The Green-Kubo formula for the spin-fermion system

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Dedicated to Jean Michel Combes on the occasion of his sixtyfifth birthday

Abstract

The spin-fermion model describes a two level quantum system $\mathcal{S}$ (spin $1/2$) coupled to finitely many free Fermi gas reservoirs $\mathcal{R}_j$ which are in thermal equilibrium at inverse temperatures β_j. We consider non-equilibrium initial conditions where not all β_j are the same. It is known that, at small coupling, the combined system $\mathcal{S}+\sum_j \mathcal{R}_j$ has a unique non-equilibrium steady state (NESS) characterized by strictly positive entropy production. In this paper we study linear response in this NESS and prove the Green-Kubo formula and the Onsager reciprocity relations for heat fluxes generated by temperature differentials.

1 Introduction

This is the third in a series of papers dealing with linear response theory in quantum statistical mechanics. In the first two papers in the series [JOP1, JOP2] we have given an abstract axiomatic derivation of the Green-Kubo formula for the heat fluxes generated by temperature differentials. In this paper we verify that this axiomatic derivation is applicable to the spin-fermion model (abbreviated SFM). We shall assume that the reader is familiar with general aspects of linear response theory discussed in the introduction of [JOP1].

The Green-Kubo formula is one of the pillars of non-equilibrium statistical mechanics and is discussed in many places in physics literature (see e.g. [KTH]). A mathematical justification of this formula is one of the outstanding open problems in mathematical physics [Si]. In the literature, most existing results concern currents induced by mechanical driving forces such as time-dependent electric or magnetic fields (see [NVW, GVV, BGKS] for references and additional information). In contrast, there are very few results dealing with fluxes generated by thermodynamical driving forces such as temperature differentials. The central difficulty is that a mathematically rigorous study of linear response to thermodynamical perturbations requires as input a detailed understanding of structural and ergodic properties of non-equilibrium steady states (NESS). In the papers [JOP1, JOP2] we have bypassed this difficulty by assuming the necessary regularity properties as *axioms*. The general axiomatic derivation of the Green-Kubo formula in [JOP1, JOP2] has led to some new insights concerning the mathematical structure of non-equilibrium quantum statistical mechanics. Concerning applications to concrete models, it reduced the proof of the Green-Kubo formula to the study of regularity properties of NESS.

In most cases, the study of NESS of physically relevant models is beyond existing mathematical techniques. The information necessary to study linear response theory has been obtained only recently and only for a handful of models [JP3, JP4, AH, AP, FMU]. To the best of our knowledge the SFM and its obvious generalizations are the first class of non-trivial models in quantum statistical mechanics for which the Green-Kubo formula and the Onsager reciprocity relations have been proven. We would also like to mention related works [AJPP1, AJPP2] where the Green-Kubo formula was established for some exactly solvable quasi-free models. Linear response theory for the quantum Markovian semigroup describing the dynamics of the SFM in the van Hove weak coupling limit was studied by Lebowitz and Spohn in [LeSp] and this work has motivated our program. The Green-Kubo formula for a class of open systems in classical non-equilibrium statistical mechanics has been established in [RBT].

The rest of this introduction is organized as follows. In Subsection 1.1 we quickly review a few basic notions and results of algebraic quantum statistical mechanics. This subsection is primarily intended for notational and reference purposes. The interested reader may consult [Ru3, JP4, FMU, AJPP1] for recent reviews of non-equilibrium algebraic quantum statistical mechanics. In Subsection 1.2 we review the abstract axiomatic derivation of the Green-Kubo formula given in [JOP1, JOP2]. In this paper we will also give a new proof of the main results of [JOP1, JOP2] (see Section 2). This new proof emphasizes the important connection between linear response theory and McLennan-Zubarev dynamical ensembles [M, Zu, ZMR1, ZMR2, TM] (this point will be further discussed in [JOPR]). In Subsection 1.3 we introduce SFM and in Subsection 1.4 we state our main results. The results of this paper can be used to refine the existing results concerning the thermodynamics of SFM and we discuss this point in Subsection 1.5. Finally, some generalizations of our model and results are discussed in Subsection 1.6.

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1.1 Basic notions

A quantum dynamical system is a triple $(\mathcal{O}, \tau, \omega)$ where $\mathcal{O}$ is a C^*-algebra (usually called the algebra of observables) with identity $\mathbb{1}$, τ is a C^*-dynamics on $\mathcal{O}$, and ω is the initial (reference) state of the system. We denote by $\mathcal{N}_\omega$ the set of all ω-normal states on $\mathcal{O}$ and by $\mathcal{I}$ the set of all τ-invariant states on $\mathcal{O}$.

An anti-linear involutive $*$-automorphism $\Theta : \mathcal{O} \to \mathcal{O}$ is called time-reversal of $(\mathcal{O}, \tau, \omega)$ if $\Theta \circ \tau^t = \tau^{-t} \circ \Theta$ for all $t \in \mathbb{R}$ and $\omega(\Theta(A)) = \omega(A^*)$ for all $A \in \mathcal{O}$. More generally, a state η on $\mathcal{O}$ is called time-reversal invariant if $\eta(\Theta(A)) = \eta(A^*)$ for all $A \in \mathcal{O}$.

Thermal equilibrium states of $(\mathcal{O}, \tau, \omega)$ are characterized by the KMS property. Let $\beta > 0$ be the inverse temperature. A state $\omega_{\rm eq}$ on $\mathcal{O}$ is called (τ, β)-KMS if for all $A, B \in \mathcal{O}$ there exists a function $F_{A,B}(z)$, analytic in the strip $0 < \operatorname{Im} z < \beta$, bounded and continuous on its closure, and satisfying the KMS-boundary condition

$$F_{A,B}(t) = \omega_{\rm eq}(A\tau^t(B)), \qquad F_{A,B}(t + \mathrm{i}\beta) = \omega_{\rm eq}(\tau^t(B)A).$$

The three line theorem yields that

$$|F_{A,B}(z)| \leq \|A\|\|B\|, \tag{1.1}$$

for z in the closed strip $0 \leq \operatorname{Im} z \leq \beta$. We shall write $\omega_{\rm eq}(A\tau^z(B)) = F_{A,B}(z)$ for such z.

If ω is a (τ, β)-KMS state one expects that

$$\mathrm{w}^* - \lim_{t\to\pm\infty} \eta \circ \tau^t = \omega,$$

for all states $\eta \in \mathcal{N}_\omega$. This property of return to equilibrium is a manifestation of the zeroth law of thermodynamics. It has been established for N-level systems coupled to free reservoirs under fairly general assumptions (see [JP6, BFS, DJ, FM])

Non-equilibrium statistical mechanics deals with the case where ω is not a KMS state (or more precisely not normal w.r.t. any KMS state of $(\mathcal{O}, \tau, \omega)$). The non-equilibrium steady states (NESS) of $(\mathcal{O}, \tau, \omega)$ are defined as the weak-$*$ limit points of the net

$$\left\{ \frac{1}{T} \int_0^T \omega \circ \tau^s \mathrm{d}s \;\Big|\; T > 0 \right\},$$

as $T \uparrow \infty$. The set of NESS, denoted by Σ_+, is non-empty and $\Sigma_+ \subset \mathcal{I}$. For information about structural properties of NESS we refer the reader to [Ru1, Ru2, Ru3, JP3, JP4, AJPP1].

In typical applications to open systems one expects that Σ_+ consists of a single NESS ω_+ and that

$$\mathrm{w}^* - \lim_{t\to+\infty} \eta \circ \tau^t = \omega_+,$$

holds for all $\eta \in \mathcal{N}_\omega$. Such strong approach to the NESS is a difficult ergodic problem and has been rigorously established only for a few models.

