Ground State of the Massless Nelson Model in a non-Fock Representation

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Abstract

We consider a model of a particle coupled to a massless scalar field (the massless Nelson model) in a non-Fock representation. We prove the existence of a ground state of the system, applying the mothod of Griesemer, Lieb and Loss.

Key words: Nelson model; ground state.

1 Introduction

The Nelson model is a quantum mechanical model which describes an interaction between some quantum mechanical particles and a Bose field. In this paper, we present a criterion for a Nelson model to have a ground state.

We consider one particle under the influence of an external potential V and coupled to a scalar Bose field. The Hilbert space of the system is given by

$$\mathcal{F} := L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\rm b}(L^2(\mathbb{R}^3)), \tag{1}$$

where $\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3))$ is the Boson Fock space over $L^2(\mathbb{R}^3)$. The standard Nelson Hamiltonian is of the form

$$H_m^V := (-\triangle + V) \otimes \mathbb{1} + \mathbb{1} \otimes H_f(m) + \lambda \phi^{\oplus}(v), \quad \text{on } \mathcal{F},$$

where $\mathbb{1}$ denotes identity, $\triangle$ is the generalized Laplacian on $L^2(\mathbb{R}^3)$, $\lambda \in \mathbb{R}$ is a coupling constant, and $H_f(m)$ and $\phi^{\oplus}(v)$ are defined by

$$H_f(m) := \int_{\mathbb{R}^3} \omega_m(k) a(k)^* a(k) \mathrm{d}k,$$

$$\phi^{\oplus}(v) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} (v(x,k) \otimes a(k)^* + v(x,k)^* \otimes a(k)) \, \mathrm{d}k,$$

with

$$\omega_m(k) := \sqrt{k^2 + m^2}, \quad v(x,k) := \frac{1}{\sqrt{(2\pi)^3}} \frac{\hat{\rho}(k)}{|k|^{1/2}} e^{-ikx},$$

where $|k|^{-1/2}\hat{\rho} \in \mathrm{Dom}(\omega_m^{-1/2})$ and $a(k)^*$, $a(k)$ are the distribution kernels of the creation and annihilation operators on $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3))$ (Dom(A) means the domain of operator A). The problem on the ground state of H_m^V can be classified as follows:

(i) the massive case : $m > 0$

(ii) the massless case: $m = 0$ $\begin{cases} |k|^{-1/2}\hat{\rho} \in \mathrm{Dom}(\omega_0^{-1}) : \text{infrared regular} \\ |k|^{-1/2}\hat{\rho} \notin \mathrm{Dom}(\omega_0^{-1}) : \text{infrared singular.} \end{cases}$

In almost all cases, to prove existence of a ground state for the massive case is easy. The first result on the ground state problem, to our knowledge, is due to Spohn [12]. In [12] he proved existence of a ground state in the case where the infrared regular(I.R.) condition holds and $(-\triangle + V + i)^{-1}$ is compact. If $(-\triangle + V + i)^{-1}$ is not compact, his theorem shows that a ground state exists if the I.R. condition holds and the coupling constant λ is small enough. After the work of Spohn [12], C. Gérard proved existence of a ground state of an extended model of the Nelson model in the case where an abstract particle Hamiiltonian K (which corresponds to $-\triangle + V$ in the above context) is compact and an I.R. like condition holds [3]. On the other hand, J. Lörinczi, R. A. Minlos and H. Spohn [7] showed that H_0^V has *no* ground state if the infrared singular(I.S.) condition holds in spite of the condition $V(x) > C|x|^\alpha (C, \alpha > 0)$ (also refer to [2] about the absence of ground states). Recently, H. Hirokawa, F. Hiroshima and H. Spohn [5] prove existence of a ground state for the renormalized Nelson model.

In the case where the I.S. condition holds, H_0^V may not has a ground state [7], but A. Arai [1] showed that a massless Nelson model in *a non-Fock representation* has a ground state.

We work with the non-Fock representation introduced in [1]. In this representation the massless Nelson model we consider is of the form :

$$\widetilde{H}^V := (-\triangle + V) \otimes \mathbb{1} + \mathbb{1} \otimes H_f(0) + \lambda\phi^{\oplus}(G) - \lambda^2 \mathcal{V}(\hat{x}) \otimes \mathbb{1} + \lambda^2 \mathcal{W}\mathbb{1},$$

where $\mathcal{V}(\hat{x})$ is the multiplication operator by $\mathcal{V}(x) := \mathrm{Re}\langle |k|^{-1/2}v(0), |k|^{-1/2}v(x)\rangle$, $\mathcal{W} := \| |k|^{-1/2}v(0)\|^2$ is a constant, and $G(x,k) := v(x,k) - v(0,k)$. If $m = 0$ and the I.R. condition holds, $\widetilde{H}^V$ is unitarily equivalent to H_0^V(Proposition 2.1). But if the I.S. condition holds, $\widetilde{H}^V$ may not be unitarily equivalent to H_0^V. If the I.S. condition holds, to consider $\widetilde{H}^V$ means to choose a non-Fock representation of the canonical commutation relations of a, a^*(see [1]). Note that, in the massless case $m = 0$, the Hamiltonian we consider is $\widetilde{H}^V$, not H_0^V.

For the non-Fock Hamiltonian $\widetilde{H}^V$, we present a criterion for $\widetilde{H}^V$ to have a ground state. The criterion is essentially the same condition as in [4], and we prove existence of a ground state without assuming the I.R. condition. Out strategy is the same as that of [4]. We, however, improved the proof of the photon derivative bound. In the proof of photon derivative bound in [4], it is difficult to prove that the integer-valued k-dependent sequence $h_l(k)$ is measurable. In our new proof of the photon derivative bound, such uncertain sequence does not appear.

This paper is organized as follows. In Sec. 2 we describe rigorous definitions of our system and state main results. In Sec. 3, we prove the main theorem. In Appendix A, we establish a formula which expresses a second quantization operator by the annihilation operators.

2 Notation and Main Results

We consider a model of one particle interacting with a scalar Bose field, and in an external potential $V : \mathbb{R}^3_x \to \mathbb{R}$ satisfying $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3_x)$. The Hilbert space for the model is given by $\mathcal{F} := L^2(\mathbb{R}^3_x) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3_k))$, where $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3_k))$ is the Boson Fock space over $L^2(\mathbb{R}^3_k)$ (see [9]). For $m \geq 0$ we define a function $\omega_m : \mathbb{R}^3_k \to \mathbb{R}$ by $\omega_m(k) := \sqrt{k^2 + m^2}$. The multiplication operator by ω_m is denoted by the same symbol. The free Hamiltonian of the scalarBose field is the second quantization of ω_m([9]):

$$H_f(m) := \mathrm{d}\Gamma_{\mathrm{b}}(\omega_m). \tag{2}$$

We set $V_\pm(x) := \max\{0, \pm V(x)\}$. Throughout this paper, we assume that the potential V has the following properties:

[N.1] There exist constants $a < 1$ and $b \in \mathbb{R}$ such that

$$\|V_-^{1/2}\psi\|^2 \le a\|(-\triangle)\psi\|^2 + b\|\psi\|^2, \quad \psi \in C_0^\infty(\mathbb{R}_x^3).$$

