A DICHOTOMY BETWEEN DISCRETE AND CONTINUOUS SPECTRUM FOR A CLASS OF SPECIAL FLOWS OVER ROTATIONS.

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Abstract. We provide sufficient conditions on a positive function so that its associated special flow over any irrational rotation is either weak mixing or L^2-conjugate to a suspension flow.

1. Introduction

In his I.C.M. address of 1954 [7], Kolmogorov raised a number of questions concerning reparameterization of irrational linear flows on $\mathbb{T}^n$, or equivalently special flows over translations. One of them was to determine what kind of spectral properties could be displayed by the unitary operator associated to the special flow built over an irrational rotation on the circle and under an analytic roof function.

Kolmogorov noticed that if the rotation angle α is not very well approximated by rational angles, e.g. α Diophantine, and if the roof function φ is a strictly positive real analytic function, then the special flow $T^t_{(\alpha,\varphi)}$ built over the rotation R_α and under the function φ is analytically conjugate to a constant time suspension over R_α, i.e. to an irrational linear flow on $\mathbb{T}^2$. The argument, based on solving an additive cohomological equation, also proves that, for any irrational angle α, if the roof function is a strictly positive trigonometric polynomial then the special flow $T^t_{(\alpha,\varphi)}$ is analytically conjugate to an irrational linear flow on $\mathbb{T}^2$.

Later, Šhklover proved that for any strictly positive real analytic function that is not a trigonometric polynomial, there exists an irrational angle α such that the special flow $T^t_{(\alpha,\varphi)}$ has continuous spectrum [10]. Thus, for analytic functions φ that are not trigonometric polynomials, both continuous and discrete spectra can be obtained depending on α.

These are not the only possibilities. In a recent work, the authors together with A. Katok, have proved that for every Liouvillean angle α there exists a strictly positive C^∞ function φ such that the special flow $T^t_{(\alpha,\varphi)}$ has

2000 *Mathematics Subject Classification.* Primary 37A30; Secondary 37A45, 37E45.

Prepared during B. Fayad's visit to the Pennsylvania State University, Fall 2001.

A. Windsor partially supported by NSF grant DMS 0071339.

The authors gratefully acknowledge Y. Katznelson for suggesting to us the use of Salem and Zygmund's paper [8], and A. Katok for suggesting the problem to us and for helpful conversations.

mixed spectrum. When the angle α is exceptionally well approximated by rational numbers the function φ can be made analytic [2]. The roof functions constructed in [2] have blocks of relatively large Fourier coefficients which appear in a lacunary progression. The possibility that mixed spectra would be precluded for roof functions with more regular decay of Fourier coefficients was raised in [5], a slightly reworked part of earlier unpublished notes [4]. In these notes the authors observe that for a function such as

$$\varphi(x) = \sum_{n\in\mathbb{Z}} 2^{-|n|}\cos(2\pi n x)$$

the special flow $T^t_{(\alpha,\varphi)}$ is conjugated to a linear flow if α is such that there exists a $c > 0$ so that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$2^q\left|\alpha - \frac{p}{q}\right| > c.$$

Conversely, they show that a sufficient condition for weak mixing is the existence of sequences $\{p_n\}$ and $\{q_n\}$ such that

$$2^{q_n} q_n\left|\alpha - \frac{p_n}{q_n}\right| \to 0.$$

To prove weak mixing they use a criterion involving the distribution of the Birkhoff sums of the roof function φ,

$$S_m\varphi(x) = \sum_{k=0}^{m-1} \varphi(x + k\alpha),$$

along a sequence m_n satisfying $R_\alpha^{m_n} \to Id$. They are able to choose a sequence where each m_n is a multiple of a single frequency q_n.

In order to bridge the gap between the conditions above and prove a full dichotomy depending on α between continuous and discrete spectrum for $T^t_{(\alpha,\varphi)}$, we consider the distribution of the Birkhoff sums $S_m\varphi(x)$ along sequences m_n which again have the property that $R_\alpha^{m_n} \to Id$ but which involve multiple frequencies q_n. We use conditions on the regularity of the decay of the Fourier coefficients (c.f. [H1]) to extract for each α a lacunary representative of the additive cohomology class of φ. If φ is not an L^2 coboundary, we use the central limit theorem for lacunary series to study the distribution of the Birkhoff sums of its lacunary representative and prove weak mixing. Our motivation for examining lacunary series was a result of M. Herman [3]:

Theorem. *If φ is a lacunary Fourier series, and the equation*

$$\psi(x + \alpha) - \psi(x) = \varphi(x)$$

has a measurable solution ψ, then in fact the equation has a solution in L^2.

Our result can be viewed as a rigidity result that covers the multiplicative equation too. Either the additive equation admits an L^2 solution or else the multiplicative equation admits no solution.

