SPECTRAL PROPERTIES OF PERTURBED MULTIVORTEX AHARONOV-BOHM HAMILTONIANS

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Abstract. The diamagnetic inequality is established for the Schrödinger operator $H_0^{(d)}$ in $L^2(\mathbb{R}^d)$, $d = 2, 3$, describing a particle moving in a magnetic field generated by finitely or infinitely many Aharonov-Bohm solenoids located at the points of a discrete set in $\mathbb{R}^2$, e.g., a lattice. This fact is used to prove the Lieb-Thirring inequality as well as CLR-type eigenvalue estimates for the perturbed Schrödinger operator $H_0^{(d)} - V$, using new Hardy type inequalities. Large coupling constant eigenvalue asymptotic formulas for the perturbed operators are also proved.

Contents

1. Introduction and main results

Consider a non-relativistic, spinless quantum particle in $\mathbb{R}^d$, $d = 2, 3$, interacting with a magnetic field B associated with finitely or infinitely many thin solenoids aligned along the x_3-axis which pass through the points λ of some discrete subset Λ of the x_1x_2 plane. The magnetic flux through each solenoid is a noninteger α_λ. If, moreover, the radii of the solenoids tend to zero, whilst the flux α_λ through each solenoid remains constant then one obtains a particle moving in $\mathbb{R}^d$ subject to a

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finite or an infinite sum of δ-type magnetic fields, so-called Aharonov-Bohm fields or magnetic vortices, located at the points of Λ which may be interpreted as infinitely small impurities within a superconductor. Setting $\Lambda^d = \Lambda \times \mathbb{R}^{d-2}$, the multiply-connected region $\mathbb{R}^d \backslash \Lambda^d$, in which the field B equals zero, represents the configuration space. In the case of a lattice (defined by $\lambda_{kl} = k\boldsymbol{\omega}_1 + l\boldsymbol{\omega}_2$, where $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ are vectors in $\mathbb{R}^2$ and k, l runs over the whole of $\mathbb{Z}$ or a subset of $\mathbb{Z}$) such a situation occurs experimentally in GaAs/AlGaAs heterostructures coated with a film of type-II superconductors [6].

The vector potential $\boldsymbol{A}(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$ associated with B is chosen such that

$$A_1(x_1, x_2) = \operatorname{Im} \mathcal{A}(x_1, x_2), \quad \text{and} \quad A_2(x_1, x_2) = \operatorname{Re} \mathcal{A}(x_1, x_2), \tag{1.1}$$

where $\mathcal{A}(z) = \mathcal{A}(x_1, x_2)$, $z = x_1 + ix_2$, is a meromorphic function having simple poles at $\lambda \in \Lambda$ with residues α_λ; existence (and examples) of such a function $\mathcal{A}(z)$ is discussed in Section 2. One easily verifies that

$$\partial_{x_1} A_2 - \partial_{x_2} A_1 = \sum_{\lambda \in \Lambda} \alpha_\lambda \delta(z - \lambda) = B$$

in the sense of distributions; as usual, it suffices to consider $\alpha \in (0, 1)$ due to gauge invariance.

The dynamics of a spinless particle moving in any of the above-mentioned configurations of Aharonov-Bohm (abbrev. A-B) solenoids in $\mathbb{R}^d$ is described by the Schrödinger operator

$$H_0^{(d)} = -(\nabla + i\boldsymbol{A})^2 \tag{1.2}$$

acting in $L^2(\mathbb{R}^d)$, where ∇ is the gradient on $\mathbb{R}^d$. Since the singularities of the A-B magnetic potential are very strong, the operator defined initially on functions with support away from the singularities is not essentially self-adjoint. In Section 2 we define the Friedrichs extension of $H_0^{(d)}$ by means of quadratic forms. In the case of a single A-B solenoid the corresponding standard A-B Schrödinger operator has been studied intensively in two dimensions and there is an ongoing discussion on the mathematical and physical reasonability of different self-adjoint extensions [31, 1, 11, 15, 35]. The Friedrichs extension considered herein corresponds to the model of solenoids being non-penetrable for electrons, and, moreover, with interaction preserving circular symmetry [1].

Within the theory of Schrödinger operators with magnetic fields $L(\boldsymbol{A}) = -(\nabla + i\boldsymbol{A})^2$ associated with a vector potential $\boldsymbol{A} = (A_1, \dots, A_d)$ satisfying $A_j \in L^2_{loc}(\mathbb{R}^d)$, one of the fundamental facts is the diamagnetic inequality [3], viz., $|e^{-tL(\boldsymbol{A})}u| \leq e^{-tL_0}|u|$ for all $t \geq 0$ and all $u \in L^2(\mathbb{R}^d)$; here L_0 denotes the negative Laplacian in $L^2(\mathbb{R}^d)$.

In Section 4 we show that this inequality is valid also for the Schrödinger operator $H_0^{(d)}$ in $L^2(\mathbb{R}^d)$ for *any* of the afore-mentioned A-B configurations.

Theorem 1.1. *The inequality*

$$|e^{-tH_0^{(d)}}u| \leq e^{-tL_0}|u|$$

holds for all $t \geq 0$ *and all* $u \in L^2(\mathbb{R}^d)$

This result does not follow directly from the known diamagnetic inequality since the components (1.1) of the vector potential do not belong to $L^2_{loc}(\mathbb{R}^d)$; this latter condition is crucial in all existing proofs of the diamagnetic inequality for Schrödinger operators with magnetic fields.

Our proof of Theorem 1.1 uses a recent criterion (see Section 3) for the domination of semigroups due to Ouhabaz [24][1]. This criterion is a generalization (from operators to forms) of the Simon-Hess-Schrader-Uhlenbrock test for domination of semigroups [14].

As the first application of the diamagnetic inequality we establish the Lieb-Thirring inequality for the perturbed Schrödinger operator $H_0^{(d)} - V$ in Section 6. Here the electrostatic potential V is a nonnegative, measurable function on $\mathbb{R}^d$ belonging to an appropriate class of functions, which guarantees that the form sum $H_0^{(d)} - V$ generates a semi-bounded, self-adjoint operator in $L^2(\mathbb{R}^d)$ with discrete spectrum below zero.

The classic Lieb-Thirring inequality [19] for a d-dimensional Schrödinger operator $L_0 - V$ in $L^2(\mathbb{R}^d)$, with $L_0 = -\Delta$ as above and $d \geq 1$, says that

$$\sum_j |\nu_j(L_0 - V)|^\gamma \leq b_d(\gamma) \int_{\mathbb{R}^d} V(x)^{\gamma + \frac{d}{2}}\, dx, \tag{1.3}$$

where $\nu_j(L_0 - V)$ denote the negative eigenvalues of $L_0 - V$, $\gamma > 0$ ($\gamma \geq 1/2$ for $d = 1$) and $V \in L^{\gamma + \frac{d}{2}}$. The constant $b_d(\gamma)$ is expressible in terms of Γ-functions. The Lieb-Thirring inequality plays a crucial role in the problem of stability of matter (see, e.g., [20]), where the exact value of the constant is important. One way of establishing (1.3) is to use the Cwikel-Lieb-Rozenblum (abbrev. CLR) estimate (see, e.g., [27]) which, in its original form, reads

$$N_-(L_0 - V) \leq C(d) \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}}\, dx, \quad d \geq 3. \tag{1.4}$$

Here N_- denotes the number of negative eigenvalues of a self-adjoint operator, provided its negative spectrum is discrete. The single assumption, under which (1.4) is valid, is the finiteness of the integral

[1] It might be possible to prove Theorem 1.1 from general results in [21] which, in their turn, are based on [24], but we prefer to give a direct proof.

on its right-hand side. In [33, p 99-100] it is shown how one can obtain (1.3) provided (1.4) holds. This, however, does not produce the optimal constant in the Lieb-Thirring inequality.

The Lieb-Thirring inequality for d-dimensional Schrödinger operators with magnetic fields $L(\boldsymbol{A}) - V$, with $d \geq 3$ and $A_j \in L^2_{loc}(\mathbb{R}^d)$, takes the same form and can be obtained from the CLR-estimate for $L(\boldsymbol{A}) - V$ which is shown by means of the diamagnetic inequality (see, e.g., [33, p 168]).

In two dimensions there exist certain CLR-type estimates both for $L_0 - V$ [34, 7] and $L(\boldsymbol{A}) - V$ [29], provided $A_j \in L^2_{loc}(\mathbb{R}^2)$ for the latter operator. However, unlike in higher dimensions, these estimates, having a different form, do not produce Lieb-Thirring inequalities. Moreover, in our case, the components (1.1) of the magnetic potential do not belong to $L^2_{loc}(\mathbb{R}^2)$. Therefore the question on Lieb-Thirring inequalities for the perturbed Schrödinger operator $H_0^{(d)} - V$ was up to now open.

In the present paper we establish the following Lieb-Thirring inequality for the perturbed Schrödinger operator $H_0^{(d)} - V$ in $L^2(\mathbb{R}^d)$, $d = 2, 3$, for any of the afore-mentioned configurations of A-B solenoids.

Theorem 1.2. *Let ν_j denote the negative eigenvalues of $H_0^{(d)} - V$, $d = 2, 3$. If, moreover, $\gamma > 0$ and $V \in L^{\gamma+(d/2)}(\mathbb{R}^d)$ then*

$$\sum_j |\nu_j|^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d}{2}}\, dx,$$

where the constant $C_{\gamma,d}$ fulfills the following upper bounds for the most interesting values of γ:

$$C_{\gamma,2} \leq \begin{cases} 0.5300 & \text{for } \gamma = 1/2, \\ 0.3088 & \text{for } \gamma = 1, \\ 0.2275 & \text{for } \gamma = 3/2. \end{cases}$$

and

$$C_{\gamma,3} \leq \begin{cases} 0.1542 & \text{for } \gamma = 1/2, \\ 0.0483 & \text{for } \gamma = 1, \\ 0.0270 & \text{for } \gamma = 3/2. \end{cases}$$

We note that the expression we obtain for the best constant in Theorem 1.2 is implicit; see (6.3).

The diamagnetic inequality is one out of the two crucial ingredients in the proof of Theorem 1.2. The other is an abstract CLR-estimate for generators of semigroups dominated by positive semigroups. To make the paper self-explanatory we formulate this rather recent result, obtained by Rozenblum and Solomyak, in Section 5.

One of the important applications of eigenvalue estimates for Schrödinger operators is to deduce asymptotic formulas for the eigenvalues when the coupling constant q is present and it tends to infinity. The technology of getting the asymptotic formulas from the estimates is well-established nowadays (see, e.g., [27] and [8, 10]), and what is required from the estimates is that they have correct order in the coupling constant. For weakly singular magnetic fields such estimates were obtained by Lieb (see [33]) and Melgaard-Rozenblum [22] in dimensions $d \geq 3$, and by Rozenblum-Solomyak [29] in dimension $d = 2$ (see also [30]). In the case of a single A-B solenoid, the only existing estimate for the corresponding A-B Schrödinger operator $H_{AB}^{(2)} - qV$, by Balinsky, Evans and Lewis [5], deals with a rather special case of a radially symmetric potential (or with one majorized by a radially symmetric potential). There were no preceding results concerning eigenvalue estimates for many solenoids.

