

Converse spectral problems for nodal domains.

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Abstract

We consider two-dimensional Schrödinger operators in bounded domains. Abstractions of nodal sets are introduced and spectral conditions for them ensuring that they are actually zero sets of eigenfunctions are given.

1 Introduction.

Consider a Schrödinger operator

$$H = -\Delta + V \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition. We assume that $\partial\Omega$ has finitely many piecewise smooth components and satisfies an interior and an exterior cone condition. Furthermore we assume that $V \in C^\infty(\overline{\Omega})$ is real valued.

The operator H is then selfadjoint if viewed as the Friedrichs extension of the quadratic form associated to H with form domain $\mathcal{H}_0^1(\Omega)$. We denote H , by $H(\Omega)$. We know that $H(\Omega)$ has compact resolvent. So the spectrum of $H(\Omega)$, $\sigma(H(\Omega))$ can be described by an increasing sequence of discrete eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots$$

tending to $+\infty$, such that the associated eigenfunctions u_k can be chosen to form an orthonormal basis in $L^2(\Omega)$. We can assume that the eigenfunctions u_k are real valued and by elliptic regularity [9] we have

$$u_k \in C^\infty(\Omega) \cap C_0^0(\overline{\Omega}). \tag{1.2}$$

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It is well known that u_1 can be chosen to be strictly positive and that the other eigenfunctions u_k , $k > 1$, must have nonempty zero sets. We define the zeroset $N(u)$ of a function u by

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}. \quad (1.3)$$

The nodal domains of an eigenfunction u , which are by definition the connected components of $\Omega \setminus N(u)$, will be denoted by D_j , $j = 1, \dots, \mu(u)$, where $\mu(u)$ denotes the number of nodal domains of u . For any open subset $D \subset \Omega$ we denote by $H(D)$ the operator $-\Delta + V$ with form domain $Q(H) = \mathcal{H}_0^1(D)$.

In [1] we considered together with A. Ancona the following situation. Supposing the Ω_ℓ are pairwise disjoint open subset of Ω , we derived inequalities relating the sum of the spectral counting functions of the $H(\Omega_\ell)$ with the counting function of $H(\Omega)$. Here the counting function attached to λ and $H(\Omega_\ell)$ is the number of eigenvalues of $H(\Omega_\ell)$ that are smaller or equal to a given λ . Also converse results were obtained. Namely for the case of equality it turned out that these Ω_ℓ already must be nodal domains or union of nodal domains.

The problem we address here is related in spirit. Let \mathcal{D} be a **covering family** of Ω , that is a family of μ connected open subsets of Ω such that

$$D_i \cap D_j = \emptyset \text{ for } i \neq j, \text{ and } \text{Int}(\overline{\cup_1^\mu D_i}) = \Omega. \quad (1.4)$$

If the μ domains happen to be the nodal domains of an eigenfunction u of $H(\Omega)$ such that $H(\Omega)u = \lambda u$, then λ is in the spectrum of any operator H_I where I is a subset of $\{1, 2, \dots, \mu\}$ and

$$H_I = H(\text{Int}(\overline{\cup_{i \in I} D_i})). \quad (1.5)$$

We are interested in the possibility of a converse statement. If we look first at the one dimensional case when Ω is an interval in \mathbb{R} , we easily see that, if we can find a $\lambda \in \mathbb{R}$ such that λ is a groundstate energy for each $H(D_i)$, then this λ should be an eigenvalue of $H(\Omega)$ and we can find a corresponding eigenfunction u such that the intervals D_i are the nodal domains of u . We have just indeed to multiply each eigenfunction u_i of $H(D_i)$ by a constant c_i in order to glue them together for getting an eigenfunction u .

In order to go further, we first observe that this is no longer true in the case of a circle S^1 . Some compatibility condition should occur and it is rather easy to find examples for which one cannot glue together the u_i 's. As we shall

see later, this phenomenon could also appear in higher dimension when Ω is not simply connected.

The second observation is that this gluing procedure does not work anymore in higher dimension. In order to explain the problem, let us first give a definition. We say that $D_i, D_j \subset \mathcal{D}$ are **neighbors**, or **neighbor each other**, or $D_i \sim D_j$ if

$$D_{ij} := \text{Int}(\overline{D_i \cup D_j}) \text{ is connected.} \quad (1.6)$$

Now if for two neighbors D_i and D_j , λ is the groundstate energy of both $H(D_i)$ and $H(D_j)$, there is no way in general to construct u_{ij} in the domain of $H(D_{ij})$ such that $u_{ij} = c_i u_i$ in D_i and $u_{ij} = c_j u_j$ in D_j . We would indeed need at $\partial D_i \cap \partial D_j$ the normal derivatives of u_i and u_j to be proportional.

So it is natural for the analysis of the converse problem to assume in higher dimension the existence of u_{ij} for all the pairs of neighbouring domains and to try then to glue those u_{ij} so that an eigenfunction of the whole problem is obtained. We are consequently led to the following definition :

Definition 1.1

We say that the covering family $\mathcal{D} = \{D_1, \dots, D_\mu\}$ satisfies the **Pair Compatibility Condition**, for short **PCC**, if, for some $\lambda \in \mathbb{R}$, and for any pair (i, j) such that $D_i \sim D_j$, there is an eigenfunction $u_{ij} \neq 0$ of $H(D_{ij})$ such that $H(D_{ij})u_{ij} = \lambda u_{ij}$ with

$$N(u_{ij}) = \partial D_i \cap \partial D_j. \quad (1.7)$$

We can associate (see Figure 1 for an example) to such a covering family \mathcal{D} a graph G or $G(\mathcal{D})$ by placing in each D_i , $i = 1, \dots, \mu$, a vertex v_i and by associating edges e_{ij} to the v_i, v_j such that the corresponding D_i, D_j satisfy $D_i \sim D_j$.

We say that \mathcal{D} is **admissible** if the associated graph $G(\mathcal{D})$ is bipartite. Bipartite graphs are just graphs whose vertices can be colored by two colors so that vertices which are joined by an edge have different colors. This is a well known notion in graph theory, see e.g. Diestel [8].

We shall see in the next section that the nodal set $N(u)$ of an eigenfunction u of $H(\Omega)$ has, under the condition that $\partial\Omega$ is C^∞ , the following **Abstract Nodal Set Property** (or shortly (ANSP)) which we now define.

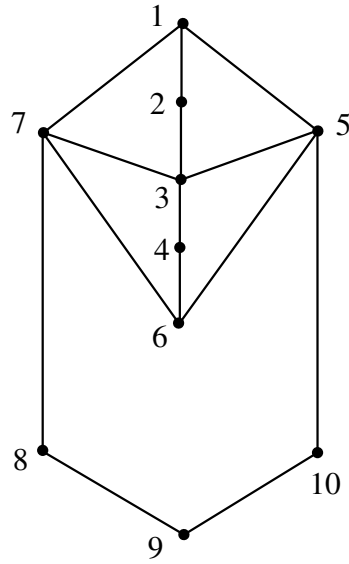
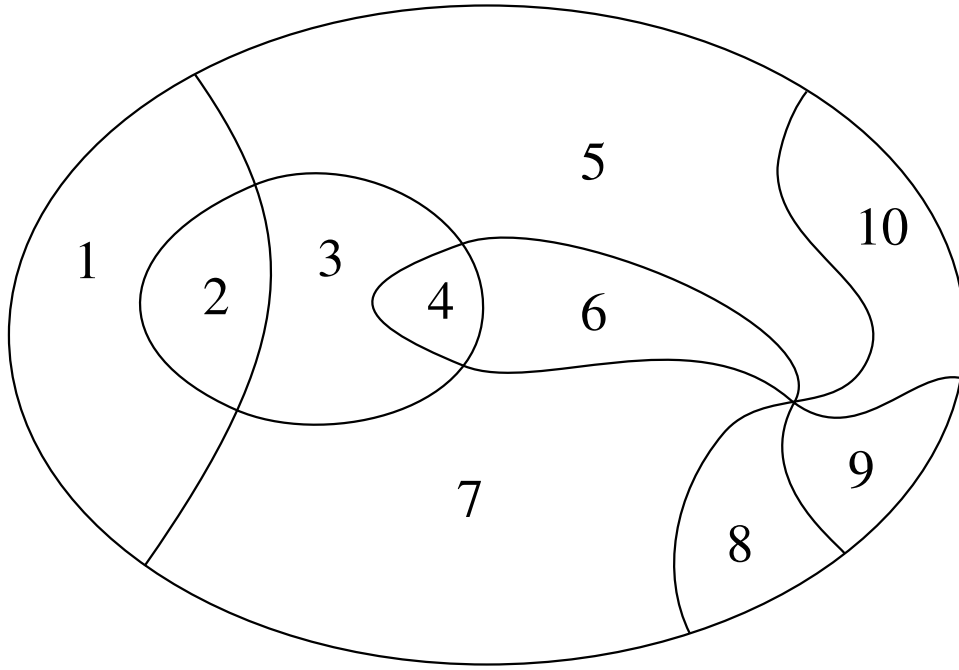


