SPECTRAL PROPERTIES OF JACOBI MATRICES AND SUM RULES OF SPECIAL FORM

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ABSTRACT. In this article, we relate the properties of elements of a Jacobi matrix from certain class to the properties of its spectral measure. The main tools we use are the so-called sum rules originally suggested by Case in [2, 3]. Later, the sum rules were efficiently applied by Killip-Simon [8] to the spectral analysis of Jacobi matrices. We use a modification of the method that permits us to work with sum rules of higher orders.

As a corollary of the main theorem, we obtain a counterpart of a result of Molchanov-Novitskii-Vainberg [11] for a "continuous" Schrödinger operator on a half-line.

INTRODUCTION

It is well-known that a bounded self-adjoint operator with simple spectrum, acting on a (separable) Hilbert space, is given by a Jacobi matrix (see (0.1)) in an appropriately chosen basis [1, 5]. The study of spectral behavior of the described self-adjoint operators hence is immediately reduced to the study of spectral structure of Jacobi matrices.

On the other hand, a Jacobi matrix is uniquely defined by its spectral measure. So, it is extremely important and interesting to connect properties of the elements of a matrix with properties of its spectral data.

In this work, we are concerned with Jacobi matrices which are compact perturbations of the free Jacobi matrix J_0 (see below). Second, we are interested in relations the spectral measure of a Jacobi matrix satisfies if the elements of the matrix possess some summability properties. The inverse implication is left out of the scope of this paper.

We introduce some notation to formulate our results. Let $a=\{a_k\}, a_k>0,\ b=\{b_k\}, b_k\in\mathbb{R}$, and

$$J=J(a,b)=\begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} \tag{0.1}$$

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be a Jacobi matrix. The free (or Chebyshev) Jacobi matrix is given by

$$J_0 = J(1,0) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix}.$$

The scalar spectral measure $\sigma = \sigma(J)$ of J is defined by the relation

$$((J-z)^{-1}e_0, e_0) = \int_{\mathbb{R}} \frac{d\sigma(x)}{x-z}, \tag{0.2}$$

where $z \in \mathbb{C}\backslash\mathbb{R}$. The density of the absolutely continuous component of σ is denoted by σ'.

As we already mentioned, we consider matrices J which are compact perturbations of J_0. In this case, the absolutely continuous spectrum $\sigma_{ac}(J)$ of J coincides with $[-2,2]$, and the discrete spectrum of J lies on two sequences $\{x_j^{\pm}\}$ with properties $x_j^- \nearrow -2, x_j^- < -2$, and $x_j^+ \searrow 2, x_j^+ > 2$.

The results we obtain in this work, essentially rely on the so-called sum rules. The sum rules were originally suggested in [2, 3]. The topic drew more interest with appearance of [4]. Later, the method of sum rules was extensively developed and efficiently applied to the spectral analysis of Jacobi matrices in [8]. The article also contains broad historical references and bibliography. The following theorem is one of the results obtained in the latter paper.

Theorem 1 ([8], Theorem 1). *Let $J = J(a,b)$ be a Jacobi matrix. Then, $J - J_0$ is Hilbert-Schmidt (see Subsection 1.3), if and only if*

$$i) \int_{-2}^{2} \log \sigma'(x)\, (4-x^2)^{1/2}\, dx > -\infty, \quad ii) \sum_j (x_j^{\pm 2} - 4)^{3/2} < \infty.$$

Note that the operator $J - J_0$ lies in the Hilbert-Schmidt class if and only if

$$\sum_j (a_j - 1)^2 + \sum_j b_j^2 < \infty.$$

Subsequent papers [9, 10, 12] concentrated on different classes of Jacobi matrices. Denoting $\alpha = a - 1$, 1 being the sequence of units, we quote one of the principal results of [10].

Theorem 2 ([10], Theorem 1.1). *Let $J = J(a,b)$ is a Jacobi matrix and $J - J_0 \in S_3$ (see Subsection 1.3). Then, for a fixed m,*

$$\sum_j (\alpha_j + \ldots + \alpha_{j+m-1})^2 + \sum_j (b_j + \ldots + b_{j+m-1})^2 < \infty$$

if and only if

$$i)\ \int_{-2}^{2} \log \sigma'(x)\, w_m(x)\, dx > -\infty, \quad ii)\ \sum_j (x_j^{\pm 2} - 4)^{3/2} < \infty,$$

where $w_m(x) = (4-x^2)^{-1/2}(1-T_m^2(x/2))$ *and* T_m *is the* m*-th Chebyshev polynomial (of the first kind).*

Other interesting questions involving sum rules are discussed in [12, 13].

The proof of the central result of the present work required us to introduce several modifications to the methods of [8, 9, 10]. First, it turns out that computations pertaining to sum rules are much simpler on the domain $\bar{\mathbb{C}}\backslash[-2,2]$ than on the unit disk $\{\zeta : |\zeta| < 1\}$. This observation is mainly borrowed from [16]. Second, we have to resort to some arguments on commutation of operators and bounds coming from relations between classes of compact operators.

The main theorem of the paper is as follows. We set $\partial a = \{a_{k+1} - a_k\}$ and $\gamma_k(a) = \{(\gamma_k(a))_j\}$, where

$$(\gamma_k(a))_j = \alpha_j^k - \alpha_j \alpha_{j+1} \dots \alpha_{j+(k-1)}. \tag{0.3}$$

Theorem 3. *Let* $J = J(a,b)$ *be a Jacobi matrix. If*

$$\begin{aligned} &i) \quad a-1, b \in l^{m+1},\ \partial a, \partial b \in l^2, \\ &ii) \quad \gamma_k(a) \in l^1,\ k = 3, [m/2+1], \end{aligned} \tag{0.4}$$

then

$$i')\ \int_{-2}^{2} \log \sigma'(x) \cdot (4-x^2)^{m-1/2}\, dx > -\infty, \quad ii')\ \sum_j (x_j^{\pm 2} - 4)^{m+1/2} < \infty. \tag{0.5}$$

With the exception of (0.4), the formulation of the theorem agrees with a conjecture from [9].

It is also worth mentioning that, unlike the quoted theorem from [10] (see Theorem 2), we do not assume any "a priori" information on the Jacobi matrix J. Furthermore, when $m = 1$, the theorem gives the "only if" direction of Theorem 1. On the other hand, the conditions $i)$, $ii)$ of the theorem are not at all necessary for $i')$, $ii')$. This follows, for example, from a result of [9]. Further open questions connected to Theorem 3, are discussed in Subsection 6.2.

As a consequence of the result, we notice that when J is a discrete Schrödinger operator (that is, when $J = J(1,b)$), condition (0.4) holds trivially.

Corollary 1. *Let* $J = J(1,b)$*. Then, if* $b \in l^{m+1}$ *and* $\partial b \in l^2$*, the relations* (0.5) *hold true.*

The corollary is a discrete counterpart of a result from [11] for a "continuous" Schrödinger operator on a half-line. We also note that assumptions of Theorem 3 can be slightly weakened in this setting. Namely, the claim is still true if $\partial b \in l^2$ and $b \in l^{m+2}$, m being even.

The article is organized in the following way. Preliminary facts are listed in Section 1. A method of deriving the sum rules on $\bar{\mathbb{C}}\backslash[-2,2]$ is explained in Section 2. Theorem 3 is proved in Section 3, the proof of the principal lemma being postponed to subsequent sections. Auxiliary facts needed for the proof of the latter lemma are proved in Section 4. Final bounds and the lemma itself are obtained in Section 5. Section 6 discusses some consequences of Theorem 3 and open questions pertaining to the subject.

We conclude the introduction saying a few words on the notation. As always, symbols $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Z}_+$ stand for the natural, integer and non-negative integer numbers, respectively. To keep the notation reasonably short, the spaces $l^p(\mathbb{Z}_+)$ and $l^p(\mathbb{Z}), p \geq 1$, are denoted by l^p. We also set $\mathbb{D}$ and $\mathbb{T}$ to be the unit disk $\{\zeta : |\zeta| < 1\}$ and the unit circle $\{\zeta : |\zeta| = 1\}$, correspondingly.

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1. Preliminaries

Information contained in this section, is well-known (see [8, 14, 15]), and is included only for reader's convenience.

1.1. **Some facts on one-sided Jacobi matrices.** Let $J = J(a,b)$ be a Jacobi matrix, defined in (0.1) and acting on $l^2(\mathbb{Z}_+)$. Let $\{e_k\}_{k\in\mathbb{Z}_+}$ be the standard basis in the space. It is easy to see that the so-called Weyl function

$$M(z) = ((J-z)^{-1}e_0, e_0),$$

associated to J, lies in the Herglotz' class (i.e., has a positive imaginary part on the upper half-plane), and, consequently, it admits representation (0.2) with a measure $\sigma = \sigma(J)$. The measure is called a spectral measure of J and is unique up to a normalization. In particular, we have

$$\sigma = \frac{1}{\pi} \text{weak-}\lim_{y\to 0+} \operatorname{Im} M(. + iy),$$

and, moreover, $\sigma'(x) = 1/\pi \lim_{y\to 0+} \operatorname{Im} M(x+iy)$ for almost all $x \in \mathbb{R}$.

