A remark on the paper: "On the existence of ground states for massless Pauli-Fierz Hamiltonians"

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1 Introduction

The purpose of this note is to correct an error in our paper [1] on the existence of ground states for massless Pauli-Fierz Hamiltonians. We will use the notation of [1]. The key point of [1] was the following lemma [1, Lemma IV.5]:

Lemma 1.1 *Let* $F \in C_0^\infty(\mathbb{R})$ *be a cutoff function with* $0 \leq F \leq 1$, $F(s) = 1$ *for* $|s| \leq \frac{1}{2}$, $F(s) = 0$ *for* $|s| \geq 1$. *Let* $F_R(x) = F(\frac{|x|}{R})$. *Then*

$$\lim_{\sigma \to 0, R \to +\infty} (\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) = 0. \tag{1.1}$$

This lemma is correct under the hypotheses in [1] but its proof was not. We explain the error in Sect. 2 and give the correct proof in Sect. 3.

2 The space $L^2(\mathbb{R}^d; B(\mathcal{H}))$

Let $\mathcal{H}$ be a separable Hilbert space. The space $\mathcal{F} := L^2(\mathbb{R}^d; B(\mathcal{H}))$ is the Banach space of weakly measurable maps:

$$T : \mathbb{R}^d \ni k \mapsto T(k) \in B(\mathcal{H}),$$

such that:

$$\|T\|_{\mathcal{F}} := (\int_{\mathbb{R}^d} \|T(k)\|^2_{B(\mathcal{H})} \mathrm{d}k)^{\frac{1}{2}} < \infty.$$

Note that such a T can be considered as an element of $B(\mathcal{H}, L^2(\mathbb{R}^d, \mathcal{H}))$ by:

$$(T\psi)(k) := T(k)\psi, \quad \psi \in \mathcal{H},$$

or equivalently:

$$(u, T\psi)_{L^2(\mathbb{R}^d, \mathcal{H})} = \int_{\mathbb{R}^d} (u(k), T(k)\psi)_{\mathcal{H}} \mathrm{d}k, \quad u \in L^2(\mathbb{R}^d, \mathcal{H}), \quad \psi \in \mathcal{H}.$$

On $L^2(\mathbb{R}^d;B(\mathcal{H}))$ we have the group $U(s)$ of isometries defined by:

$$(U(s)T)(k) := T(k-s), \text{ a.e. } k, \text{ for } s\in\mathbb{R}^d,$$

so that:

$$(u,U(s)T\psi)_{L^2(\mathbb{R}^d,\mathcal{H})} = \int_{\mathbb{R}^d}(u(k),T(k-s)\psi)_{\mathcal{H}}\mathrm{d}k.$$

Note that the function $(k,s)\mapsto(u(k),T(k-s)\psi)_{\mathcal{H}}$ is measurable and L^1 in k, so the function

$$s\mapsto(u,U(s)T\psi)_{L^2(\mathbb{R}^d,\mathcal{H})} \text{ is measurable and bounded.}$$

Hence for $F\in C_0^\infty(\mathbb{R}^d)$, one can define:

$$F(D_k)T := (2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{F}(s)U(-s)T\mathrm{d}s,$$

as a weak integral, and

$$\|F(D_k)T\|_{\mathcal{F}} \leq (2\pi)^{-d}\|T\|\int_{\mathbb{R}^d}|\widehat{F}|(s)\mathrm{d}s.$$

However the group $U(s)$ is *not strongly continuous* on $L^2(\mathbb{R}^d;B(\mathcal{H}))$ (for the same reason that the group of translations is not strongly continuous in $L^\infty(\mathbb{R}^d)$), so that if $F(0)=1$, it is not true that for an arbitrary element $T\in L^2(\mathbb{R}^d;B(\mathcal{H}))$ one has:

$$\|T-F(R^{-1}D_k)T\|_{\mathcal{F}}\to 0 \text{ when } R\to\infty. \tag{2.2}$$

We are indebted to I. Sasaki for this remark.

In the proof of Lemma IV.5, we considered the function:

$$T:\ \mathbb{R}^d\ni k\mapsto T(k)=(E-H-\omega(k))^{-1}v(k)\in B(\mathcal{H})$$

which is in $L^2(\mathbb{R}^d;B(\mathcal{H}))$ by [1, Hyp. (I2)]. We claimed that (2.2) holds for T.