Based upon the diamagnetic inequality we establish CLR-type estimates (i.e. estimates having correct order in coupling constant q) for $H_0^{(d)} - qV$ for any of the A-B configurations mentioned above. To achieve this, we derive Hardy-type inequalities for each configuration, which allows us to carry over recent CLR-type estimates for the negative eigenvalues of two-dimensional Schrödinger operators, with a regularizing positive (Hardy) term added, to the operators $H_0^{(d)} - qV$. The Hardy-type inequalities are of interest by themselves and complement the recent result by Balinsky [4]. For finitely many A-B solenoids we prove the Hardy-type inequality by using a conformal mapping. This idea belongs to Balinsky but we use another, more explicit realization, which gives a better control over the weight function in the Hardy-type inequality.

The CLR-type estimates are used to deduce the large coupling constant asymptotics for the eigenvalues of $H_0^{(d)} - qV$. The singular nature of the magnetic potential requires just a few modifications to the standard approach.

The magnetic flux parameters α_λ are nonintegers throughout the paper. If α_λ are integers, the resulting operator is gauge equivalent to the negative Laplacian in $L^2(\mathbb{R}^d)$. This, however, does not reflect itself in the Lieb-Thirring inequality but the eigenvalue estimates in Section 8 are no longer valid, as one can see, e.g., from the factor β^{-2} in formula (8.3).

2. The unperturbed Hamiltonian $H_0^{(d)}$

Choice of vector potential. As mentioned in the introduction, the vector potential $\boldsymbol{A}(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$ associated with B is chosen such that

$$A_1(x_1, x_2) = \operatorname{Im} \mathcal{A}(x_1, x_2) \text{ and } A_2(x_1, x_2) = \operatorname{Re} \mathcal{A}(x_1, x_2), \qquad (2.1)$$

where $\mathcal{A}(z) = \mathcal{A}(x_1, x_2)$, $z = x_1 + ix_2$, is a meromorphic function having (only) simple poles at $\lambda \in \Lambda$ with residues α_λ.

In the case where Λ is a finite set, say, $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, the function

$$\mathcal{A}(z) = \sum_{j=1}^{N} \frac{\alpha_{\lambda_j}}{z - \lambda_j}$$

has the desired properties. In the general case, where Λ is a discrete set with infinitely many points, without finite limit points, Mittag-Leffler's theorem guarantees the existence of a meromorphic function with the afore-mentioned properties, unique, up to an entire summand.

For an infinite regular lattice where all fluxes are equal to a noninteger α, we can construct such a function $\mathcal{A}(z)$ explicitly. Indeed let $\Phi(z)$ be an entire function such that its set of (only simple) zeros coincide with Λ. Then one can take $\mathcal{A}(z) = \alpha\Phi'(z)\Phi(z)^{-1}$. In particular, the Weierstrass function $\sigma(z)$ corresponding to the lattice can serve as $\Phi(z)$, and then $\Phi'(z)\Phi(z)^{-1}$ is the Weierstrass function $\zeta(z)$.

Magnetic quadratic forms. For $\boldsymbol{A} = (A_1, A_2)$ in (1.1) we observe that

$$A_1, A_2 \in L^\infty_{loc}(\mathbb{R}^d \backslash \Lambda^d).$$

Let

$$\Omega_n = \left(B(0,n) \times (-n,n)^{d-2}\right) \setminus \left(\cup_{\lambda\in\Lambda} B(\lambda, 1/n) \times \mathbb{R}^{d-2}\right), \ \ n \geq 2,$$

where $B(\lambda, r)$ denotes the disk with center λ and radius r. We define on $L^2(\Omega_n)$ (for each $n \geq 2$) the form

$$\mathfrak{h}_n^{(d)}[u,v] = \sum_{j=1}^{d} \int_{\Omega_n} \left(\frac{\partial u}{\partial x_j} + iA_j u\right) \overline{\left(\frac{\partial v}{\partial x_j} + iA_j v\right)} dx \qquad (2.2)$$

on the domain $\mathcal{D}(\mathfrak{h}_n^{(d)}) = H_0^1(\Omega_n)$. The form is closed since $A_1, A_2 \in L^\infty(\Omega_n)$. The associated self-adjoint, nonnegative operators are denoted by $H_n^{(d)}$.

Define, in addition, the (closed) form $\mathfrak{l}_n^{(d)}$ with the same form expression and domain as $\mathfrak{h}_n^{(d)}$ but with $A_1 = A_2 = 0$. The associated self-adjoint, nonnegative operators are denoted by $L_n^{(d)}$.

Define now the form $\mathfrak{h}^{(d)}$ by

$$\begin{aligned} \mathfrak{h}^{(d)}[u,v] &= \mathfrak{h}_n^{(d)}[u,v] \text{ if } u,v \in \mathcal{D}(\mathfrak{h}_n^{(d)}), \\ \mathcal{D}(\mathfrak{h}^{(d)}) &= \cup_n \mathcal{D}(\mathfrak{h}_n^{(d)}) = \cup_n H_0^1(\Omega_n). \end{aligned}$$

Lemma 2.1. *The form $\mathfrak{h}^{(d)}$ is closable.*

Proof. According to the definition, the form $\mathfrak{h}^{(d)}$ is closable if and only if any sequence $\{u_n\}$, $u_n \in \mathcal{D}(\mathfrak{h}^{(d)})$, for which

$$\lim_{n\to\infty} \|u_n\|_{L^2} = 0 \text{ and } \lim_{n,m\to\infty} \mathfrak{h}^{(d)}[u_n - u_m] = 0, \qquad (2.3)$$

satisfies $\lim_{n\to\infty} \mathfrak{h}^{(d)}[u_n] = 0$. First observe that (2.3) implies

$$C := \sup_n \mathfrak{h}^{(d)}[u_n]^{1/2} < \infty. \tag{2.4}$$

Take $\epsilon > 0$ and choose n_0 such that

$$\mathfrak{h}^{(d)}[u_n - u_m] \le \epsilon \text{ when } n, m \ge n_0 \tag{2.5}$$

Set, moreover, $K = \Omega_{n_0} \subset \mathbb{R}^d \backslash \Lambda^d$ such that $\operatorname{supp} u_{n_0} \subset K$. In view of (2.3),

$$\int_K |(\nabla + i\boldsymbol{A})(u_n - u_m)|^2 \, dx \le \mathfrak{h}^{(d)}[u_n - u_m] \longrightarrow 0 \text{ as } n, m \to \infty, \tag{2.6}$$

$$\int_K |u_n|^2 \, dx \longrightarrow 0 \text{ as } n \to \infty, \tag{2.7}$$

and, since $\boldsymbol{A}$ is bounded on K,

$$\int_K |\boldsymbol{A}u_n|^2 \, dx \longrightarrow 0 \text{ as } n \to \infty. \tag{2.8}$$

Now,

$$\left| \left(\int_K |\boldsymbol{A}(u_n - u_m)|^2 \, dx \right)^{1/2} - \left(\int_K |\nabla(u_n - u_m|^2 \, dx \right)^{1/2} \right| \\ \le \left(\int_K |(\nabla + i\boldsymbol{A})(u_n - u_m)|^2 \, dx \right)^{1/2}. \tag{2.9}$$

According to (2.8), the first term on the left-hand side of the latter inequality tends to zero as $n, m \to \infty$ and, due to (2.6), the same holds for the right-hand side. Thus,

$$\int_K |u_n - u_m|^2 + |\nabla(u_n - u_m)|^2 \, dx \longrightarrow 0 \text{ as } n, m \to \infty.$$

Since the form of the classical Dirichlet Laplacian is closable it follows from the latter relation, in conjunction with (2.7) that

$$\int_K |\nabla u_n|^2 \, dx \to 0, \quad \int_K |u_n|^2 \, dx \to 0, \text{ as } n \to \infty. \tag{2.10}$$

Now,

$$\begin{aligned} \mathfrak{h}^{(d)}[u_n] &= \mathfrak{h}^{(d)}[u_n, u_n - u_{n_0}] + \mathfrak{h}^{(d)}[u_n, u_{n_0}] \\ &\le \mathfrak{h}^{(d)}[u_n]^{1/2} \mathfrak{h}^{(d)}[u_n - u_{n_0}]^{1/2} + \mathfrak{h}^{(d)}[u_n, u_{n_0}]. \end{aligned} \tag{2.11}$$

It follows from (2.4) and (2.5) that

$$\mathfrak{h}^{(d)}[u_n]^{1/2} \mathfrak{h}^{(d)}[u_n - u_{n_0}]^{1/2} \le C\epsilon^{1/2} \text{ when } n \ge n_0. \tag{2.12}$$

Since $\boldsymbol{A}$ is bounded on K we infer from (2.10) and (2.8) that

$$\mathfrak{h}^{(d)}[u_n, u_{n_0}] = \int_K (\nabla + i\boldsymbol{A}) u_n \overline{(\nabla + i\boldsymbol{A}) u_{n_0}} \, dx \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Substitution of (2.12)-(2.13) into (2.11) shows that $\lim_{n\to\infty} \mathfrak{h}^{(d)}[u_n] = 0$ as desired. □

We denote the closure of $\mathfrak{h}^{(d)}$ by $\overline{\mathfrak{h}}^{(d)}$ and the associated semi-bounded (from below), self-adjoint operator by $H_0^{(d)}$. Define $\mathfrak{l}^{(d)}$ in a similar way, viz.

$$\begin{aligned} \mathfrak{l}^{(d)}[u,v] &= \mathfrak{l}_n^{(d)}[u,v] \text{ if } u,v \in \mathcal{D}(\mathfrak{l}_n^{(d)}), \\ \mathcal{D}(\mathfrak{l}^{(d)}) &= \cup_n \mathcal{D}(\mathfrak{l}_n^{(d)}) = \cup_{n\geq 2} H_0^1(\Omega_n). \end{aligned}$$

Then $\mathfrak{l}^{(d)}$ is closable. The closure $\overline{\mathfrak{l}}^{(d)}$ has domain $\mathcal{D}(\overline{\mathfrak{l}}^{(d)}) = H^1(\mathbb{R}^d)$. The associated nonnegative, self-adjoint operator is just the negative Laplacian in $L^2(\mathbb{R}^d)$; we suppress d and denote it by L_0.