Figure 1: Covering family and associated graph

Definition 1.2 (*Abstract Nodal Set Property*)

A closed set N in $\overline{\Omega}$ belongs to $\widetilde{\mathcal{N}}(\Omega)$ if N meets the following requirements :

(i):

N is the union of smooth arcs connecting points in $\partial\Omega$ and smoothly imbedded circles in Ω .

(ii):

There are finitely many distinct $x_i \in \Omega \cap N$ and associated positive integers ν_i ($\nu_i \geq 2$) such that, in a sufficiently small neighborhood of each of the x_i , N is the union of $\nu(x_i)$ C^∞ curves (non self-crossing) two by two crossing transversally at x_i (with positive angle) and such that in the complement of these points in Ω , N is locally diffeomorphic to a regular curve.

(iii):

$\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_j , such that, at each z_j , ρ_j ($\rho_j \geq 1$) nodal lines hit the boundary. Moreover, for each $z_j \in \partial\Omega$, assuming that we have rotated and translated Ω such that $z_j = \{0\}$, that $\partial\Omega$ is at the origin tangent to the x_1 -axis and that Ω lies locally above the x_1 -axis, then N is near z_j the union of ρ_j distinct C^∞ half-curves which hit the origin with strictly positive angles.

By smooth we mean as usual that each arc, respectively circle, is a component of the zeroset of a C^∞ function which has at the zero nonvanishing gradient.

The points x_i will be called “critical” points of the (abstract) nodal set.

Let us also observe that this definition implies that the family of abstract nodal domains which are by definition the components of $\Omega \setminus N$ is an admissible covering.

Our main theorem is the following

Theorem 1.3

Suppose that Ω is simply connected with smooth boundary and that, for some $N \in \widetilde{\mathcal{N}}(\Omega)$ and $\lambda \in \mathbb{R}$, the associated family $\mathcal{D} = \{D_1, \dots, D_\mu\}$ satisfies (PCC). Then there is an eigenfunction of $H(\Omega)$ with corresponding eigenvalue λ such that $\cup_i \partial D_i \setminus \partial\Omega = N(u)$.

Remarks 1.4

(i) The regularity assumptions can probably be relaxed, but we do not strive for generality here. See [1] for the type of conditions which could be given and Section 8.

(ii) If Ω is not simply connected then our result does not hold in general

as will be explained through examples in Section 6. There we also give additional conditions on the admissible covering family \mathcal{D} such that Theorem 1.3 still holds.

(iii) There are simple cases in higher dimensions for which Theorem 1.3 easily can be shown to hold (for example in some simple tree situations).

Organization of the paper.

In Section 2 we collect some well known facts about zero sets and nodal domains. In Section 3, we analyze the properties of coverings of Ω in connection with graph theory. Section 4 is devoted to the proof of the main theorem. In Sections 5, we illustrate the theorem by discussing examples. Section 6 is devoted to a general criterion for non simply connected domains. We then analyze in Section 7 the optimality of these sufficient conditions by considering families of examples for which (PCC) does not imply a general compatibility condition. In Section 8, we come back to a conjecture concerning the nodal structure of eigenfunctions in domain with corners, which can be of independent interest. Concluding remarks are given in Section 9.

2 Regularity of eigenfunctions and Abstract Nodal Set Property.

We investigate the properties of nodal domains and nodal sets. This will lead us to propose and justify the corresponding abstractions. In particular we will show that for a smooth Ω the zerosets of the eigenfunctions satisfy the Abstract Nodal Set Property introduced in Definition 1.2. First we recall some basic regularity results (cf [9]).

Proposition 2.1

Every eigenfunction u of $H(\Omega)$ belongs to $C^\infty(\Omega) \cap C_0^0(\overline{\Omega})$. Furthermore, for any eigenfunction u , any nodal domain ∂D is piecewise smooth and satisfies an interior cone condition. If in addition the boundary is C^∞ then $u \in C^\infty(\overline{\Omega})$

The next property justifies the introduction of $\tilde{\mathcal{N}}(\Omega)$.

Proposition 2.2

Suppose $\partial\Omega \in C^\infty$ and u is an eigenfunction of $H(\Omega)$. Then $N(u)$ belongs

to $\tilde{\mathcal{N}}(\Omega)$.

Moreover the nodal lines can only cross at interior points with equal angles, and at boundary points crossing nodal half-lines determine also together with the boundary equal angles.

The proof follows rather directly from the local behaviour of eigenfunctions near their zeros.

Lemma 2.3

Let u in $\mathcal{H}_0^1(\Omega \cap B(x_0, \rho_0))$ with $\Omega \cap B(x_0, \rho_0) \neq \emptyset$ where $B(x_0, \rho_0) = \{x \in \mathbb{R}^2 \mid |x - x_0| < \rho_0\}$ and $x_0 \in \overline{\Omega}$. Let us assume that :

$$(-\Delta + V - \lambda)u = 0 \text{ in } \Omega \cap B(x_0, \rho_0) .$$

(a) :

Suppose $x_0 \in N(u) \cap \Omega$ then there is ν such that,

$$u(x) = P_\nu(x - x_0) + P_{\nu+1}(x - x_0) + \mathcal{O}\left(|x - x_0|^{\nu+2}\right), \quad (2.1)$$

in a sufficiently small neighborhood of x_0 . Thereby $P_\nu \neq 0$, defined by

$$P_\nu(x) = r^\nu(a \cos \nu\omega + b \sin \nu\omega), \quad (2.2)$$

is a harmonic homogeneous polynomial of degree ν . For simplicity we have written this in polar coordinates r, ω . Note that the zero set of any P_m consists of m straight lines which intersect with equal angles.

(b) :

If $z_0 \in N(u) \cap \partial\Omega$ then there exist $\rho \geq 1$ and $r_0 > 0$ such that, in $\Omega \cap B(z_0, r_0)$,

$$u(x) = P_{\rho+1}(x - z_0) + \mathcal{O}\left(|x - z_0|^{\rho+2}\right). \quad (2.3)$$

Thereby $P_{\rho+1} \neq 0$ in (2.3) is defined as in (2.2) and has the property that the line tangent to $\partial\Omega$ at z_0 is in the zero set of $P_{\rho+1}(x - z_0)$.

About the proof of Lemma 2.3.

This lemma is well known among specialists (see for instance [12]) but it is difficult to give a reference where all the statements are proved in detail. Therefore we sketch² some of the arguments. For inner points, i.e. for **(a)** it is already in the work of Bers [3] where it is shown that locally this behaviour holds. Note that the Strong Unique Continuation Theorem gives

²The first author would like to thank L. Robbiano for useful discussions.

first that a local solution which is not identically zero cannot vanish faster than polynomially at a point inside Ω . It is then easy to see that the first non zero homogeneous term appearing in the Taylor expansion at a point x_0 in $N(u) \cap \Omega$ should be a harmonic polynomial.