By the above discussion this does not follow from the fact that $T\in L^2(\mathbb{R}^d;B(\mathcal{H}))$.

3 Proof of Lemma 1.1

As explained above, property (2.2) for $T(k)=(E-H-\omega(k))^{-1}v(k)$ does not follow from the fact that $T\in L^2(\mathbb{R}^d;B(\mathcal{H}))$.

Going over the proof of [1, Lemma IV.5], we first see that instead of $T(k)$ we can consider $\tilde{T}(k)=T(k)(K+1)^{-\frac{1}{2}}$. We will check by a direct computation that $\mathbb{R}\ni s\mapsto U(s)\tilde{T}\in\mathcal{F}$ is strongly continuous. This is done in the next two lemmas. The corrected proof of [1, LemmaIV.5] is given at the end of this section.

Lemma 3.1 *Let $\mathcal{K}$ be a separable Hilbert space and $\mathcal{H}:=\Gamma(L^2(\mathbb{R}^d,\mathrm{d}k))\otimes\mathcal{K}$. Let $\mathbb{R}^d\ni k\mapsto m(k)\in B(\mathcal{K})$ be a weakly measurable map such that:*

$$\int_{\mathbb{R}^d}\omega(k)^{-2}\|m(k)\|^2_{B(\mathcal{K})}\mathrm{d}k<\infty,$$

and let $T\in L^2(\mathbb{R}^d;B(\mathcal{H}))$ be the map:

$$\mathbb{R}^d\ni k\mapsto(E-H-\omega(k))^{-1}\mathbb{1}\otimes m(k).$$

Assume that for all $0 < C_1 < C_2$, one has:

$$\lim_{s\to 0}\int_{C_1\le |k|\le C_2} \|(K+1)^{-\frac{1}{2}}(m(k-s)-m(k))\|^2_{B(\mathcal{K})}\mathrm{d}k = 0. \tag{3.3}$$

Then:

i) $\mathrm{I\!R}^d \ni s \mapsto U(s)T \in L^2(\mathrm{I\!R}^d; B(\mathcal{H}))$ *is norm continuous.*

ii) If $F \in C_0^\infty(\mathrm{I\!R}^d)$ *satisfies* $F(0) = 1$ *and*

$$T_R = (2\pi)^{-d}\int_{\mathrm{I\!R}^d} \widehat{F}(s)U(-R^{-1}s)T\mathrm{d}s,$$

we have:

$$\|T_R - T\|_{\mathcal{F}} \to 0 \text{ when } R \to \infty.$$

Proof. To simplfy notation, in the proof below we will simply write $m(k)$ for the operator $1\!\!1 \otimes m(k)$. This should not create confusion since it will be clear from the context if $m(k)$ is considered as an operator on $\Gamma(L^2(\mathrm{I\!R}^d, \mathrm{d}k)) \otimes \mathcal{K}$ or on $\mathcal{K}$.

Set for $0 < C_1 < C_2$, $\chi_{>C_1} = 1\!\!1_{\{|k|<C_1\}}$, $\chi_{<C_2} = 1\!\!1_{\{|k|>C_2\}}$,

$$1 =: \chi_{>C_1} + \chi_{<C_2} + \chi_{C_1,C_2},$$

and $T_{<C_1}(k) = \chi_{<C_1}(k)T(k)$, $T_{>C_2}(k) = \chi_{>C_2}(k)T(k)$,

$$T =: T_{<C_1} + T_{>C_2} + R_{C_1,C_2}.$$

Then:

$$\begin{aligned}\|U(s)T - T\|_{\mathcal{F}} \le\ & \|U(s)T_{<C_1} - T_{<C_1}\|_{\mathcal{F}} + \|U(s)T_{>C_2} - T_{>C_2}\|_{\mathcal{F}} + \|U(s)R_{C_1,C_2} - R_{C_1,C_2}\|_{\mathcal{F}}\\ \le\ & 2\|T_{<C_1}\|_{\mathcal{F}} + 2\|T_{>C_2}\|_{\mathcal{F}} + \|U(s)R_{C_1,C_2} - R_{C_1,C_2}\|_{\mathcal{F}}.\end{aligned}$$

