Statistical randomization test for QCD intermittency in a single-event distribution*

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Abstract

A randomization test was developed to determine the statistical significance of QCD intermittency in single-event distributions. A total of 96 simulated intermittent distributions based on standard normal Gaussian distributions of size N=500, 1000, 1500, 2000, 4000, 8000, 16000, and 32000 containing induced holes and spikes were tested for intermittency. Non-intermittent null distributions were also simulated as part of the test. A log-linear model was developed to simultaneously test the significance of fit coefficients for the y-intercept and slope contribution to $\ln(F_2)$ vs. $\ln(M)$ from both the intermittent and null distributions. Statistical power was also assessed for each fit coefficient to reflect the proportion of times out of 1000 tests each coefficient was statistically significant, given the induced effect size and sample size of the Gaussians. Results indicate that the slope of $\ln(F_2)$ vs. $\ln(M)$ for intermittent distributions increased with decreasing sample size, due to artificially-induced holes occurring in sparse histograms. For intermittent Gaussians with 4000 variates, there was approximately 70% power to detect a slope difference of 0.02 between intermittent and null distributions. For sample sizes of 8000 and greater, there was more than 70% power to detect a slope difference of 0.01. The randomization test performed satisfactorily since the power of the test for intermittency decreased with decreasing sample size. Power was near-zero when the test was applied to null distributions. The randomization test can be used to establish the statistical significance of intermittency in empirical single-event Gaussian distributions.

Keywords: Scaled factorial moment, Intermittency, Hypothesis testing, Statistical power, Randomization tests, Permutation tests, Kernel density estimation, Rejection method, Permutation tests

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1 Introduction

Intermittency has been studied in a variety of forms including non-gaussian tails of distributions in turbulent fluid and heat transport [1,2], spikes and holes in QCD rapidity distributions [3], $1/f$ flicker noise in electrical components [4], period doubling and tangent bifurcations [5], and fractals and long-range correlations in DNA sequences [6,7]. The QCD formalism for intermittency was introduced by Bialas and Peschanski for understanding spikes and holes in rapidity distributions, which were unexpected and difficult to explain with conventional models [3,8,9]. This formalism led to the study of distributions which are discontinuous in the limit of very high resolution with spectacular features represented by a genuine physical (dynamical) effect rather than statistical fluctuation [10-19].

The majority of large QCD experiments performed to date typically sampled millions of events (distributions) to measure single-particle kinematic variables (i.e., rapidity, tranverse momentum, azimuthal angle) or event-shape variables [20-24]. These studies have employed either the horizontal scaled factorial moment (SFM)

$$F_q \equiv \frac{1}{E}\sum_{e=1}^{E}\frac{1}{M}\sum_{m=1}^{M}\frac{n_{me}(n_{me}-1)\cdots(n_{me}-q+1)}{\left(\frac{N_e}{M}\right)^{[q]}}, \tag{1}$$

which normalizes with event-specific bin average over all bins (N_e/M), the vertical SFM

$$F_q \equiv \frac{1}{M}\sum_{m=1}^{M}\frac{1}{E}\sum_{e=1}^{E}\frac{n_{me}(n_{me}-1)\cdots(n_{me}-q+1)}{\left(\frac{N_m}{E}\right)^{[q]}}, \tag{2}$$

which normalizes with bin-specific average over all events (N_m/M), or the mixed SFM

$$F_q \equiv \frac{1}{ME}\sum_{m=1}^{M}\sum_{e=1}^{E}\frac{n_{me}(n_{me}-1)\cdots(n_{me}-q+1)}{\left(\frac{N}{ME}\right)^{[q]}}, \tag{3}$$

which normalizes by the grand mean for all bin counts from all events (N/EM). In the above equations, n_{me} is the particle multiplicity in bin m for event e, M is the total number of bins, $N_e = \sum_m^M n_{me}$, $N_m = \sum_e^E n_{me}$, and N is the sum of bin counts for all bins and events. A full description of SFMs described above can be found in [10-12]. There are occassions, however, when only a single event is available for which the degree of intermittency is desired. In such cases, the mean multiplicity in the sample of events cannot be determined and the normalization must be based on the single-event SFM defined as

$$F_q \equiv \frac{1}{M}\sum_{m=1}^{M}\frac{n_m(n_m-1)\cdots(n_m-q+1)}{\left(\frac{N}{M}\right)^{[q]}}, \tag{4}$$

where n_m is the number of bin counts within bin m, and $N = \sum_m n_m$ is the total number of counts for the single-event. When intermittency is present in the distribution, F_q will be proportional to M according to the power-law

$$F_q \propto M^{\nu_q}, \qquad M \to \infty \tag{5}$$

(or $F_q \propto \delta y^{-\nu_q}, \delta y \to 0$) where ν_q is the *intermittency exponent*. By introducing a proportionality constant A into (5) and taking the natural logarithm we obtain the line-slope formula

$$\ln(F_q) = \nu_q \ln(M) + \ln(A). \tag{6}$$

If there is no intermittency in the distribution then $\ln(F_q)$ will be independent from $\ln(M)$ with slope ν_q equal to zero and $\ln(F_q)$ equal to the constant term $\ln(A)$. Another important consideration is that if ν_q is non-vanishing in the limit $\delta y \to 0$ then the distribution is discontinuous and should reveal an unusually rich structure of spikes and holes.

The goal of this study was to develop a randomization test to determine the statistical significance of QCD intermittency in intermittent single-event distributions. Application of the randomization test involved simulation of an intermittent single-event distribution, multiple simulations of non-intermittent null distributions, and a permutation-based log-linear fit method for $\ln(F_2)$ vs. $\ln(M)$ for assessing significance of individual fit coefficients. The statistical power, or probability of being statistically significant as a function of coefficient effect size, Gaussian sample size, and level of significance, was determined by repeatedly simulating each intermittent single-event distribution and performing the randomization test 1000 times. The proportion of randomizations tests that were significant out of the 1000 tests reflected the statistical power of each fit coefficient to detect its relevant slope or y-intercept value given the induced intermittency.

2 Methods

2.1 Sequential steps

A summary of steps taken for determining statistical power for each of the log-linear fit coefficients is as follows.

1. Simulate an intermittent single-event Gaussian distribution of sample size N by introducing a hole in the interval $(y, y + \Delta_h)_h$ and creating a spike by adding hole data to existing data in the interval $(y, y - \Delta_s)_s$. (This simulated intermittent distribution can be replaced with an empirical distribution for which the presence of intermittency is in question). Determine the minimum, y_{min}, maximum, y_{max}, and range $\Delta_y = y_{max} - y_{min}$ of the N variates.

2. Assume an experimental measurement error (standard deviation) such as $\epsilon = 0.01$. Determine the histogram bin counts $n(m)_\epsilon$ for the intermittent distribution using $M_{max} = \Delta y/\epsilon$ equally-spaced non-overlapping bins of width $\delta y = \epsilon$. Apply kernel density estimation (KDE) to obtain a smooth function (pdf) of the histogram containing M_{max} bins. KDE only needs be performed once when bin width $\delta y = \epsilon$.

3. Use the rejection method based on the pdf obtained from KDE to simulate a non-intermittent Gaussian. Determine histogram bin counts $n(m)_{null,\epsilon}$ over the same range Δy of the intermittent distribution.

4. For each value of M, collapse together the histogram bins determined at the experimental resolution and determine $\ln(F_2)$ and $\ln(M)$ at equal values of M for both the

intermittent and null distribution. Bin collapsing should be started with the first bin $n(1)$.

5. Perform a log-linear fit using values of $\ln(F_2)$ vs. $\ln(M)$ from both the intermittent non-intermittent null distributions. Determine the significance for each coefficient using the Wald statistic, $Z_j = \beta_j/s.e.(\beta_j)$ $(j = 1, 2, ..., 4)$ where Z_j is standard normal distributed. For the bth permutation $(b = 1, 2, ..., B)$ where $B = 10$, permute the group labels (intermittent vs. non-intermittent) in data records, refit, and calculate $Z_j^{(b)}$. After the B permutations, determine the number of times $|Z_j^{(b)}|$ exceeded $|Z|$.

6. Repeat steps 4 and 5, only this time start collapsing bins at the 2nd bin $n(2)$. This essentially keeps Δy the same but shifts the interval over which bin collapsing is performed to $(y_{min} + \epsilon, y_{max} + \epsilon)$.

7. Repeat steps 3 to 6 a total of 10 times simulating a non-intermittent null distribution each time, and determine the significance of each fit coefficient with permutation-based fits. Thus far, we have used multiple simulations combined with a permutation-based log-linear fit to determine the statistical significance of each fit coefficient for one intermittent distribution (simulated in step 1). These procedures comprise a single randomization test to determine significance of fit parameters for the single-event distribution.

8. To estimate statistical power for each fit coefficient, repeat the randomization test in steps 1-7 1000 times, each time simulating a new intermittent single-event distribution with the same sample size and spike and hole intervals. The statistical power for each fit coefficient is based on the bookkeeping to track the number of times $|Z_j^{(b)}|$ exceeds $|Z_j|$ within the permutation-based log-linear fits. Report the average values of β_j and $s.e.(\beta_j)$ obtained before permuting group labels to reflect effect size, and report the statistical power of each fit coefficient for detecting the induced effect. This power calculation step is unique to this study for assessing the proportion of randomization tests during which each coefficient was statistically significant.

2.2 Inducing spikes and holes in Gaussian distributions

An artifical hole was induced by removing bin counts in the range $y + \Delta_h$ and placing them into the interval $y - \Delta_s$ as shown in Figure 1. Figures 2 and 3 show the simulated intermittent Gaussian distributions based on 10,000 variates and plots of $\ln(F_2)$ vs. $\ln(M)$ for $y = 2.0$ and $y = 0.2$, respectively. Figure 2a shows the formation of a hole above and spike below $y = 2$ as the hole width $2 + \Delta_h$ increased and spike width $2 - \Delta_s$ decreased. The pattern in Figure 2b suggests that the level of intermittency based on the slope of $\ln(F_2)$ vs. $\ln(M)$ increases with increasing ratio Δ_h/Δ_s. The positive non-zero slopes of $\ln(F_2)$ vs. $\ln(M)$ in Figure 2b suggests intermittency at a level beyond random fluctuation of bin counts. Figure 3 shows similar results but for a hole introduced in the range $0.2+\Delta_h$ and spike in the range $0.2 - \Delta_s$. One can notice in Figure 3a that because of the larger bin counts near $y = 0.2$, the resulting spikes are larger for the same hole sizes presented in Figure 2a. This supports the rule of thumb that spikes and holes near the bulk of a

Gaussian contribute more to F_2. The overall point is that the slope of the line $\ln(F_2)$ vs. $\ln(M)$ increases with increasing size of the induced hole, decreasing spike width, and increasing value of the pdf into which hole data are piled.