2. A Weak Mixing Dichotomy for Special Flows

Theorem 1. *Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ be a C^3 function given by*

$$\varphi(x) = \sum_{m \in \mathbb{Z}} c_m e^{2\pi i m x},$$

where the coefficients satisfy the regularity conditions

[H1] *there exist C_m such that*

$$\sum_{l=2}^{\infty} |c_{lm}|^2 \leq C_m |c_m|^2$$

and

$$\sum_{m=1}^{\infty} C_m < \infty.$$

[H2] *there exists $0 < K_1 < 1/4$ such that*

$$\sum_{l=2}^{\infty} |c_{lm}| < K_1 |c_m|$$

for all m sufficiently large,

[H3] *there exists $K_2 > 0$ such that*

$$\sum_{l=2}^{\infty} |l c_{lm}| < K_2 |c_m|$$

for all m sufficiently large.

Then for all $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ we have either

(1) *the special flow $T^t_{(\alpha,\varphi)}$ is weak mixing, or*
(2) *the special flow $T^t_{(\alpha,\varphi)}$ is L^2 conjugate to a suspension flow.*

2.1. **Examples:** The hypotheses [H1], [H2], and [H3] restrict the coefficients along arithmetic progressions. Thus relatively prime frequencies do not influence each other directly.

Lemma 2.1. *A positive C^3 function φ given by*

$$\varphi(x) = c_0 + \sum_{|p| prime} c_p e^{2\pi i p x}$$

satisfies [H1], [H2], *and* [H3].

Regular exponential decay along the appropriate arithmetic progressions will also suffice.

Lemma 2.2. *A positive function φ given by*

$$\varphi(x) = \sum_{m \in \mathbb{Z}} c_m e^{2\pi i m x},$$

where the coefficients satisfy the regularity condition

$$C_1 e^{-k_1|m|} \leq |c_m| \leq C_2 e^{-k_2|m|}$$

with $1 \leq k_1/k_2 < 2$, *satisfies* [H1], [H2], *and* [H3].

3. The Tools

3.1. **Arithmetic.**

Associated to each $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ there is an infinite sequence of natural numbers $\{q_n\}$ which we call the sequence of best returns. This sequence can be computed using continued fractions as the denominators of successive convergents. We introduce the notation

$$\|x\| = \inf_{p\in\mathbb{Z}} |x - p|$$

to measure the distance of x from 0 in $\mathbb{T}$. We call $\|q\alpha\|$ the quality of the return q. We have

$$\|q_n\alpha\| < \|q\alpha\|$$

for $1 \leq q < q_n$ and $q_n < q < q_{n+1}$. This justifies our best return nomenclature.

We will use two lemmas from the theory of continued fractions, see [6]. The first relates the speed of growth of the best returns $\{q_n\}$ with the quality of returns.

Lemma 3.1. *Let* $\alpha \in \mathbb{R}^+\backslash\mathbb{Q}$ *and let* $\{q_n\}$ *be its sequence of best returns. Then*

$$\frac{1}{2q_{n+1}} < \frac{1}{q_n + q_{n+1}} < \|q_n\alpha\| \leq \frac{1}{q_{n+1}}.$$

The second lemma shows that very good returns only occur for best returns q_n and their multiples.

Lemma 3.2. *Let* $\alpha \in \mathbb{R}^+\backslash\mathbb{Q}$ *and let* $\{q_n\}$ *be its sequence of best returns.If* $q \in \mathbb{Z}\backslash\{0\}$ *satisfies*

$$\|q\alpha\| < \frac{1}{2|q|},$$

then $q = lq_n$ *for some best return* q_n *and some*

$$|l| < \sqrt{\frac{q_{n+1}}{q_n}}.$$

3.2. **Cohomological Equations.**

The behavior of the special flow $T^t_{(\alpha,\varphi)}$ is determined by the cohomology class of the function φ, see [1], [4], and [5].

We call two functions, φ_1 and φ_2, (additively) cohomologous (over R_α) if there is a measurable solution ψ to the equation

$$\psi(x + \alpha) - \psi(x) = \varphi_1(x) - \varphi_2(x).$$

We call this equation the additive cohomological equation. If a function is cohomologous to 0 then we call it an (additive) coboundary. Using this definition we can say φ_1 and φ_2 are cohomologous if their difference $\varphi_1 - \varphi_2$ is a coboundary. Coboundaries have mean 0, hence no positive function can be a coboundary. The appropriate notion of triviality for positive functions is that of being cohomologous to a constant. This corresponds to the associated special flow being conjugate to a suspension flow. Throughout our arguments we will use that fact that we can subtract coboundaries from our function φ without altering the behavior of the special flow.

Lemma 3.3. *Let $\varphi_1 : \mathbb{T} \to \mathbb{R}^+$, $\varphi_2 : \mathbb{T} \to \mathbb{R}^+$, and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. If φ_1 and φ_2 are additively cohomologous, i.e. there exists a measurable (L^2) solution ψ to the additive cohomological equation*

$$\psi(x+\alpha) - \psi(x) = \varphi_1(x) - \varphi_2(x),$$

then the special flow $T^t_{(\alpha,\varphi_1)}$ is measurably (L^2) conjugate to the special flow $T^t_{(\alpha,\varphi_2)}$. In particular, if φ_2 is a constant then the special flow $T^t_{(\alpha,\varphi)}$ is measurably (L^2) conjugate to a suspension flow.