The particle Hamiltonian $H_{\rm p}$ is a self-adjoint operator defined by

$$H_{\rm p} := -\triangle \dot{+} V, \quad \text{on } L^2(\mathbb{R}_x^3),$$

where $\dot{+}$ means the form sum. For $f \in L^2(\mathbb{R}_k^3)$ we denote by $a(f)^*$, $a(f)$, the creation and annihilation operators respectively, by $\Phi_{\rm S}(f) := \overline{[a(f) + a(f)^*]}/\sqrt{2}$ the Segal field operators (" $-$ " means closure). It is well known that $\Phi_{\rm S}(f)$ is a self-adjoint operator on $\mathcal{F}_{\rm b}(L^2(\mathbb{R}_k^3))$(see [10]). For $x \in \mathbb{R}_x^3$ and $\hat{\rho} \in L^2(\mathbb{R}_k^3) \cap \mathrm{Dom}(|k|^{-1/2})$ we define $v(x) \in L^2(\mathbb{R}_k^3)$ by

$$v(x)(k) := v(x,k) := \frac{1}{(2\pi)^{3/2}} \frac{\hat{\rho}(k)}{|k|^{1/2}} e^{-ikx}, \quad k \in \mathbb{R}_k^3.$$

The Hilbert space $\mathcal{F}$ can be identified with the fibre direct integral of $\mathcal{F}_{\rm b}(L^2(\mathbb{R}_k^3))$(see [11]):

$$\mathcal{F} = \int_{\mathbb{R}_x^3}^{\oplus} \mathcal{F}_{\rm b}(L^2(\mathbb{R}_k^3))\mathrm{d}x.$$

In this identification the opeartor

$$\phi^{\oplus}(v) := \int_{\mathbb{R}_x^3}^{\oplus} \Phi_{\rm S}(v(x))\mathrm{d}x$$

gives a self-adjoint operator on $\mathcal{F}$ ([11]).

The Hamiltonian of the standard Nelson model is defined by

$$H_m^V := H_{\rm p} \otimes 1\!\!1 + 1\!\!1 \otimes H_f(m) + \lambda\phi^{\oplus}(v).$$

Here $\lambda \in \mathbb{R}$ is a coupling constant. We set

$$H_0 := H_{\rm p} \otimes 1\!\!1 + 1\!\!1 \otimes H_f(m),$$

the free Hamiltonian of the Nelson model. By [N.1], $H_{\rm p}$ is bounded below. Therefore H_0 is self-adjoint on $D(H_0) = D(H_{\rm p} \otimes 1\!\!1) \cap D(1\!\!1 \otimes H_f(m))$ and bounded below.

The following fact is well-known:

Proposition 2.1. *Assume* $|k|^{-1/2}\hat{\rho} \in \mathrm{Dom}(\omega_m^{1/2})$ *and [N.1]. Then* H_m^V *is self-adjoint on* $\mathrm{Dom}(H_0)$ *and bounded below. Moreover* H_m^V *is essentially self-adjoint on each core for* H_0*.*

Under the assumption of Proposition 2.1, we set

$$E^V(m) := \inf \sigma(H_m^V),$$

the ground state energy of H_m^V. Where $\sigma(H_m^V)$ means the spectrum of H_m^V. If $E^V(m)$ is an eigenvalue of H_m^V, we say that H_m^V has a ground state and a eigenvector $\Phi_m \in \ker(H_m^V - E^V(m))\backslash\{0\}$ is called a ground state of H_m^V.

Let $\theta \in C_0^\infty(\mathbb{R}_x^3)$, $\widetilde{\theta} \in C^\infty(\mathbb{R}_x^3)$ be functions which satisfy the following properties (i), (ii):

$$\text{(i)} \qquad 0 \leq \theta(x), \widetilde{\theta}(x) \leq 1, \quad \theta(x)^2 + \widetilde{\theta}(x)^2 = 1, \quad (x \in \mathbb{R}_x^3).$$

$$\text{(ii)} \qquad \theta(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2. \end{cases}$$

For $R > 0$ we define particle cut-off functions θ_R, $\widetilde{\theta}_R$ as follows:

$$\theta_R(x) := \theta(x/R), \quad \widetilde{\theta}_R(x) := \widetilde{\theta}(x/R).$$

We abbreviate $\theta_R \otimes \mathbb{1}$, $\widetilde{\theta}_R \otimes \mathbb{1}$ to θ_R, $\widetilde{\theta}_R$, respectively if there is no danger of confusion. For a self-adjoint operator T, we denote by $Q(T)$ the form domain of T , and for $\Psi, \Phi \in Q(T)$, we write simply $\langle \Psi, T\Phi \rangle = \int_{\mathbb{R}} \mu \mathrm{d}\langle \Psi, E_T(\mu)\Phi \rangle$, where E_T means the spectral measure of T.

We define a quantity which physically means the minimal energy in the states where the particle is separated more than R away from the origin:

Definition 2.2.

$$E_\infty(R, m) := \inf_{\substack{\Psi \in Q(H_m^V) \\ \|\widetilde{\theta}_R \Psi\| \neq 0}} \frac{\langle \widetilde{\theta}_R \Psi, H_m^V \widetilde{\theta}_R \Psi \rangle}{\langle \Psi, \widetilde{\theta}_R^2 \Psi \rangle}.$$

Remark. For all $R > 0$, it is easy to see that $E^V(m) - E_\infty(R, m) \leq 0$.

The following condition is based on [4]:

Hypothesis I(binding condition for $m > 0$)

$$E^V(m) < \limsup_{R \to \infty} E_\infty(R, m).$$

Theorem 2.3 (Existence of ground state ($m > 0$))**.** *Let $m > 0$. Assume [N.1] and Hypothesis I. Then H_m^V has a ground state.*

Proof. This is done in the same method as in the proof of [4, Theorem 4.1]. Therefore we omit the proof. ∎

In the case $m = 0$, we need more assumptions:

[N.2] $\hat{\rho}/|k| \in L^2(\mathbb{R}^3_k)$.

Under the condition [N.1] and [N.2], the Hamiltonian of the massless Nelson model we consider is:

$$\widetilde{H}^V := H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_f(0) + \lambda\phi^{\oplus}(G) - \lambda^2 \mathcal{V}(\hat{x}) \otimes \mathbb{1} + \lambda^2 \mathcal{W}\mathbb{1},$$

where $\mathcal{W} := \|\omega_0^{-1/2} v(0)\|^2$ is a constant and $\mathcal{V}(\hat{x})$ is the multiplication operator by the function $\mathcal{V}(x) := \mathrm{Re}\langle \omega_0^{-1/2} v(0), \omega_0^{-1/2} v(x)\rangle$.