For boundary points, i.e. case **(b)**, it seems useful to explain how the Strong Unique Continuation Theorem can be extended. One can actually straighten the boundary and reflect. The change of variable should however be done carefully in order that the main term of the operator becomes, in the new coordinates $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$,

$$-\partial_{\tilde{x}_1} \tilde{a}_1(\tilde{x}) \partial_{\tilde{x}_1} - \partial_{\tilde{x}_2} \tilde{a}_2(\tilde{x}) \partial_{\tilde{x}_2} , \quad (2.4)$$

and that the boundary becomes locally defined by $\tilde{x}_1 = 0$ (with Ω locally defined by $\tilde{x}_1 > 0$).

Note that an important point in (2.4) is that no cross term $\partial_{\tilde{x}_1} \partial_{\tilde{x}_2}$ appears. Forgetting the *tilda*'s, we now reflect the solution u by

$$\begin{aligned} u^{ext}(x_1, x_2) &= u(x_1, x_2) , \quad \text{for } x_1 > 0 , \\ &= -u(-x_1, x_2) , \quad \text{for } x_1 < 0 . \end{aligned}$$

Then the boundary critical point becomes an inner point and the above argument carries over with u^{ext} appearing as a solution of a new second order operator with less regular coefficients. The fact that the Laplacian is transformed by this flattening and extension by reflection to an operator in divergence form does not cause any trouble for the application of the Strong Unique Continuation Theorem. Indeed, the principal term is Lipschitz and the lower order terms are L^∞ . We refer to [14] (p. 104) for a variant of this trick used in a similar context (Carleman estimates).

To show that the nodal arcs respectively circles are smooth also near or through the crossing points or an arc when approaching the boundary one can look up for example at the proof given in [10] (see more precisely the argument given p. 1010, lines 1-10). \square

Remarks 2.4

- (i) *Note that for each component of $\partial\Omega$ the number of nodal lines hitting this component has to be even. This is implied by the property that the graph associated to N is bipartite. Indeed, otherwise the associated graph would contain an "odd circle" (that is a circle with an odd number of vertices) and this would make it impossible to color the associated graph with two colors.*

(ii) *It should be useful to have also a description of the zeroes of eigenfunctions in the case of domains with piecewise C^∞ boundaries and to describe the local structure of the zeroes near the corners. This is discussed for example in [4] or [5]. The main difficulty for having a local structure lemma at the corner is to show that the solution cannot be flat at the corner, that is cannot decay faster than polynomially. Although quite reasonable³, this seems without additional assumptions to be open and will be discussed further in Section 8.*

3 Graphs and circulation along paths

3.1 Preliminaries

We start with a covering family $\mathcal{D} = \{D_1, \dots, D_\mu\}$ so that

$$N = \overline{\bigcup \partial D_i \setminus \partial \Omega} \in \tilde{\mathcal{N}}(\Omega).$$

We assume that for some given λ , \mathcal{D} satisfies (PCC).

We introduce various normalizations. For $i = 1, \dots, \mu$, let u_i be the positive normalized groundstate of $H(D_i)$. Similarly, for any oriented pair (i, j) such that D_i and D_j are neighbors, let u_{ij} be the eigenfunction of $H(D_{ij})$ having $\partial D_i \cap \partial D_j$ as nodal set, hence D_i and D_j as nodal domains. We normalize u_{ij} and impose that u_{ij} is strictly positive in D_i . Then u_{ij} is uniquely determined and negative in D_j . Note that with this choice

$$u_{ij} = -u_{ji} , \tag{3.1}$$

and that we can write

$$u_{ij} = d_{ij}(u_i - \gamma_{ij}u_j) , \tag{3.2}$$

with γ_{ij} and d_{ij} strictly positive.

We can then write

$$\gamma_{ij} = \exp c_{ij} , \tag{3.3}$$

and we call c_{ij} the circulation from D_i to D_j .

Having in mind (3.1), we get the relations :

$$d_{ij} = d_{ji} \gamma_{ji} , d_{ij} \gamma_{ij} = d_{ji} .$$

³We thank H. Koch for discussions.

In particular we get the important relation :

$$\gamma_{ij}\gamma_{ji} = 1 , \quad (3.4)$$

which in the circulation terminology becomes

$$c_{ij} + c_{ji} = 0 . \quad (3.5)$$

3.2 Good paths

We now consider continuous paths in Ω wandering between the nodal domains. More precisely the following notion of **good path** is useful :

Definition 3.1

We say that the path $[0, 1] \ni t \mapsto \beta(t) \in \Omega$ is **good** (with respect to \mathcal{D}) if

(i) $\beta(0), \beta(1) \in \Omega \setminus N$, $\beta(t) \cap \partial\Omega = \emptyset$ for $t \in [0, 1]$.

(ii) $\beta(t) \cap N^{(\nu)} = \emptyset$ for $\nu \geq 2$ where

$$N^{(\nu)} = \{x \in N \cap \Omega \mid \nu \text{ nodal lines locally cross in } x\}.$$

(iii) If for some $t_0 \in (0, 1)$, $\beta(t_0) \in \partial D_i \cap \partial D_j$, then there is an $\epsilon > 0$ such that, for $t \in (t_0 - \epsilon, t_0)$, $\beta(t) \in D_i$ (or D_j) and for $t_0 \in (t_0, t_0 + \epsilon)$, $\beta(t) \in D_j$ (or D_i).

To any good path, we can associate a finite sequence i_0, i_1, \dots, i_k of indices expressing the restriction of the path to the graph G . We call this restriction the associated G -path and denote it by β_G .

This simply means that the path starts from $\beta(0) \in D_{i_0}$, then leaves D_{i_0} for entering in D_{i_1} and a new index is added at each crossing of a boundary. The length of the path is then exactly the number of crossing of the path.

As usual, we say that the path is closed if $\beta(0) = \beta(1)$. Note that if $\beta(0)$ and $\beta(1)$ belongs to the same D_i then we can always close the path (using the property that D_i is arcwise connected), keeping the corresponding graph fixed, which is in any case a circle.

3.3 Circulation along β_G

We can associate to each good path $\beta(t)$ two numbers

$$\gamma^\beta = \prod_{\ell=0}^{k-1} \gamma_{i_\ell, i_{\ell+1}} ,$$

and

$$\mathcal{C}_\beta = \sum_{\ell=0}^{k-1} c_{i_\ell, i_{\ell+1}} ,$$

the second one being called the circulation along β .

Of course, we have

$$\gamma^\beta = \exp \mathcal{C}_\beta .$$

Note that \mathcal{C}_β depends only on the G -path β_G , so \mathcal{C}_β will also be called the circulation along β_G .

When we deform these paths by homotopy, it is clear that as long as the path keeps the property of being good the circulation is constant.

But one of our goals will be to follow this circulation when changing in the homotopy the corresponding G -path (that is the path in the homotopy does not remain a good path).

4 Proof of the main theorem

4.1 General Compatibility Condition

The general condition for constructing an eigenfunction of $H(\Omega)$ by gluing the u_{ij} of $H(D_{ij})$ is quite reminiscent of the problems occurring for example when analyzing the triviality of a line bundle over a manifold or, as will be seen below, of the analysis of contour integrals in complex analysis. The following criterion is quite natural :

Proposition 4.1

If we have an admissible covering \mathcal{D} in Ω generated by some N in $\tilde{\mathcal{N}}(\Omega)$ such that (PCC) holds for some $\lambda \in \mathbb{R}$, then an eigenfunction of $H(\Omega)$ associated with λ can be constructed if and only if, for any closed path on the graph of length $k > 2$, the condition

$$(GCC) : \prod_{\ell=0}^{k-1} \gamma_{i_\ell, i_{\ell+1}} = 1 , \quad (4.1)$$

is satisfied.