A cohomological equation again appears – this time a multiplicative cohomological equation – when we study the existence of eigenvalues for the special flow.

Lemma 3.4. *Let $\varphi : \mathbb{T} \to \mathbb{R}^+$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and $\lambda \in \mathbb{R}\backslash\{0\}$. If there exists an increasing sequence $\{m_n\}$ such that $\|m_n\alpha\| \to 0$ and*

$$\int \|\lambda S_{m_n}\varphi\| dx \not\to 0,$$

then λ is not an eigenvalue of the special flow $T^t_{(\alpha,\varphi)}$.

Proof. The eigenvalues of the special flow are determined by a multiplicative cohomological equation. In particular, λ is an eigenvalue of the special flow if and only if there is a measurable solution Ψ of the equation

$$e^{2\pi i\lambda\varphi(x)} = \frac{\Psi(x+\alpha)}{\Psi(x)}.$$

Iterating this we get for any m the equation

$$e^{2\pi i\lambda S_m\varphi(x)} = \frac{\Psi(x+m\alpha)}{\Psi(x)}$$

and thus

$$e^{2\pi i\lambda S_{m_n}\varphi(x)} - 1 = \frac{\Psi(x+m_n\alpha)}{\Psi(x)} - 1.$$

By the property of m_n that $\|m_n\alpha\| \to 0$ we have that the right-hand side converges to 0 in L^1. Thus the left-hand side also converges to 0. By Lemma 3.5, if λ is an eigenvalue, then

$$\int \|\lambda S_{m_n}\varphi\| dx \to 0.$$

Thus the given condition implies that λ is not an eigenvalue of the special flow. $\square$

The absence of eigenvalues other than the simple eigenvalue 0, which corresponds to the constant functions, implies weak mixing for the flow. The eigenvalues for the flow form an additive subgroup of $\mathbb{R}$. Thus, to prove the flow has a continuous spectrum, and is, hence, weak mixing, it suffices to prove that no sufficiently large λ is an eigenvalue.

3.3. **Analytical Estimates.** We will analyze the cohomological equations via Fourier techniques. We naturally arrive at considering expressions of the form $|e^{2\pi i m\alpha} - 1|$. These quantities are related to the quantities $\|m\alpha\|$ appearing in Section 3.1.

Lemma 3.5. *The two functions* $\|x\|$ *and* $|e^{2\pi i x} - 1|$ *are related by*

$$4\|x\| \le |e^{2\pi i x} - 1| \le 2\pi\|x\|.$$

In order to use the criterion for the absence of an eigenvalue, Lemma 3.4, it is necessary to control the Birkhoff sums of the function φ. When we consider these sums the following lemma will be crucial. It is an immediate consequence of Lemma 3.5 that

$$\frac{1}{2}\frac{\|mk\alpha\|}{\|k\alpha\|} < \left|\frac{e^{2\pi i m k\alpha} - 1}{e^{2\pi i k\alpha} - 1}\right| < 2\frac{\|mk\alpha\|}{\|k\alpha\|}. \tag{1}$$

4. The Structure of the Proof

We begin by looking for a conjugacy arising from the additive cohomological equation. Supposing that the additive cohomological equation has a solution given by a trigonometric sum and formally solving for the necessary coefficients yields the formal series

$$\psi(x) = \sum_{m\in\mathbb{Z}} \frac{c_m}{e^{2\pi i m\alpha} - 1} e^{2\pi i m x}.$$

Using Lemma 3.5 we estimate the coefficients by

$$\left|\frac{c_m}{e^{2\pi i m\alpha} - 1}\right| \le \frac{|c_m|}{4\|m\alpha\|}.$$

If these coefficients are square summable, then the special flow $T^t_{(\alpha,\varphi)}$ is L^2 conjugate to a suspension flow. Otherwise we must prove weak mixing. There are two different cases depending on exactly how the sequence $\{|c_m|/\|m\alpha\|\}$ behaves.

Proposition 2. *Let* φ *be a* C^3 *function satisfying the hypotheses* [H2], *and* [H3]. *If*

$$\limsup_{m\to\infty} \frac{|c_m|}{\|m\alpha\|} = C > 0,$$

then the special flow $T^t_{(\alpha,\varphi)}$ *is weak mixing.*

We call this the single frequency weak mixing case. In this case it suffices to take a sequence m_n in Lemma 3.4 that consists of multiples of single best returns as in [4]. When this is not possible we use

Proposition 3. *Let φ be a C^3 function satisfying the hypothesis* [H1]*. If*

$$\lim_{m\to\infty} \frac{|c_m|}{\|m\alpha\|} = 0 \qquad \text{and} \qquad \sum_{m\in\mathbb{Z}} \Big(\frac{|c_m|}{\|m\alpha\|}\Big)^2 = \infty$$

then the special flow $T^t_{(\alpha,\varphi)}$ is weak mixing.