By [N.2], $\mathcal{V}(x)$ is uniformly continuous and $\lim_{|x|\to 0} \mathcal{V}(x) = 0$. The relation between $\widetilde{H}^V$ and H_0^V is given by the following proposition:

Proposition 2.4. *Suppose that the infrared regular condition $\hat{\rho}/|k|^{3/2} \in L^2(\mathbb{R}^3_k)$ holds. Then $\widetilde{H}^V$ is unitarily equivalent to H_0^V.*

Proof. By the assumption, the operator $T := \exp[-i\lambda\mathbb{1} \otimes \Phi_{\mathrm{S}}(i|k|^{-1}v(0))]$ is a unitary operator on $\mathcal{F}$ and H_m^V is unitarily equivalent to $\widetilde{H}^V = TH_m^V T^*$. ∎

If the infrared singular condition $\hat{\rho}/|k|^{3/2} \notin L^2(\mathbb{R}^3_k)$ holds, this Hamiltonian $\widetilde{H}^V$ gives a Nelson Hamiltonian in a non-Fock representation (see [1]).

For the existence of ground states of $\widetilde{H}^V$, we impose some conditions on $\hat{\rho}$:

[N.3] There exists an open set $S \subset \mathbb{R}^3$, such that $\mathrm{supp}\,\hat{\rho} = \bar{S}$. Moreover, for all $n \in \mathbb{N}$

$$S_n := \{k \in S \,|\, |k| < n\}$$

has the cone-property(see [6]).

[N.4] There exists a function $\eta \in H^1(\mathbb{R}^3_k)$, such that $\hat{\rho} = \chi_S \eta$, where χ_S is the characteristic function of S.

[N.5] $\hat{\rho}$ is continuously differentiable in $S\backslash\{0\}$.

[N.6] $|k|^{-3/2}\hat{\rho}$, $|k|^{-1/2}|\nabla\hat{\rho}| \in L^p(S)$ for all p, $1 < p < 2$.

Under the condition [N.1] and [N.2], it is easy to see that $E^V(0) = \inf \sigma(\widetilde{H}^V)$. One of the most important conditions for the existence of ground states of $\widetilde{H}^V$ is

Hypothesis II(binding condition for $m = 0$)

$$E^V(0) < \limsup_{R \to \infty} E_\infty(R, 0). \tag{3}$$

Now we state the main result of this paper.

Theorem 2.5 (Existence of ground state ($m = 0$))**.** *Assume [N.1]-[N.6] and Hypothesis II. Then the massless Nelson Hamiltonian $\widetilde{H}^V$ has a ground state.*

Remark. In the case $\lim_{|x|\to\infty} V(x) = \infty$, it is easy to see that $\lim_{R\to\infty} E_\infty(R, m) = \infty$. Therefore Hypothesis II holds. On the other hand, if $\lim_{|x|\to\infty} V(x) \to 0$ and the particle Hamiltonian H_{p} has negative energy ground states, then Hypothesis I, II holds (see [4, Theorem 3.1]).

Remark. Let $\Lambda > 0$. Then $\hat{\rho} = \chi_\Lambda$ (the characteristic function of the region $|k| < \Lambda$) satisfies the above conditions [N.2]-[N.6]. Note that the function $\hat{\rho} = \chi_\Lambda$ is infrared singular, because $|k|^{-3/2}\hat{\rho}$ is not in $L^2(\mathbb{R}^3)$.

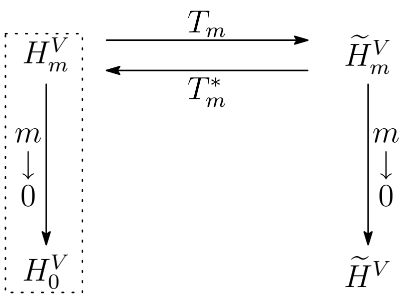
3 Proof of Theorem 2.5

Throughout this section we assume [N.1]-[N.6] and Hypothesis II. In this section, we set $\lambda = 1$, because Theorem 2.5 does not depend on λ explicitly (to restore λ, it is enough to replace $\hat{\rho}$ by $\lambda\hat{\rho}$).

For $m > 0$, $T_m := \exp[-i\mathbb{1} \otimes \Phi_{\mathrm{S}}(iv(0/\omega_m))]$ is a unitary operator on $\mathcal{F}$, and we have

$$\begin{aligned} \widetilde{H}_m^V :=& T_m H_m^V T_m^* \\ =& H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_f(m) + \phi^\oplus(G) - \mathcal{V}_m(\hat{x}) \otimes \mathbb{1} + \mathcal{W}_m \mathbb{1}, \end{aligned}$$

where $\mathcal{V}_m(\hat{x})$ is the multiplication operator by the function $\mathcal{V}_m(x) := \mathrm{Re}\langle \omega_m^{-1} v(0), v(x)\rangle$ and $\mathcal{W}_m := \|\omega_m^{-1/2} v(0)\|^2$ is a constant. In Fig.1, we show the relation to the original model.



Standard Nelson

Fig.1

The ground state energy $E^V(m)$ is monotone increasing in $m \geq 0$, and $\lim_{m\to 0} E^V(m) = E^V(0)$ (see [4, Section 5]). Therefore, by Hypothesis II, for all sufficiently small $m \geq 0$ we have $E^V(m) < \limsup_{R\to\infty} E_\infty(R,0)$. Since $E_\infty(R,m)$ is monotone increasing in $m \geq 0$, there exists a constant m such that

$$E^V(m) < \limsup_{R\to\infty} E_\infty(R,m), \quad (0 \leq m < \mathsf{m}).$$

In what follows, we consider only the case $0 < m < \mathsf{m}$. Hence, by Theorem 2.3, H_m^V has a ground state Φ_m. We set $\widetilde{\Phi}_m := T_m\Phi_m$ a ground state of $\widetilde{H}_m^V$.

Lemma 3.1 (Exponential decay). *Let $\beta > 0$ be a constant such that*

$$\beta^2 < \limsup_{R\to\infty} E_\infty(R,m) - E^V(m), \quad (0 < m < \mathsf{m}).$$

Then, for all large $R > 0$,

$$\|\exp(\beta|x|)\widetilde{\Phi}_m\|^2 \leq C\left(1 + \frac{1}{E_\infty(R,m) - E^V(m) - \beta^2 + o(1/R^0)}\right)\|\widetilde{\Phi}_m\|^2,$$

where the constant $C > 0$ does not depend on m with $C \leq \frac{3}{2}e^{4\beta R}$.

Proof. See [4]. ∎

Let $f \in \mathrm{Dom}(\omega_m)$. Since $\mathrm{Dom}(\widetilde{H}_m^V) = \mathrm{Dom}(H_\mathrm{p} \otimes 1\!\!1) \cap \mathrm{Dom}(1\!\!1 \otimes H_f(m))$, $a(f)\widetilde{\Phi}_m \in Q(\widetilde{H}_m^V)$. Hence, for all $\Psi \in \mathrm{Dom}(H_m^V)$, we have

$$\big\langle(\widetilde{H}_m^V - E^V(m))\Psi\,, a(f)\widetilde{\Phi}_m\big\rangle = -\big\langle\Psi\,, a(\omega_m f)\widetilde{\Phi}_m\big\rangle - \frac{1}{\sqrt{2}}\big\langle\Psi\,, \langle f, G(\hat{x})\rangle\widetilde{\Phi}_m\big\rangle.$$