2.3 Histogram generation at the assumed experimental resolution

It was observed that F_2 increased rapidly when the bin size was smaller than $\delta y = 0.01$. This was most likely due to round-off error during histogram generation. Because round-off error at high resolution can create artificial holes and spikes in the data, the parameter ϵ was introduced to represent a nominal level of imprecision in the intermittent data, which was assumed to be 0.01. Thus, the smallest value of δy used was ϵ, at which the greatest number of bins $M_{max} = \Delta y/\epsilon$ occurred. In addition, F_q was only calculated for $q = 2$ in order to avoid increased sensitivity to statistical fluctuations among the higher moments. For a sample of N quantiles, "base" bin counts were accumulated and stored in the vector, $n(m)_\epsilon$, which represented counts in M_{max} bins.

2.4 Simulating non-intermittent null distributions

Non-intermittent null distributions were simulated once per intermittent distribution, and the results were used to fill counts in M_{max} total bins when the bin width was ϵ. First, the underlying smooth function of the histogram for intermittent data was determined using kernel density estimation (KDE) [25] in the form

$$f(m) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{y_i - y_m}{h}\right), \tag{7}$$

where $f(m)$ is the bin count for the mth bin for non-intermittent null data, N is the total number of variates, $h = 1.06\sigma N^{-0.2}$ is the optimal bandwidth for a Gaussian [26], and σ is the standard deviation of the Gaussian. K is the Epanechnikov kernel function [27] defined as

$$K(u) = \begin{cases} \frac{3}{4}(1 - u^2) & |u| \leq 1 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where $u = (y_i - y_m)/h$ and y_m is the lower bound of the mth bin. The smooth pdf derived from KDE was used with the rejection method to build-up bin counts, $n(m)_{null,\epsilon}$, until the number of variates was equal to the number of variates in the single-event intermittent distribution. Under the rejection method, bins in the simulated non-intermittent distribution are randomly selected with the formula

$$m = (M_{max} - 1)\mathrm{U}(0, 1)_1 + 1. \tag{9}$$

where $U(0, 1)_1$ is a pseudo-random uniform distributed variate. For each m, a second pseudo-random variate is obtained and if the following criterion is met

$$\mathrm{U}(0, 1)_2 < \mathrm{pdf}(m)/\max\{\mathrm{pdf}(i)\}, \qquad i = 1, 2, ..., M_{max} \tag{10}$$

then one is added to the running sum for bin count and the running sum for the total number of simulated y values. The rejection method provided non-intermittent null distributions with attendant statistical fluctuations to determine whether non-intermittent data consistently has estimates of intermittency lower than that of a single-event distribution with simulated intermittency.

2.5 F_2 calculations and collapsing bin counts into M Bins

F_2 was calculated using (4) at varying values of $M = M_{max}/k$, where k is the number of bins at the experimental resolution collapsed together ($k = 2, 3, ..., M_{max}/30$). It follows that for M equally-spaced non-overlapping bins, the bin width is $k\epsilon$. The smallest bin width of 2ϵ ($M = M_{max}/2$) used in F_2 calculations allowed us to conservatively avoid artificial effects, whereas the greatest bin width was limited to $\Delta y/30$ ($M = 30$) since widths can become comparable to the width of the distribution. As M changed, histogram bin counts $n(m)$ were determined by collapsing together each set of k contiguous bins at an assumed experimental resolution, i.e., $n(m)_\epsilon$ or $n(m)_{null,\epsilon}$, rather than determining new lower and upper bin walls and adding up counts that fell within the walls (Table 1). This approach cut down on a tremendous amount of processor time while avoiding cumulative rounding effects from repeatedly calculating new bin walls.

2.6 Shifting the range of y

A single "shift" was performed in which the full range Δy was moved by a value of $\epsilon = 0.01$ followed by a repeat of F_2 calculations, log-linear fits, and permutations. To accomplish this, we varied the starting bin $n(shift)_\epsilon$ before collapsing bins. For example, if $M_{max} = 400$, the first value of M was $M_{max} = 400/2 = 200$, since the first value of k is 2. Bin counts for the 200 bins were based on collapsing contiguous pairs ($k = 2$) of bins starting with $n(1)_\epsilon$ when $shift = 1$. F_2 was then calculated, linear fits with permutations were made, and values of $\ln(F_2)$ and $\ln(M)$ were stored. This was repeated with $shift = 2$ so that pairs of base bin were collapsed again but by starting with base bin $n(2)_\epsilon$ when $shift = 2$. The process of shifting the range of y and repeating the randomization tests for each intermittent single-event distribution resulted in an entirely different set of bin counts, increasing the variation in intermittent and null data sets. Shifting the range and re-calculating F_2 was performed in the original Bialas and Peschanski paper (See Fig. 3, 1986).

2.7 Permutation-based log-linear regression

For intermittency and non-Gaussian distributions, it is unlikely that the null distributions are thoroughly known. Therefore, instead of using a single fit of the data to determine significance for each coefficient, permutation-based log-linear fits were used in which fit data were permuted (randomly shuffled) and refit in order to compare results before and after permutation. Permutation-based regression methods are useful when the null distributions of data used are unknown.

The presence of intermittency was determined by incorporating values of $\ln(F_2)$ and $\ln(M)$ into a log-linear regression model to obtain direct estimates for the difference in the $y-$intercept and slope of $\ln(F_2)$ vs. $\ln(M)$ for intermittent and null data. The model was in the form

$$\ln(F_2) = \beta_0 + \beta_1 \ln(M) + \beta_2 I(null) + \beta_3 \ln(M) I(null), \tag{11}$$

where β_0 is the y-intercept of the fitted line $\ln(F_2)$ vs. $\ln(M)$ for intermittent data, β_1 is the slope of the fitted line for intermittent data, β_2 is the difference in y-intercepts for intermittent and null, $I(null)$ is 1 if the record is for null data and 0 otherwise, and β_3 represents the difference in slopes for the intermittent and null data. Results of modeling suggest that β_1 can be significantly positive over a wide variety of conditions. When $\beta_3 < 0$, the slope of the intermittent fitted line for $\ln(F_2)$ vs. $\ln(M)$ is greater than the slope of $\ln(F_2)$ vs. $\ln(M)$ for null data, whereas when $\beta_3 > 0$ the slope of intermittent is less than the slope of null. Values of β_0 and β_2 were not of central importance since they were used as nuisance parameters to prevent forcing the fits through the origin. While β_1 tracks with the degree of absolute intermittency in the intermittent distribution, the focus is on the difference in slopes between the intermittent single-event distribution and the non-intermittent null distribution characterized by β_3. Again, β_3 is negative whenever the slope of $\ln(F_2)$ vs. $\ln(M)$ is greater in the intermittent than the null. Strong negative values of β_3 suggest higher levels of intermittency on a comparative basis.

For each fit, the Wald statistic was first calculated as $Z_j = \beta_j/\text{s.e.}(\beta_j)$. Next, values of $I(null)$, that is 0 and 1, were permuted within the fit data records and the fit was repeated to determine $Z_j^{(b)}$ for the bth permutation where $b = 1, 2, ..., B$. Permutations of $I(null)$ followed by log-linear fits were repeated 10 times ($B = 10$) for each intermittent distribution. Since there were 10 permutations per log-linear fit, 10 null distributions simulated via KDE and the rejection method, and 2 shifts in Δy per intermittent distribution, the total number of permutations for each intermittent distribution was $B = 200$. After B total iterations, the p-value, or statistical significance of each coefficient was

$$p_j = \frac{\#\{b : |Z_j^{(b)}| > |Z_j|\}}{B} \tag{12}$$

If the absolute value of the Wald statistic $Z_j^{(b)}$ at any time after permuting $I(null)$ and performing a fit exceeds the absolute value of the Wald statistic derived before permuting $I(null)$, then there a greater undesired chance that the coefficient is more significant as a result of the permuted configuration. What one hopes, however, is that fitting after permuting labels never results in a Wald statistic that is more significant that that based on non-permuted data. Because test statistics (e.g., Wald) are inversely proportional to variance, it was important to use Z_j as the criterion during each permutation.

2.8 Statistical significance of fit coefficients

The basis for tests of statistical significance is established by the null hypothesis. The null hypothesis states that there is no difference in intermittency among intermittent and null distributions whereas the alternative hypothesis posits that the there is indeed a difference in intermittency. Formally stated, the null hypothesis is $H_o : \beta_3 = 0$ and the one-sided alternative hypothesis is $H_a : \beta_3 < 0$, since negative values of β_3 in (11) imply intermittency. The goal is to discredit the null hypothesis. A false positive test result causing rejection of the null hypothesis when the null is in fact true is known as a Type I error. A false negative test result causing failure to reject the null hypothesis when it is not true (missed the effect) is a Type II error. The probability of making a Type I error is equal to α and the probability of making a Type II error is equal to β_{power}. Commonly used acceptable error rates in statistical hypothesis testing are $\alpha = 0.05$ and $\beta_{power} = 0.10$. The Wald statistics described above follow the standard normal distribution, such that a Z_j less than -1.645 or greater than 1.645 lies in the rejection region where $p < 0.05$. However, since permutation-based regression models were used, the significance of fit coefficients was not based on comparing Wald statistics with standard normal rejection regions, but rather the values of the empirical p-values in (12). Whenever $p_j < \alpha$, a coefficient is said to be significant and the risk of a Type I error is at least α.