We call this the multiple frequency weak mixing case. Our hypothesis [H1] ensures that the function φ is cohomologous to a function in which the only frequencies which appear are best returns. In this case, it is not sufficient to take the m_n to be a multiple of a single frequency. We will have to take m_n to be a sum of many frequencies. In this case, no frequency dominates, and, in fact, the values of $S_{m_n}\varphi$ becomes normally distributed in the limit.

5. Preliminary Reduction

The cohomology classes of those functions φ that appear in Theorem 1 admit nice representatives. We emphasize that we get different "well-adapted" representatives for each $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

Lemma 5.1. *Let φ be a C^3 function. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ have the sequence of best returns $\{q_n\}$. Then φ is cohomologous to the function φ_1 defined by*

$$\varphi_1(x) = \sum_{|m|\in M} c_m e^{2\pi i m x},$$

where $m \in M$ is either 0 or of the form $m = lq_n$, where q_n is a best return satisfying

$$q_{n+1} > q_n^2$$

and l is such that $lq_n < \frac{1}{2}q_{n+1}$.

Proof. Define the class M by

$$M := \{m \geq 0 : 2m^2\|m\alpha\| \leq 1\} \tag{2}$$

and the function ξ by

$$\xi(x) := \varphi(x) - \varphi_1(x) = \sum_{|m|\notin M} c_m e^{2\pi i m x}.$$

We need to show that ξ is an additive coboundary. If $\xi(x) = \psi(x+\alpha) - \psi(x)$ then $\psi(x)$ must be given by the formal series

$$\psi(x) := \sum_{|m|\notin M} \frac{c_m}{e^{2\pi i m\alpha} - 1} e^{2\pi i m x}.$$

The coefficients of ψ are estimated, using Lemma 3.5 and (2), by

$$\Big|\frac{c_m}{e^{2\pi i m\alpha} - 1}\Big| \leq \frac{|c_m|}{4\|m\alpha\|} \leq \frac{1}{2}m^2|c_m|$$

Since φ is C^3 these coefficients are square summable. Thus, the formal series is actually the Fourier series of an L^2 function, and hence, ξ is a coboundary.

That M contains only 0, best returns, and multiples of best returns follows from Lemma 3.2. The estimate on q_{n+1} for $q_n \in M$ follows from Lemma 3.1, and the definition of M (2), since

$$\frac{1}{2q_{n+1}} < \|q_n\alpha\| \le \frac{1}{2q_n^2}.$$

□

6. Single Frequency Weak Mixing Case

6.1. **Remarks.** Under hypothesis [H1] the proof we give is strictly only requisite for the case $C = \infty$. If $0 < C < \infty$, then hypotheses [H2] and [H3] are not necessary since in this case φ is cohomologous to a function in which only best returns appear, see Lemma 7.1.

The argument for weak mixing given here is a classical one and is similar to that given by A. Katok and E. A. Robinson in their 1983 unpublished notes [4].

6.2. **Proof of Weak Mixing.** Under hypothesis [H2] it is clear that C must be achieved along a subsequence $\{q_{s(n)}\}$ of best returns contained in M. Let $\{q_{s(n)}\}$ satisfy

$$\lim_{n\to\infty} \frac{|c_{q_{s(n)}}|}{\|q_{s(n)}\alpha\|} = C.$$

Fix $\lambda \in \mathbb{R}\backslash\{0\}$. We need to show that there exists a sequence m_n that serves in Lemma 3.4 to show that λ is not an eigenvalue. For our sequence we take

$$m_n = b_n q_{s(n)} := \left\lceil \frac{q_{s(n)+1}}{4q_{s(n)}} \right\rceil q_{s(n)}. \tag{3}$$

This sequence is chosen to isolate and inflate the terms corresponding to multiples of the best return $q_{s(n)}$. For this reason, it is natural to consider the expression

$$\phi_n(x) = \sum_{|l|q_{s(n)}\in M} c_{lq_{s(n)}} e^{2\pi i a q_{s(n)} x}. \tag{4}$$

It is technically easier deal with a function that is nearly constant. Since φ is C^3 and any trigonometric polynomial with 0 average is an additive coboundary we may discard finitely many terms from M and suppose that

$$\sum_{|m|\in M} |mc_m| < \frac{1}{16|\lambda|}. \tag{5}$$

We shall denote by φ_λ the representative of the cohomology class of φ thus obtained.

We now show that the Birkhoff sums $S_{m_n}\varphi_\lambda$ are uniformly close to the Birkhoff sums $S_{m_n}\phi_n$ for all n. The Birkhoff sums $S_{m_n}\phi_n$ are much simpler and we will be able to estimate them directly.

