Topics in Spectral Theory

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Contents

These lecture notes are an expanded version of the lectures I gave in the Summer School on Open Quantum Systems, held in Grenoble June 16—July 4, 2003. Shortly afterwards, I also lectured in the Summer School on Large Coulomb Systems—QED, held in Nordfjordeid August 11—18, 2003 [JKP]. The Nordfjordeid lectures were a natural continuation of the material covered in Grenoble, and [JKP] can be read as Section 6 of these lecture notes.

The subject of these lecture notes is spectral theory of self-adjoint operators and some of its applications to mathematical physics. This topic has been covered in many places in the literature, and in particular in [Da, RS1, RS2, RS3, RS4, Ka]. Given the clarity and precision of these references, there appears to be little need for additional lecture notes on the subject. On the other hand, the point of view adopted in these lecture notes, which has its roots in the developments in mathematical physics which primarily happened over the last decade, makes the notes different from most expositions and I hope that the reader will learn something new from them.

The main theme of the lecture notes is the interplay between spectral theory of self-adjoint operators and classical harmonic analysis. In a nutshell, this interplay can be described as follows. Consider a self-adjoint operator A on a Hilbert space $\mathcal{H}$ and a vector $\varphi \in \mathcal{H}$. The function

$$F(z) = (\varphi|(A-z)^{-1}\varphi)$$

is analytic in the upper half-plane $\mathrm{Im}\, z > 0$ and satisfies the bound $|F(z)| \leq \|\varphi\|^2/\mathrm{Im}\, z$. By a well-known result in harmonic analysis (see Theorem 2.11) there exists a positive Borel measure μ_φ on $\mathbb{R}$ such that for $\mathrm{Im}\, z > 0$,

$$F(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(x)}{x-z}.$$

The measure μ_φ is *the spectral measure* for A and φ. Starting with this definition we will develop the spectral theory of A. In particular, we will see that many properties of the spectral measure can be characterized by the boundary values $\lim_{y\downarrow 0} F(x+\mathrm{i}y)$ of the corresponding function F. The resulting theory is mathematically beautiful and has found many important applications in mathematical physics. In Section 4 we will discuss a simple and important application to the spectral theory of rank one perturbations. A related application concerns the spectral theory of the Wigner-Weisskopf atom and is discussed in the lecture notes [JKP].

Although we are mainly interested in applications of harmonic analysis to spectral theory, it is sometimes possible to turn things around and use the spectral theory to prove results in harmonic analysis. To illustrate this point, in Section 4 we will prove Boole's equality and the celebrated Poltoratskii theorem using spectral theory of rank one perturbations.

The lecture notes are organized as follows. In Section 1 we will review the results of the measure theory we will need. The proofs of less standard results are given in detail. In particular, we present detailed discussion of the differentiation of measures based on the Besicovitch covering lemma. The results of harmonic analysis we will need are discussed in Section 2. They primarily concern Poisson and Borel transforms of measures and most of them can be found in the classical references [Kat, Ko]. However, these references are not concerned with applications of harmonic analysis to spectral theory, and the reader would often need to go through a substantial body of material to extract the needed results. To aid the reader, we have provided proofs of all results discussed in Section 2. Spectral theory of self-adjoint operators is discussed in Section 3. Although this section is essentially self-contained, many proofs are omitted and the reader with no previous exposition to spectral theory would benefit by reading it in parallel with Chapters VII-VIII of [RS1] and Chapters I-II of [Da]. Spectral theory of rank one perturbations is discussed in Section 4.

Concerning the prerequisites, it is assumed that the reader is familiar with basic notions of real, functional and complex analysis. Familiarity with [RS1] or the first ten Chapters of [Ru] should suffice.

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1 Preliminaries: measure theory

1.1 Basic notions

Let M be a set and $\mathcal{F}$ a σ-algebra in M. The pair $(M, \mathcal{F})$ is called a measure space. We denote by χ_A the characteristic function of a subset $A \subset M$.

Let μ be a measure on $(M, \mathcal{F})$. We say that μ is concentrated on the set $A \in \mathcal{F}$ if $\mu(E) = \mu(E \cap A)$ for all $E \in \mathcal{F}$. If $\mu(M) = 1$, then μ is called a probability measure.

Assume that M is a metric space. The minimal σ-algebra in M that contains all open sets is called the Borel σ-algebra. A measure on the Borel σ-algebra is called a Borel measure. If μ is Borel, the complement of the largest open set $\mathcal{O}$ such that $\mu(\mathcal{O}) = 0$ is called the support of μ and is denoted by $\operatorname{supp} \mu$.

Assume that M is a locally compact metric space. A Borel measure μ is called *regular* if for every $E \in \mathcal{F}$,

$$\mu(E) = \inf\{\mu(V) \,:\, E \subset V,\, V \text{ open}\}$$
$$= \sup\{\mu(K) \,:\, K \subset E,\, K \text{ compact}\}.$$

Theorem 1.1 *Let M be a locally compact metric space in which every open set is σ-compact* (*that is, a countably union of compact sets*). *Let μ be a Borel measure finite on compact sets. Then μ is regular.*

The measure space we will most often encounter is $\mathbb{R}$ with the usual Borel σ-algebra. Throughout the lecture notes we will denote by $\mathbb{1}$ the constant function $\mathbb{1}(x) = 1 \; \forall x \in \mathbb{R}$.

1.2 Complex measures

Let $(M, \mathcal{F})$ be a measure space. Let $E \in \mathcal{F}$. A countable collection of sets $\{E_i\}$ in $\mathcal{F}$ is called a partition of E if $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E = \cup_j E_j$. A *complex measure* on $(M, \mathcal{F})$ is a function $\mu : \mathcal{F} \to \mathbb{C}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \tag{1.1}$$

for every $E \in \mathcal{F}$ and *every* partition $\{E_i\}$ of E. In particular, the series (1.1) is absolutely convergent.

Note that complex measures take only finite values. The usual positive measures, however, are allowed to take the value ∞. In the sequel, the term *positive measure* will refer to the standard definition of a measure on a σ-algebra which takes values in $[0, \infty]$.

The set function $|\mu|$ on $\mathcal{F}$ defined by

$$|\mu|(E) = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all partitions $\{E_i\}$ of E, is called the total variation of the measure μ.

Theorem 1.2 *Let μ be a complex measure. Then:*
(1) *The total variation $|\mu|$ is a positive measure on $(M, \mathcal{F})$.*
(2) $|\mu|(M) < \infty$.
(3) *There exists a measurable function $h : M \to \mathbb{C}$ such that $|h(x)| = 1$ for all $x \in M$ and*

$$\mu(E) = \int_E h(x) \mathrm{d}|\mu|(x)$$

for all $E \in \mathcal{F}$. The last relation is abbreviated $\mathrm{d}\mu = h\mathrm{d}|\mu|$.

A complex measure μ is called regular if $|\mu|$ is a regular measure. Note that if ν is a positive measure, $f \in L^1(M, \mathrm{d}\nu)$ and

$$\mu(E) = \int_E f \mathrm{d}\nu,$$

then

$$|\mu|(E) = \int_E |f| \mathrm{d}\nu.$$

The integral with respect to a complex measure is defined in the obvious way, $\int f \mathrm{d}\mu = \int f h \mathrm{d}|\mu|$.

Notation. Let μ be a complex or positive measure and $f \in L^1(M, \mathrm{d}|\mu|)$. In the sequel we will often denote by $f\mu$ the complex measure

$$(f\mu)(E) = \int_E f \mathrm{d}\mu.$$

Note that $|f\mu| = |f||\mu|$.

Every complex measure can be written as a linear combination of four finite positive measures. Let $h_1(x) = \mathrm{Re}\, h(x)$, $h_2(x) = \mathrm{Im}\, h(x)$, $h_i^+(x) = \max(h_i(x), 0)$, $h_i^-(x) = -\min(h_i(x), 0)$, and $\mu_i^\pm = h_i^\pm |\mu|$, $i = 1, 2$. Then

$$\mu = (\mu_1^+ - \mu_1^-) + \mathrm{i}(\mu_2^+ - \mu_2^-).$$

A complex measure μ which takes values in $\mathbb{R}$ is called a *signed* measure. Such a measure can be decomposed as

$$\mu = \mu^+ - \mu^-,$$

where $\mu^+ = (|\mu| + \mu)/2$, $\mu^- = (|\mu| - \mu)/2$. If $A = \{x \in M : h(x) = 1\}$, $B = \{x \in M : h(x) = -1\}$, then for $E \in \mathcal{F}$,

$$\mu^+(E) = \mu(E \cap A), \qquad \mu^-(E) = -\mu(E \cap B).$$

This fact is known as the Hahn decomposition theorem.

1.3 Riesz representation theorem

In this subsection we assume that M is a locally compact metric space.

A continuous function $f : M \to \mathbb{C}$ *vanishes at infinity* if $\forall \epsilon > 0$ there exists a compact set K_ϵ such that $|f(x)| < \epsilon$ for $x \notin K_\epsilon$. Let $C_0(M)$ be the vector space of all continuous functions that vanish at infinity, endowed with the supremum norm $\|f\| = \sup_{x \in M} |f(x)|$. $C_0(M)$ is a Banach space and we denote by $C_0(M)^*$ its dual. The following result is known as the Riesz representation theorem.

Theorem 1.3 *Let $\phi \in C_0(M)^*$. Then there exists a unique regular complex Borel measure μ such that*

$$\phi(f) = \int_M f \mathrm{d}\mu$$

for all $f \in C_0(M)$. Moreover, $\|\phi\| = |\mu|(M)$.

1.4 Radon-Nikodym theorem

Let $(M, \mathcal{F})$ be a measure space. Let ν_1 and ν_2 be complex measures concentrated on disjoint sets. Then we say that ν_1 and ν_2 are mutually singular (or orthogonal), and write $\nu_1 \perp \nu_2$. If $\nu_1 \perp \nu_2$, then $|\nu_1| \perp |\nu_2|$.

Let ν be a complex measure and μ a positive measure. We say that ν is absolutely continuous w.r.t. μ, and write $\nu \ll \mu$, if $\mu(E) = 0 \Rightarrow \nu(E) = 0$. The following result is known as the Lebesgue-Radon-Nikodym theorem.

Theorem 1.4 *Let ν be a complex measure and μ a positive σ-finite measure on $(M, \mathcal{F})$. Then there exists a unique pair of complex measures $\nu_{\rm a}$ and $\nu_{\rm s}$ such that $\nu_{\rm a} \perp \nu_{\rm s}$, $\nu_{\rm a} \ll \mu$, $\nu_{\rm s} \perp \mu$, and*

$$\nu = \nu_{\rm a} + \nu_{\rm s}.$$

Moreover, there exists a unique $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$ such that $\forall E \in \mathcal{F}$,

$$\nu_{\rm a}(E) = \int_E f \mathrm{d}\mu.$$

The Radon-Nikodym decomposition is abbreviated as $\nu = f\mu + \nu_{\rm s}$ (or $\mathrm{d}\nu = f\mathrm{d}\mu + \mathrm{d}\nu_{\rm s}$).

If $M = \mathbb{R}$ and μ is the Lebesgue measure, we will use special symbols for the Radon-Nikodym decomposition. We will denote by $\nu_{\rm ac}$ the part of ν which is absolutely continuous (abbreviated ac) w.r.t. the Lebesgue measure and by $\nu_{\rm sing}$ the part which is singular with respect to the Lebesgue measure. A point $x \in \mathbb{R}$ is called an atom of ν if $\nu(\{x\}) \neq 0$. Let A_ν be the set of all atoms of ν. The set A_ν is countable and $\sum_{x \in A_\nu} |\nu(\{x\})| < \infty$. The pure point part of ν is defined by

$$\nu_{\rm pp}(E) = \sum_{x \in E \cap A_\nu} \nu(\{x\}).$$

The measure $\nu_{\rm sc} = \nu_{\rm sing} - \nu_{\rm pp}$ is called the singular continuous part of ν.

1.5 Fourier transform of measures

Let μ be a complex Borel measure on $\mathbb{R}$. Its Fourier transform is defined by

$$\hat{\mu}(t) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx} \mathrm{d}\mu(x).$$

$\hat{\mu}(t)$ is also called the characteristic function of the measure μ. Note that

$$|\hat{\mu}(t+h) - \hat{\mu}(t)| \leq \int_{\mathbb{R}} \left|\mathrm{e}^{-\mathrm{i}hx} - 1\right| \mathrm{d}|\mu|,$$

and so the function $\mathbb{R} \ni t \mapsto \hat{\mu}(t) \in \mathbb{C}$ is uniformly continuous.

The following result is known as the Riemann-Lebesgue lemma.

Theorem 1.5 *Assume that μ is absolutely continuous w.r.t. the Lebesgue measure. Then*

$$\lim_{|t|\to\infty} |\hat{\mu}(t)| = 0. \tag{1.2}$$

The relation (1.2) may hold even if μ is singular w.r.t. the Lebesgue measure. The measures for which (1.2) holds are called Rajchman measures. A geometric characterization of such measures can be found in [Ly].

Recall that A_ν denotes the set of atoms of μ. In this subsection we will prove the Wiener theorem:

Theorem 1.6 *Let μ be a signed Borel measure. Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2 \mathrm{d}t = \sum_{x\in A_\nu} \mu(\{x\})^2.$$

Proof: Note first that

$$|\hat{\mu}(t)|^2 = \hat{\mu}(t)\overline{\hat{\mu}(t)} = \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}t(x-y)}\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Let

$$K_T(x,y) = \frac{1}{T}\int_0^T \mathrm{e}^{-\mathrm{i}t(x-y)}\mathrm{d}t = \begin{cases} (1-\mathrm{e}^{-\mathrm{i}T(x-y)})/(\mathrm{i}T(x-y)) & \text{if } x\neq y,\\ 1 & \text{if } x=y.\end{cases}$$

Then

$$\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\mathrm{d}t = \int_{\mathbb{R}^2} K_T(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Obviously,

$$\lim_{T\to\infty} K_T(x,y) = \begin{cases} 0 & \text{if } x\neq y,\\ 1 & \text{if } x=y.\end{cases}$$

Since $|K_T(x,y)|\leq 1$, by the dominated convergence theorem we have that for all x,

$$\lim_{T\to\infty}\int_{\mathbb{R}} K_T(x,y)\mathrm{d}\mu(y) = \mu(\{x\}).$$

By Fubini's theorem,

$$\int_{\mathbb{R}^2} K_T(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} K_T(x,y)\mathrm{d}\mu(y)\right]\mathrm{d}\mu(x),$$

and by the dominated convergence theorem,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\mathrm{d}t &= \int_{\mathbb{R}} \mu(\{x\})\mathrm{d}\mu(x)\\ &= \sum_{x\in A_\nu}\mu(\{x\})^2.\end{aligned}$$

1.6 Differentiation of measures

We will discuss only the differentiation of Borel measures on $\mathbb{R}$. The differentiation of Borel measures on $\mathbb{R}^n$ is discussed in the problem set.

We start by collecting some preliminary results. The first result we will need is the Besicovitch covering lemma.

Theorem 1.7 *Let A be a bounded set in $\mathbb{R}$ and, for each $x \in A$, let I_x be an open interval with center at x.*
(1) *There is a countable subcollection $\{I_j\}$ of $\{I_x\}_{x\in A}$ such that $A \subset \cup I_j$ and that each point in $\mathbb{R}$ belongs to at most two intervals in $\{I_j\}$, i.e. $\forall y \in \mathbb{R}$,*

$$\sum_j \chi_{I_j}(y) \leq 2.$$

(2) *There is a countable subcollection $\{I_{i,j}\}$, $i = 1, 2$, of $\{I_x\}_{x\in A}$ such that $A \subset \cup I_{i,j}$ and $I_{i,j} \cap I_{i,k} = \emptyset$ if $j \neq k$.*

In the sequel we will refer to $\{I_i\}$ and $\{I_{i,j}\}$ as the *Besicovitch subcollections*.
Proof. $|I|$ denotes the length of the interval I. We will use the shorthand

$$I(x, r) = (x - r, x + r).$$

Setting $r_x = |I_x|/2$, we have $I_x = I(x, r_x)$.

Let $d_1 = \sup\{r_x : x \in A\}$. Choose $I_1 = I(x_1, r_1)$ from the family $\{I_x\}_{x\in A}$ such that $r_1 > 3d_1/4$. Assume that $I_1, \ldots, I_{j-1}$ are chosen for $j \geq 1$ and that $A_j = A \setminus \cup_{i=1}^{j-1} I_i$ is nonempty. Let $d_j = \sup\{r_x : x \in A_j\}$. Then choose $I_j = I(x_j, r_j)$ from the family $\{I_x\}_{x\in A_j}$ such that $r_j > 3d_j/4$. In this way be obtain a countable (possibly finite) subcollection $I_j = I(x_j, r_j)$ of $\{I_x\}_{x\in A}$.

Suppose that $j > i$. Then $x_j \in A_i$ and

$$r_i \geq \frac{3}{4}\sup\{r_x : x \in A_i\} \geq \frac{3r_j}{4}. \tag{1.3}$$

This observation yields that the intervals $I(x_j, r_j/3)$ are disjoint. Indeed, if $j > i$, then $x_j \notin I(x_i, r_i)$, and (1.3) yields

$$|x_i - x_j| > r_i = \frac{r_i}{3} + \frac{2r_i}{3} > \frac{r_i}{3} + \frac{r_j}{3}.$$

Since A is a bounded set and $x_j \in A$, the disjointness of $I_j = I(x_j, r_j/3)$ implies that if the family $\{I_j\}$ is infinite, then

$$\lim_{j\to\infty} r_j = 0. \tag{1.4}$$

The relation (1.4) yields that $A \subset \cup_j I(x_j, r_j)$. Indeed, this is obvious if there are only finitely many I_j's. Assume that there are infinitely many I_j's and let $x \in A$. By (1.4), there is j such that $r_j < 3r_x/4$, and by the definition of r_j, $x \in \cup_{i=1}^{j-1} I_i$.

Notice that if three intervals in $\mathbb{R}$ have a common point, then one of the intervals is contained in the union of the other two. Hence, by dropping superfluous intervals from the collection $\{I_j\}$, we derive that $A \subset \cup_j I_j$ and that each point in $\mathbb{R}$ belongs to no more than two intervals I_j. This proves (1).

To prove (2), we enumerate I_j's as follows. To I_1 is associated the number 0. The intervals to right of I_1 are enumerated in succession by positive integers, and the intervals to the left by negative integers. (The "succession" is well-defined, since no point belongs simultaneously to three intervals). The intervals associated to even integers are mutually disjoint, and so are the intervals associated to odd integers. Finally, denote the interval associated to $2n$ by $I_{1,n}$, and the interval associated to $2n+1$ by $I_{2,n}$. This construction yields (2). □

Let μ be a positive Borel measure on $\mathbb{R}$ finite on compact sets and let ν be a complex measure. The corresponding maximal function is defined by

$$M_{\nu,\mu}(x) = \sup_{r>0} \frac{|\nu|(I(x,r))}{\mu(I(x,r))}, \qquad x \in \operatorname{supp} \mu.$$

If $x \notin \operatorname{supp} \mu$ we set $M_{\nu,\mu}(x) = \infty$. It is not hard (Problem 1) to show that the function $\mathbb{R} \ni x \mapsto M_{\nu,\mu}(x) \in [0, \infty]$ is Borel measurable.

Theorem 1.8 *For any $t > 0$,*

$$\mu\{x : M_{\nu,\mu}(x) > t\} \leq \frac{2}{t}|\nu|(\mathbb{R}).$$

Proof. Let $[a, b]$ be a bounded interval. Every point x in $[a, b] \cap \{x : M_{\nu,\mu}(x) > t\}$ is the center of an open interval I_x such that

$$|\nu|(I_x) \geq t\mu(I_x).$$

Let $I_{i,j}$ be the Besicovitch subcollection of $\{I_x\}$. Then,

$$[a, b] \cap \{x : M_{\nu,\mu}(x) > t\} \subset \cup I_{i,j},$$

and

$$\mu\left([a, b] \cap \{x : M_{\nu,\mu}(x) > t\}\right) \leq \sum_{i,j} \mu(I_{i,j})$$

$$\leq \frac{1}{t}\sum_{i,j} |\nu|(I_{i,j}) = \frac{1}{t}\sum_{i=1}^{2} |\nu|(\cup_j I_{i,j}) \leq \frac{2}{t}|\nu|(\mathbb{R}).$$

The statement follows by taking $a \to -\infty$ and $b \to +\infty$. □

In Problem 3 you are asked to prove:

Proposition 1.9 *Let A be a bounded Borel set. Then for any $0 < p < 1$,*

$$\int_A M_{\nu,\mu}(x)^p \mathrm{d}\mu(x) < \infty.$$

We will also need:

Proposition 1.10 *Let ν_j be a sequence of Borel complex measures such that $\lim_{j\to\infty} |\nu_j|(\mathbb{R}) = 0$. Then there is a subsequence ν_{j_k} such that*

$$\lim_{k\to\infty} M_{\nu_{j_k},\mu}(x) = 0 \qquad \text{for } \ \mu - a.e.\, x.$$

Proof. By Theorem 1.8, for each $k = 1, 2, \ldots$, we can find j_k so that

$$\mu \left\{ x : M_{\nu_{j_k},\mu}(x) > 2^{-k} \right\} \leq 2^{-k}.$$

Hence,

$$\sum_{k=1}^{\infty} \mu \left\{ x : M_{\nu_{j_k},\mu}(x) > 2^{-k} \right\} < \infty,$$

and so for μ-a.e. x, there is k_x such that for $k > k_x$, $M_{\nu_{j_k},\mu}(x) \leq 2^{-k}$. This yields the statement. □

We are now ready to prove the main theorem of this subsection.

Theorem 1.11 *Let ν be a complex Borel measure and μ a positive Borel measure finite on compact sets. Let $\nu = f\mu + \nu_{\mathrm{s}}$ be the Radon-Nikodym decomposition. Then:*
(1)

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{\mu(I(x,r))} = f(x), \qquad \text{for } \ \mu - a.e.\ x.$$

In particular, $\nu \perp \mu$ iff

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{\mu(I(x,r))} = 0, \qquad \text{for } \ \mu - a.e.\ x.$$

(2) *Let in addition ν be positive. Then*

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{\mu(I(x,r))} = \infty, \qquad \text{for } \ \nu_{\mathrm{s}} - a.e.\ x.$$

Proof. (1) We will split the proof into two steps.
Step 1. Assume that $\nu \ll \mu$, namely that $\nu = f\mu$. Let g_n be a continuous function with compact support such that $\int_{\mathbb{R}} |f - g_n| \mathrm{d}\mu < 1/n$. Set $h_n = f - g_n$. Then, for $x \in \operatorname{supp} \mu$ and $r > 0$,

$$\left| \frac{f\mu(I(x,r))}{\mu(I(x,r))} - f(x) \right| \leq \frac{|h_n|\mu(I(x,r))}{\mu(I(x,r))} + \left| \frac{g_n\mu(I(x,r))}{\mu(I(x,r))} - g_n(x) \right| + |g_n(x) - f(x)|.$$

Since g_n is continuous, we obviously have

$$\lim_{r \downarrow 0} \left| \frac{g_n\mu(I(x,r))}{\mu(I(x,r))} - g_n(x) \right| = 0,$$

and so for all n and $x \in \operatorname{supp} \mu$,

$$\limsup_{r \downarrow 0} \left| \frac{f\mu(I(x,r))}{\mu(I(x,r))} - f(x) \right| \leq M_{h_n\mu,\mu}(x) + |g_n(x) - f(x)|.$$

Let n_j be a subsequence such that $g_{n_j} \to f(x)$ for μ-a.e. x. Since $\int |h_{n_j}| \mathrm{d}\mu \to 0$ as $j \to \infty$, Proposition 1.10 yields that there is a subsequence of n_j (which we denote by the same letter) such that $M_{h_{n_j}\mu,\mu}(x) \to 0$ for μ-a.e. x. Hence, for μ-a.e. x,

$$\limsup_{r \downarrow 0} \left| \frac{f\mu(I(x,r))}{\mu(I(x,r))} - f(x) \right| = 0,$$

and (1) holds if $\nu \ll \mu$.
Step 2. To finish the proof of (1), it suffices to show that if ν is a complex measure such that $\nu \perp \mu$, then

$$\lim_{r \downarrow 0} \frac{|\nu|(I(x,r))}{\mu(I(x,r))} = 0, \qquad \text{for } \mu - a.e.\ x. \tag{1.5}$$

Let S be a Borel set such that $\mu(S) = 0$ and $|\nu|(\mathbb{R} \setminus S) = 0$. Then

$$\frac{|\nu|(I(x,r))}{(\mu + |\nu|)(I(x,r))} = \frac{\chi_S(|\nu| + \mu)(I(x,r))}{(\mu + |\nu|)(I(x,r))}. \tag{1.6}$$

By Step 1,

$$\lim_{r \downarrow 0} \frac{\chi_S(|\nu| + \mu)(I(x,r))}{(|\nu| + \mu)(I(x,r))} = \chi_S(x), \qquad \text{for } |\nu| + \mu - a.e.\ x. \tag{1.7}$$

Since $\chi_S(x) = 0$ for μ-a.e. x, (1.6) and (1.7) yield (1.5).

(2) Since ν is positive, $\nu(I(x,r)) \geq \nu_{\mathrm{s}}(I(x,r))$, and we may assume that $\nu \perp \mu$. By (1.6) and (1.7),

$$\lim_{r \downarrow 0} \frac{\nu(I(x,r))}{(\nu + \mu)(I(x,r))} = 1, \qquad \text{for } \nu - a.e.\ x,$$

and so

$$\lim_{r\downarrow 0} \frac{\mu(I(x,r))}{\nu(I(x,r))} = 0, \qquad \text{for } \nu - a.e.\ x.$$

This yields (2). □

We finish this subsection with several remarks. If μ is the Lebesgue measure, then the results of this section reduce to the standard differentiation results discussed, for example, in Chapter 7 of [Ru]. The arguments in [Ru] are based on the Vitali covering lemma which is specific to the Lebesgue measure. The proofs of this subsection are based on the Besicovitch covering lemma and they apply to an arbitrary positive measure μ. In fact, the proofs directly extend to $\mathbb{R}^n$ (one only needs to replace the intervals $I(x,r)$ with the balls $B(x,r)$ centered at x and of radius r) if one uses the following version of the Besicovitch covering lemma.

Theorem 1.12 *Let A be a bounded set in $\mathbb{R}^n$ and, for each $x \in A$, let B_x be an open ball with center at x. Then there is an integer N, which depends only on n, such that:*
(1) *There is a countable subcollection $\{B_j\}$ of $\{B_x\}_{x\in A}$ such that $A \subset \cup B_j$ and each point in $\mathbb{R}^n$ belongs to at most N balls in $\{B_j\}$, i.e. $\forall y \in \mathbb{R}$,*

$$\sum_j \chi_{B_j}(y) \leq N.$$

(2) *There is a countable subcollection $\{B_{i,j}\}$, $i = 1, \cdots, N$, of $\{B_x\}_{x\in A}$ such that $A \subset \cup B_{i,j}$ and $B_{i,j} \cap B_{i,k} = \emptyset$ if $j \neq k$.*

Unfortunately, unlike the proof of the Vitali covering lemma, the proof of Theorem 1.12 is somewhat long and complicated.