2.9 Statistical power

Randomization tests were used for assessing the level of significance for each fit coefficient, given the intermittent single-event and the simulated null distributions used. When developing a test of hypothesis, it is essential to know the power of the test for each log-linear coefficient. The power of a statistical test is equal to $1 - \beta_{power}$, or the probability of rejecting the null hypothesis when it is not true. In other words, power is the probability of detecting a true effect when it is truly present. Power depends on the sample size N of the Gaussian, effect size β_j, and level of significance (α). To determine power for each fit coefficient, 1000 intermittent single-event Gaussians with the same induced spikes and holes and sample size were simulated, and used in 1000 randomizations tests. During the 1000 randomization tests, power for each fit coefficient was based on the proportion of tests in which p_j in (12) was less than $\alpha = 0.05$. For a given effect size β_j, sample size of Gaussian N, and significance level (α), an acceptable level of power is typically greater than 0.70. Thus, the p-value for a particular log-linear fit coefficient would have to be less than 0.05 during 700 randomization tests on an assumed intermittent distribution.

2.10 Simulated intermittent distributions used

Three types in intermittent distributions were simulated, each having a single hole-spike pair at a different location. Hole location and widths are discussed first. The first simulated intermittent distribution had an induced hole starting at y =0.2 of width Δ_h=0.64, corresponding to the interval $(0.2\text{-}0.84)_h$. The second distribution had a hole starting at $y = 1.0$ of width Δ_h=0.02, corresponding to interval $(1.0\text{-}1.2)_h$. The third type of intermittent distribution had a single hole induced starting at y =2 also at a width of Δ_h=0.64, corresponding to interval $(2.0\text{-}2.64)_h$. For each hole induced, a single spike was generated by adding hole data to existing simulated data below the y-values using varying spike widths of Δ_s=0.64, 0.24, 0.08, or 0.02. Therefore, for the first distribution with a hole in interval $(0.2\text{-}0.84)_h$, spikes were either in intervals $(-0.44\text{-}0.2)_s$, $(-0.04\text{-}0.2)_s$, $(0.12\text{-}0.2)_s$, or $(0.18\text{-}0.2)_s$. For the distribution with a hole in interval $(1.0\text{-}1.2)_s$, the spike intervals were at either $(0.36\text{-}1.0)_s$, $(0.76\text{-}1.0)_s$, $(0.92\text{-}1.0)_s$, or $(0.98\text{-}1.0)_s$. Finally, for a hole in interval $(2.0\text{-}2.64)_h$, spike intervals were either at (1.36-2.0), (1.76-2.0), (1.92-2.0), or (1.98-2.0).

The combination of distribution having single hole-spike pairs at three hole locations, and four spike locations resulted in a total of 12 distributions. Each of the 12 types of intermittent distributions were simulated with 500, 1000, 1500, 2000, 4000, 8000, 16000, and 32000 variates. This resulted in 96 different simulated intermittent distributions (Table 2).

2.11 Algorithm flow

The algorithm flow necessary for performing a single randomization test to determine significance of fit coefficients for one single-event distribution is given in Figures 4 and 5. Figure 4 shows how test involves 10 simulations of non-intermittent null distributions, 2 shifts each, and $B = 10$ permutations during the log-linear fit. The 10 null distribution simulations and 10 permutations are done twice, once for the first shift based on interval y_{min} to y_{max} and once for the second shift using the interval $y_{min} + \epsilon$ to $y_{max} + \epsilon$.

Figure 5 shows a telescoping schematic of what occurs for each of the 10 null distributions. For each null distribution there are two shifts during which $\ln(F_2)$ and $\ln(M)$ are determined. Values of $\ln(F_2)$ and $\ln(M)$ as a function of M for both the intermittent single event and null distribution are used in the log-linear fit to yield Z_j for each coefficient. The values of $I(null)$ are then permuted $B = 10$ times resulting in $Z_j^{(b)}$, where $b = 1, 2, ..., 10$, for a total of 200 permutations (10*2*10). Bookkeeping is done using the counter $\texttt{numsig}_j$ which keeps track of the total number of times $|Z_j^{(b)}|$ exceeds $|Z_j|$ during the 200 permuted fits. The 200 fits represent one randomization test, from which the statistical significance of each fit coefficient is determined based on p_j in (12). Statistical power for each coefficient was determined based on 1000 randomization tests in which the same intermittent distribution (same hole and spike widths and sample size) was simulated.

2.12 Summary statistics of results

Each randomization test to determine the statistical significance of fit coefficients for a single simulated intermittent distribution employed 200 permutation-based log-linear fits. The 200 total fits per randomization test (i.e., intermittent distribution) is based on 10 simulations to generate non-intermittent null distributions, followed by 2 shifts, and then 10 permutation-based fits (10*2*10). Within each randomization test, the total 200 permutation-based fits were used for obtaining the significance of each coefficient via p_j in (12). Among the 200 total permutation-based fits, there were 20 fits for which fit data were not permuted. Averages of fit coefficients and their standard error were obtained for the 20 "non-permuted" fits. Since randomization tests were carried out 1000 times (in order to obtain power for each coefficient), the averages of coefficients and their standard errors over the 1000 tests was based on averages (over 1000 randomization tests) of averages (over 20 non-permuted fits). The global averages of coefficients and their standard errors, based on a total of 20,000 non-permuted fits, are listed in Table 2 under column headers with angle brackets $\langle\rangle$. In total, since there were 20 non-permuted and 200 permuted fits per randomization test, the use of 1000 randomization tests for obtaining power for each coefficient for the intermittent distribution considered resulted in 200,000 permutation-based fits for each distribution (row in Table 2).

3 Results

The averages of fit coefficients and their standard deviation from 20,000 non-permuted fits during 1000 randomization tests are listed in of Table 2. Each row in Table represents the result of 1000 randomization tests for a single simulated intermittent distribution. Also listed in the Table 2 is the statistical power for each coefficient reflecting the proportion of the 1000 randomization tests during which each coefficient was significant.

As described in the methods section, β_0 reflects the y-intercept of the fitted line for $\ln(F_2)$ vs. $\ln(M)$ for intermittent data. Average values of β_0 were jumpy and did not correlate consistently with the level of induced intermittency. There can be tremendous variation in the y-intercept of either an intermittent or null distribution depending on unique properties of the distributions, so this was expected.

Average values of β_1 reflected the slope of fitted line $\ln(F_2)$ vs. $\ln(M)$ for intermittent data, and ranged from 0.0007 to 0.2347 for distributions with 32000 variates. One of the most interesting observations was that, as sample size of the intermittent distributions decreased from 32000 to 500, values of β_1 increased. This was expected because the abundance of artificial holes in histograms increases as the sample size of the input distribution decreases. Accordingly, the "true" intermittency effect induced is portrayed at only the larger sample sizes where there is less likelihood for artificial holes to exist in the histograms.

Similar to β_0, β_2 reflects the difference between the y-intercepts of the intermittent and null distributions compared. But again, β_0 and β_2 are essentially important nuisance parameters introduced in the model in order not to force the fitted lines through the origin. Nevertheless, β_2 was found to be especially important when evaluating power as will be discussed later.

The most important coefficient was β_3, which reflects the difference in slopes between fitted lines $\ln(F_2)$ vs. $\ln(M)$ for intermittent and non-intermittent data. (Values of β_3 and $P(\beta_3)$ when N=32000 are listed in bold in Table 2). When $\beta_3 < 0$, the slope of the line for $\ln(F_2)$ vs. $\ln(M)$ of the intermittent data is greater than the slope of $\ln(F_2)$ vs. $\ln(M)$ for non-intermittent data, whereas when $\beta_3 > 0$, the slope of intermittent is less than the slope of non-intermittent. Average values for β_3 for distributions with 32000 variates ranged from -0.5294 to -0.0002. The statistical power $P(\beta_3)$ of β_3 for detecting the induced intermittency in various intermittent distributions is also shown in Table 2. For distributions with 32000 variates, $P(\beta_3)$ was 100% for values of β_3 that were more negative than -0.015, suggesting that at large samples sizes the randomization test had 100% power to detect a difference of 0.015 between slopes for intermittent and null distributions. It was observed that for these cases, $P(\beta_3)$ was 100% only when the y-intercept difference β_2 exceeded 0.05. In one case, β_3 was more negative than -0.015 (-0.0319), but $P(\beta_3)$ was 82.8%. For this case, β_2 was equal to 0.0385. For the remaining distributions for which N=32000, when β_3 was more positive than -0.015, $P(\beta_3)$ was less than 100%.

It was also noticed that while β_1 increased with decreasing sample size as a result of the introduction of artificial holes, $P(\beta_3)$ decreased. This is important because it suggests that the power of the randomization test to detect intermittency decreases with decreasing sample size.

Power of the randomization test was also assessed for non-intermittent null distributions (end of Table 2). The null distributions used did not contain any holes or spikes and therefore did not contain induced intermittency. Since intermittency is not present in null distributions, zero power is expected. With regard to coefficient values for null distributions, β_0 did not differ from values of β_0 for intermittent distributions, but β_1 increased from 0.0004 for N=32000 to 0.0389 for N=500. As sample sized decreased from N=32000 to N=500, β_2 increased from -0.0016 to 0.1471 and β_3 decreased from 0.00005 to -0.0294, suggesting again that even in null distributions the level of artificial intermittency increases as sample size decreased.

The value of the power calculations is reflected in results for null distributions lacking any effect, because power indicates the probability that the $p-$values of coefficients in (12) are significant (i.e., $p_j < \alpha = 0.05$). The greatest probability of obtaining a significant value for either β_2 or β_3 for a single randomization test on a null distribution was 0.062 for N=1000. At $N = 500$ the probability (power) of obtaining a significant coefficient for β_2 or β_3 was 0.04 and 0.03, respectively. At $N = 2000$ the power for β_2 or β_3 was 0.015 and 0.019, respectively. At $N = 8000$ and greater the probability (power) of obtaining a significant coefficient for β_2 or β_3 was zero. In conclusion, power was zero for null distributions having sample sizes greater than $N = 8000$.