Lemma 6.1. *Suppose φ_λ satisfies* (5) *and ϕ_n is given by* (4)*. For all n,*

$$\left|\lambda S_{m_n}\varphi_\lambda(x) - \lambda S_{m_n}\phi_n(x)\right| < \frac{1}{8}. \tag{6}$$

Proof. We will show that

$$\left|\lambda S_{m_n}\varphi_\lambda(x) - \lambda S_{m_n}\phi_n(x)\right| \leq 2|\lambda| \sum_{|m|\in M} |mc_m|$$

from which we get the required estimate using (5). We can directly compute

$$\left|\lambda S_{m_n}\varphi_\lambda(x) - \lambda S_{m_n}\phi_n(x)\right| \leq |\lambda| \sum_{|l|q_k\in M: k\neq s(n)} \left|\frac{e^{2\pi i m_n l q_k\alpha}-1}{e^{2\pi i l q_k \alpha}-1}\right| |c_{lq_k}|.$$

Using the estimate (1) yields

$$\left|\frac{e^{2\pi i m_n l q_k\alpha}-1}{e^{2\pi i l q_k \alpha}-1}\right| \leq 2\frac{\|m_n q_k\alpha\|}{\|q_k\alpha\|} \leq 2m_n. \tag{7}$$

For $k > s(n)$ we have $2m_n < |lq_k|$. For $k < s(n)$ we need to use the fact that m_n, as a multiple of $q_{s(n)}$, produces a better return than does q_k. From the definition of m_n (3) we get

$$\frac{\|m_n q_k\alpha\|}{\|q_k\alpha\|} \leq \frac{b_n q_k \|q_{s(n)}\alpha\|}{\|q_k\alpha\|}.$$

Using Lemma 3.1 and estimating $b_n < \frac{q_{s(n)+1}}{2q_{s(n)}}$ yields

$$\frac{\|m_n q_k\alpha\|}{\|q_k\alpha\|} < \frac{q_k q_{k+1}}{q_{s(n)}} < q_k.$$

For all $|lq_k| \in M$ with $k \neq s(n)$ we have

$$\left|\frac{e^{2\pi i m_n l q_k\alpha}-1}{e^{2\pi i l q_k \alpha}-1}\right| \leq 2\frac{\|m_n q_k\alpha\|}{\|q_k\alpha\|} \leq |l|q_k,$$

which proves the result. □

We estimate the Birkhoff sums $S_{m_n}\phi_n$ geometrically. In essence, we show that an appropriately renormalized version of the sum has a derivative that is of the same magnitude as the length of the range. Since the length of the range does not go to zero, this is sufficient to conclude that the integral in Lemma 3.4 does not go to zero.

Lemma 6.2. *Let φ_n, given by* (4) *satisfy hypotheses* [H2] *and* [H3]*. For $\lambda > 0$ sufficiently large there exists $\epsilon > 0$ such that*

$$\mu\left\{x : \|\lambda S_{m_n}\phi_n(x)\| \geq \frac{1}{4}\right\} > \epsilon$$

for all n sufficiently large.

Proof. Let

$$R_n := \sup\, S_{m_n}\phi_n(x) - \inf\, S_{m_n}\phi_n(x)$$
$$D_n := \sup |S_{m_n}\phi_n'(x)|$$

We can estimate R_n from below using [H2] as

$$\begin{aligned} R_n &> 4\Big(\Big|\frac{e^{2\pi i m_n q_{s(n)}\alpha}-1}{e^{2\pi i q_{s(n)}\alpha}-1}c_{q_{s(n)}}\Big| - \sum_{l=2}^{\infty}\Big|\frac{e^{2\pi i m_n l q_{s(n)}\alpha}-1}{e^{2\pi i l q_{s(n)}\alpha}-1}c_{lq_{s(n)}}\Big|\Big) \\ &> 4\frac{\|m_n q_{s(n)}\alpha\|}{\|q_{s(n)}\alpha\|}\Big(\frac{1}{2}|c_{q_{s(n)}}| - 2\sum_{l=2}^{\infty}|c_{lq_{s(n)}}|\Big) \\ &> 2\frac{\|m_n q_{s(n)}\alpha\|}{\|q_{s(n)}\alpha\|}\,|c_{q_{s(n)}}|(1-4K_1). \end{aligned}$$

Similarly, we can estimate D_n from above using [H3] as

$$\begin{aligned} D_n &< 2\sum_{l=1}^{\infty}\Big|\frac{e^{2\pi i m_n l q_{s(n)}\alpha}-1}{e^{2\pi i l q_{s(n)}\alpha}-1}\,lq_{s(n)}\,c_{aq_{s(n)}}\Big| \\ &< 4\,q_{s(n)}\,\frac{\|m_n q_{s(n)}\alpha\|}{\|q_{s(n)}\alpha\|}\,|c_{q_{s(n)}}|(1+K_2). \end{aligned}$$