Throughout the paper we will use the shorthands

$$\mathfrak{L}(A, B, t) = \frac{1}{\beta} \int_0^t \mathrm{d}s \int_0^\beta \mathrm{d}u \, \omega_{\rm eq}(\tau^s(A)\tau^{\mathrm{i}u}(B)), \tag{1.2}$$

and

$$\begin{aligned} \mathfrak{L}(A, B) &= \lim_{t\to+\infty} \mathfrak{L}(A, B, t), \\ \mathcal{L}(A, B) &= \lim_{t\to+\infty} \frac{1}{2} \int_{-t}^t \omega_{\rm eq}(\tau^s(A)B)\mathrm{d}s, \end{aligned} \tag{1.3}$$

whenever the limits exists.

We shall freely use the well-known properties of KMS-states discussed in classical references [BR1, BR2]. In particular, we will need the following result:

Theorem 1.1 *Assume that ω is a (τ, β)-KMS state such that for all $A, B \in \mathcal{O}$,*

$$\lim_{|t|\to\infty} \omega(A\tau^t(B)) = \omega(A)\omega(B).$$

Then:
(1) *For all $A, B \in \mathcal{O}$,*

$$\lim_{t\to+\infty} \int_{-t}^{t} \omega([A, \tau^s(B)])\mathrm{d}s = 0.$$

(2) *Assume in addition that $(\mathcal{O}, \tau, \omega)$ is time-reversal invariant and that $A, B \in \mathcal{O}$ are two self-adjoint observables which are both even or odd under Θ. Then*

$$\lim_{t\to+\infty} \left[\mathfrak{L}(A, B, t) - \int_{-t}^{t} \omega(A\tau^s(B))\mathrm{d}s \right] = 0.$$

The first part of this theorem is a classical result (see Theorem 5.4.12 in [BR2]). The second part is proven in [JOP1, JOP2].

In the sequel $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded operators on a Hilbert space $\mathcal{H}$.

1.2 Abstract Green-Kubo formula

In this subsection we review the abstract derivation of the Green-Kubo formula given in [JOP1, JOP2]. In view of the specific models we will study in this paper, we consider the abstract setup where a "small" (finite dimensional) quantum system $\mathcal{S}$ is coupled to finitely many reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_M$. For a more general framework we refer the reader to Section 5 in [JOP2].

The system $\mathcal{S}$ is described by the finite dimensional Hilbert space $\mathcal{H}_\mathcal{S}$ and the Hamiltonian $H_\mathcal{S}$. Its algebra of observables is $\mathcal{O}_\mathcal{S} = \mathcal{B}(\mathcal{H}_\mathcal{S})$ and its dynamics is

$$\tau_\mathcal{S}^t(A) = \mathrm{e}^{\mathrm{i}tH_\mathcal{S}} A \mathrm{e}^{-\mathrm{i}tH_\mathcal{S}}.$$

A convenient reference state of the system $\mathcal{S}$ is

$$\omega_\mathcal{S}(A) = \frac{1}{\dim \mathcal{H}_\mathcal{S}} \mathrm{Tr}(A),$$

but none of our results depends on this specific choice.

The reservoir $\mathcal{R}_j$ is described by the quantum dynamical system $(\mathcal{O}_j, \tau_j, \omega_j)$. We assume that reservoir is in thermal equilibrium at inverse temperature β_j, i.e., that ω_j is a (τ_j, β_j)-KMS state on $\mathcal{O}_j$. The complete reservoir system $\mathcal{R} = \sum_j \mathcal{R}_j$ is described by the quantum dynamical system $(\mathcal{O}_\mathcal{R}, \tau_\mathcal{R}, \omega_\mathcal{R})$ where

$$\mathcal{O}_\mathcal{R} = \otimes_{j=1}^M \mathcal{O}_j, \qquad \tau_\mathcal{R} = \otimes_{j=1}^M \tau_j, \qquad \omega_\mathcal{R} = \otimes_{j=1}^M \omega_j.$$

Since we are interested in the non-equilibrium statistical mechanics, we shall always assume that $M \geq 2$.

Notation. In the sequel, whenever the meaning is clear within the context, we will write A for the operators $A \otimes I$, $I \otimes A$.

In absence of coupling the joint system $\mathcal{S} + \mathcal{R}$ is described by the quantum dynamical system $(\mathcal{O}, \tau_0, \omega)$, where

$$\mathcal{O} = \mathcal{O}_\mathcal{S} \otimes \mathcal{O}_\mathcal{R}, \qquad \tau_0 = \tau_\mathcal{S} \otimes \tau_\mathcal{R}, \qquad \omega = \omega_\mathcal{S} \otimes \omega_\mathcal{R}.$$

We denote by δ_j the generator of τ_j and by

$$\delta_0 = \mathrm{i}[H_{\mathcal{S}},\,\cdot\,] + \sum_{j=1}^{M} \delta_j,$$

the generator of τ_0.

Let $V \in \mathcal{O}$ be a self-adjoint perturbation describing the coupling between $\mathcal{S}$ and $\mathcal{R}$ and let τ be the C^*-dynamics generated by

$$\delta = \delta_0 + \mathrm{i}[V,\,\cdot\,].$$

The coupled joint system $\mathcal{S}+\mathcal{R}$ is described by the quantum dynamical system $(\mathcal{O}, \tau, \omega)$.

Let $\beta_{\rm eq} > 0$ be a given reference (equilibrium) inverse temperature. Since we are interested in linear response theory, without loss of generality we may restrict the inverse temperatures β_j of the reservoirs to an interval $(\beta_{\rm eq} - \epsilon, \beta_{\rm eq} + \epsilon)$, where $0 < \epsilon < \beta_{\rm eq}$ is a small number. For our purposes the size of ϵ is not relevant. Our first assumption is:

(G1) The reference states of $\mathcal{R}_j$ are parametrized by $\beta_j \in (\beta_{\rm eq} - \epsilon, \beta_{\rm eq} + \epsilon)$ and ω_j is the *unique* (τ_j, β_j)-KMS state on $\mathcal{O}_j$.

We introduce the thermodynamical forces

$$X_j = \beta_{\rm eq} - \beta_j,$$

and set $X = (X_1, \ldots, X_M)$. The vector X uniquely describes the initial state of the system (note that the value $X = 0$ corresponds to the equilibrium case where all β_j are the same and equal to $\beta_{\rm eq}$). The restriction $\beta_j \in (\beta_{\rm eq} - \epsilon, \beta_{\rm eq} + \epsilon)$ is equivalent to $|X|_+ < \epsilon$, where $|X|_+ = \max |X_j|$. We set $\mathbb{I}_\epsilon = \{X \in \mathbb{R}^M \,|\, |X|_+ < \epsilon\}$, $D_\epsilon = \{X \in \mathbb{C}^M \,|\, |X|_+ < \epsilon\}$. We shall explicitly indicate the dependence of the reference states on X by denoting $\omega_{X_j} = \omega_j$, $\omega_{\mathcal{R}X} = \omega_{X_1} \otimes \cdots \otimes \omega_{X_M}$, and

$$\omega_X^{(0)} = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}X}.$$

We denote by $\mathcal{N}_X$ the set of all $\omega_X^{(0)}$-normal states on $\mathcal{O}$.