Table 3 lists the sorted values of statistical power $P(\beta_3)$ and averages of coefficients extracted from Table 2 for distributions with N=32000. The performance of β_3 for detecting the induced intemittency in terms of slope difference between intermittent and null distributions can be gleaned from this table. For most experiments, it is desired to use a test that has 80% or more power to detect a signal as a function of effect size, sample size and α. In this study, effect size is reflected in the β_j coefficients. Sample size is reflected in the number of variates N used for each simulated distribution. The significance level α is used as an acceptable probability of Type I error, which essentially means reporting that a coefficient is significant (rejecting the null hypothesis of no signal) when the coefficient is truly insignificant (false positive). One the other hand, power is one minus the Type II error rate, which is the probability of missing a signal when the signal is present (false negative). One minus this probability (i.e., power) is the probability of finding a signal when it is truly present. Looking at Table 3, one notices that when β_3 was more negative than -0.01 (greater than a 1% signal), the power of β_3 for detecting intermittency was at least 70%.

Figure 6 shows the statistical power $P(\beta_3)$ as a function of average effect size $\langle\beta_3\rangle$ and sample size of simulated intermittent Gaussian distributions. Each point represents a distribution in a row of Table 2. At the lowest sample size of N=500, an acceptable level of approximately 70% power is attainable for a 0.08 difference in slopes between the intermittent and null distributions. As sample size increases, the power to detect intermittency increases. For intermittent Gaussians with 4000 variates, there was approximately 70% power to detect a slope difference of 0.02 between intermittent and null distributions. Whereas for sample sizes of 8000 and greater, there was more than 70% power to detect a slope difference of 0.01.

4 Discussion

A basic characteristic of QCD intermittency is that if the smooth distribution for a histogram measured at the limit of experimental resolution is discontinuous, it should reveal an abundance of spikes and holes. For a discontinuous smooth distribution, QCD intermittency is a measure of non-normality in fluctuations and reflects little about the deterministic or stochastic properties of a distribution. QCD intermittency is also independent of the scale of data and scaling in spatial or temporal correlations. The lowest scale of resolution used in this paper ($\delta y = 0.01$) refers to a measure of imprecision, or standard deviation in measured data. Thus, the quantile values of Gaussians used were not assumed to be infinitely precise.

The statistically significant levels of intermittency identified in this study show how various methods from applied statistics can be assembled to form a randomization test for intermittency for single-event distributions. It is important to compare fit coefficients for the intermittent distribution versus that from a null distribution. This study employed a log-linear model that was able to extract simultaneously information on the y-intercepts and slopes for intermittent and null data separately. At the same time, the log-linear model provided a method to simultaneously test the statistical significance of each coefficient. Not surprisingly, the most important and consistent coefficient for identifying intermittency was β_3, or the difference in slopes of the line $\ln(F_2)$ vs. $\ln(M)$ for intermittent and null distributions.

Generally speaking, the power to detect intermittency exceeds 70% when β_3 is less than (more negative than) -0.01 assuming a Gaussian sample size of 32000. As the sample size decreases, artificial holes are introduced which obscures any real intermittency. This was reflected in an increase of β_1 with decreasing sample size. As the sample size decreased, wider artificial holes were introduced into the histograms for both the intermittent and non-intermittent null distributions. Since both distributions were affected equally on a random basis, the difference in slopes measured by β_3 did not increase. Power naturally increases with sample size, so there was no reason to expect an increase in power with decreasing sample size. Nevertheless, a positive finding was that the power of the randomization test to detect intermittency decreased with decreasing sample size, in spite of the increase of β_1. This indicates that the ability to appropriately identify the presence of intermittency in a single-event distribution depends on more than a single linear fit of $\ln(F_2)$ vs. $\ln(M)$ for the intermittent distribution.

The finding that power was at a maximum of 0.06 for β_3 from a null distribution of size 1000 suggests that there is at most a 6% chance of obtaining a p-value for β_3 below 0.05. For null distributions of sample size 4000, the probability (power) of obtaining a significant β_3 coefficient was 0.007. When sample size was 8000 or more, power for β_3 was zero. No attempt was made to relate hole and spike widths and their locations in simulated intermittent Gaussians with resultant power, since the effect size needed for power calculations was captured by the fit coefficients. This was confirmed by the change in coefficients with changing sample size. While the widths and locations of induced holes and spikes were fixed, coefficients changed with sample size and the induced effect. The purpose of power is to establish the consistency of a coefficient for detecting an effect when it is truly present. It follows that null distributions are also not used for establishing power of a test, since power depends on a known effect size β_j, sample size N of a Gaussian, and the level of significance α used for determining when each coefficient is significant.

There were several limitations encountered in this study. First, there is an infinitely large number of ways one can induce intermittency in Gaussian distributions. Given the size and duration of the study, it was assumed that the 96 intermittent distributions considered would adequately reflect the robustness of the randomization test over a range of induced levels of intermittency. Additional research is needed to assess the power of the randomization test for horizontal, vertical, and mixed SFMs, number of null distributions, number of shifts, number of permutations during fits, distribution sample sizes, imprecision, skewness and kurtosis of multimodal distributions, and variations in the choice of bandwidth and kernel smoother, etc. It was impossible to address the majority of these issues in this investigation, and therefore the intent of this paper was to introduce results for the limited set of conditions used.

Over the course of this investigation, there were many evaluations on how best to make a smooth distribution for the non-intermittent null distributions. The most promising was the combined approach using KDE and the rejection method, which is also probably the most robust. Parametric methods were problematic when there were large holes or spikes, and when there was a significant level of kurtosis or skewness in the data. Long tails present another challenge for simulating an appropriate null distribution with parametric fitting methods. KDE is non-parametric and by altering the bandwidth settings one can closely obtain the original histogram, or more smoothed histograms. Because the simulated distributions were known be standard normal Gaussian distributions, the optimal bandwith $h = 1.06\sigma N^{-1/5}$ was used [26].

If an experimenter has a single-event distribution for which intermittency is in question, then application of sequential methods listed in steps 2-7 skipping simulation in step 1 would be used. This corresponds to running a single randomization test on the intermittent data. If the p-value in (12) for any of the coefficients in (11) are less than 0.05, then the coefficient is statistically significant. Specifically, intermittency would be statistically significant if the p-value for β_3 was less than 0.05. Power of the randomization test for truly detecting intermittency could be looked up in Figure 6, for the specific coefficient value of β_3 and sample size of the distribution. An unacceptable value of power below 0.70 would suggest that a greater effect size or greater sample size is needed for detecting a statistically significant level of intermittency.

5 Conclusions

Results indicate that the slope of $\ln(F_2)$ vs. $\ln(M)$ for intermittent distributions increased with decreasing sample size, due to artificially-induced holes occurring in sparse histograms.

When the average difference in slopes between intermittent distributions and null distributions was greater than 0.01, there was at least 70% power for detecting the effect for distributions of size 8000 and greater. The randomization test performed satisfactorily since the power of the test for intermittency decreased with decreasing sample size.

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Table 1: Example of collapsing bins to calculate bin counts $n(m)$, as the total number of histogram bins (M) change with each change of scale δy. k is the number of bins added together to obtain bin counts for each of M total bins

δy	k	M	Bin counts, $n(m)$										
0.06	6	$M_{max}/6$						23					68
0.05	5	$M_{max}/5$					14					60	
0.04	4	$M_{max}/4$				7				44			
0.03	3	$M_{max}/3$			3			20			40		
0.02	2	$M_{max}/2$		1		6		16		28		23	
0.01=ϵ*	1	M_{max}	1	0	2	4	7	9	13	15	12	11	8

* ϵ is the assumed experimental imprecision (standard deviation) equal to 0.01.

Table 2: Averages of log-linear fit coefficients and their standard errors and statistical power for each coefficient. Averages based on 20,000 fits and power based on 1000 tests.

Interval[a]	N[b]	$\langle\beta_0\rangle$[c]	$\langle\sigma_{\beta_0}\rangle$	$\langle\beta_1\rangle$	$\langle\sigma_{\beta_1}\rangle$	$\langle\beta_2\rangle$	$\langle\sigma_{\beta_2}\rangle$	$\langle\beta_3\rangle$	$\langle\sigma_{\beta_3}\rangle$	P(β_0)[d]	P(β_1)	P(β_2)	P(β_3)
$(0.2\text{-}0.84)_h$	500	0.6668	0.1675	0.0698	0.0219	0.2654	0.1584	-0.0793	0.0337	0.700	0.939	0.495	0.606
$(-0.44\text{-}0.2)_s$	1000	0.8117	0.1867	0.0510	0.0273	0.2178	0.2083	-0.0662	0.0438	0.901	0.924	0.506	0.661
	1500	0.9020	0.1851	0.0389	0.0274	0.1417	0.2094	-0.0484	0.0440	0.971	0.951	0.373	0.635
	2000	0.9642	0.1724	0.0318	0.0248	0.0955	0.1914	-0.0375	0.0402	0.987	0.976	0.260	0.676
	4000	1.0757	0.0968	0.0191	0.0117	0.0083	0.0867	-0.0163	0.0180	1.000	0.996	0.043	0.761
	8000	1.1318	0.0593	0.0169	0.0041	0.0002	0.0223	-0.0121	0.0044	1.000	0.999	0.011	0.810
	16000	1.1723	0.0507	0.0176	0.0036	0.0087	0.0170	-0.0118	0.0032	1.000	1.000	0.046	0.792
	32000	1.2085	0.0475	0.0181	0.0036	0.0150	0.0176	**-0.0113**	0.0033	1.000	1.000	0.099	**0.763**
$(0.2\text{-}0.84)_h$	500	0.7117	0.1858	0.1062	0.0267	0.2630	0.1936	-0.1218	0.0418	1.000	0.998	0.830	0.893
$(-0.04\text{-}0.2)_s$	1000	0.8872	0.2136	0.0821	0.0322	0.1813	0.2473	-0.0969	0.0522	1.000	0.999	0.565	0.922
	1500	0.9654	0.2046	0.0715	0.0315	0.1242	0.2433	-0.0802	0.0513	1.000	1.000	0.411	0.920
	2000	1.0315	0.1821	0.0624	0.0280	0.0677	0.2166	-0.0647	0.0454	1.000	1.000	0.264	0.919
	4000	1.1476	0.1111	0.0488	0.0161	-0.0129	0.1129	-0.0403	0.0233	1.000	1.000	0.051	0.903
	8000	1.1910	0.0700	0.0488	0.0100	-0.0004	0.0521	-0.0356	0.0099	1.000	1.000	0.027	0.899
	16000	1.2280	0.0621	0.0505	0.0095	0.0228	0.0505	-0.0340	0.0094	1.000	1.000	0.087	0.883
	32000	1.2571	0.0654	0.0519	0.0098	0.0385	0.0520	**-0.0319**	0.0097	1.000	1.000	0.146	**0.828**
$(0.2\text{-}0.84)_h$	500	0.5614	0.1444	0.2141	0.0295	0.2647	0.1263	-0.2058	0.0316	0.814	1.000	0.542	0.995
$(0.12\text{-}0.2)_s$	1000	0.6253	0.1759	0.2101	0.0345	0.3105	0.1670	-0.2055	0.0348	0.757	1.000	0.556	0.999
	1500	0.6645	0.2046	0.2090	0.0399	0.3331	0.2040	-0.2038	0.0401	0.623	1.000	0.445	0.999
	2000	0.6588	0.2176	0.2124	0.0419	0.3599	0.2161	-0.2033	0.0416	0.533	1.000	0.407	0.999
	4000	0.6761	0.2264	0.2173	0.0430	0.4184	0.2277	-0.2036	0.0427	0.373	1.000	0.377	0.997
	8000	0.6746	0.2062	0.2250	0.0396	0.4855	0.2139	-0.2051	0.0398	0.260	1.000	0.460	0.998
	16000	0.6908	0.2023	0.2289	0.0384	0.5225	0.2070	-0.2005	0.0383	0.238	1.000	0.548	1.000
	32000	0.6795	0.2215	0.2367	0.0416	0.5684	0.2108	**-0.1974**	0.0386	0.180	1.000	0.663	**1.000**