Now consider the number I of intervals of the form $\big[p+\frac{1}{4}, p+\frac{3}{4}\big]$ contained in the range of $\lambda S_{m_n}\phi_n$. This can be estimated from below by $I > \lfloor|\lambda|R_n\rfloor - 1$. For λ and n sufficiently large, $|\lambda|R_n > 4$ and hence $I > |\lambda|R_n/2$. By continuity, $\lambda S_{m_n}\phi_n$ must cross each interval at least once and hence, by periodicity, it must cross each interval at least $q_{s(n)}$ times. Therefore, we have for each interval contained in the range of $\lambda S_{m_n}\phi_n$

$$\mu\Big\{x : \lambda S_{m_n}\phi_n(x) \in \big[p+\frac{1}{4}, p+\frac{3}{4}\big]\Big\} > \frac{q_{s(n)}}{2|\lambda|D_n}.$$

Multiplying this by our estimate for the number of intervals we get

$$\mu\big\{x : \|\lambda S_{m_n}\phi_n(x)\| \geq \frac{1}{4}\big\} > \frac{q_{s(n)}R_n}{4D_n} > \frac{1-4K_1}{8(1+K_2)} > 0.$$

□

Thus, combining Lemma 6.1 and Lemma 6.2 we have, for λ sufficiently large, that

$$\mu\big\{x : \|\lambda S_{m_n}\varphi_\lambda(x)\| \geq \frac{1}{8}\big\} > \epsilon$$

for all n. Using our criterion for the absence of an eigenvalue, Lemma 3.4, this shows that λ is not an eigenvalue of the special flow. Since λ was any sufficiently large number this shows, by the remark following Lemma 3.4, that the special flow is weak mixing.

7. Multiple Frequency Weak Mixing Case

We simplify our problem by extracting an even simpler representative of the cohomology class of φ. At this point the multiples of the best returns are used. It is at this point that we use hypothesis [H1].

Lemma 7.1. *Let φ be a C^3 function satisfying hypothesis* [H1]. *If*

$$\sup \frac{|c_m|}{\|m\alpha\|} = K_3 < \infty,$$

then the function φ is cohomologous to the function φ_2 given by

$$\varphi_2(x) = \sum_{|m| \in M'} c_m e^{2\pi i m x},$$

where $m \in M'$ is either 0 *or a best return q_n satisfying $q_{n+1} > q_n^2$.*

Proof. Applying Lemma 5.1 we see it suffices to prove that we may exclude the multiples of best returns. Let ξ be the trigonometric series generated by the multiples of best returns,

$$\xi(x) = \sum_{|l|q_n \in M : |l| \geq 2} c_{lq_n} e^{2\pi i l q_n x}.$$

If $\xi(x) = \psi(x + \alpha) - \psi(x)$, then ψ must be given by the formal series

$$\psi(x) = \sum_{|l|q_n \in M : |l| \geq 2} \frac{c_{lq_n}}{e^{2\pi i l q_n \alpha} - 1} e^{2\pi i l q_n x}.$$

Using [H1] we get

$$\sum_{a=2}^{\infty} \left| \frac{c_{lq_n}}{e^{2\pi i l q_n \alpha} - 1} \right|^2 \leq \frac{1}{16 \|q_n \alpha\|^2} \sum_{l=2}^{\infty} \frac{|c_{lq_n}|^2}{a^2} < \frac{C_{q_n}}{16} K_3^2.$$

Thus, since by [H1] the C_m are summable, we have have ψ is actually an L^2 function and hence ξ is an additive coboundary. □

No individual frequency contributes enough to prove weak mixing. In order to apply our criterion we need to take a group of frequencies together. Our weak mixing sequence is of the form

$$m_n := \sum_{k=l_n}^{u_n} b_k q_{s(k)},$$

where

$$b_k := \left\lceil \frac{q_{s(k)+1}}{4 q_{s(k)}} \right\rceil$$

and l_n is an increasing sequence. This satisfies the requirement that $\|m_n \alpha\| \to 0$ regardless of the exact choices of l_n and u_n.

We now prove a lemma analogous to Lemma 6.1 for our more complicated situation. Our sequence m_n is chosen to isolate and inflate those coefficients corresponding to the frequencies $\{q_{s(k)}\}_{k=l_n}^{u_n}$. For this reason it is natural to define

$$\phi_n(x) := c_{-q_{s(n)}} e^{-2\pi i q_{s(n)} x} + c_0 + c_{q_{s(n)}} e^{2\pi i q_{s(n)} x}. \tag{8}$$

These functions asymptotically capture all the behavior in $S_{m_n}\varphi(x)$. Unfortunately, their behavior is not as easy to control as in the single frequency case.

Lemma 7.2. *Let*

$$\Delta_n = \Big\| S_{m_n}\varphi(x) - \sum_{k=l_n}^{u_n} S_{b_k q_{s(k)}} \phi_k\big(x + \sum_{j=l_n}^{k-1} b_j q_{s(j)} \alpha\big) \Big\|_\infty. \tag{9}$$

Then $\Delta_n \to 0$ as $n \to \infty$.

