_0^3 -slow relaxations: $\tau_3^\varepsilon(f^\varepsilon(t|x),\gamma) = 0$ for any (x,ε)-motion $f^\varepsilon(t|x)$ and $\gamma > 0$. This is associated with the fact that for any (x,ε)-motion and arbitrary $t_0 \geq 0$ the following function

$$f^*(t) = \begin{cases} f^\varepsilon(t|x), & \text{if } 0 \leq t \leq t_0; \\ f^\varepsilon(t_0|x), & \text{if } t \geq t_0 \end{cases}$$

is (x,ε)-motion too, consequently, each (x,ε)-trajectory consists of the points of $\omega_f^\varepsilon(x)$.

For connected X the existence of $\omega^0(x,k)$-bifurcations is equivalent to the existence of τ_3^0-slow relaxations.

Theorem 4.2. Let X be connected. In this case the system (1) possesses τ_3^0 -slow relaxations if and only if it possesses $\omega^0(x,k)$-bifurcations.

One part of the theorem 4.2 (only if) follows from the theorem 4.1. Let us put off the proof of the other part of the theorem 4.2 till subsection 4.4, and the remained part of the present subsection devote to the study of the set of singularities of the relaxation time τ_2 for perturbed motions.

Theorem 4.3. Let $\gamma > 0,\ \varepsilon_i > 0,\ \varepsilon_i \to 0,\ (x_i,k_i) \in X \times K,\ f^{\varepsilon_i}(t|x_i,k_i)$ be (k_i,x_i,ε_i)-motions, $\tau_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i,k_i),\gamma) \to \infty$. Then any limit point of the sequence $\{(x_i,k_i)\}$ is a point of $\omega^0(x,k)$-bifurcation with the discontinuity $\tilde{\gamma} \geq \gamma$.

Proof. Let (x_0, k_0) be limit point of the sequence $\{(x_i, k_i)\}$. Turning to subsequence and preserving the denotations, let us write down $(x_i, k_i) \to (x_0, k_0)$. Let $X = \bigcup_{j=1}^n V_j$ be a finite open covering of X. Note that

$$\tau_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i, k_i)\gamma) \leq$$
$$\leq \sum_{j=1}^{n} \overline{\text{mes}}\{t \geq 0 \mid f^{\varepsilon_j}(t|x_i, k_i) \in V_j, \rho^*(f^{\varepsilon_i}(t|x_i, k_i), \omega^{\varepsilon_i}(x_i, k_i)) \geq \gamma\}.$$

Using this remark, consider a sequence of reducing coverings. Let us find (similarly to the proof of the theorem 3.1) such $y_0 \in X$ and subsequence in $\{(x_i, k_i)\}$ (preserving for it the previous denotation) that for any neighbourhood V of the point y_0

$$\overline{\text{mes}}\{t \geq 0 \mid f^{\varepsilon_i}(t|x_i, k_i) \in V, \ \rho(f^{\varepsilon_i}(t|x_i, k_i), \omega^{\varepsilon_i}(x_i, k_i)) \geq \gamma\} \to \infty.$$

Let us show that $y_0 \in \omega^0(x_0, k_0)$. Let $\varepsilon > 0$. Construct (k_0, x_0, ε)-motion with y_0 as its ω-limit point. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. From some i_0 (for $i > i_0$) the following inequalities are true: $\rho(x_i, x_0) < \delta$, $\rho_K(k_i, k_0) < \delta$, $\varepsilon_i < \delta$, and

$$\overline{\text{mes}}\{t \geq 0 \mid \rho(f^{\varepsilon_i}(t|x_i, k_i), y_0) < \delta, \ \rho^*(f^{\varepsilon_i}(t|x_i, k_i), \ \omega^{\varepsilon_i}(x_i, k_i)) \geq \gamma\} > T.$$

On account of the last of these inequalities for every $i > i_0$ there are such $t_1, t_2 > 0$ that $t_2 - t_1 > T$ and $\rho(f^{\varepsilon_i}(t_{1,2}|x_i, k_i), y_0) < \delta$. Let $i > i_0$. Suppose

$$f^*(t) = \begin{cases} x_0, & \text{if } t = 0; \\ f^{\varepsilon_i}(t|x_i, k_i), & \text{if } 0 < t < t_2, \ t \neq t_1; \\ y_0, & \text{if } t = t_1. \end{cases}$$

If $t \geq t_1$, then $f^*(t + n(t_2 - t_1)) = f^*(t)$, $n = 0, 1, \ldots$. By virtue of the construction f^* is (k_0, x_0, ε)-motion, $y_0 \in \omega(f^*)$. Consequently (due to the arbitrary choice of $\varepsilon > 0$), $y_0 \in \omega^0(x_0, k_0)$. Our choice of the point y_0 guarantees that $y_0 \notin \omega^{\varepsilon_i}(x_i, k_i)$ from some i. Furthermore, for any $\varkappa > 0$ exists such $i = i(\varkappa)$ that for $i > i(\varkappa)$ $\rho^*(y_0, \omega^{\varepsilon_i}(x_i, k_i) > \gamma - \varkappa$. Consequently, (x_0, k_0) is the point of $\omega^0(x, k)$-bifurcation with the discontinuity $\tilde{\gamma} \geq \gamma$.