Suppose that $\operatorname{rank}(J - J_0) < \infty$. As we mentioned in the introduction, the function M is meromorphic on $\bar{\mathbb{C}}\backslash[-2,2]$. It is often convenient to uniformize the domain with the help of maps $\zeta(z) = 1/2(z - \sqrt{z^2-4}),\ z \in \bar{\mathbb{C}}\backslash[-2,2]$, and $z(\zeta) = \zeta + 1/\zeta,\ \zeta \in \mathbb{D}$. It is clear that $\zeta : \bar{\mathbb{C}}\backslash[-2,2] \to \mathbb{D},\ z : \mathbb{D} \to \bar{\mathbb{C}}\backslash[-2,2]$, and the maps are mutually inverse.

Let us consider a generalized eigenvector $u(\zeta) = \{u_j(\zeta)\}$ of J (that is, $Ju(\zeta) = (\zeta + 1/\zeta)u(\zeta)$) with the property

$$\lim_{j\to+\infty} \zeta^{-j} u_j(\zeta) = 1.$$

The vector u and the function u_0 are called the Jost solution and the Jost function, respectively. We have the following theorem.

Theorem 4 ([8], Theorem 2.16, 2.19)**.** *Let* $\operatorname{rank}(J - J_0) < \infty$. *Then*

$$u_0(z) = u_0(\zeta(z)) = \frac{1}{A_0'} \det(J - z)(J_0 - z)^{-1},$$

where $A_0' = \prod_j a_j$ *and* $z \in \bar{\mathbb{C}} \backslash [-2, 2]$. *Furthermore,*

$$|u_0(x)|^2 = \frac{\sqrt{4 - x^2}}{\sigma'(x)}$$

almost everywhere on $[-2, 2]$.

More information on the Jost solution and the Jost function of a Jacobi matrix J can be found in [14], Ch. 10.

We also need a result connecting properties of the discrete spectrum $\{x_j^{\pm}\}$ of J to the properties of the sequences $a = \{a_k\}$ and $b = \{b_k\}$.

Theorem 5 ([7], Theorem 3)**.** *Let* $J = J(a, b)$ *(see* (0.1)*), and* $a, b \in l^{m+1}$. *Then*

$$\sum_j (x_j^{\pm 2} - 4)^{m+1/2} \le C_0\Big(\sum_j |a - 1|^{m+1} + \sum_j |b|^{m+1}\Big)$$

with a constant C_0 *depending on* m.

1.2. **Facts on two-sided Jacobi matrices.** In this subsection, we are interested in two-sided Jacobi matrices. The exposition mainly follows [14], Ch. 10, and [15]. The second paper also discusses very interesting and deep aspects of the scattering theory for Jacobi matrices.

Let $a = \{a_k\}_{k\in\mathbb{Z}}$, $b = \{b_k\}_{k\in\mathbb{Z}}$, and $J = J(a, b)$ be a Jacobi matrix, acting on $l^2(\mathbb{Z})$. As before, $\{e_k\}_{k\in\mathbb{Z}}$ is the standard basis in $l^2(\mathbb{Z})$. We set $\mathcal{E}$ to be the operator of orthogonal projection from the space on $\operatorname{lin}\{e_{-1}, e_0\}$, where $\operatorname{lin}\{.\}$ refers to the linear span of indicated vectors.

We define a 2×2-matrix-valued function M with the help of the formula

$$M(z) = \mathcal{E}(J - z)^{-1}\mathcal{E}^*.$$

As in the previous subsection, M is in the matrix-valued Herglotz' class (see [14], Appendix B, [15]), and, consequently, it can be represented as

$$M(z) = \int_{\mathbb{R}} \frac{d\Sigma(x)}{x - z},$$

where Σ is a 2×2-matrix-valued measure Σ. The density of its absolutely continuous component is denoted by Σ'.

Let $J_0 = J(1, 0)$, where 1 and 0 are two-sided sequences of 1's and 0's. Assume that rank $(J - J_0) < \infty$. In this case the absolutely continuous spectrum $\sigma_{ac}(J)$ of J coincides with $[-2, 2]$ (and is of multiplicity two). The discrete spectrum of J lies on sequences $\{x_j^\pm\}$ with properties $x_j^- \nearrow -2, x_j^- < -2$, and $x_j^+ \searrow 2, x_j^+ > 2$.

We introduce the so-called transmission coefficient of the Jacobi matrix J now. Consider the Jost solutions $u_\pm$ satisfying the relations

$$Ju_\pm(\zeta) = (\zeta + 1/\zeta)u_\pm(\zeta), \quad \lim_{j\to\infty} \zeta^{\mp j} u_\pm(\zeta) = 1,$$

where $\zeta \in \mathbb{D}\backslash(-1, 1)$. It is not difficult to see that vectors $u_\pm(\zeta), u_\pm(1/\zeta), \zeta \in \mathbb{T}$, are linearly independent (see [14], Sect. 10.2), and we have for some functions $s, s_\pm$ that

$$u_\pm(\zeta) = s(\zeta)u_\mp(1/\zeta) + s_\mp(\zeta)u_\mp(\zeta),$$

where $\zeta \in \mathbb{T}\backslash\{-1, 1\}$. The functions possess a number of remarkable properties, and play an important role in the spectral analysis of Jacobi matrices. For instance, they can be extended up to analytic functions on $\mathbb{D}$. The function s is called the transmission coefficient of J.

Theorem 6 ([15], Theorem 1.1). *Let $J = J(a, b)$, $a_0 = 1$, and s be the transmission coefficient of J. Then*

$$\det(2\pi\Sigma'(x)) = |s(\zeta(x))|^2$$

for almost all $x \in [-2, 2]$.

The theorem suggests that the Jost function u_0 for one-sided Jacobi matrices is a right counterpart of the transmission coefficient for two-sided Jacobi matrices and vice versa.

1.3. **Compact operators and commutators.** Now, we discuss some properties of certain classes of compact operators; see [6] in this connection.

Let A be a compact operator on a (separable) Hilbert space H. The singular values $\{s_k(A)\}, s_k(A) \searrow 0, s_k(A) \geq 0$, are defined as $s_k(A) = \lambda_k(A^*A)^{1/2}$, where $\lambda_k(A^*A)$ is the k-th eigenvalue of operator A^*A. The Schatten–von Neumann classes are given by the relations

$$S_p = \Big\{A - \text{ compact } : ||A||_{S_p}^p = \sum_k s_k(A)^p < \infty\Big\},$$

where $p \geq 1$. In particular, S_1 and S_2 describe classes of nuclear and Hilbert-Schmidt operators, respectively.

The sets S_p are ideals, that is,

$$||BAC||_{S_p} \leq ||B||\,||A||_{S_p}\,||C||,$$

for any bounded operators B, C on H and $A \in S_p$. We also have the Hölder inequality for S_p's, i.e.

$$||A_1 \dots A_n||_{S_1} \leq ||A_1||_{S_{p_1}} \dots ||A_n||_{S_{p_n}},$$

where $A_j \in S_{p_j}, j = 1, n$, and $\sum_j 1/p_j = 1$.

Suppose now that A, B are some operators on H. For the sake of simplicity, we suppose A to be of finite rank. Let, furthermore, $\{e_j\}$ be a fixed basis in the space. By $\operatorname{tr} A$ we mean

$$\operatorname{tr} A = \sum_j (Ae_j, e_j),$$

and, clearly, $|\operatorname{tr} A| \leq ||A||_{S_1}$.

We define the commutator $[A, B]$ of A and B by $[A, B] = AB - BA$. The following simple lemma holds.

Lemma 1. *Let A, B be some operators. Then*

$$[A^k, B] = \sum_{j=0}^{k-1} A^{k-1-j}[A, B]A^j. \tag{1.1}$$

The proof of the lemma immediately follows by induction from the equality

$$[AB, C] = A[B, C] + [A, C]B.$$

Of course, the lemma also implies that

$$[A, B^k] = \sum_{j=0}^{k-1} B^{k-1-j}[A, B]B^j. \tag{1.2}$$

2. A special approach to the sum rules

The contents of the first two subsections of this section closely follows [16] and is quoted only for completeness of exposition.

2.1. Sum rules as identities involving residues.

We suppose first that rank $(J - J_0) < \infty$. We have the following proposition.