3. Semigroup criterion

Throughout this section $\mathcal{H}$ denotes our Hilbert space $L^2(\mathbb{R}^d)$. For a given $u \in \mathcal{H}$ we denote by $\overline{u} := \mathrm{Re}\, u - i\mathrm{Im}\, u$ the conjugate function of u. By $|u|$ we denote the absolute value of u (i.e. the function $x \mapsto |u(x)| := \sqrt{u(x)\cdot\overline{u(x)}}$) and by $\mathrm{sign}\, u$ the function defined by

$$\mathrm{sign}\, u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Let $\mathfrak{s}$ be a sesquilinear form which satisfies

$$\mathcal{D}(\mathfrak{s}) \text{ is dense in } \mathcal{H}, \tag{3.1}$$

$$\mathrm{Re}\, \mathfrak{s}[u,u] \geq 0, \quad \forall u \in \mathcal{D}(\mathfrak{s}), \tag{3.2}$$

$$|\mathfrak{s}[u,v]| \leq C\|u\|_{\mathfrak{s}}\|v\|_{\mathfrak{s}}, \quad \forall u,v \in \mathcal{D}(\mathfrak{s}), \tag{3.3}$$

where C is a constant and $\|u\|_{\mathfrak{s}} = \sqrt{\mathrm{Re}\, \mathfrak{s}[u,u] + \|u\|^2}$, and, moreover,

$$\langle \mathcal{D}(\mathfrak{s}), \|\cdot\|_{\mathfrak{s}}\rangle \text{ is a complete space.} \tag{3.4}$$

Definition 3.1. Let $\mathcal{K}$ and $\mathcal{L}$ be two subspaces of $\mathcal{H}$. We shall say that $\mathcal{K}$ is an ideal of $\mathcal{L}$ if the following two assertions are fulfilled:
1) $u \in \mathcal{K}$ implies $|u| \in \mathcal{L}$.
2) If $u \in \mathcal{K}$ and $v \in \mathcal{L}$ such that $|v| \leq |u|$ then $v \cdot \mathrm{sign}\, u \in \mathcal{K}$.

Let $\mathfrak{s}$ and $\mathfrak{t}$ be two sesquilinear forms both of which satisfy (3.1)-(3.4). The semigroups associated to corresponding self-adjoint operators S, T will be denoted by e^{-tS} and e^{-tT}, respectively.

The following result was established by Ouhabaz [24, Theorem 3.3 and its Corollary].

Theorem 3.2 (Ouhabaz'96). *Assume that the semigroup e^{-tT} is positive. The following assertions are equivalent:*
1) $|e^{-tS}f| \leq e^{-tT}|f|$ for all $t \geq 0$ and all $f \in \mathcal{H}$.
2) $\mathcal{D}(\mathfrak{s})$ is an ideal of $\mathcal{D}(\mathfrak{t})$ and

$$\mathrm{Re}\, \mathfrak{s}[u, |v|\mathrm{sign}\, u] \geq \mathfrak{t}[|u|, |v|] \tag{3.5}$$

for all $(u, v) \in \mathcal{D}(\mathfrak{s}) \times \mathcal{D}(\mathfrak{t})$ *such that* $|v| \leq |u|$.
3) $\mathcal{D}(\mathfrak{s})$ *is an ideal of* $\mathcal{D}(\mathfrak{t})$ *and*

$$\operatorname{Re} \mathfrak{s}[u, v] \geq \mathfrak{t}[|u|, |v|] \tag{3.6}$$

for all $u, v \in \mathcal{D}(\mathfrak{s})$ *such that* $u \cdot \overline{v} \geq 0$.

The following lemma is useful in applications when one wishes to apply the criteria in Theorem 3.2.

Lemma 3.3. *Let* $\Omega \subset \mathbb{R}^d$ *be an open set and* $u, v \in H^1(\Omega)$ *be functions satisfying* $u(x) \cdot \overline{v(x)} \geq 0$ *for a.e.* $x \in \Omega$. *Then*
1. $\operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \overline{v}\right) = |v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign} \overline{u}\right)$.
2. $|v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign} \overline{u}\right) = |u| \operatorname{Im}\left(\frac{\partial v}{\partial x_j} \cdot \operatorname{sign} \overline{u}\right)$.

Proof. Let $\chi_{\{u=0\}}$ denote the characteristic function of the set $\{\, x \mid u(x) = 0\}$. Since $(\partial u/\partial x_j) \cdot \chi_{\{u=0\}} = 0$, we have that

$$\frac{\partial u}{\partial x_j} \cdot \overline{v} = \frac{\partial u}{\partial x_j} \cdot \overline{v} \cdot \frac{v \cdot \overline{u}}{|u| \cdot |v|} \chi_{\{u \neq 0\}} \chi_{\{v \neq 0\}} = |v| \cdot \frac{\partial u}{\partial x_j} \cdot \frac{\overline{u}}{|u|} \cdot \chi_{\{u \neq 0\}}.$$

By taking the imaginary part on both sides of the latter equality, we obtain that

$$\operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \overline{v}\right) = |v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign} \overline{u}\right),$$

which verifies the first assertion. To prove the second assertion we start from

$$|v| \cdot u = |v| \cdot u \cdot \frac{v \cdot \overline{u}}{|u| \cdot |v|} \chi_{\{u \neq 0\}} \chi_{\{v \neq 0\}} = |u| \cdot v.$$

Hence,

$$\frac{\partial |v|}{\partial x_j} \cdot u + |v| \cdot \frac{\partial u}{\partial x_j} = \frac{\partial |u|}{\partial x_j} \cdot v + |u| \cdot \frac{\partial v}{\partial x_j}.$$

We multiply both sides by $\operatorname{sign} \overline{u} = (\overline{u}/|u|)\chi_{\{u \neq 0\}}$ and take the imaginary parts on both sides to obtain

$$|v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign} \overline{u}\right) = \operatorname{Im}\left(\frac{\partial v}{\partial x_j} \cdot \overline{u} \chi_{\{u \neq 0\}}\right) = \operatorname{Im}\left(\frac{\partial v}{\partial x_j} \cdot \overline{u}\right).$$

The latter in combination with the first assertion (with u substituted by v and vice-versa) shows the second assertion. ☐

4. Diamagnetic inequality for $H_0^{(d)}$

The usual diamagnetic inequality is established for vector potentials which belong to L^2_{loc} (see, e.g., [3]). In this section we establish the diamagnetic inequality for the Schrödinger operator $H_0^{(d)}$, i.e. when $A_j \notin L^2_{loc}$, $j = 1, 2$.

Denote by $e^{-tH_n^{(d)}}$ (resp. $e^{-tL_n^{(d)}}$) the semigroup associated with $H_n^{(d)}$ (resp. $L_n^{(d)}$) introduced in Section 2. For each n the diamagnetic inequality holds for these pairs of semigroups.

Proposition 4.1. *The inequality*

$$|e^{-tH_n^{(d)}} f| \le e^{-tL_n^{(d)}} |f|$$

holds for all $t \ge 0$ *and all* $f \in L^2(\Omega_n)$ $(n \ge 2)$*.*

Proof. We give the proof for $d = 2$ and suppress the upper index in $\mathfrak{h}_n^{(2)}$. With a few obvious modifications the proof for $d = 3$ is the same. By the domination criterion in Theorem 3.2, assertion 3, it suffices to prove that

$$\mathrm{Re}\ \mathfrak{h}_n[u, v] \ge \mathfrak{l}_n[|u|, |v|] \tag{4.1}$$

for all $u, v \in \mathcal{D}(\mathfrak{h}_n) = H_0^1(\Omega_n)$ obeying $u \cdot \overline{v} \ge 0$.

Let $u, v \in H_0^1(\Omega_n)$ be such that $u \cdot \overline{v} \ge 0$. We have that

$$\begin{aligned}
I_1 \;:=\; & \mathrm{Re} \int_{\Omega_n} \left\{ \frac{\partial u}{\partial x_1} \cdot \frac{\partial \overline{v}}{\partial x_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial \overline{v}}{\partial x_2} \right\} dx \\
= & \int_{\Omega_n} \left\{ \mathrm{Re} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_1} \cdot \mathrm{sign}\, \overline{v} \right) \right. \\
& \left. + \mathrm{Re} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_2} \cdot \mathrm{sign}\, \overline{v} \right) \right\} dx \\
& + \int_{\Omega_n} \left\{ \mathrm{Im} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial v}{\partial x_1} \cdot \mathrm{sign}\, \overline{v} \right) \right. \\
& \left. + \mathrm{Im} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial v}{\partial x_2} \cdot \mathrm{sign}\, \overline{v} \right) \right\} dx \\
= & \int_{\Omega_n} \left\{ \mathrm{Re} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_1} \cdot \mathrm{sign}\, \overline{v} \right) \right. \\
& + \mathrm{Re} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_2} \cdot \mathrm{sign}\, \overline{v} \right) \\
& + \mathrm{Im} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \\
& \left. + \mathrm{Im} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right\} dx,
\end{aligned}$$

where we applied Lemma 3.3, part 2, in the last equality. From [23, Lemma 4.1] we have that

$$\frac{\partial |u|}{\partial x_1} = \mathrm{Re} \left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right), \quad \forall u \in H^1(\Omega_n) \supset H_0^1(\Omega_n).$$

Using this, we find that

$$I_1 = \int_{\Omega_n} \left\{ \frac{\partial |u|}{\partial x_1} \cdot \frac{\partial |v|}{\partial x_1} + \frac{\partial |u|}{\partial x_2} \cdot \frac{\partial |v|}{\partial x_2} + \left[\mathrm{Im}\left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right) \right]^2 \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right.$$
$$\left. + \left[\mathrm{Im}\left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right) \right]^2 \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right\} dx. \tag{4.2}$$

Next, let $u, v \in H_0^1(\Omega_n)$ with $u \cdot \overline{v} \geq 0$. Using $\mathrm{Re}\, u \overline{\frac{\partial v}{\partial x_1}} = \mathrm{Re}\, \overline{u} \frac{\partial v}{\partial x_1}$ we have that

$$\begin{aligned} I_2 &:= \mathrm{Re} \int_{\Omega_n} \left\{ -iA_1 \frac{\partial u}{\partial x_1} \overline{v} - iA_2 \frac{\partial u}{\partial x_2} \overline{v} + iA_1 u \overline{\frac{\partial v}{\partial x_1}} + iA_2 u \overline{\frac{\partial v}{\partial x_2}} \right\} dx \\ &= \int_{\Omega_n} \left\{ -\mathrm{Im}\,(-iA_1)\, \mathrm{Im}\left(\frac{\partial u}{\partial x_1} \overline{v} \right) - \mathrm{Im}\,(-iA_2)\, \mathrm{Im}\left(\frac{\partial u}{\partial x_2} \overline{v} \right) \right. \\ &\quad \left. - \mathrm{Im}\,(iA_1)\, \mathrm{Im}\left(\overline{u} \frac{\partial v}{\partial x_1} \right) - \mathrm{Im}\,(iA_2)\, \mathrm{Im}\left(\overline{u} \frac{\partial v}{\partial x_2} \right) \right\} dx. \end{aligned}$$

Using the first part of Lemma 3.3 we may rewrite I_2 as

$$\begin{aligned} I_2 = \int_{\Omega_n} &\left\{ -\mathrm{Im}\,(-iA_1)\, \mathrm{Im}\left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right) |v| - \mathrm{Im}\,(-iA_2) \right. \\ &\times \mathrm{Im}\left(\frac{\partial u}{\partial x_2} \mathrm{sign}\, \overline{u} \right) |v| - \mathrm{Im}\,(iA_1)\, \mathrm{Im}\left(\frac{\partial v}{\partial x_1} \mathrm{sign}\, \overline{v} \right) |u| \\ &\left. - \mathrm{Im}\,(iA_2)\, \mathrm{Im}\left(\frac{\partial v}{\partial x_2} \mathrm{sign}\, \overline{v} |u| \right) \right\} dx. \end{aligned}$$

Next we apply the second part of Lemma 3.3 to the last two terms in I_2. It follows that

$$\begin{aligned} I_2 \;=\; \int_{\Omega_n} &\left\{ (A_1 - A_1)\, \mathrm{Im}\left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right) |v| \right. \\ &\left. + (A_2 - A_2)\, \mathrm{Im}\left(\frac{\partial u}{\partial x_2} \mathrm{sign}\, \overline{u} \right) |v| \right\} dx = 0. \end{aligned} \tag{4.3}$$

For the last term in $\mathfrak{h}_n$, we have that

$$I_3 := \mathrm{Re} \int_{\Omega_n} (A_1^2 + A_2^2) u \cdot \overline{v} dx = \int_{\Omega_n} (A_1^2 + A_2^2) |u|\, |v| dx \tag{4.4}$$

for all $u, v \in H_0^1(\Omega_n)$ such that $u \cdot \overline{v} \geq 0$.