1.7 Problems

[1] *Prove that the maximal function $M_{\nu,\mu}(x)$ is Borel measurable.*

[2] *Let μ be a positive σ-finite measure on $(M, \mathcal{F})$ and let f be a measurable function. Let*

$$m_f(t) = \mu\{x \,:\, |f(x)| > t\}.$$

Prove that for $p \geq 1$,

$$\int_M |f(x)|^p \mathrm{d}\mu(x) = p \int_0^\infty t^{p-1} m_f(t) \mathrm{d}t.$$

This result can be generalized as follows. Let $\alpha : [0,\infty] \mapsto [0,\infty]$ be monotonic and absolutely continuous on $[0,T]$ for every $T < \infty$. Assume that $\alpha(0) = 0$ and $\alpha(\infty) = \infty$. Prove that

$$\int_M (\alpha \circ f)(x) \mathrm{d}\mu(x) = \int_0^\infty \alpha'(t) m_f(t) \mathrm{d}t.$$

Hint: See Theorem 8.16 in [Ru].

[3] *Prove Proposition 1.9. Hint: Use Problem 2.*

[4] *Prove the Riemann-Lebesgue lemma (Theorem 1.5).*

[5] *Let μ be a complex Borel measure on $\mathbb{R}$. Prove that $|\mu_{\mathrm{sing}}| = |\mu|_{\mathrm{sing}}$.*

[6] *Let μ be a positive measure on $(M, \mathcal{F})$. A sequence of measurable functions f_n converges in measure to zero if*

$$\lim_{n\to\infty} \mu(\{x : |f_n(x)| > \epsilon\}) = 0$$

for all $\epsilon > 0$. The sequence f_n converges almost uniformly to zero if for all $\epsilon > 0$ there is a set $M_\epsilon \in \mathcal{F}$, such that $\mu(M_\epsilon) < \epsilon$ and f_n converges uniformly to zero on $M \setminus M_\epsilon$.

Prove that if f_n converges to zero in measure, then there is a subsequence f_{n_j} which converges to zero almost uniformly.

[7] *Prove Theorem 1.12.* (*The proof can be found in* [EG]).

[8] *State and prove the analog of Theorem 1.11 in $\mathbb{R}^n$.*

[9] *Let μ be a positive Borel measure on $\mathbb{R}$ finite on compact sets and $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$. Prove that*

$$\lim_{r\downarrow 0} \frac{1}{\mu(I(x,r))} \int_{I(x,r)} |f(t) - f(x)| \mathrm{d}\mu(t) = 0, \qquad \text{for } \mu - a.e.\ x.$$

Hint: You may follow the proof of Theorem 7.7 in [Ru].

[10] *Let $p \geq 1$ and $f \in L^p(\mathbb{R}, \mathrm{d}x)$. The maximal function of f, M_f, is defined by*

$$M_f(x) = \sup_{r>0} \frac{1}{2r} \int_{I(x,r)} |f(t)| \mathrm{d}t.$$

(1) *If $p > 1$, prove that $M_f \in L^p(\mathbb{R}, \mathrm{d}x)$. Hint: See Theorem 8.18 in* [Ru].
(2) *Prove that if f and M_f are in $L^1(\mathbb{R}, \mathrm{d}x)$, then $f = 0$.*

[11] *Denote by $B_{\mathrm{b}}(\mathbb{R})$ the algebra of the bounded Borel functions on $\mathbb{R}$. Prove that $B_{\mathrm{b}}(\mathbb{R})$ is the smallest algebra of functions which includes $C_0(\mathbb{R})$ and is closed under pointwise limits of uniformly bounded sequences.*

2 Preliminaries: harmonic analysis

In this section we will deal only with Borel measures on $\mathbb{R}$. We will use the shorthand $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$. We denote the Lebesgue measure by m and write $\mathrm{d}m = \mathrm{d}x$.

Let μ be a complex measure or a positive measure such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}|\mu|(t)}{1+|t|} < \infty.$$

The Borel transform of μ is defined by

$$F_\mu(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t-z}, \qquad z \in \mathbb{C}_+. \tag{2.8}$$

The function $F_\mu(z)$ is analytic in $\mathbb{C}_+$.

Let μ be a complex measure or positive measure such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}|\mu|(t)}{1+t^2} < \infty. \tag{2.9}$$

The Poisson transform of μ is defined by

$$P_\mu(x+\mathrm{i}y) = y \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(x-t)^2+y^2}, \qquad y > 0.$$

The function $P_\mu(z)$ is harmonic in $\mathbb{C}_+$. If μ is the Lebesgue measure, then $P_\mu(z) = \pi$ for all $z \in \mathbb{C}_+$. If μ is a positive or signed measure, then $\operatorname{Im} F_\mu = P_\mu$. Note also that F_μ and P_μ are linear functions of μ, i.e. for $\lambda_1, \lambda_2 \in \mathbb{C}$, $F_{\lambda_1\mu_1+\lambda_2\mu_2} = \lambda_1 F_{\mu_1} + \lambda_2 F_{\mu_2}$, $P_{\lambda_1\mu_1+\lambda_2\mu_2} = \lambda_1 P_{\mu_1} + \lambda_2 P_{\mu_2}$,

Our goal in this section is to study the boundary values of $P_\mu(x+\mathrm{i}y)$ and $F_\mu(x+\mathrm{i}y)$ as $y \downarrow 0$. More precisely, we wish to study how these boundary values reflect the properties of the measure μ.

Although we will restrict ourselves to the radial limits, all the results discussed in this section hold for the non-tangential limits (see the problem set). The non-tangential limits will not be needed for our applications.

2.1 Poisson transforms and Radon-Nikodym derivatives

This subsection is based on [JL1]. Recall that $I(x,r) = (x-r, x+r)$.

Lemma 2.1 *Let μ be a positive measure. Then for all $x \in \mathbb{R}$ and $y > 0$,*

$$\frac{1}{y} P_\mu(x+\mathrm{i}y) = \int_0^{1/y^2} \mu(I(x, \sqrt{u^{-1} - y^2}))\mathrm{d}u.$$

Proof. Note that

$$\begin{aligned}\int_0^{1/y^2} \mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u &= \int_0^{1/y^2}\left[\int_{\mathbb{R}} \chi_{I(x,\sqrt{u^{-1}-y^2})}(t)\mathrm{d}\mu(t)\right]\mathrm{d}u \\ &= \int_{\mathbb{R}}\left[\int_0^{1/y^2} \chi_{I(x,\sqrt{u^{-1}-y^2})}(t)\mathrm{d}u\right]\mathrm{d}\mu(t).\end{aligned} \tag{2.10}$$

Since

$$|x-t| < \sqrt{u^{-1}-y^2} \qquad \Longleftrightarrow \qquad 0 \le u < ((x-t)^2+y^2)^{-1},$$

we have

$$\chi_{I(x,\sqrt{u^{-1}-y^2})}(t) = \chi_{[0,((x-t)^2+y^2)^{-1})}(u),$$

and

$$\int_0^{1/y^2} \chi_{I(x,\sqrt{u^{-1}-y^2})}(t)\mathrm{d}u = ((x-t)^2+y^2))^{-1}.$$

Hence, the result follows from (2.10). □

Lemma 2.2 *Let ν be a complex and μ a positive measure. Then for all $x \in \mathbb{R}$ and $y > 0$,*

$$\frac{|P_\nu(x+\mathrm{i}y)|}{P_\mu(x+\mathrm{i}y)} \le M_{\nu,\mu}(x).$$

Proof. Since $|P_\nu| \le P_{|\nu|}$, w.l.o.g. we may assume that ν is positive. Also, we may assume that $x \in \operatorname{supp}\mu$ (otherwise $M_{\nu,\mu}(x) = \infty$ and there is nothing to prove). Since

$$\begin{aligned}\int_0^{1/y^2} \nu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u &= \int_0^{1/y^2} \frac{\nu(I(x,\sqrt{u^{-1}-y^2}))}{\mu(I(x,\sqrt{u^{-1}-y^2}))}\mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u \\ &\le M_{\nu,\mu}(x)\int_0^{1/y^2} \mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u,\end{aligned}$$

the result follows from Lemma 2.1. □

Lemma 2.3 *Let μ be a positive measure. Then for μ-a.e. x,*

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(x-t)^2} = \infty. \tag{2.11}$$

The proof of this lemma is left for the problem set.

Lemma 2.4 *Let μ be a positive measure and $f \in C_0(\mathbb{R})$. Then for μ-a.e. x,*

$$\lim_{y\downarrow 0} \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = f(x). \tag{2.12}$$

Remark. The relation (2.12) holds for all x for which (2.11) holds. For example, if μ is the Lebesgue measure, then (2.12) holds for all x.

Proof. Note that

$$\left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x) \right| \le \frac{P_{|f-f(x)|\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be such that $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \epsilon$. Let $M = \sup|f(t)|$ and

$$C = 2M \int_{|x-t|\ge\delta} \frac{\mathrm{d}\mu(t)}{(x-t)^2}.$$

Then

$$P_{|f-f(x)|\mu}(x+\mathrm{i}y) \le \epsilon P_\mu(x+\mathrm{i}y) + Cy,$$

and

$$\left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x) \right| \le \epsilon + \frac{Cy}{P_\mu(x+\mathrm{i}y)}.$$

Let x be such that (2.11) holds. The monotone convergence theorem yields that

$$\lim_{y\downarrow 0} \frac{y}{P_\mu(x+iy)} = \left(\int \frac{\mathrm{d}\mu(t)}{(x-t)^2} \right)^{-1} = 0$$

and so for all $\epsilon > 0$,

$$\limsup_{y\downarrow 0} \left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x) \right| \le \epsilon.$$

This yields the statement. □

The main result of this subsection is:

Theorem 2.5 *Let ν be a complex measure and μ a positive measure. Let $\nu = f\mu + \nu_{\mathrm{s}}$ be the Radon-Nikodym decomposition. Then:*

(1)

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = f(x), \qquad \text{for } \mu - a.e.\ x.$$

In particular, $\nu \perp \mu$ iff

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = 0, \qquad \text{for } \mu - a.e.\ x.$$

(2) *Assume in addition that ν is positive. Then*

$$\lim_{y\downarrow 0}\frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}=\infty,\qquad \text{for }\ \nu_{\mathrm{s}}-a.e.\ x.$$

Proof. The proof is very similar to the proof of Theorem 1.11 in Section 1.

(1) We will split the proof into two steps.

Step 1. Assume that $\nu\ll\mu$, namely that $\nu=f\mu$. Let g_n be a continuous function with compact support such that $\int_{\mathbb{R}}|f-g_n|\mathrm{d}\mu<1/n$. Set $h_n=f-g_n$. Then,

$$\left|\frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}-f(x)\right|\le\frac{P_{|h_n|\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}+\left|\frac{P_{g_n\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}-g_n(x)\right|+|g_n(x)-f(x)|.$$

It follows from Lemmas 2.2 and 2.4 that for μ-a.e. x,

$$\limsup_{y\downarrow 0}\left|\frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y))}-f(x)\right|\le M_{|h_n|\mu,\mu}(x)+|g_n(x)-f(x)|.$$

As in the proof of Theorem 1.11, there is a subsequence $n_j\to\infty$ such that $g_{n_j}(x)\to f(x)$ and $M_{|h_n|\mu,\mu}(x)\to 0$ for μ-a.e. x, and (1) holds if $\nu\ll\mu$.

Step 2. To finish the proof of (1), it suffices to show that if ν is a finite positive measure such that $\nu\perp\mu$, then

$$\lim_{y\downarrow 0}\frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}=0,\qquad \text{for }\ \mu-\ a.e.\ x. \tag{2.13}$$

Let S be a Borel set such that $\mu(S)=0$ and $\nu(\mathbb{R}\setminus S)=0$. Then

$$\frac{P_\nu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y)+P_\mu(x+\mathrm{i}y)}=\frac{P_{\chi_S(\nu+\mu)}(x+\mathrm{i}y)}{P_{\nu+\mu}(x+\mathrm{i}y)}. \tag{2.14}$$

By Step 1,

$$\lim_{y\downarrow 0}\frac{P_{\chi_S(\nu+\mu)}(x+\mathrm{i}y)}{P_{\nu+\mu}(x+\mathrm{i}y)}=\chi_S(x),\qquad \text{for }\ \nu+\mu-a.e.\ x. \tag{2.15}$$

Since $\chi_S(x)=0$ for μ-a.e. x,

$$\lim_{y\downarrow 0}\frac{P_\nu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y)+P_\mu(x+\mathrm{i}y)}=0,\qquad \text{for }\ \mu-a.e.\ x,$$

and (2.13) follows.

(2) Since ν is positive, $\nu(I(x,r))\ge\nu_{\mathrm{s}}(I(x,r))$, and we may assume that $\nu\perp\mu$. By (2.14) and (2.15),

$$\lim_{y\downarrow 0}\frac{P_\nu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y)+P_\mu(x+\mathrm{i}y)}=1,\qquad \text{for }\ \nu-a.e.\ x,$$

and so

$$\lim_{y\downarrow 0}\frac{P_\mu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y)}=0,\qquad \text{for }\ \nu-a.e.\ x.$$

This yields part (2). □

2.2 Local L^p norms, $0 < p < 1$.

In this subsection we prove Theorem 3.1 of [Si1]. ν is a complex measure, μ is a positive measure and $\nu = f\mu + \nu_{\rm s}$ is the Radon-Nikodym decomposition.

Theorem 2.6 *Let A be a bounded Borel set and $0 < p < 1$. Then*

$$\lim_{y\downarrow 0} \int_A \left|\frac{P_\nu(x+{\rm i}y)}{P_\mu(x+{\rm i}y)}\right|^p {\rm d}\mu(x) = \int_A |f(x)|^p {\rm d}\mu(x).$$

(*Both sides are allowed to be ∞*). *In particular, $\nu \restriction A \perp \mu \restriction A$ iff for some $p \in (0,1)$,*

$$\lim_{y\downarrow 0} \int_A \left|\frac{P_\nu(x+{\rm i}y)}{P_\mu(x+{\rm i}y)}\right|^p {\rm d}\mu(x) = 0.$$

Proof. By Theorem 2.5,

$$\lim_{y\downarrow 0} \left|\frac{P_\nu(x+{\rm i}y)}{P_\mu(x+{\rm i}y)}\right|^p = |f(x)|^p \qquad \text{for } \mu - \ a.e. \, x.$$

By Lemma 2.2,

$$\left|\frac{P_\nu(x+{\rm i}y)}{P_\mu(x+{\rm i}y)}\right|^p \leq M_{\nu,\mu}(x)^p.$$

Hence, Proposition 1.9 and the dominated convergence theorem yield the statement. □

2.3 Weak convergence

Let ν be a complex or positive measure and

$${\rm d}\nu_y(x) = \frac{1}{\pi} P_\nu(x+{\rm i}y){\rm d}x. \tag{2.16}$$

Theorem 2.7 *For any $f \in C_{\rm c}(\mathbb{R})$* (*continuous functions of compact support*),

$$\lim_{y\downarrow 0} \int_{\mathbb{R}} f(x){\rm d}\nu_y(x) = \int_{\mathbb{R}} f(x){\rm d}\nu(x). \tag{2.17}$$

In particular, $P_{\nu_1} = P_{\nu_2} \Rightarrow \nu_1 = \nu_2$.

Proof. Note that

$$\int_{\mathbb{R}} f(x){\rm d}\nu_y(x) = \int_{\mathbb{R}} \left[\frac{y}{\pi}\int_{\mathbb{R}} \frac{f(x){\rm d}x}{(x-t)^2+y^2}\right] {\rm d}\nu(t),$$

and so

$$\left|\int_{\mathbb{R}} f(x)\mathrm{d}\nu_y(x) - \int_{\mathbb{R}} f(x)\mathrm{d}\nu(x)\right| \leq \int_{\mathbb{R}} \frac{|L_y(t)|\mathrm{d}|\nu|(t)}{1+t^2}, \tag{2.18}$$

where

$$L_y(t) = (1+t^2)\left(f(t) - \frac{y}{\pi}\int_{\mathbb{R}} \frac{f(x)\mathrm{d}x}{(x-t)^2+y^2}\right).$$

Clearly, $\sup_{y>0, t\in\mathbb{R}} |L_y(t)| < \infty$. By Lemma 2.4 and Remark after it, $\lim_{y\downarrow 0} L_y(t) = 0$ for all $t \in \mathbb{R}$ (see also Problem 2). Hence, the statement follows from the estimate (2.18) and the dominated convergence theorem. □

2.4 Local L^p-norms, $p > 1$

In this subsection we will prove Theorem 2.1 of [Si1].

Let ν be a complex or positive measure and let $\nu = fm + \nu_{\text{sing}}$ be its Radon-Nikodym decomposition w.r.t. the Lebesgue measure.

Theorem 2.8 *Let $A \subset \mathbb{R}$ be open, $p > 1$, and assume that*

$$\sup_{0<y<1} \int_A |P_\nu(x+\mathrm{i}y)|^p \mathrm{d}x < \infty.$$

Then:
(1) $\nu_{\text{sing}} \restriction A = 0$.
(2) $\int_A |f(x)|^p \mathrm{d}x < \infty$.
(3) *For any* $[a,b] \subset A$, $\pi^{-1}P_\nu(x+\mathrm{i}y) \to f(x)$ *in* $L^p([a,b], \mathrm{d}x)$ *as* $y \downarrow 0$.

Proof. We will prove (1) and (2); (3) is left to the problems.

Let g be a continuous function with compact support contained in A and let q be the index dual to p, $p^{-1} + q^{-1} = 1$. Then, by Theorem 2.7,

$$\int_A g\mathrm{d}\nu = \lim_{y\downarrow 0} \pi^{-1} \int_A g(x) P_\nu(x+\mathrm{i}y)\mathrm{d}x,$$

and

$$\begin{aligned}\left|\int_A g(x)P_\nu(x+\mathrm{i}y)\mathrm{d}x\right| &\leq \left(\int_A |g(x)|^q \mathrm{d}x\right)^{1/q} \left(\int_A |P_\nu(x+\mathrm{i}y)|^p \mathrm{d}x\right)^{1/p} \\ &\leq C\left(\int_A |g(x)|^q \mathrm{d}x\right)^{1/q}.\end{aligned}$$

Hence, the map $g \mapsto \int_A g(x)\mathrm{d}\nu(x)$ is a continuous linear functional on $L^q(A, \mathrm{d}x)$, and there is a function $\tilde{f} \in L^p(A, \mathrm{d}\mu)$ such that

$$\int_A g(x)\mathrm{d}\nu(x) = \int_A g(x)\tilde{f}(x)\mathrm{d}x.$$

This relation implies that $\nu \restriction A$ is absolutely continuous w.r.t. the Lebesgue measure and that $f(x) = \tilde{f}(x)$ for Lebesgue a.e. x. (1) and (2) follow. □

Theorem 2.8 has a partial converse which we will discuss in the problem set.

2.5 Local version of the Wiener theorem

In this subsection we prove Theorem 2.2 of [Si1].

Theorem 2.9 *Let ν be a signed measure and A_ν be the set of atoms of ν. Then for any finite interval* $[a, b]$,

$$\lim_{y\downarrow 0} y \int_a^b P_\nu(x + \mathrm{i}y)^2 \mathrm{d}x = \frac{\pi}{2}\left(\frac{\nu(\{a\})^2}{2} + \frac{\nu(\{b\})^2}{2} + \sum_{x\in(a,b)\cap A_\nu} \nu(\{x\})^2\right),$$

Proof.

$$P_\nu(x + \mathrm{i}y)^2 = y^2 \int_{\mathbb{R}^2} \frac{\mathrm{d}\nu(t)\mathrm{d}\nu(t')}{((x-t)^2 + y^2)((x-t')^2 + y^2)}.$$

and

$$y\int_a^b P_\nu(x+\mathrm{i}y)^2\mathrm{d}x = \int_{\mathbb{R}^2} g_y(t,t')\mathrm{d}\nu(t)\mathrm{d}\nu(t'),$$

where

$$g_y(t,t') = \int_a^b \frac{y^3\mathrm{d}x}{((x-t)^2 + y^2)((x-t')^2 + y^2)}.$$

Notice now that:

(1) $0 \le g_y(t,t') \le \pi$.

(2) $\lim_{y\downarrow 0} g_y(t,t') = 0$ if $t \ne t'$, or $t \notin [a,b]$, or $t' \notin [a,b]$.

(3) If $t = t' \in (a,b)$, then

$$\lim_{y\downarrow 0} g_y(t,t) = \lim_{y\downarrow 0} y^3 \int_{\mathbb{R}} \frac{\mathrm{d}x}{(x^2+y^2)^2} = \frac{\pi}{2}$$

(compute the integral using the residue calculus).

(4) If $t = t' = a$ or $t = t' = b$, then

$$\lim_{y\downarrow 0} g_y(t,t) = \lim_{y\downarrow 0} y^3 \int_0^\infty \frac{\mathrm{d}x}{(x^2+y^2)^2} = \frac{\pi}{4}$$

The result follows from these observations and the dominated convergence theorem. □

Corollary 2.10 *A signed measure ν has no atoms in $[a,b]$ iff*

$$\lim_{y\downarrow 0} y \int_a^b P_\nu(x+\mathrm{i}y)^2 \mathrm{d}x = 0.$$

2.6 Poisson representation of harmonic functions

Theorem 2.11 *Let $V(z)$ be a positive harmonic function in $\mathbb{C}_+$. Then there is a constant $c \geq 0$ and a positive measure μ on $\mathbb{R}$ such that*

$$V(x+\mathrm{i}y) = cy + P_\mu(x+\mathrm{i}y).$$

The c and μ are uniquely determined by V.

Remark 1. The constant c is unique since $c = \lim_{y\to\infty} V(\mathrm{i}y)/y$. By Theorem 2.7, μ is also unique.

Remark 2. Theorems 2.5 and 2.11 yield that if V is a positive harmonic function in $\mathbb{C}_+$ and $\mathrm{d}\mu = f(x)\mathrm{d}x + \mu_{\mathrm{sing}}$ is the associated measure, then for Lebesgue a.e. x

$$\lim_{y\downarrow 0} \pi^{-1} V(x+\mathrm{i}y) = f(x).$$

Let $D = \{z : |z| < 1\}$ and $\Gamma = \{z : |z| = 1\}$. For $z \in D$ and $w \in \Gamma$ let

$$p_z(w) = \mathrm{Re}\, \frac{w+z}{w-z} = \frac{1-|z|^2}{|w-z|^2}.$$

We shall first prove:

Theorem 2.12 *Let U be a positive harmonic function in D. Then there exists a finite positive Borel measure ν on Γ such that for all $z \in D$,*

$$U(z) = \int_\Gamma p_z(w)\mathrm{d}\nu(w).$$

Proof. By the mean value property of harmonic functions, for any $0 < r < 1$,

$$U(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

In particular,

$$\sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} U(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta = U(0) < \infty.$$

For $f \in C(\Gamma)$ set

$$\Phi_r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(re^{\mathrm{i}\theta})f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

Each Φ_r is a continuous linear functional on the Banach space $C(\Gamma)$ and $\|\Phi_r\| = U(0)$. The standard diagonal argument yields that there is a sequence $r_j \to 1$ and a bounded linear functional Φ on $C(\Gamma)$ such that $\Phi_{r_j} \to \Phi$ weakly, that is, for all $f \in C(\Gamma)$, $\Phi_{r_j}(f) \to \Phi(f)$. Obviously, $\|\Phi\| = U(0)$. By the Riesz representation theorem there exists a complex measure ν on Γ such that $|\nu|(\Gamma) = U(0)$ and

$$\Phi(f) = \int_\Gamma f(w)\mathrm{d}\nu(w).$$

Since $\Phi_{r_j}(f) \geq 0$ if $f \geq 0$, the measure ν is positive. Finally, let $z \in D$. If $r_j > |z|$, then

$$U(zr_j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(r_j\mathrm{e}^{\mathrm{i}\theta})p_z(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta = \Phi_{r_j}(p_z). \tag{2.19}$$

(the proof of this relation is left for the problems—see Theorem 11.8 in [Ru]). Taking $j \to \infty$ we derive

$$U(z) = \Phi(p_z) = \int_\Gamma p_z(w)\mathrm{d}\nu(w).$$

□

Before proving Theorem 2.11, I would like to make a remark about change of variables in measure theory. Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be measure spaces. A map $T : M_1 \to M_2$ is called a measurable tranformation if for all $F \in \mathcal{F}_2$, $T^{-1}(F) \in \mathcal{F}_1$. Let μ be a positive measure on $(M_1, \mathcal{F}_1)$, and let μ_T be a positive measure on $(M_2, \mathcal{F}_2)$ defined by $\mu_T(F) = \mu(T^{-1}(F))$. If f is a measurable function on $(M_2, \mathcal{F}_2)$, then $f_T = f \circ T$ is a measurable function on $(M_1, \mathcal{F}_1)$. Moreover, $f \in L^1(M_2, \mathrm{d}\mu_T)$ iff $f_T \in L^1(M_1, \mathrm{d}\mu)$, and in this case

$$\int_{M_2} f\mathrm{d}\mu_T = \int_{M_1} f_T\mathrm{d}\mu.$$

This relation is easy to check if f is a characteristic function. The general case follows by the usual approximation argument through simple functions.

If T is a bijection, then $g \in L^1(M_1, \mathrm{d}\mu)$ iff $g_{T^{-1}} \in L^1(M_2, \mathrm{d}\mu_T)$, and in this case

$$\int_{M_1} g \mathrm{d}\mu = \int_{M_2} g_{T^{-1}} \mathrm{d}\mu_T.$$

Proof of Theorem 2.11. We define a map $S : \mathbb{C}_+ \to D$ by

$$S(z) = \frac{\mathrm{i} - z}{\mathrm{i} + z}. \tag{2.20}$$

This is the well-known conformal map between the upper half-plane and the unit disc. The map S extends to a homeomorphism $S : \overline{\mathbb{C}}_+ \mapsto \overline{D} \setminus \{-1\}$. Note that $S(\mathbb{R}) = \Gamma \setminus \{-1\}$. If

$$K_z(t) = \frac{y}{(x-t)^2 + y^2}, \qquad z = x + \mathrm{i}y \in \mathbb{C}_+,$$

then

$$(1 + t^2) K_z(t) = p_{S(z)}(S(t)).$$

Let $T = S^{-1}$. Explicitly,

$$T(\xi) = \frac{\xi - 1}{\mathrm{i}\xi + \mathrm{i}}. \tag{2.21}$$

Let $U(\xi) = V(T(\xi))$. Then there exists a positive finite Borel measure ν on Γ such that

$$U(\xi) = \int_\Gamma p_\xi(w) \mathrm{d}\nu(w).$$

The map $T : \Gamma \setminus \{-1\} \to \mathbb{R}$ is a homeomorphism. Let ν_T be the induced Borel measure on $\mathbb{R}$. By the previous change of variables,

$$\int_{\Gamma \setminus \{-1\}} p_\xi(w) \mathrm{d}\nu(w) = \int (1 + t^2) K_{T(\xi)}(t) \mathrm{d}\nu_T(t).$$

Hence, for $z \in \mathbb{C}_+$,

$$V(z) = p_{S(z)}(-1)\nu(\{-1\}) + \int_\mathbb{R} (1 + t^2) K_z(t) \mathrm{d}\nu_T(t).$$

Since $p_{S(z)}(-1) = y$, setting $c = \nu(\{-1\})$ and $\mathrm{d}\mu(t) = (1+t^2)\mathrm{d}\nu_T(t)$, we derive the statement.
□

2.7 The Hardy class $H^\infty(\mathbb{C}_+)$

The Hardy class $H^\infty(\mathbb{C}_+)$ is the vector space of all functions V analytic in $\mathbb{C}_+$ such that

$$\|V\| = \sup_{z\in\mathbb{C}_+} |V(z)| < \infty. \tag{2.22}$$

$H^\infty(\mathbb{C}_+)$ with norm (2.22) is a Banach space. In this subsection we will prove two basic properties of $H^\infty(\mathbb{C}_+)$.