In other words, **the circulation along any closed path on the graph must be 0.**

Remarks 4.2

(i) *At this stage, it is not necessary to assume that Ω is simply connected.*

(ii) *Note that because we are, in a bipartite graph, k has to be even in (4.1).*

Proof of Proposition 4.1

One starts from one domain D_{i_0} and from its groundstate u_{i_0} . Then the extension of u_{i_0} to all neighboring domains is obtained by using assumption (PCC). One can then propagate the extension to the next neighbors till Ω is covered. (GCC) just permits a construction which is independent of the path used for the extension. This gives a global construction of an element u in $\mathcal{H}_0^1(\Omega)$ belonging to $\mathcal{H}_{loc}^2(\Omega \setminus N^c)$ where N^c is the set of critical points of N lying in Ω , and satisfying in the distribution sense

$$(-\Delta + V - \lambda)u = 0 \text{ in } \Omega \setminus N^c .$$

But $(-\Delta + V - \lambda)u$ belongs to $\mathcal{H}^{-1}(\Omega)$ so it remains to show that a distribution in \mathcal{H}^{-1} with support in N^c is 0.

But N^c is a finite set and it is enough to recall the standard

Lemma 4.3

If ω is an open set in \mathbb{R}^2 and $x_0 \in \omega$, there are no distribution in $\mathcal{H}_{loc}^{-1}(\omega)$ with support in $\{x_0\}$.

This achieves the proof of the proposition.

So the proof of our main theorem consists in showing that Condition (4.1) is always satisfied when Ω is simply connected.

In particular, it is not too difficult to see that the condition is automatically satisfied in the case when the graph associated to the covering is a tree.

4.2 Further reduction

We now explain how to reduce the computation of the circulation along a good closed path to the case where a given good path encloses only one or no critical points. The proof is by induction. Suppose that a closed good path γ parametrized by $t \in [0, 1]$ encloses exactly k critical points (with $k \geq 2$). The claim is that we can find two points (inside nodal domains) on this closed path corresponding to times t_0 and t_1 , and construct a continuous curve ℓ_{01} going from $\gamma(t_0)$ to $\gamma(t_1)$ avoiding the critical set such that the path γ_1 defined by $\gamma_1 = \gamma$ on $[0, t_0]$, by ℓ_{01} on $[t_0, t_1]$ (after reparametrization) and by $\gamma_1 = \gamma$ on $[t_1, 1]$ is a good path containing only in its interior one critical point.

If now ℓ_{10} denotes the opposite path to ℓ_{01} , we can consider the closed good path γ_2 such that $\gamma_2 = \ell_{10}$ on $[0, t_0]$, $\gamma_2 = \gamma$ on $[t_0, t_1]$.

It is clear that γ_2 encloses in its interior $(k - 1)$ critical points and that the circulation along γ is the sum of the circulation along γ_1 and of the circulation along γ_2 . If $(k - 1) \geq 2$, we can iterate the procedure till each path encloses at most one critical point.

So the general proof is reduced to the analysis of the condition (4.1), in the case when a good closed path either encloses no critical point or one critical point. This will be the object of the two next subsections.

We will actually show that in these two situations the circulation along the path is zero, when the path is homotopic in Ω to a point, **which is automatically the case if Ω is simply connected.**

4.3 Proof when a good closed path does not enclose any critical point and is homotopic to a point

In this case we can find a homotopy $\gamma(s, t)$ such that $\gamma(0, t)$ is the initial path, $\gamma(1, t)$ is a single point living in some D_i .

Of course the graph trace of the path, i.e. β_G is changing with this homotopy. But since each nodal domain has only finitely many critical points in its boundary a continuity argument shows that, by modifying the homotopy, we can get one for which there are only finitely many s_k for which the paths are no more good paths. Moreover one can pick this homotopy in such a way that, at these s_k , the paths $\gamma(s_k, \cdot)$ have still the property that they are

good except at one point t_k . So the transition near the point⁴ $\gamma(s_k, t_k)$ is the following (or the converse). There exists some pair of neighboring D_i and D_j such that $\gamma(s_k, t_k) \in \partial D_i \cap \partial D_j$. For $s < s_k$ ((s, t) near (s_k, t_k)) the path $\gamma(s, t)$ (see Figure 2) is contained in D_{i_ℓ} , with $i_\ell = i$. For $s = s_k$ and t near t_k , then $\gamma(s_k, t)$ belongs to D_{i_ℓ} except at $\gamma(s_k, t_k)$. For $s > s_k$ (s near s_k), the path enters the neighbor $D_j = D_{i_{\ell+1}^{new}}$, before returning to D_{i_ℓ} and entering $D_{i_{\ell+1}}$. In particular we can pick the homotopy always so that the path avoids any critical point.

So the initial corresponding G -path $i_1, i_2, i_3, \dots, i_{\ell-1}, i_\ell, i_{\ell+1}, \dots, i_k$ becomes $i_1, i_2, i_3, \dots, i_{\ell-1}, i_\ell, i_{\ell+1}^{new}, i_\ell, i_{\ell+1}, \dots, i_k$. The fact that the circulation is conserved in this transformation is an immediate consequence of (3.4).

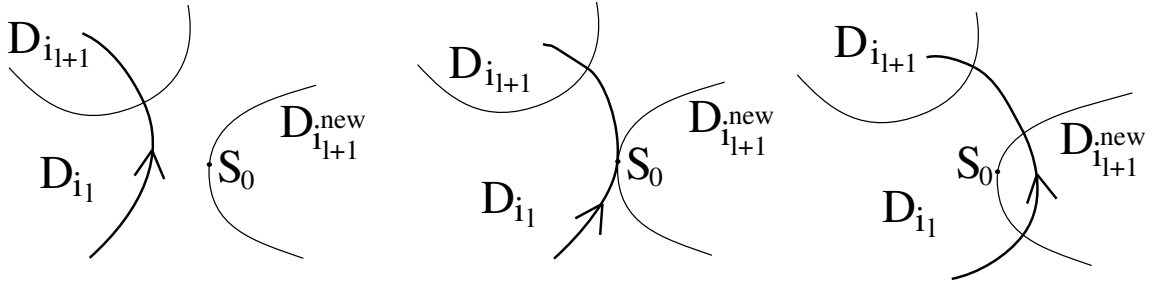


Figure 2: Deformation argument in three pictures : before, at the touching and after.

For the converse transition, we just replace a sequence $i_\ell, i_{\ell+1}^{new}, i_\ell, i_{\ell+1}$ by i_ℓ . After finitely many operations of this type, we will obtain a path reduced to a single point whose G -path is also a point.

Actually the main point here is that the associated closed G -path is a path on a tree !

4.4 Proof when a good closed path encloses a unique critical point and is homotopic to a point.

Let us now consider the case of a closed path encloses a unique critical point. We can reduce the computation to the case when this closed good path is a small circle turning once and positively around this point x_0 . There exists $\nu \geq 2$ so that the ν nodal arcs $\sigma_0, \sigma_1, \dots, \sigma_{\nu-1}$ pass locally through x_0 . This

⁴denoted by S_0 in the figure,

means that there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$

$$N \cap B(x_0, \epsilon) = \cup_{j=0}^{\nu-1} \sigma_j \cap B(x_0, \epsilon)$$

where $B(x_0, \epsilon) = \{x \in \mathbb{R}^2 \mid |x - x_0| < \epsilon\}$. We shall also use the (2ν) half arcs σ_ℓ^+ (such that $\sigma_\ell = \sigma_\ell^+ \cup \sigma_{\ell+\nu}^+$). Without loss we might assume that $x_0 = \{0\}$ and that σ_0 is tangent to the x_1 -axis at x_0 . The arc σ_0 splits $B(x_0, \epsilon)$ in two parts and we denote by $B^+(x_0, \epsilon)$ the upper part which lies "above" σ_0 .

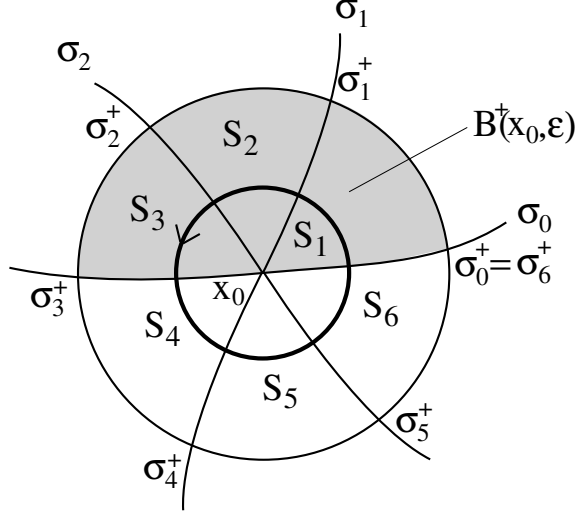


Figure 3: Picture in the case when $\nu = 3$.

