WHITNEY REGULARITY OF THE IMAGE OF THE CHEVALLEY MAP.

by Gérard P. BARBANÇON

- Regularité au sens de Whitney de l'image de l'application de Chevalley.

Résumé : Un fermé F est 1-régulier au sens de Whitney si pour tout compact $K \subset F$, il existe $C > 0$ tel que si x et x' sont dans K, ils peuvent etre joints par un arc rectifiable tracé dans K de longueur $L \leq C \mid x - x' \mid$. Dans cette note on montre la regularité de l'image de $\mathbf{R}^n$ par l'application de Chevalley définie par une base de la sous-algèbre des polynomes invariants par un groupe fini engendré par des reflexions. La démonstration repose sur une version d'un théorème de prolongement de Lojasiewicz adaptée aux jets r-reguliers d'ordre $m \geq r$ et sur une caractérisation des ensembles 1-réguliers au sens de Whitney donnée par Glaeser.

- Whitney regularity of the image of the Chevalley map.

Abstract : A closed set F is Whitney 1-regular if for all compact $K \subset F$, there exists a $C > 0$ such that any two points x and x' in K can be joined by a path of length $L \leq C \mid x - x' \mid$. In this note we prove the Whitney regularity of the image of $\mathbf{R}^n$ by the Chevalley map defined by an integrity basis of the subalgebra of polynomials invariant by a finite orthogonal group generated by reflections. The proof relies upon a Glaeser characterization of Whitney regular sets and a version of a Lojasiewicz extension theorem adjusted to r-regular jets of order $m \geq r$.

1. Introduction

Definition 1. ([24], [23])*A compact set $K \subset \mathbf{R}^n$, connected by rectifiable arcs, is Whitney 1-regular or has the Whitney property $\mathbf{P}_1$ if the geodesic distance in K is equivalent to the Euclidean distance. That is*:
$\exists k_K > 0$, $\forall (x, x') \in K^2$, *there exists a rectifiable arc from x to x' in K with length* $l(x, x') \leq k_K |x - x'|$.

A closed set $\mathcal{F}$ is Whitney 1-regular when any compact set in $\mathcal{F}$ is Whitney 1-regular.
The condition for property $\mathbf{P}_\rho$ would be the inequality $l(x, x') \leq k_K \mid x - x' \mid^\rho$ for all x and x' in K. Any semi-analytic set is ρ-regular for a small enough ρ.

The interest of property $\mathbf{P}_1$ lies in the:

Theorem 1. [24] *Let $\mathcal{O}$ be an open set of $\mathbf{R}^n$ and assume that its closure $\mathcal{F} = \overline{\mathcal{O}}$ has property $\mathbf{P_1}$. If $f \in C^m(\mathcal{O})$ is such that whenever $\mid k \mid = m$, $\dfrac{\partial^{|k|} f}{\partial x^k}$ has a continuous extension to $\mathcal{F}$, then f has an extension $\tilde{f} \in C^m(\mathbf{R}^n)$.*

A function f invariant on the fibers of a polynomial mapping $\theta : \mathbf{R}^n \to \mathbf{R}^n$, may be written as $f = F \circ \theta$. In [5] we considered symmetric functions that may be written as

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$f = F \circ N$ where N is the Newton mapping. If f is of class $\mathcal{C}^r$, F is of class $\mathcal{C}^r$ on the regular image where it is obtained by local inversion. It may be shown that F and its derivatives of order less than or equal to the integer part of r/n, say $[r/n]$, are continuous on $N(\mathbf{R}^n)$. Then theorem 1 shows that F is of class $\mathcal{C}^{[r/n]}$ if $N(\mathbf{R}^n)$ is 1-regular. The property $\mathbf{P}_1$ of $N(\mathbf{R}^n)$ conjectured in [4] and [5], see also [3], was proved by Kostov [17], using results of Arnold [1], and Givent'al [12].

Let $W \subset O(n)$ be a finite reflection group. A theorem of Chevalley ([9]) states that the algebra of W-invariant polynomials is generated by n algebraically independent W-invariant homogeneous polynomials, say the basic invariants or an integrity basis. Let $p_1(x), \ldots, p_n(x)$ be these basic invariants and P be the mapping $x \mapsto (p_1(x), \ldots, p_n(x))$, say the 'Chevalley' mapping. This note gives a proof of the Whitney property $\mathbf{P}_1$ for the image of the 'Chevalley' mapping associated with any finite reflection group. This property was implicitly conjectured in [12].

The proof relies upon a characterization of Whitney 1-regular sets given by Glaeser [13] and uses a version of the Łojasiewicz extension theorem ([18], [23]) fitted to r-regular jets of order $m \geq r$.

Besides allowing the use of theorem 1 to get an easier proof for a theorem of Chevalley in finite class of differentiability, the Whitney regularity property provides a geometric insight of the image sets of the Chevalley maps and their discriminants that may be of interest in the theory of singularities ([2]). Anyway, this is an example of Whitney 1-regular set which is natural but not trivial.

2. The Chevalley map.

The reader familiar with these questions may omit this section. Proofs and details may be found in [8], [11], and [15].

Let W be a finite orthogonal group generated by reflections and let the polynomial mapping $P : \mathbf{R}^n \ni x \mapsto P(x) = (p_1(x), \ldots, p_n(x)) \in \mathbf{R}^n$ be its associated Chevalley mapping. P is proper and separates the W-orbits ([22]), but it is neither injective nor surjective. For $i = 1, \ldots, n$ the degree of p_i will be denoted by k_i.