Since $\int \|T(k)\|^2_{B(\mathcal{H})}\mathrm{d}k < \infty$, we have:

$$\lim_{C_1\to 0}\|T_{<C_1}\|_{\mathcal{F}} = \lim_{C_2\to+\infty}\|T_{>C_2}\|_{\mathcal{F}} = 0. \tag{3.4}$$

Let us now fix $0 < C_1 < C_2$. We have:

$$U(s)R_{C_1,C_2} - R_{C_1,C_2} = (U(s)\chi_{C_1,C_2} - \chi_{C_1,C_2})T + U(s)\chi_{C_1,C_2}(U(s)T - T),$$

and

$$\begin{aligned}&\|(U(s)\chi_{C_1,C_2} - \chi_{C_1,C_2})T\|^2_{\mathcal{F}}\\ =\ & \textstyle\int_{\mathrm{I\!R}^d} |\chi_{C_1,C_2}(k-s) - \chi_{C_1,C_2}(k)|^2\|T(k)\|^2_{B(\mathcal{H})}\mathrm{d}k \to 0 \text{ when } s \to 0,\end{aligned} \tag{3.5}$$

by dominated convergence. Now:

$$\begin{aligned}\|U(s)\chi_{C_1,C_2}(U(s)T - T)\|^2_{\mathcal{F}} =\ & \textstyle\int_{\mathrm{I\!R}^d} \chi^2_{C_1,C_2}(k-s)\|T(k-s) - T(k)\|^2_{B(\mathcal{H})}\mathrm{d}k\\ \le\ & \textstyle\int_{C_1/2\le|k|\le 2C_2} \|T(k-s) - T(k)\|^2_{B(\mathcal{H})}\mathrm{d}k,\end{aligned} \tag{3.6}$$

if $|s| < C_1/4$. Next we have:

$$\begin{aligned} T(k-s) - T(k) = \;& (E - H - \omega(k))^{-1}(m(k-s) - m(k)) \\ & + (E - H - \omega(k))^{-1}(E - H - \omega(k-s))^{-1} m(k-s)(\omega(k-s) - \omega(k)), \end{aligned}$$

so:

$$\begin{aligned} & \|T(k-s) - T(k)\|_{B(\mathcal{H})} \\ \leq\; & \|(E - H - \omega(k))^{-1}(K+1)^{\frac{1}{2}}\|_{B(\mathcal{H})} \|(K+1)^{-\frac{1}{2}}(m(k-s) - m(k))\|_{B(\mathcal{K})} \\ & + \tfrac{1}{\omega(k-s)} \|(E - H - \omega(k))^{-1}(K+1)^{\frac{1}{2}}\|_{B(\mathcal{H})} \|(K+1)^{-\frac{1}{2}} m(k-s)\|_{B(\mathcal{K})} |s| \\ \leq\; & C_{C_1,C_2} \|(K+1)^{-\frac{1}{2}}(m(k-s) - m(k))\|_{B(\mathcal{K})} + C_{C_1,C_2} |s| \|(K+1)^{-\frac{1}{2}} m(k-s)\|_{B(\mathcal{K})}, \end{aligned}$$

uniformly for $C_1/2 \leq |k| \leq 2C_2$ and $|s| < C_1/4$.

By (3.3), we obtain:

$$\lim_{s \to 0} \int_{C_1/2 \leq |k| \leq 2C_2} \|T(k-s) - T(k)\|^2_{B(\mathcal{H})} \mathrm{d}k = 0. \tag{3.7}$$

To prove *i)* we first fix $C_1 \ll 1$ and $C_2 \gg 1$ and then let $s \to 0$ using (3.5) and (3.7).