Proposition 1 ([16]). *Let u_0 be the Jost function of J and p be a real entire function. Then*

$$\int_{x_1^-}^{x_1^+} \lambda(x)\, dx = \operatorname{Res}_\infty \left\{ \frac{p(z)}{\sqrt{z^2 - 4}} \log u_0(z) \right\}, \tag{2.1}$$

where λ is a function defined by relations

$$\lambda(x) = \begin{cases} \dfrac{p(x)}{\sqrt{x^2 - 4}} \lambda_0(x), & x \notin [-2, 2], \\ -\dfrac{p(x)}{2\pi\sqrt{4 - x^2}} \log \dfrac{\sqrt{4 - x^2}}{\sigma'(x)}, & x \in [-2, 2], \end{cases}$$

and

$$\lambda_0(x) = \begin{cases} \#\{x_j^+ : \ x_j^+ > x\}, & x > 2, \\ \#\{x_j^- : \ x_j^- < x\}, & x < -2, \\ 0, & x \in [-2, 2]. \end{cases}$$

As always, symbol # means the number of elements in a set and $\text{Res}_{\infty}(.)$ refers to the residue of a function at $z = \infty$.

Proof. Let

$$F(z) = \frac{p(z)}{\sqrt{z^2 - 4}} \log u_0(z).$$

We choose the branch of $\sqrt{z^2 - 4}$ with the properties $\sqrt{z^2 - 4} > 0$, when $z < -2$, $\sqrt{z^2 - 4} \in i\mathbb{R}_+$, when $z \in [-2, 2]$, and $\sqrt{z^2 - 4} < 0$, when $z > 2$. We readily see that the function F is analytic on $\bar{\mathbb{C}} \backslash [x_1^-, x_1^+]$. The function also has well-defined boundary values on the upper and lower edges of $[x_1^-, x_1^+]$. We denote them by $F_{\pm}$, respectively.

For a sufficiently big $r > 0$, we have by definition of the residue at $z = \infty$ that

$$-\frac{1}{2\pi i} \int_{z:\,|z|=r} F(z)\, dz = \text{Res}_{\infty} F(z).$$

We have at the left-hand side of the equality

$$-\frac{1}{2\pi i} \int_{z:\,|z|=r} F(z)\, dz = \frac{1}{2\pi i} \left(\int_{x_1^-}^{x_1^+} F(x)_+ \, dx - \int_{x_1^-}^{x_1^+} F(x)_- \, dx \right).$$

Since $F(x)_- = \overline{F(x)_+}, x \in [x_1^-, x_1^+]$, we continue as

$$\frac{1}{2\pi i} \int_{x_1^-}^{x_1^+} (F(x)_+ - F(x)_-)\, dx = \frac{1}{\pi} \int_{x_1^-}^{x_1^+} \text{Im}\, F(x)_+ \, dx. \tag{2.2}$$

We note that $(\sqrt{x^2 - 4})_+ = i\sqrt{4 - x^2}$ for $x \in [-2, 2]$, and, by Theorem 4,

$$\text{Re} \log u_0(x)_+ = \log |u_0(x)| = \frac{1}{2} \log \frac{\sqrt{4 - x^2}}{\sigma'(x)}.$$

Furthermore,

$$\text{Im}\, (\log u_0(x))_+ = \pi \begin{cases} -\#\{x_j^+ : \ x_j^+ > x\}, & x > 2, \\ \ \ \#\{x_j^- : \ x_j^- < x\}, & x < -2. \end{cases}$$

Consequently,

$$\text{Im} \left(\frac{p(z)}{\sqrt{z^2 - 4}} \log u_0(z) \right)_+ = \begin{cases} -\dfrac{p(x)}{2\sqrt{4 - x^2}} \log \dfrac{\sqrt{4 - x^2}}{\sigma'(x)}, & x \in [-2, 2], \\ \\ \dfrac{\pi p(x)}{\sqrt{x^2 - 4}} \lambda_0(x), & x \notin [-2, 2]. \end{cases}$$

Plugging this expression in (2.2), we obtain

$$\operatorname{Res}_{\infty} F(z) = -\frac{1}{2\pi}\int_{-2}^{2} \frac{p(x)}{\sqrt{4-x^2}} \log\frac{\sqrt{4-x^2}}{\sigma'(x)}\,dx + \int_{x_1^-}^{x_1^+} \frac{p(x)}{\sqrt{x^2-4}}\lambda_0(x)\,dx.$$

The proposition is proved. □

It is worth mentioning that assertions of this type may be proved for functions of more general form than real entire functions.

2.2. **A special sum rule.** We are particularly concerned with the case

$$p(z) = p_m(z) = (-1)^{m+1}(z^2-4)^m,$$

where $m \in \mathbb{N}$. We have

$$\lambda_m(x) = \begin{cases} \dfrac{1}{2\pi}(4-x^2)^{m-1/2}\log\dfrac{\sqrt{4-x^2}}{\sigma'(x)}, & x \in [-2,2], \\ (-1)^{m+1}(x^2-4)^{m-1/2}\lambda_0(x), & x \notin [-2,2]. \end{cases}$$

We put $\mu_{0\pm} = \sum \delta_{x_j^\pm}$, δ_{x_0} being Dirac's delta centered at x_0. We notice that $\lambda_0(x) = \int_2^x d\mu_{0+}(s)$ for $x > 2$, and we get integrating by parts

$$\int_2^{x_1^+} (x^2-4)^{m-1/2}\lambda_0(x)\,dx = \int_2^{x_1^+} G_m(x)\,d\mu_{0+}(x) = \sum_j G_m(x_j^+),$$

where

$$G_m(x) = \int_2^x (s^2-4)^{m-1/2}\,ds. \tag{2.3}$$

We extend G_m to $x < -2$ in even way and, carrying out similar computation for μ_{0-}, we see that

$$\int_{x_1^-}^{-2} \lambda_m(x)\,dx + \int_2^{x_1^+} \lambda_m(x)\,dx = (-1)^{m+1}\sum_j G_m(x_j^\pm).$$

Furthermore, the inequality $C_1(x\pm 2)^{m-1/2} \le (x^2-4)^{m-1/2} \le C_2(x\pm 2)^{m-1/2}$ for x in $[x_1^-,-2)$ or $(2,x_1^+]$, respectively, and some constants C_1, C_2, implies that

$$G_m(x) = C_3(x^2-4)^{m+1/2} + O((x^2-4)^{m+3/2}). \tag{2.4}$$

Summing up, we obtain that the left-hand side of (2.1) is given by the formula

$$\begin{aligned} \Phi_m(\sigma) &= \Phi_{m,1}(\sigma) + \Phi_{m,2}(\sigma) \\ &= \frac{1}{2\pi}\int_{-2}^{2}(4-x^2)^{m-1/2}\log\frac{\sqrt{4-x^2}}{\sigma'(x)}\,dx + (-1)^{m+1}\sum_j G_m(x_j^\pm). \end{aligned} \tag{2.5}$$

Observe that $\Phi_{m,2}(\sigma) \ge 0$ when m is odd and $\Phi_{m,2}(\sigma) \le 0$ when m is even.

2.3. **Evaluation of the residue.** Let us compute the right-hand side of equality (2.1) now. In a neighborhood of $z=\infty$, we have

$$\begin{aligned}\log u_0(z) &= \operatorname{tr}\left(\log(z-J)-\log(z-J_0)\right)-\log A_0' \\ &= \operatorname{tr}\left(\log(I-J/z)-\log(I-J_0/z)\right)-\log A_0' \\ &= -\left\{\log A_0' + \sum_{k=1}^{\infty}\frac{1}{k}\operatorname{tr}\left(J^k-J_0^k\right)\frac{1}{z^k}\right\}.\end{aligned}$$

It is convenient to set $A_0=\operatorname{diag}\{a_k\}$, and so $\log A_0' = \operatorname{tr}\,\log A_0$.