Theorem 2.13 *Let $V \in H^\infty(\mathbb{C}_+)$. Then for Lebesgue a.e. $x \in \mathbb{R}$, the limit*

$$V(x) = \lim_{y\downarrow 0} V(x + \mathrm{i}y) \tag{2.23}$$

exists. Obviously, $V \in L^\infty(\mathbb{R}, \mathrm{d}x)$.

Theorem 2.14 *Let $V \in H^\infty(\mathbb{C}_+)$, $V \not\equiv 0$, and let $V(x)$ be given by (2.23). Then*

$$\int_{\mathbb{R}} \frac{|\log|V(x)||}{1+x^2}\mathrm{d}x < \infty.$$

In particular, if $\alpha \in \mathbb{C}$, then either $V(z) \equiv \alpha$ or the set $\{x \in \mathbb{R} : V(x) = \alpha\}$ has zero Lebesgue measure.

A simple and important consequence of Theorems 2.13 and 2.14 is:

Theorem 2.15 *Let F be an analytic function on $\mathbb{C}_+$ with positive imaginary part. Then:*
(1) *For Lebesgue a.e. $x \in \mathbb{R}$ the limit*

$$F(x) = \lim_{y\downarrow 0} F(x + \mathrm{i}y),$$

exists and is finite.
(2) *If $\alpha \in \mathbb{C}$, then either $F(z) \equiv \alpha$ or the set $\{x \in \mathbb{R} : F(x) = \alpha\}$ has zero Lebesgue measure.*

Proof. To prove (1), apply Theorem 2.13 to the function $(F(z) + \mathrm{i})^{-1}$. To prove (2), apply Theorem 2.14 to the function $(F(z) + \mathrm{i})^{-1} - (\alpha + \mathrm{i})^{-1}$. $\square$

Proof of Theorem 2.13. Let $\mathrm{d}\nu_y(t) = V(t + \mathrm{i}y)\mathrm{d}t$. Then, for $f \in L^1(\mathbb{R}, \mathrm{d}t)$,

$$\left|\int_{\mathbb{R}} f(t)\mathrm{d}\nu_y(t)\right| \leq \|V\| \int_{\mathbb{R}} |f(t)|\mathrm{d}t.$$

The map $\Phi_y(f) = \int_{\mathbb{R}} f\mathrm{d}\nu_y$ is a linear functional on $L^1(\mathbb{R}, \mathrm{d}t)$ and $\|\Phi_y\| \leq \|V\|$. By the Banach-Alaoglu theorem, there a bounded linear functional Φ and a sequence $y_n \downarrow 0$ such that for all $f \in L^1(\mathbb{R}, \mathrm{d}t)$,

$$\lim_{n\to\infty} \Phi_{y_n}(f) = \Phi(f).$$

Let $V \in L^\infty(\mathbb{R}, \mathrm{d}t)$ be such that $\Phi(f) = \int_\mathbb{R} V(t) f(t) \mathrm{d}t$. Let $f(t) = \pi^{-1} y((x-t)^2 + y^2)^{-1}$. A simple residue calculation yields

$$\Phi_{y_n}(f) = \pi^{-1} y \int_\mathbb{R} \frac{V(t + \mathrm{i}y_n)\mathrm{d}t}{(x-t)^2 + y^2} = V(x + \mathrm{i}(y + y_n)).$$

Taking $n \to \infty$, we get

$$V(x + \mathrm{i}y) = \pi^{-1} y \int_\mathbb{R} \frac{V(t)\mathrm{d}t}{(x-t)^2 + y^2}, \tag{2.24}$$

and Theorem 2.5 yields the statement. □

Remark 1. Theorem 2.13 can be also proven using Theorem 2.11. The above argument has the advantage that it extends to any Hardy class $H^p(\mathbb{C}_+)$.

Remark 2. In the proof we have also established the Poisson representation of V (the relation (2.24)).

Proposition 2.16 (Jensen's formula) *Assume that $U(z)$ is analytic for $|z| < 1$ and that $U(0) \neq 0$. Let $r \in (0, 1)$ and assume that U has no zeros on the circle $|z| = r$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of $U(z)$ in the region $|z| < r$, listed with multiplicities. Then*

$$|U(0)| \prod_{j=1}^n \frac{r}{|\alpha_j|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |U(r\mathrm{e}^{\mathrm{i}t})| \mathrm{d}t \right\}. \tag{2.25}$$

Remark. The Jensen formula holds even if U has zeros on $|z| = r$. We will only need the above elementary version.

Proof. Set

$$V(z) = U(z) \prod_{j=1}^n \frac{r^2 - \overline{\alpha}_j z}{r(\alpha_j - z)}.$$

Then for some $\epsilon > 0$ $V(z)$ has no zeros in the disk $|z| < r + \epsilon$ and the function $\log |V(z)|$ is harmonic in the same disk (see Theorem 13.12 in [Ru]). By the mean value theorem for harmonic functions,

$$\log |V(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |V(r\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}\theta.$$

The substitution yields the statement. □

Proof of Theorem 2.14. Setting $U(\mathrm{e}^{\mathrm{i}t}) = V(\tan(t/2))$, we have that

$$\int_\mathbb{R} \frac{|\log |V(x)||}{1 + x^2} \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} |\log |U(\mathrm{e}^{\mathrm{i}t})|| \mathrm{d}t. \tag{2.26}$$

Hence, it suffices to show that the integral on the r.h.s. is finite.

In the rest of the proof we will use the same notation as in the proof of Theorem 2.11. Recall that S and T are defined by (2.20) and (2.21). Let $U(z) = V(T(z))$. Then, U is holomorphic in D and $\sup_{z\in D}|U(z)| < \infty$. Moreover, a change of variables and the formula (2.24) yield that

$$U(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}U(\mathrm{e}^{\mathrm{i}t})\mathrm{d}t.$$

(The change of variables exercise is done in detail in [Ko], pages 106-107.) The analog of Theorem 2.5 for the circle yields that for Lebesgue a.e. θ

$$\lim_{r\to 1}U(r\mathrm{e}^{\mathrm{i}\theta}) = U(\mathrm{e}^{\mathrm{i}\theta}). \tag{2.27}$$

The proof is outlined in the problem set.

We will now make use of the Jensen formula. If $U(0) = 0$, let m be such $U_m(z) = z^{-m}U(z)$ satisfies $U_m(0) \neq 0$ (if $U(0) \neq 0$, then $m = 0$). Let $r_j \to 1$ be a sequence such that U has no zeros on $|z| = r_j$. Set

$$J_{r_j} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|U_m(r_j\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t = -m\log r_j + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|U(r_j\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t.$$

The Jensen formula (applied to U_m) yields that $J_{r_i} \leq J_{r_j}$ if $r_i \leq r_j$. Write $\log^+ x = \max(\log x, 0)$, $\log^- x = -\min(\log x, 0)$. Note that

$$\sup_t \log^+|U(\mathrm{e}^{\mathrm{i}t})| \leq \sup_t |U(\mathrm{e}^{\mathrm{i}t})| < \infty.$$

Fatou's lemma, the dominated convergence theorem and (2.27) yield that

$$J_{r_1} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^-|U(\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|U(\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t < \infty.$$

Hence,

$$\int_{-\pi}^{\pi}|\log|U(\mathrm{e}^{\mathrm{i}t})||\mathrm{d}t < \infty,$$

and the identity (2.26) yields the statement. □

2.8 The Borel tranform of measures

Recall that the Borel transform $F_\mu(z)$ is defined by (2.8).

Theorem 2.17 *Let μ be a complex or positive measure. Then:*
(1) *For Lebesgue a.e. x the limit*

$$F_\mu(x) = \lim_{y\downarrow 0}F_\mu(x+\mathrm{i}y)$$

exists and is finite.
(2) *If $F_\mu \not\equiv 0$, then*

$$\int_{\mathbb{R}} \frac{|\log|F_\mu(x)||}{1+x^2}\mathrm{d}x < \infty. \tag{2.28}$$

(3) *If $F_\mu \not\equiv 0$, then for any complex number α the set*

$$\{x \in \mathbb{R} : F_\mu(x) = \alpha\} \tag{2.29}$$

has zero Lebesgue measure.

Remark. It is possible that

$$\mu \neq 0 \qquad \text{and} \qquad F_\mu \equiv 0. \tag{2.30}$$

For example, this is the case if $\mathrm{d}\mu = (x-2\mathrm{i})^{-1}(x-\mathrm{i})^{-1}\mathrm{d}x$. By the theorem of F. $\&$ M. Riesz (see, e.g., [Ko]), if (2.30) holds, then $\mathrm{d}\mu = h(x)\mathrm{d}x$, where $h(x) \neq 0$ for Lebesgue a.e. x. We will prove the F. $\&$ M. Riesz theorem in Section 4.

Proof. We will first show that

$$F_\mu(z) = \frac{R(z)}{G(z)} \tag{2.31}$$

where $R, G \in H^\infty(\mathbb{C}_+)$ and G has no zeros in $\mathbb{C}_+$. If μ is positive, set

$$G(z) = \frac{1}{\mathrm{i} + F_\mu(z)}.$$

Then, $G(z)$ is holomorphic in $\mathbb{C}_+$, $|G(z)| \leq 1$ (since $\mathrm{Im}\, F_\mu(z) \geq 0$), and

$$F_\mu(z) = \frac{1 - \mathrm{i}G(z)}{G(z)}.$$

If μ is a complex measure, we first decompose $\mu = (\mu_1 - \mu_2) + \mathrm{i}(\mu_3 - \mu_4)$, where the μ_i's are positive measures, and then decompose

$$F_\mu(z) = (F_{\mu_1}(z) - F_{\mu_2}(z)) + \mathrm{i}(F_{\mu_3}(z) - F_{\mu_4}(z)).$$

Hence, (2.31) follows from the corresponding result for positive measures.

Proof of (1): By Theorems 2.13 and 2.14, the limits $R(x) = \lim_{y\downarrow 0} R(x+\mathrm{i}y)$ and $G(x) = \lim_{y\downarrow 0} G(x+\mathrm{i}y)$ exist and $G(x)$ is non-zero for Lebesgue a.e. x. Hence, for Lebesgue a.e. x,

$$F_\mu(x) = \lim_{y\downarrow 0} F_\mu(x+\mathrm{i}y) = \frac{R(x)}{G(x)}.$$

Proof of (2): $F_\mu(x)$ is zero on a set of positive measure iff $R(x)$ is, and if this is the case, $R \equiv 0$ and then $F_\mu \equiv 0$. Hence, if $F_\mu \not\equiv 0$, then $R \not\equiv 0$. Obviously,

$$|\log|F_\mu(x)|| \leq |\log|R(x)|| + |\log|G(x)||,$$

and (2.28) follows from Theorem 2.14.

Proof of (3): The sets $\{x : F_\mu(x) = \alpha\}$ and $\{x : R(x) - \alpha G(x) = 0\}$ have the same Lebesgue measure. If the second set has positive Lebesgue measure, then by Theorem 2.14, $R(z) = \alpha G(z)$ for all $z \in \mathbb{C}_+$, and $F_\mu(z) \equiv \alpha$. Since $\lim_{y\to\infty} |F_\mu(x + \mathrm{i}y)| = 0$, $\alpha = 0$, and so $F_\mu \equiv 0$. Hence, if the set $\{x : F_\mu(x) = \alpha\}$ has positive Lebesgue measure, then $\alpha = 0$ and $\mu = 0$. □

The final result we would like to mention is the theorem of Poltoratskii [Po1].

Theorem 2.18 *Let ν be a complex and μ a positive measure. Let $\nu = f\mu + \nu_{\mathrm{s}}$ be the Radon-Nikodym decomposition. Let μ_{sing} be the part of μ singular with respect to the Lebesgue measure. Then*

$$\lim_{y\downarrow 0} \frac{F_\nu(x + \mathrm{i}y)}{F_\mu(x + \mathrm{i}y)} = f(x) \qquad \text{for } \mu_{\mathrm{sing}} - a.e.\, x. \tag{2.32}$$

This theorem has played an important role in the recent study of the spectral structure of Anderson type Hamiltonians [JL2, JL3].

Poltoratskii's proof of Theorem 2.18 is somewhat complicated, partly since it is done in the framework of a theory that is also concerned with other questions. A relatively simple proof of Poltoratskii's theorem has recently been found in [JL1]. This new proof is based on the spectral theorem for self-adjoint operators and rank one perturbation theory, and will be discussed in Section 4.

2.9 Problems

[1] (1) *Prove Lemma 2.3.*
(2) *Assume that μ satifies (2.9). Prove that the set of x for which (2.11) holds is G_δ* (*countable intersection of open sets*) *in* $\operatorname{supp}\mu$.

[2] (1) *Let $C_0(\mathbb{R})$ be the usual Banach space of continuous function on $\mathbb{R}$ vanishing at infinity with norm $\|f\| = \sup |f(x)|$. For $f \in C_0(\mathbb{R})$ let*

$$f_y(x) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + y^2} \mathrm{d}t.$$

Prove that $\lim_{y\downarrow 0} \|f_y - f\| = 0$.
(2) *Prove that the linear span of the set of functions* $\{((x-a)^2 + b^2)^{-1} : a \in \mathbb{R}, b > 0\}$ *is dense in* $C_0(\mathbb{R})$.
(3) *Prove that the linear span of the set of functions* $\{(x-z)^{-1} : z \in \mathbb{C} \setminus \mathbb{R}\}$ *is dense in* $C_0(\mathbb{R})$.

Hint: To prove (1), you may argue as follows. Fix $\epsilon > 0$. *Let* $\delta > 0$ *be such that* $|t - x| < \delta \Rightarrow |f(t) - f(x)| < \epsilon$. *The estimates*

$$
\begin{aligned}
|f_y(x) - f(x)| &\le \frac{y}{\pi}\int_{\mathbb{R}} \frac{|f(t) - f(x)|}{(x-t)^2 + y^2}\mathrm{d}t \\
&\le \epsilon + 2\|f\|\frac{y}{\pi}\int_{|t-x|>\delta} \frac{1}{(x-t)^2+y^2}\mathrm{d}t \\
&\le \epsilon + 4\pi^{-1}\|f\|y/\delta,
\end{aligned}
$$

yield that $\limsup_{y\downarrow 0}\|f_y - f\| \le \epsilon$. *Since* ϵ *is arbitrary, (1) follows. Approximating* f_y *by Riemann sums deduce that (1)* $\Rightarrow$ *(2). Obviously, (2)* $\Rightarrow$ *(3).*

[3] *Prove Part (3) of Theorem 2.8.*

[4] *Prove the following converse of Theorem 2.8: If (1) and (2) hold, then for* $[a, b] \subset A$,

$$
\sup_{0<y<1}\int_a^b |P_\nu(x + \mathrm{i}y)|^p \mathrm{d}x < \infty.
$$

[5] *The following extension of Theorem 2.9 holds: Let* ν *be a finite positive measure. Then for any* $p > 1$,

$$
\lim_{y\downarrow 0} y^{p-1}\int_a^b P_\nu(x+\mathrm{i}y)^p \mathrm{d}x = C_p\left(\frac{\nu(\{a\})^p}{2} + \frac{\nu(\{b\})^p}{2} + \sum_{x\in(a,b)}\nu(\{x\})^p\right).
$$

Prove this and compute C_p *in terms of gamma functions. Hint: See Remark 1 after Theorem 2.2 in* [Si1].

[6] *Prove the formula (2.19).*

[7] *Let* μ *be a complex measure. Prove that* $F_\mu \equiv 0 \Rightarrow \mu = 0$ *if either one of the following holds:*
(a) μ *is real-valued.*
(b) $|\mu|(S) = 0$ *for some open set* S.
(c) $\int_{\mathbb{R}} \exp(p|x|)\mathrm{d}|\mu| < \infty$ *for some* $p > 0$.

[8] *Let* μ *be a complex or positive measure on* $\mathbb{R}$ *and*

$$
H_\mu(z) = \pi^{-1}(\mathrm{i}P_\mu(z) - F_\mu(z)) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{(x-t)\mathrm{d}\mu(t)}{(x-t)^2+y^2}.
$$

By Theorems 2.5 and 2.17, for Lebesgue a.e. x the limit

$$H_\mu(x) = \lim_{y\downarrow 0} H_\mu(x + \mathrm{i}y)$$

exists and is finite. If $\mathrm{d}\mu = f\mathrm{d}x$, we will denote $H_\mu(z)$ and $H_\mu(x)$ by $H_f(z)$ and $H_f(x)$. The function $H_\mu(x)$ is called the Hilbert transform of the measure μ (H_f is called the Hilbert transform of the function f).
(1) *Prove that for Lebesgue a.e. x the limit*

$$\lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|t-x|>\epsilon} \frac{\mathrm{d}\mu(t)}{x - t}$$

exists and is equal to $H_\mu(x)$.
(2) *Assume that $f \in L^p(\mathbb{R}, \mathrm{d}x)$ for some $1 < p < \infty$. Prove that*

$$\sup_{y>0} \int_{\mathbb{R}} |H_f(x + \mathrm{i}y)|^p \mathrm{d}x < \infty$$

and deduce that $H_f \in L^p(\mathbb{R}, \mathrm{d}x)$.
(3) *If $f \in L^2(\mathbb{R}, \mathrm{d}x)$, prove that $H_{H_f} = -f$ and deduce that*

$$\int_{\mathbb{R}} |H_f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x.$$

[9] *Let $1 \le p < \infty$. The Hardy class $H^p(\mathbb{C}_+)$ is the vector space of all analytic functions f on $\mathbb{C}_+$ such that*

$$\|f\|_p^p = \sup_{y>0} \int_{\mathbb{R}} |f(x + \mathrm{i}y)|^p \mathrm{d}x < \infty.$$

(1) *Prove that $\|\cdot\|_p$ is a norm and that $H^p(\mathbb{C}_+)$ is a Banach space.*
(2) *Let $f \in H^p(\mathbb{C}_+)$. Prove that the limit*

$$f(x) = \lim_{y\downarrow 0} f(x + \mathrm{i}y)$$

exists for Lebesgue a.e. x and that $f \in L^p(\mathbb{R}, \mathrm{d}x)$. Prove that

$$f(x + \mathrm{i}y) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x - t)^2 + y^2} \mathrm{d}t.$$

(3) *Prove that $H^2(\mathbb{C}_+)$ is a Hilbert space and that*

$$\sup_{y>0} \int_{\mathbb{R}} |f(x + \mathrm{i}y)|^2 \mathrm{d}x = \int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x.$$

Hence, $H^2(\mathbb{C}_+)$ can be identified with a subspace of $L^2(\mathbb{R}, \mathrm{d}x)$ which we denote by the same letter. Let $\overline{H}^2(\mathbb{C}_+) = \{f \in L^2(\mathbb{R}, \mathrm{d}x) : \overline{f} \in H^2(\mathbb{C}_+)\}$. Prove that

$$L^2(\mathbb{R}, \mathrm{d}x) = H^2(\mathbb{C}_+) \oplus \overline{H}^2(\mathbb{C}_+).$$

[10] *In this problem we will study the Poisson transform on the circle. Let $\Gamma = \{z : |z| = 1\}$ and let μ be a complex measure on Γ. The Poisson transform of the measure μ is*

$$P_\mu(z) = \int_\Gamma \frac{1 - |z|^2}{|z - w|^2} \mathrm{d}\mu(w).$$

If we parametrize Γ by $w = \mathrm{e}^{\mathrm{i}t}$, $t \in (-\pi, \pi]$ and denote the induced complex measure by $\mu(t)$, then

$$P_\mu(r\mathrm{e}^{\mathrm{i}\theta}) = \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} \mathrm{d}\mu(t).$$

Note also that if $\mathrm{d}\mu(t) = \mathrm{d}t$, then $P_\mu(z) = 2\pi$. For $w \in \Gamma$ we denote by $I(w, r)$ the arc of length $2r$ centered at w. Let ν be a complex measure and μ a finite positive measure on Γ. The corresponding maximal function is defined by

$$M_{\nu,\mu}(w) = \sup_{r>0} \frac{|\nu|(I(w, r))}{\mu(I(w, r))}$$

if $x \in \operatorname{supp} \mu$, otherwise $M_{\nu,\mu}(w) = \infty$.
(1) *Formulate and prove the Besicovitch covering lemma for the circle.*
(2) *Prove the following bound: For all $r \in [0, 1)$ and $\theta \in (-\pi, \pi]$,*

$$\frac{|P_\nu(r\mathrm{e}^{\mathrm{i}\theta})|}{P_\mu(r\mathrm{e}^{\mathrm{i}\theta})} \le M_{\nu,\mu}(\mathrm{e}^{\mathrm{i}\theta}).$$

You may either mimic the proof of Lemma 2.2, or follow the proof of Theorem 11.20 in [Ru].
(3) *State and prove the analog of Theorem 2.5 for the circle.*
(4) *State and prove the analogs of Theorems 2.7 and 2.13 for the circle.*

[11] *In Part (4) of the previous problem you were asked to prove the relation (2.27). This relation could be also proved like follows: show first that*

$$\limsup_{r\to 1} |U(r\mathrm{e}^{\mathrm{i}\theta}) - U(\mathrm{e}^{\mathrm{i}\theta})| \le \limsup_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} |U(\mathrm{e}^{\mathrm{i}t}) - U(\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}t$$

$$\le \limsup_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \int_{I(\theta,\epsilon)} |U(\mathrm{e}^{\mathrm{i}t}) - U(\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}t,$$

and then use Problem 9 of Section 1.

[12] *The goal of this problem is to extend all the results of this section to non-tangential limits. Our description of non-tangential limits follows* [Po1]. *Let again* $\Gamma = \{z : |z| = 1\}$ *and* $D = \{z : |z| < 1\}$. *Let* $w \in \Gamma$. *We say that* z *tends to* w *non-tangentially, and write*

$$z \underset{\angle}{\to} w$$

if z *tends to* w *inside the region*

$$\Delta_w^\varphi = \{z \in D \,:\, |\mathrm{Arg}(1 - z\overline{w})| < \varphi\}$$

for all $\varphi \in (0, \pi/2)$ *(draw a picture).* $\mathrm{Arg}(z)$ *is the principal branch of the argument with values in* $(-\pi, \pi]$. *In the sector* Δ_w^φ *inscribe a circle centered at the origin (we denote it by* Γ_φ*). The two points on* $\Gamma_\varphi \cap \{z \,:\, \mathrm{Arg}(1 - z\overline{w}) = \pm\varphi\}$ *divide the circle into two arcs. The open region bounded by the shorter arc and the rays* $\mathrm{Arg}(1 - z\overline{w}) = \pm\varphi$ *is denoted* C_w^φ. *Let* ν *and* μ *be as in Problem 10.*
(1) *Let* $\varphi \in (0, \pi/2)$ *be given. Then there is a constant* C *such that*

$$\sup_{z \in \mathrm{C}_w^\varphi} \frac{|P_\nu(z)|}{P_\mu(z)} \leq C M_{\nu,\mu}(w) \qquad \text{for } \mu - \, a.e.\, w. \tag{2.33}$$

This is the key result which extends the radial estimate of Part (2) of Problem 10. The passage from the radial estimate to (2.33) is similar to the proof of Harnack's lemma. Write the detailed proof following Lemma 1.2 of [Po1].
(2) *Let* $\nu = f\mu + \nu_{\mathrm{s}}$ *be the Radon-Nikodym decomposition. Prove that*

$$\lim_{z \underset{\angle}{\to} w} \frac{P_\nu(z)}{P_\mu(z)} = f(w) \qquad \text{for } \mu - \, a.e.w.$$

If ν *is a positive measure, prove that*

$$\lim_{z \underset{\angle}{\to} w} \frac{P_\nu(z)}{P_\mu(z)} = \infty \qquad \text{for } \nu_{\mathrm{s}} - \, a.e.\, w.$$

(3) *Extend Parts (3) and (4) of Problem 10 to non-tangential limits.*
(4) *Consider now* $\mathbb{C}_+$. *We say that* z *tends to* x *non-tangentially if for all* $\varphi \in (0, \pi/2)$ z *tends to* x *inside the cone* $\{z : |\mathrm{Arg}(z - x) - \pi/2| < \varphi\}$. *Let* T *be the conformal mapping (2.21). Prove that* $z \to w$ *non-tangentially in* D *iff* $T(z) \to T(w)$ *non-tangentially in* $\mathbb{C}_+$. *Using this observation extend all the results of this section to non-tangential limits.*

3 Self-adjoint operators, spectral theory

3.1 Basic notions

Let $\mathcal{H}$ be a Hilbert space. We denote the inner product by $(\cdot|\cdot)$ (the inner product is linear w.r.t. the second variable).

A linear operator on $\mathcal{H}$ is a pair $(A, \mathrm{Dom}\,(A))$, where $\mathrm{Dom}\,(A) \subset \mathcal{H}$ is a vector subspace and $A : \mathrm{Dom}\,(A) \to \mathcal{H}$ is a linear map. We set

$$\mathrm{Ker}\, A = \{\psi \in \mathrm{Dom}\,(A) \;:\; A\psi = 0\}, \qquad \mathrm{Ran}\, A = \{A\psi \;:\; \psi \in \mathrm{Dom}\,(A)\}.$$

An operator A is densely defined if $\mathrm{Dom}\,(A)$ is dense in $\mathcal{H}$. If A and B are linear operators, then $A + B$ is defined on $\mathrm{Dom}\,(A + B) = \mathrm{Dom}\,(A) \cap \mathrm{Dom}\,(B)$ in the obvious way. For any $z \in \mathbb{C}$ we denote by $A + z$ the operator $A + z\mathbf{1}$, where $\mathbf{1}$ is the identity operator. Similarly, $\mathrm{Dom}\,(AB) = \{\psi \,:\, \psi \in \mathrm{Dom}\,(B), B\psi \in \mathrm{Dom}\,(A)\}$, and $(AB)\psi = A(B\psi)$. $A = B$ if $\mathrm{Dom}\,(A) = \mathrm{Dom}\,(B)$ and $A\psi = B\psi$. The operator B is called an extension of A (one writes $A \subset B$) if $\mathrm{Dom}\,(A) \subset \mathrm{Dom}\,(B)$ and $A\psi = B\psi$ for $\psi \in \mathrm{Dom}\,(A)$.

The operator A is called bounded if $\mathrm{Dom}\,(A) = \mathcal{H}$ and

$$\|A\| = \sup_{\|\psi\|=1} \|A\psi\| < \infty. \tag{3.34}$$

We denote by $\mathcal{B}(\mathcal{H})$ the vector space of all bounded operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ with the norm (3.34) is a Banach space. If A is densely defined and there is a constant C such that for all $\psi \in \mathrm{Dom}\,(A)$, $\|A\psi\| \leq C\|\psi\|$, then A has a unique extension to a bounded operator on $\mathcal{H}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is called a projection if $P^2 = P$. An operator $U \in \mathcal{B}(\mathcal{H})$ is called unitary if U is onto and $(U\phi|U\psi) = (\phi|\psi)$ for all $\phi, \psi \in \mathcal{H}$.

The graph of a linear operator A is defined by

$$\Gamma(A) = \{(\psi, A\psi) \;:\; \psi \in \mathrm{Dom}\,(A)\} \subset \mathcal{H} \oplus \mathcal{H}.$$

Note that $A \subset B$ if $\Gamma(A) \subset \Gamma(B)$. A linear operator A is called *closed* if $\Gamma(A)$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$.