Corollary 5.6. Let $\gamma > 0$. The set of all points $(x, k) \in X \times K$, for which there are such sequences of numbers $\varepsilon_i > 0$, $\varepsilon \to 0$, of points $(x_i, k_i) \to (x, k)$, and of $(k_i, x_i, \varepsilon_i)$-motions $f^{\varepsilon_i}(t|x_i, k_i)$ that $\tau_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i, k_i), \gamma) \to \infty$, is nowhere dense in $X \times K$. The union of all $\gamma > 0$ these sets (for all $\gamma > 0$) is a set of first category in $X \times K$.

4.3 Equivalence and Preorder Relations, Generated by Semiflow

Everywhere in this subsection one semiflow of homeomorphisms f on X is studied.

Definition 4.5. Let $x_1, x_2 \in X$. Say that points x_1 and x_2 are f-*equivalent* (denotation $x_1 \sim x_2$), if for any $\varepsilon > 0$ there are such (x_1, ε)- and (x_2, ε)-motions $f^\varepsilon(t|x_1)$ and $f^\varepsilon(t|x_2)$ that for some $t_1, t_2 > 0$

$$f^\varepsilon(t_1|x_1) = x_2, \ f^\varepsilon(t_2|x_2) = x_1.$$

Proposition 4.17. The relation $\sim$ is a closed f-invariant equivalence relation: the set of pairs (x_1, x_2), for which $x_1 \sim x_2$ is closed in $X \times K$; if $x_1 \sim x_2$ and $x_1 \neq x_2$, then x_1- and x_2-motions are whole and for any $t \in (-\infty, \infty)$ $f(t, x_1) \sim f(t, x_2)$. If $x_1 \neq x_2$, then $x_1 \sim x_2$ if and only if $\omega^0_f(x_1) = \omega^0_f(x_2),\ x_1 \in \omega^0_f(x_1),\ x_2 \in \omega^0_f(x_2)$ (compare with [52, ch.6, sec.1], where analogous propositions are proved for equivalence relation defined by action functional).

Proof. Symmetry and reflexivity of the relation $\sim$ are evident. Let us prove its transitivity. Let $x_1 \sim x_2,\ x_2 \sim x_3,\ \varepsilon > 0$. Construct ε-motions which go from x_1 to x_3, and from x_3 to x_1, gluing together δ-motions, going from x_1 to x_2, from x_2 to x_3 and from x_3 to x_2, from x_2 to x_1. Suppose that $\delta = \varepsilon/4$. Then, according to the proposition 4.2, as a result of the gluing we obtain ε-motions with required properties. Therefore $x_1 \sim x_3$.

Let us consider the closure of the relation $\sim$. Let $\varepsilon > 0,\ x_i, y_i \in X,\ x_i \to x,\ y_i \to y,\ x_i \sim y_i$. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. There is such i that $\rho(x_i, x) < \delta$ and $\rho(y_i, y) < \delta$. Since $x_i \sim y_i$, there are binding them δ-motions $f^\delta(t|y_i)$ and $f^\delta(t|x_i)$: $f^\delta(t_{1i}|x_i) = y_i,\ f^\delta(t_{2i}|y_i) = x_i$, for which $t_{1i}, t_{2i} > 0$. Suppose that

$$f_1^*(t) = \begin{cases} x, & \text{if } t = 0; \\ y, & \text{if } t = t_{1i}; \\ f^\delta(t|x_i), & \text{if } t \neq 0, t_{1i}, \end{cases}$$

$$f^*(t) = \begin{cases} y, & \text{if } t = 0; \\ x, & \text{if } t = t_{2i}; \\ f^\delta(t|y_i), & \text{if } t \neq 0, t_{2i}, \end{cases}$$

here f_1^* and f_2^* are correspondingly (x, ε)- and (y, ε)-motions, $f_1^*(t_{1i}) = y,\ f^*(t_{2i}) = x$. Since was chosen arbitrarily, it is proved that $x \sim y$. Let $x_{1,2} \in X,\ x_1 \sim x_2,\ x_1 \neq x_2$. Show that $\omega^0_f(x_1) = \omega^0_f(x_2)$ and $x_{1,2} \in \omega^0_f(x_1)$. Let $\varepsilon > 0,\ y \in \omega^0_f(x_2)$. Prove that $y \in \omega^\varepsilon_f(x_1)$. Really, let $f^{\varepsilon/3}(t|x_1)$ be $(x_1, \varepsilon/3)$-motion, $f^{\varepsilon/3}(t_0|x_1) = x_2,\ f^{\varepsilon/3}(t|x_2)$ be $(x_2, \varepsilon/3)$-motion, $y \in \omega(f^{\varepsilon/3}(f^{\varepsilon/3}(t|x_2))$. Suppose