Let $\mathcal{R}$ be the set of reflections different from identity in W. The number of these reflections is $\mathcal{R}^{\#} = d = \sum_{i=1}^{n}(k_i - 1)$. For each $\tau \in \mathcal{R}$, let λ_τ be a linear form with kernel $H_\tau = \{x \in \mathbf{R}^n | \tau(x) = x\}$. The jacobian of P is $J_P = c \prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through $\mathcal{R}$.

A Weyl Chamber C is a connected component of the regular set. All of the other connected components are obtained by the action of W and the regular set is $\bigcup_{w \in W} w(C)$. There is a stratification of $\mathbf{R}^n$ by the regular set, the reflecting hyperplanes H_τ and their intersections. The mapping P induces an analytic diffeomorphism of C onto the interior of $P(\mathbf{R}^n)$. It also induces an homeomorphism that carries the stratification from the fundamental domain $\overline{C}$ onto $P(\mathbf{R}^n)$.

When W is reducible, it is a direct product of its irreducible components, $W = W^1 \times \ldots \times W^s$ and we may write $\mathbf{R}^n$ as an orthogonal direct sum $\mathbf{R}^{n_0} \oplus \mathbf{R}^{n_1} \oplus \ldots \oplus \mathbf{R}^{n_s}$ where $\mathbf{R}^{n_0}$ is the subspace of W-invariant vectors and for $i = 1, \ldots, s$, W^i is an irreducible finite Coxeter group acting on $\mathbf{R}^{n_i}$.

Any Weyl Chamber C for W is of the form $\mathbf{R}^{n_0} \times C_1 \times \ldots \times C_s$ where C_i is a chamber for W^i in $\mathbf{R}^{n_i}$. If $w = w_1 \ldots w_s \in W$ with $w_i \in W^i$, $1 \leq i \leq s$, for all $x \in \mathbf{R}^n$, in coordinates that fits the orthogonal direct sum, we have $w(x) = w(x_0, x_1, \ldots, x_s) = (x_0, w_1(x_1), \ldots, w_s(x_s))$. The direct product of the identity on $\mathbf{R}^{n_0}$ and of Chevalley mappings P^i associated with W^i acting on $\mathbf{R}^{n_i}$, $1 \leq i \leq s$, is the Chevalley mapping $P = Id_0 \times P^1 \times \ldots \times P^s$ associated with the action of W on $\mathbf{R}^n$.

For an irreducible W (or for an irreducible component W^i) we may assume that the degrees of the coordinate polynomials $p_1, \ldots, p_n$ are in increasing order. We have $k_1 = 2$, $k_n = h$, the Coxeter number of W. In the reducible case, for each W^i, $i = 1, \ldots, s$, we assume the degrees of the p^i_j to be in increasing order: $2 = k^i_1 \leq \ldots \leq k^i_{n_i} = h_i$, the Coxeter number of W^i. We may have $h_i = h_j$, either $W^i = W^j$ or not. Considering for an example $A_9 \times A_9 \times H_3$, $h_1 = h_2 = h_3 = 10$. Anyway we will denote by h the degree of the coordinate polynomial of highest degree, equal to the highest Coxeter number of the irreducible components. We may also observe that if $j \neq i$, P^i and P^j have no monomial in common since P^i acts on $\mathbf{R}^{n_i}$ and $P^i(X)$ contains only the indeterminate $X^i = (X^i_1, \ldots, X^i_{n_i})$, while P^j acts on $\mathbf{R}^{n_j}$ and $P^j(X)$ contains only the indeterminate $X^j = (X^j_1, \ldots, X^j_{n_j})$.

Let us recall that there are only finitely many types of irreducible finite Coxeter groups defined by their connected graph types. Even though these groups are Weyl groups of root systems or of Lie algebras, we will follow the general usage and denote them with upper case letters: $A_n, B_n, D_n, I_2(n), H_3, H_4, F_4, E_6, E_7, E_8$ (we omit C_n and G_2 since the Weyl groups of B_n and C_n are the same and $G_2 = I_2(6)$). For all of these groups an integrity basis or system of basic invariants is explicitly given in [21]. In each case, an invariant set of linear forms $\{L_1, \ldots, L_v\}$ is chosen. Symmetric functions of the L_i are W-invariant and we may choose $p_i(X) = \sum_{j=1}^{v} [L_j(X)]^{k_i}$ with k_i as determined in [10]. As usual D_n does not follow the general line, but an integrity basis for D_n is known ([8]). With the exception of D_n we have $k_1 < \ldots < k_n$. In D_n with the usual choice $p_i(X) = \sum_{j=1}^{n} X_j^{2i}$, $1 \leq i \leq n-1$, $\quad p_n(X) = X_1 \ldots X_n$, the greatest degree is not $k_n = n$ but $k_{n-1} = 2(n-1)$. Additionally if $n = 2m$, there are two polynomials of the same degree $2m$, $p_{2m}(X) = X_1 \ldots X_{2m}$ and $p_m(X) = \sum_{i=1}^{2m} X_i^{2m}$.

The Whitney regularity property $\mathbf{P}_1$ is preserved by diffeomorphism. So, it does not depend on the choice of the set of basic invariants, since a change of basic invariants is an invertible polynomial map on $\mathbf{R}^n$. Therefore when W is reducible, we may and will choose a coordinate system fitted to the decomposition of W in irreducible factors and the corresponding Chevalley mapping as described above. We will also assume that for each W^i, the P^i are as given in [21].