Assume now (PCC) for \mathcal{D} . The abstract nodal set (intersected with $B(x_0, \epsilon)$) defines 2ν curved sectors S_ℓ (also intersected with $B(x_0, \epsilon)$)⁵ ($\ell = 1, \dots, 2\nu$) delimited⁶ by $\sigma_{\ell-1}^+$ and σ_ℓ^+ , each one belonging to some D_{i_ℓ} . Take the first domain S_1 starting from σ_0^+ . Starting from u_{i_1} , the pair compatibility condition can be used iteratively to extend the restriction of u_{i_1} to S_1 as a local solution $v_1 \in C^\infty(B^+(x_0, \epsilon))$ such that $(-\Delta + V)v_1 = \lambda v_1$ in $B^+(x_0, \epsilon)$, $v_1 = 0$ for $N \cap B^+(x_0, \epsilon)$ and $v_1 = u_{i_1}$ on S_1 . We can now apply Lemma 2.3 (b) to v_1 in B^+ so that⁷

$$v_1 = c_1 r^\nu \sin \nu \omega + \mathcal{O}(r^{\nu+1})$$

⁵We omit from now on recalling the fact that we are always in a small ball around x_0 . But the whole proof is local.

⁶By convention, $\sigma_{2\nu}^+$ is σ_0^+ .

⁷Here there are two possibilities in the choice of polar coordinates. We can either flatten the boundary of B^+ and take polar coordinates after flattening or keep the initial ones. In any case, the two choices lead to the same main term.

for some constant $c_1 \neq 0$. In particular this means that though we have not assumed that the nodal lines cross at the point x_0 under equal angles this is enforced by Lemma 2.3. The second point is that by restriction to S_1 , we get

$$u_{i_1} = c_1 r^\nu \sin \nu \omega + \mathcal{O}(r^{\nu+1}) \text{ in } S_1 .$$

and a similar expansion is true for $u_{i_1 i_2}$:

$$u_{i_1 i_2} = c_{12} r^\nu \sin \nu \omega + \mathcal{O}(r^{\nu+1}) \text{ in } \text{Int}(\overline{S_1} \cup \overline{S_2}) .$$

Of course we can do the same thing starting from any S_ℓ . So, for each ℓ , we have shown the existence of $c_\ell > 0$ such that u_{i_ℓ} has the asymptotics

$$u_{i_\ell} = -c_\ell r^\nu \sin(\nu(\omega - \omega_{\ell, \ell+1})) + \mathcal{O}(r^{\nu+1}) , \quad (4.2)$$

in S_ℓ , where $\omega_{\ell, \ell+1}$ is the argument of the tangent to $\partial S_\ell \cap \partial S_{\ell+1}$ at 0 and a similar expansion holds for $u_{i_\ell i_{\ell+1}}$.

But reusing (PCC) (through (3.2)) gives that

$$\gamma_{i_\ell, i_{\ell+1}} = \frac{c_{\ell+1}}{c_\ell} . \quad (4.3)$$

Coming back to the definition of the circulation and using (4.3) we get the vanishing of the circulation along the good path enclosing the critical point. This completes the consideration of this case and finishes also the proof of the main theorem.

Remark 4.4

In this section we could also have tried to consider directly $\text{Int}(\overline{S_1} \cup \overline{S_2})$. But then we would need to know more about the structure of the nodal set and the behavior of an eigenfunction near a corner. This will be discussed in Section 8.

5 Examples.

We apply the general constructions above for the analysis of specific examples.

5.1 Three examples whose corresponding graph is a tree

A simple covering.

The left subfigure in Figure 4 presents a covering by five domains, whose

corresponding graph is a tree. Moreover, there are no critical points inside the domain.

The eight.

Take the case of the eight as in the central subfigure of Figure 4 with $(0, 0)$ as critical point. Let D_2 and D_3 the two interior "abstract" nodal domains and D_1 the "exterior" nodal domain meeting $\partial\Omega$. Then $D_1 \sim D_2$ and $D_1 \sim D_3$. Suppose that, for some λ , we have (PCC). We apply Lemma 2.3 (case **(b)**) locally to u_{12} in $D_{12}^{r_0, \pm}$, ($r_0 > 0$ small enough), where $D_{12}^{r, \pm} = \text{Int} \left(\overline{D_2^r} \cup \overline{D_1^{r, \pm}} \right)$, $D_1^{r, \pm}$ being the two components (for r small enough) of $D_1 \cap B(0, r)$. This shows that the groundstate u_1 (which is up to a multiplicative constant the restriction to D_1 of u_{12}) satisfies

$$u_1 = c_1 r^2 \sin(2(\omega - \omega_{\pm})) + \mathcal{O}(r^3) ,$$

near $x_0 = 0$, where the constant c_1 is the same for the two "opposite" sectors describing D_1 near x_0 . But the associate graph is a tree. It is trivial in this case that the trace of a good closed path on the graph has always zero circulation. We do not need to use the information given by the local analysis around the critical point. The only additional information given by this analysis is that u_1 has the same asymptotics near 0 in the two opposite sectors.

The clover leaf intersection.

This example (right subfigure in Figure 4) does not lead to any difficulty. The graph is a tree. On can directly extend from D_1 toward respectively D_2 , D_3 and D_4 .

5.2 Examples with circles

The cross.

When Ω is the disk $B(0, 1)$, the cross (say $\{x_1 = 0\} \cup \{x_2 = 0\}$) determines four nodal domains D_j ($j = 1, 2, 3, 4$) (see the left subfigure in Figure 5 and its corresponding graph below) so we have a "circle" $(1, 2, 3, 4)$. The corresponding graph can be represented by a square. Here we cannot avoid the local analysis around the center.

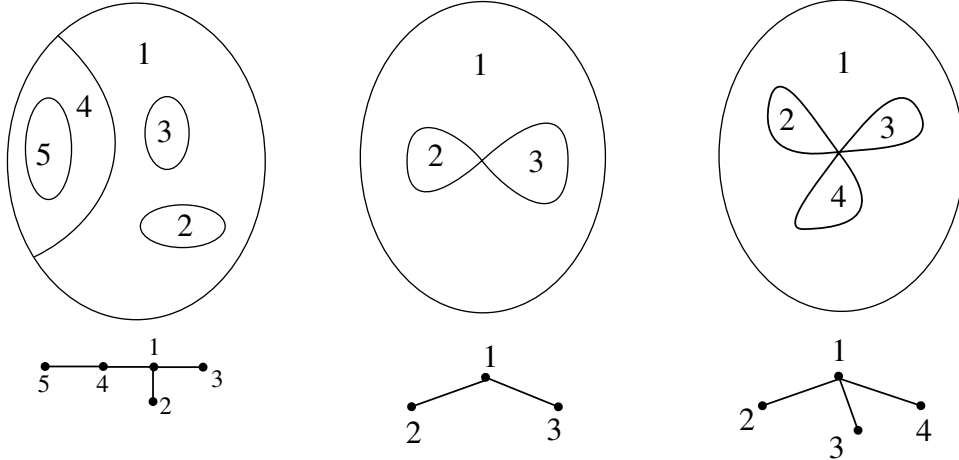


Figure 4: Three examples with associated graph below : simple, eight and clover.

Two intersecting circles.

Although the corresponding graph is the same, the nodal structure is different. There are two critical points. There are three typical paths enclosing critical points. One is living in D_1 and turning positively and once around $D_2 \cup D_3 \cup D_4$. By definition, the circulation along this path is trivial. The second one is a small positively oriented circle around the left critical point. Its corresponding trace in the graph is the sequence $(1, 2, 3, 4)$. Here we need to perform the local analysis.