$$f^*(t) = \begin{cases} f^{\varepsilon/3}(t|x_1), & \text{if } 0 \le t \le t_0; \\ f^{\varepsilon/3}(t_0|x_2), & \text{if } t > t_0, \end{cases}$$

here f^* is (x_1, ε)-motion (in accordance with the proposition 4.2), $y \in \omega(f^*)$. Consequently, $y \in \omega^\varepsilon(x_1)$ and, due to arbitrary choice of $\varepsilon > 0,\ y \in \omega^0(x_1)$. Similarly $\omega^0(x_1) \subset \omega^0(x_2)$, therefore $\omega^0(x_1) = \omega^0(x_2)$. It can be shown that $x_1 \in \omega^0_f(x_1),\ x_2 \in \omega^0_f(x_2)$. According to the proposition 4.8, the sets $\omega^0_f(x_{1,2})$ are invariant and $x_{1,2}$-motions are whole.

Now, let us show that if $x_2 \in \omega^0_f(x_1)$ and $x_1 \in \omega^0_f(x_2)$ then $x_1 \sim x_2$. Let $x_2 \in \omega^0_f(x_1),\ \varepsilon > 0$. Construct ε-motion going from x_1 to x_2. Suppose that $\delta = \delta = \frac{1}{2}\delta(\frac{\varepsilon}{2}, T)$. There is (x_1, δ)-motion with x_2 as its ω-limit point: $f^\delta(t_1|x_1) \to x_2,\ t_1 \to \infty$. There is such $t_0 > 0$ that $\rho(f^\delta(t_0|x_1), x_2) < \delta$. Suppose that

$$f^*(t) = \begin{cases} f^\delta(t|x_1), & \text{if } t \neq t_0; \\ x_2, & \text{if } t = t_0, \end{cases}$$

where $f^*(t)$ is (x_1, ε)-motion and $f^*(t_0) = x_2$. Similarly, if $x_1 \in \omega^0_f(x_2)$, then for any $\varepsilon > 0$ exists (x_2, ε)-motion which goes from x_2 to x_1. Thus, if $x_1 \neq x_2$, then $x_1 \sim x_2$ if and only if $x_1 \in \omega^0(x_2)$ and $x_2 \in \omega^0_f(x_1)$. In this case $\omega^0_f(x_1) = \omega^0_f(x_2)$. The invariance of the relation $\sim$ follows now from the invariance of the sets $\omega^0_f(x)$ and the fact that $\omega^0_f(x) = \omega^0_f(f(t, x))$ if $f(t, x)$ is defined. The proposition is proved.

Let us remind, that topological space is called *totally disconnected* if there exist a base of topology, consisting of sets which are simultaneously open and closed. Simple examples of such spaces are discrete space and Cantor discontinuum.

Proposition 4.18. Factor space $\omega_f^0/\sim$ is compact and totally disconnected.

Proof. This directly follows from the propositions 4.11, 4.17 and the corollary 4.4.

Definition 4.6. (*Preorder, generated by semiflow*). Let $x_1, x_2 \in X$. Let say $x_1 \succsim x_2$ if for any $\varepsilon > 0$ exists such (x_1, ε)-motion $f^\varepsilon(t|x_1)$ that $f^\varepsilon(t_0|x_1) = x_2$ for some $t_0 \geq 0$.

Proposition 4.19. The relation $\succsim$ is a closed preorder relation on X.

Proof. Transitivity of $\succsim$ easily follows from the proposition 4.2 about gluing of ε-motions. The reflexivity is evident. The closure can be proved similarly to the proof of the closure of $\sim$ (proposition 4.17, practically literal coincidence).

Proposition 4.20. Let $x \in X$. Then

$$\omega_f^0(x) = \{y \in \omega_f^0 \mid x \succsim y\}.$$

Proof. Let $y \in \omega_f^0(x)$. Let us show that $x \succsim y$. Let $\varepsilon > 0$. Construct ε-motion going from x to y. Suppose $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. There is (x, δ)-motion $f^\delta(t|x)$ with y as its ω-limit point: $f^\delta(t_j|x) \to$ for some sequence $t_j \to \infty$. There is such $t_0 > 0$ that $\rho(f^\delta(t_0|x), y) < \delta$. Suppose that

$$f^*(t) = \begin{cases} f^\delta(t|x), & \text{if } t \neq t_0; \\ y, & \text{if } t = t_0, \end{cases}$$