A is called closable if it has a closed extension. If A is closable, its smallest closed extension is called the closure of A and is denoted by $\overline{A}$. It is not difficult to show that A is closable iff $\overline{\Gamma(A)}$ is the graph of a linear operator and in this case $\Gamma(\overline{A}) = \overline{\Gamma(A)}$.

Let A be closed. A subset $D \subset \mathrm{Dom}\,(A)$ is called a *core* for A if $\overline{A \restriction D} = A$.

Let A be a densely defined linear operator. Its adjoint, A^*, is defined as follows. $\mathrm{Dom}\,(A^*)$ is the set of all $\phi \in \mathcal{H}$ for which there exists a $\psi \in \mathcal{H}$ such that

$$(A\varphi|\phi) = (\varphi|\psi) \qquad \text{for all } \varphi \in \mathrm{Dom}\,(A).$$

Obviously, such ψ is unique and $\mathrm{Dom}\,(A^*)$ is a vector subspace. We set $A^*\phi = \psi$. It may happen that $\mathrm{Dom}\,(A^*) = \{0\}$. If $\mathrm{Dom}\,(A^*)$ is dense, then $A^{**} = (A^*)^*$, etc.

Theorem 3.1 *Let A be a densely defined linear operator. Then:*
(1) *A^* is closed.*
(2) *A is closable iff $\mathrm{Dom}\,(A^*)$ is dense, and in this case $\overline{A} = A^{**}$.*
(3) *If A is closable, then $\overline{A}^* = A^*$.*

Let A be a closed densely defined operator. We denote by $\rho(A)$ the set of all $z \in \mathbb{C}$ such that

$$A - z : \mathrm{Dom}\,(A) \to \mathcal{H}$$

is a bijection. By the closed graph theorem, if $z \in \rho(A)$, then $(A-z)^{-1} \in \mathcal{B}(\mathcal{H})$. The set $\rho(A)$ is called the resolvent set of A. The spectrum of A, $\mathrm{sp}(A)$, is defined by

$$\mathrm{sp}(A) = \mathbb{C} \setminus \rho(A).$$

A point $z \in \mathbb{C}$ is called an eigenvalue of A if there is a $\psi \in \mathrm{Dom}\,(A)$, $\psi \neq 0$, such that $A\psi = z\psi$. The set of all eigenvalues is called the point spectrum of A and is denoted by $\mathrm{sp}_{\mathrm{p}}(A)$. Obviously, $\mathrm{sp}_{\mathrm{p}}(A) \subset \mathrm{sp}(A)$. It is possible that $\mathrm{sp}(A) = \mathrm{sp}_{\mathrm{p}}(A) = \mathbb{C}$. It is also possible that $\mathrm{sp}(A) = \emptyset$. (For simple examples see [RS1], Example 5 in Chapter VIII).

Theorem 3.2 *Assume that $\rho(A)$ is non-empty. Then $\rho(A)$ is an open subset of $\mathbb{C}$ and the map*

$$\rho(A) \ni z \mapsto (A-z)^{-1} \in \mathcal{B}(\mathcal{H}),$$

is (*norm*) *analytic. Moreover, if $z_1, z_2 \in \rho(A)$, then*

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}.$$

The last relation is called the resolvent identity.

3.2 Digression: The notions of analyticity

Let $\Omega \subset \mathbb{C}$ be an open set and X a Banach space. A function $f : \Omega \to X$ is called norm analytic if for all $z \in \Omega$ the limit

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists in the norm of X. f is called weakly analytic if $x^* \circ f : \Omega \to \mathbb{C}$ is analytic for all $x^* \in X^*$. Obviously, if f is norm analytic, then f is weakly analytic. The converse also holds and we have:

Theorem 3.3 *f is norm analytic iff f is weakly analytic.*

For the proof, see [RS1].

The mathematical theory of Banach space valued analytic functions parallels the classical theory of analytic functions. For example, if γ is a closed path in a simply connected domain Ω, then

$$\oint_\gamma f(z)\mathrm{d}z = 0. \tag{3.35}$$

(The integral is defined in the usual way by the norm convergent Riemann sums.) To prove (3.35), note that for $x^* \in X^*$,

$$x^*\left(\oint_\gamma f(z)\mathrm{d}z\right) = \oint_\gamma x^*(f(z))\mathrm{d}z = 0.$$

Since X^* separates points in X, (3.35) holds. Starting with (3.35) one obtains in the usual way the Cauchy integral formula,

$$\frac{1}{2\pi\mathrm{i}}\oint_{|w-z|=r}\frac{f(w)}{w-z}\mathrm{d}w = f(z).$$

Starting with the Cauchy integral formula one proves that for $w \in \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n, \tag{3.36}$$

where $a_n \in X$. The power series converges and the representation (3.36) holds in the largest open disk centered at w and contained in Ω, etc.

3.3 Elementary properties of self-adjoint operators

Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. A is called symmetric if $\forall\phi,\psi \in \mathrm{Dom}\,(A)$,

$$(A\phi|\psi) = (\phi|A\psi).$$

In other words, A is symmetric if $A \subset A^*$. Obviously, any symmetric operator is closable.

A densely defined operator A is called *self-adjoint* if $A = A^*$. A is self-adjoint iff A is symmetric and $\mathrm{Dom}\,(A) = \mathrm{Dom}\,(A^*)$.

Theorem 3.4 *Let A be a symmetric operator on $\mathcal{H}$. Then the following statements are equivalent:*
(1) *A is self-adjoint.*
(2) *A is closed and $\mathrm{Ker}\,(A^* \pm \mathrm{i}) = \{0\}$.*
(3) $\mathrm{Ran}\,(A \pm \mathrm{i}) = \mathcal{H}$.

A symmetric operator A is called essentially self-adjoint if $\overline{A}$ is self-adjoint.

Theorem 3.5 *Let A be a symmetric operator on $\mathcal{H}$. Then the following statements are equivalent:*
(1) *A is essentially self-adjoint.*
(2) $\mathrm{Ker}\,(A^* \pm \mathrm{i}) = \{0\}$.
(3) $\mathrm{Ran}\,(A \pm \mathrm{i})$ *are dense in $\mathcal{H}$.*

Remark. In Parts (2) and (3) of Theorems 3.4 and 3.5 $\pm\mathrm{i}$ can be replaced by $z, \overline{z}$, for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 3.6 *Let A be self-adjoint. Then:*
(1) *If $z = x + \mathrm{i}y$, then for $\psi \in \mathrm{Dom}\,(A)$,*

$$\|(A - z)\psi\|^2 = \|(A - x)\psi\|^2 + y^2\|\psi\|^2.$$

(2) $\mathrm{sp}(A) \subset \mathbb{R}$ *and for $z \in \mathbb{C} \setminus \mathbb{R}$, $\|(A - z)^{-1}\| \le |\mathrm{Im}\,z|^{-1}$.*
(3) *For any $x \in \mathbb{R}$ and $\psi \in \mathcal{H}$,*

$$\lim_{y \to \infty} \mathrm{i}y(A - x - \mathrm{i}y)^{-1}\psi = -\psi.$$

(4) *If $\lambda_1, \lambda_2 \in \mathrm{sp_p}(A)$, $\lambda_1 \neq \lambda_2$, and ψ_1, ψ_2 are corresponding eigenvectors, then $\psi_1 \perp \psi_2$.*

Proof. (1) follows from a simple computation:

$$\begin{aligned}\|(A - x - \mathrm{i}y)\psi\|^2 &= ((A - x - \mathrm{i}y)\psi|(A - x - \mathrm{i}y)\psi) \\ &= \|(A - x)\psi\|^2 + y^2\|\psi\|^2 + \mathrm{i}y((A - x)\psi|\psi) - \mathrm{i}y((A - x)\psi|\psi) \\ &= \|(A - x)\psi\|^2 + y^2\|\psi\|^2.\end{aligned}$$

(2) Let $z \in \mathbb{C} \setminus \mathbb{R}$. By (1), if $(A - z)\psi = 0$, then $\psi = 0$, and so $A - z : \mathrm{Dom}\,(A) \to \mathcal{H}$ is one-one. $\mathrm{Ran}\,(A - z) = \mathcal{H}$ by Theorem 3.4. Let us prove this fact directly. We will show first that $\mathrm{Ran}\,(A - z)$ is dense. Let $\psi \in \mathcal{H}$ such that $((A - z)\phi|\psi) = 0$ for all $\phi \in \mathrm{Dom}\,(A)$. Then $\psi \in \mathrm{Dom}\,(A)$ and $(\psi|A\psi) = \overline{z}\|\psi\|^2$. Since $(\psi|A\psi) \in \mathbb{R}$ and $\mathrm{Im}\,z \neq 0$, $\psi = 0$. Hence, $\mathrm{Ran}\,(A - z)$ is dense. Let $\psi_n = (A - z)\phi_n$ be a Cauchy sequence. Then, by (1), ϕ_n is also a Cauchy sequence, and since A is closed, $\mathrm{Ran}\,(A - z)$ is closed. Hence, $\mathrm{Ran}\,(A - z) = \mathcal{H}$ and $z \in \rho(A)$. Finally, the estimate $\|(A - z)^{-1}\| \le |\mathrm{Im}\,z|^{-1}$ is an immediate consequence of (1).

(3) By replacing A with $A - x$, w.l.o.g. we may assume that $x = 0$. We consider first the case $\psi \in \mathrm{Dom}\,(A)$. The identity

$$\mathrm{i}y(A - \mathrm{i}y)^{-1}\psi + \psi = (A - \mathrm{i}y)^{-1}A\psi$$

and (2) yield that $\|\mathrm{i}y(A - \mathrm{i}y)^{-1}\psi + \psi\| \le \|A\psi\|/y$, and so (3) holds. If $\psi \notin \mathrm{Dom}\,(A)$, let $\psi_n \in \mathrm{Dom}\,(A)$ be a sequence such that $\|\psi_n - \psi\| \le 1/n$. We estimate

$$\begin{aligned}\|\mathrm{i}y(A - \mathrm{i}y)^{-1}\psi + \psi\| &\le \|\mathrm{i}y(A - \mathrm{i}y)^{-1}(\psi - \psi_n)\| + \|\psi - \psi_n\| \\ &\quad + \|(\mathrm{i}y(A - \mathrm{i}y)^{-1}\psi_n + \psi_n\| \\ &\le 2\|\psi - \psi_n\| + \|(\mathrm{i}y(A - \mathrm{i}y)^{-1}\psi_n + \psi_n\| \\ &\le 2/n + \|A\psi_n\|/y.\end{aligned}$$

Hence,

$$\limsup_{y\to\infty} \|\mathrm{i}y(A-\mathrm{i}y)^{-1}\psi+\psi\| \leq 2/n.$$

Since n is arbitary, (3) follows.

(4) $\lambda_1(\psi_1|\psi_2) = (A\psi_1|\psi_2) = (\psi_1|A\psi_2) = \lambda_2(\psi_1|\psi_2)$. Since $\lambda_1 \neq \lambda_2$, $(\psi_1|\psi_2) = 0$. □

A self-adjoint operator A is called positive if $(\psi|A\psi) \geq 0$ for all $\psi \in \mathrm{Dom}\,(A)$. If A and B are bounded and self-adjoint, then obviously $A \pm B$ are also self-adjoint; we write $A \geq B$ if $A - B \geq 0$.

A self-adjoint projection P is called an orthogonal projection. In this case $\mathcal{H} = \mathrm{Ker}\,P \oplus \mathrm{Ran}\,P$. We write $\dim P = \dim \mathrm{Ran}\,P$.

Let A be a bounded operator on $\mathcal{H}$. The real and the imaginary part of A are defined by

$$\mathrm{Re}A = \frac{1}{2}(A + A^*), \qquad \mathrm{Im}A = \frac{1}{2\mathrm{i}}(A - A^*).$$

Clearly, $\mathrm{Re}A$ and $\mathrm{Im}A$ are self-adjoint operators and $A = \mathrm{Re}A + \mathrm{i}\mathrm{Im}A$.

3.4 Direct sums and invariant subspaces

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and A_1, A_2 self-adjoint operators on $\mathcal{H}_1, \mathcal{H}_2$. Then, the operator $A = A_1 \oplus A_2$ with the domain $\mathrm{Dom}\,(A) = \mathrm{Dom}\,(A_1) \oplus \mathrm{Dom}\,(A_2)$ is self-adjoint. Obviously, $(A-z)^{-1} = (A_1 - z)^{-1} \oplus (A_2 - z)^{-1}$.

This elementary construction has a partial converse. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $\mathcal{H}_1$ be a closed subspace of $\mathcal{H}$. The subspace $\mathcal{H}_1$ is *invariant* under A if for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(A-z)^{-1}\mathcal{H}_1 \subset \mathcal{H}_1$. Obviously, if $\mathcal{H}_1$ is invariant under A, so is $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$. Set $\mathrm{Dom}\,(A_i) = \mathrm{Dom}\,(A) \cap \mathcal{H}_i$, $A_i\psi = A\psi$, $i = 1, 2$. A_i is a self-adjoint operator on $\mathcal{H}_i$ and $A = A_1 \oplus A_2$. We will call A_1 the restriction of A to the invariant subspace $\mathcal{H}_1$ and write $A_1 = A \restriction \mathcal{H}_1$.

Let Γ be a countable set and $\mathcal{H}_n$, $n \in \Gamma$, a collection of Hilbert spaces. The direct sum of this collection,

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n,$$

is the set of all sequences $\{\psi_n\}_{n\in\Gamma}$ such that $\psi_n \in \mathcal{H}_n$ and

$$\sum_{n\in\Gamma} \|\psi_n\|^2_{\mathcal{H}_n} < \infty.$$

$\mathcal{H}$ is a Hilbert space with the inner product

$$(\phi|\psi) = \sum_{n\in\Gamma} (\phi_n|\psi_n)_{\mathcal{H}_n}.$$

Let $B_n \in \mathcal{B}(\mathcal{H}_n)$ and assume that $\sup_n \|B_n\| < \infty$. Then $B\{\psi_n\}_{n\in\Gamma} = \{B_n\psi_n\}_{n\in\Gamma}$ is a bounded operator on $\mathcal{H}$ and $\|B\| = \sup_n \|B_n\|$.

Proposition 3.7 *Let A_n be self-adjoint operators on $\mathcal{H}_n$. Set*

$$\mathrm{Dom}\,(A) = \{\psi = \{\psi_n\} \in \mathcal{H} \,:\, \psi_n \in \mathrm{Dom}\,(A_n),\ \sum_n \|A_n\psi_n\|^2_{\mathcal{H}_n} < \infty\},$$

$A\psi = \{A_n\psi_n\}$. Then A is a self-adjoint operator on $\mathcal{H}$. We write $A = \oplus_{n\in\Gamma} A_n$. Moreover:
(1) *For $z \in \mathbb{C}\setminus\mathbb{R}$, $(A-z)^{-1} = \oplus_n (A_n - z)^{-1}$.*
(2) $\mathrm{sp}(A) = \overline{\cup_n \mathrm{sp}(A_n)}$.

The proof of Proposition 3.7 is easy and is left to the problems.

3.5 Cyclic spaces and the decomposition theorem

Let $\mathcal{H}$ be a separable Hilbert space and A a self-adjoint operator on $\mathcal{H}$. A collection of vectors $\mathcal{C} = \{\psi_n\}_{n\in\Gamma}$, where Γ is a countable set, is called *cyclic* for A if the closure of the linear span of the set of vectors

$$\{(A-z)^{-1}\psi_n \,:\, n \in \Gamma, z \in \mathbb{C}\setminus\mathbb{R}\}$$

is equal to $\mathcal{H}$. A cyclic set for A always exists (take an orthonormal basis for $\mathcal{H}$). If $\mathcal{C} = \{\psi\}$, then ψ is called a cyclic vector for A.

Theorem 3.8 (The decomposition theorem) *Let $\mathcal{H}$ be a separable Hilbert space and A a self-adjoint operator on $\mathcal{H}$. Then there exists a countable set Γ, a collection of mutually orthogonal closed subspaces $\{\mathcal{H}_n\}_{n\in\Gamma}$ of $\mathcal{H}$ ($\mathcal{H}_n \perp \mathcal{H}_m$ if $n \neq m$), and self-adjoint operators A_n on $\mathcal{H}_n$ such that:*
(1) *For all $n \in \Gamma$ there is a $\psi_n \in \mathcal{H}_n$ cyclic for A_n.*
(2) *$\mathcal{H} = \oplus_n \mathcal{H}_n$ and $A = \oplus_n A_n$.*

Proof. Let $\{\phi_n : n = 1, 2, \cdots\}$ be a given cyclic set for A. Set $\psi_1 = \phi_1$ and let $\mathcal{H}_1$ be the cyclic space generated by A and ψ_1 ($\mathcal{H}_1$ is the closure of the linear span of the set of vectors $\{(A-z)^{-1}\psi_1 : z \in \mathbb{C}\setminus\mathbb{R}\}$). By Theorem 3.6, $\psi_1 \in \mathcal{H}_1$. Obviously, $\mathcal{H}_1$ is invariant under A and we set $A_1 = A \restriction \mathcal{H}_1$.

We define ψ_n, $\mathcal{H}_n$ and A_n inductively as follows. If $\mathcal{H}_1 \neq \mathcal{H}$, let ϕ_{n_2} be the first element of the sequence $\{\phi_2, \phi_3, \cdots\}$ which is not in $\mathcal{H}_1$. Decompose $\phi_{n_2} = \phi'_{n_2} + \phi''_{n_2}$, where $\phi'_{n_2} \in \mathcal{H}_1$ and $\phi''_{n_2} \in \mathcal{H}_1^{\perp}$. Set $\psi_2 = \phi''_{n_2}$ and let $\mathcal{H}_2$ be the cyclic space generated by A and ψ_2. It follows from the resolvent identity that $\mathcal{H}_1 \perp \mathcal{H}_2$. Set $A_2 = A \restriction \mathcal{H}_2$. In this way we inductively define $\psi_n, \mathcal{H}_n, A_n$, $n \in \Gamma$, where Γ is a finite set $\{1, \cdots, N\}$ or $\Gamma = \mathbb{N}$. By the construction, $\{\phi_n\}_{n\in\Gamma} \subset \cup_{n\in\Gamma}\mathcal{H}_n$. Hence, (1) holds and $\mathcal{H} = \oplus_n \mathcal{H}_n$.

To prove the second part of (2), note first that by the construction of A_n,

$$(A-z)^{-1} = \bigoplus_n (A_n - z)^{-1}.$$

If $\tilde{A} = \oplus A_n$, then by Proposition 3.7, $\tilde{A}$ is self-adjoint and $(\tilde{A}-z)^{-1} = \oplus_n (A_n - z)^{-1}$. Hence $A = \tilde{A}$. $\square$

3.6 The spectral theorem

We start with:

Theorem 3.9 *Let $(M, \mathcal{F})$ be a measure space and μ a finite positive measure on $(M, \mathcal{F})$. Let $f : M \to \mathbb{R}$ be a measurable function and let A_f be a linear operator on $L^2(M, \mathrm{d}\mu)$ defined by*

$$\mathrm{Dom}\,(A_f) = \{\psi \in L^2(M, \mathrm{d}\mu) \,:\, f\psi \in L^2(M, \mathrm{d}\mu)\}, \qquad A_f\psi = f\psi.$$

Then:
(1) A_f *is self-adjoint.*
(2) A_f *is bounded iff* $f \in L^\infty(M, \mathrm{d}\mu)$*, and in this case* $\|A_f\| = \|f\|_\infty$.
(3) $\mathrm{sp}(A_f)$ *is equal to the essential range of* f*:*

$$\mathrm{sp}(A_f) = \{\lambda \in \mathbb{R} \,:\, \mu(f^{-1}(\lambda - \epsilon, \lambda + \epsilon)) > 0 \text{ for all } \epsilon > 0\}.$$

The proof of this theorem is left to the problems.

The content of the spectral theorem for self-adjoint operators is that *any* self-adjoint operator is unitarily equivalent to A_f for some f.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. A linear bijection $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called unitary if for all $\phi, \psi \in \mathcal{H}_1$, $(U\phi|U\psi)_{\mathcal{H}_2} = (\phi|\psi)_{\mathcal{H}_1}$. Let A_1, A_2 be linear operators on $\mathcal{H}_1, \mathcal{H}_2$. The operators A_1, A_2 are unitarily equivalent if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\mathrm{Dom}\,(A_1) = \mathrm{Dom}\,(A_2)$ and $UA_1U^{-1} = A_2$.

Theorem 3.10 (Spectral theorem for self-adjoint operators) *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a measure space $(M, \mathcal{F})$, a finite positive measure μ and measurable function $f : M \to \mathbb{R}$ such that A is unitarily equivalent to the operator A_f on $L^2(M, \mathrm{d}\mu)$.*

We will prove the spectral theorem only for separable Hilbert spaces.

In the literature one can find many different proofs of Theorem 3.10. The proof below is constructive and allows to explicitly identify M and f while the measure μ is directly related to $(A - z)^{-1}$.

3.7 Proof of the spectral theorem—the cyclic case

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $\psi \in \mathcal{H}$.

Theorem 3.11 *There exists a unique finite positive Borel measure μ_ψ on $\mathbb{R}$ such that $\mu_\psi(\mathbb{R}) = \|\psi\|^2$ and*

$$(\psi|(A - z)^{-1}\psi) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi(t)}{t - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.37}$$

The measure μ_ψ is called the spectral measure for A and ψ.

Proof. Since $(A - \overline{z})^{-1} = (A - z)^{-1*}$, we need only to consider $z \in \mathbb{C}_+$. Set $U(z) = (\psi|(A - z)^{-1}\psi)$ and $V(z) = \operatorname{Im} U(z)$, $z \in \mathbb{C}_+$. It follows from the resolvent identity that

$$V(x + \mathrm{i}y) = y\|(A - x - \mathrm{i}y)^{-1}\psi\|^2, \tag{3.38}$$

and so V is harmonic and strictly positive in $\mathbb{C}_+$. Theorem 2.11 yields that there is a constant $c \geq 0$ and a unique positive Borel measure μ_ψ on $\mathbb{R}$ such that for $y > 0$,

$$V(x + \mathrm{i}y) = cy + P_{\mu_\psi}(x + \mathrm{i}y) = cy + y\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi(t)}{(x - t)^2 + y^2}. \tag{3.39}$$

By Theorem 3.6,

$$V(x + \mathrm{i}y) \leq \|\psi\|^2/y \qquad \text{and} \qquad \lim_{y\to\infty} yV(x + \mathrm{i}y) = \|\psi\|^2.$$

The first relation yields that $c = 0$. The second relation and the dominated convergence theorem yield that $\mu_\psi(\mathbb{R}) = \|\psi\|^2$.

The functions

$$F_{\mu_\psi}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi(t)}{t - z}$$

and $U(z)$ are analytic in $\mathbb{C}_+$ and have equal imaginary parts. The Cauchy-Riemann equations imply that $F_{\mu_\psi}(z) - U(z)$ is a constant. Since $F_{\mu_\psi}(z)$ and $U(z)$ vanish as $\operatorname{Im} z \to \infty$, $F_{\mu_\psi}(z) = U(z)$ for $z \in \mathbb{C}_+$ and (3.37) holds. □

Corollary 3.12 *Let $\varphi, \psi \in \mathcal{H}$. Then there exists a unique complex measure $\mu_{\varphi,\psi}$ on $\mathbb{R}$ such that*

$$(\varphi|(A - z)^{-1}\psi) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_{\varphi,\psi}(t)}{t - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.40}$$

Proof. The uniqueness is obvious (the set of functions $\{(x - z)^{-1} : z \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in $C_0(\mathbb{R})$). The existence follows from the polarization identity:

$$\begin{aligned} 4(\varphi|(A - z)^{-1}\psi) &= (\varphi + \psi|(A - z)^{-1}(\varphi + \psi)) - (\varphi - \psi|(A - z)^{-1}(\varphi - \psi)) \\ &\quad + \mathrm{i}(\varphi - \mathrm{i}\psi|(A - z)^{-1}(\varphi - \mathrm{i}\psi)) - \mathrm{i}(\varphi + \mathrm{i}\psi|(A - z)^{-1}(\varphi + \mathrm{i}\psi)). \end{aligned}$$

In particular,

$$\mu_{\varphi,\psi} = \frac{1}{4}\left(\mu_{\varphi+\psi} - \mu_{\varphi-\psi} + \mathrm{i}(\mu_{\varphi-\mathrm{i}\psi} - \mu_{\varphi+\mathrm{i}\psi})\right). \tag{3.41}$$

□

The main result of this subsection is:

Theorem 3.13 *Assume that ψ is a cyclic vector for A. Then A is unitarily equivalent to the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. In particular,* $\mathrm{sp}(A) = \mathrm{supp}\,\mu_\psi$.

Proof. Clearly, we may assume that $\psi \neq 0$. Note that $(A-z)^{-1}\psi = (A-w)^{-1}\psi$ iff $z = w$.

For $z \in \mathbb{C} \setminus \mathbb{R}$ we set $r_z(x) = (x-z)^{-1}$. $r_z \in L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ and the linear span of $\{r_z\}_{z\in\mathbb{C}\setminus\mathbb{R}}$ is dense in $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Set

$$U(A-z)^{-1}\psi = r_z. \tag{3.42}$$

If $\overline{z} \neq w$, then

$$\begin{aligned}(r_z|r_w)_{L^2(\mathbb{R},\mathrm{d}\mu_\psi)} &= \int_{\mathbb{R}} r_{\overline{z}} r_w \mathrm{d}\mu_\psi = \frac{1}{\overline{z}-w}\int_{\mathbb{R}} (r_{\overline{z}} - r_w)\mathrm{d}\mu_\psi \\ &= \frac{1}{\overline{z}-w}\left[(\psi|(A-\overline{z})^{-1}\psi) - (\psi|(A-w)^{-1}\psi)\right] \\ &= ((A-z)^{-1}\psi|(A-w)^{-1}\psi).\end{aligned}$$

By a limiting argument, the relation

$$(r_z|r_w)_{L^2(\mathbb{R},\mathrm{d}\mu_\psi)} = ((A-z)^{-1}\psi|(A-w)^{-1}\psi)$$

holds for all $z, w \in \mathbb{C} \setminus \mathbb{R}$. Hence, the map (3.42) extends to a unitary $U : \mathcal{H} \to L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Since

$$U(A-z)^{-1}(A-w)^{-1}\psi = r_z(x)r_w(x) = r_z(x)U(A-w)^{-1}\psi,$$

$(A-z)^{-1}$ is unitarily equivalent to the operator of multiplication by $(x-z)^{-1}$ on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. For any $\phi \in \mathcal{H}$,

$$\begin{aligned}UA(A-z)^{-1}\phi &= U\phi + zU(A-z)^{-1}\phi = (1 + z(x-z)^{-1})U\phi \\ &= x(x-z)^{-1}U\phi = xU(A-z)^{-1}\phi,\end{aligned}$$

and so A is unitarily equivalent to the operator of multiplication by x. □

We finish this subsection with the following remark. Assume that ψ is a cyclic vector for A and let A_x be the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Then, by Theorem 3.13, there exists a unitary $U : \mathcal{H} \to L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ such that

$$UAU^{-1} = A_x. \tag{3.43}$$

However, a unitary satisfying (3.43) is not unique. If U is such a unitary, then $U\psi$ is a cyclic vector for A_x. On the other hand, if $f \in L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ is a cyclic vector for A_x, then there is a unique unitary $U : \mathcal{H} \to L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ such that (3.43) holds and $U\psi = f\|\psi\|/\|f\|$. The unitary constructed in the proof of Theorem 3.13 satisfies $U\psi = \mathbb{1}$.