Here we use the canonical commutation relations of a, a^*, and $\langle f, G(\hat{x})\rangle$ is the multiplication operator by the function $\langle f, G(x)\rangle$. Since $\Psi \in \mathrm{Dom}(\widetilde{H}_m^V)$ is arbitrary, $a(f)\widetilde{\Phi}_m \in \mathrm{Dom}(\widetilde{H}_m^V)$, and hence,

$$\big\langle a(f)\widetilde{\Phi}_m\,, a(\omega_m f)\widetilde{\Phi}_m\big\rangle + \frac{1}{\sqrt{2}}\big\langle a(f)\widetilde{\Phi}_m\,, \langle f, G(\hat{x})\rangle\widetilde{\Phi}_m\big\rangle \leq 0. \tag{4}$$

Lemma 3.2 (Photon number bound)**.** *For all* $0 < m < \mathsf{m}$*, we have*

$$\|a(k)\widetilde{\Phi}_m\|^2 \leq \frac{1}{2(2\pi)^3}\frac{|k|}{\omega_m(k)^2}|\hat{\rho}(k)|^2\||x|\widetilde{\Phi}_m\|^2, \quad \text{a.e. } k \in \mathbb{R}^3. \tag{5}$$

Proof. Let $q(k)$ be a bounded real-valued measurable function. We choose some complete orthonormal system $\{f_i\}_{i=1}^\infty \subset \mathrm{Dom}(\omega_m)$. By (4), we have

$$\sum_{i=1}^\infty \langle a(\omega_m^{-1/2}qf_i)\widetilde{\Phi}_m\,, a(\omega_m^{1/2}qf_i)\widetilde{\Phi}_m\rangle + \frac{1}{\sqrt{2}}\sum_{i=1}^\infty \langle a(\langle\omega_m^{-1/2}qf_i, G(\hat{x})\rangle\omega_m^{-1/2}qf_i)\widetilde{\Phi}_m\,, \widetilde{\Phi}_m\rangle \leq 0.$$

By Lemma A.1 in Appendix, we have

$$\begin{aligned}\langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(q^2)\widetilde{\Phi}_m\rangle &\leq -\frac{1}{\sqrt{2}}\langle a(\omega_m^{-1}q^2G(\hat{x}))\widetilde{\Phi}_m, \widetilde{\Phi}_m\rangle \\ &\leq \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\mathrm{d}k\frac{q(k)^2}{\omega_m(k)}|\langle G(\hat{x},k)^*a(k)\widetilde{\Phi}_m, \widetilde{\Phi}_m\rangle|.\end{aligned}$$

Note that q is arbitrary. Hence, we obtain

$$\|a(k)\widetilde{\Phi}_m\|^2 \leq \frac{1}{\sqrt{2}}\frac{1}{\omega_m(k)}\|a(k)\widetilde{\Phi}_m\|\,\|G(\hat{x};k)\widetilde{\Phi}_m\|, \quad \text{a.e. } k.$$

By the definition of G, we have $|G(x,k)|^2 \leq |\hat{\rho}(k)|^2|k||x|^2/(2\pi)^3$. Therefore, (5) holds. ∎

We write $\widetilde{\Phi}_m = (\widetilde{\Phi}_m^{(n)})_{n=0}^\infty$ with $\widetilde{\Phi}_m^{(n)} \in L^2(\mathbb{R}_x^3) \otimes (\otimes_s^n L^2(\mathbb{R}_k^3))$, $n \geq 0$, where $\otimes_s^n L^2(\mathbb{R}_k^3)$ is the n-fold symmetric tensor product of $L^2(\mathbb{R}_k^3)$.

Lemma 3.3 (Photon derivative bound)**.** *Let* $0 < m < \mathsf{m}$*. Then, for all* $\widetilde{\Phi}_m^{(n)}$ *is in the Sobolev space* $H^1(\mathbb{R}_x^3 \times S^{3n})$*, and* $\mathcal{F}$*-valued function* $a(k)\widetilde{\Phi}_m$ *is strongly differentiable in* $k \in S\backslash\{0\}$ *for all directions with*

$$\partial_j a(k)\widetilde{\Phi}_m = \left(\partial_j\widetilde{\Phi}_m^{(1)}(k), \sqrt{2}\partial_j\widetilde{\Phi}_m^{(2)}(k,\cdot), \ldots, \sqrt{n}\partial_j\widetilde{\Phi}_m^{(n)}(k,\cdot), \ldots\right), \quad j = 1,2,3,$$

$$\|\nabla_k a(k)\widetilde{\Phi}_m\|^2 \leq \frac{1}{(2\pi)^3}\frac{1}{\omega_m(k)^2}\left[3\frac{|\hat{\rho}(k)|^2}{|k|} + |k||\nabla\hat{\rho}(k)|^2\right]\||\hat{x}|\widetilde{\Phi}_m\|^2,$$

where ∂_j *and* ∇_k *means the differential operator for* j*-th component of* k *and the nabla operator for the coordinate* k*.*

Proof. For $h \in \mathbb{R}^3$ and a function $f(k)$, we define

$$(\Delta_h f)(k) := f(k+h) - f(k).$$

We consider (4) with f replaced by $\Delta_{-h}\omega_m^{-1/2}qf_i$. Here q and f_i are the same function as in the proof of the above Lemma. By Lemma A.1, we have

$$\sum_{i=1}^{\infty}\left\langle a(\Delta_{-h}\omega_m^{-1/2}qf_i)\widetilde{\Phi}_m\,,\, a(\omega_m\Delta_{-h}\omega_m^{-1/2}qf_i)\widetilde{\Phi}_m\right\rangle = \left\langle \widetilde{\Phi}_m\,,\, \mathrm{d}\Gamma(\Delta_{-h}\omega_m^{-1}q^2\Delta_h\omega_m)\widetilde{\Phi}_m\right\rangle. \tag{6}$$

We introduce an operator $(T_hf)(k) := f(k+h)$. It is easy to see that $\Delta_h\omega_m = (\Delta_h\omega_m)T_h + \omega_m\Delta_h$. therefore, we have

$$(6) = \langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{-h}q^2\Delta_h)\widetilde{\Phi}_m\rangle + \langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{-h}q^2\omega_m^{-1}(\Delta_h\omega_m)T_h)\widetilde{\Phi}_m\rangle.$$

On the other hand,

$$\sum_{i=1}^{\infty}\left\langle a(\Delta_{-h}\omega_m^{-1/2}qf_i)\widetilde{\Phi}_m\,,\, \langle\Delta_{-h}\omega_m^{-1/2}qf_i, G(\hat{x})\rangle\widetilde{\Phi}_m\right\rangle = \left\langle a(\Delta_{-h}\omega_m^{-1}q^2\Delta_hG(\hat{x}))\widetilde{\Phi}_m\,,\,\widetilde{\Phi}_m\right\rangle.$$

Therefore, we obtain

$$\begin{aligned}\left\langle\widetilde{\Phi}_m\,,\,\mathrm{d}\Gamma(\Delta_{-h}q^2\Delta_h)\widetilde{\Phi}_m\right\rangle \leq &-\left\langle\widetilde{\Phi}_m\,,\,\mathrm{d}\Gamma(\Delta_{-h}q^2\omega_m^{-1}(\Delta_h\omega_m)T_{-h})\widetilde{\Phi}_m\right\rangle\\ &-\frac{1}{\sqrt{2}}\langle a(\Delta_{-h}\omega_m^{-1}q^2\Delta_hG(\hat{x}))\widetilde{\Phi}_m\,,\,\widetilde{\Phi}_m\rangle.\end{aligned} \tag{7}$$