here $f^*(t)$ is (x, ε)-motion and $f^*(t_0) = y$. Consequently, $x \succsim y$. Now suppose that $y \in \omega_f^0$, $x \succsim y$. Let us show that $y \in \omega_f^0(x)$. Let $\varepsilon > 0$. Construct (x, ε)-motion with y as its ω-limit point. To do this, use the proposition 4.9 and the corollary 4.1 and construct a periodical $(y, \varepsilon/3)$-motion $f^{\varepsilon/3}$: $f^{\varepsilon/3}(nt_0|y) = y$, $n = 0, 1, \ldots$.. Glue it together with $(x, \varepsilon/3)$-motion going from x to $y(f^{\varepsilon/3}(t_1|x) = y)$:

$$f^*(t) = \begin{cases} f^{\varepsilon/3}(t|x), & \text{if } 0 \leq t \leq t_1; \\ f^{\varepsilon/3}(t - t_1|y), & \text{if } t \geq t_1, \end{cases}$$

where $f^*(t)$ is (x, ε)-motion, $y \in \omega(f^*)$, consequently ($\varepsilon > 0$ is arbitrary), $y \in \omega_f^0(x)$. The proposition is proved.

We say that the set $Y \subset \omega_f^0$ is *saturated downwards*, if for any $y \in Y$

$$\{x \in \omega_f^0 | y \succsim x\} \subset Y.$$

It is evident that every saturated downwards subset in ω_f^0 is saturated also for the equivalence relation $\sim$.

Proposition 4.21. Let $Y \subset \omega_f^0$ be open (in ω_f^0) saturated downwards set. Then the set $At^0(Y) = \{x \in X \mid \omega_f^0(x) \subset Y\}$ is open in X.

Proof. Suppose the contrary. Let $x \in At^0(Y)$, $x_i \to x$ and for every $i = 1, 2, \ldots$ there is $y_i \in \omega_f^0(x_i) \setminus Y$. On account of the compactness of $\omega_f^0 \setminus Y$ there is a subsequence in $\{y_i\}$, which converges to $y^* \in \omega_f^0 \setminus Y$. Let us turn to corresponding subsequences in $\{x_i\}$, $\{y_i\}$, preserving the denotations: $y_i \to y^*$. Let us show that $y \in \omega_f^0(x)$. Let $\varepsilon > 0$. Construct ε-motion going from x to y. Suppose that $\delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$. From some i_0 $\rho(x_i, x) < \delta$ and $\rho(y_i, y^*) < \delta$. Let $i > i_0$. There is (x_i, δ)-motion going from x_i to y_i : $f^\delta(t_0|x_i) = y_i$ (according to the proposition 4.20). Suppose that

$$f^*(t) = \begin{cases} f^*, & \text{if } t = 0; \\ y^*, & \text{if } t = t_0; \\ f^\delta(t|x_i), & \text{if } t \neq 0, t_0, \end{cases}$$

where f^* is $(x,)$-motion going from x to y^*. Since $\varepsilon > 0$ is arbitrary, from this follows that $x \gtrsim y^*$ and, according to the proposition 4.20, $y^* \in \omega_f^0(x)$. The obtained contradiction ($y^* \in \omega_f^0(x) \setminus Y$, but $\omega_f^0(x) subset Y$) proves the proposition.

Theorem 4.4. Let $x \in X$ be a point of $\omega_f^0(x)$-bifurcation. Then there is such open in ω_f^0 saturated downwards set W that $x \in \partial At^0(W)$.

Proof. Let $x \in X$ be a point of ω_f^0-bifurcation: there are such sequence $x_i \to x$ and such $y^* \in \omega_f^0(x)$ that $\rho^*(y^*, \omega_f^0(x_i)) > \alpha > 0$ for all $i = 1, 2, \ldots$. Let us consider the set $\omega = \bigcup_{i=1}^{\infty} \omega_f^0(x_i)$. The set ω is saturated downwards (according to the proposition 4.20). We have to prove that it possesses open (in ω_f^0) saturated downwards neighbourhood which does not contain y^*. Beforehand let us prove the following lemma.

Lemma 4.6. Let $y_1, y_2 \in \omega_f^0$, $y_1 \notin \omega_f^0(y_2)$. Then there is open saturated downwards set Y containing y_2 but not containing y_1. $Y \subset \omega_f^\varepsilon(y_2)$ for some $\varepsilon > 0$.

Proof. $\omega_f^0(y_2) = \bigcap_{\varepsilon > 0} \omega^\varepsilon(y_2)$ (according to the proposition 4.7). There are such $\varepsilon_0 > 0$, $\tau > 0$ that if $0 < \varepsilon \leq \varepsilon_0$ then $\rho^*(y_1, \omega^\varepsilon(y_2)) > \lambda$. This follows from the compactness of X and so called Shura-Bura's lemma [66, p.171-172]: let a subset V of compact space be intersection of some family of closed sets. Then for any neighbourhood of V exists a finite collection of sets from that family, intersection of which contains in given neighbourhood. Note now that if $\rho^*(z, \omega_f^0(y_2)) < \delta = \frac{1}{2}\delta(\frac{\varepsilon}{3}, T)$ and $z \in \omega_f^0$, then $z \in \omega_f^\varepsilon(y_2)$. Really, in this case there are such $p \in \omega_f^0(y_2)$, (y_2, δ)-motion $f^\delta(t|y_2)$, and monotone sequence $t_i \to \infty$ that $\rho(z, p) < \delta$, $t_{j+1} - t_j > T$ and $\rho(f^\delta(t_i|y_2), p) < \delta$. Suppose