3.8 Proof of the spectral theorem—the general case

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{H}_n$, A_n, ψ_n, $n \in \Gamma$ be as in the decomposition theorem (Theorem 3.8). Let $U_n : \mathcal{H}_n \to L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n})$ be unitary such that A_n is unitarily equivalent to the operator of multiplication by x. We denote this last operator by $\tilde{A}_n$. Let $U = \oplus_n U_n$. An immediate consequence of the decomposition theorem and Theorem 3.13 is

Theorem 3.14 *The map $U : \mathcal{H} \to \bigoplus_{n\in\Gamma} L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n})$ is unitary and A is unitarily equivalent to the operator $\bigoplus_{n\in\Gamma} \tilde{A}_n$. In particular,*

$$\mathrm{sp}(A) = \overline{\bigcup_{n\in\Gamma} \mathrm{supp}\, \mu_{\psi_n}}.$$

Note that if $\phi \in \mathcal{H}$ and $U\phi = \{\phi_n\}_{n\in\Gamma}$, then $\mu_\phi = \sum_{n\in\Gamma} \mu_{\phi_n}$.

Theorem 3.10 is a reformulation of Theorem 3.14. To see that, choose cyclic vectors ψ_n so that $\sum_{n\in\Gamma} \|\psi_n\|^2 < \infty$. For each $n \in \Gamma$, let $\mathbb{R}_n$ be a copy of $\mathbb{R}$ and

$$M = \bigcup_{n\in\Gamma} \mathbb{R}_n.$$

You may visualize M as follows: enumerate Γ so that $\Gamma = \{1, \ldots, N\}$ or $\Gamma = \mathbb{N}$ and set $\mathbb{R}_n = \{(n, x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Hence, M is just a collection of lines in $\mathbb{R}^2$ parallel to the y-axis and going through the points $(n, 0), n \in \Gamma$. Let $\mathcal{F}$ be the collection of all sets $F \subset M$ such that $F \cap \mathbb{R}_n$ is Borel for all n. Then $\mathcal{F}$ is a σ-algebra and

$$\mu(F) = \sum_{n\in\Gamma} \mu_{\psi_n}(F \cap \mathbb{R}_n)$$

is a finite measure on M ($\mu(M) = \sum_{n\in\Gamma} \|\psi_n\|^2 < \infty$). Let $f : M \to \mathbb{R}$ be the identity function ($f(n, x) = x$). Then

$$L^2(M, \mathrm{d}\mu) = \bigoplus_{n\in\Gamma} L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n}), \qquad A_f = \bigoplus_{n\in\Gamma} \tilde{A}_n,$$

and Theorem 3.10 follows.

Set

$$\mu_{\mathrm{ac}}(F) = \sum_{n\in\Gamma} \mu_{\psi_n,\mathrm{ac}}(F \cap \mathbb{R}_n),$$

$$\mu_{\mathrm{sc}}(F) = \sum_{n\in\Gamma} \mu_{\psi_n,\mathrm{sc}}(F \cap \mathbb{R}_n),$$

$$\mu_{\mathrm{pp}}(F) = \sum_{n\in\Gamma} \mu_{\psi_n,\mathrm{pp}}(F \cap \mathbb{R}_n).$$

Then $L^2(M, \mathrm{d}\mu_{\mathrm{ac}})$, $L^2(M, \mathrm{d}\mu_{\mathrm{sc}})$, $L^2(M, \mathrm{d}\mu_{\mathrm{pp}})$ are closed subspace of $L^2(M, \mathrm{d}\mu)$ invariant under A_f and

$$L^2(M, \mathrm{d}\mu) = L^2(M, \mathrm{d}\mu_{\mathrm{ac}}) \oplus L^2(M, \mathrm{d}\mu_{\mathrm{sc}}) \oplus L^2(M, \mathrm{d}\mu_{\mathrm{pp}}).$$

Set

$$\mathcal{H}_{\mathrm{ac}} = U^{-1} L^2(M, \mathrm{d}\mu_{\mathrm{ac}}), \qquad \mathcal{H}_{\mathrm{sc}} = U^{-1} L^2(M, \mathrm{d}\mu_{\mathrm{sc}}), \qquad \mathcal{H}_{\mathrm{pp}} = U^{-1} L^2(M, \mathrm{d}\mu_{\mathrm{pp}}).$$

These subspaces are invariant under A. Moreover, $\psi \in \mathcal{H}_{\mathrm{ac}}$ iff the spectral measure μ_ψ is absolutely continuous w.r.t. the Lebesgue measure, $\psi \in \mathcal{H}_{\mathrm{sc}}$ iff μ_ψ is singular continuous and $\psi \in \mathcal{H}_{\mathrm{pp}}$ iff μ_ψ is pure point. Obviously,

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{pp}}.$$

The spectra

$$\mathrm{sp}_{\mathrm{ac}}(A) = \mathrm{sp}(A \restriction \mathcal{H}_{\mathrm{ac}}) = \overline{\bigcup_{n \in \Gamma} \mathrm{supp} \mu_{\psi_n, \mathrm{ac}}},$$

$$\mathrm{sp}_{\mathrm{sc}}(A) = \mathrm{sp}(A \restriction \mathcal{H}_{\mathrm{sc}}) = \overline{\bigcup_{n \in \Gamma} \mathrm{supp} \mu_{\psi_n, \mathrm{sc}}},$$

$$\mathrm{sp}_{\mathrm{pp}}(A) = \mathrm{sp}(A \restriction \mathcal{H}_{\mathrm{pp}}) = \overline{\bigcup_{n \in \Gamma} \mathrm{supp} \mu_{\psi_n, \mathrm{pp}}}$$

are called, respectively, the absolutely continuous, the singular continuous, and the pure point spectrum of A. Note that

$$\mathrm{sp}(A) = \mathrm{sp}_{\mathrm{ac}}(A) \cup \mathrm{sp}_{\mathrm{sc}}(A) \cup \mathrm{sp}_{\mathrm{pp}}(A),$$

and $\overline{\mathrm{sp}_{\mathrm{p}}(A)} = \mathrm{sp}_{\mathrm{pp}}(A)$. The singular and the continuous spectrum of A are defined by

$$\mathrm{sp}_{\mathrm{sing}}(A) = \mathrm{sp}_{\mathrm{sc}}(A) \cup \mathrm{sp}_{\mathrm{pp}}(A), \qquad \mathrm{sp}_{\mathrm{cont}}(A) = \mathrm{sp}_{\mathrm{ac}}(A) \cup \mathrm{sp}_{\mathrm{sc}}(A).$$

The subspaces $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}$, $\mathcal{H}_{\mathrm{pp}}$ are called the spectral subspaces associated, respectively, to the absolutely continuous, singular continuous, and pure point spectrum. The projections on these subspaces are denoted by $\mathbf{1}_{\mathrm{ac}}(A)$, $\mathbf{1}_{\mathrm{sc}}(A)$, $\mathbf{1}_{\mathrm{pp}}(A)$. The spectral subspaces associated to the singular and the continuous spectrum are $\mathcal{H}_{\mathrm{sing}} = \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{pp}}$ and $\mathcal{H}_{\mathrm{cont}} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}$. The corresponding projections are $\mathbf{1}_{\mathrm{sing}}(A) = \mathbf{1}_{\mathrm{sc}}(A) + \mathbf{1}_{\mathrm{pp}}(A)$ and $\mathbf{1}_{\mathrm{cont}}(A) = \mathbf{1}_{\mathrm{ac}}(A) + \mathbf{1}_{\mathrm{sc}}(A)$.

When we wish to indicate the dependence of the spectral subspaces on the operator A, we will write $\mathcal{H}_{\mathrm{ac}}(A)$, etc.

3.9 Harmonic analysis and spectral theory

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, $\psi \in \mathcal{H}$, and μ_ψ the spectral measure for A and ψ. Let F_{μ_ψ} and P_{μ_ψ} be the Borel and the Poisson transform of μ_ψ. The formulas

$$(\psi|(A-z)^{-1}\psi) = F_{\mu_\psi}(z),$$

$$\mathrm{Im}\,(\psi|(A-z)^{-1}\psi) = P_{\mu_\psi}(z)$$

provide the key link between the harmonic analysis (the results of Section 2) and the spectral theory. Recall that $\mu_{\psi,\mathrm{sing}} = \mu_{\psi,\mathrm{sc}} + \mu_{\psi,\mathrm{pp}}$.

Theorem 3.15 (1) *For Lebesgue a.e.* $x \in \mathbb{R}$ *the limit*

$$(\psi|(A-x-\mathrm{i}0)^{-1}\psi) = \lim_{y\downarrow 0}(\psi|(A-x-\mathrm{i}y)^{-1}\psi)$$

exists and is finite and non-zero.
(2) $\mathrm{d}\mu_{\psi,\mathrm{ac}} = \pi^{-1}\mathrm{Im}(\psi|(A-x-\mathrm{i}0)^{-1}\psi)\mathrm{d}x$.
(3) $\mu_{\psi,\mathrm{sing}}$ *is concentrated on the set*

$$\{x\,:\,\lim_{y\downarrow 0}\mathrm{Im}\,(\psi|(A-x-\mathrm{i}y)^{-1}\psi) = \infty\}.$$

Theorem 3.15 is an immediate consequence of Theorems 2.5 and 2.17. Similarly, Theorems 2.6, 2.8 and Corollary 2.10 yield:

Theorem 3.16 *Let* $[a,b]$ *be a finite interval.*
(1) $\mu_{\psi,\mathrm{ac}}([a,b]) = 0$ *iff for some* $p \in (0,1)$

$$\lim_{y\downarrow 0}\int_a^b \left[\mathrm{Im}\,(\psi|(A-x-\mathrm{i}y)^{-1}\psi)\right]^p \mathrm{d}x = 0.$$

(2) *Assume that for some* $p > 1$

$$\sup_{0<y<1}\int_a^b \left[\mathrm{Im}\,(\psi|(A-x-\mathrm{i}y)^{-1}\psi)\right]^p \mathrm{d}x < \infty.$$

Then $\mu_{\psi,\mathrm{sing}}([a,b]) = 0$.
(3) $\mu_{\psi,\mathrm{pp}}([a,b]) = 0$ *iff*

$$\lim_{y\downarrow 0} y\int_a^b \left[\mathrm{Im}\,(\psi|(A-x-\mathrm{i}y)^{-1}\psi)\right]^2 \mathrm{d}x = 0.$$

Let $\mathcal{H}_\psi$ be the cyclic subspace spanned by A and ψ. W.l.o.g. we may assume that $\|\psi\| = 1$. By Theorem 3.13 there exists a (unique) unitary $U_\psi : \mathcal{H}_\psi \to L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ such that $U_\psi \psi = \mathbb{1}$ and $U_\psi A U_\psi^{-1}$ is the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. We extend U_ψ to $\mathcal{H}$ by setting $U_\psi \phi = 0$ for $\phi \in \mathcal{H}_\psi^\perp$. Recall that

$$\mathrm{Im}\,(A - z)^{-1} = \frac{1}{2\mathrm{i}}((A - z)^{-1} - (A - \overline{z})^{-1}).$$

The interplay between spectral theory and harmonic analysis is particularly clearly captured in the following result.

Theorem 3.17 *Let $\phi \in \mathcal{H}$. Then:*

(1)

$$(U_\psi \mathbf{1}_{\mathrm{ac}} \phi)(x) = \lim_{y \downarrow 0} \frac{(\psi | \mathrm{Im}\,(A - x - \mathrm{i}y)^{-1} \phi)}{\mathrm{Im}\,(\psi | (A - x - \mathrm{i}y)^{-1} \psi)} \qquad \text{for } \mu_{\psi,\mathrm{ac}} - a.e.\ x.$$

(2)

$$(U_\psi \mathbf{1}_{\mathrm{sing}} \phi)(x) = \lim_{y \downarrow 0} \frac{(\psi | (A - x - \mathrm{i}y)^{-1} \phi)}{(\psi | (A - x - \mathrm{i}y)^{-1} \psi)} \qquad \text{for } \mu_{\psi,\mathrm{sing}} - a.e.\ x.$$

Proof. Since

$$\frac{(\psi | \mathrm{Im}\,(A - x - \mathrm{i}y)^{-1} \phi)}{\mathrm{Im}\,(\psi | (A - x - \mathrm{i}y)^{-1} \psi)} = \frac{P_{(U_\psi \phi)\mu_\psi}(x + \mathrm{i}y)}{P_{\mu_\psi}(x + \mathrm{i}y)},$$

(1) follows from Theorem 2.5. Similarly, since

$$\frac{(\psi | (A - x - \mathrm{i}y)^{-1} \phi)}{(\psi | (A - x - \mathrm{i}y)^{-1} \psi)} = \frac{F_{(U_\psi \phi)\mu_\psi}(x + \mathrm{i}y)}{F_{\mu_\psi}(x + \mathrm{i}y)},$$

(2) follows from the Poltoratskii theorem (Theorem 2.18). □

3.10 Spectral measure for A

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and let $\{\phi_n\}_{n \in \Gamma}$ be a cyclic set for A. Let $\{a_n\}_{n \in \Gamma}$ be a sequence such that $a_n > 0$ and

$$\sum_{n \in \Gamma} a_n \|\phi_n\|^2 < \infty.$$

The spectral measure for A, μ_A, is a Borel measure on $\mathbb{R}$ defined by

$$\mu_A(\cdot) = \sum_{n \in \Gamma} a_n \mu_{\phi_n}(\cdot).$$

Obviously, μ_A depends on the choice of $\{\phi_n\}$ and a_n. Two positive Borel measures ν_1 and ν_2 on $\mathbb{R}$ are called equivalent (we write $\nu_1 \sim \nu_2$) iff ν_1 and ν_2 have the same sets of measure zero.

Theorem 3.18 *Let* μ_A *and* ν_A *be two spectral measures for* A*. Then* $\mu_A \sim \nu_A$*. Moreover,* $\mu_{A,\mathrm{ac}} \sim \nu_{A,\mathrm{ac}}$*,* $\mu_{A,\mathrm{sc}} \sim \nu_{A,\mathrm{sc}}$*, and* $\mu_{A,\mathrm{pp}} \sim \nu_{A,\mathrm{pp}}$*.*

Theorem 3.19 *Let* μ_A *be a spectral measure for* A*. Then*

$$\mathrm{supp}\,\mu_{A,\mathrm{ac}} = \mathrm{sp}_{\mathrm{ac}}(A), \qquad \mathrm{supp}\,\mu_{A,\mathrm{sc}} = \mathrm{sp}_{\mathrm{sc}}(A), \qquad \mathrm{supp}\,\mu_{A,\mathrm{pp}} = \mathrm{sp}_{\mathrm{pp}}(A).$$

The proofs of these two theorems are left to the problems.

3.11 The essential support of the ac spectrum

Let B_1 and B_2 be two Borel sets in $\mathbb{R}$. Let $B_1 \sim B_2$ iff the Lebesgue measure of the symmetric difference $(B_1 \setminus B_2) \cup (B_2 \setminus B_1)$ is zero. $\sim$ is an equivalence relation. Let μ_A be a spectral measure of a self-adjoint operator A and $f(x)$ its Radon-Nikodym derivative w.r.t. the Lebesgue measure ($\mathrm{d}\mu_{A,\mathrm{ac}} = f(x)\mathrm{d}x$). The equivalence class associated to $\{x : f(x) > 0\}$ is called the essential support of the absolutely continuous spectrum and is denoted by $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$. With a slight abuse of terminology we will also refer to a particular element of $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ as an essential support of the ac spectrum (and denote it by the same symbol $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$). For example, the set

$$\left\{x : 0 < \lim_{r \downarrow 0} (2r)^{-1} \mu_A(I(x,r)) < \infty \right\}$$

is an essential support of the absolutely continuous spectrum.

Note that the essential support of the ac spectrum is independent on the choice of μ_A.

Theorem 3.20 *Let* $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ *be an essential support of the absolutely continuous spectrum. Then* $\mathrm{cl}(\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A) \cap \mathrm{sp}_{\mathrm{ac}}(A)) = \mathrm{sp}_{\mathrm{ac}}(A)$*.*

The proof is left to the problems.

Although the closure of an essential support $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A) \subset \mathrm{sp}_{\mathrm{ac}}(A)$ equals $\mathrm{sp}_{\mathrm{ac}}(A)$, $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ could be substantially "smaller" than $\mathrm{sp}_{\mathrm{ac}}(A)$; see Problem 6.

3.12 The functional calculus

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Let $U : \mathcal{H} \to L^2(M, \mathrm{d}\mu)$, f, and A_f be as in the spectral theorem. Let $B_{\mathrm{b}}(\mathbb{R})$ be the vector space of all bounded Borel functions on $\mathbb{R}$. For $h \in B_{\mathrm{b}}(\mathbb{R})$, consider the operator $A_{h \circ f}$. This operator is bounded and $\|A_{h\circ f}\| \le \sup h(x)$. Set

$$h(A) = U^{-1} A_{h \circ f} U. \tag{3.44}$$

Let $\Phi : B_{\mathrm{b}}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ be given by $\Phi(h) = h(A)$. Recall that $r_z(x) = (x - z)^{-1}$.

Theorem 3.21 (1) *The map Φ is an algebraic $*$-homomorphism.*
(2) $\|\Phi(h)\| \le \max|h(x)|$.
(3) $\Phi(r_z) = (A-z)^{-1}$ *for all* $z \in \mathbb{C} \setminus \mathbb{R}$.
(4) *If* $h_n(x) \to h(x)$ *for all* x, *and* $\sup_{n,x}|h_n(x)| < \infty$, *then* $h_n(A)\psi \to h(A)\psi$ *for all* ψ.
The map Φ is uniquely specified by (1)-(4). Moreover, it has the following additional properties:
(5) *If* $A\psi = \lambda\psi$, *then* $\Phi(h)\psi = h(\lambda)\psi$.
(6) *If* $h \ge 0$, *then* $\Phi(h) \ge 0$.

We remark that the uniqueness of the functional calculus is an immediate consequence of Problem 11 in Section 1.

Let $\mathcal{K} \subset \mathcal{H}$ be a closed subspace. If $\mathcal{K}$ is invariant under A, then for all $h \in B_{\mathrm{b}}(\mathbb{R})$, $h(A)\mathcal{K} \subset \mathcal{K}$.

For any Borel function $h : \mathbb{R} \to \mathbb{C}$ we define $h(A)$ by (3.44). Of course, $h(A)$ could be an unbounded operator. Note that $h_1(A)h_2(A) \subset h_1 \circ h_2(A)$, $h_1(A) + h_2(A) \subset (h_1 + h_2)(A)$. Also, $h(A)^* = \overline{h}(A)$ and $h(A)$ is self-adjoint iff $h(x) \in \mathbb{R}$ for μ_A-a.e. $x \in M$.

In fact, to define $h(A)$, we only need that $\mathrm{Ran}\, f \subset \mathrm{Dom}\, h$. Hence, if $A \ge 0$, we can define $\sqrt{A}$, if $A > 0$ we can define $\ln A$, etc.

The two classes of functions, characteristic functions and exponentials, play a particularly important role.

Let F be a Borel set in $\mathbb{R}$ and χ_F its characteristic function. The operator $\chi_F(A)$ is an orthogonal projection, called the spectral projection on the set F. In these notes we will use the notation $\mathbf{1}_F(A) = \chi_F(A)$ and $\mathbf{1}_{\{e\}}(A) = \mathbf{1}_e(A)$. Note that $\mathbf{1}_e(A) \ne 0$ iff $e \in \mathrm{sp_p}(A)$. By definition, the multiplicity of the eigenvalue e is $\dim \mathbf{1}_e(A)$.

The subspace $\mathrm{Ran}\,\mathbf{1}_F(A)$ is invariant under A and

$$\mathrm{cl}(\mathrm{int}(F) \cap \mathrm{sp}(A)) \subset \mathrm{sp}(A \restriction \mathrm{Ran}\,\mathbf{1}_F(A)) \subset \mathrm{sp}(A) \cap \mathrm{cl}(F). \tag{3.45}$$

Note in particular that $e \in \mathrm{sp}(A)$ iff for all $\epsilon > 0$ $\mathrm{Ran}\,\mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) \ne \{0\}$. The proof of (3.45) is left to the problems.

Theorem 3.22 (Stone's formula) *For* $\psi \in \mathcal{H}$,

$$\lim_{y\downarrow 0} \frac{y}{\pi} \int_a^b \mathrm{Im}(A - x - \mathrm{i}y)^{-1}\psi \mathrm{d}x = \frac{1}{2}\left[\mathbf{1}_{[a,b]}(A)\psi + \mathbf{1}_{(a,b)}(A)\psi\right].$$

Proof. Since

$$\lim_{y\downarrow 0} \frac{y}{\pi} \int_a^b \frac{1}{(t-x)^2 + y^2} \mathrm{d}x = \begin{cases} 0 & \text{if } t \notin [a,b], \\ 1/2 & \text{if } t = a \text{ or } t = b, \\ 1 & \text{if } t \in (a,b), \end{cases}$$

the Stone formula follows from Theorem 3.21. □

Another important class of functions are exponentials. For $t \in \mathbb{R}$, set $U(t) = \exp(\mathrm{i}tA)$. Then $U(t)$ is a group of unitary operators on $\mathcal{H}$. The group $U(t)$ is strongly continuous, i.e. for all $\psi \in \mathcal{H}$,

$$\lim_{s \to t} U(s)\psi = U(t)\psi.$$

For $\psi \in \mathrm{Dom}\,(A)$ the function $\mathbb{R} \ni t \mapsto U(t)\psi$ is strongly differentiable and

$$\lim_{t \to 0} \frac{U(t)\psi - \psi}{t} = \mathrm{i}A\psi. \tag{3.46}$$

On the other hand, if the limit on the l.h.s. exists for some ψ, then $\psi \in \mathrm{Dom}\,(A)$ and (3.46) holds.

The above results have a converse:

Theorem 3.23 (Stone's theorem) *Let $U(t)$ be a strongly continuous group on a separable Hilbert space $\mathcal{H}$. Then there is a self-adjoint operator A such that $U(t) = \exp(\mathrm{i}tA)$.*

3.13 The Weyl criteria and the RAGE theorem

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$.

Theorem 3.24 (Weyl criterion 1) $e \in \mathrm{sp}(A)$ *iff there exists a sequence of unit vectors* $\psi_n \in \mathrm{Dom}\,(A)$ *such that*

$$\lim_{n \to \infty} \|(A - e)\psi_n\| = 0. \tag{3.47}$$

Remark. A sequence of unit vectors for which (3.47) holds is called a Weyl sequence.

Proof. Recall that $e \in \mathrm{sp}(A)$ iff $\mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) \neq 0$ for all $\epsilon > 0$.

Assume that $e \in \mathrm{sp}(A)$. Let $\psi_n \in \mathrm{Ran}\,\mathbf{1}_{(e-1/n,e+1/n)}(A)$ be unit vectors. Then, by the functional calculus,

$$\|(A - e)\psi_n\| \leq \sup_{x \in (e-1/n,e+1/n)} |x - e| \leq 1/n.$$

On the other hand, assume that there is a sequence ψ_n such that (3.47) holds and that $e \notin \mathrm{sp}(A)$. Then

$$\|\psi_n\| = \|(A - e)^{-1}(A - e)\psi_n\| \leq C\|(A - e)\psi_n\|,$$

and so $1 = \|\psi_n\| \to 0$, contradiction. □

The discrete spectrum of A, denoted $\mathrm{sp}_{\mathrm{disc}}(A)$, is the set of all isolated eigenvalues of finite multiplicity. Hence $e \in \mathrm{sp}_{\mathrm{disc}}(A)$ iff $1 \leq \dim \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) < \infty$ for all ϵ small enough. The essential spectrum of A is defined by

$$\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}(A) \setminus \mathrm{sp}_{\mathrm{disc}}(A).$$

Hence, $e \in \mathrm{sp}_{\mathrm{ess}}(A)$ iff for all $\epsilon > 0$ $\dim \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) = \infty$. Obviously, $\mathrm{sp}_{\mathrm{ess}}(A)$ is a closed subset of $\mathbb{R}$.

Theorem 3.25 (Weyl criterion 2) $e \in \mathrm{sp}_{\mathrm{ess}}(A)$ *iff there exists an orthonormal sequence* $\psi_n \in \mathrm{Dom}\,(A)$ *(*$\|\psi_n\| = 1$*,* $(\psi_n|\psi_m) = 0$ *if* $n \neq m$*), such that*

$$\lim_{n\to\infty} \|(A-e)\psi_n\| = 0. \tag{3.48}$$

Proof. Assume that $e \in \mathrm{sp}_{\mathrm{ess}}(A)$. Then $\dim \mathbf{1}_{(e-1/n,e+1/n)}(A) = \infty$ for all n, and we can choose an orthonormal sequence ψ_n such that $\psi_n \in \mathrm{Ran}\,\mathbf{1}_{(e-1/n,e+1/n)}(A)$. Clearly,

$$\|(A-e)\psi_n\| \leq 1/n$$

and (3.48) holds.

On the other hand, assume that there exists an orthonormal sequence ψ_n such that (3.48) holds and that $e \in \mathrm{sp}_{\mathrm{disc}}(A)$. Choose $\epsilon > 0$ such that $\dim \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) < \infty$. Then, $\lim_{n\to\infty} \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A)\psi_n = 0$ and

$$\lim_{n\to\infty} \|(A-e)\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n\| = 0.$$

Since $(A-e) \restriction \mathrm{Ran}\,\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)$ is invertible and the norm of its inverse is $\leq 1/\epsilon$, we have that

$$\|\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n\| \leq \epsilon^{-1}\|(A-e)\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n\|,$$

and so $\lim_{n\to\infty} \mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n = 0$. Hence $1 = \|\psi_n\| \to 0$, contradiction. □

Theorem 3.26 (RAGE) (1) *Let* K *be a compact operator. Then for all* $\psi \in \mathcal{H}_{\mathrm{cont}}$*,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|K\mathrm{e}^{-\mathrm{i}tA}\psi\|^2 \mathrm{d}t = 0. \tag{3.49}$$

(2) *The same result holds if* K *is bounded and* $K(A+\mathrm{i})^{-1}$ *is compact.*

Proof. (1) First, recall that any compact operator is a norm limit of finite rank operators. In other words, there exist vectors $\phi_n, \varphi_n \in \mathcal{H}$ such that $K_n = \sum_{j=1}^n (\phi_j|\cdot)\varphi_j$ satisfies $\lim_{n\to\infty}\|K - K_n\| = 0$. Hence, it suffices to prove the statement for the finite rank operators K_n. By induction and the triangle inequality, it suffices to prove the statement for the rank one operator $K = (\phi|\cdot)\varphi$. Thus, it suffices to show that for $\phi \in \mathcal{H}$ and $\psi \in \mathcal{H}_{\mathrm{cont}}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |(\phi|\mathrm{e}^{-\mathrm{i}tA}\psi)|^2 \mathrm{d}t = 0.$$

Moreover, since

$$(\phi|\mathrm{e}^{-\mathrm{i}tA}\psi) = (\phi|\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_{\mathrm{cont}}(A)\psi) = (\mathbf{1}_{\mathrm{cont}}(A)\phi|\mathrm{e}^{-\mathrm{i}tA}\psi),$$

w.l.o.g we may assume that $\phi \in \mathcal{H}_{\rm cont}$. Finally, by the polarization identity, we may assume that $\varphi = \psi$. Since for $\psi \in \mathcal{H}_{\rm cont}$ the spectral measure μ_ψ has no atoms, the result follows from the Wiener theorem (Theorem 1.6 in Section 1).