[a] Interval of artificial hole $()_h$ and spike $()_s$ in simulated intermittent Gaussians. Hole data were added to existing data in spike interval.
[b] Number of standard normal variates in each simulated Gaussian (sample size).
[c] Averages in angle brackets $\langle\rangle$ based on total of 20,000 non-permuted fits (20 non-permuted fits per randomization test times 1000 tests).
[d] Statistical power for each fit coefficient based on proportion times each coefficient was significant among the 1000 tests.
Power calculations employed 220,000 permutation-based fits (20 non-permuted fits and 200 fits with permutation per test times 1000 tests).

Table 2: (cont'd).

Interval[a]	N^b	$\langle\beta_0\rangle^c$	$\langle\sigma_{\beta_0}\rangle$	$\langle\beta_1\rangle$	$\langle\sigma_{\beta_1}\rangle$	$\langle\beta_2\rangle$	$\langle\sigma_{\beta_2}\rangle$	$\langle\beta_3\rangle$	$\langle\sigma_{\beta_3}\rangle$	$P(\beta_0)^d$	$P(\beta_1)$	$P(\beta_2)$	$P(\beta_3)$
$(0.2\text{-}0.84)_h$	500	-0.0340	0.3669	0.4017	0.0838	0.2730	0.6358	-0.2756	0.1302	0.000	1.000	0.174	1.000
$(0.18\text{-}0.2)_s$	1000	-0.2038	0.4992	0.4452	0.1110	0.7156	0.8545	-0.3561	0.1739	0.000	1.000	0.401	1.000
	1500	-0.3956	0.5029	0.4900	0.1110	1.0951	0.8552	-0.4278	0.1749	0.001	1.000	0.612	1.000
	2000	-0.5100	0.4780	0.5153	0.1041	1.3164	0.7992	-0.4676	0.1623	0.000	1.000	0.725	1.000
	4000	-0.7414	0.2587	0.5693	0.0554	1.7832	0.3768	-0.5496	0.0770	0.000	1.000	0.964	1.000
	8000	-0.7410	0.1711	0.5729	0.0351	1.8679	0.1845	-0.5518	0.0377	0.000	1.000	0.998	1.000
	16000	-0.7276	0.1581	0.5743	0.0318	1.9082	0.1589	-0.5442	0.0324	0.000	1.000	1.000	1.000
	32000	-0.6976	0.1617	0.5721	0.0326	1.9112	0.1547	**-0.5294**	0.0321	0.000	1.000	1.000	**1.000**
$(1\text{-}1.2)_h$	500	0.3973	0.1247	0.0447	0.0142	0.1609	0.0879	-0.0377	0.0186	0.207	0.754	0.077	0.097
$(0.36\text{-}1)_s$	1000	0.5503	0.1461	0.0262	0.0173	0.1374	0.1272	-0.0331	0.0268	0.459	0.507	0.091	0.134
	1500	0.6061	0.1407	0.0198	0.0172	0.1040	0.1297	-0.0261	0.0273	0.552	0.489	0.059	0.137
	2000	0.6564	0.1299	0.0147	0.0159	0.0698	0.1191	-0.0188	0.0250	0.664	0.514	0.045	0.132
	4000	0.7505	0.0783	0.0061	0.0069	0.0086	0.0508	-0.0057	0.0106	0.832	0.643	0.003	0.144
	8000	0.7997	0.0559	0.0051	0.0023	0.0015	0.0108	-0.0037	0.0022	0.921	0.793	0.000	0.220
	16000	0.8456	0.0507	0.0051	0.0017	0.0027	0.0051	-0.0035	0.0009	0.972	0.885	0.001	0.270
	32000	0.8848	0.0475	0.0052	0.0016	0.0039	0.0046	**-0.0033**	0.0008	0.983	0.917	0.001	**0.248**
$(1\text{-}1.2)_h$	500	0.3856	0.1254	0.0463	0.0143	0.1537	0.0878	-0.0397	0.0188	0.233	0.793	0.084	0.163
$(0.76\text{-}1)_s$	1000	0.5216	0.1394	0.0301	0.0166	0.1429	0.1223	-0.0377	0.0259	0.542	0.665	0.178	0.284
	1500	0.5956	0.1376	0.0223	0.0170	0.0970	0.1272	-0.0280	0.0267	0.734	0.695	0.166	0.329
	2000	0.6477	0.1206	0.0167	0.0150	0.0597	0.1126	-0.0200	0.0236	0.836	0.742	0.104	0.324
	4000	0.7302	0.0757	0.0096	0.0068	0.0090	0.0486	-0.0087	0.0102	0.980	0.888	0.023	0.485
	8000	0.7844	0.0543	0.0085	0.0026	0.0040	0.0136	-0.0067	0.0027	0.998	0.968	0.015	0.648
	16000	0.8261	0.0521	0.0088	0.0022	0.0061	0.0093	-0.0063	0.0018	1.000	0.991	0.022	0.650
	32000	0.8624	0.0481	0.0091	0.0021	0.0086	0.0086	**-0.0060**	0.0016	1.000	0.995	0.038	**0.664**
$(1\text{-}1.2)_h$	500	0.3518	0.1040	0.0588	0.0130	0.1571	0.0618	-0.0473	0.0145	0.437	0.970	0.270	0.492
$(0.92\text{-}1)_s$	1000	0.4804	0.1186	0.0437	0.0144	0.1574	0.0903	-0.0475	0.0197	0.928	0.976	0.604	0.786
	1500	0.5425	0.1174	0.0379	0.0136	0.1297	0.0916	-0.0416	0.0196	0.985	0.991	0.531	0.894
	2000	0.5827	0.1110	0.0347	0.0130	0.1087	0.0856	-0.0370	0.0185	0.998	0.995	0.507	0.914
	4000	0.6531	0.0762	0.0300	0.0090	0.0754	0.0519	-0.0292	0.0110	1.000	1.000	0.482	0.982
	8000	0.7001	0.0598	0.0299	0.0064	0.0743	0.0310	-0.0280	0.0062	1.000	1.000	0.607	0.994
	16000	0.7402	0.0550	0.0304	0.0056	0.0787	0.0275	-0.0277	0.0053	1.000	1.000	0.709	1.000
	32000	0.7741	0.0499	0.0312	0.0050	0.0821	0.0252	**-0.0270**	0.0048	1.000	1.000	0.785	**1.000**

Table 2: (cont'd).