Since $\mathrm{Re}\ \mathfrak{h}_n[u, v] = \sum_{j=1}^{3} I_j$, we obtain from (4.2), (4.3) and (4.4) that

$$\mathrm{Re}\ \mathfrak{h}_n[u, v] = \int_{\Omega_n} \left\{ \frac{\partial |u|}{\partial x_1} \cdot \frac{\partial |v|}{\partial x_1} + \frac{\partial |u|}{\partial x_2} \cdot \frac{\partial |v|}{\partial x_2} + \left[\mathrm{Im}\left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right) \right]^2 \right.$$
$$\left. \times \frac{|v|}{|u|} \chi_{\{u \neq 0\}} + \left[\mathrm{Im}\left(\frac{\partial u}{\partial x_1} \mathrm{sign}\, \overline{u} \right) \right]^2 \frac{|v|}{|u|} \chi_{\{u \neq 0\}} + (A_1^2 + A_2^2) |u|\, |v| \right\} dx.$$

In this expression, the sum of last three terms is nonnegative, so we infer that

$$\begin{aligned}\operatorname{Re}\, \mathfrak{h}_n[u,v] &\geq \int_{\Omega_n} \left\{ \frac{\partial |u|}{\partial x_1} \cdot \frac{\partial |v|}{\partial x_1} + \frac{\partial |u|}{\partial x_2} \cdot \frac{\partial |v|}{\partial x_2} + (A_1^2 + A_2^2)|u|\,|v| \right\} \\ &\geq \int_{\Omega_n} \left\{ \frac{\partial |u|}{\partial x_1} \cdot \frac{\partial |v|}{\partial x_1} + \frac{\partial |u|}{\partial x_2} \cdot \frac{\partial |v|}{\partial x_2} \right\} = \mathfrak{l}_n[|u|,|v|]\end{aligned}$$

for all $u, v \in H_0^1(\Omega_n)$ obeying $u \cdot \overline{v} \geq 0$. This verifies (4.1). □

The semigroups associated with $H_0^{(d)}$ and L_0, introduced in Section 2, are denoted by $e^{-tH_0^{(d)}}$ and e^{-tL_0}, resp. By means of Proposition 4.1 we are ready to prove Theorem 1.1, i.e., the diamagnetic inequality for the operator $H_0^{(d)}$.

Proof of Theorem 1.1. Bear in mind that when $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are closed forms bounded from below then $\mathfrak{s}_1 \geq \mathfrak{s}_2$ means that $\mathcal{D}(\mathfrak{s}_1) \subset \mathcal{D}(\mathfrak{s}_2)$ and $\mathfrak{s}_1[u,u] \geq \mathfrak{s}_2[u,u]$ for $u \in \mathcal{D}(\mathfrak{s}_1)$. A sequence $\{\mathfrak{s}_n\}$ of closed forms bounded from below is nonincreasing if $\mathfrak{s}_n \geq \mathfrak{s}_{n+1}$ for all n.

The forms $\{\mathfrak{h}_n^{(d)}\}$ defined in (2.2) on the domains $\mathcal{D}(\mathfrak{h}_n^{(d)}) = H_0^1(\Omega_n)$ in $L^2(\Omega_n)$, $n \geq 2$, compose a nonincreasing sequence

$$\cdots \leq \mathfrak{h}_{n+1}^{(d)} \leq \mathfrak{h}_n^{(d)} \leq \mathfrak{h}_{n-1}^{(d)} \leq \ldots,$$

of closed, *non-densely defined* forms in $L^2(\mathbb{R}^d)$. The monotone convergence theorem for closed forms is also valid for non-densely defined forms [32, Theorem 4.1]. Hence, the latter theorem in conjunction with Lemma 2.1 yields that $\mathfrak{h}_n^{(d)} \to \mathfrak{h}^{(d)}$ in strong resolvent sense or, equivalently,

$$e^{-tH_0^{(d)}} = \text{s}-\lim_{n\to\infty} e^{-tH_n^{(d)}}. \tag{4.5}$$

A similar argument yields

$$e^{-tL_0^{(d)}} = \text{s}-\lim_{n\to\infty} e^{-tL_n^{(d)}}. \tag{4.6}$$

Thus we can pass to the limit $n \to \infty$ in the diamagnetic inequality for operators H_n^d, L_n^d, Proposition 4.1 and therefore

$$\begin{aligned}\left| e^{-tH_0^{(d)}} f \right| &= \lim_{n\to\infty} \left| e^{-tH_n^{(d)}} f \right| \\ &\leq \lim_{n\to\infty} e^{-tL_n^{(d)}} |f| \\ &= e^{-tL_0} |f|,\end{aligned}$$

which proves the assertion. □

Remark 4.2. Our proof of the diamagnetic inequality also applies to the case where we have another metric, that is, the result holds also for operators of the type $(\nabla + i\boldsymbol{A})M(x)(\nabla + i\boldsymbol{A})$ where $M(x) = (a_{kj}(x))$ is a symmetric matrix with real-valued and bounded measurable coefficients (satisfying the classical ellipticity condition). The semigroup generated by this operator is dominated by the semigroup generated by the elliptic operator $\nabla M(x)\nabla$.

5. Abstract CLR eigenvalue estimates and semigroup domination

In this section we recall Rozenblum's and Solomyak's abstract CLR-estimate for generators of positively dominated semigroup.

Let Ω be a space with σ-finite measure μ, $L^2 = L^2(\Omega, \mu)$. Let T be a nonnegative, self-adjoint operator in L^2, generating a positivity preserving semigroup $Q(t) = e^{-tT}$. We suppose also that $Q(t)$ is an integral operator with bounded kernel $Q(t; x, y)$ subject to

$$M_T(t) := \text{ess sup}_x Q(t; x, x),\ M_T(t) = O(t^{-\beta}) \text{ as } t \to 0 \text{ for some } \beta > 0. \tag{5.1}$$

We will write $T \in \mathcal{P}$ if T satisfies the afore-mentioned assumptions[2].

If $T \in \mathcal{P}$, the operator $T_\mu = T+\mu$ also belongs to $\mathcal{P}$. The corresponding semigroup is $Q_{T_\mu}(t) = e^{-\mu t} Q_T(t)$ and thus $M_{T_\mu}(t) = e^{-\mu t} M_T(t)$.

We say that the semigroup $P(t) = e^{-tS}$ is dominated by $Q(t)$ if the diamagnetic inequality holds, i.e., if any $u \in L^2$ satisfies

$$|P(t)u| \le Q(t)|u| \text{ a.e. on } \Omega. \tag{5.2}$$

In the latter case we write $S \in \mathcal{PD}(T)$.

Let now G be a nonnegative, continuous, convex function on $[0, \infty)$. To such a function we associate

$$g(\lambda) = \mathcal{L}(G)(\lambda) := \int_0^\infty z^{-1} G(z) e^{-z/\lambda}\, dz, \quad \lambda > 0, \tag{5.3}$$

provided the latter integral converges. In other words, $g(1/\lambda)$ is the Laplace transform of $z^{-1}G(z)$.

For a nonnegative, measurable function V such that the operator of multiplication by V is form-bounded with respect to T with a bound less than one, we associate the operators $T - V$, $S - V$ by means of quadratic forms (see [26, Theorem X.17]). The number of negative eigenvalues (counting multiplicity) of $T - V$ is denoted by $N_-(T - V)$; if there is some essential spectrum below zero, we set $N_-(T - V) = \infty$.

Rozenblum and Solomyak [28, Theorem 2.4] have established the following abstract CLR-estimate.

Theorem 5.1. *Let G, g and $T \in \mathcal{P}$ be as above and suppose that $\int_a^\infty M_T(t)\, dt < \infty$ for some $a > 0$. If $S \in \mathcal{PD}(T)$ then*

$$N_-(S - V) \le \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_\Omega M_T(t) G(tV(x))\, dx, \tag{5.4}$$

as long as the expression on the right-hand side is finite.

[2]Although the diagonal in $\Omega \times \Omega$ may be a set with measure zero in $\Omega \times \Omega$, the semigroup property defines $Q(t, \cdot, \cdot)$ as a function in $L^\infty(\Omega)$, see [2], [28]

The assumption that V is form-bounded with respect to T with a bound smaller than one in conjunction with $S \in \mathcal{PD}(T)$ implies that V is form-bounded with respect to S with a bound less than one, thus $N_-(S-V)$ is well defined.

In Section 6 we shall apply Theorem 5.1 to prove the Lieb-Thirring inequality for $H_0^{(d)} - V$.

Rozenblum has also developed an abstract machinery which, in our situation, allows us to carry over *any*, sufficiently regular, bound for $N_-(T-V)$ to $N_-(S-V)$, as soon as the diamagnetic inequality (5.2) is valid for S, T [30, Theorem 4]. We customize it to our situation.

Theorem 5.2. *Assume that $T \in \mathcal{P}$, $S \in \mathcal{PD}(T)$ and $V \geq 0$ is a measurable function infinitesimally form-bounded with respect to T. Suppose that, for some $p > 0$,*

$$N_-(T - qV) \leq Kq^p \tag{5.5}$$

for all $q > 0$ and some positive constant K. Then

$$N_-(S - qV) \leq eC_pKq^p. \tag{5.6}$$

6. Lieb-Thirring inequality for $H_0^{(d)} - V$

Having the diamagnetic inequality in Theorem 1.1 as well as the abstract CLR-estimate in Theorem 5.1 at our disposal, we are ready to prove Theorem 1.2.

Before proceeding with the proof, observe that the assumption $V \in L^p(\mathbb{R}^d)$, $p > 1$ for $d = 2$ and $p \geq 3/2$ for $d = 3$, in Theorem 1.2 implies that V is infinitesimally L_0- form-bounded; Theorem 1.1 then implies that V is infinitesimally $H_0^{(d)}$- form-bounded. Thus, according to the KLMN Theorem [26, Theorem X.17], the form sum $H_0^{(d)} - V$ generates a lower semi-bounded, self-adjoint operator in $L^2(\mathbb{R}^d)$.

Proof of Theorem 1.2. As it is well known, $L_0 \in \mathcal{P}$ and the kernel of its semigroup e^{-tL_0} on the diagonal is given by $Q(t; x, x) = (4\pi)^{-d/2} t^{-d/2}$ [12]. From Theorem 1.1 we have that $H_0^{(d)} \in \mathcal{PD}(L_0)$ and the kernel P of its semigroup obeys $|P(t; x, x)| \leq (4\pi)^{-d/2} t^{-d/2}$.