$$f^*(t) = \begin{cases} f^\delta(t|y_2), & \text{if } t \neq t_j; \\ z, & \text{if } t = t_j, \end{cases}$$

here $f^*(t)$ is (y_2, ε)-motion and $z \in \omega(f^*) \subset \omega^\varepsilon(y_2)$. Strengthen somewhat this statement. Let $z \in \omega_f^0$ and for some $n > 0$ exist such chain $\{z_1, z_2, \ldots, z_n\} \in \omega_f^0$ that $y_2 = z_1$, $z = z_n$ and for any $i = 1, 2, \ldots, n-1$ either $z_i \gtrsim z_{i+1}$ or $\rho(z_i, z_{i+1}) < \delta = \frac{1}{2}\delta(\frac{\varepsilon}{7}, T)$. Then $z \in \omega^\varepsilon(y_2)$ and (y_2, ε)-motion with z as its ω-limit point is constructed as follows. If $z_i \gtrsim z_{i+1}$, then find (z_i, δ)-motion going from z_i to z_{i+1}, and for every $i = 1, \ldots, n$ find a periodical (z_i, δ)-motion. If $z_1 \gtrsim z_2$, then suppose that f_1^* is (z_1, δ)-motion going from z_1 to z_2, $f_1^*(t_1) = z_2$; and if $\rho(z_1, z_2) < \delta$, $z_1 \gtrsim z_2$, then suppose that f_1^* is a periodical (z_2, δ)-motion and $t_1 > 0$ is such a number that $t_1 > T$ and $f_1^*(t_1) = z_2$. Let $f_1^*, \ldots, f_k^*$, $t_1, \ldots, t_k$ be already determined. Determine f_{k+1}^*. Four variants are possible:

1) f^* is periodical (z_i, δ)-motion, $i < n$, $z_i \gtrsim z_{i+1}$, then f_{k+1}^* is (z_i, δ)-motion going from z_i to z_{i+1} $f_{k+1}^*(t_{k+1}) = z_{i+1}$;

2) f_k^* is periodical (z_i, δ)-motion, $i < n$, $\rho(z_i, z_{i+1}) < \delta$, then f_{k+1}^* is periodical (z_{i+1}, δ)-motion, $f_{k+1}^*(t_{k+1}) = z_{i+1}$, $t_{k+1} > T$;

3) f_k^* is (z_i, δ)-motion going from z_i to z_{i+1}, then f_{k+1}^* is periodical (z_{i+1}, δ)-motion, $f_{k+1}^*(t_{k+1}) = z_{i+1}$, $t_{k+1} > T$;

4) f_k^* is periodical (z_n, δ)-motion, then the constructing is finished.

Having constructed the whole chain of δ-motions f_k^* and time moments t_k, denote the number of its elements by q and suppose that

$$f^*(t) = \begin{cases} z_1, & \text{if } t = 0; \\ f_1^*(t), & \text{if } 0 < t \leq t_1; \\ f_k^*\left(t - \sum_{j=1}^{k-1} t_j\right), & \text{if } \sum_{j=1}^{k-1} t_j < t \leq \sum_{j=1}^{k} t_j (k < q); \\ f_q^*\left(t - \sum_{j=1}^{q-1} t_j\right), & \text{if } t > \sum_{j=1}^{q-1} t_j, \end{cases}$$

here $f^*(t)$ is (y_2, ε)-motion with $z_n = z$ as its ω-limit point. The set of those $z \in \omega_f^0$ for which exist such chains $z_1, \ldots, z_n$ $(n = 1, 2, \ldots)$ is an openly-closed (in ω_f^0) subset of ω_f^0, saturated downwards. Supposing $0 < \varepsilon \leq \varepsilon_0$, we obtain the needed result. Even more strong statement was proved: Y can be chosen openly-closed (in ω_f^0), not only open.

Let us return to the proof of the theorem 4.4. Since $\omega = \bigcup_{i=1}^{\infty} \omega_f^0(x_i)$ and each $z \in \omega_f^0(x_i)$ has an open (in ω_f^0) saturated downwards neighbourhood W_z which does not contain y^*, then the union of these neighborhoods is an open (in ω_f^0) saturated downwards set which includes ω but does not contain y^*. Denote this set by W: $W = \bigcup_{z \in \omega} W_z$. Since $x_i \in At^0(W)$, $x \notin At^0(W)$ and $x_i \to x$, then $x \in \partial At^0(W)$. The theorem is proved.

The following proposition will be used in 4.4 when studying slow relaxations of one perturbed system.