The third one is a small positively oriented circle around the right critical point. Its corresponding trace in the graph is the sequence $(4, 3, 2, 1)$. One can perform the local analysis but also observe that the circulation along this path is just the opposite of the previous one.

6 Sufficient conditions in the non simply connected case

In the non simply connected case, what remains from the previous proof can be formulated as follows.

Proposition 6.1

In each homotopy class of Ω , all the good paths have the same circulation.

In particular we can speak of a circulation attached to a homotopy class.

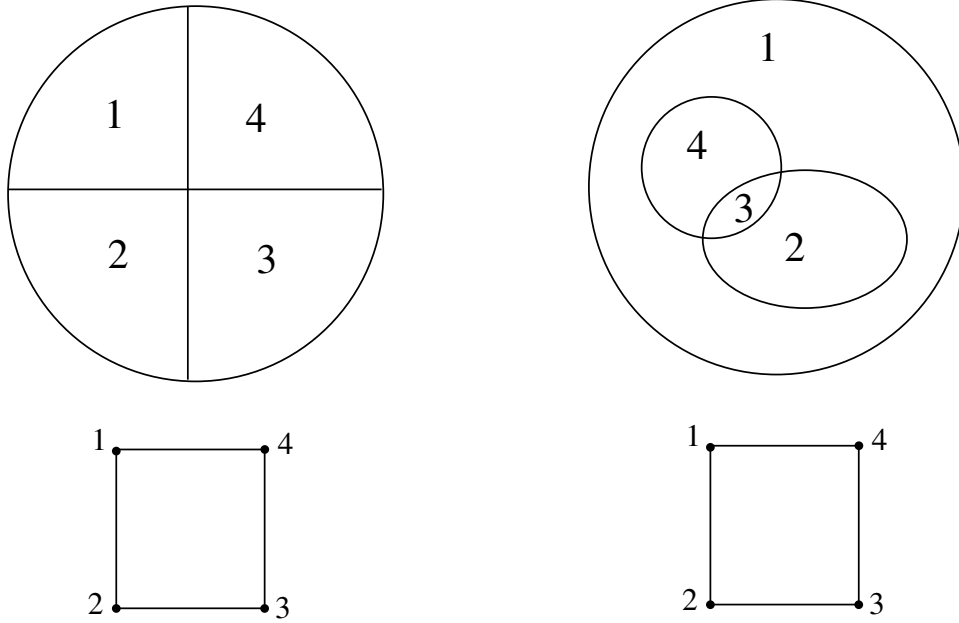


Figure 5: Two other examples in simply connected domains with square graph.

Remark 6.2

As a consequence of Proposition 6.1, for a given covering family satisfying (ANSP) and (PCC), the proof that (GCC) holds is reduced to the proof that in each homotopy class there is a representative with circulation 0. Of course we recover in the simply connected case Theorem 1.3.

Remark 6.3

It is actually enough (using the properties of the fundamental group of Ω) to verify (GCC) for a set of generators of this group.

In this spirit the case of one hole can be treated in greater detail. The homotopy group is generated by the (class of) simple path (s) turning once and anticlockwise around the hole. This leads to the following sufficient condition :

Proposition 6.4

We assume that Ω has just one hole and that, for some covering \mathcal{D} and λ , (PCC) is satisfied. If there is a good path of index 1 around the hole, which intersects the abstract nodal set at at most two points, then the conclusions of Theorem 1.3 hold.

The proof is immediate using Proposition 6.1, Remark 6.2 and the fact that the circulation along a good path with no crossing is 0 by definition and is also 0 in the case of two crossings by (PCC).

Typically the assumptions are satisfied when the hole or the exterior boundary are hit by no or two nodal lines. The left subfigure in Figure 10 can be treated in the same way (its associated graph is actually a tree). We believe that the last criterion is in the case of one hole optimal.

7 On the optimality in the non simply connected case

Let us consider the case with one hole and let λ be an eigenvalue of multiplicity 1 such that the corresponding eigenfunction u has at least four nodal domains. We would like to present a family of examples for which one can then construct a new potential so that the main theorem does not hold any more in spite of (PCC).

7.1 Examples where (PCC) is not sufficient

We first discuss here a preliminary case where the proof is easier to analyze.

Assumption 7.1

We assume that there exists one nodal domain D such that $\Omega \setminus D$ becomes simply connected and such that the boundary of D with each of its neighbors is connected. We also assume that $\partial D \cap \partial\Omega$ meets the regular parts of the exterior boundary and of the interior boundary.

Then the claim is

Proposition 7.2

Under the previous assumptions, we can find a new potential V_ϵ such that λ satisfies the pair compatibility condition corresponding to $N(u)$ and such that λ is not an eigenvalue of $-\Delta + V_\epsilon$ with an eigenfunction having the same nodal domain as u .

The proof is inspired by the analysis of the case of the circle presented in the introduction.

By assumption ∂D contains two distinct non crossing continuous curves L_{\pm} joining the two boundaries.

We proceed by constructing a C^{∞} function b in D , such that the support of ∇b does not meet L_{\pm} , $b = 0$ near L_+ , and $b = b_-$ near L_- , where $b_- \in \mathbb{R} \setminus \{0\}$. Moreover, we can require that

$$\nabla b \cdot n = 0 \text{ on } \partial\Omega \cap \partial D, \quad (7.1)$$

where n is the outward normal to $\partial\Omega$.

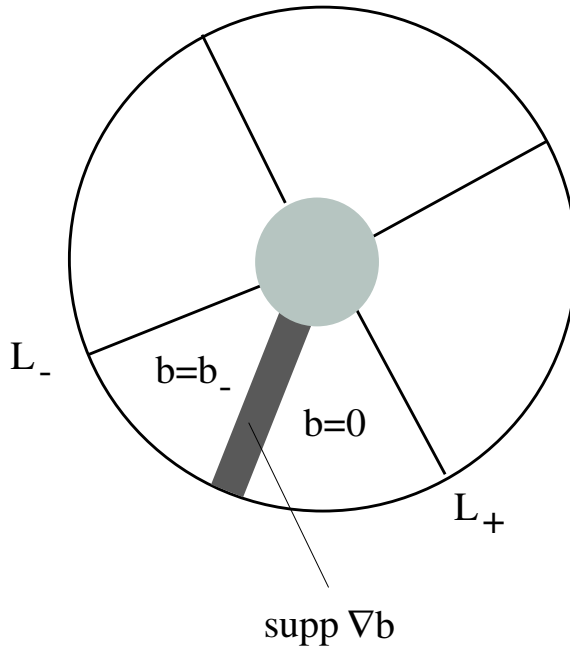


Figure 6: Construction of b in the simplest case

We note that b can be extended by 0 outside D to $\Omega \setminus L_-$ but **NOT** to Ω . On the contrary, ∇b and Δb can be extended to the whole Ω !

We now introduce

$$u_{\epsilon} = (1 + \epsilon b)u. \quad (7.2)$$

We note that u_{ϵ} is well defined in $\Omega \setminus L_-$ and can also admit an extension, when crossing L_- , by $(1 + \epsilon b_-)u$ to the neighbors of D touching D in L_- . Moreover, if $\epsilon \neq 0$, it cannot be extended to a C^{∞} function in Ω .

We now claim that, for small enough $\epsilon \neq 0$, u_ϵ is an eigenfunction of $-\Delta + V_\epsilon$ in $\Omega \setminus L_-$, with eigenvalue λ and with

$$V_\epsilon := V + \epsilon \left[\frac{\Delta b}{1 + \epsilon b} + 2 \frac{\nabla b \cdot \nabla u}{u_\epsilon} \right]. \quad (7.3)$$

We observe, using the property (7.1) and that ∇u is not vanishing on the boundary on the support of ∇b , by Hopf's boundary point Lemma (see [9]), that V_ϵ admits a C^∞ extension to Ω .