We now describe a particular state in $\mathcal{N}_X$ which will play a central role in our study of linear response theory. Consider the C^*-dynamics $\sigma_X^{(0)}$ generated by

$$\delta_X^{(0)} = \sum_j (1 - X_j/\beta_{\rm eq})\delta_j.$$

The state $\omega_X^{(0)}$ is the unique $(\sigma_X^{(0)}, \beta_{\rm eq})$-KMS state on $\mathcal{O}$. Let σ_X be the C^*-dynamics on $\mathcal{O}$ generated by

$$\delta_X = \delta_X^{(0)} + \mathrm{i}[H_{\mathcal{S}} + V,\,\cdot\,] = \delta - \sum_j \frac{X_j}{\beta_{\rm eq}} \delta_j.$$

The Araki perturbation theory [Ar, BR2, DJP] yields that there exists a unique $(\sigma_X, \beta_{\rm eq})$-KMS state on $\mathcal{O}$. We denote this state by ω_X. The states ω_X and $\omega_X^{(0)}$ are mutually normal. Note that $\omega_{X=0}$ is the unique $(\tau, \beta_{\rm eq})$-KMS state on $\mathcal{O}$. We denote this state by $\omega_{\rm eq}$ and assume:

(G2) For all $A, B \in \mathcal{O}$,

$$\lim_{|t|\to\infty} \omega_{\rm eq}(\tau^t(A)B) = \omega_{\rm eq}(A)\omega_{\rm eq}(B).$$

In the next assumption we postulate the existence of NESS w.r.t. the reference state ω_X

(G3) For all $X \in \mathbb{I}_\epsilon$ there exists a state ω_{X+} on $\mathcal{O}$ such that for all $A \in \mathcal{O}$,

$$\lim_{t\to+\infty} \omega_X(\tau^t(A)) = \omega_{X+}(A).$$

As we have already remarked in Subsection 1.1, under normal conditions one expects that the NESS is independent of the choice of reference state in $\mathcal{N}_X$, i.e., that for all $\eta \in \mathcal{N}_X$ and $A \in \mathcal{O}$, $\lim_{t\to+\infty} \eta(\tau^t(A)) = \omega_{X+}(A)$. We however do not need such an assumption in our derivation of the abstract Green-Kubo formula.

Our next assumption deals with time-reversal invariance.

(G4) There exists a time-reversal Θ of $(\mathcal{O}, \tau_0)$ such that $\Theta(V) = V$ and $\Theta \circ \tau_j^t = \tau_j^{-t} \circ \Theta$ for all j.

To define heat fluxes observables we need

(G5) For all j, $V \in \mathrm{Dom}\,(\delta_j)$.

The observable describing the heat flux out of $\mathcal{R}_j$ is

$$\Phi_j = \delta_j(V).$$

It is not difficult to show (see [JP4]) that

$$\sum_{j=1}^M \omega_{X+}(\Phi_j) = 0,$$

which is the first law of thermodynamics (conservation of energy). The entropy production of the NESS ω_{X+} is defined by

$$\mathrm{Ep}(\omega_{X+}) = \sum_{j=1}^M X_j \omega_{X+}(\Phi_j),$$

and

$$\mathrm{Ep}(\omega_{X+}) \geq 0,$$

see [Ru2, JP2]. The heat flux observables are odd under time-reversal, i.e., if (G4) holds, then

$$\Theta(\Phi_j) = -\Phi_j. \tag{1.4}$$

An observable $A \in \mathcal{O}$ is called *centered* if $\omega_X(A) = 0$ for all $X \in \mathbb{I}_\epsilon$. We denote by $\mathcal{C}$ the set of all centered observables. If (G1) and (G4) hold, then it is not difficult to show that the state ω_X is time-reversal invariant (see Lemma 3.1 in [JOP1]). This fact and (1.4) imply $\omega_X(\Phi_j) = -\omega_X(\Phi_j)$, and so $\Phi_j \in \mathcal{C}$.

It is an important fact that the heat flux observables are centered irrespectively of the time-reversal assumption. The following result was proven in [JOP2].

Proposition 1.2 *If (G1) and (G5) hold, then $\Phi_j \in \mathcal{C}$ for all j.*

The key result in the abstract derivation of the Green-Kubo formula is the following *finite time linear response formula* proven in [JOP1, JOP2]. Recall that $\mathfrak{L}(A, B, t)$, $\mathfrak{L}(A, B)$, $\mathcal{L}(A, B)$, are defined by (1.2) and (1.3). Set

$$\mathcal{O}_{\mathrm{c}} = \left(\cap_{j=1}^M \mathrm{Dom}\,(\delta_j)\right) \cap \mathcal{C}.$$

Theorem 1.3 *Suppose that Assumptions (G1) and (G5) hold and let $A \in \mathcal{O}_{\rm c}$. Then for all $t \in \mathbb{R}$ the function*

$$X \mapsto \omega_X(\tau^t(A)),$$

is differentiable at $X = 0$ and

$$\partial_{X_j}\omega_X(\tau^t(A))\big|_{X=0} = \mathfrak{L}(A, \Phi_j, t).$$

In Section 2 we shall give a new proof of Theorem 1.3 which is different then the original argument in [JOP1, JOP2] and which will play an important role in future developments [JOPR].

To derive the Green-Kubo formula from Theorem 1.3 we need the concept of regular observable. An observable A is called regular if the limit and derivative in the expressions

$$\lim_{t\to+\infty} \partial_{X_j}\omega_X(\tau^t(A))\big|_{X=0},$$

can be interchanged. More precisely:

Definition 1.4 *Suppose that (G1) and (G3) hold. Let $A \in \mathcal{O}$ be an observable such that the function*

$$X \mapsto \omega_X(\tau^t(A)),$$

is differentiable at $X = 0$ for all t. We call such an observable regular if the function

$$X \mapsto \omega_{X+}(A),$$

is also differentiable at $X = 0$ and for all j,

$$\lim_{t\to+\infty} \partial_{X_j}\omega_X(\tau^t(A))\big|_{X=0} = \partial_{X_j}\omega_{X+}(A)\big|_{X=0}.$$

In study of concrete models one of the key steps is verification that physically relevant observables like heat fluxes are regular. Our justification of this step will be based on the following general result.

Proposition 1.5 *Suppose that Assumptions (G1) and (G3) hold. Let $A \in \mathcal{O}$ be an observable such that for some $\epsilon > 0$ and all $t \geq 0$ the functions*

$$X \mapsto \omega_X(\tau^t(A)), \tag{1.5}$$

have an analytic extension to D_ϵ satisfying

$$\sup_{X\in D_\epsilon, t\geq 0} \left|\omega_X(\tau^t(A))\right| < \infty.$$

Then for all $X \in D_\epsilon$ the limit

$$h(X) = \lim_{t\to+\infty} \omega_X(\tau^t(A)),$$

exists and is an analytic function on D_ϵ. Moreover, as $t \to +\infty$, all derivatives of the functions (1.5) converge uniformly on compact subsets of D_ϵ to the corresponding derivatives of $h(X)$.

Proof. This result follows from the multivariable Vitali theorem. We sketch the proof for the reader convenience.