Proposition 4.22. Let X be connected, ω_f^0 be disconnected. Then there is such $x \in X$ that x-motion is whole and $x \notin \omega_f^0$. There is also such partition of ω_f^0 into openly-closed (in ω_f^0) subsets:

$$\omega_f^0 = W_1 \bigcup W_2,\ W \bigcap W_2 = \varnothing, \alpha_f(x) \subset W_1 \mathrm{but} \omega_f^0(x) \subset W_2.$$

Proof. Repeating the proof of the lemma 3.3 (the repetition is practically literal, ω_f^0 should be substituted instead of $\overline{\omega_f}$), we obtain that ω_f^0 is not Lyapunov stable. Then, according to the lemma 3.2, there is such $x \in X$ that x-motion is whole and $x \notin \omega_f^0$. Note now that the set $\alpha_f(x)$ lies in equivalence class by the relation $\sim$, and the set ω_f^0 is saturated by the relation $\sim$ (proposition 4.17, lemma 4.5). $\alpha_f(x) \bigcap \omega_f^0(x) = \varnothing$, otherwise, according to the proposition 4.17 and the lemma 4.4, $x \in \omega_f^0$. Since $\omega_f^0 / \sim$ is totally disconnected space (proposition 4.18), there exists partition of it into openly-closed subsets, one of which contains image of $\alpha_f(x)$ and the other contains image of $\omega_f^0(x)$ (under natural projection $\omega_f^0 \to \omega_f^0 / \sim$). Prototypes of these openly-closed sets form the needed partition of ω_f^0. The proposition is proved.

4.4 Slow Relaxations in One Perturbed System

In this subsection, as in the preceding one, we investigate one semiflow of homeomorphisms f over a compact space X.

Theorem 4.5. η_1^0- and η_2^0-slow relaxations are impossible for one semiflow.

Proof. It is enough to show that η_2^0-slow relaxations are impossible. Suppose the contrary: there are such $\gamma > 0$ and such sequences of numbers $\varepsilon_i > 0$ $\varepsilon_i \to 0$, of points $x_i \in X$ and of (x_i, ε_i)-motions $f^{\varepsilon_i}(t|x_i)$ that $\eta_2^{\varepsilon_i}(f^{\varepsilon_i}(t|x_i), \gamma) \to \infty$. Similarly to the proofs of the theorems 4.3 and 3.1, find a subsequence in $\{f^{\varepsilon_i}(t|x_i)\}$ and such $y^* \in X$ that $\rho^*(y^*, \omega_f^0) \geq \gamma$ and, whatever be the neighbourhood V of the point y^* in X, $\overline{\mathrm{mes}}\{t \geq 0 \mid f^{\varepsilon_i}(t|x_i) \in V\} \to \infty$ $(i \to \infty, f^{\varepsilon_i}(t|x_i)$ belongs to the chosen subsequence). As in the proof of the theorem 4.3, we have $y^* \in \omega_f^0(y^*) \subset \omega_f^0$. But, according to the constructing, $\rho^*(y^*, \omega_f^0) \geq \gamma > 0$. The obtained contradiction proves the absence of η_2^0-slow relaxations.

Theorem 4.6. Let X be connected. Then, if ω_f^0 is connected then the semiflow f has not $\tau_{1,2,3}^0$- and η_3^0-slow relaxations. If ω_f^0 is disconnected, then f possesses $\tau_{1,2,3}^0$- and η_3^0-slow relaxations.

Proof. Let X and ω_f^0 be connected. Then, according to the propositions 4.17 and 4.18, $\omega_f^0(x) = \omega_f^0$ for any $x \in X$. Consequently, $\omega^0(x)$-bifurcations are absent. Therefore (theorem 4.1) τ_3-slow relaxations are absent. Consequently, there are not other τ_i^0- and

η_i^0-slow relaxations due to the inequalities $\tau_i^\varepsilon \leq \tau_3^\varepsilon$ and $\eta_i^\varepsilon \leq \tau_3^\varepsilon$ $(i = 1, 2, 3)$ 1,2,3) (see proposition 4.16). The first part of the theorem is proved.