Now (PCC) is satisfied for λ , $-\Delta + V_\epsilon$ in Ω and the family associated to $N(u)$. If λ was an eigenvalue of the Dirichlet realization of $-\Delta + V_\epsilon$ in Ω with a corresponding eigenfunction v_ϵ with nodal set $N(u)$, then comparing v_ϵ and u_ϵ in D , we would get $v_\epsilon = c_\epsilon u_\epsilon$.

Remark 7.3

Note that we do not know if λ is an eigenvalue of the Dirichlet realization of $-\Delta + V_\epsilon$ in Ω .

But v_ϵ is C^∞ in Ω and u_ϵ has a discontinuity ! Hence a contradiction.

Remark 7.4

One could ask naturally if replacing (PCC) by the Triple Compatibility Condition (TCC) will lead to other results. Using the same ideas as above, it is easy to construct examples for which (TCC) does not imply (GCC).

7.2 Further examples and analysis of the optimality

We still discuss the case with one hole. The construction of the previous subsection can be extended in the following more general situation.

We keep the same starting point and assume now that we have found a closed set F in $\overline{\Omega}$ which has the following properties :

- (i) There exists a set of indices I_F such that :

$$F = \cup_{i \in I_F} \overline{D}_i .$$

- (ii) $\Omega \setminus F$ is simply connected and the intersection of its boundary with $\partial\Omega$ has two components contained respectively in each connected component of $\partial\Omega$, each one being an arc with non empty interior.

- (iii) $\partial F \setminus \partial \Omega$ is the union of two distinct non crossing continuous nodal lines (possibly touching at isolated critical points) L_- and L_+ , joining the two components of $\partial \Omega$.
- (iv) For any pair of neighboring nodal domains D_i, D_j such that $D_i \subset F$ and $D_j \subset \Omega \setminus F$, $\partial D_i \cap \partial D_j$ meets only L_- or L_+ .

Under these assumptions, the proof given in Subsection 7.1 goes through with D replaced by $\Omega \setminus F$. The generalization is that $\Omega \setminus F$ can contain nodal lines. So in addition with the previous constraints, the construction of b should be done more carefully, with the additional conditions that ∇b vanishes in the neighborhood of critical points and is orthogonal to ∇u when $u = 0$. This can be done by first constructing a smooth line in $\overline{\Omega} \setminus F$ starting from the interior boundary and ending at the exterior boundary, with the properties that it crosses orthogonally the boundaries, avoids all the critical points, has no selfintersection, and intersects nodal lines orthogonally. The function b can then be constructed by integrating along a vector field obtained by extending in a tube around this line the normal vector field to the line. This implies that the potential V_ϵ is C^∞ .

The last condition (iv) in the above enumeration permits to verify that (PCC) is satisfied as in the previous case. The function u_ϵ is defined as before in $\Omega \setminus L_-$ and cannot be extended as a C^∞ function in Ω . Looking at (PCC) for λ and $-\Delta + V_\epsilon$ in Ω , we first see that the compatibility condition is clearly satisfied for the pairs of neighboring nodal domains contained simultaneously in F or $\Omega \setminus F$. Then the condition permits to treat the case when the neighbors belong respectively to F and $\Omega \setminus F$. One has just to observe that the property that only L_- or L_+ is involved in the extension in D_{ij} permits to use u or $(1 + \epsilon b_-)u$ for the extension.

Conjecture 7.5

If any good path turning once anticlockwise around the hole meets the nodal set at at least three points (hence four), one can always construct F with the above properties.

Towards the construction of \mathbf{F}

We color our covering with two colours, (+) and (-), and consider $\overline{\Omega}_+$ the closure of the nodal domains corresponding to the positive sign and define similarly $\overline{\Omega}_-$. We now take in $\overline{\Omega}_+$ a minimal closed, arcwise connected, set satisfying (i) and touching the two components of $\partial \Omega$. This gives us some

F_+ . We note that the assumption of minimality with respect to these properties implies that F_+ can only contain two distinct (that is non touching except at critical points) non crossing nodal lines L_- and L_+ joining the two components of the boundary of Ω and they are necessarily contained in the boundary of F_+ . Then the hope is that $\Omega \setminus F$ should be the exterior connected component \mathcal{C}_- of $\Omega \setminus F_+$. Examples show that it does not always work. Although, $\overline{\mathcal{C}_-}$ contains by assumption and construction four distinct continuous nodal lines joining the two boundaries, $\Omega \setminus F_+$ could be not connected (in the case these four lines have a common touching point in Ω) and also in the case when it is connected, we could still have problem with condition (ii) if these four lines touched the interior boundary at the same point. If this does not work, the next try consists in exchanging the roles of $+$ and $-$ by looking for an F in $\overline{\mathcal{C}_-}$. The analysis of examples shows that it works very often but we have no proof that this works in full generality.

Note also that a sufficient condition for the construction is the existence of one continuous nodal line joining the two boundaries and not touching simultaneously any point of L_- and L_+ .

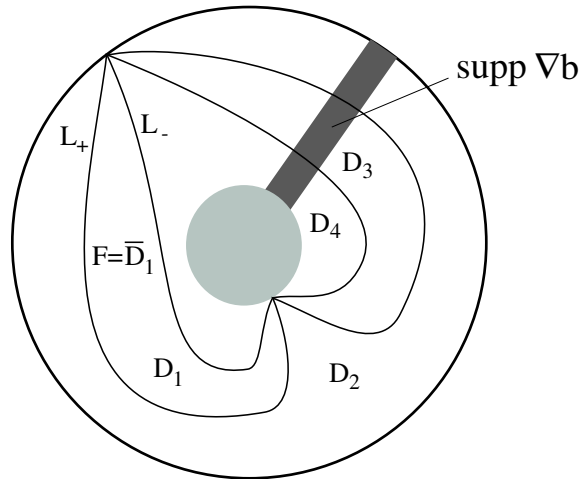


Figure 7: A new example

In the example described by Figure 7, we can take $I_F = \{1\}$, hence in this example $F = \overline{D_1}$. The assumptions of the previous subsection are not satisfied but the construction of this subsection can be applied.

Another example which did not enter in the domain of validity of the

analysis in Subsection 7.1, is given by the right subfigure of Figure 8 (take $I_F = \{1, 4\}$ and see the right subfigure of Figure 9 for the description of the support of ∇b).

7.3 On the sufficiency of (PCC) in the case of holes.

Although it is very likely that the same techniques work in the case of many holes, we do not have (except what was said in Remarks 6.2 and 6.3), general compact assumptions or conjectures like in the case of one hole.

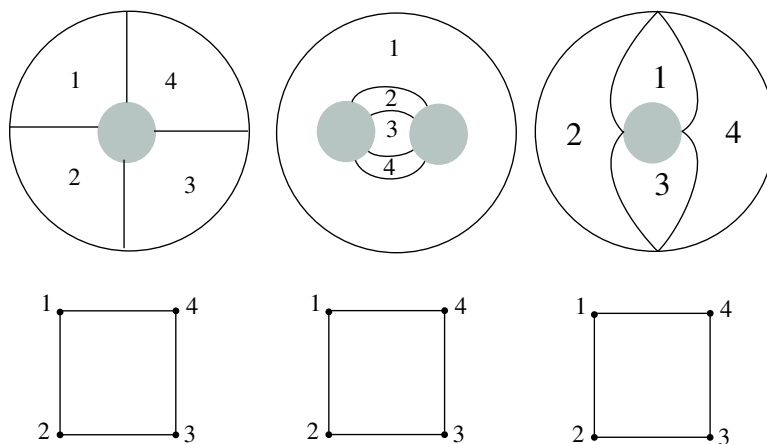


Figure 8: Non simply connected examples with square graph

For example, in the middle subfigure of Figure 8, we can follow the previous proof by constructing b such that ∇b is supported in D_3 as described in the middle subfigure of Figure 9.