(2) Since $\mathrm{Dom}\,(A) \cap \mathcal{H}_{\rm cont}$ is dense in $\mathcal{H}_{\rm cont}$, it suffices to prove the statement for $\psi \in \mathrm{Dom}\,(A) \cap \mathcal{H}_{\rm cont}$. Write

$$\|K\mathrm{e}^{-\mathrm{i}tA}\psi\| = \|K(A+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tA}(A+\mathrm{i})\psi\|$$

and use (1). □

3.14 Stability

We will first discuss stability of self-adjointness—if A and B are self-adjoint operators, we wish to discuss conditions under which $A+B$ is self-adjoint on $\mathrm{Dom}\,(A) \cap \mathrm{Dom}\,(B)$. One obvious sufficient condition is that either A or B is bounded. A more refined result involves the notion of relative boundedness.

Let A and B be densely defined linear operators on a separable Hilbert space $\mathcal{H}$. The operator B is called A*-bounded* if $\mathrm{Dom}\,(A) \subset \mathrm{Dom}\,(B)$ and for some positive constants a and b and all $\psi \in \mathrm{Dom}\,(A)$,

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\|. \tag{3.50}$$

The number a is called a relative bound of B w.r.t. A.

Theorem 3.27 (Kato-Rellich) *Suppose that A is self-adjoint, B is symmetric, and B is A-bounded with a relative bound $a < 1$. Then:*
(1) $A+B$ *is self-adjoint on* $\mathrm{Dom}\,(A)$.
(2) $A+B$ *is essentially self-adjoint on any core of* A.
(3) *If A is bounded from below, then $A+B$ is also bounded from below.*

Proof. We will prove (1) and (2); (3) is left to the problems. In the proof we will use the following elementary fact: if V is a bounded operator and $\|V\| < 1$, then $0 \notin \mathrm{sp}(\mathbf{1}+V)$. This is easily proven by checking that the inverse of $\mathbf{1}+V$ is given by $\mathbf{1}+\sum_{k=1}^\infty(-1)^k V^k$.

By Theorem 3.4 (and the Remark after Theorem 3.5), to prove (1) it suffices to show that there exists $y > 0$ such that $\mathrm{Ran}(A+B\pm \mathrm{i}y) = \mathcal{H}$. In what follows $y = (1+b)/(1-a)$. The relation (3.50) yields

$$\|B(A\pm \mathrm{i}y)^{-1}\| \leq a\|A(A\pm\mathrm{i}y)^{-1}\| + b\|(A\pm\mathrm{i}y)^{-1}\| \leq a + by^{-1} < 1,$$

and so $\mathbf{1} + B(A\pm\mathrm{i}y)^{-1} : \mathcal{H} \to \mathcal{H}$ are bijections. Since $A \pm \mathrm{i}y : \mathrm{Dom}\,(A) \to \mathcal{H}$ are also bijections, the identity

$$A+B\pm\mathrm{i}y = (\mathbf{1}+B(A\pm\mathrm{i}y)^{-1})(A\pm\mathrm{i}y)$$

yields $\mathrm{Ran}(A+B\pm\mathrm{i}y) = \mathcal{H}$.

The proof of (2) is based on Theorem 3.5. Let D be a core for A. Then the sets $(A \pm \mathrm{i}y)(D) = \{(A \pm \mathrm{i}y)\psi : \psi \in D\}$ are dense in $\mathcal{H}$, and since $\mathbf{1} + B(A \pm \mathrm{i}y)^{-1}$ are bijections,

$$(A + B \pm \mathrm{i}y)(D) = (\mathbf{1} + B(A \pm \mathrm{i}y)^{-1})(A \pm \mathrm{i}y)(D)$$

are also dense in $\mathcal{H}$. □

We now turn to stability of the essential spectrum. The simplest result in this direction is:

Theorem 3.28 (Weyl) *Let A and B be self-adjoint and B compact. Then* $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}_{\mathrm{ess}}(A+B)$.

Proof. By symmetry, it suffices to prove that $\mathrm{sp}_{\mathrm{ess}}(A+B) \subset \mathrm{sp}_{\mathrm{ess}}(A)$. Let $e \in \mathrm{sp}_{\mathrm{ess}}(A+B)$ and let ψ_n be an orthonormal sequence such that

$$\lim_{n\to\infty} \|(A + B - e)\psi_n\| = 0.$$

Since ψ_n converges weakly to zero and B is compact, $B\psi_n \to 0$. Hence, $\|(A-e)\psi_n\| \to 0$ and $e \in \mathrm{sp}_{\mathrm{ess}}(A)$. □

Section XIII.4 of [RS4] deals with various extensions and generalizations of Theorem 3.28.

3.15 Scattering theory and stability of ac spectra

Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$. Assume that for all $\psi \in \mathrm{Ran}\, \mathbf{1}_{\mathrm{ac}}(A)$ the limits

$$\Omega^{\pm}(A, B)\psi = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{-\mathrm{i}tB}\psi \tag{3.51}$$

exist. The operators $\Omega^{\pm}(A, B) : \mathrm{Ran}\, \mathbf{1}_{\mathrm{ac}}(A) \to \mathcal{H}$ are called *wave operators*.

Proposition 3.29 *Assume that the wave operators exist. Then*
(1) $(\Omega^{\pm}(A, B)\phi|\Omega^{\pm}(A, B)\psi) = (\phi|\psi)$.
(2) *For any* $f \in B_{\mathrm{b}}(\mathbb{R})$, $\Omega^{\pm}(A, B)f(A) = f(B)\Omega^{\pm}(A, B)$.
(3) $\mathrm{Ran}\, \Omega^{\pm}(A, B) \subset \mathrm{Ran}\, \mathbf{1}_{\mathrm{ac}}(B)$.
The wave operators $\Omega^{\pm}(A, B)$ *are called complete if* $\mathrm{Ran}\, \Omega^{\pm}(A, B) = \mathrm{Ran}\, \mathbf{1}_{\mathrm{ac}}(B)$*;*
(4) *Wave operators* $\Omega^{\pm}(A, B)$ *are complete iff* $\Omega^{\pm}(B, A)$ *exist.*

The proof of this proposition is simple and is left to the problems (see also [RS3]).

Let $\mathcal{H}$ be a separable Hilbert space and $\{\psi_n\}$ an orthonormal basis. A bounded positive self-adjoint operator A is called *trace class* if

$$\mathrm{Tr}(A) = \sum_n (\psi_n|A\psi_n) < \infty,$$

The number $\mathrm{Tr}(A)$ does not depend on the choice of orthonormal basis. More generaly, a bounded self-adjoint operator A is called trace class if $\mathrm{Tr}(|A|) < \infty$. A trace class operator is compact.

Concerning stability of the ac spectrum, the basic result is:

Theorem 3.30 (Kato-Rosenblum) *Let A and B be self-adjoint and B trace class. Then the wave operators $\Omega^{\pm}(A+B, A)$ exist and are complete. In particular,* $\mathrm{sp}_{\mathrm{ac}}(A+B) = \mathrm{sp}_{\mathrm{ac}}(A)$ *and* $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A+B) = \Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$.

For the proof of Kato-Rosenblum theorem see [RS3], Theorem XI.7.

The singular and the pure point spectra are in general unstable under perturbations—they may appear or dissapear under the influence of a rank one perturbation. We will discuss in Section 4 criteria for "generic" stability of the singular and the pure point spectra.

3.16 Notions of measurability

In mathematical physics one often encounters self-adjoint operators indexed by elements of some measure space $(M, \mathcal{F})$, namely one deals with functions $M \ni \omega \mapsto A_\omega$, where A_ω is a self-adjoint operator on some fixed separable Hilbert space $\mathcal{H}$. In this subsection we address some issues concerning measurability of such functions.

Let $(M, \mathcal{F})$ be a measure space and X a topological space. A function $f : M \to X$ is called measurable if the inverse image of every open set is in $\mathcal{F}$.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the vector space of all bounded operators on $\mathcal{H}$. We distinguish three topologies in $\mathcal{B}(\mathcal{H})$, the uniform topology, the strong topology, and the weak topology. The uniform topology is induced by the operator norm on $\mathcal{B}(\mathcal{H})$. The strong topology is the minimal topology w.r.t. which the functions $\mathcal{B}(\mathcal{H}) \ni A \mapsto A\psi \in \mathcal{H}$ are continuous for all $\psi \in \mathcal{H}$. The weak topology is the minimal topology w.r.t. which the functions $\mathcal{B}(\mathcal{H}) \ni A \mapsto (\phi|A\psi) \in \mathbb{C}$ are continuous for all $\phi, \psi \in \mathcal{H}$. The weak topology is weaker than the strong topology, and the strong topology is weaker than the uniform topology.

A function $f : M \to \mathcal{B}(\mathcal{H})$ is uniform/strong/weak measurable if it is measurable with respect to the uniform/strong/weak topology. Obviously, uniform measurability $\Rightarrow$ strong measurability $\Rightarrow$ weak measurability. Note that f is weakly measurable iff the function $M \ni \omega \mapsto (\phi|f(\omega)\psi) \in \mathbb{C}$ is measurable for all $\phi, \psi \in \mathcal{H}$.

Theorem 3.31 *A function $f : M \to \mathcal{B}(\mathcal{H})$ is uniform measurable iff it is weakly measurable.*

The proof of this theorem is left to the problems. A function $f : M \to \mathcal{B}(\mathcal{H})$ is *measurable* iff it is weakly measurable (which is equivalent to requiring that f is strongly or uniform measurable).

Let $\omega \mapsto A_\omega$ be a function with values in (possibly unbounded) self-adjoint operators on $\mathcal{H}$. We say that A_ω is *measurable* if for all $z \in \mathbb{C} \setminus \mathbb{R}$ the function

$$\omega \mapsto (A_\omega - z)^{-1} \in \mathcal{B}(\mathcal{H})$$

is measurable.

Until the end of this subsection $\omega \mapsto A_\omega$ is a given measurable function with values in self-adjoint operators.

Theorem 3.32 *The function $\omega \mapsto h(A_\omega)$ is measurable for all $h \in B_{\rm b}(\mathbb{R})$.*

Proof. Let $\mathcal{T} \subset B_{\rm b}(\mathbb{R})$ be the class of functions such that $\omega \mapsto h(A_\omega)$ is measurable By definition, $(x-z)^{-1} \in \mathcal{T}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Since the linear span of $\{(x-z)^{-1} : z \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in the Banach space $C_0(\mathbb{R})$, $C_0(\mathbb{R}) \subset \mathcal{T}$. Note also that if $h_n \in \mathcal{T}$, $\sup_{n,x} |h_n(x)| < \infty$, and $h_n(x) \to h(x)$ for *all* x, then $h \in \mathcal{T}$. Hence, by Problem 11 in Section 1, $\mathcal{T} = B_{\rm b}(\mathbb{R})$. □

From this theorem it follows that the functions $\omega \mapsto \mathbf{1}_B(A_\omega)$ (B Borel) and $\omega \mapsto \exp({\rm i}tA_\omega)$ are measurable. One can also easily show that if $h : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary Borel measurable real valued function, then $\omega \mapsto h(A_\omega)$ is measurable.

We now turn to the measurability of projections and spectral measures.

Proposition 3.33 *The function $\omega \mapsto \mathbf{1}_{\rm cont}(A_\omega)$ is measurable.*

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and let P_n be the orthogonal projection on the subspace spanned by $\{\phi_k\}_{k\geq n}$. The RAGE theorem yields that for $\varphi, \psi \in \mathcal{H}$,

$$(\varphi|\mathbf{1}_{\rm cont}(A_\omega)\psi) = \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T(\varphi|{\rm e}^{{\rm i}tA_\omega}P_n{\rm e}^{-{\rm i}tA_\omega}\psi){\rm d}t \tag{3.52}$$

(the proof of (3.52) is left to the problems), and the statement follows. □

Proposition 3.34 *The function $\omega \mapsto \mathbf{1}_{\rm ac}(A_\omega)$ is measurable.*

Proof. By Theorem 2.6, for all $\psi \in \mathcal{H}$,

$$(\psi|\mathbf{1}_{\rm ac}(A_\omega)\psi) = \lim_{M\to\infty}\lim_{p\uparrow 1}\lim_{\epsilon\downarrow 0}\frac{1}{\pi^p}\int_{-M}^{M}\left[{\rm Im}\,(\psi|(A_\omega - x - {\rm i}\epsilon)^{-1}\psi)\right]^p{\rm d}x,$$

and so $\omega \mapsto (\psi|\mathbf{1}_{\rm ac}(A_\omega)\psi)$ is measurable. The polarization identity yields the statement. □

Proposition 3.35 *The functions $\omega \mapsto \mathbf{1}_{\rm sc}(A_\omega)$ and $\omega \mapsto \mathbf{1}_{\rm pp}(A_\omega)$ are measurable.*

Proof. $\mathbf{1}_{\rm sc}(A_\omega) = \mathbf{1}_{\rm cont}(A_\omega) - \mathbf{1}_{\rm ac}(A_\omega)$ and $\mathbf{1}_{\rm pp}(A_\omega) = \mathbf{1} - \mathbf{1}_{\rm cont}(A_\omega)$. □

Let $M(\mathbb{R})$ be the Banach space of all complex Borel measures on $\mathbb{R}$ (the dual of $C_0(\mathbb{R})$). A map $\omega \mapsto \mu^\omega \in M(\mathbb{R})$ is called measurable iff for all $f \in B_{\rm b}(\mathbb{R})$ the map $\omega \mapsto \mu^\omega(f)$ is measurable.

We denote by μ_ψ^ω the spectral measure for A_ω and ψ.

Proposition 3.36 *The functions $\omega \mapsto \mu_{\psi,{\rm ac}}^\omega$, $\omega \mapsto \mu_{\psi,{\rm sc}}^\omega$, $\omega \mapsto \mu_{\psi,{\rm pp}}^\omega$ are measurable.*

Proof. Since for any Borel set B, $(\mathbf{1}_B(A_\omega)\psi|\mathbf{1}_{\rm ac}(A_\omega)\psi) = \mu^\omega_{\psi,\rm ac}(B)$, $(\mathbf{1}_B(A_\omega)\psi|\mathbf{1}_{\rm sc}(A_\omega)\psi) = \mu^\omega_{\psi,\rm sc}(B)$, $(\mathbf{1}_B(A_\omega)\psi|\mathbf{1}_{\rm pp}(A_\omega)\psi) = \mu^\omega_{\psi,\rm pp}(B)$, the statement follows from Propositions 3.34 and 3.35. □

Let $\{\psi_n\}$ be a cyclic set for A_ω and let $a_n > 0$ be such that $\sum_n a_n\|\psi_n\|^2 < \infty$. We denote by

$$\mu^\omega = \sum_n a_n \mu^\omega_{\psi_n}$$

the corresponding spectral measure for A_ω. Proposition 3.36 yields

Proposition 3.37 *The functions $\omega \mapsto \mu^\omega_{\rm ac}$, $\omega \mapsto \mu^\omega_{\rm sc}$, $\omega \mapsto \mu^\omega_{\rm pp}$ are measurable.*

Let $\Sigma^{{\rm ess},\omega}_{\rm ac}$ be the essential support of the ac spectrum of A_ω. The map

$$\omega \mapsto (1+x^2)^{-1}\chi_{\Sigma^{{\rm ess},\omega}_{\rm ac}}(x) \in L^1(\mathbb{R}, {\rm d}x) \tag{3.53}$$

does not depend on the choice of representative in $\Sigma^{{\rm ess},\omega}_{\rm ac}$.

Proposition 3.38 *The function (3.53) is weakly measurable, namely for all $h \in L^\infty(\mathbb{R}, {\rm d}x)$, the function*

$$\omega \mapsto \int_{\mathbb{R}} h(x)(1+x^2)^{-1}\chi_{\Sigma^{{\rm ess},\omega}_{\rm ac}}(x){\rm d}x$$

is measurable.

Proof. It suffices to prove the statement for $h(x) = (1+x)^2\chi_B(x)$ where B is a bounded interval. Let μ^ω be a spectral measure for A_ω. By the dominated convergence theorem

$$\int_B \chi_{\Sigma^{{\rm ess},\omega}_{\rm ac}}(x){\rm d}x = 2\lim_{\epsilon\downarrow 0}\lim_{\delta\downarrow 0}\int_B \frac{P_{\mu^\omega_{\rm ac}}(x+{\rm i}\delta)}{P_{\mu^\omega_{\rm ac}}(x+{\rm i}\epsilon) + P_{\mu^\omega_{\rm ac}}(x+{\rm i}\delta)}{\rm d}x, \tag{3.54}$$

and the statement follows. □

3.17 Non-relativistic quantum mechanics

According to the usual axiomatization of quantum mechanics, a physical system is described by a Hilbert space $\mathcal{H}$. Its observables are described by bounded self-adjoint operators on $\mathcal{H}$. Its states are described by density matrices on $\mathcal{H}$, i.e. positive trace class operators with trace 1. If the system is in a state ρ, then the expected value of the measurement of an observable A is

$$\langle A\rangle_\rho = {\rm Tr}(\rho A).$$

The variance of the measurement is

$$D_\rho(A) = \langle (A - \langle A\rangle_\rho)^2\rangle_\rho = \langle A^2\rangle_\rho - \langle A\rangle_\rho^2.$$

The Cauchy-Schwarz inequality yields the *Heisenberg principle*: For self-adjoint $A, B \in \mathcal{B}(\mathcal{H})$,

$$|\mathrm{Tr}(\rho \mathrm{i}[A, B])| \le D_\rho(A)^{1/2} D_\rho(B)^{1/2}.$$

Of particular importance are the so called pure states $\rho = (\varphi|\cdot)\varphi$. In this case, for a self-adjoint A,

$$\langle A\rangle_\rho = \mathrm{Tr}(\rho A) = (\varphi|A\varphi) = \int_{\mathbb{R}} x \mathrm{d}\mu_\varphi(x),$$

$$D_\rho(A) = \int_{\mathbb{R}} x^2 \mathrm{d}\mu_\varphi - \left(\int_{\mathbb{R}} x \mathrm{d}\mu_\varphi\right)^2,$$

where μ_φ is the spectral measure for A and φ. If the system is in a pure state described by a vector φ, the possible results R of the measurement of A are numbers in $\mathrm{sp}(A)$ randomly distributed according to

$$\mathrm{Prob}(R \in [a, b]) = \int_{[a,b]} \mathrm{d}\mu_\varphi$$

(recall that μ_φ is supported on $\mathrm{sp}(A)$). Obviously, in this case $\langle A\rangle_\rho$ and $D_\rho(A)$ are the usual expectation and variance of the random variable R.

The dynamics is described by a strongly continuous unitary group $U(t)$ on $\mathcal{H}$. In the Heisenberg picture, one evolves observables and keeps states fixed. Hence, if the system is initially in a state ρ, then the expected value of A at time t is

$$\mathrm{Tr}(\rho[U(t)AU(t)^*]).$$

In the Schrödinger picture, one keeps observables fixed and evolves states—the expected value of A at time t is $\mathrm{Tr}([U(t)^*\rho U(t)]A)$. Note that if $\rho = |\varphi)(\varphi|$, then

$$\mathrm{Tr}([U(t)^*\rho U(t)]A) = \|AU(t)\varphi\|^2.$$

The generator of $U(t)$, H, is called the Hamiltonian of the system. The spectrum of H describes the possible energies of the system. The discrete spectrum of H describes energy levels of bound states (the eigenvectors of H are often called bound states). Note that if φ is an eigenvector of H, then $\|AU(t)\varphi\|^2 = \|A\varphi\|^2$ is independent of t.

By the RAGE theorem, if $\varphi \in \mathcal{H}_{\mathrm{cont}}(H)$ and A is compact, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|AU(t)\varphi\|^2 \mathrm{d}t = 0. \tag{3.55}$$

Compact operators describe what one might call sharply localized observables. The states associated to $\mathcal{H}_{\mathrm{cont}}(H)$ move to infinity in the sense that after a sufficiently long time the sharply localized observables are not seen by these states. On the other hand, if $\varphi \in \mathcal{H}_{\mathrm{pp}}(H)$, then for any bounded A,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|AU(t)\varphi\|^2 \mathrm{d}t = \sum_{e\in \mathrm{sp}_{\mathrm{pp}}(H)} \|\mathbf{1}_e(H)A\mathbf{1}_e(H)\varphi\|^2.$$

The mathematical formalism sketched above is commonly used for a description of non-relativistic quantum systems with finitely many degrees of freedom. It can be used, for example, to describe non-relativistic matter—a finite assembly of interacting non-relativistic atoms and molecules. In this case H is the usual N-body Schrödinger operator. This formalism, however, is not suitable for a description of quantum systems with infinitely many degrees of freedom like non-relativistic QED, an infinite electron gas, quantum spin-systems, etc.

3.18 Problems

[1] *Prove Proposition 3.7*

[2] *Prove Theorem 3.9.*

[3] *Prove Theorem 3.18.*

[4] *Prove Theorem 3.19.*

[5] *Prove Theorem 3.20.*

[6] *Let* $0 < \epsilon < 1$. *Construct an example of a self-adjoint operator* A *such that* $\mathrm{sp}_{\mathrm{ac}}(A) = [0, 1]$ *and that the Lebesgue measure of* $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ *is equal to* ϵ.

[7] *Prove that* $\psi \in \mathcal{H}_{\mathrm{cont}}$ *iff (3.49) holds for all compact* K.

[8] *Prove that* $A \geq 0$ *iff* $\mathrm{sp}(A) \subset [0, \infty)$.

[9] *Prove Relation (3.45).*

[10] *Prove Part (3) of Theorem 3.27.*

[11] *Prove Proposition 3.29.*

[12] *Prove Theorem 3.31.*

[13] *Let* $M \ni \omega \mapsto A_\omega$ *be a function with values in self-adjoint operators on* $\mathcal{H}$. *Prove that the following statements are equivalent:*
(1) $\omega \mapsto (A_\omega - z)^{-1}$ *is measurable for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) $\omega \mapsto \exp(\mathrm{i}tA_\omega)$ *is measurable for all* $t \in \mathbb{R}$.
(3) $\omega \mapsto \mathbf{1}_B(A_\omega)$ *is measurable for all Borel sets* B.

[14] *Prove Relation (3.52).*

[15] *In the literature, the proof of the measurability of* $\omega \mapsto \mathbf{1}_{\mathrm{sc}}(A_\omega)$ *is usually based on Carmona's lemma: Let* μ *be a finite, positive Borel measure on* $\mathbb{R}$*, and let* $\mathcal{I}$ *be the set of finite unions of open intervals, each of which has rational endpoints. Then, for any Borel set* B*,*

$$\mu_{\mathrm{sing}}(B) = \lim_{n\to\infty} \sup_{I\in\mathcal{I},|I|<1/n} \mu(B\cap I).$$

Prove Carmona's lemma and using this result show that $\omega \mapsto \mathbf{1}_{\mathrm{sc}}(H_\omega)$ *is measurable. Hint: See* [CL] *or Section 9.1 in* [CFKS].

[16] *Recall that* $M(\mathbb{R})$ *is the Banach space of all complex measures on* $\mathbb{R}$*. Assume that* $\omega \mapsto \mu_\omega \in M(\mathbb{R})$ *is measurable. Prove that this function is also measurable w.r.t. the uniform topology of* $M(\mathbb{R})$*.*

[17] *Assume that* $\omega \mapsto H_\omega$ *is self-adjoint measurable. Prove that* $\omega \mapsto \mathbf{1}_{\mathrm{pp}}(H_\omega)$ *is measurable by using Simon's local Wiener theorem (Theorem 2.9).*

The next set of problems deals with spectral theory of a closed operator A *on a Hilbert space* $\mathcal{H}$*.*

[18] *Let* $F \subset \mathrm{sp}(A)$ *be an isolated, bounded subset of* $\mathrm{sp}(A)$*. Let* γ *be a closed simple path in the complex plane that separates* F *from* $\mathrm{sp}(A)\setminus F$*. Set*

$$\mathbf{1}_F(A) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z-A)^{-1}\mathrm{d}z.$$

(1) *Prove that* $\mathbf{1}_F(A)$ *is a (not necessarily orthogonal) projection.*
(2) *Prove that* $\mathrm{Ran}\,\mathbf{1}_F(A)$ *and* $\mathrm{Ker}\,\mathbf{1}_F(A)$ *are complementary (not necessarily orthogonal) subspaces:* $\mathrm{Ran}\,\mathbf{1}_F(A) + \mathrm{Ker}\,\mathbf{1}_F(A) = \mathcal{H}$ *and* $\mathrm{Ran}\,\mathbf{1}_F(A) \cap \mathrm{Ker}\,\mathbf{1}_F(A) = \{0\}$.
(3) *Prove that* $\mathrm{Ran}\,\mathbf{1}_F(A) \subset \mathrm{Dom}\,(A)$ *and that* $A : \mathrm{Ran}\,\mathbf{1}_F(A) \to \mathrm{Ran}\,\mathbf{1}_F(A)$*. Prove that* $A \restriction \mathrm{Ran}\,\mathbf{1}_F(A)$ *is a bounded operator and that its spectrum is* F*.*
(4) *Prove that* $\mathrm{Ker}\,\mathbf{1}_F(A) \cap \mathrm{Dom}\,(A)$ *is dense and that*

$$A \restriction (\mathrm{Ker}\,\mathbf{1}_F(A) \cap \mathrm{Dom}\,(A)) \to \mathrm{Ker}\,\mathbf{1}_F(A). \tag{3.56}$$

Prove that the operator (3.56) is closed and that its spectrum is $\mathrm{sp}(A)\setminus F$*.*
(5) *Assume that* $F = \{z_0\}$ *and that* $\mathrm{Ran}\,\mathbf{1}_{z_0}(A)$ *is finite dimensional. Prove that if* $\psi \in \mathrm{Ran}\,\mathbf{1}_{z_0}(A)$*, then* $(A-z_0)^n\psi = 0$ *for some* n*.*
Hint: Consult Theorem XII.5 in [RS4].
Remark. *The set of* $z_0 \in \mathrm{sp}(A)$ *which satisfy (5) is called the discrete spectrum of the*

closed operator operator A and is denoted $\mathrm{sp}_{\mathrm{disc}}(A)$. The essential spectrum is defined by $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}(A) \setminus \mathrm{sp}_{\mathrm{disc}}(A)$

[19] *Prove that $\mathrm{sp}_{\mathrm{ess}}(A)$ is a closed subset of $\mathbb{C}$.*

[20] *Prove that $z \mapsto (A - z)^{-1}$ is a meromorphic function on $\mathbb{C} \setminus \mathrm{sp}_{\mathrm{ess}}(A)$ with singularities at points $z_0 \in \mathrm{sp}_{\mathrm{disc}}(A)$. Prove that the negative coefficents of of the Laurent expansion at $z_0 \in \mathrm{sp}_{\mathrm{disc}}(A)$ are finite rank operators. Hint: See Lemma 1 in* [RS4]*, Section XIII.4.*

[21] *The numerical range of A is defined by $N(A) = \{(\psi|A\psi) : \psi \in \mathrm{Dom}\,(A)\}$. In general, $N(A)$ is neither open nor closed subset of $\mathbb{C}$. It is a deep result of Hausdorff that $N(A)$ is a convex subset of $\mathbb{C}$. Prove that if $\mathrm{Dom}\,(A) = \mathrm{Dom}\,(A^*)$, then $\mathrm{sp}(A) \subset \overline{N(A)}$. For additional information about numerical range, the reader may consult* [GR].