Furthermore, we expand $p_m(z)/\sqrt{z^2-4}=(-1)^{m+1}(z^2-4)^{m-1/2}$ in the Laurent series centered at $z=\infty$. That is, we have

$$(1-x)^{m-1/2}=\sum_{k=0}^{m-1}(-1)^k\tilde C_{2m-1}^{2k}x^k+(-1)^m\frac{(2m-1)!!}{(2m)!!}x^m+O(x^{m+1}) \tag{2.6}$$

for small $|x|$, $\tilde C_{2m-1}^{2k}=\frac{(2m-1)!!}{(2m-1-2k)!!(2k)!!}$, and $k!!$ refers to "even" or "odd" factorials. Consequently,

$$(z^2-4)^{m-1/2}=z^{2m-1}(1-(4/z^2))^{m-1/2},$$

and, making use of (2.6) together with $\tilde C_{2m-1}^{2m-2k-2}=\tilde C_{2m-1}^{2k+1}$, we see that

$$(-1)^{m+1}(z^2-4)^{m-1/2}=2^{2m}\left\{\sum_{k=0}^{m-1}\frac{(-1)^k}{2^{2(k+1)}}\tilde C_{2m-1}^{2k+1}z^{2k+1}-\frac{(2m-1)!!}{(2m)!!}\frac{1}{z}\right\}+O\left(\frac{1}{z^3}\right).$$

For the sake of brevity, we put

$$\log u_0(z)=\sum_{k=0}^{\infty}c_k\frac{1}{z^k},\qquad (z^2-4)^{m-1/2}=\sum_{k=-1}^{m-1}d_{2k+1}z^{2k+1}+O\left(\frac{1}{z^3}\right).$$

Then

$$\operatorname{Res}_\infty\left(\frac{p_m(z)}{\sqrt{z^2-4}}\log u_0(z)\right)=-\sum_{k=0}^{m}d_{2k-1}c_{2k}.$$

An elementary computation shows that

$$\begin{aligned}\Psi_m(J) &= \operatorname{Res}_\infty\left(\frac{p_m(z)}{\sqrt{z^2-4}}\log u_0(z)\right) \\ &= -\left\{\sum_{k=1}^{m}\frac{(-1)^k}{2^{2k+1}k}\tilde C_{2m-1}^{2k-1}\operatorname{tr}\left(J^{2k}-J_0^{2k}\right)+\frac{(2m-1)!!}{(2m)!!}\operatorname{tr}\,\log A_0\right\}.\end{aligned} \tag{2.7}$$

Comparing (2.1), (2.5), and the latter relation, we obtain

$$\frac{1}{2\pi}\int_{-2}^{2}(4-x^2)^{m-1/2}\cdot\log\frac{\sqrt{4-x^2}}{\sigma'(x)}\,dx+(-1)^{m+1}\sum_j G_m(x_j^{\pm})=\Psi_m(J). \tag{2.8}$$

This is precisely the sum rule we are interested in.

3. Proof of Theorem 3.

The proof of the theorem depends on a number of auxiliary lemmas which are proved in subsequent sections. In this section we prove the theorem modulo these facts.

The key role in the argument is played by the following lemma.

Main Lemma. *Let $J = J(a, b)$ satisfy the assumption of Theorem 3. Then*
(3.1)

$$|\Psi_m(J)|\le C_4\big(||a-1||_{m+1}^{m+1}+||b||_{m+1}^{m+1}+||\partial a||_2^2+||\partial b||_2^2+\sum_{k=3}^{[m/2+1]}||\gamma_k(a)||_1\big),$$

where sequences $\gamma_k(a)$ are defined in (0.3)*, and the constant C_4 depends on $||a-1||_\infty, ||b||_\infty, ||\partial a||_\infty$, and $||\partial b||_\infty$.*

The norms $||.||_p$ refer to usual norms in l^p-spaces. With exception of the lemma, the proof of Theorem 3 is quite standard (see [8, 9, 10]).

Proof of Theorem 3. We have to prove that quantities $\Phi_{m,1}(\sigma)$ and $\Phi_{m,2}(\sigma)$, defined in (2.5), are finite.

We put $a_N = \{(a_N)_k\}$ and $a'_N = \{(a'_N)_k\}$, where

$$(a_N)_k=\begin{cases} a_k, & k\le N,\\ 1, & k>N,\end{cases}\qquad (a'_N)_k=\begin{cases} 1, & k\le N,\\ a_k, & k>N.\end{cases}$$

Define sequences b_N, b'_N in the same way (of course, with 1's replaced by 0's). Let $J_N = J(a_N, b_N)$ and σ_N be the spectral measure of J_N. As we readily see, $a'_N - 1, b'_N \to 0$, $\partial a'_N, \partial b'_N \to 0$, and $\gamma_k(a'_N) \to 0$ in corresponding norms as $N \to \infty$. By the Main Lemma, we have for $N' = N - m$

$$\begin{aligned}|\Psi_m(J)-\Psi_m(J_N)| &\le |\Psi_m(a'_{N'},b'_{N'})|\le C_4(||a'_{N'}-1||_{m+1}^{m+1}+||b'_{N'}||_{m+1}^{m+1}\\ &+ ||\partial a'_{N'}||_2^2+||\partial b'_{N'}||_2^2+\textstyle\sum_k||\gamma_k(a'_{N'})||_1),\end{aligned}$$

or, $\Psi_m(J_N) \to \Psi_m(J)$ as $N \to \infty$. On the other hand, $(J_N - z)^{-1} \to (J - z)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, and, consequently, $\sigma_N \to \sigma$ weakly. Looking at [8], Corollary 5.3 and Theorem 6.2, we get

$$\Phi_{m,1}(\sigma)\le\liminf_N \Phi_{m,1}(\sigma_N)$$

and

$$\lim_{N\to\infty} \Phi_{m,2}(\sigma_N) = \Phi_{m,2}(\sigma).$$

It is important that always $\Phi_{m,1}(\sigma) > -\infty$ (see [8], Lemma 5.1 or [10], Sect. 3.1).

We consider cases when m is odd or even separately. First, assume m to be odd. Then $\Phi_{m,2}(\sigma) \geq 0$ and

$$\begin{aligned}\Phi_m(\sigma) &= \Phi_{m,1}(\sigma) + \Phi_{m,2}(\sigma) \leq \limsup_N \Phi_m(\sigma_N) = \limsup_N \Psi_m(J_N) \\ &= \lim_{N\to\infty} \Psi_m(J_N) = \Psi_m(J) < \infty.\end{aligned}$$

Since $\Phi_{m,1}(J) > -\infty$, we see that $\Phi_{m,1}(\sigma)$ and $\Phi_{m,2}(\sigma)$ are finite. Together with (2.4) and (2.5), this proves the theorem in this particular case.

Now, suppose that m is even. We have $\Phi_{m,2} \leq 0$, and Theorem 5 implies that

$$|\Phi_{m,2}(J)| = \sum_j G_m(x_j^\pm) \leq C_0(||a-1||_{m+1}^{m+1} + ||b||_{m+1}^{m+1}).$$

Furthermore,

$$\begin{aligned}\Phi_{m,1}(\sigma) &\leq \limsup_N \Phi_{m,1}(\sigma_N) = \limsup_N (\Psi_m(J_N) - \Phi_{m,2}(J_N)) \\ &\leq \lim_{N\to\infty} \Psi_m(J_N) - \lim_{N\to\infty} \Phi_{m,2}(J_N) = \Psi_m(J) - \Phi_{m,2}(J) < \infty.\end{aligned}$$

The proof of the theorem is complete. □

So, when m is even, the proof of Theorem 3 leans on Theorem 5 besides other facts. On the other hand, we can prove a version of Theorem 5 when m is odd.

Corollary 2. *Let $J = J(a,b)$ satisfy assumptions of Theorem 3 and m is odd. Then*

$$\sum_j (x_j^{\pm 2} - 4)^{m+1/2} < \infty.$$

Proof. Keeping in mind relation (2.4), we have to prove that

$$\Phi_{m,2}(\sigma) = \sum_j G_m(x_j^\pm) < \infty.$$

Recall that $\Phi_{m,1}(\sigma) > -\infty$. Slightly modifying arguments from the proof of Theorem 3, we obtain

$$\begin{aligned}\Phi_{m,2}(\sigma) &= \lim_{N\to\infty} \Phi_{m,2}(\sigma_N) = \lim_{N\to\infty} (\Psi_m(J_N) - \Phi_{m,1}(\sigma_N)) \\ &\leq \Psi_m(J) - \liminf_N \Phi_{m,1}(\sigma_N) < \infty.\end{aligned}$$

The corollary is proved. □

Nevertheless, we have to say that the above corollary is less sharp with regard to the discrete spectrum $\{x_j^\pm\}$ than Theorem 5.

4. Commutation of certain operators and pertaining estimates.

In this section we begin discussion of the facts needed for the proof of the Main Lemma. Namely, let us look at expressions $\operatorname{tr}(J^{2k} - J_0^{2k})$ appearing in (2.7). Let $V = J - J_0$, or $V = J(\alpha, b)$, where $\alpha = a - 1$ and 1 is a sequence consisting of units. Obviously,

$$J^{2k} = (J_0 + V)^{2k} = \sum_{p=0}^{2k} \sum_{i_0+\ldots+i_p=2k-p} J^{i_0} V J_0^{i_1} \ldots V J_0^{i_p},$$

and, consequently,

$$\operatorname{tr}(J^{2k} - J_0^{2k}) = \operatorname{tr} \sum_{p=1}^{2k} \sum_{i_1+\ldots+i_p=2k-p} V J_0^{i_1} \ldots V J_0^{i_p}.$$

It occurs that, under assumptions of Theorem 3, the quantity $\operatorname{tr} V J_0^{i_1} \ldots V J_0^{i_p}$ behaves, up to complementary bounded terms, as $\operatorname{tr} V^p J_0^{2k-p}$. In other words, we may assume operators V and J_0 to commute modulo "good" summands. The proof of this observation constitutes the contents of the present section (see Lemma 5).