Statement *ii)* follows from *i)*, using:

$$\|T_R - T\|_{\mathcal{F}} \leq (2\pi)^{-d} \int |\widehat{F}|(s) \|U(-R^{-1}s)T - T\|_{\mathcal{F}} \mathrm{d}s. \quad \Box$$

Lemma 3.2 *Let $\mathbb{R}^d \ni k \mapsto m(k) \in B(\mathcal{K})$ be a weakly measurable map such that for all $0 < C_1 < C_2$ one has:*

$$\int_{C_1 \leq |k| \leq C_2} \|m(k)\|^2_{B(\mathcal{K})} \mathrm{d}k < \infty.$$

Let $R \geq 0$ be a compact selfadjoint operator on $\mathcal{K}$. Then for all $0 < C_1 < C_2$ one has:

$$\lim_{s \to 0} \int_{C_1 \leq |k| \leq C_2} \|R(m(k-s) - m(k))\|^2_{B(\mathcal{K})} \mathrm{d}k = 0.$$

Proof. Let us fix $0 < C_1 < C_2$ and let $\chi = \mathbb{1}_{\{C_1/2 \leq |k| \leq 2C_2\}}$, $\tilde{m}(k) := \chi(k) m(k)$. We have:

$$\tilde{m}(k-s) - \tilde{m}(k) = \chi(k-s)(m(k-s) - m(k)) + (\chi(k-s) - \chi(k)) m(k).$$

If $C_1 \leq |k| \leq C_2$ and $|s| \leq C_1/2$, we have $C_1/2 \leq |k-s| \leq 2C_2$ and hence:

$$\begin{aligned} & \|R(m(k-s) - m(k))\| \\ =\; & \|\chi(k-s) R(m(k-s) - m(k))\| \\ \leq\; & \|R(\tilde{m}(k-s) - \tilde{m}(k))\| + |\chi(k-s) - \chi(k)| \|R m(k)\|. \end{aligned} \tag{3.8}$$

By dominated convergence we have:

$$\lim_{s \to 0} \int_{C_1 \leq |k| \leq C_2} |\chi(k-s) - \chi(k)|^2 \|R m(k)\|^2 \mathrm{d}k = 0,$$

so using (3.8) it suffices to prove:

$$\lim_{s\to 0}\int \|R(\tilde{m}(k-s)-\tilde{m}(k))\|^2_{B(\mathcal{K})}\mathrm{d}k = 0. \tag{3.9}$$

Since $k\mapsto \tilde{m}(k)$ is weakly measurable, so is $k\to \tilde{m}^*(k)$. So we can consider the map $M\in B(\mathcal{K}, L^2(\mathbb{R}^d;\mathcal{K}))$ defined by:

$$(M\psi)(k) := \tilde{m}^*(k)\psi,\quad \psi\in\mathcal{K},$$

and

$$\|M\psi\|^2 = \int \|\tilde{m}^*(k)\psi\|^2_{\mathcal{K}}\mathrm{d}k \le \|\psi\|^2\int \|\tilde{m}(k)\|^2_{B(\mathcal{K})}\mathrm{d}k.$$

The group $U(s)$ of translations on $L^2(\mathbb{R}^d;\mathcal{K})$ defined by:

$$U(s)u(k) := u(k-s),\quad u\in L^2(\mathbb{R}^d;\mathcal{K})$$

is strongly continuous. Hence for each $\psi\in\mathcal{K}$, we have:

$$\lim_{s\to 0}\|U(s)M\psi - M\psi\|^2 = \lim_{s\to 0}\int \|(\tilde{m}^*(k-s)-\tilde{m}^*(k))\psi\|^2_{\mathcal{K}}\mathrm{d}k = 0. \tag{3.10}$$

Let us fix $\epsilon>0$. Since R is compact, we can write:

$$R = \sum_{i=1}^{N}\lambda_i|e_i\rangle\langle e_i| + R_\epsilon,$$

where $\lambda_i\ge 0$, $\{e_i\}_{i\in\mathbb{N}}$ is an o.n. basis of $\mathcal{K}$ and $\|R_\epsilon\|_{B(\mathcal{K})}\le\epsilon$. This yields:

$$\begin{aligned} &\|R(\tilde{m}(k-s)-\tilde{m}(k))\|_{B(\mathcal{K})}\\ \le\ & \textstyle\sum_{i=1}^{N}\lambda_i\|(\tilde{m}^*(k-s)-\tilde{m}^*(k))e_i\|_{\mathcal{K}} + \|R_\epsilon\|_{B(\mathcal{K})}(\|\tilde{m}(k-s)\|_{B(\mathcal{K})}+\|\tilde{m}(k)\|_{B(\mathcal{K})}).\end{aligned}$$