Proof. Define

$$\delta_k := \big\| S_{b_k q_{s(k)}} \varphi(x) - S_{b_k q_{s(k)}} \phi_k(x) \big\|_\infty$$

and observe that

$$\Delta_n \le \sum_{k=l_n}^{u_n} \delta_k.$$

If we show that δ_k is a summable sequence then $\lim_{n\to\infty} \Delta_n = 0$ follows from $l_n \to \infty$ and is independent of the choice of u_n. Using the triangle inequality and Lemma 3.5 we get

$$\delta_k \le 4 \sum_{j \in \mathbb{N}\setminus\{k\}} \frac{\|\!| q_{s(j)} b_k q_{s(k)} \alpha |\!\|}{\|\!| q_{s(j)} \alpha |\!\|} |c_{q_{s(j)}}| := 4 \sum_{j \in \mathbb{N}\setminus\{k\}} Q_j$$

We break the sum into two pieces, which we will estimate separately,

$$\delta_k \le 4 \sum_{j=1}^{k-1} Q_j + 4 \sum_{j=k+1}^{\infty} Q_j$$

For $j > k$, using the bound for $|c_m| / \|\!| m\alpha |\!\|$, we produce

$$Q_j = b_k q_{s(k)} \|\!| q_{s(j)} \alpha |\!\| \frac{|c_{q_{s(j)}}|}{\|\!| q_{s(j)} \alpha |\!\|} \le \frac{K q_{s(k)+1}}{2 q_{s(j)+1}}.$$

Using the condition $q_{s(k)+1} > q_{s(k)}^2$ yields

$$\sum_{j=k+1}^{\infty} \frac{K q_{s(k)+1}}{2 q_{s(j)+1}} \le \sum_{j=k+1}^{\infty} \frac{K q_{s(k+1)}}{2 q_{s(j)}^2} \le \frac{K}{q_{s(k+1)}^2}.$$

This is summable in k. For $j < k$ we produce an estimate analogous to the one above using the fact that q_k produces a better return than does q_j,

$$Q_j = b_k q_{s(j)} \|\!| q_{s(k)} \alpha |\!\| \frac{|c_{q_{s(j)}}|}{\|\!| q_{s(j)} \alpha |\!\|} \le \frac{K_3 q_{s(j)}}{2 q_{s(k)}}.$$

Again, using the condition $q_{s(k)+1} > q_{s(k)}^2$, we obtain

$$\sum_{j=1}^{k-1} \frac{K_3 q_{s(j)}}{2q_{s(k)}} \leq \frac{K_3 k q_{s(k-1)}}{2q_{s(k)}} \leq \frac{K_3 k}{2q_{s(k-1)}}$$

which is summable in k. □

Thus, $S_{m_n}\varphi(x)$ is asymptotic to $\sum_{k=l_n}^{u_n} S_{b_k q_{s(k)}}\phi_k(x)$. Ignoring the constant term, this sum is of the form

$$\sum_{k=l_n}^{u_n} S_{b_k q_{s(k)}}\phi_k(x) = \sum_{k=l_n}^{u_n} d_k \cos(q_k x + r_k).$$

We choose a sequence (l_n, u_n) with $l_n \to \infty$ and such that

$$\lim_{n\to\infty} \sum_{k=l_n}^{u_n} d_k^2 = 1. \tag{10}$$

For lacunary series q_n, the random variables $\cos(q_n x + r_n)$ are only weakly dependent. This observation will allow us to compute the asymptotic distribution of the sums $S_{m_n}\phi_n(x)$.

7.1. **The Distribution of the Birkhoff Sums.** The proof of the usual central limit theorem for lacunary trigonometric sum was carried out by Salem and Zygmund in 1947 [8]. Unfortunately, the convergence of normalized sums is not exactly what is needed. What is needed is a version of the "series" central limit theorem [9] for lacunary series. Fortunately, the proof of Salem and Zygmund carries through with no changes to prove this theorem.

Theorem 4. *Let*

$$X_n(x) = \sum_{k=1}^{u_n} c_{k,n} \cos(q_{k,n} x + r_{k,n})$$

for some sequence $q_{k+1,n} \geq \lambda q_{k,n}$ *with* $\lambda > 1$ *and some coefficients satisfying*

$$\operatorname{var}(X_n(x)) = \sum_{k=1}^{u_n} c_{k,n}^2 = 1$$

and

$$c_{k,n} \to 0 \textit{ uniformly as } n \to \infty.$$

Then

$$X_n \xrightarrow{dist} N(0,1),$$

where $N(0,1)$ *is the normal distribution with mean* 0 *and variance* 1.

Proof. We use the method of characteristic functions developed by Lyapunov to prove the central limit theorem. For simplicity we only prove the case where $q_{k+1,n} > 2q_{k,n}$ since this is sufficient for us.