4.1. Commuting operators V and J_0.

We agree to write $\tilde{O}(A^2)$ instead of $\sum_k^N B_k A C_k A D_k$ with some bounded operators B_k, C_k, D_k. We also introduce multi-indices $\mathbf{k} = (k_1, \ldots, k_p)$, $\mathbf{p} = (p_1, p_2, p_3)$, and $\mathbf{l} = (l_1, l_2, l_3)$. Set $|\mathbf{k}| = \sum_i k_i$.

The following lemmas hold.

Lemma 2. *Let $|\mathbf{k}| = N$. Then*

(4.1)
$$V J_0^{k_1} \ldots V J_0^{k_p} = V^p J_0^N + \sum_{|\mathbf{l}|=p,\, |\mathbf{p}|=N} c_{\mathbf{l},\mathbf{p}}\, J_0^{p_1} V^{l_1} [V^{l_2}, J_0^{p_2}] V^{l_3} J_0^{p_3} + \tilde{O}([V, J_0]^2).$$

Proof. We prove the lemma by induction on p. The claim of the lemma is trivial when $p = 1$. We suppose that the lemma is valid for p and we prove it for $p+1$. We have $\mathbf{k} = (k_1, \ldots, k_{p+1})$ and

$$V J_0^{k_1} \ldots V J_0^{k_{p+1}} = V J_0^{k_1} \big(V^p J_0^{N'} + \sum_{|\mathbf{l}|=p,\, |\mathbf{p}|=N'} c_{\mathbf{l},\mathbf{p}} J_0^{p_1} V^{l_1} [V^{l_2}, J_0^{p_2}] V^{l_3} J_0^{p_3} + \tilde{O}([V, J_0]^2) \big),$$

where $N' = k_2 + \ldots + k_{p+1}$. Furthermore,

$$V J_0^{k_1} V^p J_0^{N'} = V(V^p J_0^{k_1} + [J_0^{k_1}, V^p]) J_0^{N'} = V^{p+1} J_0^{k_1+N'} + V[J_0^{k_1}, V^p] J_0^{N'}.$$

Then, taking $p_1' = k_1 + p_1$, we get

$$\begin{aligned}
VJ_0^{k_1}J_0^{p_1}V^{l_1}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} &= VJ_0^{p_1'}V^{l_1}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} \\
&= V(V^{l_1}J_0^{p_1'} + [J_0^{p_1'}, V^{l_1}])[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} \\
&= V^{l_1+1}J_0^{p_1'}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} + \tilde{O}([V, J_0]^2) \\
&= (J_0^{p_1'}V^{l_1+1} + [V^{l_1+1}, J_0^{p_1}])[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} + \tilde{O}([V, J_0]^2) \\
&= J_0^{p_1'}V^{l_1+1}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} + \tilde{O}([V, J_0]^2).
\end{aligned}$$

Above, we repeatedly used formulas (1.1) and (1.2) with $A = V^k, B = J_0$ and $A = V, B = J_0^k$, respectively. Finally, it is plain that

$$VJ_0^{k_1}\tilde{O}([V, J_0]^2) = \tilde{O}([V, J_0]^2).$$

The proof is complete. □

Lemma 3. *Let* $|\mathbf{k}| = N$. *Then*

$$\operatorname{tr} VJ_0^{k_1} \dots VJ_0^{k_p} = \operatorname{tr} V^p J_0^N + C_5 \operatorname{tr} V^{p-1}[V, J_0]J_0^{N-1} + \operatorname{tr} \tilde{O}([V, J_0]^2), \tag{4.2}$$

where C_5 *is a constant depending on* p *and* N.

Proof. Employing Lemma 2, we immediately get the first and the last terms on the right-hand side of the latter equality. As for the second term, we see that

$$\begin{aligned}
\operatorname{tr} J_0^{p_1}V^{l_1}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} &= \operatorname{tr} V^{l_1}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p'} \\
&= \operatorname{tr} V^{l_1}[V^{l_2}, J_0^{p_2}]J_0^{p_1'}V^{l_3} + \operatorname{tr} \tilde{O}([V, J_0]^2) \\
&= \operatorname{tr} V^{l_1'}[V^{l_2}, J_0^{p_2}]J_0^{p_1'} + \operatorname{tr} \tilde{O}([V, J_0]^2),
\end{aligned}$$

where $p' = p_1 + p_3$ and $l_1' = l_1 + l_3$. Recalling (1.1) and (1.2), we obtain that

$$\begin{aligned}
\operatorname{tr} V^{l_1'}[V^{l_2}, J_0^{p_2}]J_0^{p_1'} &= \operatorname{tr} V^{l_1'}\big(\textstyle\sum_j V^j[V, J_0^{p_2}]V^{l_2-j-1}\big)J_0^{p_1'} \\
&= \textstyle\sum_j \operatorname{tr} V^{l_1'+j}[V, J_0^{p_2}](J_0^{p_1'}V^{l_2-j-1} + [V^{l_2-j-1}, J_0^{p_1'}]) \\
&= \textstyle\sum_j V^{l_1'+l_2-1}[V, J_0^{p_2}]J_0^{p_1'} + \operatorname{tr} \tilde{O}([V, J_0]^2),
\end{aligned}$$

where $l_1' + l_2 - 1 = l_1 + l_2 + l_3 - 1 = p - 1$. Transforming expressions $[V, J_0^{p_2}]$ in the same way, we finish the proof of the lemma. □

4.2. **Bounds on** $\operatorname{tr} \tilde{O}([V, J_0]^2)$ **and** $\operatorname{tr} V^{p-1}[V, J_0]J_0^{N-1}$**.** In this subsection we estimate quantities

$$|\operatorname{tr} \tilde{O}([V, J_0]^2)|, \quad |\operatorname{tr} (V^{p-1}[V, J_0]J_0^{N-1})| \tag{4.3}$$

under assumptions of Theorem 3.

To start with, we extend Jacobi matrices $J = J(a, b)$ and J_0, acting on $l^2(\mathbb{Z}_+)$, to Jacobi matrices on $l^2(\mathbb{Z})$, by joining 1's on the auxiliary diagonals and 0's elsewhere. For the sake of simplicity, the two-sided matrices generated by J and J_0 in this way, are also denoted by J and J_0. We note that the quantity $\Psi_m(J)$ in (2.7), up to tr $\log A_0$, is a trace of a polynomial with respect to J and J_0. Consequently, $\Psi_m(J)$ for the two-sided Jacobi matrix J differs from $\Psi_m(J)$ for the one-sided matrix only by a finite number of terms. These terms depend only on a finite number of elements of the sequences $\{a_k\}$ and $\{b_k\}$. So, we shall obtain bound (3.1) for one-sided matrices if we prove it for two-sided ones. Hence, *we work with two-sided Jacobi matrices from now on.*

Let $\{e_k\}_{k\in\mathbb{Z}}$ be the standard orthonormal basis in the space $l^2(\mathbb{Z})$. We define a shift operator S as $Se_k = e_{k+1}$. Obviously, its inverse S^{-1} is $S^{-1}e_k = e_{k-1}$.

We identify a sequence $a = \{a_k\}$ with diagonal operator diag $\{a_k\}$. *This agreement is in force through the rest of the paper.* Let, furthermore, $a_{(s)} = \{a_{k+s}\}$ for an integer s. So, we have

$$J = J(a, b) = Sa + aS^{-1} + b, \quad J_0 = S + S^{-1}, \tag{4.4}$$

and $V = S\alpha + \alpha S^{-1} + b$, where $\alpha = a - 1$. We have for an integer k

$$S^k a = a_{(k)}S^k, \quad aS^k = Sa_{(-k)}. \tag{4.5}$$

A straightforward computation with the help of the latter formulas shows that

$$\begin{aligned} VJ_0 &= S^2\alpha_{(-1)} + \alpha S^{-2} + Sb_{(-1)} + bS^{-1} + \alpha + \alpha_{(1)}, \\ J_0V &= S^2\alpha + \alpha_{(-1)}S^{-2} + Sb + b_{(-1)}S^{-1} + \alpha + \alpha_{(1)}, \end{aligned}$$

and, consequently,

$$[V, J_0] = -S^2\gamma + \gamma S^{-2} - S\beta + \beta S^{-1}, \tag{4.6}$$

where, for the sake of brevity, $\gamma = \alpha - \alpha_{(-1)} = a - a_{(-1)} = \partial a$, and $\beta = b - b_{(-1)} = \partial b$. Notice, for instance, that

$$\operatorname{tr} [V, J_0]^2 = -2\operatorname{tr} (\gamma^2 + \beta^2) = -2(||\partial a||_2^2 + ||\partial b||_2^2). \tag{4.7}$$

So,

$$
\begin{aligned}
|\mathrm{tr}\,\tilde{O}([V,J_0]^2)| &\le \sum_k^M |\mathrm{tr}\,(B_k[V,J_0]C_k[V,J_0]D_k)| \\
&\le \sum_k^M ||B_k[V,J_0]C_k[V,J_0]D_k||_{S_1} \le \sum_k^M ||B_k[V,J_0]C_k||_{S_2}\cdot||[V,J_0]D_k||_{S_2} \\
&\le \sum_k^M ||B_k||\,||C_k||\,||D_k||\,||[V,J_0]||_{S_2}^2, \qquad (4.8)
\end{aligned}
$$

and, by (4.7), we obtain an estimate for the first expression in (4.3).