By the Schwarz inequality, we have

$$\begin{aligned}&|\langle a(\Delta_{-h}\omega_m^{-1}q^2\Delta_hG(\hat{x}))\widetilde{\Phi}_m\,,\,\widetilde{\Phi}_m\rangle|\\ &=\left\langle\Phi\,,\,\mathrm{d}\Gamma(\Delta_{-h}q^2\Delta_h)\Phi\right\rangle^{1/2}\left[\int_{\mathbb{R}^3}\mathrm{d}k\frac{q(k)^2}{\omega_m(k)^2}\left\|(\Delta_hG(\hat{x}))(k)\widetilde{\Phi}_m\right\|^2\right]^{1/2}.\end{aligned}$$

By using the general inequality $|\langle\Phi, \mathrm{d}\Gamma(S^*T)\Psi\rangle| \leq \langle\Phi, \mathrm{d}\Gamma(S^*S)\Phi\rangle^{1/2}\langle\Phi, \mathrm{d}\Gamma(T^*T)\Psi\rangle^{1/2}$, we have

$$\begin{aligned}\left|\left\langle\widetilde{\Phi}_m\,,\,\mathrm{d}\Gamma(\Delta_{-h}q^2\omega_m^{-1}(\Delta_h\omega_m)T_h)\widetilde{\Phi}_m\right\rangle\right| \leq &\left\langle\widetilde{\Phi}_m\,,\,\mathrm{d}\Gamma((\Delta_{-h}q)(\Delta_{-h}q)^*)\widetilde{\Phi}_m\right\rangle^{1/2}\\ &\times\left\langle\widetilde{\Phi}_m\,,\,\mathrm{d}\Gamma(T_{-h}(\Delta_h\omega_m)q^2\omega_m^{-2}(\Delta_h\omega_m)T_h)\widetilde{\Phi}_m\right\rangle^{1/2}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{-h}q^2\Delta_h)\widetilde{\Phi}_m\rangle\\ &\leq\int_{\mathbb{R}^3}\mathrm{d}k\frac{q(k)^2}{\omega_m(k)^2}\left\|(\Delta_hG(\hat{x}))(k)\widetilde{\Phi}_m\right\|^2 + 2\left\langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(T_{-h}(\Delta_h\omega_m)q^2\omega_m^{-2}(\Delta_h\omega_m)T_h)\widetilde{\Phi}_m\right\rangle\\ &=\int_{\mathbb{R}^3}\mathrm{d}k\frac{q(k)^2}{\omega_m(k)^2}\left\|(\Delta_hG(\hat{x}))(k)\widetilde{\Phi}_m\right\|^2 + 2\int_{\mathbb{R}^3}\mathrm{d}k\frac{q(k-h)^2}{\omega_m(k-h)^2}|(\Delta_h\omega_m)(k-h)|^2\|a(k)\widetilde{\Phi}_m\|^2.\end{aligned}$$

By (5), this is dominated by

$$\int_{\mathbb{R}^3} \mathrm{d}k \frac{q(k)^2}{\omega_m(k)^2} \big\|(\Delta_h G(\hat{x}))(k)\widetilde{\Phi}_m\big\|^2 + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}k \frac{q(k)^2}{\omega_m(k)^2} |(\Delta_h \omega_m)(k)|^2 \frac{|k+h|\hat{\rho}(k+h)^2}{\omega_m(k+h)^2} \||x|\widetilde{\Phi}_m\|^2$$

Since the function q is arbitrary, we have

$$\|\Delta_h a(k)\widetilde{\Phi}_m\|^2 \leq \frac{\|(\Delta_h G(\hat{x}))(k)\widetilde{\Phi}_m\|^2}{\omega_m(k)^2} + \frac{|k+h|\hat{\rho}(k+h)^2|(\Delta_h\omega_m)(k)|^2}{(2\pi)^3\omega_m(k+h)^2\omega_m(k)^2}\||x|\widetilde{\Phi}_m\|^2,$$

for a.e. $k \in \mathbb{R}^3$. By using the definition of $G(x,k)$, we have

$$\begin{aligned}&\sqrt{(2\pi)^3}|\Delta_h G(x,k)| \\ &\leq \frac{|h||k+h||x|}{|k+h|^{1/2}|k|}|\hat{\rho}(k+h)| + \frac{|h||x|}{|k|^{1/2}}|\hat{\rho}(k+h)| + |k|^{1/2}|x||\hat{\rho}(k+h)-\hat{\rho}(k)|,\end{aligned}$$

and it is easy to see that $|(\Delta_h\omega_m)(k)| \leq |h|$. Therefore we obtain

$$\begin{aligned}\|\Delta_h a(k)\widetilde{\Phi}_m\|^2 \leq &\frac{1}{(2\pi)^3}\frac{1}{\omega_m(k)^2}\Bigg[3\frac{|h|^2}{|k|^2}|k+h||\hat{\rho}(k+h)|^2 + 3\frac{|h|^2}{|k|}|\hat{\rho}(k+h)|^2 \\ &+ 3|k||\hat{\rho}(k+h)-\hat{\rho}(k)|^2 + \frac{|k+h||\hat{\rho}(k+h)|^2|h|^2}{\omega_m(k+h)^2}\Bigg]\||x|\widetilde{\Phi}_m\|^2. \qquad (8)\end{aligned}$$

By this inequality with [N.5], we see that $\mathcal{F}$-valued function $a(k)\widetilde{\Phi}_m$ is strongly continuous in $k \in S\backslash\{0\}$. Next, we show that $a(k)\widetilde{\Phi}_m$ is strongly differentiable. For this purpose, we introduce the operator $\Delta_{h,\ell}$ by

$$(\Delta_{h,\ell}f)(k) = \frac{f(k+h)-f(k)}{|h|} - \frac{f(k+\ell)-f(k)}{|\ell|}, \quad k,\ell \in \mathbb{R}^3.$$

We define $\Delta^*_{h,\ell} := \Delta_{-h,-\ell}$. Returning to (4) with f replaced by $\Delta^*_{h,\ell}\omega_m^{-1/2}qf_i$ and summing over $i = 1,\ldots,\infty$ we have

$$\big\langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta^*_{h,\ell}q^2\omega_m^{-1}\Delta_{h,\ell}\omega_m)\widetilde{\Phi}_m\big\rangle + \frac{1}{\sqrt{2}}\big\langle a(\Delta^*_{h,\ell}\omega_m^{-1}q^2\Delta_{h,\ell}G(\hat{x}))\widetilde{\Phi}_m, \widetilde{\Phi}_m\big\rangle \leq 0.$$