The mapping P is the restriction to $\mathbf{R}^n$ of a complex mapping from $\mathbf{C}^n$ to $\mathbf{C}^n$, still denoted by P. The linear mappings defined by the action of W on $\mathbf{R}^n$ are restrictions of $\mathbf{C}$-automorphisms of $\mathbf{C}^n$ and we will still denote by W the group of these automorphisms. The complex P is W-invariant and thus is not injective, but it is surjective ([16]).
On its regular set, the mapping P is a local analytic isomorphism. The critical set where the jacobian vanishes is the union of the complex hyperplanes H_τ, kernels of the complex forms λ_τ. These hyperplanes and their intersections provide a stratification of $\mathbf{C}^n$. The

points of each stratum are stabilized by the same isotropy subgroup of W which is generated by the reflections in the hyperplanes containing the stratum.
The critical image is the algebraic set $\{u \in \mathbf{C}^n | \Delta(u) = J_P^2(z) = 0\}$, onto which P carries the stratification.

3. Continuous-regular polynomial fields

For a complete presentation of these questions see [23].

Let $k = (k_1, ..., k_n) \in \mathbf{N}^n$ and $x = (x_1, ..., x_n) \in \mathbf{R}^n$. We put: $|k| = k_1 + ... + k_n$, $k! = k_1!...k_n!$ and $x^k = x_1^{k_1}...x_n^{k_n}$. Analogously for the indeterminate $X = (X_1, ..., X_n)$, we put $X^k = X_1^{k_1}...X_n^{k_n}$. In $\mathbf{N}^n$, we write $k \leq l$, if and only if for all j, $k_j \leq l_j$, and in this case $l - k = (l_1 - k_1, ..., l_n - k_n)$. The Euclidean norm of x will be denoted by $| x |$.

Definition 2. *Let E be a subset of $\mathbf{R}^n$. A polynomial field on E is a mapping $A : E \to R[X]$ that assigns to each $x \in E$ the polynomial $A_x = \sum_k \frac{1}{k!} a_k(x) X^k$, where $k \in \mathbf{N}^n$. The field A is of degree m if its image is in the subspace of polynomials of degree $\leq m$.*

Let us now assume that E is a closed set of an open set $\Omega \subset \mathbf{R}^n$.

Definition 3. *The field A of degree m is m-continuous, if its coefficients $a_k : E \to R$ of order $| k | \leq m$, are continuous.*
An m-continuous polynomial field, may be identified with a jet of order m, $A \simeq (a_k)_{|k| \leq m} \in J^m(E)$.
At each point $x \in E$ the field A induces a polynomial mapping still denoted by A_x that acts upon vectors $x' - x$ tangent to $\mathbf{R}^n$ at x.

$$A_x : x' \mapsto A_x(x') = a_0(x) + \sum_{|k|>0} \frac{1}{k!} a_k(x) \; (x' - x)^k.$$

After formal derivation with respect to the indeterminates we obtain polynomial fields $\left(\partial^{|q|} A / \partial X^q\right)$ inducing polynomial mappings:

$$\left(\frac{\partial^{|q|} A}{\partial X^q}\right)_x (x') = a_q(x) + \sum_{k>q} \frac{1}{(k-q)!} a_k(x) \; (x' - x)^{k-q}.$$

In particular $A_x(x) = a_0(x)$ and $\left(\partial^{|q|} A / \partial X^q\right)_x (x) = a_q(x)$.
For $| q | \leq r \leq m$, we put:

$$(R_x A)^q(x') = \left(\frac{\partial^{|q|} A}{\partial X^q}\right)_{x'} (x') - \left(\frac{\partial^{|q|} A}{\partial X^q}\right)_x (x').$$

Definition 4. *Let A be an m-continuous field of polynomials of degree m on E. For $r \leq m$, A is r-regular on E, if and only if it satisfies the Whitney conditions:*
for all compact set K in E, for all x and x' in K and for all $q \in \mathbf{N}^n$ with $| q | \leq r$,
$(\mathcal{W}_q^r)$ $\qquad | (R_x A)^q(x') | = o(| x' - x |^{r-|q|})$, *when* $|x - x'| \to 0$.

Example. Let f be a function of class $\mathcal{C}^r$ on an open set of $\Omega \subset \mathbf{R}^n$, and let E be a closed set in Ω. The restriction to E of the field of Taylor polynomials of f, $T^r_E f$ say, defines on E an r-regular field of degree r, or Whitney field of order r.

Remark. When necessary we denote by A^r the field A truncated at order r, but A being m-continuous ($m \geq r$) there is no need to consider A^r in stead of $A\,(= A^m)$ in the conditions $(\mathcal{W}^r_q)$, since for $m > r, (R_x A^r)^q(x')$ and $(R_x A)^q(x')$ differ by a sum of terms $[a_k(x)/(k-q)!]\,(x-x')^{k-q}$, with a_k uniformly continuous on K and $|k| - |q| > r - |q|$.