A bound for the second expression in (4.3) is more complex and essentially relies on the specifics of the situation. To begin with, we observe that $\mathrm{tr}\,V^p[V,J_0]J_0^N=0$ trivially, when $p=0$ or $N=0$.

Lemma 4. *Let $p,N\in\mathbb{N}$. Then*

$$|\mathrm{tr}\,V^p[V,J_0]J_0^N|\le C_6(||\partial a||_2^2+||\partial b||_2^2),$$

where the constant depends on $p,N,||\alpha||_\infty,||b||_\infty,||\partial a||_\infty$, and $||\partial b||_\infty$.

Proof. In essential, the proof is based on the antisymmetry property of $[V,J_0]$. We have

$$V^p=\sum_{k=0}^p(S^kp_{p,k}(a,b)+p_{p,k}(a,b)S^{-k}), \qquad (4.9)$$

where $p_{p,k}(a,b)=p_{p,k}(a,a_{(1)},\dots,a_{(p-1)};b,b_{(1)},\dots b_{(p-1)})$ are homogeneous polynomials of degree p. In a similar way,

$$J_0^N=\sum_{k=0}^N c_{N,k}(S^k+S^{-k})$$

for some coefficients $c_{N,k}$. Applying (4.5) and (4.6), we compute

$$
\begin{aligned}
[V,J_0](S^k+S^{-k}) &= \left(-S^{k+2}\gamma_{(-k)}+\gamma S^{-(k+2)}\right)+\left(S^{k-2}\gamma_{(-(k-2))}-\gamma_{(2)}S^{-(k-2)}\right) \\
&+ \left(-S^{k+1}\beta_{(-k)}+\beta S^{-(k+1)}\right)+\left(S^{k-1}\beta_{(-(k-1))}-\beta_{(1)}S^{-(k-1)}\right).
\end{aligned}
$$

Above, the terms in the brackets have the same form. So, we shall obtain the required bound if we prove it for a summand of the form

$$(S^kp_{p,k}(a,b)+p_{p,k}(a,b)S^{-k})(-S^k\gamma_{(-(k-2))}+\gamma S^{-k}).$$

Recalling commutation relations (4.5), we get

$$
\begin{aligned}
(S^kp_{p,k}(a,b) &+ p_{p,k}(a,b)S^{-k})(-S^k\gamma_{(-(k-2))}+\gamma S^{-k}) \\
&= \left\{\gamma p_{p,k}(a,b)\right\}_{(k)}-\gamma_{(-(k-2))}p_{p,k}(a,b)+\text{off-diagonal terms.}
\end{aligned}
$$

This shows that the trace of the latter expression equals

$$\operatorname{tr}\left\{(\gamma p_{p,k}(a,b))_{(k)} - \gamma_{(-(k-2))} p_{p,k}(a,b)\right\} = \operatorname{tr}\gamma\left\{p_{p,k}(a,b) - p_{p,k}(a,b)_{(k-2)}\right\}.$$

We prove now that, for a fixed k,

$$p_{p,k}(a,b) - p_{p,k}(a,b)_{(k-2)} = \sum_s q_{1,s}(a,b)\gamma_{(s)} + \sum_s q_{2,s}(a,b)\beta_{(s)}, \tag{4.10}$$

and this equality will readily imply the claim of the lemma. Above, $q_{1,s}, q_{2,s}$ are some polynomials, $\gamma = \partial a, \beta = \partial b$ (see (4.6) and below), and the sums contain a finite number of terms. Indeed, similarly to (4.8)

$$\begin{aligned} |\operatorname{tr}\gamma\{p_{p,k}(a,b) - p_{p,k}(a,b)_{(k-2)}\}| &\leq \textstyle\sum_s ||q_{1,s}(a,b)|| \cdot ||\gamma||_{S_2}\, ||\gamma_{(s)}||_{S_2} \\ &+ \textstyle\sum_s ||q_{2,s}(a,b)|| \cdot ||\gamma||_{S_2}\, ||\beta_{(s)}||_{S_2}. \end{aligned}$$

Furthermore, $||\gamma_{(s)}||_{S_2} = ||\gamma||_{S_2} = ||\partial a||_2$ and $||\beta_{(s)}|| = ||\beta||_{S_2} = ||\partial b||_2$, so

$$\begin{aligned} |\operatorname{tr}\gamma\{p_{p,k}(a,b) - p_{p,k}(a,b)_{(k-2)}\}| &\leq ||\partial a||_2^2 \left(\textstyle\sum_s ||q_{1,s}(a,b)|| + \frac{1}{2}||q_{2,s}(a,b)||\right) \\ &+ \tfrac{1}{2}||\partial b||_2^2 \textstyle\sum_s ||q_{2,s}(a,b)||. \end{aligned}$$

To prove (4.10), we consider the terms composing $p_{p,k}(a,b)$. For the sake of simplicity we assume that the term we look at involves $a, a_{(1)}, \dots, a_{(p-1)}$ only. The general case is dealt with similarly. By induction on p, we show that

$$a^{n_1}a_{(1)}^{n_2}\dots a_{(p-1)}^{n_p} - (a^{n_1}a_{(1)}^{n_2}\dots a_{(p-1)}^{n_p})_{(k)} = \sum_s r_s(a)\gamma_{(s)}. \tag{4.11}$$

For $p = 1$ the claim is trivial. For an arbitrary m, we have

$$\begin{aligned} a^{n_1}a_{(1)}^{n_2}\dots a_{(p-1)}^{n_p} &- (a^{n_1}a_{(1)}^{n_2}\dots a_{(p-1)}^{n_p})_{(k)} \\ &= \textstyle\sum_{j=1}^k \left((a^{n_1}a_{(1)}^{n_2}\dots a_{(p-1)}^{n_p})_{(j-1)} - (a^{n_1}a_{(1)}^{n_2}\dots a_{(p-1)}^{n_p})_{(j)}\right). \end{aligned}$$

Denoting $a' = a_{(j-1)}$, we continue as

$$\begin{aligned} a'^{n_1}a'^{n_2}_{(1)}\dots a'^{n_p}_{(p-1)} &- a'^{n_1}_{(1)}a'^{n_2}_{(2)}\dots a'^{n_p}_{(p)} \\ &= \left(a'^{n_1} - a'^{n_1}_{(1)}\right) a'^{n_2}_{(1)}\dots a'^{n_p}_{(p-1)} \\ &- a'^{n_1}_{(1)}\left(a'^{n_2}_{(1)}\dots a'^{n_p}_{(p-1)} - a'^{n_2}_{(2)}\dots a'^{n_p}_{(p)}\right). \end{aligned}$$

We note that factor $a'^{n_1} - a'^{n_1}_{(1)}$ can be represented in the form (4.10). Consequently, the first term in the above equality has form (4.10) as well. The factor in the brackets entering in the second term is of required form by induction. Thus, relation (4.10) is proved, and so is the lemma. □

4.3. **Estimating** $\operatorname{tr}(V^pJ_0^N - VJ_0^{k_1}\dots VJ_0^{k_p})$**.** The discussion of the previous subsection yield the following lemma.

Lemma 5. *Let* $\mathbf{k}=(k_1,\dots,k_p)$ *and* $|\mathbf{k}|=N$*. Then*

$$|\operatorname{tr}(V^pJ_0^N - VJ_0^{k_1}\dots VJ_0^{k_p})|\le C_7(||\partial a||_2^2+||\partial b||_2^2)$$

with constant C_7 *depending on* $\mathbf{k}, ||a-1||_\infty, ||b||_\infty, ||\partial a||_\infty,$ *and* $||\partial b||_\infty$.

The proof is immediate from comparison of Lemma 3, Lemma 4, and the discussion before it (in particular, see (4.8)).