It is easy to see that $\Delta_{h,\ell}\omega_m = \omega_m\Delta_{h,\ell} + F_h - F_\ell$, where $F_h := (\Delta_h\omega_m)|h|^{-1}T_h$. Hence, we have

$$\big\langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta^*_{h,\ell}q^2\Delta_{h,\ell})\widetilde{\Phi}_m\big\rangle \leq -\frac{1}{\sqrt{2}}\langle a(\Delta^*_{h,\ell}\omega_m^{-1}q^2\Delta_{h,\ell}G(\hat{x}))\widetilde{\Phi}_m, \widetilde{\Phi}_m\rangle \qquad (9)$$

$$+ \langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta^*_{h,\ell}q^2\omega_m^{-1}F_h)\widetilde{\Phi}_m\rangle + \langle\widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta^*_{h,\ell}q^2\omega_m^{-1}F_\ell)\widetilde{\Phi}_m\rangle. \qquad (10)$$

By the Schwarz inequality, we have

$$|\text{r.h.s of (9)}| \leq \big\langle \widetilde{\Phi}_m\,,\, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \Delta_{h,\ell})\widetilde{\Phi}_m \big\rangle^{1/2} \left[\int_{\mathbb{R}^3} \mathrm{d}k \frac{q(k)^2}{\omega_m(k)^2} \|(\Delta_{h,\ell} G(\hat{x}))(k)\widetilde{\Phi}_m\|^2\right]^{1/2},$$

$$|(10)| \leq \big|\big\langle \widetilde{\Phi}_m\,,\, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \omega_m^{-1}[\Delta_{h,\ell}^*\omega_m]T_h)\widetilde{\Phi}_m\big\rangle\big| \tag{11}$$

$$+ \left|\left\langle \widetilde{\Phi}_m, \mathrm{d}\Gamma\Big(\Delta_{h,\ell}^* q^2 \omega_m^{-1}\Big[\frac{\Delta_\ell}{|\ell|}\omega_m\Big](T_h - T_\ell)\Big)\widetilde{\Phi}_m\right\rangle\right|. \tag{12}$$

Moreover,

$$\begin{aligned}
&\text{r.h.s. of (11)}\\
&\leq \big\langle \widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \Delta_{h,\ell})\widetilde{\Phi}_m\big\rangle^{1/2} \big\langle \widetilde{\Phi}_m, \mathrm{d}\Gamma\big(T_{-h} q^2 \omega_m^{-2}[\Delta_{h,\ell}^*\omega_m]^2 T_h\big)\widetilde{\Phi}_m\big\rangle^{1/2}\\
&= \big\langle \widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \Delta_{h,\ell})\widetilde{\Phi}_m\big\rangle^{1/2} \left[\int_{\mathbb{R}^3} \mathrm{d}k q(k)^2 \omega_m^{-2}(k)[\Delta_{h,\ell}^*\omega_m]^2(k)\|a(k+h)\widetilde{\Phi}_m\|^2\right]^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\text{r.h.s. of (12)}\\
&\leq \langle \widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \Delta_{h,\ell})\widetilde{\Phi}_m\rangle^{1/2} \left\langle \widetilde{\Phi}_m, \mathrm{d}\Gamma\Big((T_{-h} - T_{-\ell})\Big[\frac{\Delta_\ell}{|\ell|}\omega_m\Big]^2 q^2 \omega_m^{-2}(T_h - T_\ell)\Big)\right\rangle^{1/2}\\
&\leq \langle \widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \Delta_{h,\ell})\widetilde{\Phi}_m\rangle^{1/2} \left[\int_{\mathbb{R}^3} \mathrm{d}k\, q(k)^2 \omega_m(k)^{-2} \|a(k+h)\widetilde{\Phi}_m - a(k+\ell)\widetilde{\Phi}_m\|^2\right]^{1/2}.
\end{aligned}$$

These inequality yields

$$\begin{aligned}
\langle \widetilde{\Phi}_m, \mathrm{d}\Gamma(\Delta_{h,\ell}^* q^2 \Delta_{h,\ell})\widetilde{\Phi}_m\rangle \leq & \frac{3}{2}\int_{\mathbb{R}^3} \mathrm{d}k \frac{q(k)^2}{\omega_m(k)^2}\|(\Delta_{h,\ell} G(\hat{x}))(k)\widetilde{\Phi}_m\|^2\\
&+ 3\int_{\mathbb{R}^3} \mathrm{d}k \frac{q(k)^2}{\omega_m(k)^2}|\Delta_{h,\ell}^*\omega_m|^2(k)\|a(k+h)\widetilde{\Phi}_m\|^2\\
&+ 3\int_{\mathbb{R}^3} \mathrm{d}k \frac{q(k)^2}{\omega_m(k)^2}\|a(k+h)\widetilde{\Phi}_m - a(k+\ell)\widetilde{\Phi}_m\|^2
\end{aligned}$$

Since the function q is arbitrary, we have

$$\begin{aligned}
\|\Delta_{h,\ell} a(k)\widetilde{\Phi}_m\|^2 \leq & \frac{3}{\omega_m(k)^2}\left[\frac{1}{2}\|(\Delta_{h,\ell} G(\hat{x}))(k)\widetilde{\Phi}_m\|^2 + |\Delta_{h,\ell}^*\omega_m|^2(k)\|a(k+h)\widetilde{\Phi}_m\|^2\right.\\
&\left. + \|a(k+h)\widetilde{\Phi}_m - a(k+\ell)\widetilde{\Phi}_m\|^2\right], \quad \text{a.e.}\, k \in \mathbb{R}^3.
\end{aligned} \tag{13}$$

Remembering the condition [N.5] and that $a(k)\widetilde{\Phi}_m$ is continuous, we get,

$$\lim_{h,\ell \to 0} \|\Delta_{|h|\boldsymbol{e},|\ell|\boldsymbol{e}} a(k)\widetilde{\Phi}_m\|^2 = 0, \quad \text{a.e. } k \in S\backslash\{0\},$$

for all $\boldsymbol{e} \in \mathbb{R}^3$. Therefore the $\mathcal{F}$-valued function $\Delta_{|h|\boldsymbol{e}} a(k)\widetilde{\Phi}_m/|h|$ is a Cauchy sequence in $|h|$ as $|h| \to 0$. Namely, for all directions, $a(k)\widetilde{\Phi}_m$ is strongly differentiable in $k \in S\backslash\{0\}$. Let e_j $(j = 1, 2, 3)$ be the unit vectors of the j-th direction, and let

$$v_j(k) := \text{s-lim}_{|h|\to 0} \frac{1}{|h|} \Delta_{|h|e_j} a(k)\widetilde{\Phi}_m, \quad \text{a.e. } k \in S.$$