The space of polynomial fields m-continuous and r-regular on E or space of r-regular jets of order m on E, is naturally provided with the Fréchet topology defined by the family of semi-norms:

$$\|A\|^{K_n}_{r,m} = \sup_{\substack{x\in K_n\\ |k|\leq m}} \mid \frac{1}{k!} a_k(x) \mid + \sup_{\substack{(x,x')\in K_n^2\\ x\neq x', |k|\leq r}} \left(\frac{\mid (R_x A)^k(x') \mid}{\mid x - x' \mid^{r-|k|}} \right)$$

where K_n runs through a countable collection of compact sets of E, with $E = \bigcup_n K_n$. Provided with this topology the space of r-regular, m-continuous polynomial fields on E will be denoted by $\mathcal{E}^{r,m}(E)$.

If $r = m$, $\mathcal{E}^r(E)$ is the space of Whitney fields of order r or Whitney functions of class $\mathcal{C}^r$ on E. If $r = 0$, $\mathcal{E}^{0,m}(E) \simeq J^m(E)$ is the space of m-continuous polynomial fields or jets of order m, with the topology defined by the semi-norms: $|A|^{K_n}_m = \sup_{\substack{x\in K_n\\ |k|\leq m}} \mid \frac{1}{k!} a_k(x) \mid$.

In general the norms $\| \;\|^K_{r,m}$ and $| \;|^K_m$ are not equivalent on $\mathcal{E}^{r,m}(K)$. Nevertheless, we have the following result of Whitney:

Proposition 1. ([23], [24]) *If K is 1-regular, the norms $\| \;\|^K_{r,m}$ and $| \;|^K_m$ are equivalent on $\mathcal{E}^{r,m}(K)$.*

Conversely, assuming that the compact K is connected by rectifiable paths (or is a finite union of sets connected by rectifiable paths), Glaeser has proved :

Proposition 2.([13], [23]) *If the norms $\| \;\|^K_1$ and $| \;|^K_1$ are equivalent on $\mathcal{E}^1(K)$, then K is 1-regular.*

There is a well-known relevant Whitney extension theorem.

Theorem 3. *The restriction mapping $T^r_E : f \to T^r_E f$, of the space $\mathcal{E}^r(\mathbf{R}^n)$ of functions of class $\mathcal{C}^r$ to the space $\mathcal{E}^r(E)$ of Whitney fields of order r on E, is surjective. There is a linear section, continuous when the spaces are provided with their natural Fréchet topologies.*

This theorem states that if $A \in \mathcal{E}^r(E)$, there exists a function $f \in \mathcal{C}^r(\mathbf{R}^n)$ such that $A = T^r_E f$. However, when $A \in \mathcal{E}^{r,m}(E)$ the Whitney extension erases any information that might be carried by the terms of degree higher than r.

Example. The Newton mapping, N induces

$$N^* : \mathcal{C}^r(\mathbf{R}^n, \mathbf{R}) \ni F \mapsto f = F \circ N \in \mathcal{C}^r(\mathbf{R}^n, \mathbf{R}).$$

For any $(a, x) \in \mathbf{R}^n \times \mathbf{R}^n$, we have:

$$f(x) = F[N(x)] = \sum_{\mid k \mid \leq r} \frac{1}{k!} D^k F[N(a)]\ (N(x) - N(a))^k + o(|N(x) - N(a)|^r)$$

which is the Taylor formula for F between $N(a)$ and $N(x)$. Expanding $N(x) - N(a)$ by the polynomial Taylor formula we get a polynomial in $x - a$ of degree n and thus

$$f(x) = \sum_{\mid k \mid \leq nr} \frac{1}{k!} a_k(x)\ (x - a)^k + o(|N(x) - N(a)|^r).$$

If for some closed set E we consider $T^r_E f$, its extension given by the theorem of Whitney will not be in $N^*(\mathcal{C}^r(\mathbf{R}^n, \mathbf{R}))$. It is the field A with $A_x = \sum_{\mid k \mid \leq nr} \frac{1}{k!} a_k(x)\ (x - a)^k$ that carries the informations about the fact that $f = N^*(F)$ and not the truncated field A^r. This is the reason why we are considering r-regular, m-continuous fields and not the usual Whitney fields of order r. $\diamondsuit$

Let E be a closed subset of an open set $\Omega \subset \mathbf{C}^n \simeq \mathbf{R}^{2n}$, we may consider polynomial fields on E with complex coefficients:

$$A : E \ni z \mapsto A_z = \sum_{|k|+|l| \leq m} \frac{1}{k!l!} a_{k,l}(z)\ X^k Y^l \in \mathbf{C}[X, Y].$$

The questions of continuity and regularity discussed in the real case may be reproduced here and we may define the Fréchet space $\mathcal{E}^{r,m}(E; \mathbf{C})$. For a polynomial field $A = Re(A) + iIm(A)$, where $Re(A)$ and $Im(A)$ the real and imaginary part of A are both in $\mathcal{E}^{r,m}(E; \mathbf{R})$, we have the semi-norms:

$$\|A\|^K_{r,m} = \|Re(A)\|^K_{r,m} + \|Im(A)\|^K_{r,m}$$

.

Definition 5. [18] *A polynomial field $A \in \mathcal{E}^{r,m}(E; \mathbf{C})$ is formally holomorphic if it satisfies the Cauchy-Riemann equalities:*

$$i \frac{\partial A}{\partial X_j} = \frac{\partial A}{\partial Y_j},\ j = 1, ..., n.$$

This means that $\frac{\partial A}{\partial \overline{Z}_j} = 0,\ j = 1, ..., n$, and that for all $z \in E$ the polynomial A_z belongs to $\mathbf{C}[Z]$ and is of the form $A_z(Z) = \sum_k \frac{1}{k!} a_k(z) Z^k$.