Let $\mu > 0$ and define the auxiliary operators $S_\mu = H_0^{(d)} + \mu$ and $T_\mu = L_0 + \mu$. Now $L_0 \in \mathcal{P}$ and $H_0^{(d)} \in \mathcal{PD}(L_0)$ imply that $T_\mu \in \mathcal{P}$ and $S_\mu \in \mathcal{PD}(T_\mu)$. For the kernel $P_\mu(t; x, x) = e^{-\mu t} P(t; x, x)$ of the semigroup generated by S_μ we have therefore that $|P_\mu(t; x, x)| \leq Q_\mu(t; x, x) = e^{-\mu t} Q(t; x, x) = (4\pi)^{-d/2} t^{-d/2} e^{-\mu t}$. Thus we may apply Theorem 5.1 which yields

$$N_-(S_\mu - V) \leq \frac{1}{(4\pi)^{d/2}} \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_{\mathbb{R}^d} t^{-d/2} e^{-\mu t} G(tV(x))\, dx. \tag{6.1}$$

We will not evaluate the integral in (6.1) as one might be inclined to do. Instead, for $\gamma > 0$, we recall that (see e.g. [20])

$$LT_{\gamma,d} := \sum_j |\nu_j(H_0^{(d)} - V)|^\gamma = -\int \mu^\gamma \, dN_\mu = \gamma \int_0^\infty \mu^{\gamma-1} N_-(S_\mu - V)\, d\mu. \quad (6.2)$$

We substitute (6.1) into (6.2) and get that

$$LT_{\gamma,d} \le \frac{1}{(4\pi)^{d/2}} \frac{\gamma}{g(1)} \int_{R^d} dx \int_0^\infty \mu^{\gamma-1}\, d\mu \int_0^\infty t^{-d/2} e^{-\mu t} G(tV(x))\, \frac{dt}{t}.$$

Making first the change of variables $s = V(x)t$ and then the change of variables $\tau = \mu/V(x)$ we obtain that

$$LT_{\gamma,d} \le \tilde{L}_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d}{2}}\, dx,$$

where

$$\tilde{L}_{\gamma,d} = \frac{1}{(4\pi)^{d/2}} \frac{\gamma}{g(1)} \int_0^\infty \int_0^\infty s^{-\frac{d}{2}-1} e^{-\tau s} G(s) \tau^{\gamma-1}\, ds\, d\tau.$$

Now, $\int_0^\infty \tau^{\gamma-1} e^{-\tau s}\, d\tau = s^{-\gamma}\Gamma(\gamma)$, where $\Gamma(\gamma)$ is the Gamma-function evaluated at γ. Choose $G(s) = (s-k)_+$ for some $k > 0$; this is Lieb's original choice. Then

$$\int_0^\infty s^{-\gamma} s^{-\frac{d}{2}-1} (s-k)_+\, ds = \frac{1}{\left(\gamma + \frac{d-2}{2}\right)\left(\gamma + \frac{d}{2}\right) k^{\gamma \frac{d}{2}}}.$$

Moreover,

$$g(1) = \int_1^\infty e^{-ks} s^{-2}\, ds \ge \frac{e^{-k}}{k} - \frac{2}{k} g(1),$$

i.e., $1/g(1) \le e^k(k+2)$. Thus

$$\tilde{L}_{\gamma,d} \le C_{\gamma,d} := \frac{\Gamma(\gamma) e^k (k+2)}{(4\pi)^{d/2}(\gamma + \frac{d-2}{2})(\gamma + \frac{d}{2}) k^{\gamma + \frac{d-2}{2}}}. \quad (6.3)$$

The optimization problem for the expression in (6.3) does not admit an exact solution. For the three most interesting values of γ, namely $1, 1/2$ and $3/2$, one easily finds the numerical values of $C_{\gamma,d}$ given in the Theorem. □

Remark 6.1. In the case of a single A-B solenoid, A. Laptev pointed out to the authors that the Lieb-Thirring inequality can be derived *without* using the diamagnetic inequality [18]. His argument goes as follows. When $\boldsymbol{A} = \alpha(-x_2/|x|^2, x_1/|x|^2)$ we may use the decomposition $L^2(\mathbb{R}^2) = L^2(\mathbb{R}^+, r\,dr) \otimes L^2(\mathbb{S}^1) = \oplus_{n\in\mathbb{Z}}\{L^2(\mathbb{R}^+, r\,dr)[e^{in\theta}/2\pi]\}$ ($[\cdot]$ denotes the linear span) to express the A-B Schrödinger operator as

$$H_{AB}^{(2)} = \oplus_{n\in\mathbb{Z}}\{H_n \otimes I_n\},$$

where H_n is the Friedrichs operator in $L^2(\mathbb{R}^+, r\,dr)$ associated with the quadratic form

$$\mathfrak{h}_n[u_n] = \int_0^\infty \left(|u_n^{'}(r)|^2 + \frac{(n+\alpha)^2}{r^2}|u_n(r)|^2 \right) r\,dr.$$

Thus, with a slight abuse of notation, the quadratic form associated with H_{AB} is given by $\mathfrak{h}[u] = \sum_{n\in\mathbb{Z}} \mathfrak{h}_n[u_n]$. Taking $\alpha \in (0, 1/2)$, we note that $|n+\alpha|^2 \geq |1-\alpha|^2$ provided $n \neq 0$. As a consequence, we have that

$$\mathfrak{h}_n[u_n] \geq |1-\alpha|^2 \int_0^\infty \left(|u_n^{'}(r)|^2 + \frac{n^2}{r^2}|u_n(r)|^2 \right) r\,dr = |1-\alpha|^2 \mathfrak{l}_n[u_n],$$

where $\mathfrak{l}[u] = \sum_{n\in\mathbb{Z}} \mathfrak{l}_n[u_n]$ is the quadratic form of the negative Laplacian in $L^2(\mathbb{R}^2)$. In conclusion, $\mathfrak{h}[u] \geq |1-\alpha|^2 \mathfrak{l}[u]$. The latter inequality immediately implies that the usual Lieb-Thirring inequalities for $-\Delta - V$ carry over to the A-B Schrödinger operator $H_{AB}^{(2)} - V$ with a constant $L_{2,\gamma}/|1-\alpha|^2$, where $L_{2,\gamma}$ is the usual Lieb-Thirring constant. A similar reasoning was used in [5]. This argument, however, does not work for many A-B solenoids.

7. Hardy-type inequalities

In order to establish eigenvalue estimates in the two-dimensional case for various configurations of A-B solenoids (or magnetic vortices), we require certain Hardy-type inequalities which we will obtain in this section. Generally, a Hardy-type inequality is an estimate where the integral involving the gradient of the function majorizes the weighted integral of the square of the function itself.

The classical Hardy inequality

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx \leq \text{ const. } \int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx, \quad u \in C_0^\infty(\mathbb{R}^d \setminus \{0\}),$$

does not hold for $d = 2$. It was discovered by Laptev and Weidl [17], however, that the presence of a magnetic field can improve this situation. In particular, if the gradient ∇ is replaced by the "magnetic" gradient $\nabla + i\boldsymbol{A}$, where $\boldsymbol{A}$ is the standard A-B vector potential (see (7.5) below), and the flux $\alpha = \frac{1}{2\pi}\int_{\mathbb{S}^1} \boldsymbol{A}\,dx$ is noninteger then ([17, Theorem 3])

$$\int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx \leq \rho(\alpha)^{-2} \int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \tag{7.1}$$

where $\rho(\alpha) = \min_{k\in\mathbb{Z}} |k - \alpha|$.

We are going to find analogies of this fact for configurations of magnetic solenoids considered in Section 2. In what follows, we will freely interchange real and complex picture in description of our magnetic object. Thus $x = (x_1, x_2), z = x_1 + ix_2$, $\boldsymbol{A} = (A_1, A_2), \mathcal{A} = (A_1 + iA_2), dx = \frac{1}{2}dzd\overline{z}$ etc.

Finitely many solenoids. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_J\}$ with $\lambda_j = (\lambda_{1,j}, \lambda_{2,j})$. For finitely many A-B solenoids located at the points of Λ the corresponding A-B vector potential is given by

$$\boldsymbol{A}(x) = \sum_{j=1}^{J} \frac{\alpha_j}{|x-\lambda_j|^2} \begin{pmatrix} -x_2 + \lambda_{2,j} \\ x_1 - \lambda_{1,j} \end{pmatrix} \tag{7.2}$$

for $x = (x_1, x_2) \in \mathbb{R}^2 \backslash \Lambda$ and α_j being the flux through the j-th solenoid.

The aim is to establish the following Hardy-type inequality.

Proposition 7.1. *Suppose that $\alpha_j \notin \mathbb{Z}$, $j = 1, 2, \ldots, J$, and that $\alpha_s := \sum_{j=1}^{J} \alpha_j \notin \mathbb{Z}$. Define*

$$W(x) = \min\{\rho(\alpha_j)^2, \rho(\alpha_s)^2\} \sum_{j=1}^{J} |x - \lambda_j|^{-2}. \tag{7.3}$$

Then there exists a constant C such that

$$\int_{\mathbb{R}^2} W(x)|u(x)|^2\, dx \le C \int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx \tag{7.4}$$

is valid for all $u \in C^\infty(\mathbb{R}^2 \backslash \Lambda)$.

Note that the constant C above may depend on the configuration of the solenoids. As usual, the inequality where the right-hand side is infinite, is automatically true.

We begin by showing a slightly modified version of [17, Theorem 3].

Lemma 7.2. *Assume that $\alpha_0 \notin \mathbb{Z}$ and let*

$$\boldsymbol{A}_0(x) = \frac{\alpha_0}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \tag{7.5}$$

Let $\Omega = B_R(0)$. Then

$$\rho(\alpha_0)^2 \int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx \le \int_\Omega |(\nabla + i\boldsymbol{A}_0)u(x)|^2\, dx \tag{7.6}$$

holds for any $u \in C^\infty(\Omega \backslash \{0\})$.

Proof. In polar co-ordinates (r, θ), we have that

$$\nabla + i\boldsymbol{A}_0 = -\boldsymbol{e}_r(\partial/\partial_r) + (1/r)\boldsymbol{e}_\theta[(-\partial/\partial_\theta) + i\alpha_0]. \tag{7.7}$$

Therefore, for any function $f(r)e^{in\theta}$, $n \in \mathbb{Z}$, we have that

$$\begin{aligned}
&\int_\Omega |(\nabla + i\boldsymbol{A}_0)f(r)e^{in\theta}|^2\, r\, drd\theta \\
&\quad= \int_\Omega \Big(|f_r'|^2 + (1/r^2)|f(r)|^2(n+\alpha_0)^2\Big)\, r\, drd\theta \\
&\quad\geq \int_\Omega \frac{1}{r^2}|f(r)|^2(n+\alpha_0)^2\, r\, drd\theta \\
&\quad\geq \rho(\alpha_0)^2 \int_\Omega \frac{|f(r)e^{in\theta}|^2}{r^2}\, r\, drd\theta.
\end{aligned}$$

This proves (7.6) for spherical functions and thus for any $u \in C^\infty(\Omega)$ since the left-hand side and the right-hand side of (7.6) are both sums of contributions of spherical functions. ☐

In a similar way we establish the following result.