[22] *Let $z \in \mathrm{sp}(A)$. A sequence $\psi_n \in \mathrm{Dom}\,(A)$ is called a Weyl sequence if $\|\psi_n\| = 1$ and $\|(A - z)\psi_n\| \to 0$. If A is not self-adjoint, then a Weyl sequence may not exist for some $z \in \mathrm{sp}(A)$. Prove that a Weyl sequence exists for every z on the topological boundary of $\mathrm{sp}(A)$. Hint: See Section XIII.4 of* [RS4] *or* [VH].

[23] *Let A and B be densely defined linear operators. Assume that B is A-bounded with a relative bound $a < 1$. Prove that $A + B$ is closable iff A is closable, and that in this case the closures of A and $A + B$ have the same domain. Deduce that $A + B$ is closed iff A is closed.*

[24] *Let A and B be densely defined linear operators. Assume that A is closed and that B is A-bounded with constants a and b. If A is invertible* (*that is, $0 \notin \mathrm{sp}(A)$*)*, and if a and b satisfy*

$$a + b\|A^{-1}\| < 1,$$

prove that $A + B$ is closed, invertible, and that

$$\|(A + B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - a - b\|A^{-1}\|},$$

$$\|(A + B)^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\|(a + b\|A^{-1}\|)}{1 - a - b\|A^{-1}\|}.$$

Hint: See Theorem IV.1.16 in [Ka].

[25] *In this problem we will discuss the regular perturbation theory for closed operators. Let A be a closed operator and let B be A-bounded with constants a and b. For $\lambda \in \mathbb{C}$ we set $A_\lambda = A + \lambda B$. If $|\lambda|a < 1$, then A_λ is a closed operator and $\mathrm{Dom}\,(A_\lambda) = \mathrm{Dom}\,(A)$. Let F be an isolated, bounded subset of A and γ a simple closed path that separates F and $\mathrm{sp}(H) \setminus F$.*
(1) *Prove that for $z \in \gamma$,*

$$\|B(A - z)^{-1}\| \leq a + (a|z| + b)\|(A - z)^{-1}\|.$$

(2) *Prove that if* A *is self-adjoint and* $z \notin \mathrm{sp}(A)$, *then* $\|(A-z)^{-1}\| = 1/\mathrm{dist}\{z, \mathrm{sp}(A)\}$.
(3) *Assume that* $\mathrm{Dom}\,(A) = \mathrm{Dom}\,(A^*)$ *and let* $N(A)$ *be the numerical range of* A. *Prove that for all* $z \notin \overline{N(A)}$, $\|(A-z)^{-1}\| \le 1/\mathrm{dist}\{z, \overline{N(A)}\}$.
(4) *Let* $\Lambda = \left[a + \sup_{z\in\gamma}(a|z|) + b)\|(A-z)^{-1}\|\right]^{-1}$ *and assume that* $|\lambda| < \Lambda$. *Prove that* $\mathrm{sp}(A_\lambda) \cap \gamma = \emptyset$ *and that for* $z \in \gamma$,

$$(z - A_\lambda)^{-1} = \sum_{n=0}^{\infty} \lambda^n (z-A)^{-1}\left[B(z-A)^{-1}\right]^n.$$

Hint: Start with $z - A_\lambda = (1 - \lambda B(z-A)^{-1})(z-A)$.
(5) *Let* F_λ *be the spectrum of* A_λ *inside* γ *(so* $F_0 = F$*). For* $|\lambda| < \Lambda$ *the projection of* A_λ *onto* F_λ *is given by*

$$P_\lambda \equiv \mathbf{1}_{F_\lambda}(A_\lambda) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z - A_\lambda)^{-1} \mathrm{d}z.$$

Prove that the projection-valued function $\lambda \mapsto P_\lambda$ *is analytic for* $|\lambda| < \Lambda$.
(6) *Prove that the differential equation* $U'_\lambda = [P'_\lambda, P_\lambda]U_\lambda$, $U_0 = \mathbf{1}$, *(the derivatives are w.r.t.* λ *and* $[A,B] = AB - BA$*) has a unique solution for* $|\lambda| < \Lambda$, *and that* U_λ *is an analytic family of bounded invertible operators such that* $U_\lambda P_0 U_\lambda^{-1} = P_\lambda$.
Hint: See [RS4], *Section XII.2.*
(7) *Set* $\tilde{A}_\lambda = U_\lambda^{-1} A_\lambda U_\lambda$ *and* $\Sigma_\lambda = P_0 \tilde{A}_\lambda P_0$. Σ_λ *is a bounded operator on the Hilbert space* $\mathrm{Ran}\,P_0$. *Prove that* $\mathrm{sp}(\Sigma_\lambda) = F_\lambda$ *and that the operator-valued function* $\lambda \mapsto \Sigma_\lambda$ *is analytic for* $|\lambda| < \Lambda$. *Compute the first three terms in the expansion*

$$\Sigma_\lambda = \sum_{n=0}^{\infty} \lambda^n \Sigma_n. \tag{3.57}$$

The term Σ_1 *is sometimes called the Feynman-Hellman term. The term* Σ_2, *often called the Level Shift Operator (LSO), plays an important role in quantum mechanics and quantum field theory. For example, the formal computations in physics involving Fermi's Golden Rule are often best understood and most easily proved with the help of LSO.*
(8) *Assume that* $\dim P_0 = \dim \mathrm{Ran}\,P_0 < \infty$. *Prove that* $\dim P_\lambda = \dim P_0$ *for* $|\lambda| < \Lambda$ *and conclude that the spectrum of* A_λ *inside* γ *is discrete and consists of at most* $\dim P_0$ *distinct eigenvalues. Prove that the eigenvalues of* A_λ *inside* γ *are all the branches of one or more multi-valued analytic functions with at worst algebraic singularities.*
(9) *Assume that* $F_0 = \{z_0\}$ *and* $\dim P_0 = 1$ *(namely that the spectrum of* A *inside* γ *is a semisimple eigenvalue* z_0*). In this case* $\Sigma_\lambda = z(\lambda)$ *is an analytic function for* $|\lambda| < \Lambda$. *Compute the first five terms in the expansion* $z(\lambda) = \sum_{n=0}^{\infty} \lambda^n z_n$.

4 Spectral theory of rank one perturbations

The Hamiltonians which arise in non-relativistic quantum mechanics typically have the form

$$H_V = H_0 + V, \tag{4.58}$$

where H_0 and V are two self-adjoint operators on a Hilbert space $\mathcal{H}$. H_0 is the "free" or "reference" Hamiltonian and V is the "perturbation". For example, the Hamiltonian of a free non-relativistic quantum particle of mass m moving in $\mathbb{R}^3$ is $-\frac{1}{2m}\Delta$, where Δ is the Laplacian in $L^2(\mathbb{R}^3)$. If the particle is subject to an external potential field $V(x)$, then the Hamiltonian describing the motion of the particle is

$$H_V = -\frac{1}{2m}\Delta + V, \tag{4.59}$$

where V denotes the operator of multiplication by the function $V(x)$. Operators of this form are called Schrödinger operators.

We will not study in this section the spectral theory of Schrödinger operators. Instead, we will keep H_0 general and focus on simplest non-trivial perturbations V. More precisely, let $\mathcal{H}$ be a Hilbert space, H_0 a self-adjoint operator on $\mathcal{H}$ and $\psi \in \mathcal{H}$ a given unit vector. We will study spectral theory of the family of operators

$$H_\lambda = H_0 + \lambda(\psi|\cdot)\psi, \qquad \lambda \in \mathbb{R}. \tag{4.60}$$

This simple model is of profound importance in mathematical physics. The classical reference for the spectral theory of rank one perturbations is [Si2].

The cyclic subspace generated by H_λ and ψ does not depend on λ and is equal to the cyclic subspace generated by H_0 and ψ which we denote $\mathcal{H}_\psi$ (this fact is an immediate consequence of the formulas (4.62) below). Let μ^λ be the spectral measure for H_λ and ψ. This measure encodes the spectral properties of $H_\lambda \restriction \mathcal{H}_\psi$. Note that $H_\lambda \restriction \mathcal{H}_\psi^\perp = H_0 \restriction \mathcal{H}_\psi^\perp$. In this section we will always assume that $\mathcal{H} = \mathcal{H}_\psi$, namely that ψ is a cyclic vector for H_0.

The identities

$$\begin{aligned}(H_\lambda - z)^{-1} - (H_0 - z)^{-1} &= (H_\lambda - z)^{-1}(H_0 - H_\lambda)(H_0 - z)^{-1} \\ &= (H_0 - z)^{-1}(H_0 - H_\lambda)(H_\lambda - z)^{-1}\end{aligned} \tag{4.61}$$

yield

$$\begin{aligned}(H_\lambda - z)^{-1}\psi &= (H_0 - z)^{-1}\psi - \lambda(\psi|(H_0 - z)^{-1}\psi)(H_\lambda - z)^{-1}\psi, \\ (H_0 - z)^{-1}\psi &= (H_\lambda - z)^{-1}\psi + \lambda(\psi|(H_\lambda - z)^{-1}\psi)(H_0 - z)^{-1}\psi.\end{aligned} \tag{4.62}$$

Let

$$F_\lambda(z) = (\psi|(H_\lambda - z)^{-1}\psi) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu^\lambda(t)}{t - z}.$$

Note that if $z \in \mathbb{C}_+$, then $F_\lambda(z)$ is the Borel transform and $\operatorname{Im} F_\lambda(z)$ is the Poisson transform of μ^λ.

The second identity in (4.62) yields

$$F_0(z) = F_\lambda(z)(1 + \lambda F_0(z)),$$

and so

$$F_\lambda(z) = \frac{F_0(z)}{1 + \lambda F_0(z)}, \tag{4.63}$$

$$\operatorname{Im} F_\lambda(z) = \frac{\operatorname{Im} F_0(z)}{|1 + \lambda F_0(z)|^2}. \tag{4.64}$$

These elementary identities will play a key role in our study. The function

$$G(x) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu^0(t)}{(x-t)^2}$$

will also play an important role. Recall that $G(x) = \infty$ for μ^0-a.e. x (Lemma 2.3).

In this section we will occasionally denote by $|B|$ the Lebesgue measure of a Borel set B.

4.1 Aronszajn-Donoghue theorem

Recall that the limit

$$F_\lambda(x) = \lim_{y \downarrow 0} F_\lambda(x + \mathrm{i}y)$$

exists and is finite and non-zero for Lebesgue a.e. x.

For $\lambda \neq 0$ define

$$S_\lambda = \{x \in \mathbb{R} \,:\, F_0(x) = -\lambda^{-1},\ G(x) = \infty\},$$

$$T_\lambda = \{x \in \mathbb{R} \,:\, F_0(x) = -\lambda^{-1},\ G(x) < \infty\},$$

$$L = \{x \in \mathbb{R} \,:\, \operatorname{Im} F_0(x) > 0\}.$$

In words, S_λ is the set of all $x \in \mathbb{R}$ such that $\lim_{y\downarrow 0} F_0(x + \mathrm{i}y)$ exists and is equal to $-\lambda^{-1}$, etc. Any two sets in the collection $\{S_\lambda, T_\lambda, L\}_{\lambda \neq 0}$ are disjoint. By Theorem 3.11, $|S_\lambda| = |T_\lambda| = 0$.

As usual, $\delta(y)$ denotes the delta-measure of $y \in \mathbb{R}$; $\delta(y)(f) = f(y)$.

Theorem 4.1 (1) T_λ *is the set of eigenvalues of* H_λ*. Moreover,*

$$\mu^\lambda_{\mathrm{pp}} = \sum_{x_n \in T_\lambda} \frac{1}{\lambda^2 G(x_n)} \delta(x_n).$$

(2) $\mu^\lambda_{\mathrm{sc}}$ *is concentrated on* S_λ.
(3) *For all* λ, L *is the essential support of the ac spectrum of* H_λ *and* $\mathrm{sp}_{\mathrm{ac}}(H_\lambda) = \mathrm{sp}_{\mathrm{ac}}(H_0)$.
(4) *The measures* $\{\mu^\lambda_{\mathrm{sing}}\}_{\lambda \in \mathbb{R}}$ *are mutually singular. In other words, if* $\lambda_1 \neq \lambda_2$*, then the measures* $\mu^{\lambda_1}_{\mathrm{sing}}$ *and* $\mu^{\lambda_2}_{\mathrm{sing}}$ *are concentrated on disjoint sets.*

Proof. (1) The eigenvalues of H_λ are precisely the atoms of μ^λ. Let $\tilde{T}_\lambda = \{x \in \mathbb{R} : \mu^\lambda(\{x\}) \neq 0\}$. Since

$$\mu^\lambda(\{x\}) = \lim_{y\downarrow 0} y \mathrm{Im}\, F_\lambda(x+\mathrm{i}y) = \lim_{y\downarrow 0} \frac{y \mathrm{Im}\, F_0(x+\mathrm{i}y)}{|1+\lambda F_0(x+\mathrm{i}y)|^2}, \tag{4.65}$$

$\tilde{T}_\lambda \subset \{x : F_0(x) = -\lambda^{-1}\}$. The relation (4.65) yields

$$\mu^\lambda(\{x\}) \leq \lambda^{-2} \lim_{y\downarrow 0} \frac{y}{\mathrm{Im}\, F_0(x+\mathrm{i}y)} = \frac{1}{\lambda^2 G(x)},$$

and so $\tilde{T}_\lambda \subset \{x : F_0(x) = -\lambda^{-1}, G(x) < \infty\} = T_\lambda$. On the other hand, if $F_0(x) = -\lambda^{-1}$ and $G(x) < \infty$, then

$$\lim_{y\downarrow 0} \frac{F_0(x+\mathrm{i}y) - F_0(x)}{\mathrm{i}y} = G(x) \tag{4.66}$$

(the proof of this relation is left to the problems). Hence, if $x \in T_\lambda$, then

$$F_0(x+\mathrm{i}y) = \mathrm{i}yG(x) - \lambda^{-1} + o(y),$$

and

$$\mu^\lambda(\{x\}) = \lim_{y\downarrow 0} \frac{y \mathrm{Im}\, F_0(x+\mathrm{i}y)}{|1+\lambda F_0(x+\mathrm{i}y)|^2} = \frac{1}{\lambda^2 G(x)} > 0.$$

Hence $T_\lambda = \tilde{T}_\lambda$, and for $x \in \tilde{T}_\lambda$, $\mu^\lambda(\{x\}) = 1/\lambda^2 G(x)$. This yields (1).
(2) By Theorem 2.5, $\mu^\lambda_{\mathrm{sing}}$ is concentrated on the set

$$\{x \,:\, \lim_{y\downarrow 0} \mathrm{Im}\, F_\lambda(x+\mathrm{i}y) = \infty\}.$$

The formula (4.64) yields that $\mu^\lambda_{\mathrm{sing}}$ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}\}$. If $F_0(x) = -\lambda^{-1}$ and $G(x) < \infty$, then by (1) x is an atom of μ^λ. Hence, $\mu^\lambda_{\mathrm{sc}}$ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}, G(x) = \infty\} = S_\lambda$.
(3) By Theorem 2.5,

$$\mathrm{d}\mu^\lambda_{\mathrm{ac}}(x) = \pi^{-1} \mathrm{Im}\, F_\lambda(x)\mathrm{d}x.$$

On the other hand, by the formula (4.64), the sets $\{x : \mathrm{Im}\, F_0(x) > 0\}$ and $\{x : \mathrm{Im}\, F_\lambda(x) > 0\}$ coincide up to a set of Lebesgue measure zero. Hence, L is the essential support of the ac spectrum of H_λ for all λ. Since μ^0_{ac} and $\mu^\lambda_{\mathrm{ac}}$ are equivalent measures, $\mathrm{sp}_{\mathrm{ac}}(H_0) = \mathrm{sp}_{\mathrm{ac}}(H_\lambda)$.
(4) By (1) and (2), for $\lambda \neq 0$, $\mu^\lambda_{\mathrm{sing}}$ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}\}$. By Theorem 2.5, μ^0_{sing} is concentrated on $\{x : \mathrm{Im}\, F_0(x) = \infty\}$. This yields the statement. □

4.2 The spectral theorem

By Theorem 3.13, for all λ there exists a unique unitary $U_\lambda : \mathcal{H}_\psi \to L^2(\mathbb{R}, \mathrm{d}\mu_\lambda)$ such that $U_\lambda \psi = \mathbb{1}$ and $U_\lambda H_\lambda U_\lambda^{-1}$ is the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu_\lambda)$. In this subsection we describe U_λ.

For $\phi \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ let

$$M_\phi(z) = (\psi|(H_0 - z)^{-1}\phi),$$

and

$$M_\phi(x \pm \mathrm{i}0) = \lim_{y \downarrow 0} (\psi|(H_0 - x \mp \mathrm{i}y)^{-1}\phi)$$

whenever the limits exist. By Theorem 2.17 the limits exist and are finite for Lebesgue a.e. x.

For consistency, in this subsection we write $F_0(x + \mathrm{i}0) = \lim_{y\downarrow 0} F_0(x + \mathrm{i}y)$.

Theorem 4.2 *Let $\phi \in \mathcal{H}$.*
(1) *For all λ and for $\mu_{\lambda,\mathrm{ac}}$-a.e. x,*

$$(U_\lambda \mathbf{1}_{\mathrm{ac}}\phi)(x) = \frac{1}{2\mathrm{i}} \frac{M_\phi(x + \mathrm{i}0) - M_\phi(x - \mathrm{i}0)}{\mathrm{Im}\, F_0(x + \mathrm{i}0)} - \lambda M_\phi(x + \mathrm{i}0)$$
$$+ \frac{\lambda}{2\mathrm{i}} \frac{(M_\phi(x + \mathrm{i}0) - M_\phi(x - \mathrm{i}0))F_0(x + \mathrm{i}0)}{\mathrm{Im}\, F_0(x + \mathrm{i}0)}.$$

(2) *Let $\lambda \neq 0$. Then for $\mu_{\lambda,\mathrm{sing}}$-a.e. x the limit $M_\phi(x + \mathrm{i}0)$ exists and*

$$(U_\lambda \mathbf{1}_{\mathrm{sing}}\phi)(x) = -\lambda M_\phi(x + \mathrm{i}0).$$

Proof. The identities (4.61) yield

$$(\psi|(H_\lambda - z)^{-1}\phi) = \frac{M_\phi(z)}{1 + \lambda F_0(z)}.$$

Combining this relation with (4.63) and (4.64) we derive

$$\frac{(\psi|\mathrm{Im}\,(H_\lambda - z)^{-1}\phi)}{\mathrm{Im}\,(\psi|(H_\lambda - z)^{-1}\psi)} = \frac{1}{2\mathrm{i}} \frac{M_\phi(z) - M_\phi(\overline{z}) + \lambda(F_0(\overline{z})M_\phi(z) - F_0(z)M_\phi(\overline{z}))}{\mathrm{Im}\, F_0(z)}. \tag{4.67}$$

Similarly,

$$\frac{(\psi|(H_\lambda - z)^{-1}\phi)}{(\psi|(H_\lambda - z)^{-1}\psi)} = \frac{M_\phi(z)}{F_0(z)}. \tag{4.68}$$

(1) follows from the identity (4.67) and Part 1 of Theorem 3.17. Since $\mu_{\lambda,\mathrm{sing}}$ is concentrated on the set $\{x : \lim_{y\downarrow 0} F_0(x + \mathrm{i}0) = -\lambda^{-1}\}$, the identity (4.68) and Part 2 of Theorem 3.17 yield (2). □

Note that Part 2 of Theorem 4.2 yields that for every eigenvalue x of H_λ (i.e. for all $x \in T_\lambda$),

$$(U_\lambda \mathbf{1}_{\mathrm{pp}}\phi)(x) = -\lambda M_\phi(x + \mathrm{i}0). \tag{4.69}$$

This special case (which can be easily proven directly) has been used in the proofs of dynamical localization in the Anderson model; see [A, DJLS]. The extension of (4.69) to singular continuous spectrum depends critically on the full strength of the Poltoratskii theorem. For some applications of this result see [JL3].

4.3 Spectral averaging

In the sequel we will freely use the measurability results established in Subsection 3.16.

Let

$$\overline{\mu}(B) = \int_{\mathbb{R}} \mu^{\lambda}(B)\mathrm{d}\lambda,$$

where $B \subset \mathbb{R}$ is a Borel set. Obviously, $\overline{\mu}$ is a Borel measure on $\mathbb{R}$. The following (somewhat surprising) result is often called *spectral averaging*:

Theorem 4.3 *The measure $\overline{\mu}$ is equal to the Lebesgue measure and for all $f \in L^1(\mathbb{R}, \mathrm{d}x)$,*

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x)\mathrm{d}\mu^{\lambda}(x) \right] \mathrm{d}\lambda.$$

Proof. For any positive Borel function f,

$$\int_{\mathbb{R}} f(t)\mathrm{d}\overline{\mu}(t) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(t)\mathrm{d}\mu^{\lambda}(t) \right] \mathrm{d}\lambda$$

(both sides are allowed to be infinity). Let

$$f(t) = \frac{y}{(t-x)^2 + y^2},$$

where $y > 0$. Then

$$\int_{\mathbb{R}} f(t)\mathrm{d}\mu^{\lambda}(t) = \operatorname{Im} F_{\lambda}(x + \mathrm{i}y) = \frac{\operatorname{Im} F_0(x + \mathrm{i}y)}{|1 + \lambda F_0(x + \mathrm{i}y)|^2}.$$

By the residue calculus,

$$\int_{\mathbb{R}} \frac{\operatorname{Im} F_0(x + \mathrm{i}y)}{|1 + \lambda F_0(x + \mathrm{i}y)|^2} \mathrm{d}\lambda = \pi, \tag{4.70}$$

and so the Poisson transform of $\overline{\mu}$ exists and is identically equal to π, the Poisson transform of the Lebesgue measure. By Theorem 2.7, $\overline{\mu}$ is equal to the Lebesgue measure. □

Spectral averaging is a mathematical gem which has been rediscovered by many authors. A detailed list of references can be found in [Si3].

4.4 Simon-Wolff theorems

Theorem 4.4 *Let $B \subset \mathbb{R}$ be a Borel set. Then the following statements are equivalent:*
(1) $G(x) < \infty$ *for Lebesgue a.e.* $x \in B$.
(2) $\mu^\lambda_{\rm cont}(B) = 0$ *for Lebesgue a.e.* λ.

Proof. (1) ⇒ (2). If $G(x) < \infty$ for Lebesgue a.e. $x \in B$, then $\mathrm{Im}\, F_0(x) = 0$ for Lebesgue a.e. $x \in B$. Hence, for all λ, $\mathrm{Im}\, F_\lambda(x) = 0$ for Lebesgue a.e. $x \in B$, and

$$\mu^\lambda_{\rm ac}(B) = \pi^{-1} \int_B \mathrm{Im}\, F_\lambda(x) \mathrm{d}x = 0.$$

By Theorem 4.1, the measure $\mu^\lambda_{\rm sc} \restriction B$ is concentrated on the set $A = \{x \in B : G(x) = \infty\}$. Since A has Lebesgue measure zero, by spectral averaging,

$$\int_{\mathbb{R}} \mu^\lambda_{\rm sc}(A) \mathrm{d}\lambda \leq \int_{\mathbb{R}} \mu^\lambda(A) \mathrm{d}\lambda = |A| = 0.$$

Hence, $\mu^\lambda_{\rm sc}(A) = 0$ for Lebesgue a.e. $\lambda \in \mathbb{R}$, and so $\mu^\lambda_{\rm sc}(B) = 0$ for Lebesgue a.e. λ.
(2) ⇒ (1). Assume that the set $A = \{x \in B : G(x) = \infty\}$ has positive Lebesgue measure. By Theorem 4.1, $\mu^\lambda_{\rm pp}(A) = 0$ for all $\lambda \neq 0$. By spectral averaging,

$$\int_{\mathbb{R}} \mu^\lambda_{\rm cont}(A) \mathrm{d}\lambda = \int_{\mathbb{R}} \mu^\lambda(A) \mathrm{d}\lambda = |A| > 0.$$

Hence, for a set of λ of positive Lebesgue measure, $\mu^\lambda_{\rm cont}(B) > 0$. □

Theorem 4.5 *Let B be a Borel set. Then the following statements are equivalent:*
(1) $\mathrm{Im}\, F_0(x) > 0$ *for Lebesgue a.e.* $x \in B$.
(2) $\mu^\lambda_{\rm sing}(B) = 0$ *for Lebesgue a.e.* λ.

Proof. (1) ⇒ (2). By Theorem 4.1, for $\lambda \neq 0$ the measure $\mu^\lambda_{\rm sing} \restriction B$ is concentrated on the set $A = \{x \in B : \mathrm{Im}\, F_0(x) = 0\}$. Since A has Lebesgue measure zero, by spectral averaging,

$$\int_{\mathbb{R}} \mu^\lambda_{\rm sing}(A) \mathrm{d}\lambda \leq \int_{\mathbb{R}} \mu^\lambda(A) \mathrm{d}\lambda = 0.$$

Hence, for Lebesgue a.e. λ, $\mu^\lambda_{\rm sing}(B) = 0$.
(2) ⇒ (1). Assume that the set $A = \{x \in B : \mathrm{Im}\, F_0(x) = 0\}$ has positive Lebesgue measure. Clearly, $\mu^\lambda_{\rm ac}(A) = 0$ for all λ, and by spectral averaging,

$$\int_{\mathbb{R}} \mu^\lambda_{\rm sing}(A) \mathrm{d}\lambda = \int_{\mathbb{R}} \mu^\lambda(A) \mathrm{d}\lambda = |A| > 0.$$

Hence, for a set of λ of positive Lebesgue measure, $\mu^\lambda_{\rm sing}(B) > 0$. □

Theorem 4.6 *Let B be a Borel set. Then the following statements are equivalent:*
(1) $\operatorname{Im} F_0(x) = 0$ *and* $G(x) = \infty$ *for Lebesgue a.e.* $x \in B$.
(2) $\mu_{\rm ac}^\lambda(B) + \mu_{\rm pp}^\lambda(B) = 0$ *for Lebesgue a.e.* λ.

The proof of Theorem 4.6 is left to the problems.

Theorem 4.4 is the celebrated result of Simon-Wolff [SW]. Although Theorems 4.5 and 4.6 are well known to the workers in the field, I am not aware of a convenient reference.

4.5 Some remarks on spectral instability

By the Kato-Rosenblum theorem, the absolutely continuous spectrum is stable under trace class perturbations, and in particular under rank one perturbations. In the rank one case this result is also an immediate consequence of Theorem 4.1.

The situation is more complicated in the case of the singular continuous spectrum. There are examples where sc spectrum is stable, namely when H_λ has purely singular continuous spectrum in (a, b) for all $\lambda \in \mathbb{R}$. There are also examples where H_0 has purely sc spectrum in (a, b), but H_λ has pure point spectrum for all $\lambda \neq 0$.

A. Gordon [Gor] and del Rio-Makarov-Simon [DMS] have proven that pp spectrum is *always* unstable for generic λ.

Theorem 4.7 *The set*

$$\{\lambda \,:\, H_\lambda \text{ has no eigenvalues in } \operatorname{sp}(H_0)\}$$

is dense G_δ in $\mathbb{R}$.