Set $h_t(X) = \omega_X(\tau^t(A))$. For $0 < \rho < \epsilon$ we denote

$$\mathbb{T}_\rho = \{X \in \mathbb{C}^M \,|\, |X_j| = \rho \text{ for all } j\}.$$

The Cauchy integral formula for polydisk yields that for $X \in D_\rho$,

$$h_t(X) = \frac{1}{(2\pi \mathrm{i})^M} \int_{\mathbb{T}_\rho} \frac{h_t(\xi_1, \ldots, \xi_M)}{(\xi_1 - X_1) \cdots (\xi_M - X_M)} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_M. \tag{1.6}$$

It follows that the family of functions $\{h_t\}_{t \geq 0}$ is equicontinuous on $D_{\rho'}$ for any $\rho' < \rho$. Hence, by the Arzela-Ascoli theorem, for any $\rho' < \rho$ the set $\{h_t\}_{t \geq 0}$ is precompact in the Banach space $C(\overline{D}_{\rho'})$ of all continuous functions on $\overline{D}_{\rho'}$ equipped with the sup norm. The Cauchy integral formula (1.6), where now $X \in D_{\rho'}$ and the integral is over $\mathbb{T}_{\rho'}$, yields that any limit in $C(\overline{D}_{\rho'})$ of the net $\{h_t\}$ as $t \to +\infty$ is an analytic function in $D_{\rho'}$. Assumption (G3) implies that any two limit functions coincide for X real, and hence they are identical. This yields the first part of the theorem. The convergence of partial derivatives of $h_t(X)$ is an immediate consequence of the Cauchy integral formula. □

The next two theorems are an immediate consequence of Theorem 1.3.

Theorem 1.6 *Suppose that Assumptions (G1), (G3) and (G5) hold.*
(1) *Let $A \in \mathcal{O}_{\rm c}$ be a regular observable. Then*

$$\partial_{X_j} \omega_{X+}(A)\big|_{X=0} = \mathfrak{L}(A, \Phi_j). \tag{1.7}$$

(2) *If in addition (G2) and (G4) hold and $A \in \cap_j \mathrm{Dom}\,(\delta_j)$ is a regular self-adjoint observable such that $\Theta(A) = -A$, then*

$$\partial_{X_j} \omega_{X+}(A)\big|_{X=0} = \mathcal{L}(A, \Phi_j). \tag{1.8}$$

Relation (1.7) is the Green-Kubo formula without the time reversal assumption. Relation (1.8), which follows from (1.7) and Part (2) of Theorem 1.1, is the Green-Kubo formula in the standard form.

Specializing Theorem 1.6 to the heat-flux observables we derive

Theorem 1.7 *Suppose that Assumptions (G1), (G3) and (G5) hold and that $\Phi_k \in \cap_j \mathrm{Dom}\,(\delta_j)$. Then:*
(1) *The kinetic transport coefficients*

$$L_{kj} = \partial_{X_j} \omega_{X+}(\Phi_k)\big|_{X=0},$$

satisfy

$$L_{kj} = \mathfrak{L}(\Phi_k, \Phi_j).$$

(2) *If in addition (G2) and (G4) hold, then*

$$L_{kj} = \mathcal{L}(\Phi_k, \Phi_j), \tag{1.9}$$

and

$$L_{kj} = L_{jk}. \tag{1.10}$$

The Onsager reciprocity relations (1.10) follow from (1.9) and Part (1) of Theorem 1.1.

1.3 Spin-fermion model

The spin-fermion model is an example of abstract $\mathcal{S} + \mathcal{R}$ model which describes a two level quantum system (spin $1/2$) coupled to M free Fermi gas reservoirs. This model—a paradigm of open quantum system—has been much studied and we shall be brief in its description. The reader not familiar with the model may consult [JP3] or any of the references [Da, BR1, BR2, LeSp, JP4] for additional information.

The small system $\mathcal{S}$ is described by the Hilbert space $\mathcal{H}_\mathcal{S} = \mathbb{C}^2$ and the Hamiltonian $H_\mathcal{S} = \sigma_z$ (we denote the usual Pauli matrices by $\sigma_x, \sigma_y, \sigma_z$).

The reservoir $\mathcal{R}_j$ is a free Fermi gas in thermal equilibrium at inverse temperature β_j. It is described by the quantum dynamical system $(\mathcal{O}_j, \tau_j, \omega_j)$, where $\mathcal{O}_j = \mathrm{CAR}(\mathfrak{h}_j)$ is the CAR algebra over a single fermion Hilbert space $\mathfrak{h}_j$, the C^*-dynamics τ_j^t is the group of Bogoliubov $*$-automorphisms generated by a single particle Hamiltonian h_j and ω_j is the unique (τ_j, β_j)-KMS state on $\mathcal{O}_j$. The assumption (G1) is automatically satisfied.

Let

$$V_j = \sigma_x \otimes \varphi_j(\alpha_j), \tag{1.11}$$

where $\alpha_j \in \mathfrak{h}_j$ is a given vector (sometimes called "form-factor"), and

$$\varphi_j(\alpha_j) = \frac{1}{\sqrt{2}}(a_j(\alpha_j) + a_j^*(\alpha_j)) \in \mathcal{O}_j,$$

is the field operator associated to α_j. The interaction of $\mathcal{S}$ with $\mathcal{R}_j$ is described by λV_j where $\lambda \in \mathbb{R}$ is the coupling constant. The complete interaction between $\mathcal{S}$ and $\mathcal{R}$ is described by

$$V_\lambda = \lambda \sum_{j=1}^{M} V_j.$$

In the sequel we shall explicitly indicate the λ-dependence by writing $\delta_\lambda = \delta$, $\tau_\lambda = \tau$, $\omega_{\lambda X} = \omega_X$, etc.

The spin-fermion system is time-reversal invariant. Indeed, for all j there exists a complex conjugation c_j on $\mathfrak{h}_j$ which commutes with h_j and satisfies $c_j\alpha_j = \alpha_j$. The map $\Theta_j(a(f_j)) = a(c_j f_j)$ uniquely extends to an involutive anti-linear $*$-automorphism of $\mathcal{O}_j$ such that $\Theta_j \circ \tau_j^t = \tau_j^{-t} \circ \Theta_j$. Let $\Theta_\mathcal{S}$ be the standard complex conjugation on $\mathcal{O}_\mathcal{S}$. Obviously, $\Theta_\mathcal{S}(\sigma_z) = \sigma_z$, $\Theta_\mathcal{S}(\sigma_x) = \sigma_x$, and in particular $\Theta_\mathcal{S} \circ \tau_\mathcal{S}^t = \tau_\mathcal{S}^{-t} \circ \Theta_\mathcal{S}$. Let $\Theta = \Theta_\mathcal{S} \otimes \Theta_1 \otimes \cdots \otimes \Theta_M$. Then $\Theta(V_j) = V_j$ for all j, and $\Theta \circ \tau_\lambda^t = \tau_\lambda^{-t} \circ \Theta$ for all $\lambda \in \mathbb{R}$. Hence, Assumption (G4) holds.

Concerning Assumptions (G2) and (G3), we need to recall several results concerning non-equilibrium thermodynamics of $\mathcal{S} + \mathcal{R}$ established in [JP3]. We first list technical conditions needed for these results.

(A1) $\mathfrak{h}_j = L^2(\mathbb{R}_+, \mathrm{d}s\,; \mathfrak{H}_j)$ for some auxiliary Hilbert space $\mathfrak{H}_j$ and h_j is the operator of multiplication by $s \in \mathbb{R}_+$.

Let $I(\delta) = \{z \in \mathbb{C} \,|\, |\mathrm{Im}\, z| < \delta\}$ and let $H_j^2(\delta)$ be the usual Hardy class of analytic functions $f : I(\delta) \to \mathfrak{H}_j$.

(A2) For some $\delta > 0$, $\kappa > \beta_{\mathrm{eq}}$, and all j, $\mathrm{e}^{-\kappa s}\alpha_j(|s|) \in H_j^2(\delta)$.

(A3) For all j, $\|\alpha_j(2)\|_{\mathfrak{H}_j} > 0$.

(A1) and (A2) are regularity assumptions needed for the spectral theory of NESS developed in [JP3]. Assumption (A3) is the "Fermi Golden Rule" condition which ensures that $\mathcal{S}$ is effectively coupled to each reservoir $\mathcal{R}_j$.

The following result was proven in [JP3].