Lemma 7.3. *Suppose that $\alpha_s := \sum \alpha_j \notin \mathbb{Z}$. Then, provided $R > 0$ is sufficiently large, $\Omega_0 = \{|x| > R\}$, the inequality*

$$\int_{\Omega_0} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx \geq \rho(\alpha_s)^2 \int_{\Omega_0} \frac{|u(x)|^2}{|x|^2}\, dx \tag{7.8}$$

holds for any $u \in C^\infty(\Omega_0)$.

Proof. First we note that there exists a function φ such that $\boldsymbol{A}(x) - \boldsymbol{A}_s(x) = (\nabla\varphi)(x)$,

$$\boldsymbol{A}_s(x) = \frac{\alpha_s}{|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{7.9}$$

for any $x \in B_R(0)^c$ provided $R > 0$ is large enough. Since the right-hand side of (7.8) is gauge invariant, it suffices to show (7.8) for the vector potential $\boldsymbol{A}_s$. Now we switch to polar co-ordinates and repeat the reasoning in Lemma 7.2. ☐

Lemma 7.4 (Local Hardy inequality)**.** *Let D be a bounded, simply-connected domain in $\mathbb{C}$ with smooth boundary and let $z_0 \in D$. Let $\mathcal{A}(z) = A_1(z) + iA_2(z)$, $z = x_1 + ix_2$, be a (complex) magnetic vector potential such that $\mathcal{A}(z)$ is analytic in $D\backslash\{z_0\}$ and has a simple pole at z_0 with residue equal to μ_0, $\boldsymbol{A} = (A_1, A_2)$. Then, there exists a constant $C > 0$ such that for any $u \in C^\infty(D\backslash\{z_0\})$,*

$$\int_D \frac{|u(z)|^2}{|z-z_0|^2}\, dx \leq \rho(\mu_0)^{-2} C \int_D |(\nabla + i\boldsymbol{A})u(z)|^2\, dz, \tag{7.10}$$

Proof. Let $w = y_1 + iy_2 = F(z)$, $F : D \to B_1(0)$ be a conformal mapping of D onto the unit disk $B_1(0)$ so that x_0 is mapped to the origin. Since D has a smooth boundary, F is smooth up to the boundary [25, p 49], together with its inverse. For latter purpose we note that there exists c such that

$$\frac{c}{|z-z_0|} \leq \left|\frac{F_z'(z)}{F(z)}\right|. \tag{7.11}$$

Indeed, since F is smoothly invertible, F' is bounded away from 0. Therefore F'/F has the order of $1/F$ near z_0. Since F has a simple zero at z_0, it has the order of $|z-z_0|$, which verifies (7.11).

Let $\omega_{\boldsymbol{A}}$ denote the differential 1-form $A_1(z)dx_1 + A_2(z)dx_2$ and let $\boldsymbol{A}^F(w) = (A_1^F(w), A_2^F(w))$ be the transformed magnetic vector potential in $B_1(0)$ such that $F^*(\omega_{\boldsymbol{A}^F}) = \omega_{\boldsymbol{A}}$ (F^* denotes the pull-back), i.e.,

$$A_1^F(w)dy_1 + A_2^F(w)dy_2 = A_1(z)dx_1 + A_2(z)dx_2.$$

In particular, $\mathcal{A}^F$ has a simple pole at the origin with residue equal to μ_0. Since F is a conformal mapping it follows that

$$\int_D |(\nabla_x + i\boldsymbol{A})u(x)|^2\, dx = \int_{B_1(0)} |(\nabla_y + i\boldsymbol{A}^F)u(y)|^2\, dy \tag{7.12}$$

for any $u \in C^\infty(D)$.

Next we gauge away the regular part of $\mathcal{A}^F = A_1^F + iA_2^F$ (as we did in the proof of Lemma 7.3). From Lemma 7.2 we immediately get that

$$\int_{B_1(0)} \frac{|u(y)|^2}{|y|^2}\, dy \leq \rho(\mu_0)^{-2} \int_{B_1(0)} |(\nabla_y + i\boldsymbol{A}_0)u(y)|^2\, dy, \tag{7.13}$$

where $\boldsymbol{A}_0$ is the pure A-B vector potential given in (7.5). Finally, we return to the domain D by making the inverse transform F^{-1} : $B_1(0) \to D$. Clearly

$$\int_{B_1(0)} \frac{|u(w)|^2}{|w|^2}\, dy = \int_D |u(F(z))|^2 \left|\frac{F_z'(z)}{F(z)}\right|^2\, dx. \tag{7.14}$$

Using (7.11) in conjunction with (7.12) and (7.13) we arrive at (7.10). □

We are ready to give the proof of Proposition 7.1

Proof of Proposition 7.1. We make the following covering of $\mathbb{R}^2$. Let $B_R(0)$ be a disk centered at the origin with a radius $R > 0$ so large that all the points of Λ are in $B_R(0)$. Cover the disk $B_R(0)$ with simply connected domains Ω_j having smooth boundaries in such a way that Ω_j contains λ_j but no other point from Λ. Let $\varkappa$ be the multiplicity of the covering of $B_R(0)$ and let Ω_0 be the exterior of $B_R(0)$.

We clearly have that

$$\int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx$$
$$\geq (1+\varkappa)^{-1} \left(\int_{\Omega_0} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx + \sum_{j=1}^J \int_{\Omega_j} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx \right).$$

The first term on the right-hand side is estimated by the inequality in Lemma 7.3 and each of the terms in the sum on the right-hand side is estimated by the local Hardy inequality in Lemma 7.4. In this way, we obtain that

$$\int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2 \, dx$$
$$\geq \rho(\alpha_s)^2 \int_{\Omega_0} \frac{|u(x)|^2}{|x|^2} \, dx + (1+\varkappa)^{-1} \sum_{j=1}^{J} c_j \rho(\alpha_j)^2 \int_{\Omega_j} \frac{|u(x)|^2}{|x-\lambda_j|^2} \, dx.$$

Since, inside Ω_j, $j > 0$, we have $|x-\lambda_j|^{-2} \geq C \sum_{k=1}^{J} |x-\lambda_k|^{-2}$, and inside Ω_0, we have $|x|^{-2} \geq C \sum_{k=1}^{J} |x-\lambda_k|^{-2}$, this proves (7.4). □

Remark 7.5. Using a conformal mapping was inspired by A. Balinsky [4]. He has recently derived a Hardy-type inequality for an A-B Schrödinger operator on general punctured domains. His result, however, does not give sufficient control over the Hardy weight, in particular, does not guarantee strict positivity of the weight everywhere. This does not fit our purpose and, consequently, we have derived a slightly modified Hardy-type inequality.

The inequality (7.4) has a shortcoming: if just one of the fluxes is very close to an integer, the weight on the left-hand side deteriorates. The following version of the Hardy inequality takes care of this situation: if the sum of fluxes is non-integer, we can exclude any solenoids we wish, from the expression in (7.4).

Proposition 7.6. *Suppose that* $\alpha_j \notin \mathbb{Z}$, $j = 1, 2, \ldots, J$, *and that* $\alpha_s := \sum_{j=1}^{J} \alpha_j \notin \mathbb{Z}$. *Let* $\mathcal{J}_0$ *be a subset in* $\{1, \ldots, J\}$. *Set*

$$W(x) = \min_{\{j \in \mathcal{J}_0\}} \{\rho(\alpha_j)^2, \rho(\alpha_s)^2\} \sum_{j \in \mathcal{J}_0} |x-\lambda_j|^{-2}, \tag{7.15}$$

if $\mathcal{J}_0$ *is nonempty, and*

$$W(x) = \rho(\alpha_s)^2 (1+|x|^2)^{-1}. \tag{7.16}$$

otherwise. Then there exists a constant C *such that (7.4) is satisfied.*

Proof. We consider the case of empty $\mathcal{J}_0$ first. Let, as in the proof of Proposition 7.1, the ball $B_R(0)$ contain all points z_j. Lemma 7.3 gives us the required estimate for integrals over $\Omega_0 = \mathbb{R}^2 \setminus B_R(0)$. Now we will take care of the integral over the ball $B_R(0)$. Let φ be a smooth function, $\varphi \in C_0^\infty(B_{2R}(0))$, $1-\varphi \in C_0^\infty(\Omega_0)$, $|\nabla\varphi| < 2/R$. Then for any u,

$$J_{\boldsymbol{A}}(u) := \int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2 \, dx = J_{\boldsymbol{A}}(\varphi u + (1-\varphi)u)$$
$$\geq C_1 J_{\boldsymbol{A}}(\varphi u) - C_2 R^{-2} \int_{R<|x|<2R} |u|^2 dx - C_3 J_{\boldsymbol{A}}((1-\varphi)u). \tag{7.17}$$

Now we use the well known fact (see, e.g., [33], page 2) that (for any magnetic potential $\boldsymbol{A}$),

$$J_{\boldsymbol{A}}(v) \geq \int |\nabla |v||^2 dx. \tag{7.18}$$

Applying (7.18) to the function $v = \varphi u$ and substituting the result into (7.17), we obtain

$$J_{\boldsymbol{A}}(u) \geq \int |\nabla |\varphi u||^2 dx - CR^{-2} \int_{R<|x|<2R} |u|^2 dx - C_3 J_{\boldsymbol{A}}((1-\varphi)u). \tag{7.19}$$

To the first integral in (7.19) we apply the Friedrichs inequality, the second term can be estimated from both sides by $\int_{R<|x|<2R} |x|^{-2}|u|^2 dx$. The third term on the right-hand side in (7.19) is majorized by $\int_{\Omega_0} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx + \int_{R<|x|<2R} |x|^{-2}|u|^2 dx$. Thus we have

$$\begin{aligned} J_{\boldsymbol{A}}(u) &\geq C_1 \int |\varphi u|^2 dx - C_2 \int_{R<|x|<2R} |x|^{-2}|u|^2 dx \\ &\quad - C_3 \int_{\Omega_R} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx \\ &\geq C_1 \int_{B_R} |u|^2 dx - C_2 \int_{\Omega_0} |x|^{-2}|u|^2 dx - C_3 J_{\boldsymbol{A}}(u). \end{aligned}$$

For some $\epsilon > 0$, multiply the latter inequality by ϵ and add to (7.8), multiplied by $1 - \epsilon$. We obtain

$$\begin{aligned} (1 + \epsilon + \epsilon C_3) J_{\boldsymbol{A}}(u) \geq C_1 \epsilon \int_{B_R} |u|^2 dx - C_2 \epsilon \int_{\Omega_0} |x|^{-2}|u|^2 dx \\ + C_3 (1-\epsilon)\rho(\alpha_s)^2 \int_{\Omega_0} |x|^{-2}|u|^2 dx. \end{aligned} \tag{7.20}$$

Choosing ϵ small enough (this smallness depends only on $\rho(\alpha_s)$ and R), we can arrange that the third integral in (7.20) absorbs the second one, and we get the required inequality.