The set of m-continuous, r-regular, formally holomorphic polynomial fields on the closed set E of $\Omega \subset \mathbf{C}^n$ will be denoted by $\mathcal{H}^{r,m}(E)$. It is a closed sub-algebra of $\mathcal{E}^{r,m}(E; \mathbf{C})$ and therefore a Fréchet space when provided with the induced topology. In practice we shall define the semi-norms $\|A\|_{r,m}^{K_n}$ on $\mathcal{H}^{r,m}(E)$ by the same formulas as in $\mathcal{E}^{r,m}(E;\mathbf{R})$, only using moduli instead of absolute values.

4. An extension operation.

Definition 6. *A real form* ([20]) *or a really situated subspace* ([18], [23]) *of* $\mathbf{C}^n$ *is a real vector subspace* E *of dimension* n *such that* $E \oplus iE = \mathbf{C}^n$.

Example. For any involution α, the real subspace $\Gamma_\alpha = \{z \in \mathbf{C}^n | z_{\alpha(i)} = \overline{z_i}\}$, is a real form of $\mathbf{C}^n$.

The reciprocal image $P^{-1}(\mathbf{R}^n)$ is a W-invariant finite union of real forms of $\mathbf{C}^n$. This is a particular case of a property which is true for any finite group. (*)

A classical theorem of Hilbert states that for any finite subgroup G of $O(n)$ the algebra of G-invariant polynomials on $\mathbf{R}^n$ is finitely generated. There is a finite number $d \geq n$ of G-invariant homogeneous polynomials, say $q_1, \ldots, q_d$, and for all G-invariant polynomial function $f : \mathbf{R}^n \to \mathbf{R}$ there exists a polynomial function $F : \mathbf{R}^d \to \mathbf{R}$ such that $f(x) = F(q_1(x), \ldots, q_d(x))$.
The polynomial mapping $Q : \mathbf{R}^n \ni x \mapsto Q(x) = (q_1(x), \ldots, q_d(x)) \in \mathbf{R}^d$ is the restriction of a complex mapping from $\mathbf{C}^n$ to $\mathbf{C}^d$, still denoted by Q.

Lemma 1. *Let* G *be a finite group acting orthogonally on* $\mathbf{R}^n$ *and* Q *be the associated polynomial mapping as above. The reciprocal image* $Q^{-1}(\mathbf{R}^d) \subset \mathbf{C}^n$ *is a* G*-invariant finite union of real forms of* $\mathbf{C}^n$.

Proof. For $g \in G$, let us put $S_g = \{u + iv \mid gu = u \text{ and } gv = -v\}$.
Let $Q(z)$ be real for some $z \in \mathbf{C}^n$, that is $Q(z) = \overline{Q(z)}$. Since the coefficients of Q are real, we have $\overline{Q(z)} = Q(\overline{z})$ and thus $Q(z) = Q(\overline{z})$. The fibers of $Q : \mathbf{C}^n \to \mathbf{C}^d$ are the orbits of G and there is a $g \in G$ such that $\overline{z} = gz$. Therefore $z \in S_g$.
Conversely if $z \in S_g$ for some $g \in G$, then $gz = \overline{z}$ and $Q(z) = Q(gz) = Q(\overline{z})$. Since Q has real coefficients $Q(\overline{z}) = \overline{Q(z)}$, and thus $Q(z) = \overline{Q(z)}$ which means that $Q(z)$ is real.

S_g is a real subspace and $S_g \cap iS_g = 0$. If g is not an involution $S_g = \{0\}$. If g is an involution (including the identity in which case $S_{\mathrm{Id}} = \mathbf{R}^n$), then S_g is defined by n real equations, $2q$ of the form $\mathrm{Im}(z_j + (gz)_j) = \mathrm{Re}(z_j - (gz)_j) = 0$, and $n - 2q$ of the form $\mathrm{Im}(z_i) = 0$, for $i = 2q+1, \ldots, n$. Therefore the S_g that are not reduced to $\{0\}$, are real forms of $\mathbf{C}^n$ and $Q^{-1}(\mathbf{R}^d) = \bigcup_{g\in G} S_g$ is a finite union of real forms.

Since $g'S_g = \{u' + iv' = g'(u + iv) | \; gu = u, gv = -v\}$ so that $g'gg'^{-1}u' = u'$ and $g'gg'^{-1}v' = -v'$, we can see that $g'S_g = S_{g'gg'^{-1}}$. Therefore $\bigcup_{g\in G} S_g$ is G-stable. $\diamondsuit$

Definition 7.([19], [23]) *Two closed sets* E *and* F *of an open set* $\Omega \subseteq R^n$ *are 1-regularly separated if either* $E \cap F$ *is empty or if for all* $x_0 \in E \cap F$ *there exists a neighborhood* U *of* x_0 *and a constant* $C > 0$ *such that for all* $x \in U$,

$$d(x, E) + d(x, F) \geq C d(x, E \cap F).$$

(*) I am indebted to the referee for the general form and the smart proof of lemma 1.

One can prove that E and F are 1-regularly separated if and only if the 0-sequence:

$$0 \to \mathcal{H}^{r,m}(E \cup F) \to \mathcal{H}^{r,m}(E) \oplus \mathcal{H}^{r,m}(F) \to \mathcal{H}^{r,m}(E \cap F) \to 0$$

is exact ([23]).