Fixing first $\epsilon>0$ and letting then $s\to 0$ using (3.10) we obtain (3.9). This completes the proof of the lemma. □

Proof of Lemma 1.1 Recall that if B is a bounded operator on $\mathfrak{h}$ with distribution kernel $b(k,k')$, we have

$$(u,\mathrm{d}\Gamma(B)u) = \int\int b(k,k')(a(k)u, a(k')u)\mathrm{d}k\mathrm{d}k',\ u\in D(N^{\frac{1}{2}}).$$

Using this identity, we obtain

$$(\psi_\sigma, \mathrm{d}\Gamma(1-F_R)\psi_\sigma) = (a(\cdot)\psi_\sigma, (1-F(\frac{|D_k|}{R}))a(\cdot)\psi_\sigma)_{L^2(\mathbb{R}^d,\mathrm{d}k;\mathcal{H})}.$$

By [1, Prop. IV.4], we have:

$$a(\cdot)\psi_\sigma = (E-H-\omega(\cdot))^{-1}v(\cdot)\psi_\sigma + o(\sigma^0)\text{ in } L^2(\mathbb{R}^d;\mathcal{H}),$$

hence:

$$(\psi_\sigma, \mathrm{d}\Gamma(1-F_R)\psi_\sigma) = ((E-H-\omega(\cdot))^{-1}v(\cdot)\psi_\sigma, (1-F(\frac{|D_k|}{R}))(E-H-\omega(\cdot))^{-1}v(\cdot)\psi_\sigma) + o(\sigma^0),$$

uniformly in R. This yields:

$$
\begin{aligned}
& (\psi_\sigma, \mathrm{d}\Gamma(1-F_R)\psi_\sigma) \\
\leq \; & \|(E-H-\omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d,B(\mathcal{H}))}\times \\
& \|(1-F(\tfrac{|D_k|}{R}))(E-H-\omega(\cdot))^{-1}v(\cdot)(K+1)^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^d,B(\mathcal{H}))}\times\|(K+1)^{\frac{1}{2}}\psi_\sigma\|_{\mathcal{H}}+o(\sigma^0).
\end{aligned}
$$

By [1, Lemma IV.1], we have:

$$
\|(K+1)^{\frac{1}{2}}\psi_\sigma\|_{\mathcal{H}} \leq (\psi_\sigma, H_0\psi_\sigma)^{\frac{1}{2}} \leq C, \text{ uniformly in } \sigma > 0,
$$

hence:

$$
(\psi_\sigma, \mathrm{d}\Gamma(1-F_R)\psi_\sigma) \leq C\|(1-F(\frac{|D_k|}{R}))(E-H-\omega(\cdot))^{-1}v(\cdot)(K+1)^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^d,B(\mathcal{H}))}+o(\sigma^0),
$$

uniformly in σ, R.

We apply now Lemma 3.1 to $m(k) = v(k)(K+1)^{-\frac{1}{2}}$, checking its hypotheses: first by [1, Hyp. (I1)], the map $k \mapsto m(k) \in B(\mathcal{K})$ is weakly measurable, and by [1, Hyp. (I3)], we have

$$
\int \omega(k)^{-2}\|m(k)\|^2_{B(\mathcal{K})}\mathrm{d}k < \infty.
$$

Moreover again by [1, Hyp. (I1)], we have:

$$
\int_{C_1\leq|k|\leq C_2} \|m(k)\|^2_{B(\mathcal{K})}\mathrm{d}k < \infty, \quad \forall 0 < C_1 < C_2.
$$

By [1, Hyp. (H0)], we can hence apply Lemma 3.2 to $m(k)$ for $R = (K+1)^{-\frac{1}{2}}$. It follows from Lemma 3.2 that hypothesis (3.3) of Lemma 3.1 is satisfied. Applying then Lemma 3.1, we obtain that:

$$
\lim_{R\to\infty} \|(1-F(\frac{|D_k|}{R}))(E-H-\omega(\cdot))^{-1}v(\cdot)(K+1)^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^d,B(\mathcal{H}))} = 0,
$$

which completes the proof of Lemma 1.1. □

References

[1] On the existence of ground states for massless Pauli-Fierz Hamiltonians. Ann. Henri Poincaré 1 (2000), no. 3, p 443-45.