Theorem 1.8 *Assume that (A1)-(A3) hold. Then there exist $\Lambda > 0$, $\epsilon > 0$ and states $\omega_{\lambda X+}$ on $\mathcal{O}$ such that for $0 < |\lambda| < \Lambda$, $X \in I_\epsilon$, $\eta \in \mathcal{N}_X$, and $A \in \mathcal{O}$,*

$$\lim_{t \to +\infty} \eta(\tau_\lambda^t(A)) = \omega_{\lambda X+}(A). \tag{1.12}$$

The states $\omega_{\lambda X+}$ are the NESS of the joint system $\mathcal{S} + \mathcal{R}$ and are the central objects of the non-equilibrium statistical mechanics of this system. We remark that $\omega_{\lambda X=0+}$ is the unique $(\tau_\lambda, \beta_{\mathrm{eq}})$-KMS state on $\mathcal{O}$ (hence,

$\omega_{\lambda \mathrm{eq}} = \omega_{\lambda X=0+}$), and in this case Relation (1.12) is the statement of the zeroth law of thermodynamics. In particular, Theorem 1.8 implies that for $0 < |\lambda| < \Lambda$ and all $A, B \in \mathcal{O}$,

$$\lim_{|t|\to\infty} \omega_{\lambda \mathrm{eq}}(A\tau_\lambda^t(B)) = \omega_{\lambda \mathrm{eq}}(A)\omega_{\lambda \mathrm{eq}}(B).$$

Note also that (A1)-(A2) imply (G5). The observable describing the heat flux out of $\mathcal{R}_j$ is

$$\Phi_j = \lambda\delta_j(V_j) = \lambda\sigma_x \otimes \varphi_j(\mathrm{i}h_j\alpha_j).$$

We summarize:

Theorem 1.9 *Suppose that Assumptions (A1)-(A3) are satisfied. Then there exists $\epsilon > 0$ and $\Lambda > 0$ such that Assumptions (G1)-(G5) hold for $0 < |\lambda| < \Lambda$.*

If the thermodynamical forces X_j are not all the same, then one expects that the NESS $\omega_{\lambda X+}$ is thermodynamically non-trivial and has strictly positive entropy production. This result was also established in [JP3] (see also [JP4]). If (A1)-(A3) hold and the X_j's are not all the same, then for λ non-zero and small enough, $\mathrm{Ep}(\omega_{\lambda X+}) > 0$. We will return to this topic in Subsection 1.5.

1.4 Green-Kubo formula for the spin-fermion system

In this subsection we state our main results concerning linear response of $\omega_{\lambda X+}$ to the thermodynamical forces X_j.

Suppose that (A1) and (A2) hold and let $\tilde{\mathfrak{h}}_j = L^2(\mathbb{R}, \mathrm{d}s; \mathfrak{H}_j)$. To any $f_j \in \mathfrak{h}_j$ we associate $\tilde{f}_j \in \tilde{\mathfrak{h}}_j$ by

$$\tilde{f}_j(s) = \begin{cases} f_j(s) & \text{if } s \geq 0, \\ (c_j f_j)(|s|) & \text{if } s < 0. \end{cases} \tag{1.13}$$

Let δ and κ be as in (A2) and

$$\mathcal{A}_j = \{ f_j \in \mathfrak{h}_j \mid e^{-bs}\tilde{f}_j(s) \in H_j^2(\delta) \text{ for some } b > (\kappa + \beta_{\mathrm{eq}})/2 \}.$$

Let $\tilde{\mathcal{O}}$ be a $*$-subalgebra of $\mathcal{O}$ generated by

$$\{ Q \otimes a_j^{\#}(f_j) \mid Q \in \mathcal{O}_{\mathcal{S}},\ f_j \in \mathcal{A}_j,\ j = 1, \ldots, M \},$$

where $a^{\#}$ stands either for a or a^*. Let

$$\tilde{\mathcal{O}}_{\mathrm{c}} = \tilde{\mathcal{O}} \cap \mathcal{C}.$$

Obviously, $\tilde{\mathcal{O}}_{\mathrm{c}}$ is a vector subspace of $\mathcal{O}$. In addition, we have

Proposition 1.10 *Suppose that (A1) and (A2) hold. Then*
(1) $\tilde{\mathcal{O}}_{\mathrm{c}} \subset \cap_{j=1}^M \mathrm{Dom}\,(\delta_j)$.
(2) $\Phi_j \in \tilde{\mathcal{O}}_{\mathrm{c}}$.
(3) *The algebra $\tilde{\mathcal{O}}$ is dense in $\mathcal{O}$ and for all $A \in \tilde{\mathcal{O}}$, $A - \Theta(A^*) \in \tilde{\mathcal{O}}_{\mathrm{c}}$.*
(4) *Suppose in addition that (A3) holds. Then there exists $\Lambda > 0$ such that for $0 < |\lambda| < \Lambda$ and all $A, B \in \tilde{\mathcal{O}}_{\mathrm{c}}$,*

$$\omega_{\lambda \mathrm{eq}}(\tau_\lambda^t(A)B) = O(\mathrm{e}^{-\gamma(\lambda)|t|}),$$

where $\gamma(\lambda) > 0$. In particular, $\mathcal{L}(A, B)$ is well-defined for all $A, B \in \tilde{\mathcal{O}}_{\mathrm{c}}$.

Proof. Part (1) is obvious. One easily checks that $\mathrm{i}h_j\alpha_j \in \mathcal{A}_j$ and this yields (2). Let $\phi_j \in \mathfrak{h}_j$ be given. Write $\phi_j = \phi_{j+} + \phi_{j-}$, where $c_j(\phi_{j\pm}) = \pm\phi_{j\pm}$. Then

$$\{\mathrm{e}^{-\alpha s^2}\phi_{j+} \,|\, \alpha > 0\} \subset \mathcal{A}_j, \qquad \{\mathrm{ie}^{-\alpha s^2}\phi_{j-} \,|\, \alpha > 0\} \subset \mathcal{A}_j,$$

and so the linear span of $\mathcal{A}_j$ is dense in $\mathfrak{h}_j$. This yields that $\tilde{\mathcal{O}}$ is dense in $\mathcal{O}$. Since $\omega_{\lambda X}$ is time-reversal invariant (see [JOP1]), $\omega_{\lambda X}(A - \Theta(A^*)) = 0$. Hence, $A - \Theta(A^*) \in \mathcal{C}$ for all $A \in \mathcal{O}$ and the second part of (3) follows. Part (4) was proven in [JP3]. □

The main technical result of this paper is:

Theorem 1.11 *Suppose that (A1) and (A2) hold. Then there exist $\Lambda > 0$ and $\epsilon > 0$ such that for $0 < |\lambda| < \Lambda$, $t \geq 0$ and $A \in \tilde{\mathcal{O}}$ the function*

$$X \mapsto \omega_{\lambda X}(\tau_\lambda^t(A)),$$

has an analytic extension to D_ϵ such that

$$\sup_{X\in D_\epsilon, t\geq 0} \left|\omega_{\lambda X}(\tau_\lambda^t(A))\right| < \infty.$$

Combining Theorem 1.11 with Propositions 1.5, 1.10 and Theorems 1.6, 1.7, 1.9, we derive our main result:

Theorem 1.12 *Suppose that Assumptions (A1)-(A3) are satisfied. Then there exists $\Lambda > 0$ and $\epsilon > 0$ such that for $0 < |\lambda| < \Lambda$ the following holds.*
(1) *For all $A \in \tilde{\mathcal{O}}$ the map*

$$\mathbb{I}_\epsilon \ni X \mapsto \omega_{\lambda X+}(A),$$

extends to an analytic function on D_ϵ.
In the remaining statements we assume that $A \in \tilde{\mathcal{O}}_{\mathrm{c}}$.
(2) *For all j,*

$$\partial_{X_j}\omega_{\lambda X+}(A)\big|_{X=0} = \frac{1}{\beta_{\mathrm{eq}}}\int_0^\infty \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_{\lambda\mathrm{eq}}(\tau_\lambda^s(A)\tau_\lambda^{\mathrm{i}u}(\Phi_j)).$$

(3) *If in addition A is a self-adjoint observable such that $\Theta(A) = -A$, then*

$$\partial_{X_j}\omega_{\lambda X+}(A)\big|_{X=0} = \frac{1}{2}\int_{-\infty}^\infty \omega_{\lambda\mathrm{eq}}(\tau_\lambda^t(A)\Phi_j)\mathrm{d}t.$$

(4) *The kinetic transport coefficients*

$$L_{\lambda kj} = \partial_{X_j}\omega_{\lambda X+}(\Phi_k)\big|_{X=0}, \tag{1.14}$$

satisfy

$$L_{\lambda kj} = \frac{1}{2}\int_{-\infty}^\infty \omega_{\lambda\mathrm{eq}}(\tau_\lambda^t(\Phi_k)\Phi_j)\mathrm{d}t, \tag{1.15}$$

and

$$L_{\lambda kj} = L_{\lambda jk}. \tag{1.16}$$

Our final result is:

Theorem 1.13 *Assume that (A1)-(A3) hold. Then there is $\Lambda > 0$ such that the functions $\lambda \mapsto L_{\lambda kj}$ are analytic for $|\lambda| < \Lambda$ and have power expansions*

$$L_{\lambda kj} = \sum_{n=2}^{\infty} \lambda^n L_{kj}^{(n)}. \tag{1.17}$$

Moreover, for $k \neq j$,

$$L_{kj}^{(2)} = -\frac{\pi}{(\cosh \beta_{\rm eq})^2} \frac{\|\alpha_k(2)\|_{\mathfrak{H}_k}^2 \|\alpha_j(2)\|_{\mathfrak{H}_j}^2}{\sum_i \|\alpha_i(2)\|_{\mathfrak{H}_i}^2}, \tag{1.18}$$

and $L_{jj}^{(2)} = -\sum_{k \neq j} L_{kj}^{(2)}$.

Remark. Starting with formula (1.15), this theorem can be proven by an explicit computation based on the spectral theory of the standard Liouvillean. Our proof in Section 4 is somewhat indirect and emphasizes the important connection between $L_{kj}^{(2)}$ and the weak coupling Green-Kubo formula established in [LeSp]. This connection is discussed in more detail in Subsection 1.6

1.5 Thermodynamics of the SFM revisited

The results established in this paper could be used to improve existing results concerning the thermodynamics of the SFM. In this subsection we do not assume that ϵ is small and $\beta_{\rm eq}$ does not play any particular role. For this reason, in this subsection we replace the subscripts X by $\vec{\beta} = (\beta_1, \ldots, \beta_M)$. Hence, $\omega_{\beta_j} = \omega_j$ is the initial state of the reservoir $\mathcal{R}_j$, $\omega_{\mathcal{R}\vec{\beta}} = \omega_{\beta_1} \otimes \cdots \otimes \omega_{\beta_M}$, $\omega_{\vec{\beta}} = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}\vec{\beta}}$ is the reference state of the joint system, $\mathcal{N}_{\vec{\beta}}$ is the set of all $\omega_{\vec{\beta}}$ -normal states on $\mathcal{O}$, etc. For $0 < \gamma_1 < \gamma_2$ we denote $\mathbb{I}_{\gamma_1\gamma_2} = [\gamma_1, \gamma_2]^M \subset \mathbb{R}^M$. In this subsection we will always assume the constant κ in Assumption (A2) satisfies $\kappa > \gamma_2$.

The following results hold:

Theorem 1.14 *Let $0 < \gamma_1 < \gamma_2$ be given and assume that (A1)-(A3) hold. Then there exist $\Lambda > 0$ and states $\omega_{\lambda\vec{\beta}+}$ on $\mathcal{O}$ such that:*
(1) *For all $0 < |\lambda| < \Lambda$, $\vec{\beta} \in \mathbb{I}_{\gamma_1\gamma_2}$, $\eta \in \mathcal{N}_{\vec{\beta}}$, and $A \in \mathcal{O}$,*

$$\lim_{t \to +\infty} \eta(\tau_\lambda^t(A)) = \omega_{\lambda\vec{\beta}+}(A). \tag{1.19}$$

(2) *The limit (1.19) is exponentially fast in the following sense: There exist $\rho_{\lambda\vec{\beta}} > 0$, a norm dense set of states $\mathcal{N}_{0\vec{\beta}} \subset \mathcal{N}_{\vec{\beta}}$, and a norm-dense $*$-subalgebra $\mathcal{O}_0 \subset \mathcal{O}$ such that for $\eta \in \mathcal{N}_{0\vec{\beta}}$, $A \in \mathcal{O}_0$, and $t > 0$,*

$$|\eta(\tau_\lambda^t(A)) - \omega_{\lambda\vec{\beta}+}(A)| \leq C_{A,\eta,\lambda} {\rm e}^{-\rho_{\lambda\vec{\beta}} t}. \tag{1.20}$$

Moreover, $\omega_{\vec{\beta}} \in \mathcal{N}_{0\vec{\beta}}$, $\Phi_j \in \mathcal{O}_0$, and

$$\rho_{\lambda\vec{\beta}} = \frac{\pi}{2} \left(\sum_j \|\alpha_j(2)\|_{\mathfrak{H}_j}^2 \right) \lambda^2 + O(\lambda^4), \tag{1.21}$$

where the remainder is uniform in $\vec{\beta} \in \mathbb{I}_{\gamma_1\gamma_2}$.
(3) *There exists a neighborhood $O_{\gamma_1\gamma_2}$ of $\mathbb{I}_{\gamma_1\gamma_2}$ in $\mathbb{C}^M$ such that for all $A \in \mathcal{O}_0$ the functions*

$$(\lambda, \vec{\beta}) \mapsto \omega_{\lambda\vec{\beta}+}(A), \tag{1.22}$$

extend to analytic functions on $\{\lambda \,|\, |\lambda| < \Lambda\} \times O_{\gamma_1\gamma_2}$.

Remark. Parts (1) and (2) are proven in [JP3] and are stated here for reference purposes. The new result is (3)—in [JP3] the analyticity of the functions (1.22) was discussed only w.r.t. λ.

We denote by $\hat{\mathbb{I}}_{\gamma_1\gamma_2}$ the "off-diagonal" part of $\mathbb{I}_{\gamma_1\gamma_2}$, i.e.,

$$\hat{\mathbb{I}}_{\gamma_1\gamma_2} = \mathbb{I}_{\gamma_1\gamma_2} \setminus \{\vec{\beta}\,|\,\beta_1 = \ldots = \beta_M\}.$$

Theorem 1.15 *Let $0 < \gamma_1 < \gamma_2$ be given and assume that (A1)-(A3) hold. Then there exists $\Lambda > 0$ such that for $0 < |\lambda| < \Lambda$ and $\vec{\beta} \in \hat{\mathbb{I}}_{\gamma_1\gamma_2}$ the following holds:*
(1) $\mathrm{Ep}(\omega_{\lambda\vec{\beta}+}) > 0$.
(2) *There are no τ_λ-invariant states in $\mathcal{N}_{\vec{\beta}}$.*

Remark 1. Statements (1) and (2) are equivalent. Indeed, the exponentially fast approach to NESS (Part (2) of Theorem 1.14) and Theorem 1.1 in [JP3] yield that (2) implies (1). On the other hand, if η is a normal τ_λ-invariant state in $\mathcal{N}_{\vec{\beta}}$, then, by Part (1) of Theorem 1.14, $\eta = \omega_{\lambda\vec{\beta}+}$. This fact and Theorem 1.3 in [JP5] yield that $\mathrm{Ep}(\omega_{\lambda\vec{\beta}+}) = 0$, and so (2) implies (1).
Remark 2. Theorem 1.15 was proven in [JP3] under the additional assumption that for some $\delta > 0$,

$$\sum_{i,j} |\beta_i - \beta_j| > \delta.$$

The constant Λ was dependent on δ.
Remark 3. A result related to Part (2) of Theorem 1.15 was recently established in [MMS].