5. Estimates for $\Psi_m(J)$ and the proof of the Main Lemma.

5.1. **Simplifying the expression for** $\Psi_m(J)$**.** Let us turn back to equality (2.7), where the quantity $\Psi_m(J)$ is introduced. By Lemma 5, we see that

$$\begin{aligned}\operatorname{tr}(J^{2k}-J_0^{2k}) &= \operatorname{tr}((J_0+V)^{2k}-J_0^{2k})\\ &= \operatorname{tr}\textstyle\sum_{p=1}^{2k} C_{2k}^p V^pJ_0^{2k-p} + H_{2k}(a,b),\end{aligned}$$

and $|H_{2k}(a,b)|\le C_8(||\partial a||_2^2+||\partial b||_2^2)$. Consequently,

$$\begin{aligned}\Psi_m(J) &= \Psi_{m,1}(J)+\Psi_{m,2}(J)\\ &= -\operatorname{tr}\left\{\sum_{k=1}^m \frac{(-1)^k}{2^{2k+1}k}\tilde C_{2m-1}^{2k-1}\Big(\sum_{p=1}^{2k} C_{2k}^pV^pJ_0^{2k-p}\Big)+\frac{(2m-1)!!}{(2m)!!}\log A_0\right\} \qquad (5.1)\\ &+ \Psi_{m,2}(J),\end{aligned}$$

where $|\Psi_{m,2}(a,b)|\le C_9(||\partial a||_2^2+||\partial b||_2^2)$. With notation we accepted (see the beginning of Subsection 4.2 and (4.4))

$$\log A_0=\log(I+\alpha)=\sum_{p=1}^{2m}\frac{(-1)^{p+1}}{p}\alpha^p+O(\alpha^{2m+1}).$$

Changing the order of summation in (5.1) gives

(5.2)

$$\Psi_{m,1}(J)=-\operatorname{tr}\left\{\sum_{p=1}^{2m}V^pF_p(J_0)+\frac{(2m-1)!!}{(2m)!!}\sum_{p=1}^{2m}\frac{(-1)^{p+1}}{p}\alpha^p+O(\alpha^{2m+1})\right\},$$

where, by definition,

$$F_p(J_0)=\sum_{k=[(p+1)/2]}^{m}\frac{(-1)^k}{2^{2k+1}k}\tilde C_{2m-1}^{2k-1}C_{2k}^pJ_0^{2k-p}. \qquad (5.3)$$

It turns out that polynomials $F_p, p=\overline{1,m}$, have a very special structure. We denote by $J_{0,k}$ a symmetric matrix with 1's on the k-th auxiliary diagonals and

0's elsewhere. Differently, $J_{0,k} = S^k + S^{-k}$ (see Subsection 4.2). The following lemma holds.

Lemma 6. *We have for* $p = 1, m$

$$F_p(J_0) = (-1)^p \frac{(2m-1)!!}{2p(2m)!!} J_{0,p} + \sum_{s=p+1}^{2m-p} d_{p,s} J_{0,s}$$

with some coefficients $d_{p,s}$.

Before going into the proof of the assertion, we formulate and prove a claim of a technical nature.

Lemma 7. *If* $p_1 + 2j - 1 < m - p_1$ *and* $p_1 > 0$, *then*

$$\sum_{k=2j}^{m-p_1} (-1)^k C_{m-p_1}^k \frac{(k+p_1-j)!}{(k-2j)!} = 0.$$

If $p_1 = 0$, *the above expression equals* $(2p-1)!$.

Proof. Obviously,

$$x^{p_1-1}(1-x)^{m-p_1} = \sum_{k=0}^{m-p_1} (-1)^k C_{m-p_1}^k x^{k+p_1-1},$$

and, consequently,

$$\frac{d^{p_1+2j-1}}{dx^{p_1+2j-1}} \left(x^{p_1-1}(1-x)^{m-p_1} \right) = \sum_{k=2j}^{m-p_1} (-1)^k C_{m-p_1}^k \frac{(k+p_1-1)!}{(k-2j)!} x^{k-2j}.$$

We set $x = 1$ and notice that, since $p_1 + 2j - 1 < m - p_1$, the left-hand side of the equality equals zero. This proves the first claim of the lemma. The proof of the second one follows the same lines. □

Proof of Lemma 6. The lemma is proved by a straightforward computation. We have

$$F_{p'}(J_0) = \sum_{s=1}^{2m-p'} d_{p',s} J_{0,s}.$$

We compute the first p' coefficients $d_{p',s}$ in this sum. It is convenient to split the computation in two particular cases – when p' is even or odd. Indeed, when p' is even (odd), the odd (even) auxiliary diagonals of $F_{p'}$ (see (5.3)) vanish, and we have to compute $d_{p',s}$ only for even (odd) $s = 0, p'$, respectively.

We give the details only for even p', that is, when $p' = 2p$. The other case can be dealt with in the same spirit. Recalling that $J_0 = S + S^{-1}$, we obtain

$$J_0^{2k-2p} = \sum_{l=0}^{2k-2p} C_{2k-2p}^l S^{2l-2(k-p)} = \sum_{j=0}^{k-p} C_{2(k-p)}^{j+(k-p)} J_{0,2j}.$$

Combining the latter equality with (5.3), we come to

$$d_{2p,2j} = \sum_{k=p+j}^{m} \frac{(-1)^k}{2^{2k+1}k} \tilde{C}_{2m-1}^{2k-1} C_{2k}^{2p} C_{2(k-p)}^{j+(k-p)}.$$

We simplify this expression using definitions of $\tilde{C}_{2m-1}^{2k-1}$ (see Subsection 2.3) and C_k^p. Letting $j = 0, p$ and $p_1 = p - j$, we get

$$d_{2p,2j} = \frac{(-1)^{p_1}(2m-1)!!}{2^{m+1}(2p)!(m-p_1)!} \sum_{k=2j}^{m-p_1} (-1)^k C_{m-p_1}^k \frac{(k+p_1-1)!}{(k-2j)!}.$$

Since $p_1 + 2j - 1 < m - p_1$, or, equivalently, $2p - 1 < m$, Lemma 7 shows that $d_{2p,2j} = 0$ for $j = 0, p-1$. By the same lemma

$$d_{2p,2p} = \frac{(2m-1)!!}{2^{m+1}(2p)!m!}(2p-1)! = \frac{(2m-1)!!}{2(2p)(2m)!!},$$

when $j = p$ and $p_1 = 0$. The proof is complete. □

5.2. **Proof of the Main Lemma.** First, we bound summands corresponding to $p = m+1, 2m$ in (5.2). We get

$$|\mathrm{tr}\,(V^p F_p(J_0))| \le ||V^p F_p(J_0)||_{S_1} \le ||F_p(J_0)|| \cdot ||V^p||_{S_1},$$

and for these p's

$$||V^p||_{S_1} \le C_{10}||V^{m+1}||_{S_1} \le C_{10}(||a-1||_{m+1}^{m+1} + ||b||_{m+1}^{m+1}), \tag{5.4}$$

with the constant depending on $||V||$. Similarly, $|\mathrm{tr}\,\alpha^p| \le C_{11}||a-1||_{m+1}^{m+1}$.

Let $p = 3, m$ now. As we already mentioned (see (4.9)),

$$V^p = \sum_{j=0}^{p} (S^j p_{p,j}(a,b) + p_{p,j}(a,b)S^{-j}).$$

It is easy to show by induction that the polynomials $p_{p,p}(a,b)$ are particularly simple. Namely,

$$p_{p,p}(a,b) = \alpha\alpha_{(1)} \dots \alpha_{(p-1)}.$$

Lemma 6 yields that

$$\begin{aligned}
\mathrm{tr}\, V^p F_p(J_0) &= (-1)^p \frac{(2m-1)!!}{2p(2m)!!} \mathrm{tr}\, V^p J_{0,p} \\
&= (-1)^p \frac{(2m-1)!!}{2p(2m)!!} \mathrm{tr}\,(p_{p,p}(a,b) + p_{p,p}(a,b)_{(p)}) \\
&= (-1)^p \frac{(2m-1)!!}{p(2m)!!} \sum_j \alpha_j \alpha_{j+1} \dots \alpha_{j+(p-1)},
\end{aligned}$$

since $\operatorname{tr} V^p J_{0,s} = 0$ for $s \geq p+1$. Hence,

$$\begin{aligned}\operatorname{tr}(V^p F_p(J_0) \quad &+ \quad (-1)^{p+1}\frac{(2m-1)!!}{p(2m)!!}\alpha^p) \\ &= \quad (-1)^{p+1}\frac{(2m-1)!!}{p(2m)!!}\sum_j(\alpha_j^p - \alpha_j\alpha_{j+1}\dots\alpha_{j+(p-1)}),\end{aligned}$$

and we obtain that

$$\left|\operatorname{tr}\left(V^p F_p(J_0) + (-1)^{p+1}\frac{(2m-1)!!}{p(2m)!!}\alpha^p\right)\right| \leq C_{12}||\gamma_p(a)||_1, \tag{5.5}$$

where C_{12} depends on p, m, and sequences $\gamma_k(a)$ are defined in (0.3).