Let F_n denote the distribution functions for the random variables X_n. Let $\varphi_n(t)$ be the characteristic function of the distribution F_n,

$$\varphi_n(t) = \int_{-\infty}^{+\infty} e^{ity} dF_n(y).$$

Our goal is to show that these characteristic functions converge to that of the normal distribution,

$$\lim_{n\to\infty} \varphi_n(t) = e^{-\frac{t^2}{2}}.$$

Passing from the integral with respect to the distribution to the integral with the random variable X_n we get

$$\varphi_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \exp\bigl(it\sum_{k=1}^{u_n} \cos(q_{k,n}x + r_{k,n})\bigr)dx.$$

We use the relation

$$e^z = (1+z)e^{\frac{1}{2}z^2+o(|z^2|)}$$

and the fact $\sum_{k=1}^{u_n} c_{k,n}^2 = 1$ to obtain

$$\varphi_n(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{o(1)}\prod_{k=1}^{u_n}\bigl(1+itc_{k,n}\cos(q_{k,n}x + r_{k,n})\bigr) \exp\bigl(-\frac{1}{2}t^2c_{k,n}^2\cos(q_{k,n}x + r_{k,n})\bigr)dx.$$

First we show that the first term is bounded.

$$\Bigl|\prod_{k=1}^{u_n}\bigl(1+itc_{k,n}\cos(q_{k,n}x + r_{k,n})\bigr)\Bigr| \le \prod_{k=1}^{u_n}\bigl(1+t^2c_{k,n}^2\bigr)^{\frac{1}{2}} \le e^{\lambda^2}$$

The exponent of the second term can be rewritten using the double angle formula as

$$\sum_{k=1}^{u_n} c_{k,n}^2\cos^2(q_{k,n}x + r_{k,n}) = 1 + \sum_{k=1}^{u_n}\frac{c_{k,n}^2}{2}\cos(2q_{k,n}x + 2r_{k,n}) = 1 + \xi_n(x).$$

The Lebesgue measure of the set of points where $|\xi_n(x)| \ge \delta > 0$ can be estimated by

$$\frac{1}{\delta^2}\int_0^{2\pi}\xi_n^2(x) = \frac{1}{4\delta^2}\pi\sum_{k=1}^{u_n} c_{k,n}^4 \overset{N\to\infty}{\to} 0$$

since $c_{k,n} \to 0$. Since ξ_n is bounded we have that the convergence in measure implies L^1 convergence from which we get that the integral is asymptotic to

$$e^{-\frac{t^2}{2}}\int_0^{2\pi}\prod_{k=1}^{u_n}\bigl(1+itc_{k,n}\cos(q_{k,n}x + r_{k,n})\bigr)dx.$$

Since the first term is the characteristic function of the Gaussian random variable with mean 0 and variance 1 we simply need to show that

$$\int_0^{2\pi} \prod_{k=1}^{u_n} \big(1 + itc_{k,n}\cos(q_{k,n}x + r_{k,n})\big)dx = 1. \tag{11}$$

Using the fact that

$$\cos(mx)\cos(nx) = \frac{1}{2}\big(\cos\big((m+n)x\big) + \cos\big((m-n)x\big)\big),$$

we can rewrite the product in the integral in the form

$$\prod_{k=1}^{u_n} \big(1 + itc_{k,n}\cos(q_{k,n}x)\big) = a_{0,n} + \sum_{l \in L_n} a_{l,n}\cos(lx + s_l),$$

where

$$L_n := \Big\{l \geq 0 \,:\, l = \sum_{k=1}^{u_n} b_k q_{k,n} \text{ with } b_k \in \{-1,0,1\}\Big\}.$$

Given that $q_{k+1,n} \geq 2q_{k,n}$, we immediately observe $\sum_{k=1} l - 1 q_{k,n} < q_{l,n}$ from which it immediately follows that the representation of 0 in the form $\sum b_k q_{k,n}$ is impossible. Thus we have $a_{0,n} = 1$. Integrating, we immediately get the proof of the theorem. □

7.2. **Proof of Weak Mixing.** Let $z_n = m_n \lambda c_0 \mod 1$. By passing to a subsequence we may assume that $z_n \to z$. Using (10), yields

$$\lambda S_{m_n}\phi_n \xrightarrow{d} N(z,1).$$

Finally,

$$\lim_{n\to\infty} \int \|\lambda S_{m_n}\varphi(x)\| dx = \lim_{n\to\infty} \int \|z_n + \lambda S_{m_n}\phi_n(x)\| = \int \|x\| dw(x) > 0,$$

where w is the measure corresponding to the $N(z,1)$ distribution. This proves that λ is not an eigenvalue. Since $\lambda \in \mathbb{R}\backslash\{0\}$ was arbitrary this proves that the special flow has continuous spectrum and is hence weak mixing.

8. Final Comments

Almost all of our arguments require very weak hypotheses. Every statement with the exception of Lemma 7.1 holds for C^3 functions satisfying the appropriate regularity of decay properties. We should be able to considerably enlarge the class of functions for which our dichotomy holds by using appropriate uniform estimates similar to Lemma 6.1 rather than the stronger statements of Lemma 7.1 and Lemma 7.2.

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