In the case of nonempty $\mathcal{J}_0$, we split $J_{\boldsymbol{A}}(u) = \frac{1}{2} J_{\boldsymbol{A}}(u) + \frac{1}{2} J_{\boldsymbol{A}}(u)$. To the first term here we use the inequality we have just established. To estimate from below the second term, we act as in the proof of Proposition 7.1, i.e., consider the covering of the disk $B_R(0)$ by domains Ω_j but we write the local Hardy inequalities only for $j \in \mathcal{J}_0$. Summing such estimates, we arrive at (7.4). □

Regular lattice of solenoids. For a regular lattice of A-B solenoids we establish the following Hardy-type inequality.

Proposition 7.7. *Let $\mathcal{A}(z) = \mathcal{A}(x_1 + ix_2) = A_1 + iA_2$ be a magnetic potential such that $\mathcal{A}$ is analytical in $\mathbb{C}$ with exception of the points*

$z_{kl} = k\boldsymbol{\omega}_1 + il\boldsymbol{\omega}_2$, $k, l \in \mathbb{Z}$, and in these points $\mathcal{A}$ has simple poles with residue equal to some non-integer α. Then, for any $u \in C^\infty(\mathbb{C} \setminus \cup z_{jk})$,

$$J_{\boldsymbol{A}}(u) = \int |(\nabla + i\boldsymbol{A})u|^2 \, dx_1 dx_2 \geq C\rho(\alpha)^2 \int |u|^2 W(z)^{-2} \, dx_1 dx_2,$$

where $C > 0$, $\rho(\alpha) = \min_{k\in\mathbb{Z}} |k - \alpha|$ and $W(z)$ is the distance from $z = x_1 + ix_2$ to the nearest lattice point.

Proof. We consider first the case of a lattice Λ with $\boldsymbol{\omega}_1 = 1, \boldsymbol{\omega}_2 = i$. Write $J_{\boldsymbol{A}}(u) = 4 \times \frac{1}{4} J_{\boldsymbol{A}}(u)$. Split the lattice Λ into four sublattices, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$, where Λ_1 consists of the points $(2k, 2l)$ and, $\Lambda_2 = \{(2k+1, 2l)\}$, $\Lambda_3 = \{(2k, 2l+1)\}$ and $\Lambda_4 = \{(2k+1, 2l+1)\}$.

Around each point $z_{kl} \in \Lambda_j$, draw a disk D_{kl} with radius 0.8. Such a disk does not contain other points in the lattice. For this disk D_{kl} and any $u \in C_0^\infty(\mathbb{C} \setminus \cup z_{jk})$, we can apply the inequality (7.1),

$$\frac{1}{4} \int_{D_{kl}} |(\nabla + i\boldsymbol{A})u|^2 \, dx_1 dx_2 \geq \frac{1}{4} \rho(\alpha)^2 \int_{D_{kl}} |u|^2 \mathrm{dist}(z, z_{kl})^{-2} dx_1 dx_2, \tag{7.21}$$

since, in the punctured disk $D_{kl} \setminus \{z_{kl}\}$, the vector potential $\boldsymbol{A}$ is gauge equivalent to the potential $\alpha/|z|$.

For j fixed ($j = 1, 2, 3, 4$) we now sum (7.21) over all $z_{kl} \in \Lambda_j$. The size of the disks is selected in such way that for j fixed, the corresponding disks are disjoint and therefore one can sum (7.21) termwise and get

$$\frac{1}{4} J_{\boldsymbol{A}}(u) \geq \frac{1}{4} \rho(\alpha)^2 \sum_{z_{kl} \in \Lambda_j} \int_{D_{kl}} |u|^2 \mathrm{dist}(z, z_{kl})^{-2} \, dx_1 dx_2.$$

Next we sum the latter inequality over $j = 1, 2, 3, 4$, which yields

$$J_{\boldsymbol{A}}(u) \geq \frac{1}{4} \rho(\alpha)^2 \sum_{z_{kl} \in \Lambda} \int_{D_{kl}} |u|^2 \mathrm{dist}(z, z_{kl})^{-2} \, dx_1 dx_2 \tag{7.22}$$

Now we note that

$$\mathrm{dist}(z, z_{kl})^{-2} \geq C W(z)^{-2} \text{ for } z \in D_{kl}$$

for some $C > 0$ and, moreover, the disks D_{kl} cover the plane. Therefore the expression in (7.22) majorizes $\rho(\alpha)^2 \int |u|^2 W(z)^{-2} \, dx dx_2$.

For an arbitrary lattice we perform the same reasoning, just with disks with radius 0.8 being replaced by equal ellipses of proper size, covering the plane, and with the local Hardy inequality in the ellipse used instead for the one in the disk. □

The weight function $W(z)$ in Proposition 7.7 is positive and separated from zero, $W(z) \geq W_0 > 0$. This implies, in particular, that the spectrum of the operator $H_0^{(2)}$ is separated from zero, i.e., the magnetic field produces a spectral gap. It is remarkable to compare this with the result of Geyler-Grishanov [13] who have shown that for another self-adjoint realization of the A-B operator corresponding to an infinite regular lattice of solenoids, the lowest point of the spectrum is zero and, moreover, an eigenvalue with infinite multiplicity.[3]

8. CLR-type estimates and large coupling constant asymptotics

In three dimensions the CLR-inequality for $H_0^{(3)} - V$ takes its standard form for any of the configurations of A-B solenoids considered in Section 2, viz.

$$N_-(H_0^{(3)} - V) \leq C_3 \int_{\mathbb{R}^3} V(x)^{\frac{3}{2}}\, dx.$$

This follows automatically from the non-magnetic inequality and domination, Theorem 5.1; the constant $C_3 > 0$ only depends on the dimension.

In the two-dimensional case the presence of the magnetic field improves the non-magnetic estimates.

The aim of this section is *not* to obtain the most general nor the best possible bounds for the number of negative eigenvalues for the two-dimensional perturbed A-B Schrödinger operator; rather we just want to show that *any* CLR-type eigenvalue estimate (existing or obtained in the future) for the (nonmagnetic) two-dimensional perturbed Schrödinger operator, *with a proper Hardy term added*, automatically produces a similar estimate for the A-B Schrödinger operator.

A single solenoid. We suppose $\alpha \in (0,1)$. The (closed) quadratic form $\mathfrak{h}^{(2)}$ of the unperturbed A-B Schrödinger operator $H_0^{(2)}$ can be written as

$$\mathfrak{h}^{(2)}[u] = \frac{\mathfrak{h}^{(2)}[u]}{2} + \frac{\mathfrak{h}^{(2)}[u]}{2}. \tag{8.1}$$

Let $\beta = \min(\alpha, 1-\alpha)$. To one of the two terms in (8.1), we apply the Hardy type inequality (7.1). This yields

$$\mathfrak{h}^{(2)}[u] \geq \frac{\mathfrak{h}^{(2)}[u]}{2} + \beta^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\, dx. \tag{8.2}$$

Let $H_0^{(2)}(\beta^2 r^{-2})$, $r = |x|^{-2}$, denote the operator generated by the form on the right-hand side of (8.2). Since $H_0^{(2)}$ obeys the diamagnetic inequality, it follows from, e.g., the Trotter-Kato formula that $H_0^{(2)}(\beta^2 r^{-2})$ fulfills the diamagnetic inequality as well, in shorthand, $H_0^{(2)}(\beta^2 r^{-2}) \in \mathcal{PD}(L_0 + A^2 r^{-2})$. The latter fact in conjunction with Theorem 5.2 allows us to carry over all bounds for the two-dimensional Schrödinger operator $L_0 + r^{-2} - V$ to the A-B Schrödinger operator

[3] It is an interesting question, whether the lowest point of the spectrum of our operator is an eigenvalue.

$H_0^{(2)} - V$. In order to take into account the influence of the value of β, we can write

$$L_0 + \beta^2 r^{-2} - V \geq \beta^2 L_0 + \beta^2 r^{-2} - V = \beta^2(L_0 + r^{-2} - \beta^{-2}V).$$

Therefore, to estimate the number of eigenvalues for the operator with given β, we may use the existing estimates for the operator $L_0 + r^{-2}$ with potential $\beta^{-2}V$.

Estimates of the number of negative eigenvalues for the Schrödinger operator $L(r^{-2}, q) := L_0 + r^{-2} - qV$ in $L^2(\mathbb{R}^2)$, where we have introduced a coupling constant $q > 0$, have been studied in [34, 7, 16]. Following Solomyak [34], suppose that V belongs to the Orlicz space $L \ln(1 + L)$ locally and let $\{\zeta_j\}$, $j \geq 0$, be the sequence of averaged Orlicz norms over the annuli $r \in (2^{j-1}, 2^j)$, $j > 0$, and over the unit disk for $j = 0$, and let $S(V) = \sum \zeta_j$. Then, according to [34], $N_-(L(r^{-2}, q)) \leq CqS(V)$. Due to domination, this estimate is carried over to $H_0^{(2)} - qV$:

Theorem 8.1 (A single solenoid). *If $V \in L \ln(1 + L)(\mathbb{R}^2)$ locally then*

$$N_-(H_0^{(2)} - qV) \leq Cq\beta^{-2}S(V). \tag{8.3}$$

In a similar way, all estimates obtained in [7] and [16] hold for $H_0^{(2)} - qV$. Of course, the factor β^{-2} must arise in the estimates, as it was explained above.

Another important case is the one of the radially symmetric potential considered in [16, Theorem 1.2]. We immediately get the following result (re-producing [5, Theorem 2]).

Theorem 8.2 (A single solenoid). *If V is radially symmetric and $V \in L^1(\mathbb{R}^2)$ then*

$$N_-(H_0^{(2)} - V) \leq C\beta^{-2} \int_{\mathbb{R}^2} V(x)\, dx. \tag{8.4}$$

Finitely many solenoids. Let $\mathfrak{h}^{(2)}$ be the (closed) quadratic form generating the unperturbed magnetic Schrödinger operator $H_0^{(2)}$ associated with finitely many A-B solenoids. Again, write (8.1) and apply the Hardy type inequality established in Proposition 7.1 to one of the two terms in (8.1). We get

$$\mathfrak{h}^{(2)}[u] \geq \frac{\mathfrak{h}^{(2)}[u]}{2} + \int_{\mathbb{R}^2} W(x)|u(x)|^2\, dx, \quad u \in C_0^\infty(\mathbb{R}^2 \backslash \Lambda). \tag{8.5}$$

where $W(x)$ is given in (7.3). Let $H_0^{(2)}(W(x))$ denote the operator generated by the form on the right-hand side of (8.5). Since $H_0^{(2)}$ obeys the diamagnetic inequality, it follows from, e.g., the Trotter-Kato formula that $H_0^{(2)}(W(x))$ fulfills the diamagnetic inequality as well. The latter fact in conjunction with Theorem 5.2 allows us to carry over all bounds for the two-dimensional Schrödinger operator $L_0 + W(x) - V$ to the A-B Schrödinger operator $H_0^{(2)} - V$.

Thus our task is to estimate $N_-(L_0 + W(x) - qV)$. From the Weyl inequality we infer that

$$N_-(L_0 + W(x) - qV) \leq \sum_j^J N_-(J^{-1}L_0 + C|x - \lambda_j|^{-2} - qV_j) \quad (8.6)$$

where $V = \sum_{j=1}^J V_j$ is an appropriate splitting of V into nonnegative potentials; e.g. we can set $V_j = V$ on $\tilde{\Omega}_j$ and zero outside, where $\tilde{\Omega}_j$ contains λ_j but no other points from Λ and $\mathbb{R}^2 = \cup_j \tilde{\Omega}_j$.