Interval[a]	N^b	$\langle\beta_0\rangle^c$	$\langle\sigma_{\beta_0}\rangle$	$\langle\beta_1\rangle$	$\langle\sigma_{\beta_1}\rangle$	$\langle\beta_2\rangle$	$\langle\sigma_{\beta_2}\rangle$	$\langle\beta_3\rangle$	$\langle\sigma_{\beta_3}\rangle$	P(β_0)[d]	P(β_1)	P(β_2)	P(β_3)
$(1\text{-}1.2)_h$	500	0.2511	0.0894	0.0865	0.0217	0.1590	0.0947	-0.0541	0.0212	0.242	1.000	0.272	0.713
$(0.98\text{-}1)_s$	1000	0.3316	0.0846	0.0813	0.0200	0.2345	0.0966	-0.0705	0.0220	0.706	1.000	0.878	1.000
	1500	0.3595	0.0825	0.0821	0.0190	0.2609	0.0908	-0.0756	0.0208	0.779	1.000	0.975	1.000
	2000	0.3709	0.0867	0.0842	0.0180	0.2827	0.0860	-0.0799	0.0196	0.764	1.000	0.987	1.000
	4000	0.4043	0.0810	0.0863	0.0139	0.3095	0.0605	-0.0847	0.0143	0.825	1.000	1.000	1.000
	8000	0.4559	0.0765	0.0858	0.0105	0.3140	0.0457	-0.0842	0.0105	0.935	1.000	1.000	1.000
	16000	0.4990	0.0654	0.0851	0.0087	0.3140	0.0389	-0.0827	0.0087	0.980	1.000	1.000	1.000
	32000	0.5362	0.0616	0.0843	0.0079	0.3112	0.0360	**-0.0807**	0.0078	0.993	1.000	1.000	**1.000**
$(2\text{-}2.64)_h$	500	0.3690	0.1315	0.0366	0.0140	0.1488	0.0838	-0.0296	0.0182	0.117	0.613	0.029	0.027
$(1.36\text{-}2)_s$	1000	0.5061	0.1396	0.0217	0.0162	0.1431	0.1170	-0.0298	0.0248	0.303	0.272	0.058	0.055
	1500	0.5742	0.1380	0.0149	0.0166	0.1019	0.1219	-0.0214	0.0258	0.419	0.207	0.032	0.042
	2000	0.6220	0.1269	0.0100	0.0149	0.0683	0.1132	-0.0145	0.0237	0.448	0.166	0.011	0.016
	4000	0.7129	0.0792	0.0019	0.0070	0.0090	0.0512	-0.0023	0.0107	0.384	0.120	0.000	0.005
	8000	0.7691	0.0565	0.0007	0.0017	-0.0015	0.0049	-0.0001	0.0010	0.264	0.120	0.000	0.001
	16000	0.8131	0.0523	0.0007	0.0015	-0.0012	0.0027	-0.0001	0.0005	0.192	0.132	0.000	0.000
	32000	0.8532	0.0483	0.0007	0.0014	-0.0008	0.0017	**-0.0002**	0.0003	0.150	0.136	0.000	**0.000**
$(2\text{-}2.64)_h$	500	0.3681	0.1320	0.0375	0.0139	0.1484	0.0838	-0.0298	0.0183	0.131	0.632	0.037	0.037
$(1.76\text{-}2)_s$	1000	0.5099	0.1401	0.0212	0.0160	0.1358	0.1169	-0.0287	0.0247	0.329	0.288	0.051	0.059
	1500	0.5696	0.1350	0.0151	0.0163	0.1010	0.1225	-0.0216	0.0257	0.387	0.214	0.024	0.034
	2000	0.6203	0.1262	0.0102	0.0149	0.0655	0.1113	-0.0144	0.0234	0.489	0.197	0.010	0.027
	4000	0.7107	0.0780	0.0024	0.0070	0.0090	0.0518	-0.0026	0.0108	0.430	0.155	0.001	0.007
	8000	0.7621	0.0562	0.0012	0.0020	-0.0009	0.0095	-0.0006	0.0020	0.319	0.173	0.000	0.000
	16000	0.8057	0.0500	0.0012	0.0015	-0.0008	0.0030	-0.0006	0.0005	0.265	0.201	0.000	0.000
	32000	0.8463	0.0478	0.0013	0.0014	-0.0003	0.0019	**-0.0005**	0.0003	0.188	0.198	0.000	**0.000**
$(2\text{-}2.64)_h$	500	0.3563	0.1282	0.0398	0.0130	0.1506	0.0768	-0.0314	0.0168	0.127	0.707	0.043	0.040
$(1.92\text{-}2)_s$	1000	0.5059	0.1385	0.0235	0.0155	0.1365	0.1121	-0.0302	0.0236	0.396	0.449	0.103	0.121
	1500	0.5677	0.1345	0.0177	0.0155	0.1049	0.1155	-0.0238	0.0244	0.471	0.435	0.072	0.120
	2000	0.6161	0.1225	0.0128	0.0140	0.0705	0.1047	-0.0167	0.0220	0.559	0.465	0.053	0.129
	4000	0.7002	0.0783	0.0061	0.0066	0.0192	0.0473	-0.0061	0.0099	0.701	0.645	0.035	0.213
	8000	0.7521	0.0569	0.0048	0.0019	0.0100	0.0067	-0.0041	0.0014	0.731	0.743	0.018	0.304
	16000	0.7927	0.0509	0.0049	0.0017	0.0111	0.0050	-0.0042	0.0010	0.833	0.861	0.032	0.443
	32000	0.8373	0.0498	0.0050	0.0016	0.0120	0.0041	**-0.0041**	0.0008	0.897	0.903	0.071	**0.533**

Table 2: (cont'd).

Interval[a]	N^b	$\langle\beta_0\rangle^c$	$\langle\sigma_{\beta_0}\rangle$	$\langle\beta_1\rangle$	$\langle\sigma_{\beta_1}\rangle$	$\langle\beta_2\rangle$	$\langle\sigma_{\beta_2}\rangle$	$\langle\beta_3\rangle$	$\langle\sigma_{\beta_3}\rangle$	P(β_0)[d]	P(β_1)	P(β_2)	P(β_3)
$(2\text{-}2.64)_h$	500	0.3408	0.1183	0.0452	0.0128	0.1492	0.0631	-0.0327	0.0144	0.210	0.838	0.117	0.127
$(1.98\text{-}2)_s$	1000	0.4720	0.1233	0.0314	0.0136	0.1518	0.0854	-0.0350	0.0186	0.720	0.846	0.450	0.513
	1500	0.5277	0.1203	0.0271	0.0132	0.1317	0.0892	-0.0311	0.0191	0.852	0.912	0.561	0.682
	2000	0.5689	0.1113	0.0231	0.0122	0.1074	0.0831	-0.0260	0.0178	0.939	0.961	0.634	0.792
	4000	0.6468	0.0777	0.0177	0.0067	0.0672	0.0406	-0.0176	0.0088	0.994	0.999	0.893	0.992
	8000	0.7027	0.0596	0.0165	0.0037	0.0597	0.0155	-0.0158	0.0036	1.000	1.000	0.996	1.000
	16000	0.7473	0.0545	0.0161	0.0025	0.0584	0.0086	-0.0154	0.0021	1.000	1.000	1.000	1.000
	32000	0.7888	0.0508	0.0159	0.0021	0.0580	0.0071	**-0.0150**	0.0017	1.000	1.000	1.000	**1.000**
Null	500	0.3584	0.1195	0.0389	0.0137	0.1471	0.0809	-0.02935	0.0174	0.120	0.694	0.040	0.030
	1000	0.4967	0.1332	0.0219	0.0159	0.1387	0.1152	-0.02867	0.0243	0.320	0.310	0.054	0.062
	1500	0.5694	0.1365	0.0143	0.0163	0.0965	0.1205	-0.02010	0.0253	0.452	0.216	0.031	0.038
	2000	0.6239	0.1197	0.0080	0.0138	0.0533	0.1042	-0.01126	0.0218	0.504	0.155	0.015	0.019
	4000	0.7026	0.0787	0.0018	0.0069	0.0084	0.0502	-0.00200	0.0105	0.431	0.098	0.001	0.007
	8000	0.7591	0.0571	0.0003	0.0016	-0.0019	0.0048	0.00009	0.0009	0.261	0.085	0.000	0.000
	16000	0.8033	0.0511	0.0004	0.0015	-0.0016	0.0028	0.00003	0.0005	0.194	0.120	0.000	0.000
	32000	0.8434	0.0496	0.0004	0.0014	-0.0016	0.0018	**0.00005**	0.0002	0.141	0.103	0.000	**0.000**

Table 3: Sorted values of statistical power $P(\beta_3)$ and averages of coefficients from Table 2 for distributions with N=32000.

$P(\beta_3)$	$\langle\beta_0\rangle$	$\langle\beta_1\rangle$	$\langle\beta_2\rangle$	$\langle\beta_3\rangle$
0.000	0.8463	0.0013	-0.0003	-0.0005
0.000	0.8532	0.0007	-0.0008	-0.0002
0.248	0.8848	0.0052	0.0039	-0.0033
0.533	0.8373	0.0050	0.0120	-0.0041
0.664	0.8624	0.0091	0.0086	-0.0060
0.763	1.2085	0.0181	0.0150	-0.0113
0.828	1.2571	0.0519	0.0385	-0.0319
1.000	-0.6976	0.5721	1.9112	-0.5294
1.000	0.6795	0.2367	0.5684	-0.1974
1.000	0.5362	0.0843	0.3112	-0.0807
1.000	0.7741	0.0312	0.0821	-0.0270
1.000	0.7888	0.0159	0.0580	-0.0150

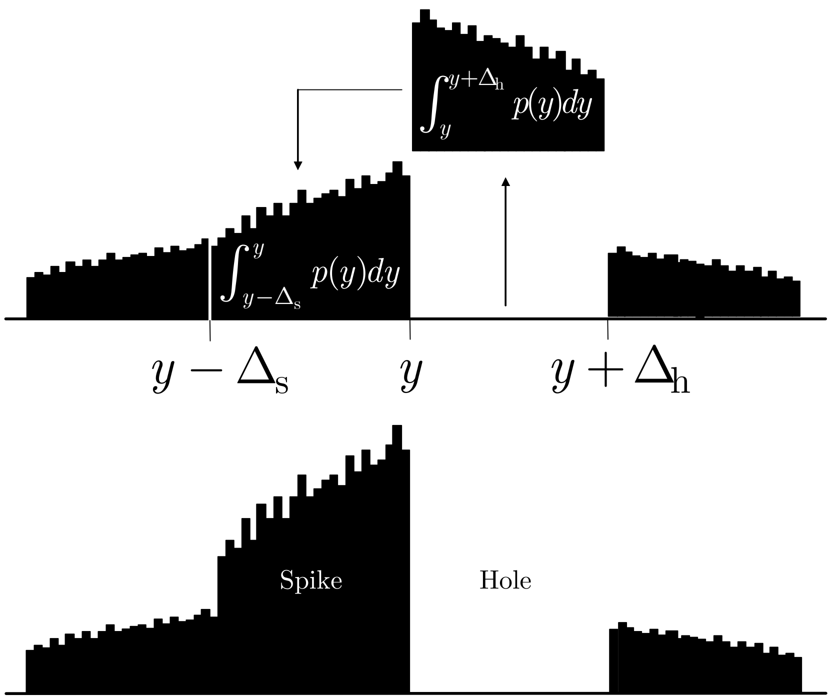


Figure 1: Artificially-induced hole and spike in frequency distribution caused by moving data between y and $y + \Delta_h$ and adding it to existing data between y and $y - \Delta_s$. Δ_h is the width of the hole and Δ_s is the width of the spike formed by adding hole data.