Assume that $(a, b) \subset \operatorname{sp}(H_0)$ and that for Lebesgue a.e. $x \in (a, b)$, $G(x) < \infty$. Then the spectrum of H_λ in (a, b) is pure point for Lebesgue a.e. λ. However, by Theorem 4.7, there is a dense G_δ set of λ's such that H_λ has purely singular continuous spectrum in (a, b) (of course, H_λ has no ac spectrum in (a, b) for all λ).

4.6 Boole's equality

So far we have used the rank one perturbation theory and harmonic analysis to study spectral theory. In the last three subsections we will turn things around and use rank one perturbation theory and spectral theory to reprove some well known results in harmonic analysis. This subsection deals with Boole's equality and is based on [DJLS] and [Po2].

Let ν be a finite positive Borel measure on $\mathbb{R}$ and $F_\nu(z)$ its Borel transform. As usual, we denote

$$F_\nu(x) = \lim_{y \downarrow 0} F_\nu(x + \mathrm{i}y).$$

The following result is known as Boole's equality:

Proposition 4.8 *Assume that ν is a pure point measure with finitely many atoms. Then for all $t>0$*

$$|\{x\,:\,F_\nu(x)>t\}|=|\{x\,:\,F_\nu(x)<-t\}|=\frac{\nu(\mathbb{R})}{t}.$$

Proof. We will prove that $|\{x\,:\,F_\nu(x)>t\}|=\nu(\mathbb{R})/t$. Let $\{x_j\}_{1\leq j\leq n}$, $x_1<\cdots<x_n$, be the support of ν and $\alpha_j=\nu(\{x_j\})$ the atoms of ν. W.l.o.g. we may assume that $\nu(\mathbb{R})=\sum_j\alpha_j=1$. Clearly,

$$F_\nu(x)=\sum_{j=1}^n\frac{\alpha_j}{x_j-x}.$$

Set $x_0=-\infty$, $x_{n+1}=\infty$. Since $F_\nu'(x)>0$ for $x\neq x_j$, the function $F_\nu(x)$ is strictly increasing on (x_j,x_{j+1}), with vertical asymptots at $x_j, 1\leq j\leq n$. Let $r_1<\cdots<r_n$ be the solutions of the equation $F_\nu(x)=t$. Then

$$|\{x\,:\,F_\nu(x)>t\}|=\sum_{j=1}^n(x_j-r_j).$$

On the other hand, the equation $F_\nu(x)=t$ is equivalent to

$$\sum_{k=1}^n\alpha_k\prod_{j\neq k}(x_j-x)=t\prod_{j=1}^n(x_j-x),$$

or

$$\prod_{j=1}^n(x_j-x)-t^{-1}\sum_{k=1}^n\alpha_k\prod_{j\neq k}(x_j-x)=0.$$

Since $\{r_j\}$ are all the roots of the polynomial on the l.h.s.,

$$\sum_{j=1}^n r_j=-t^{-1}+\sum_{j=1}^n x_j$$

and this yields the statement. □

Proposition 4.8 was first proven by G. Boole in 1867. The Boole equality is another gem that has been rediscovered by many authors; see [Po2] for the references.

The rank one perturbation theory allows for a simple proof of the optimal version of the Boole equality.

Theorem 4.9 *Assume that ν is a purely singular measure. Then for all $t>0$*

$$|\{x\,:\,F_\nu(x)>t\}|=|\{x\,:\,F_\nu(x)<-t\}|=\frac{\nu(\mathbb{R})}{t}.$$

Proof. W.l.o.g. we may assume that $\nu(\mathbb{R}) = 1$. Let H_0 be the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\nu)$ and $\psi \equiv 1$. Let $H_\lambda = H_0 + \lambda(\psi|\cdot)\psi$ and let μ^λ be the spectral measure for H_λ and ψ. Obviously, $\mu^0 = \nu$ and $F_0 = F_\nu$. Since ν is a singular measure, μ^λ is singular for all $\lambda \in \mathbb{R}$.

By Theorem 4.1, for $\lambda \neq 0$, the measure μ^λ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}\}$. Let

$$\Gamma_t = \{x \,:\, F_0(x) > t\}.$$

Then for $\lambda \neq 0$,

$$\mu^\lambda(\Gamma_t) = \begin{cases} 1 & \text{if } -t^{-1} < \lambda < 0, \\ 0 & \text{if } \lambda \leq -t^{-1} \text{ or } \lambda > 0. \end{cases}$$

By the spectral averaging,

$$|\Gamma_t| = \int_{\mathbb{R}} \mu^\lambda(\Gamma_t)\mathrm{d}\lambda = t^{-1}.$$

A similar argument yields that $|\{x : F_\nu(x) < -t\}| = t^{-1}$. □

The Boole equality fails if ν is not a singular measure. However, in general we have

Theorem 4.10 *Let ν be a finite positive Borel measure on $\mathbb{R}$. Then*

$$\lim_{t\to\infty} t\,|\{x : |F_\nu(x)| > t\}| = 2\nu_{\text{sing}}(\mathbb{R}).$$

Theorem 4.10 is due to Vinogradov-Hruschev. Its proof (and much additional information) can be found in the paper of Poltoratskii [Po2].

4.7 Poltoratskii's theorem

This subsection is devoted to the proof of Theorem 2.18. We follow [JL1].

We first consider the case $\nu_{\mathrm{s}} = 0$, μ compactly supported, $f \in L^2(\mathbb{R}, \mathrm{d}\mu)$ real valued. W.l.o.g. we may assume that $\mu(\mathbb{R}) = 1$.

Consider the Hilbert space $L^2(\mathbb{R}, \mathrm{d}\mu)$ and let H_0 be the operator of multiplication by x. Note that

$$F_\mu(z) = (\,\mathbb{1}\,|(H_0 - z)^{-1}\mathbb{1}\,), \qquad F_{f\mu}(z) = (\,\mathbb{1}\,|(H_0 - z)^{-1}f\,).$$

For $\lambda \in \mathbb{R}$, let

$$H_\lambda = H_0 + \lambda(\,\mathbb{1}\,|\cdot\,)\mathbb{1},$$

and let μ^λ be the spectral measure for H_λ and $\mathbb{1}$. To simplify the notation, we write

$$F_\lambda(z) = (\,\mathbb{1}\,|(H_\lambda - z)^{-1}\mathbb{1}\,) = F_{\mu^\lambda}(z).$$

Note that with this notation, $F_0 = F_\mu$!

By Theorem 4.1, the measures $\{\mu_{\rm sing}^\lambda\}_{\lambda\in\mathbb{R}}$ are mutually singular. By Theorem 2.5, the measure $\mu_{\rm sing} = \mu_{\rm sing}^0$ is concentrated on the set

$$\{x \in \mathbb{R} : \lim_{y\downarrow 0} \operatorname{Im} F_0(x + \mathrm{i}y) = \infty\}.$$

We also recall the identity

$$F_\lambda(z) = \frac{F_0(z)}{1 + \lambda F_0(z)}. \tag{4.71}$$

By the spectral theorem, there exists a unitary

$$U_\lambda : L^2(\mathbb{R}, \mathrm{d}\mu) \to L^2(\mathbb{R}, \mathrm{d}\mu^\lambda)$$

such that $U_\lambda \mathbb{1} = \mathbb{1}$ and $U_\lambda H_\lambda U_\lambda^{-1}$ is the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu^\lambda)$. Hence

$$(\mathbb{1}|(H_\lambda - z)^{-1} f) = \int_{\mathbb{R}} \frac{(U_\lambda f)(x)}{x - z} \mathrm{d}\mu^\lambda(x) = F_{(U_\lambda f)\mu^\lambda}(z).$$

In what follows we set $\lambda = 1$ and write $U = U_1$.

For $a \in \mathbb{R}$ and $b > 0$ let $h_{ab}(x) = 2b((x - a)^2 + b^2)^{-1}$, $w = a + \mathrm{i}b$, and $r_w(x) = (x - w)^{-1}$ (hence $h_{ab} = \mathrm{i}^{-1}(r_w - r_{\overline{w}})$). The relation

$$Uh_{ab} = h_{ab} + \lambda \mathrm{i}^{-1}(F_0(w)r_w - F_0(\overline{w})r_{\overline{w}}) \tag{4.72}$$

yields that Uh_{ab} is a real-valued function. The proof of (4.72) is simple and is left to the problems. Since the linear span of $\{h_{ab} : a \in \mathbb{R}, b > 0\}$ is dense in $C_0(\mathbb{R})$, U takes real-valued functions to real-valued functions. In particular, Uf is a real-valued function.

The identity

$$(\mathbb{1}|(H_0 - z)^{-1} f) = (1 + (\mathbb{1}|(H_0 - z)^{-1}\mathbb{1}))(\mathbb{1}|(H_1 - z)^{-1} f)$$

can be rewritten as

$$(\mathbb{1}|(H_0 - z)^{-1} f) = (1 + F_0(z))F_{(Uf)\mu^1}(z). \tag{4.73}$$

It follows that

$$\frac{\operatorname{Im}(\mathbb{1}|(H_0 - z)^{-1} f)}{\operatorname{Im} F_0(z)} = \operatorname{Re} F_{(Uf)\mu^1}(z) + L(z), \tag{4.74}$$

where

$$L(z) = \frac{\operatorname{Re}(1 + F_0(z))}{\operatorname{Im} F_0(z)} \operatorname{Im} F_{(Uf)\mu^1}(z).$$

We proceed to prove that

$$\lim_{y\downarrow 0} \operatorname{Im} F_{(Uf)\mu^1}(x + \mathrm{i}y) = 0 \qquad \text{for } \ \mu_{\rm sing} - a.e.\ x, \tag{4.75}$$

$$\lim_{y\downarrow 0} L(x+\mathrm{i}y) = 0 \qquad \text{for } \mu_{\mathrm{sing}} - a.e.\ x. \tag{4.76}$$

We start with (4.75). Using first that Uf is real-valued and then the Cauchy-Schwarz inequality, we derive

$$\mathrm{Im}\, F_{(Uf)\mu^1}(x+\mathrm{i}y) = P_{(Uf)\mu^1}(x+\mathrm{i}y) \le \sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_{(Uf)^2\mu^1}(x+\mathrm{i}y)}.$$

Since the measures $(Uf)^2\mu^1_{\mathrm{sing}}$ and μ_{sing} are mutually singular,

$$\lim_{y\downarrow 0} \frac{P_{(Uf)^2\mu^1}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = 0 \qquad \text{for } \mu_{\mathrm{sing}} - a.e.\ x$$

(see Problem 4). Hence,

$$\lim_{y\downarrow 0} \frac{\mathrm{Im}\, F_{(Uf)\mu^1}(x+\mathrm{i}y)}{\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_\mu(x+\mathrm{i}y)}} = 0 \qquad \text{for } \mu_{\mathrm{sing}} - a.e.\ x. \tag{4.77}$$

Since

$$P_{\mu^1}(x+\mathrm{i}y)P_\mu(x+\mathrm{i}y) = \mathrm{Im}\, F_1(x+\mathrm{i}y)\mathrm{Im}\, F_0(x+\mathrm{i}y) = \frac{(\mathrm{Im}\, F_0(x+\mathrm{i}y))^2}{|1+F_0(x+\mathrm{i}y)|^2} \le 1$$

for all $x \in \mathbb{R}$, (4.77) yields (4.75).

To prove (4.76), note that

$$|L(x+\mathrm{i}y)| = \frac{\mathrm{Im}\, F_{(Uf)\mu^1}(x+\mathrm{i}y)}{\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_\mu(x+\mathrm{i}y)}} \frac{|\mathrm{Re}\,(1+F_0(x+\mathrm{i}y))|}{\mathrm{Im}\, F_0(x+\mathrm{i}y)} \frac{\mathrm{Im}\, F_0(x+\mathrm{i}y)}{|1+F_0(x+\mathrm{i}y)|}$$

$$\le \frac{\mathrm{Im}\, F_{(Uf)\mu^1}(x+\mathrm{i}y)}{\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_\mu(x+\mathrm{i}y)}}.$$

Hence, (4.77) yields (4.76).

Rewrite (4.74) as

$$F_{(Uf)\mu^1}(z) = \frac{\mathrm{Im}\,(\mathbb{1}|(H_0-z)^{-1}f)}{\mathrm{Im}\, F_0(z)} + \mathrm{Im}\, F_{(Uf)\mu^1}(z) - L(z). \tag{4.78}$$

By Theorem 2.5,

$$\lim_{y\downarrow 0} \frac{\mathrm{Im}\,(\mathbb{1}|(H_0-x-\mathrm{i}y)^{-1}f)}{\mathrm{Im}\, F_0(x+\mathrm{i}y)} = \lim_{y\downarrow 0} \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = f(x) \qquad \text{for } \mu - a.e.\ x.$$

Hence, (4.78), (4.75), and (4.76) yield that

$$\lim_{y\downarrow 0} F_{(Uf)\mu^1}(x+\mathrm{i}y) = f(x) \qquad \text{for } \mu_{\mathrm{sing}} - a.e.\ x. \tag{4.79}$$

Rewrite (4.73) as

$$\frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = \left(\frac{1}{F_0(x+\mathrm{i}y)}+1\right) F_{(Uf)\mu^1}(x+\mathrm{i}y). \tag{4.80}$$

Since $|F_0(x+\mathrm{i}y)| \to \infty$ as $y \downarrow 0$ for μ_{sing}-a.e. x, (4.79) and (4.80) yield

$$\lim_{y\downarrow 0} \frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = f(x) \qquad \text{for } \ \mu_{\mathrm{sing}} - a.e.\ x.$$

This proves the Poltoratskii theorem in the special case where $\nu_{\mathrm{s}} = 0$, μ is compactly supported, and $f \in L^2(\mathbb{R}, \mathrm{d}\mu)$ is real-valued.

We now remove the assumptions $f \in L^2(\mathbb{R}, \mathrm{d}\mu)$ and that f is real valued (we still assume that μ is compactly supported and that $\nu_{\mathrm{s}} = 0$). Assume that $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$ and that f is *positive*. Set $g = 1/(1+f)$ and $\rho = (1+f)\mu$. Then

$$\lim_{y\downarrow 0} \frac{F_\mu(x+\mathrm{i}y)}{F_{(1+f)\mu}(x+\mathrm{i}y)} = \lim_{y\downarrow 0} \frac{F_{g\rho}(x+\mathrm{i}y)}{F_\rho(x+\mathrm{i}y)} = \frac{1}{1+f(x)},$$

for μ_{sing}-a.e. x. By the linearity of the Borel transform,

$$\lim_{y\downarrow 0} \frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = \lim_{y\downarrow 0} \frac{F_{(1+f)\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} - 1 = f(x),$$

for μ_{sing}-a.e. x. Since every $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$ is a linear combination of four positive functions in $L^1(\mathbb{R}, \mathrm{d}\mu)$, the linearity of the Borel transform implies the statement for all $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$.

Assume that μ is not compactly supported (we still assume $\nu_{\mathrm{s}} = 0$) and let $[a, b]$ be a finite interval. Decompose $\mu = \mu_1 + \mu_2$, where $\mu_1 = \mu \restriction [a, b]$, $\mu_2 = \mu \restriction \mathbb{R} \setminus [a, b]$. Since

$$\frac{F_{f\mu}(z)}{F_\mu(z)} = \frac{F_{f\mu_1}(z) + F_{f\mu_2}(z)}{F_{\mu_1}(z)(1 + F_{\mu_2}(z)/F_{\mu_1}(z))}$$

and $\lim_{y\downarrow 0} |F_{\mu_1}(x+\mathrm{i}y)| \to \infty$ for $\mu_{1,\mathrm{sing}}$-a.e. $x \in [a, b]$,

$$\lim_{y\downarrow 0} \frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = f(x) \qquad \text{for } \ \mu_{\mathrm{sing}}\text{-a.e. } x \in (a, b).$$

Since $[a, b]$ is arbitrary, we have removed the assumption that μ is compactly supported.

Finally, to finish the proof we need to show that if $\nu \perp \mu$, then

$$\lim_{y\downarrow 0} \frac{F_\nu(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = 0 \tag{4.81}$$

for $\mu_{\rm sing}$-a.e. x. Since ν can be written as a linear combination of four positive measures each of which is singular w.r.t. μ, w.l.o.g. me may assume that ν is positive. Let S be a Borel set such that $\mu(S) = 0$ and that ν is concentrated on S. Then

$$\lim_{y\downarrow 0} \frac{F_{\chi_S(\mu+\nu)}(x+{\rm i}y)}{F_{\mu+\nu}(x+{\rm i}y)} = \chi_S(x),$$

for $\mu_{\rm sing} + \nu_{\rm sing}$-a.e. x. Hence,

$$\lim_{y\downarrow 0} \frac{F_\nu(x+{\rm i}y)}{F_\mu(x+{\rm i}y) + F_\nu(x+{\rm i}y)} = 0$$

for $\mu_{\rm sing}$- a.e. x, and this yields (4.81). The proof of the Poltoratskii theorem is complete.

The Poltoratskii theorem also holds for complex measures μ:

Theorem 4.11 *Let ν and μ be complex Borel measures and $\nu = f\mu + \nu_{\rm s}$ be the Radon-Nikodym decomposition. Let $|\mu|_{\rm sing}$ be the part of $|\mu|$ singular with respect to the Lebesgue measure. Then*

$$\lim_{y\downarrow 0} \frac{F_\nu(x+{\rm i}y)}{F_\mu(x+{\rm i}y)} = f(x) \qquad \text{for } \ |\mu|_{\rm sing} - \ a.e.\, x.$$

Theorem 4.11 follows easily from Theorem 2.18.

4.8 F. & M. Riesz theorem

The celebrated theorem of F. & M. Riesz states:

Theorem 4.12 *Let $\mu \neq 0$ be a complex measure and $F_\mu(z)$ its Borel transform. If $F_\mu(z) = 0$ for all $z \in \mathbb{C}_+$, then $|\mu|$ is equivalent to the Lebesgue measure.*

In the literature one can find many different proofs of this theorem (for example, three different proofs are given in [Ko]). However, it has been only recently noticed that F. & M. Riesz theorem is an easy consequence of the Poltoratskii theorem. The proof below follows [JL3].

Proof. For $z \in \mathbb{C} \setminus \mathbb{R}$ we set

$$F_\mu(z) = \int_{\mathbb{R}} \frac{{\rm d}\mu(t)}{t-z}$$

and write

$$F_\mu(x \pm {\rm i}0) = \lim_{y\downarrow 0} F_\mu(x \pm {\rm i}y).$$

By Theorem 2.17 (and its obvious analog for the lower half-plane), $F_\mu(x \pm {\rm i}0)$ exists and is finite for Lebesgue a.e. x.

Write $\mu = h|\mu|$, where $|h(x)| = 1$ for all x. By the Poltoratskii theorem,

$$\lim_{y\downarrow 0} \frac{|F_\mu(x+\mathrm{i}y)|}{|F_{|\mu|}(x+\mathrm{i}y)|} = |h(x)| = 1$$

for $|\mu|_{\rm sing}$-a.e. x. Since by Theorem 2.5, $\lim_{y\downarrow 0}|F_{|\mu|}(x+\mathrm{i}y)| = \infty$ for $|\mu|_{\rm sing}$-a.e. x, we must have $\lim_{y\downarrow 0}|F_\mu(x+\mathrm{i}y)| = \infty$ for $|\mu|_{\rm sing}$-a.e. x. Hence, if $|\mu|_{\rm sing} \neq 0$, then $F_\mu(z)$ cannot vanish on $\mathbb{C}_+$.

It remains to prove that $|\mu|$ is equivalent to the Lebesgue measure. By Theorem 2.5, $\mathrm{d}|\mu| = \pi^{-1}\mathrm{Im}\, F_{|\mu|}(x+\mathrm{i}0)\mathrm{d}x$, so we need to show that $\mathrm{Im}\, F_{|\mu|}(x+\mathrm{i}0) > 0$ for Lebesgue a.e. x. Assume that $\mathrm{Im}\, F_{|\mu|}(x+\mathrm{i}0) = 0$ for $x \in S$, where S has positive Lebesgue measure. The formula

$$F_\mu(x+\mathrm{i}y) = \int_{\mathbb{R}} \frac{(t-x)\mathrm{d}\mu(t)}{(t-x)^2+y^2} + \mathrm{i}\int_{\mathbb{R}} \frac{y\mathrm{d}\mu(t)}{(t-x)^2+y^2}$$

and the bound

$$\left|\int_{\mathbb{R}} \frac{y\mathrm{d}\mu(t)}{(t-x)^2+y^2}\right| \leq \mathrm{Im}\, F_{|\mu|}(x+\mathrm{i}y)$$

yield that for $x \in S$,

$$\lim_{y\to 0} F_\mu(x+\mathrm{i}y) = \lim_{y\to 0}\int_{\mathbb{R}} \frac{(t-x)\mathrm{d}\mu(t)}{(t-x)^2+y^2}.$$

Hence,

$$F_\mu(x-\mathrm{i}0) = F_\mu(x+\mathrm{i}0) = 0 \qquad \text{for Lebesgue a.e. } x \in S. \tag{4.82}$$

Since F_μ vanishes on $\mathbb{C}_+$, F_μ does not vanish on $\mathbb{C}_-$ (otherwise, since the linear span of the set of functions $\{(x-z)^{-1} : z \in \mathbb{C}\setminus\mathbb{R}\}$ is dense in $C_0(\mathbb{R})$, we would have $\mu = 0$). Then, by Theorem 2.17 (i.e., its obvious analog for the lower half-plane), $F_\mu(x-\mathrm{i}0) \neq 0$ for Lebesgue a.e $x \in \mathbb{R}$. This contradicts (4.82). □

4.9 Problems and comments

[1] *Prove Relation (4.66). Hint: See Theorem I.2 in* [Si2].

[2] *Prove Theorem 4.6.*

[3] *Prove Relation (4.72).*

[4] *Let ν and μ be positive measures such that $\nu_{\rm sing} \perp \mu_{\rm sing}$. Prove that for $\mu_{\rm sing}$-a.e. x*

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = 0.$$

Hint: Write

$$\frac{P_\nu(z)}{P_\mu(z)} = \frac{\dfrac{P_{\nu_{\rm ac}}(z)}{P_{\mu_{\rm sing}}(z)} + \dfrac{P_{\nu_{\rm sing}}(z)}{P_{\mu_{\rm sing}}(z)}}{\dfrac{P_{\mu_{\rm ac}}(z)}{P_{\mu_{\rm sing}}(z)} + 1}$$

and use Theorem 2.5.

[5] *Prove the Poltoratskii theorem in the case where ν and μ are positive pure point measures.*

[6] *In the Poltoratskii theorem one cannot replace $\mu_{\rm sing}$ by μ. Find an example justifying this claim.*

The next set of problems deals with various examples involving rank one perturbations. Note that the model (4.60) is completely determined by a choice of a Borel probability measure μ^0 on $\mathbb{R}$. Setting $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}\mu^0)$, $H_0 =$ operator of multiplication by x, $\psi \equiv 1$, we obtain a class of Hamiltonians $H_\lambda = H_0 + \lambda(\psi|\cdot)\psi$ of the form (4.60). On the other hand, by the spectral theorem, any family Hamiltonians (4.60), when restricted to the cyclic subspace $\mathcal{H}_\psi$, is unitarily equivalent to such a class.

[7] *Let μ_C be the standard Cantor measure (see Example 3 in Section I.4 of* [RS1]*) and $\mathrm{d}\mu^0 = (\mathrm{d}x \restriction [0,1] + \mathrm{d}\mu_C)/2$. The ac spectrum of H_0 is $[0,1]$. The singular continuous part of μ^0 is concentrated on the Cantor set C. Since C is closed, $\mathrm{sp}_{\rm sing}(H_0) = C$. Prove that for $\lambda \neq 0$ the spectrum of H_λ in $[0,1]$ is purely absolutely continuous. Hint: See the last example in Section XIII.7 of* [RS4].

[8] *Assume that $\mu^0 = \mu_C$. Prove that for all $\lambda \neq 0$, H_λ has only pure point spectrum. Compute the spectrum of H_λ. Hint: This is Example 1 in* [SW]. *See also Example 3 in Section II.5 of* [Si2].

[9] *Let*

$$\mu_n = 2^{-n} \sum_{j=1}^{2^n} \delta(j/2^n),$$

and $\mu = \sum_n a_n \mu_n$, where $a_n > 0$, $\sum_n a_n = 1$, $\sum_n 2^n a_n = \infty$. The spectrum of H_0 is pure point and equal to $[0,1]$. Prove that the spectrum of H_λ in $[0,1]$ is purely singular continuous for all $\lambda \neq 0$. Hint: This is Example 2 in [SW]. *See also Example 4 in Section II.5 of* [Si2].

[10] *Let $\nu_{j,n}(A) = \mu_C(A + j/2^n)$ and*

$$\mu^0 = c\chi_{[0,1]} \sum_{n=1}^{\infty} n^{-2} \sum_{j=1}^{2^n} \nu_{j,n},$$

where c is the normalization constant. Prove that the spectrum of H_λ on $[0,1]$ is purely singular continuous for all λ. Hint: This is Example 5 in Section II.5 of [Si2].

[11] *Find μ^0 such that:*
(1) *The spectrum of H_0 is purely absolutely continuous and equal to* $[0,1]$.
(2) *For a set of λ's of positive Lebesgue measure, H_λ has embedded point spectrum in* $[0,1]$.
Hint: See [DS] *and Example 7 in Section II.5 of* [Si2].

[12] *Find μ^0 such that:*
(1) *The spectrum of H_0 is purely absolutely continuous and equal to* $[0,1]$.
(2) *For a set of λ's of positive Lebesgue measure, H_λ has embedded singular continuous spectrum in* $[0,1]$.
Hint: See [DS] *and Example 8 in Section II.5 of* [Si2].

[13] *del Rio and Simon* [DS] *have shown that there exists μ^0 such that:*
(1) *For all λ* $\mathrm{sp}_{\mathrm{ac}}(H_\lambda) = [0,1]$.
(2) *For a set of λ's of positive Lebesgue measure, H_λ has embedded point spectrum in* $[0,1]$.
(3) *For a set of λ's of positive Lebesgue measure, H_λ has embedded singular continuous spectrum in* $[0,1]$.

[14] *del Rio-Fuentes-Poltoratskii* [DFP] *have shown that there exists μ^0 such that:*
(1) *For all λ* $\mathrm{sp}_{\mathrm{ac}}(H_\lambda) = [0,1]$. *Moreover, for all $\lambda \in [0,1]$, the spectrum of H_λ is purely absolutely continuous.*
(2) *For all* $\lambda \notin [0,1]$, $[0,1] \subset \mathrm{sp}_{\mathrm{sing}}(H_\lambda)$.

[15] *Let μ^0 be a pure point measure with atoms $\mu^0(\{x_n\}) = a_n$, $n \in \mathbb{N}$, where $x_n \in [0,1]$. Clearly,*

$$G(x) = \sum_{n=1}^{\infty} \frac{a_n}{(x-x_n)^2}.$$

(1) *Prove that if $\sum_n \sqrt{a_n} < \infty$, then $G(x) < \infty$ for Lebesgue a.e. $x \in [0,1]$.*
(2) *Assume that $x_n = x_n(\omega)$ are independent random variables uniformly distributed on $[0,1]$ (we keep a_n deterministic). Assume that $\sum_n \sqrt{a}_n = \infty$. Prove that for a.e. ω, $G(x) = \infty$ for Lebesgue a.e. $x \in [0,1]$.*
(3) *What can you say about the spectrum of H_λ in the cases (1) and (2)?*
Hint: (1) and (2) are proven in [How].

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