Suppose now that X is connected and ω_f^0 is disconnected. Let us use the proposition 4.22. Find such $x \in X$ that x-motion is whole, $x \notin \omega_f^0$, and such partition of ω_f^0 into openly-closed subsets $\omega_f^0 = W_1 \bigcup W_2$, $W_1 \bigcap W_2 = \varnothing$ that $\alpha_f(x) \subset W_1$, $\omega_f^0(x) \subset W_2$. Suppose $\gamma = \frac{1}{3} r(W_1, W_2)$. There is such t_0 that for $t < t_0$ $\rho^*(f(t,x), W_2) > 2\gamma$. Let $p \in \alpha_f(x)$, $t_j < t_0$, $t_j \to -\infty$, $f(t_j, x) \to p$. For each $j = 1, 2, \dots$ exists such $\delta_j > 0$ that for $\varepsilon < \delta_j$ $d(\omega_f^\varepsilon(f(t_j,x)), \omega_f^0(f(t_i,x))) < \gamma$ (this follows from Shura-Bura's lemma and the proposition 4.8). Since $\omega_f^0(f(t_j,x)) = \omega_f^0(x)$ (corollary 4.5), for $\varepsilon < \delta_j$ $d(\omega_f^\varepsilon(f(t_j,x)), W_2) < \gamma$. Therefore $\rho^*(f(t,x), \omega_f^\varepsilon(f(t_j,x))) > \gamma$ if $t \in [t_j, t_0]$, $\varepsilon > \delta_j$. Suppose $x_i = f(t_j,\ x)$, $\varepsilon_j > 0$, $\varepsilon_j < \delta_j$, $\varepsilon_j \to 0$, $f^{\varepsilon_j}(t|x_j) = f(t, x_j)$. Then $\tau_1^{\varepsilon_j}(f^{\varepsilon_j}(t|x_j), \gamma) \geq t_0 - t_i \to \infty$. The existence of τ_1- (and consequently of $\tau_{2,3}$-)-slow relaxations is proved. To prove the existence of η_3-slow relaxations we need the following lemma.

Lemma 4.7. For any $\varepsilon > 0$, $\varkappa > 0$

$$\overline{\omega_f^\varepsilon} \subset \omega_f^{\varepsilon+\varkappa}.$$

Proof. Let $y \in \overline{\omega_f^\varepsilon}$: there are such sequences of points $x_i \in X$, $y_i \in \omega_f^\varepsilon$, of (x_i, ε)-motions $f^\varepsilon(t|x_i)$, of numbers $t_j^i > 0$, $t_j^i \to \infty$ as $j \to \infty$ that $y_i \to y$, $f^\varepsilon(t_j^i|x_i) \to y_i$ as $j \to \infty$. Suppose that $\delta = \frac{1}{2}\delta(\frac{\varkappa}{3}, T)$. There is such y_i that $\rho(y_i, y) < \delta$. For this y_i there is such monotone sequence $t_j \to \infty$ that $t_j - t_{j-1} > T$ and $\rho(y_i, f^\varepsilon(t_j|x_i)) < \delta$. Suppose

$$f^*(t) = \begin{cases} f^\varepsilon(t|x_i), & \text{if } t \neq t_j; \\ y, & \text{if } t = t_j \ (j = 1, 2, \dots), \end{cases}$$

where $f^*(t)$ is $(x_i, \varkappa + \varepsilon)$-motion and $y \in \omega(f^*)$. Consequently, $y \in \omega_f^{\varepsilon+\varkappa}$. The lemma is proved.

Corollary 4.7. $\omega_f^0 = \bigcap_{\varepsilon > 0} \overline{\omega_f^\varepsilon}$.

Let us return to the proof of the theorem 4.6 and show the existence of η_3^0-slow relaxations if X is connected and ω_f^0 is not. Suppose that $\gamma = \frac{1}{5} r(W_1, W_2)$. Find such $\varepsilon_0 > 0$ that for $\varepsilon < \varepsilon_0$ $d(\omega_f^\varepsilon, \omega_f^0) > \gamma$ (it exists according to the corollary 4.7 and Shura-Bura's lemma). There is t_1 for which $d(f(t_1, x), \omega_f^0) > 2\gamma$. Let $t_j < t_1$, $t_j \to -\infty$, $x_j = f(x_j, x)$. As (x_j, ε)-motions let choose true motions $f(t, x_j)$. Suppose that $\varepsilon_j \to 0$, $0 < \varepsilon_j < \varepsilon_0$. Then $\eta^{\varepsilon_j}(f(t, x_j), \gamma) > t_1 - t_j \to \infty$ and, consequently, η_3^0-slow relaxations exist. The theorem is proved.

In conclusion of this subsection let us give *the proof of the theorem 4.2.* We again consider the family of parameter depending semiflows.

Proof. Let X be connected and $\omega^0(x,k)$-bifurcations exist. Even if for one $k \in K$ $\omega^0(k)$ is disconnected, then, according to the theorem 4.6, τ_3^0-slow relaxations exist. Let $\omega^0(k)$ be connected for any $k \in K$. Then $\omega^0(x,k) = \omega^0(k)$ for any $x \in X$, $k \in K$. Therefore from the existence of $\omega^0(x,k)$-bifurcations follows in this case the existence of $\omega^0(k)$-bifurcations. Thus, the theorem 4.2 follows from the following lemma which is of interest by itself too.

Lemma 4.8. If the system (1) possesses $\omega^0(k)$-bifurcations, then it possesses τ_3^0- and η_3^0-slow relaxations.