Next, we show that $\widetilde{\Phi}_m^{(n)} \in H^1(\mathbb{R}_x^3 \times S^{3n})$ for all $n \in \mathbb{N}$. Let $\psi \in C_0^\infty(\mathbb{R}_x^3) \times S^{3n}$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{3(n+1)}} (\partial_j \psi)(x, k, K)\widetilde{\Phi}_m^{(n)}(x, k, K)\mathrm{d}x\mathrm{d}k\mathrm{d}K \\
&= \lim_{h\to 0} \frac{1}{|h|} \int_{\mathbb{R}^{3(n+1)}} [\psi(x, k, K) - \psi(x, k - |h|e_j, K)]\widetilde{\Phi}_m^{(n)}(x, k, K)\mathrm{d}x\mathrm{d}k\mathrm{d}K \\
&= -\lim_{h\to 0} \frac{1}{|h|} \int_{\mathbb{R}^3} \mathrm{d}k \left[\int_{\mathbb{R}^{3(n+1)}} \psi(x, k, K) \left[\widetilde{\Phi}_m^{(n)}(x, k + |h|e_j, K) - \widetilde{\Phi}_m^{(n)}(x, k, K)\right] \mathrm{d}x\mathrm{d}K\right],
\end{aligned}$$

where $K = (k_1, k_2, \ldots, k_{n-1}) \in \mathbb{R}^{3(n-1)}$. On the other hand,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\mathrm{d}k\left[\int_{\mathbb{R}^{3n}}\mathrm{d}x\mathrm{d}K\psi(x, k, K)\left\{\frac{1}{|h|}[\widetilde{\Phi}_m^{(n)}(x, k + |h|e_j, K)+\widetilde{\Phi}_m^{(n)}(x, k, K)] - v_j^{(n)}(x, k, K)\right\}\right]\right| \\
&\leq \int_{\mathbb{R}^3} \mathrm{d}k\|\psi(k, \cdot)\|_{L^2(\mathbb{R}^{3n})}\left\|\frac{1}{|h|}(\Delta_{|h|e_j} a(k)\widetilde{\Phi}_m)^{(n)} - v_j^{(n)}(k)\right\| \\
&\leq \int_{\mathbb{R}^3} \mathrm{d}k\|\psi(k, \cdot)\|_{L^2(\mathbb{R}^{3n})}\left\|\frac{1}{|h|}\Delta_{|h|e_j} a(k)\widetilde{\Phi}_m - v_j(k)\right\|. \qquad (14)
\end{aligned}$$

Returning to (13) with $h \to |h|e_j$, $\ell \to |\ell|e_j$ and $\lim_{|\ell|\to 0}$, we have

$$\begin{aligned}
&\left\|\frac{1}{|h|}\Delta_h a(k)\widetilde{\Phi}_m - v_j(k)\right\|^2 \\
&\leq \frac{3}{\omega_m(k)^2}\left[\frac{1}{2}\left\|\left(\frac{1}{h}(\Delta_{he_j} G)(\hat{x}, k) - \partial_j G(\hat{x}, k)\right)\widetilde{\Phi}_m\right\|^2\right. \\
&\quad \left. + \frac{2|h|}{\omega_m(k)}\|a(k + h)\widetilde{\Phi}_m\|^2 + \|a(k + h)\widetilde{\Phi}_m - a(k)\widetilde{\Phi}_m\|^2\right], \quad \text{a.e. } k \in \mathbb{R}^3,
\end{aligned}$$

where we use the elementary inequality $|\frac{1}{h}\Delta_{he_j}\omega_m(k) - \partial_j\omega_m(k)| \leq 2|h|/\omega_m(k)$. Since the set $S_\psi := \operatorname{supp}\|\psi(k, \cdot)\|$ is a subset of S, $k + h \in S$ for all h and $k \in S_\psi$ with $|h| < \operatorname{dist}\{S_\psi, S\}$. Using this fact and (8), we obtain

$$\lim_{|h|\to 0} \int_{\mathbb{R}^3} \mathrm{d}k\|\psi(k, \cdot)\|_{L^2(\mathbb{R}^{3n})} \left[|h|\frac{\|a(k + h)\widetilde{\Phi}_m\|}{\omega_m(k)^2} + \frac{\|a(k + h)\widetilde{\Phi}_m - a(k)\widetilde{\Phi}_m\|}{\omega_m(k)}\right] = 0.$$

By condition [N.4] and the dominated convergence theorem, we have

$$\begin{aligned}\|\chi_{S_\psi}|h|^{-1}\Delta_h\hat{\rho}-\partial_j\hat{\rho}\|^2 &\le \||h|^{-1}\Delta_h\eta-\partial_j\eta\|^2_{L^2(\mathbb{R}^3)}\\ &\le \int_{\mathbb{R}^3_y}\left|\frac{e^{-i|h|y_j}-1}{|h|y_j}+i\right|^2 y_j^2|(\mathsf{F}\eta)(y)|^2\mathrm{d}y\to 0,(|h|\to 0),\end{aligned}$$

where F means Fourier transformation. By this formula and simple but tedious estimates, we can show that

$$\lim_{|h|\to 0}\int_{\mathbb{R}^3}\mathrm{d}k\|\psi(k,\cdot)\|_{L^2(\mathbb{R}^{3n})}\cdot\frac{1}{\omega_m}\left\|\left(\frac{1}{h}(\Delta_{he_j}G)(\hat{x},k)-\partial_j G(\hat{x},k)\right)\widetilde{\Phi}_m\right\|=0.$$

These facts mean that

$$\lim_{h\to 0}(14)=0.$$

Therefore, $\widetilde{\Phi}_m^{(n)}\in H^1(\mathbb{R}^3_x\times S^{3n})$. ∎

Pick a sequence $m_1>m_2>\cdots$ tending to zero and we set

$$\widetilde{\Phi}_j:=\widetilde{\Phi}_{m_j},\quad j=1,2,\ldots.$$

Since $\widetilde{\Phi}_j$'s are normalized, a subsequence of $\{\widetilde{\Phi}_j\}_j$ has a weak limit $\widetilde{\Phi}$ (the subsequence denoted by the same symbol).

Lemma 3.4. $\widetilde{\Phi}\in \mathrm{Dom}(\widetilde{H}^V)$ *and,*

$$\widetilde{H}^V\widetilde{\Phi}=E^V(0)\widetilde{\Phi}. \tag{15}$$

Proof. First, we show that $\widetilde{\Phi}\in Q(\widetilde{H}^V)=\mathrm{Dom}(H_f(0)^{1/2})\cap Q(H_\mathrm{p})$. For all $\Psi\in\mathrm{Dom}(H_f(0)^{1/2})$, we have

$$|\langle\widetilde{\Phi},H_f(0)^{1/2}\Psi\rangle|=\lim_{j\to\infty}|\langle H_f(0)^{1/2}\widetilde{\Phi}_j,\Psi\rangle|=\limsup_{j\to\infty}\|H_f(0)^{1/2}\widetilde{\Phi}_j\|\|\Psi\|.$$

Since H_p is bounded below, we have

$$\|H_f(0)^{1/2}\widetilde{\Phi}_j\|^2\le \text{const.}\left\langle\widetilde{\Phi}_j,(\widetilde{H}^V(m_j)-E^V(m_j)+1)\widetilde{\Phi}_j\right\rangle\le\text{const.},$$

where const. is a constant independent of j. Hence $\widetilde{\Phi}\in\mathrm{Dom}(H_f(0)^{1/2})$. Similarly we have $\widetilde{\Phi}\in Q(H_\mathrm{p})$. Since $E^V(m_j)\to E^V(0)\,(j\to\infty)$, we have

$$\|(\widetilde{H}^V-E^V(0))^{1/2}\widetilde{\Phi}_j\|^2\le\|(\widetilde{H}^V(m_j)-E^V(0))^{1/2}\widetilde{\Phi}_j\|^2\le(E^V(m_j)-E^V(0))\|\widetilde{\Phi}_j\|^2\to 0,$$

as $j\to\infty$. Therefore $(\widetilde{H}^V-E^V(0))^{1/2}\widetilde{\Phi}=0$. This means $\widetilde{\Phi}\in\mathrm{Dom}(\widetilde{H}^V)$ and $\widetilde{H}^V\widetilde{\Phi}=E^V(0)\widetilde{\Phi}$. ∎

By this lemma, if $\widetilde{\Phi}\neq 0$ then $\widetilde{\Phi}$ is a ground state of $\widetilde{H}^V$. This proof is essentially same as [4, 7 Proof of Theorem 2.1], so we omit it (Notice that the condition [N.3] and [N.6] were used there).