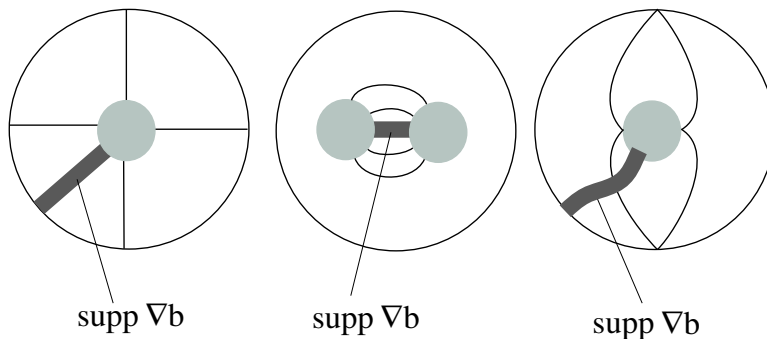


Figure 9: ... and the corresponding localization of ∇b

Figure 10 below gives two examples where Theorem 1.3 holds. The left subfigure corresponds to a tree. For the right subfigure there is a circle (2543). But following Remark 6.3, we observe that the fundamental group is generated by the trivial path, a simple path turning once around the right hole and a simple path turning once around the two holes. In each of these last two homotopy classes, we can find a representative with respectively two crossings and no crossing.

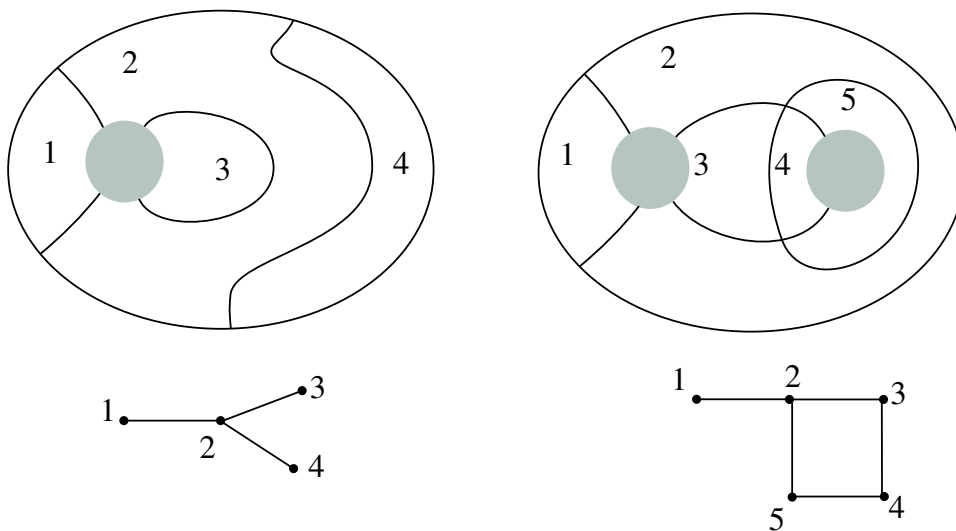


Figure 10: Non simply connected examples where Theorem 1.3 holds

8 On the local structure of the nodal set near the corner

Though for our results a detailed investigation of the local behaviour near the boundary of Ω was not necessary we collect in this section a few results which are of independent interest and present a natural conjecture. These results could actually permit to propose weaker assumptions in the Abstract Nodal Set Property.

We consider a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$ which satisfies a uniform interior cone condition. A uniform interior cone condition means for the two dimensional case that we can inscribe at each corner $y_i \in \partial\Omega$ a sector with opening angle say, $\alpha_i \in (0, 2\pi)$, and sufficiently small radius $\rho(y_i)$. More precisely, we have :

Proposition 8.1

Let u be an eigenfunction of $H(\Omega)$ and assume that u has near a corner y_i a nodal domain D such that $y_i \in \partial\Omega$ and so that D satisfies a uniform interior cone condition at y_i . Then u cannot be flat at y_i .

This is an immediate consequence of a local version of the following theorem of Davies-Simon [7] (see also B. Davies [6] – Theorem 4.6.8 and Remark 4.6.10–) which states that the ground state of the Dirichlet Laplacian (or more generally the Dirichlet realization of a Schrödinger operator with regular coefficients) in a bounded domain Ω satisfying an interior cone condition, then there exists $c > 0$ and $\alpha > 0$ such that

$$u(x) \geq cd(x, \partial\Omega)^\alpha .$$

The proof relies on Harnack's inequality⁸ combined with a comparison theorem⁹.

When u is not flat at y_i , a classical theorem of Kondratiev [13] gives (see [4] and [5]) :

Proposition 8.2

Suppose $\Omega \subset \mathbb{R}^2$ is locally diffeomorphic¹⁰ near the corner to a sector of angle

⁸See also [2] or [15], which permit extensions to less regular cases

⁹See also [11], permitting a better control of the non-flatness

¹⁰We take here a local diffeomorphism which preserves the angle at the corner. So the derivative of the diffeomorphism at the corner is assumed to be a rotation and the angle α is well defined.

α . Let $W \in C^\infty(\overline{\Omega})$ and let u in $\mathcal{H}_0^1(\Omega \cap B(0, r_0))$ be a distribution solution of $(-\Delta + W)u = 0$ in $\Omega \cap B(0, r_0)$. Assume also that u is not flat at the corner. Then, after a local diffeomorphism sending Ω on a sector, there exists ℓ and $C \neq 0$ such that :

$$u(r, \omega) = C r^{\frac{\ell\pi}{\alpha}} \sin \frac{\ell(\omega - \omega_0)\pi}{\alpha} + o(r^{\frac{\ell\pi}{\alpha}}). \quad (8.1)$$

Note that the leading term is harmonic and homogeneous in the sector. Kondratiev gives actually a complete expansion for which (because we are in the case of Schrödinger) only higher order terms can contain logarithmic terms.

Corollary 8.3

Let $\Omega \subset \mathbb{R}^2$ be again a bounded domain as above. Then the groundstate of $H(\Omega)$ satisfies (8.1) with $\ell = 1$ near each corner.

For excited states the proof also goes through as long as we assume that one of the nodal domains at each corner satisfies an interior cone condition¹¹ or that the eigenfunction we look at is not flat near the corners.

From the above it is natural to expect that the following strengthening of Proposition 8.2 holds true.

Conjecture 8.4

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary satisfying a uniform interior cone condition. Let $V \in C^\infty(\Omega)$ be real valued, then any real valued eigenfunction $u \not\equiv 0$ satisfying $H(\Omega)u = \lambda u$ has in the neighborhood of a corner with opening angle α locally the behaviour given by (8.1). In particular each nodal domain of u satisfies an interior cone condition.

Remarks 8.5

- (i) The conjecture holds if we assume that V is real analytic and Ω is a polygon, see [4].
- (ii) What is missing for a possible proof of Conjecture 8.4 is for example a suitable version of strong unique continuation in the neighborhood of a corner.

¹¹This was for example the case in the previous sections when considering u_{ij} .

9 Final remarks

In this paper we have analyzed some of the properties (local, global, spectral) satisfied by a family of sets formed by nodal domains of an eigenfunction.

We have then proposed a sufficient natural Pair Compatibility Condition permitting to glue together eigenfunctions attached to each pair of neighboring domains.

We have shown its sufficiency in the case when Ω is a simply connected open set in \mathbb{R}^2 and described how one can extend the analysis in the non simply connected situation. The analysis of a family of examples shows that the sufficient conditions we have proposed are in some sense not far from optimal.

Except trivial cases, where no circle in the corresponding graph can occur, the analysis of the same question in dimension > 2 is completely open. A precise description of the structure of the nodal set of the eigenfunction near its critical point is indeed missing.

Let us finally mention that the same problem can be considered for a Schrödinger operator on a surface either with or without boundary. Then the genus has to play a role. While for the flat case and actually also for the sphere $N \in \tilde{\mathcal{N}}(\Omega)$ automatically guarantees that the nodal domains created by $\Omega \setminus N$ lead to an admissible family \mathcal{D} this has to be required for the case of surfaces in general. Take for instance the torus : then an N which is just a simple closed loop which is not zero-homotopic creates just one nodal domain, hence not an admissible \mathcal{D} .

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