The proofs of Theorems 1.14 and 1.15 are given in Section 5.

1.6 Some generalizations

All our results easily extend to more general models where $\mathcal{S}$ is an N-level atom described by the Hilbert space $\mathcal{H}_\mathcal{S} = \mathbb{C}^N$ and the Hamiltonian $H_\mathcal{S}$. Each V_j is a finite sum of terms of the form

$$Q_{j,k} \otimes \varphi_j(\alpha_{j,k,1}) \cdots \varphi_j(\alpha_{j,k,n_{j,k}}) + \mathrm{h.c.},$$

where $n_{j,k} \geq 1$, $Q_{j,k} \in \mathcal{O}_\mathcal{S} = M(\mathbb{C}^N)$ and $\alpha_{j,k,n} \in \mathfrak{h}_j$ satisfy:

(A0) If $k \neq l$ or $n \neq m$, then $(\alpha_{j,k,n}, \mathrm{e}^{\mathrm{i}th_j}\alpha_{j,l,m}) = 0$ for all $t \in \mathbb{R}$.

We shall call this model *the general spin-fermion model* (abbreviated GSFM). The GSFM may not be time-reversal invariant. Assume that (A1) holds. Let c_j be a distinguished complex conjugation on $\mathfrak{h}_j$ and let $\tilde{\alpha}_{j,k,n}(s)$ be defined by (1.13).

(A4) For some $\delta > 0$, $\kappa > \beta_{\mathrm{eq}}$, and all j, k, n, $\mathrm{e}^{-\kappa s}\tilde{\alpha}_{j,k,n}(s) \in H_j^2(\delta)$.

The general "Fermi Golden Rule" non-degeneracy condition is formulated as follows. Assumptions (A0), (A1) and (A4) ensure that for all X there exists a linear map $K_X : \mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{S}$ such that for all $A, B \in \mathcal{O}_\mathcal{S}$,

$$\lim_{t\to+\infty} \omega_X^{(0)}(A\,\tau_0^{-t/\lambda^2} \circ \tau_\lambda^{t/\lambda^2}(B)) = \frac{1}{N}\mathrm{Tr}(A\mathrm{e}^{tK_X}(B)). \tag{1.23}$$

As usual, we write $K_{\mathrm{eq}} = K_{X=0}$. This relation (the quantum Markovian semigroup approximation of the dynamics of an open quantum system in the van Hove weak coupling limit) is a celebrated result of Davies [Da] who has proven it under very general technical conditions (see also [De, JP3, JP4]). The result of Davies was the starting point of numerous studies of thermodynamics of open quantum systems in weak coupling limit (see [LeSp, AJPP1] for references and additional information). We will return to this point at the end of this subsection.

We recall that the generator K_X has the form

$$K_X = \sum_{j=1}^{M} K_{X_j},$$

where K_{X_j} is the generator obtained by considering the weak coupling limit of the system $\mathcal{S} + \mathcal{R}_j$ w.r.t. the initial state $\omega_{\mathcal{S}} \otimes \omega_{X_j}$. By construction, the spectrum of K_{X_j} is contained in $\{z \,|\, \mathrm{Re}\, z \le 0\}$ and $0 \in \sigma(K_{X_j})$. Assumption (A3) is replaced with

(A5) For all j and $|X_j| < \epsilon$, $\sigma(K_{X_j}) \cap \mathrm{i}\mathbb{R} = \{0\}$ and 0 is a simple eigenvalue of K_{X_j}.

In the literature one can find various algebraic characterizations of (A5) (see [Sp, De] for references and additional information).

If Assumptions (A1), (A4) and (A5) hold, then Theorem 1.8 holds for the GSFM. The heat fluxes are again defined by $\Phi_j = \lambda \delta_j(V_j)$, and if not all X_j's are the same, the entropy production of $\omega_{\lambda X+}$ is strictly positive for small λ (see [JP3, JP4]).

Our next assumption concerns time-reversal invariance.

(A6) The complex conjugations c_j commute with h_j and satisfy $c_j \alpha_{j,k,n} = \alpha_{j,k,n}$ for all j, k, n. Moreover, the matrices $H_{\mathcal{S}}$ and $Q_{j,k}$ are real w.r.t. the usual complex conjugation on $\mathcal{B}(\mathcal{H}_{\mathcal{S}})$.

This assumption ensures that there exists an involutive, anti-linear $*$-automorphism (time-reversal) Θ of $\mathcal{O}$ such that for all j, $\Theta(V_j) = V_j$, $\Theta \circ \tau_j^t = \tau_j^{-t} \circ \Theta$, and $\Theta \circ \tau_{\mathcal{S}}^t = \tau_{\mathcal{S}}^{-t} \circ \Theta$. In particular, $\Theta \circ \tau_\lambda^t = \tau_\lambda^{-t} \circ \Theta$ for all $\lambda \in \mathbb{R}$.

Theorem 1.9 holds for the GSFM under the Assumptions (A0), (A1), (A4), (A5), (A6). The definition of $\tilde{\mathcal{O}}$ and $\mathcal{O}_{\mathrm{c}}$ and Proposition 1.10 holds under the Assumptions (A0), (A1), (A4) (obviously, in the second part of Part (3) we also need (A6)). Theorem 1.11 holds under the Assumption (A0), (A1), (A4). Finally, Parts (1) and (2) of Theorem 1.12 hold for the GSFM under the Assumptions (A0), (A1), (A4), (A5). Parts (3) and (4) require in addition the time reversal assumption (A6).

Before discussing the generalization of Theorem 1.13 we recall a few basic definitions and results of the weak coupling (sometimes also called Fermi Golden Rule or FGR) thermodynamics of open quantum systems. Assumption (A5) ensures that there exists a density matrix $\omega_{\mathcal{S}X+}$ on $\mathcal{H}_{\mathcal{S}}$ such that for any initial density matrix ρ on $\mathcal{H}_{\mathcal{S}}$ and $A \in \mathcal{O}_{\mathcal{S}}$,

$$\lim_{t \to +\infty} \mathrm{Tr}(\rho\, \mathrm{e}^{tK_X}(A)) = \mathrm{Tr}(\omega_{\mathcal{S}X+} A) \equiv \omega_{\mathcal{S}X+}(A).$$

The density matrix $\omega_{\mathcal{S}X+}$ is the weak coupling NESS of the open quantum system $\mathcal{S} + \sum_j \mathcal{R}_j$. Clearly,

$$\omega_{\mathcal{S}X=0+} = \mathrm{e}^{-\beta_{\mathrm{eq}} H_{\mathcal{S}}} / \mathrm{Tr}(\mathrm{e}^{-\beta_{\mathrm{eq}} H_{\mathcal{S}}}),$$

and we will write $\omega_{\mathcal{S}X=0+} = \omega_{\mathcal{S}\mathrm{eq}}$. Weak coupling heat flux observables are defined by $\overline{\Phi}_{jX} = K_{X_j}(H_{\mathcal{S}})$ and we denote $\overline{\Phi}_{j\mathrm{eq}} = \overline{\Phi}_{jX=0}$. The weak coupling entropy production is

$$\overline{\mathrm{Ep}} = \sum_{j=1}^{M} X_j \omega_{\mathcal{S}X+}(\overline{\Phi}_{jX}).$$