Example. Any two linear subspaces are regularly separated. In particular any two real forms in $\mathbf{C}^n \simeq \mathbf{R}^{2n}$ are 1-regularly separated. Moreover the closed strata of the stratification of $P^{-1}(\mathbf{R}^n)$ by the reflecting hyperplanes and their intersections are regularly separated.

We have the following consequences for regular continuous fields of polynomials:

Proposition 3.([19], [23]) *Let E and F be 1-regularly separated closed sets and A be a field on $E \cup F$. If the restrictions A_E and A_F are respectively in $\mathcal{H}^{r,m}(E)$ and $\mathcal{H}^{r,m}(F)$, then A is in $\mathcal{H}^{r,m}(E \cup F)$.*

Proposition 4([18], [23]): *Let $\Pi_1, \ldots, \Pi_s, \ldots, \Pi_t$ be real forms in $\mathbf{C}^n$. There exists a linear and continuous extension from $\mathcal{H}^{r,m}(\bigcup_{i=1}^s \Pi_i)$ to $\mathcal{H}^{r,m}(\bigcup_{i=1}^t \Pi_i)$.*

In particular:

Proposition 5. *Let W be a finite sub group of $O(n)$, generated by reflections. Let $P : \mathbf{R}^n \ni x \mapsto (p_1(x), \ldots, p_n(x)) \in \mathbf{R}^n$ be the Chevalley mapping defined by a basis of W-invariant polynomials. There exists a linear and continuous extension operator*

$$\mathcal{H}^{r,m}(\mathbf{R}^n)^W \to \mathcal{H}^{r,m}(P^{-1}(\mathbf{R}^n))^W.$$

Proof. Let $A \in \mathcal{H}^{r,m}(\mathbf{R}^n)^W$. By Proposition 4 we get an extension to $P^{-1}(\mathbf{R}^n)$ which is a W-invariant union of real forms and we just have to average on W. Since the initial field was in $(\mathcal{H}^{r,m}(\mathbf{R}^n))^W$, it is not altered by the averaging. All the operations involved are linear and continuous. $\diamondsuit$

5. 1-regularity of the image of the Chevalley mapping.

Let F be a function of class $\mathbf{C}^1$ on the interior of $P(\mathbf{R}^n)$. We assume that F and its first derivatives have continuous extensions to $P(\mathbf{R}^n)$. This function induces on $P(\mathbf{R}^n)$ a field still denoted by F which is in $\mathcal{E}^{0,1}(P(\mathbf{R}^n))$ and is 1-regular on the interior of $P(\mathbf{R}^n)$. In the reducible case, when using fitted coordinates, we have:

$$F_u(U) = F_0(u) + F_1^0(u)U_1^0 + \ldots + F_{n_0}^0(u)U_{n_0}^0 + F_1^1(u)U_1^1 + \ldots + F_{n_1}^1(u)U_{n_1}^1 +$$

$$+ \ldots + F_1^s(u)U_1^s + \ldots + F_{n_s}^s(u)U_{n_s}^s$$

We then consider the field $f = F \circ P$ defined on $\mathbf{R}^n$ by:

$$f_x(x') = F_0(P(x)) + F_1^0(P(x))(p_1^0(x') - p_1^0(x)) + \ldots + F_{n_0}^0(P(x))(p_{n_0}^0(x') - p_{n_0}^0(x))$$

$$+F_1^1(P(x))(p_1^1(x')-p_1^1(x))+\ldots+F_{n_1}^1(P(x))(p_{n_1}^1(x')-p_{n_1}^1(x))+\ldots$$

$$\ldots+F_1^s(P(x))(p_1^s(x')-p_1^s(x))+\ldots+F_{n_s}^s(P(x))(p_{n_s}^s(x')-p_{n_s}^s(x))$$

and, using the Taylor's polynomial expansion of the $p_i^j(x')$ at x:

$$=F_0(P(x))+\sum_{i=1}^{n_0}F_i^0(P(x))((x')_i^0-x_i^0)+\sum_{j=1}^{s}\sum_{i=1}^{n_j}F_i^j(P(x))\Big(\sum_{|\alpha|=1}^{k_i^j}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}p_i^j}{\partial x^\alpha}((x')^j-x^j)^\alpha\Big).$$

The definition of f entails that it is W-invariant, h-continuous, and 1-regular on the regular set of P. Moreover, since the critical set is the set where the polynomial $\prod_{\tau\in\mathcal{R}}\lambda_\tau$ vanishes, f is in $\mathcal{E}^{1,h}(\mathbf{R}^n)^W$ by a real version of the following consequence of the mean value theorem:

Lemma 2.([18]) *Let Γ be a finite union of real forms in $\mathbf{C}^n$. Let $P\neq 0$ be a complex polynomial and $X=\{z\in\mathbf{C}^n\mid P(z)=0\}$. If $A\in\mathcal{H}^{0,m}(\Gamma)\cap\mathcal{H}^{r,m}(\Gamma\backslash X)$ then $A\in\mathcal{H}^{r,m}(\Gamma)$.*

This field f induces a formally holomorphic field in $\mathcal{H}^{1,h}(\mathbf{R}^n)^W$. By proposition 5 this field has an extension $\tilde{f}$ in $\mathcal{H}^{1,h}(P^{-1}(\mathbf{R}^n))^W$. On the complement in Γ of $\Gamma\cap\bigcup_{\tau\in\mathcal{R}}H_\tau$, the mapping P is a local analytic isomorphism and this yields the construction of a 1-regular $\tilde{F}=\tilde{f}\circ P^{-1}$, unambiguously since both $\tilde{f}$ and P are W-invariant.