Observe that $\gamma_p(a) = 0$ when $p = 1$. Furthermore, we have for $p = 2$ that

$$\begin{aligned}\textstyle\sum_j(\alpha_j^2 - \alpha_j\alpha_{j+1}) \quad &= \quad \tfrac{1}{2}\textstyle\sum_j(\alpha_j^2 - 2\alpha_j\alpha_{j+1} + \alpha_{j+1}^2) \\ &= \quad \tfrac{1}{2}\textstyle\sum_j(\alpha_j - \alpha_{j+1})^2 = \tfrac{1}{2}||\partial a||_2^2.\end{aligned}$$

So, the left hand-side of (5.5) for $p = 2$ can be estimated by $C_{13}||\partial a||_2^2$.

It is also clear that inclusions $\alpha \in l^{m+1}$ and $\partial a \in l^2$ give that $\gamma_p(a) \in l^1$ for $p > m/2 + 1$. Indeed, we have

$$\alpha^p - \alpha\alpha_{(1)}\dots\alpha_{(p-1)} = \sum_{k=1}^{p}\alpha^{p-k}(\alpha - \alpha_{(p-k)})\alpha_{(p-(k-1))}\dots\alpha_{(p-1)}.$$

The terms in the latter sum look like $\alpha_{(i_1)}\dots\alpha_{(i_{p-1})}(\alpha - \alpha_{(i_p)})$ for some $\mathbf{i} = (i_1, \dots, i_p)$. Obviously, $\alpha - \alpha_{i_p} = a - a_{i_p} = \partial a \in l^2$. Applying the Hölder inequality

$$\sum_k a_{1,k}\dots a_{p,k} \leq \sum_k\left(\sum_{j=1}^{p}\frac{1}{q_j}a_{j,k}^{q_j}\right),$$

with $a_{j,k} = |(\alpha_{(i_j)})_k|$, $q_j = 2(p-1)$ for $j = 1, p-1$, and $a_{p,k} = |(\alpha - \alpha_{(i_p)})_k|, q_p = 1/2$, we get that

$$||\alpha^p - \alpha\alpha_{(1)}\dots\alpha_{(p-1)}||_1 \leq C_{14}\left(||\partial a||_2^2 + ||\alpha||_{2(p-1)}^{2(p-1)}\right),$$

which is finite for $p > m/2 + 1$.

Thus, gathering the above argument with (5.1), (5.2), (5.4), and (5.5), we complete the proof of the lemma. □

6. Discussion of the results

6.1. **Corollaries of the theorem.** Note that condition (0.4) appearing in Theorem 3 is essentially non-linear and is not easy to check. It seems useful to have simple relations implying (0.4).

We define a sequence $A_k(a)$ as

$$(A_k(a))_j = \alpha_{j+1} + \ldots + \alpha_{j+(k-1)} - (k-1)\alpha_j.$$

The proof of the lemma is close in spirit to the reasoning at the end of Subsection 5.2.

Lemma 8. *If $\alpha \in l^{m+1}, \partial a \in l^2$ and $A_k(a) \in l^{q(m,k)}$, where $q(m,k) = (m+1)/(m+2-k)$, then $\gamma_k(a) \in l^1$ for $k = \overline{3, [m/2+1]}$.*

Proof. Let $\delta^i = \{\delta^i_j\}$, where $\delta^i_j = \alpha_{j+i} - \alpha_j$, or $\alpha_{j+i} = \alpha_j + \delta^i_j, i = \overline{1, k-1}$. Obviously, $\partial a \in l^2$ yields $\delta^i \in l^2$. We also have

$$\begin{aligned} (\gamma_k(a))_j &= \alpha_j^k - \alpha_j \alpha_{j+1} \ldots \alpha_{j+(k-1)} \\ &= \alpha_j^k - \alpha_j(\alpha_j + \delta^1_j) \ldots (\alpha_j + \delta^{k-1}_j) \\ &= -\alpha_j^{k-1} (\textstyle\sum_{i=1}^{k-1} \delta^i_j) + \text{additional terms.} \end{aligned}$$

Furthermore, we have $(A_k(a))_j = \sum_{i=1}^{k-1} \delta^i_j$ and

$$||\gamma_k(a)||_1 \le \sum_j |\alpha_j|^{k-1} \, |\sum_{i=1}^{k-1} \delta^i_j| + \sum_j O((\partial a)^2_j).$$

Using inequality $ab \le (1/p)a^p + (1/q)b^q$ with $p = p(m,k) = (m+1)/(k-1)$ and $q = q(m,k) = (m+1)/(m+2-k)$, we obtain

$$||\gamma_k(a)||_1 \le \frac{1}{p(m,k)} ||a-1||^{m+1}_{m+1} + \frac{1}{q(m,k)} ||A_k(a)||^{q(m,k)}_{q(m,k)} + C_{15}||\partial a||^2_2.$$

The quantity on the right hand-side of the inequality is finite by the assumptions of the lemma. □

Of course, it is easy to obtain other sufficient conditions providing $\gamma_k(a) \in l^1$.

Corollary 3. *Theorem 3 holds, if condition (0.4) is replaced with $A_k(a) \in l^{q(m,k)}$, $k = \overline{3, [m/2+1]}$.*

It is not difficult to observe (see Subsection 5.2) that assumptions (0.4) of the theorem can be slightly relaxed. More precisely, we have the following corollary.

Corollary 4. *Theorem 3 holds, if we require the series $\sum_j (\gamma_k(a))_j$, $k = \overline{3, [m/2+1]}$, to converge instead of (0.4).*

Concluding the subsection, we point out that a counterpart of Theorem 3 is true for two-sided Jacobi matrices as well.

Theorem 7. *Let $J = J(a,b)$ is a two-sided Jacobi matrix. If*

$$\begin{array}{ll} i) & a-1, b \in l^{m+1}(\mathbb{Z}),\ \partial a, \partial b \in l^2(\mathbb{Z}), \\ ii) & \gamma_k(a) \in l^1(\mathbb{Z}),\ k = \overline{3, [m/2+1]}, \end{array}$$

then

$$i') \int_{-2}^{2} \log\det\Sigma'(x)\cdot(4-x^2)^{m-1/2}\,dx > -\infty, \quad ii') \sum_j (x_j^{\pm 2}-4)^{m+1/2} < \infty.$$

This result follows from a representation of the transmission coefficient s of J in terms of the spectral data (see Theorem 6) and from word-by-word repeating the proof of Theorem 3.

6.2. **Some open questions.** We discuss several open questions pertaining to Theorem 3.

It seems that, despite some progress, the structure of the sum rules of high order is far from being well-understood. For instance, we still do not know conditions necessary and sufficient for (0.5). Despite the fact that the question is very natural and immediate after the proof of Theorem 3, answer to it presently appears to be out of reach. Nevertheless, even partial information on it would be extremely important for understanding the general sum rules (see below).

A part of the above question is, of course, sharpening conditions $\partial a \in l^2$ and $\partial b \in l^2$ arising in the theorem (see [9] in this connection).

On the other hand, it is well-known that the Schatten-von Neumann classes $S_p, p \geq 1$ (see Subsection 1.3) have strong interpolation properties. So, it would be interesting to obtain counterparts of Theorem 3 for non-integer $m \geq 1$. In particular, it is not clear what condition has to replace inclusions (0.4).

Let w be a measurable function on $[-2,2]$ such that $w(x) > 0$ for almost all $x \in [-2,2]$. Furthermore, let $\psi_- : (-\infty,-2) \to \mathbb{R}_+$ and $\psi_+ : (2,+\infty) \to \mathbb{R}_+$ be continuous decreasing (increasing) functions with properties $\psi_-(-2) = \psi_+(2) = 0$. Suppose that $J = J(a,b)$ and σ is its spectral measure. Questions on necessary and sufficient conditions (or either necessary or sufficient ones) for $\{a_k\}, \{b_k\}$ implying

$$i) \int_{-2}^{2} \log\sigma'(x)\,w(x)\,dx > -\infty, \quad ii) \sum_j \psi_\pm(x_j^\pm) < \infty$$

seem to be a more distant perspective. Interesting results in this direction were obtained in [12]. Otherwise, everything appears to be open on this level of generality. For instance, it would be useful to see what happens for the sum rules of "negative" order, that is, when $w(x) \to +\infty$ sufficiently fast and $\psi_\pm(x) \to 0$ sufficiently slowly as $x \to \pm 2$.

We also mention that all situations studied up to now (see [8, 9, 10]) are strongly symmetric. In other words, a weight w is even and $\psi_+(x) = \psi_-(-x), x > 2$. It is natural to ask, how the conditions on $\{a_k\}$ and $\{b_k\}$ change when the symmetry disappears.

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