Estimates of the number of negative eigenvalues for the Schrödinger operator $L(r^{-2}, q) := L_0 + r^{-2} - qV$ in $L^2(\mathbb{R}^2)$ have already been discussed above. Applying, in particular, Solomyak's estimates from [34] and (8.6) we immediately get the following result.

Theorem 8.3 (Finitely many solenoids). *If $V \in L\ln(1+L)(\mathbb{R}^2)$ locally then*

$$N_-(H_0^{(2)} - qV) \leq Cq \sum_j^J S(V_j), \quad (8.7)$$

where $S(V_j)$ on the right-hand side is the 'norm' of V_j from the similar result above for a single solenoid.

In a similar way, all estimates obtained in [7] and [16] can be carried over to $H_0^{(2)} - qV$.

Another interesting case is the one of several radially symmetric potentials. Following the reasoning of the previous theorem, we immediately get the following result.

Theorem 8.4 (Finitely many solenoids). *If $V = \sum_j^J V_j(|x - \lambda_j|)$ and $V \in L^1(\mathbb{R}^2)$ then*

$$N_-(H_0^{(2)} - V) \leq C \int_{\mathbb{R}^2} V(x)\, dx. \quad (8.8)$$

Regular lattice of solenoids. We consider the case with infinitely many solenoids located at the points of the lattice $\Lambda = \{\lambda_{kl} = (k, l) \in \mathbb{R}^2 : k, l \in \mathbb{Z}\}$. As it is typical for the two-dimensional case, one can here, as for the previous configurations, give different types of CLR estimates. We restrict ourselves to the two, most simple versions.

Theorem 8.5. *Let $V \in L\ln(1 + L)(\mathbb{R}^2)$ locally. Consider a partition of $\mathbb{R}^2$ into unit cubes Q_j. Then, for some constant C,*

$$N_-(H_0^{(2)} - V) \leq C \sum_j \|V\|_{Q_j, L\ln(1+L)}, \quad (8.9)$$

where the norms involved are the usual Orlicz norms over the cubes.

Proof. We use the inequality, already mentioned, $W(x) \geq W_0 > 0$, where $W(x)$ is the weight function in Proposition 7.7. Thus, for the (nonmagnetic) two-dimensional Schrödinger operator $L_0 + W - V$, the estimate of the type (8.9) follows from [34]. Then the Hardy inequality in Proposition 7.7 and the diamagnetic inequality, together with Theorem 5.2, imply that the same kind of estimate, with some other constant, holds for $H_0^{(2)} - V$. □

Another estimate we give here is in flavour of Theorem 8.4.

Theorem 8.6. *Suppose that $V(x)$ is the sum of radial, nonnegative functions V_{kl} centered at the points $\lambda_{kl} \in \Lambda$, viz.*

$$V(x) = \sum_{kl} V_{kl}(|x - \lambda_{kl}|).$$

Then, for some constant C,

$$N_-(H_0^{(2)} - V) \leq C \left\{ \sum_{kl} \left(\int_0^\infty V_{kl}(r)\, r dr \right)^{1/2} \right\}^2, \tag{8.10}$$

as long as the quantity on the right-hand side is finite.

Proof. Similar to the reasoning in the previous proof, the diamagnetic inequality, the Hardy inequality, and Theorem 5.2 reduce our task to establishing (8.10) for the two-dimensional Schrödinger operator $L_0 + W - V$.

Denote $\int_0^\infty V_{kl}(r) r\, dr$ by R_{kl} and suppose that the series $\sum R_{kl}^{1/2}$ converges to some number M.

Set $\delta_{kl} = R_{kl}^{1/2} M^{-1}$, $\sum_{kl} \delta_{kl} = 1$. Then the series

$$\tilde{W}(z) = \sum_{kl} \delta_{kl} |x - \lambda_{kl}|^{-2}$$

converges for any $x \notin \Lambda$. Moreover, for some constant C, not depending on V, $\tilde{W}(x) \leq CW(x)$. Thus, due to the max-min principle, it suffices to prove the estimate (8.10) for the operator $L_0 + C\tilde{W}(z) - V$. From Weyl's inequality it follows that

$$\begin{aligned} N_-(L_0 + C\tilde{W}(z) - V) &= N_- \left(\sum_{kl} (\delta_{kl} L_0 + C\delta_{kl} |x - \lambda_{kl}|^{-2} - V_{kl}) \right) \\ &\leq \sum_{kl} N_-(\delta_{kl} L_0 + C\delta_{kl} |x - \lambda_{kl}|^{-2} - V_{kl}(|x - \lambda_{kl}|)) \\ &= \sum_{kl} N_-(L_0 + C|x - \lambda_{kl}|^{-2} - \delta_{kl}^{-1} V_{kl}(|x - \lambda_{kl}|)). \end{aligned}$$

To each term in the latter sum we apply the estimate by Laptev-Netrusov [16, Theorem 1.2], getting

$$\begin{aligned} N_-(L_0 + C\tilde{W}(z) - V) &\leq C \sum_{kl} \delta_{kl}^{-1} \int_0^\infty V_{kl}(r) r\, dr \\ &= C \sum_{kl} \delta_{kl}^{-1} R_{kl} = CM \sum_{kl} R_{kl}^{1/2}, \end{aligned}$$

which coincides with the expression in (8.10). □

Large coupling constant asymptotics for $H_0^{(d)} - qV$. The task of establishing large coupling constant asymptotics for Schrödinger-like operators is nowadays a routine matter as soon as correct estimates are obtained, see e.g., [8, 27, 29]. Therefore, in the case of a singular magnetic field we just indicate those (minor) modifications one has to make.

Having the Schrödinger operator $H_0^{(d)}$ and a potential $V \in L^1_{loc}$, we define the Birman-Schwinger operator $K = K_V = K_{H_0^{(d)},V}$ as the operator defined in the domain of the quadratic form of the operator $H_0^{(d)}$ by the quadratic form of the operator V. According to the Birman-Schwinger principle, the eigenvalue distribution of the operator $K = K_V$ is closely related to the negative spectrum of the Schrödinger-like operator $H_{qV} = H_0^{(d)} - qV$, $n(t, K_V) = N_-(H_{qV}), t = q^{-1}$.

We consider the cases described in the previous sections. If $d = 3$, we set $S(V) = \int V(x)^{\frac{3}{2}} dx$. In the case $d = 2$, we denote by $S(V)$ the quantity entering in the eigenvalue estimate for the particular configuration of A-B solenoids above.

Theorem 8.7. *Suppose that* $d = 2, 3$ *and let* $H_0^{(d)}$ *be the (multivortex) Aharonov-Bohm Schrödinger operator for any of the solenoid configurations in Section 2. Assume that the conditions for the corresponding CLR-type estimate are satisfied and, moreover, assume that* $S(V)$ *is finite. Then, for the negative eigenvalues of* $H_{qV} = H_0^{(d)} - qV$, *the following asymptotic formula holds*

$$N_-(H_{qV}) \sim c_d q^{\frac{d}{2}} \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}} \text{ as } q \to \infty, \tag{8.11}$$

where c_d *is the standard coefficient,* $c_d = (2\pi)^{-d}\omega_d$ *and* ω_d *is the volume, resp. area, of the unit ball, resp. disk, in* $\mathbb{R}^d$.

Note that, similar to the nonmagnetic case, the asymptotic formula in dimension $d = 2$ may require some additional restrictions compared with just finiteness of the asymptotic coefficient in (8.11).

Proof. We are going to use the asymptotic perturbation lemma from [8]. If for any positive ϵ one can represent the operator K under consideration as a sum of two operators, $K = K' + K''$, so that for the eigenvalue counting function $n(t, K')$ an asymptotic formula with the required order ($t^{-\frac{d}{2}}$) holds and for K'' an estimate $n(t, K'') \leq \epsilon t^{-\frac{d}{2}}$ is satisfied then the required asymptotic formula holds for the operator K.

Consider the case of a finite or infinite system of solenoids, in the conditions of Theorem 8.3 So, for ϵ fixed, we choose R large enough so that $S(\chi_{\{|x|>R\}}V) < \epsilon/3$, where χ_U is the characteristic function of the set U. If there are some A-B solenoids inside the ball $B(0, R)$, consider disks (for $d = 2$) or cylinders (for $d = 3$) U_j around these solenoids, chosen so small that $S(\chi_{\cup U_j}V) < \epsilon/3$. Finally, in the bounded domain $\Omega = \Omega_\epsilon = \mathbb{R}^d \backslash (\{|x| > R\} \cup (\cup U_j))$ we find a function V_ϵ such that $V - V_\epsilon$ is bounded in this domain and $S(V_\epsilon) < \epsilon/3$. Set $V'' = \chi_{\{|x|>R\}}V + \chi_{\cup U_j}V + V_\epsilon$, $V' = V - V''$.

Correspondingly, the operator K splits into the sum $K = K' + K''$. For the operator K'', according to our choice of the function V'', the results of the previous section hold, which gives $n(t, K'') \leq \epsilon t^{\frac{d}{2}}$. We consider now the operator K'. For the study of its spectrum we can apply the standard Dirichlet-Neumann bracketing, like in [27, page...]. Since the function V' equals zero outside Ω, the eigenvalue counting function for K' is bracketed between the spectrum of similar operator in Ω with Dirichlet, from below, and Neumann, from above, boundary conditions on the boundary of Ω. In both cases, since the magnetic potential is bounded and smooth in Ω, the magnetic field term in the quadratic form of the operator is compact with respect to the leading term. Thus, as it follows from Lemma 1.3 in [9], this term can be dropped, without changing the asymptotics of the spectrum of K'. For the resulting operators, both with Dirichlet and Neumann conditions, the asymptotics of the spectrum is found, say, in [27, Sect. XIII.15], and therefore $n(t, K') \sim c_d t^{-\frac{d}{2}} \int V'^{\frac{d}{2}} dx$. Finally, applying the asymptotic perturbation lemma, we arrive at the statement of the theorem.

In the case when our potential V is the sum of radial ones, the splitting of V into 'small' and 'regular' parts, which we have just performed, goes in a little bit different way (we show it for the infinite system of solenoids; for the finite system, obvious changes are to be made).

In the conditions of Theorem 8.6 , for a fixed ϵ, find N so that $\sum_{|j|+|k|\geq N} (\int V_{jk}(r) r dr)^{1/2} < \epsilon$. With remaining terms in V, $V_{jk}, |j| + |k| < N$, we perform the following. For a fixed σ, we denote $V'_{jk}(r) = V_{jk}(r)\chi_{\{\sigma<r<\sigma^{-1}\}}$, and set $V'(x) = \sum_{|j|+|k|<N} V'_{jk}(x - \lambda_{jk})$; V' is a bounded function with compact support, $V'' = V - V'$. If σ is chosen small enough, we have, according to Theorem 8.6, for the potential V'', an eigenvalue estimate with arbitrarily (depending on ϵ) small coefficient, and for V', supported in a bounded domain, with bounded magnetic potential, the previous reasoning holds without changes - thus we can again apply the asymptotic perturbation lemma. $\square$

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