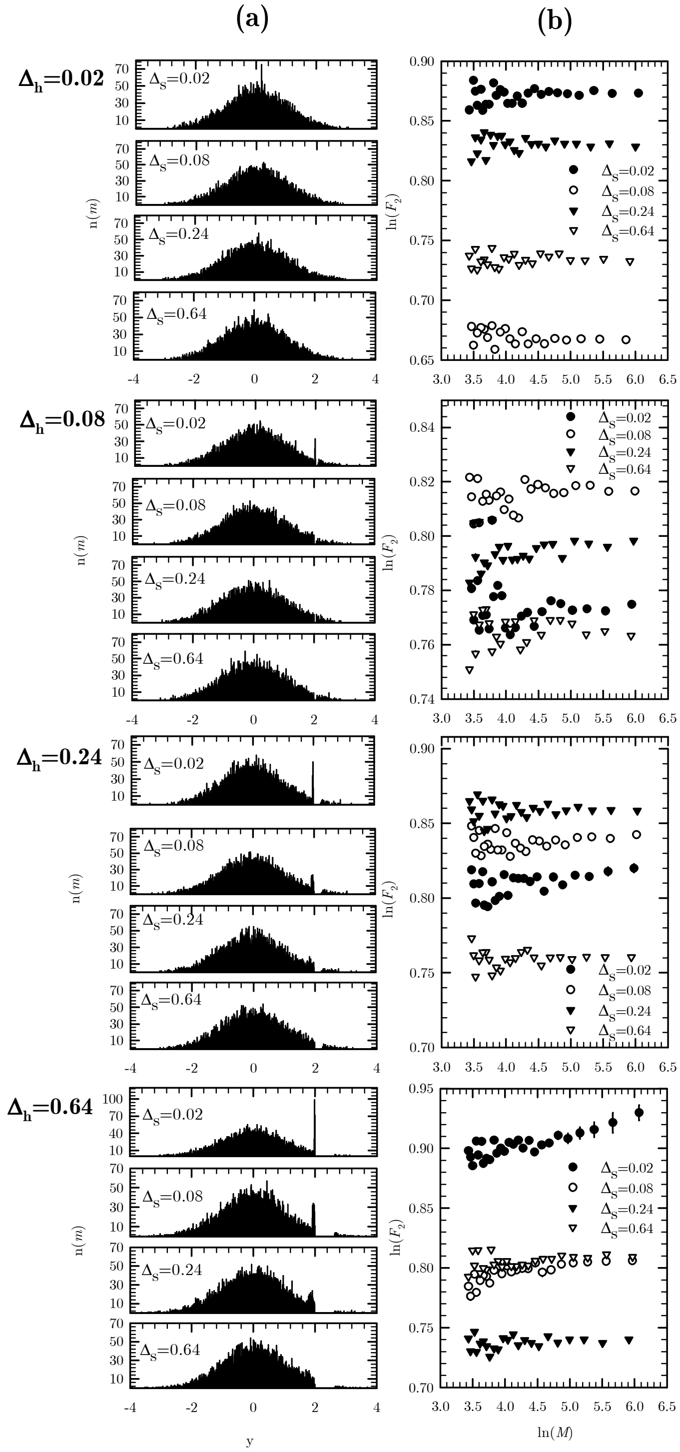


Figure 2: Scaled factorial moments (F_2) resulting from an artificially induced hole of width Δ_h beginning at y=2.0 in a frequency histogram of 10,000 standard normal variates. (a) Bin counts $n(m)$ when bin width $\delta y = \epsilon = 0.01$, Δ_s is the interval width into which bin counts from the hole were randomly distributed during replacement. (b) Characteristics of scaled factorial moments (F_2) as a function of total bins (M) based on varying δy.

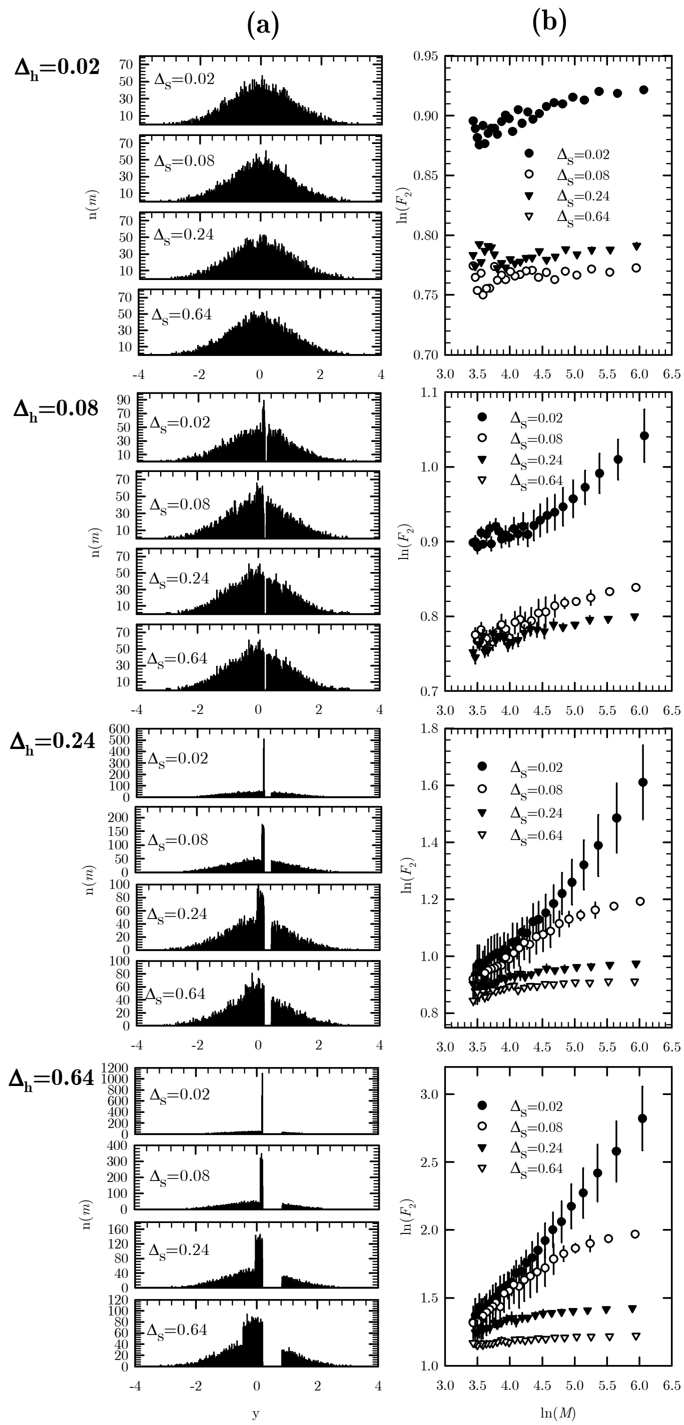


Figure 3: Scaled factorial moments (F_2) resulting from an artificially induced hole of width Δ_h beginning at y=0.2 in a frequency histogram of 10,000 standard normal variates. (a) Bin counts $n(m)$ when bin width $\delta y = \epsilon = 0.01$, Δ_s is the interval width into which bin counts from the hole were randomly distributed during replacement. (b) Characteristics of scaled factorial moments (F_2) as a function of total bins (M) based on varying δy.