A Parseval's Equality for the Annihilation Operators

Let $\mathcal{K}$ be a complex separable Hilbert space, and let $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$ be the Boson Fock space over $\mathcal{K}$. We denote by N_{b} the number operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{K})$. Let S and T be densely defined closed linear operators on $\mathcal{K}$, such that $\mathrm{Dom}(S) \cap \mathrm{Dom}(T)$ is dense.

Lemma A.1 (Parseval's equality for the annihilation operators)**.** *Assume that, for vectors $\Psi, \Phi \in \mathrm{Dom}(N_{\mathrm{b}}^{1/2})$, there exist constants α, β ($\alpha + \beta = 1$, $\alpha, \beta \geq 0$) such that,*

$$N_{\mathrm{b}}^{\alpha-1}\Phi \in \mathrm{Dom}(\mathrm{d}\Gamma(T^*)), \quad N_{\mathrm{b}}^{\beta-1}\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(S^*)).$$

Then, for all complete orthonormal basis $\{f_j\}_{j=1}^{\infty} \subset \mathrm{Dom}(S) \cap \mathrm{Dom}(T)$, the following equality holds:

$$\sum_{j=1}^{\infty} \langle a(Sf_j)\Psi, a(Tf_j)\Phi\rangle = \sum_{n=1}^{\infty} n \langle S^* \otimes \mathbb{1}_{n-1}\Psi^{(n)}, T^* \otimes \mathbb{1}_{n-1}\Phi^{(n)}\rangle_{\otimes^n \mathcal{K}}. \tag{16}$$

In particular, if $\Phi \in \mathrm{Dom}(\mathrm{d}\Gamma(ST^*))$, then

$$\sum_{j=1}^{\infty} \langle a(Sf_j)\Psi, a(Tf_j)\Phi\rangle = \langle \Psi, \mathrm{d}\Gamma(ST^*)\Phi\rangle. \tag{17}$$

Proof. It is enough to show in the case that $\mathcal{K}$ is L^2-space on a measurable space. For simplicity, we prove (16) only in the case $\mathcal{K} = L^2(\mathbb{R}^3)$. Using the definition of $a(f)$, we have

$$\begin{aligned} \big\langle a(Sf_j)\Psi\,, a(Tf_j)\Phi\big\rangle &= \int \mathrm{d}k \int \mathrm{d}k' (Sf_j)(k)(Tf_j)^*(k') \big\langle a(k)\Psi\,, a(k')\Phi\big\rangle \\ &= \int \mathrm{d}k \int \mathrm{d}k' \sum_{n=1}^{\infty} n \int \mathrm{d}K (Sf_j)(k)(Tf_j)^*(k')\Psi^{(n)}(k,K)^*\Phi^{(n)}(k',K), \end{aligned}$$

where $K = (k_2, \ldots, k_n)$, $\mathrm{d}K = \mathrm{d}k_2 \cdots \mathrm{d}k_n$. In the above equation, the integral and the summation commute, because

$$\int \mathrm{d}k \int \mathrm{d}k' \sum_{n=1}^{\infty} n \int \mathrm{d}K |(Sf_j)(k)(Tf_j)^*(k')\Psi^{(n)}(k,K)^*\Phi^{(n)}(k',K)|$$
$$\leq \int \mathrm{d}k\mathrm{d}k' |(Sf_j)(k)(Tf_j)(k')| \left[\sum_{n=1}^{\infty} n \int \mathrm{d}K |\Psi^{(n)}(k,K)|^2\right]^{1/2} \left[\sum_{n=1}^{\infty} n \int \mathrm{d}K |\Phi^{(n)}(k',K)|^2\right]^{1/2}$$
$$= \int \mathrm{d}k\mathrm{d}k' |(Sf_j)(k)| \cdot |(Tf_j)(k')| \cdot \|a(k)\Psi\| \|a(k')\Phi\|$$
$$\leq \|Sf_j\| \|Tf_j\| \left[\int \mathrm{d}k \|a(k)\Psi\|^2\right]^{1/2} \left[\int \mathrm{d}k \|a(k')\Phi\|^2\right]^{1/2}$$
$$= \|Sf_j\| \|Tf_j\| \|N_{\mathrm{b}}^{1/2}\Psi\| \|N_{\mathrm{b}}^{1/2}\Phi\| < \infty,$$

and hence on can apply Fubini's theorem. Hence,

$$\sum_{j=1}^{\infty} \langle a(Sf_j)\Psi\,, a(Tf_j)\Phi \rangle =$$
$$\sum_{j=1}^{\infty} \sum_{n=0}^{\infty} n \int \mathrm{d}K \langle (T^* \otimes 1\!\!1 \Phi^{(n)})(\cdot, K)\,, f_j(\cdot)\rangle \langle f_j(\cdot)\,, (S^* \otimes 1\!\!1 \Psi^{(n)})(\cdot, K)\rangle.$$

Using Bessel's inequality, we have

$$\left|\sum_{j=1}^{N} \langle (T^* \otimes 1\!\!1 \Phi^{(n)})(\cdot, K), f_j(\cdot)\rangle \langle f_j(\cdot), (S^* \otimes 1\!\!1 \Psi^{(n)})(\cdot, K)\rangle\right|$$
$$\leq \|(T^* \otimes 1\!\!1 \Phi^{(n)})(\cdot, K)\| \|(S^* \otimes 1\!\!1 \Psi^{(n)})(\cdot, K)\|$$
$$\leq \frac{1}{2n} \left\{ n^{2\alpha} \|(T^* \otimes 1\!\!1 \Phi^{(n)})(\cdot, K)\|^2 + n^{2\beta} \|(S^* \otimes 1\!\!1 \Psi^{(n)})(\cdot, K)\|^2 \right\}, \quad \text{a.e. } K \in \mathbb{R}^{3(n-1)}. \tag{18}$$

By assumpsion for Ψ, Φ, we have

$$\sum_{n=1}^{\infty} n \int \mathrm{d}K (\text{r.h.s of (18)}) = \frac{1}{2} \sum_{n=1}^{\infty} \|n^{\alpha} T^* \otimes 1\!\!1 \Phi^{(n)}\|^2 + \frac{1}{2} \sum_{n=1}^{\infty} \|n^{\beta} S^* \otimes 1\!\!1 \Psi^{(n)}\|^2 < \infty.$$

Hence, by applying the dominated convergence theorem and the standard parseval equality, we obtain (16). ∎

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