Using lemma 2, if we get a $\tilde{F}$ continuous on $\mathbf{R}^n$ it will be 1-regular since the critical image is the null set of the discriminant polynomial. Then if we get a $\tilde{F}\in\mathcal{H}^1(\mathbf{R}^n)$ this will mean that the function F we started with was the restriction to $P(\mathbf{R}^n)$ of a function of class $\mathcal{C}^1$ on $\mathbf{R}^n$.

In practice we can get $\tilde{F}$ by identifying

$$\tilde{f}_z(z')=\tilde{f}_0(z)+\sum_{1\leq|\alpha|\leq k_n}\frac{1}{\alpha!}\tilde{f}_\alpha(z)(z_1'-z_1)^{\alpha_1}...(z_n'-z_n)^{\alpha_n}\qquad\text{and}$$

$$\tilde{F}_0(P(z))+\sum_{i=1}^{n_0}\tilde{F}_i^0(P(z))((z')_i^0-z_i^0)+\sum_{j=1}^{s}\sum_{i=1}^{n_j}\tilde{F}_i^j(P(z))\Big(\sum_{|\alpha|=1}^{k_i^j}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}p_i^j}{\partial z^\alpha}((z')^j-z^j)^\alpha\Big).$$

This will obviously return F on $P(\mathbf{R}^n)$ where f was obtained that way. On the regular set of P where $\tilde{F}$ is of class $\mathcal{C}^1$, we will check that this process is consistent with the chain rule $D^1f=D^1FD^1P$.

In the chosen coordinates, P preserves the $\mathbf{R}^{n_i}$. Therefore the cross derivatives of $\tilde{f}$ between two spaces $\mathbf{R}^{n_i}$ and $\mathbf{R}^{n_j}$ vanish on a dense set and by continuity everywhere. Actually we just have to perform the identification for each P_i on $\mathbf{R}^{n_i}$ and in fact consider the irreducible case by identifying:

$$\tilde{f}_z(z')=\tilde{f}_0(z)+\sum_{1\leq|\alpha|\leq k_n}\frac{1}{\alpha!}\tilde{f}_\alpha(z)(z_1'-z_1)^{\alpha_1}...(z_n'-z_n)^{\alpha_n}$$

$$\text{and} \qquad \tilde{F}_0(P(z)) + \sum_1^n \tilde{F}_i \circ P(z)(\sum_{|\alpha|=1}^{k_i} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}p_i}{\partial z^\alpha}(z'-z)^\alpha).$$

Since P is proper the continuity of the $\tilde{F}_\alpha \circ P$ entails the continuity of the $\tilde{F}_\alpha$ themselves. So we just have to check the continuity of the $\tilde{F}_\alpha \circ P$ for $\alpha = 0$ and $|\alpha| = 1$. Clearly $\tilde{F}_0 \circ P = \tilde{f}_0$ is continuous.

Disregarding D_n for a while, for all the other groups the k_i are distinct and $p_i(Z)$ contains a monomial of the form $Z_1^{k_i}$. There exists ([21]) an invariant set of real linear forms $\{L_1, \ldots, L_v\}$ and we put $p_i(X) = \sum_{j=1}^v [L_j(X)]^{k_i}$. At least one of the $L_j(X)$ contains a monomial in X_1, bringing in $p_i(X)$ a monomial in $X_1^{k_i}$ that cannot be cancelled since the p_i are defined on the reals and the k_i are even, with 2 exceptions: A_n and $I_2(p)$. For $I_2(p)$ we may choose $p_1(X) = X_1^2 + X_2^2$ and $p_2(X) = \sum_{i=1}^p (X_1 \cos 2i\theta + X_2 \sin 2i\theta)^p$ in which the coefficient of X_1^p is $\sum_{i=1}^p (\cos 2i\theta)^p \neq 0$. For A_n we may either get the result from the symmetric group $\mathcal{S}_{n+1}$ an integrity basis of which is provided by $\sum_{i=1}^{n+1} X_i^k,\ k = 1, \ldots, n+1$ or directly substituting $-\sum_{i=1}^n X_i$ to X_{n+1}, and using other monomials such as $X_1^{k-1}X_2$ when k is odd. We will not make a special study for A_n. By the way in [21] we can find for E_6, E_7, and E_8 linear forms L_i acting on $\mathbf{R}^6, \mathbf{R}^7$, and $\mathbf{R}^8$ respectively without the usual additional variables.

So $\dfrac{\partial^{k_n} p_n}{\partial z_1^{k_n}} = k_n! c_n$ for some coefficient $c_n \neq 0$, while for $j \neq n$, $\dfrac{\partial^{k_n} p_j}{\partial z_1^{k_n}} = 0$, since the greatest exponent of z_1 in $p_j(z)$ is $k_j < k_n$. Then the identification shows that $c_n \tilde{F}_n \circ P(z) = \dfrac{1}{k_n!}\tilde{f}_{k_n,0,\ldots,0}(z)$ with $c_n \neq 0$, which brings the continuity of $\tilde{F}_n \circ P$.
Assuming that the $\tilde{F}_s \circ P$ are continuous when $s > i$, since $p_i(Z)$ contains a monomial in $Z_1^{k_i}$, we have $\dfrac{\partial^{k_i} p_i}{\partial z_1^{k_i}} = k_i! c_i$ for some coefficient $c_i \neq 0$, while as above for $j < i$, $\dfrac{\partial^{k_i} p_j}{\partial z_1^{k_i}} = 0$.
The identification now gives: $\dfrac{1}{k_i!}\tilde{f}_{k_i,0,\ldots,0} = c_i \tilde{F}_i \circ P + \sum_{s>i} \tilde{F}_s \circ P \, \dfrac{1}{k_i!}\,\dfrac{\partial^{k_i} p_s}{\partial z_1^{k_i}}$.
By using the induction assumption it brings the continuity of $\tilde{F}_i \circ P$, and by decreasing induction of all the $\tilde{F}_j \circ P,\ j = 1, \ldots, n$.