Proof. Let k^* be a point of $\omega^0(k)$-bifurcation: and there are such $\alpha > 0$, $y^* \in \omega^0(k^*)$ that $\rho^*(y^*, \omega^0(k_i)) > \alpha > 0$ for any $i = 1, 2, \dots$. According to the corollary 4.7 and Shura-Bura's lemma, for every i exists $\delta_i > 0$ for which $\rho^*(y^*, \omega^{\delta_i}(k_i)) > 2\alpha/3$. Suppose that

$0 < \varepsilon_i \le \delta_i$, $\varepsilon_i \to 0$. As ε-motions appearing in the definition of slow relaxations take the real (k_i, y_i)-motions, where $y_i = f(-t_i, y^*, k^*)$, and t_i are determined as follows:

$$t_i = \sup\{t > 0 \mid \rho(f(t', x, k), f(t'x, k')) < \alpha/3\}$$

under the conditions $t' \in [0, t],\ x \in X,\ \rho_K(k, k') < \rho_K(k^*, k_i)\}$.

Note that $\rho^*(f(t_i, y_i, k_i), \omega^{\varepsilon_i}(k_i)) \ge \alpha/3$, consequently, $\eta_3^{\varepsilon_i}(f(t, y_i, k_i), \alpha/4) > t_i$ and $t_i \to \infty$ as $i \to \infty$. The last follows from the compactness of X and K (see the proof of the proposition 4.1). Thus, η_3^0-slow relaxations exist and then τ_3^0-slow relaxations exist too. The lemma 4.8 and the theorem 4.2 are proved.

* *

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In the sections 1-4 the fundamental notions of the theory of transition processes and slow relaxations are stated. Two directions of further development of the theory are possible: introduction of new relaxation times and performing the same studies for them or widening the circle of solved problems and supplementing the obtained existence theorems with analytical results.

Among interesting but unsufficiently explored relaxation times let us mention the approximation time

$$\tau(x, k, \varepsilon) = \inf\{t \ge 0 \mid d(\omega(x, k), f([0, t], x, k)) < \varepsilon\}$$

and the averaging time

$$\tau_v(x, k, \varepsilon, \varphi) = \inf\left\{t \ge 0 \ \middle|\ |\frac{1}{t'}\int_0^{t'} \varphi(f(\tau, x, k))d\tau - \langle\varphi\rangle_{x,k}| < \varepsilon \text{ for } t' > t\right\},$$

here $\varepsilon > 0$, φ is a continuous function over the phase space X,
$\langle\varphi\rangle_{x,k} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(f(t, x, k))d\tau$ (if the limit exists).

The approximation time indicates the time necessary for the motion to visit ε-neighbourhood of each its ω-limit point. The averaging time depends on continuous function φ and shows the time necessary for establishing the average value of φ with accuracy ε along the trajectory.

As the most important problem of analytical research should be considered the problem of studying the asymptotical behaviour under $T \to \infty$ of "retardation domains" – the sets of those pairs (x, k) (the initial condition, parameter) for which $\tau_i(x, k, \varepsilon) > T$ (or $\eta_i(x, k, \varepsilon) > T$). The first (and simple) problem here is to study typical one-parameter families of two-dimensional dynamical systems. Such estimations for particular two-dimensional system are given in the work [67].

"Structurally stable systems are not dense." It would not be exaggeration to say that so titled work by Smeil [39] opened a new era in the understanding of dynamics. Structurally stable (rough) systems are those whose phase portraits do not change qualitatively under small perturbations (accurate definitions with detailed motivation see in [5]). Smeil constructed such structurally unstable system that any other system close enough to it is also structurally unstable. This result broke the hopes to classify if not all then "almost all" dynamical systems. Such hopes were associated with the successes of classification of two-dimensional dynamical systems [13,68] among which structurally stable ones are dense.

There are quite a number of attempts to correct the catastrophic situation with structural stability: to invent such natural notion of stability, for which almost all systems would be stable. The most successful (to authors' opinion) attempt was undertaken in the works [69–71] where is shown that if to weaken the definition of structural stability in such a way: stable are the systems whose almost all trajectories change little under small perturbations, then this stability will be already typical, almost all systems are stable in this sense.

The other way to get rid of the "Smeil nightmare" (the existence of domains of structurally unstable systems) is to consider ε-motions, subsequently considering (or not) the limit $\varepsilon \to 0$. The obtained picture (even at limit $\varepsilon \to 0$) is more stable than the phase portrait (the accurate formulation see above in sec. 4). It seems evident that at first should be studied those (more rough) details of dynamics, which do not disappear under of small perturbations.

The approach based on consideration of limit sets of ε-motions, in stated here final form was proposed in the paper [22]. It is necessary to note conceptual proximity of this approach to the method of quasi-averages in statistical physics [72]. By analogy, the stated approach could be called the method of "quasi-limit" sets.

Unfortunately, elaborated analytical or numerical methods of studying (constructing or, wider, localizing) limit sets of ε-motions for dynamical systems of general type are absent. Author do not give up the hope for the possibility of elaboration of such methods.

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