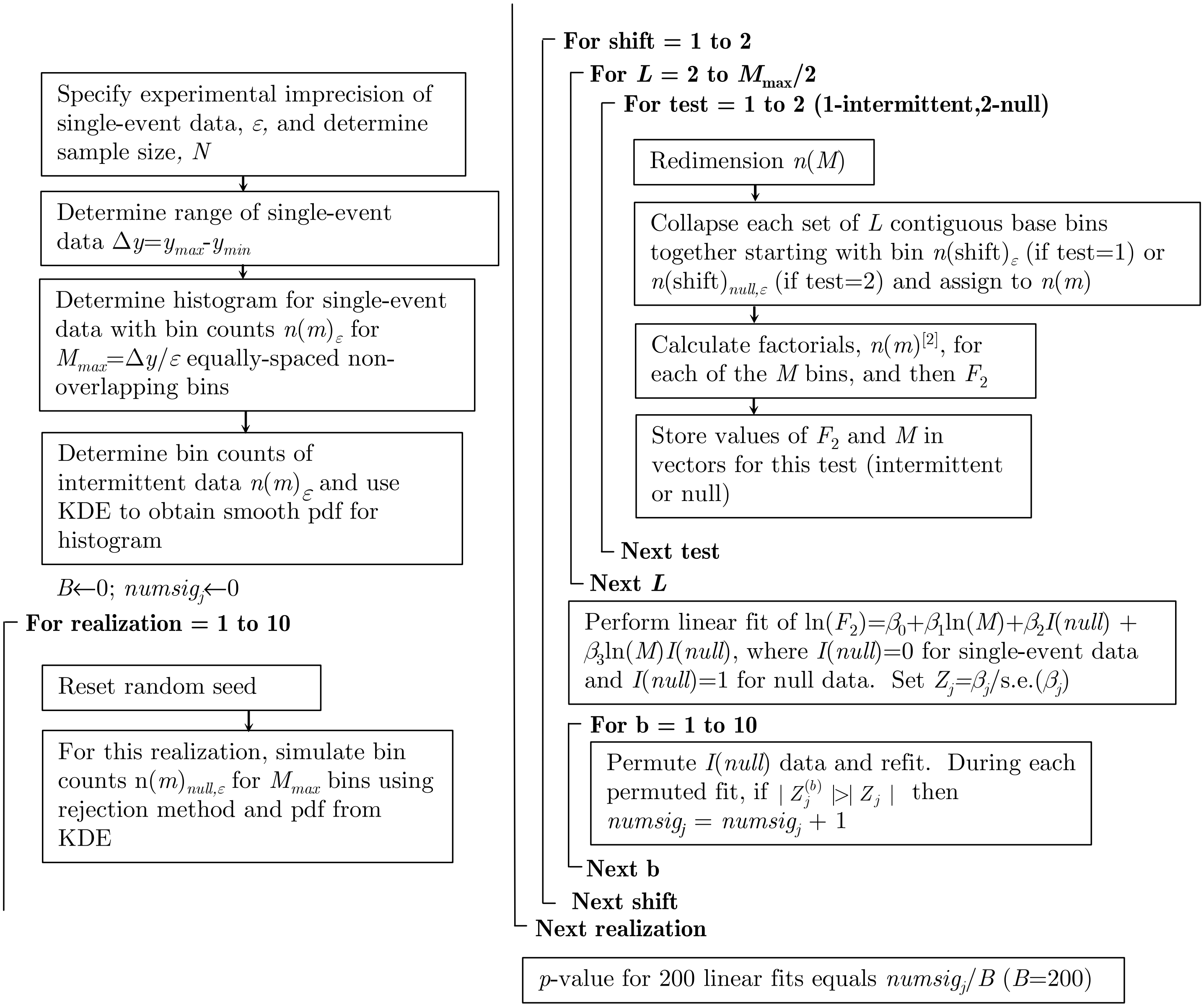
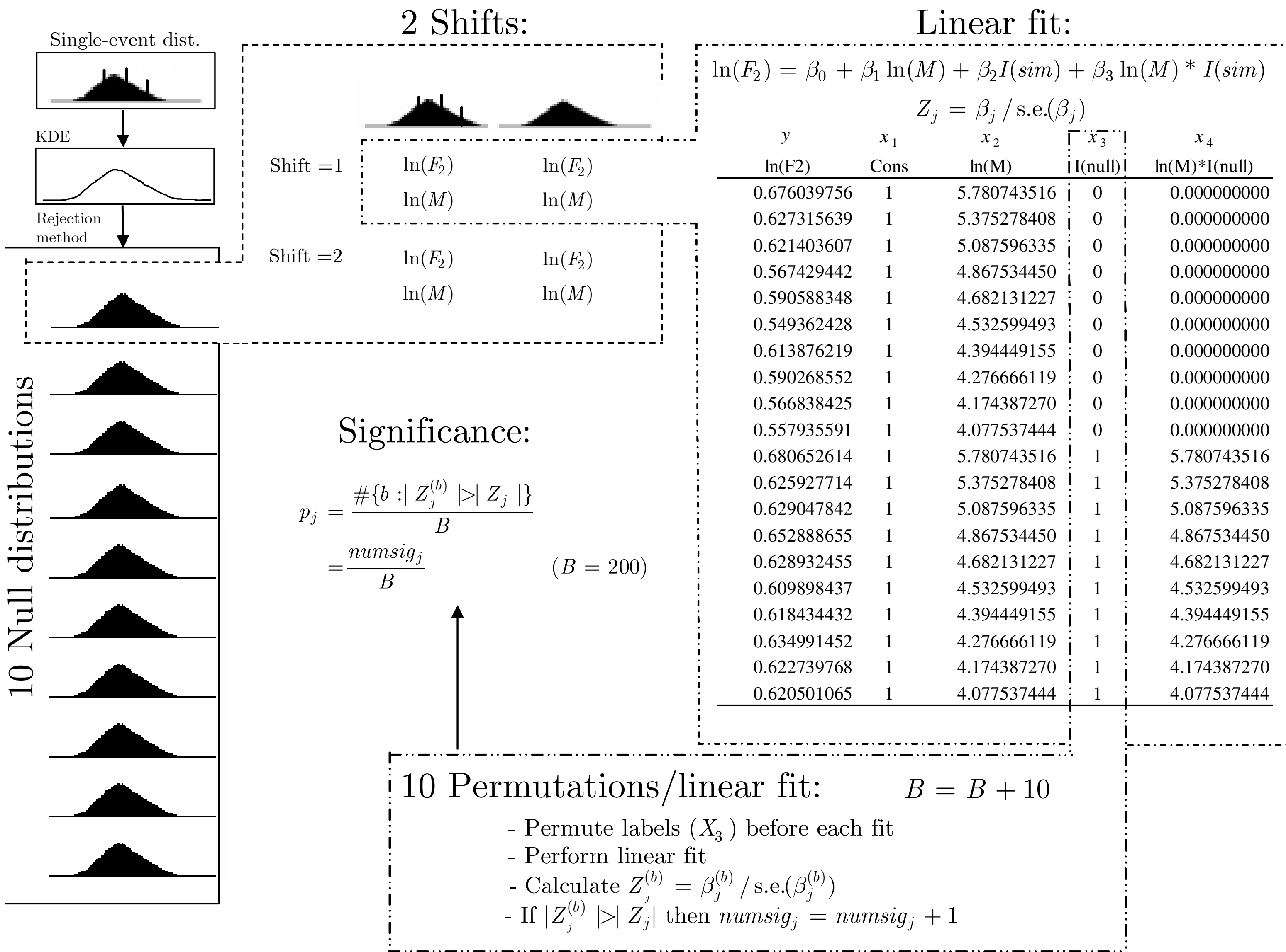


Figure 4: Algorithm flow for a single randomization test. The result of a randomization test is the number of times (i.e., $\texttt{numsig}_j$) $|Z_j^{(b)}| > |Z_j|$ for each coefficient.



y	x_1	x_2	x_3	x_4
ln(F2)	Cons	ln(M)	I(null)	ln(M)*I(null)
0.676039756	1	5.780743516	0	0.000000000
0.627315639	1	5.375278408	0	0.000000000
0.621403607	1	5.087596335	0	0.000000000
0.567429442	1	4.867534450	0	0.000000000
0.590588348	1	4.682131227	0	0.000000000
0.549362428	1	4.532599493	0	0.000000000
0.613876219	1	4.394449155	0	0.000000000
0.590268552	1	4.276666119	0	0.000000000
0.566838425	1	4.174387270	0	0.000000000
0.557935591	1	4.077537444	0	0.000000000
0.680652614	1	5.780743516	1	5.780743516
0.625927714	1	5.375278408	1	5.375278408
0.629047842	1	5.087596335	1	5.087596335
0.652888655	1	4.867534450	1	4.867534450
0.628932455	1	4.682131227	1	4.682131227
0.609898437	1	4.532599493	1	4.532599493
0.618434432	1	4.394449155	1	4.394449155
0.634991452	1	4.276666119	1	4.276666119
0.622739768	1	4.174387270	1	4.174387270
0.620501065	1	4.077537444	1	4.077537444

Figure 5: Detailed schematic showing complete methodology for a single randomization test. Each test includes analysis for each of the 10 null distributions.

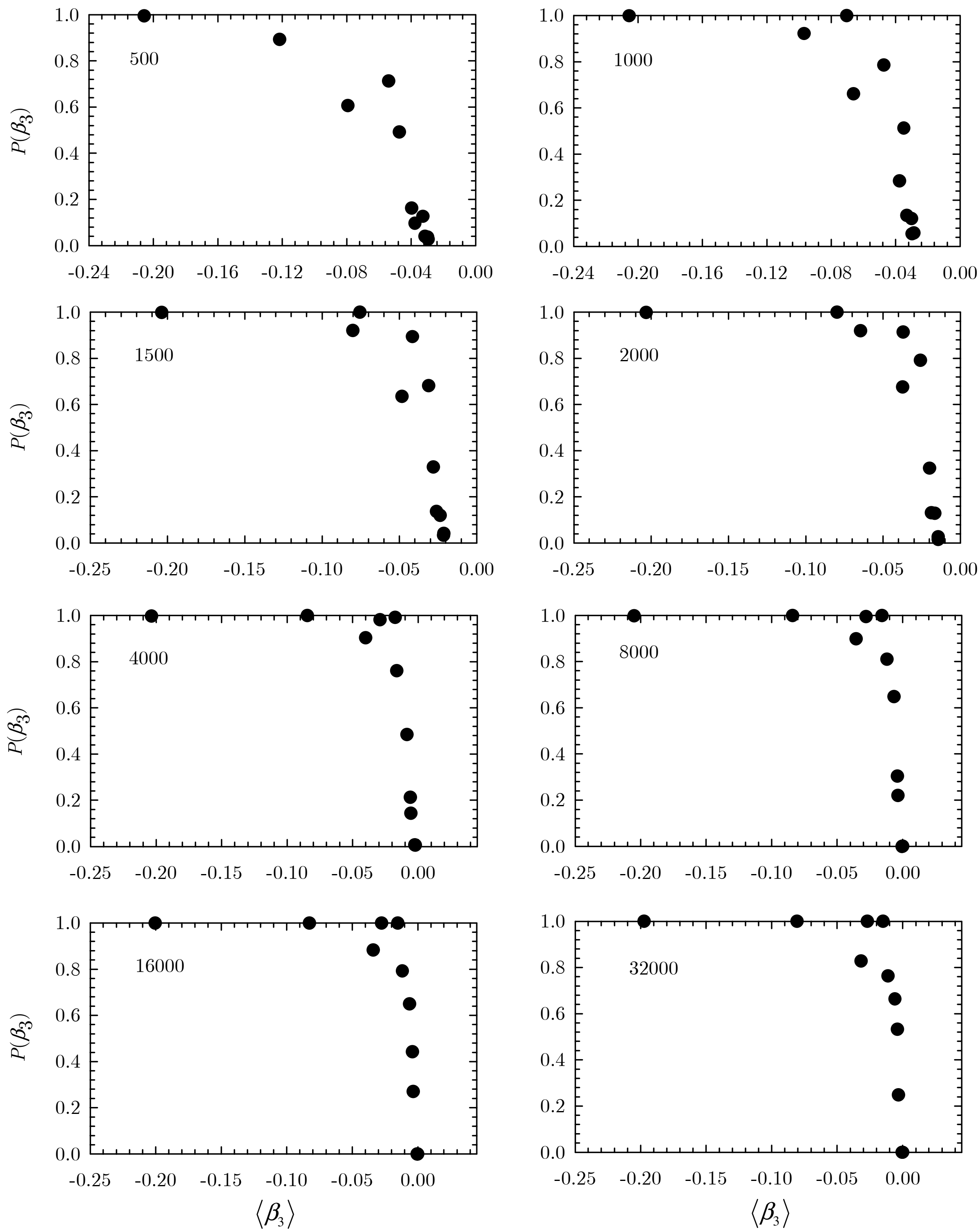


Figure 6: Statistical power $P(\beta_3)$ to detect a significant fit coefficient β_3 as a function of average effect size $\langle\beta_3\rangle$ and sample size of simulated intermittent Gaussian distribution.