Observe that at the last step we get:

$$\tilde{f}_{1,0,\ldots,0} = \frac{\partial p_1}{\partial z_1}\tilde{F}_1 \circ P + \sum_{s>1} \tilde{F}_s \circ P \, \frac{\partial p_s}{\partial z_1}.$$

Of course we might choose any $Z_i,\ i = 2, \ldots, n$ instead of Z_1, so that for $i = 1, \ldots, n$,

$$\frac{\partial \tilde{f}}{\partial z_i} = \frac{\partial p_1}{\partial z_i}\tilde{F}_1 \circ P + \sum_{s>1} \tilde{F}_s \circ P \, \frac{\partial p_s}{\partial z_i} = \sum_{s=1}^n \tilde{F}_s \circ P \, \frac{\partial p_s}{\partial z_i}.$$

On the regular image this is the condition $D^1\tilde{f} = D^1\tilde{F}D^1P$ which was to be checked.

As far as D_n is concerned if we take as basic invariant polynomials $p_j(z) = \sum_{i=1}^{n} z_i^{2j}$, $j = 1, \dots, n-1$ and $p_n(z) = z_1 z_2 \dots z_n$, we may use the above method when $1 \le j \le n-1$, and consider $\dfrac{\partial^n p_n}{\partial z_1 \dots \partial z_n} = 1$ to get the continuity of $\tilde{F}_n \circ P$.

For any finite reflection group, the coefficients $\tilde{F}_0$ and $\tilde{F}_\alpha$, $\mid \alpha \mid = 1$, belong to $\mathcal{H}^0(\mathbf{R}^n)$. By lemma 2 the field $\tilde{F}$ is in $\mathcal{H}^1(\mathbf{R}^n)$, its restriction $F = \tilde{F}_{|P(\mathbf{R}^n)}$ is 1-regular on $P(\mathbf{R}^n)$ and we may state:

Proposition 6. *Let W be a finite reflection group and P be the associated Chevalley map. If $F \in \mathcal{E}^{0,1}(P(\mathbf{R}^n))$ is 1-regular on the interior of $P(\mathbf{R}^n)$, then $F \in \mathcal{E}^1(P(\mathbf{R}^n))$.*

Remark. The coefficients of $\tilde{F}$ might be complex valued while the initial F was real valued. If we write $\tilde{F} = Re\tilde{F} + iIm\tilde{F}$, with real $Re\tilde{F}$ and $Im\tilde{F}$, on the interior of $P(\mathbf{R}^n)$ and on $P(\mathbf{R}^n)$ itself we have $Re\tilde{F} = F$ and $Im\tilde{F} = 0$. In fact $Re\tilde{F}$ is already an extension of F to $\mathbf{R}^n$ and we may replace $\tilde{F}$ by $Re\tilde{F}$, and come back from $\mathcal{H}^1(\mathbf{R}^n)$ to $\mathcal{E}^1(\mathbf{R}^n)$ by replacing the indeterminate Z by X.

Theorem 4. *The image $P(\mathbf{R}^n)$ of the Chevalley mapping has the Whitney regularity property $\mathbf{P_1}$.*

Proof. By proposition 2 it is sufficient to show that on each compact set K of $P(\mathbf{R}^n)$, the norms $|\ |_1^K$ and $\|\ \|_1^K$ are equivalent on $\mathcal{E}^1(K)$.

Let us consider a sequence $(F_k)_{k \in \mathbf{N}}$ of 1-regular Whitney fields on $P(\mathbf{R}^n)$ which is a Cauchy's sequence for the topology defined by the semi-norms $|\ |_1^K$. This sequence converges to some $F \in \mathcal{E}^{0,1}(P(\mathbf{R}^n))$.
On open subsets we may identify Whitney fields of order 1 and functions of class $\mathcal{C}^1$ and this class of differentiability is preserved by the compact convergence of functions and their first derivatives. So F is 1-regular on the interior of $P(\mathbf{R}^n)$ and satisfies the hypotheses of proposition 6. Therefore F is in $\mathcal{E}^1(P(\mathbf{R}^n))$.
This shows that $\mathcal{E}^1(P(\mathbf{R}^n))$ is complete when provided with the topology induced by the semi-norms $|\ |_1^K$. The Banach isomorphism theorem then shows the equivalence of the norms $|\ |_1^K$ and $\|\ \|_1^K$. $\diamondsuit$

Gérard P. Barbançon

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Institut de Recherche Mathematique Avancee,
7 rue Rene Descartes,
67084 Strasbourg Cedex, France

University of Texas at Austin,
Department of Mathematics,
1 University Station C1200
Austin, TX 78712-0257, USA

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