One always has $\overline{\mathrm{Ep}} \ge 0$. Lebowitz and Spohn [LeSp] have shown that if (A4) holds then $\overline{\mathrm{Ep}} > 0$ whenever X_j are not all equal. In the same paper they have also proven the Green-Kubo formula for weak coupling heat fluxes: If (A5) holds, then the functions $X \mapsto \omega_{\mathcal{S}X+}(\overline{\Phi}_{kX})$ are differentiable at $X = 0$ and

$$\overline{L}_{kj} \equiv \partial_{X_j} \omega_{\mathcal{S}X+}(\Phi_{kX})\big|_{X=0} = \int_0^\infty \omega_{\mathcal{S}\mathrm{eq}}(\mathrm{e}^{tK_{\mathrm{eq}}}(\overline{\Phi}_{k\mathrm{eq}})\overline{\Phi}_{j\mathrm{eq}})\mathrm{d}t. \tag{1.24}$$

These results are very robust and can be derived under very mild technical conditions. If in addition (A6) holds, then $\overline{L}_{kj} = \overline{L}_{jk}$, that is, the weak coupling Onsager reciprocity relations hold.

One naturally expects that the weak coupling thermodynamics is the first non-trivial contribution (in λ) to the microscopic thermodynamics. Indeed, it was proven in [JP3, JP4] that if (A0), (A4) and (A5) hold, then for $A \in \mathcal{O}_\mathcal{S}$ and λ small enough,

$$\begin{aligned} \omega_{\lambda X+}(A) &= \omega_{\mathcal{S}X+}(A) + O(\lambda), \\ \omega_{\lambda X+}(\Phi_j) &= \lambda^2 \omega_{\mathcal{S}X+}(\overline{\Phi}_{jX}) + O(\lambda^3), \\ \mathrm{Ep}(\omega_{\lambda X+}) &= \lambda^2 \overline{\mathrm{Ep}} + O(\lambda^3). \end{aligned} \tag{1.25}$$

In the next theorem we relate $L_{\lambda kj}$ and $\overline{L}_{kj}$ and complete the link between the microscopic and the weak coupling thermodynamics for this class of models.

Theorem 1.16 *Assume that (A0), (A1), (A4) and (A5) hold. Then there is $\Lambda > 0$ such that the functions $\lambda \mapsto L_{\lambda kj}$ are analytic for $|\lambda| < \Lambda$ and have power expansions*

$$L_{\lambda kj} = \sum_{n=2}^{\infty} \lambda^n L_{kj}^{(i)}.$$

Moreover,

$$L_{kj}^{(2)} = \overline{L}_{kj}.$$

Remark 1. It follows immediately from this result, the Green-Kubo formula and Relation (1.24) that

$$\lim_{\lambda \to 0} \lambda^{-2} \frac{1}{\beta_{\mathrm{eq}}} \int_0^\infty \mathrm{d}t \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u \, \omega_{\lambda \mathrm{eq}}(\tau_\lambda^t(\Phi_k)\tau_\lambda^{\mathrm{i}u}(\Phi_j))\mathrm{d}t = \int_0^\infty \omega_{\mathcal{S}\mathrm{eq}}(\mathrm{e}^{tK_{\mathrm{eq}}}(\overline{\Phi}_{k\mathrm{eq}})\overline{\Phi}_{j\mathrm{eq}})\mathrm{d}t.$$

If in addition (A6) holds, then we also get

$$\lim_{\lambda \to 0} \lambda^{-2} \frac{1}{2} \int_{-\infty}^\infty \omega_{\lambda \mathrm{eq}}(\tau_\lambda^t(\Phi_k)\Phi_j)\mathrm{d}t = \int_0^\infty \omega_{\mathcal{S}\mathrm{eq}}(\mathrm{e}^{tK_{\mathrm{eq}}}(\overline{\Phi}_{k\mathrm{eq}})\overline{\Phi}_{j\mathrm{eq}})\mathrm{d}t,$$

i.e. the rescaled microscopic flux-flux correlation functions converge to the corresponding weak coupling correlation functions.

Remark 2. The relation between the microscopic and the weak coupling thermodynamics is discussed in detail in the lecture notes [AJPP1] in the context of an exactly solvable quasi-free model.

The proofs of the results described in this subsection are only notationally different from the proofs of Theorems 1.12 and 1.13 and details can be found in the forthcoming review article [JP7].

Theorems 1.14 and 1.15 also hold for the GSFM under the Assumptions (A0), (A1), (A4) with $\kappa > \gamma_2$, and (A5) for all $\vec{\beta} \in \mathbb{I}_{\gamma_1\gamma_2}$. The only parts that need to be modified are Relations (1.20) and (1.21). In general, the constant $C_{A,\eta,\lambda}$ is replaced by a polynomial in t. The leading term in the expansion (1.21) is equal to the absolute value of the real part of the non-zero eigenvalue of $K_{\vec{\beta}}$ closest to $\mathrm{i}\mathbb{R}$ and in general depends on $\vec{\beta}$. For additional discussion of these points we refer the reader to [JP7].

2 Abstract Green-Kubo formula

In this section we give a new proof of Theorem 1.3 and hence a new derivation of the abstract Green-Kubo formula.

To motivate the argument, we shall first prove Theorem 1.3 in the case where the reservoirs $\mathcal{R}_j$ are finite dimensional. The interested reader should compare this argument with the finite dimensional computation given in the introduction of [JOP1].

2.1 Finite dimensional case

We shall identify the finite dimensional states with associated density matrices and write $\omega(A) = \mathrm{Tr}(A\omega)$.

Suppose that $\mathcal{R}_j$ is described by the finite dimensional Hilbert space $\mathcal{H}_j$ and the Hamiltonian H_j. Hence, $\mathcal{O}_j = \mathcal{B}(\mathcal{H}_j)$,

$$\tau_j^t(A) = \mathrm{e}^{\mathrm{i}tH_j} A \mathrm{e}^{-\mathrm{i}tH_j},$$

and $\omega_j = \mathrm{e}^{-\beta_j H_j}/Z_j$ where Z_j is the normalization constant. The complete reservoir system is described by the Hilbert space $\mathcal{H}_\mathcal{R} = \otimes_j \mathcal{H}_j$ and the Hamiltonian $H_\mathcal{R} = \sum_j H_j$. Finally, the interacting joint system $\mathcal{S} + \mathcal{R}$ is described by the Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R}$ and the Hamiltonian $H = H_\mathcal{S} + H_\mathcal{R} + V$. We set

$$H_X = H - \sum_{j=1}^M \frac{X_j}{\beta_{\mathrm{eq}}} H_j.$$

Clearly, $\mathcal{O} = \mathcal{B}(\mathcal{H})$ and

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH},$$

$$\sigma_X^t(A) = \mathrm{e}^{\mathrm{i}tH_X} A \mathrm{e}^{-\mathrm{i}tH_X},$$

$$\omega_X = \mathrm{e}^{-\beta_{\mathrm{eq}} H_X}/Z_X.$$

Note also that

$$\Phi_j = \mathrm{i}[H_j, V] = -\frac{\mathrm{d}}{\mathrm{d}t}\tau^t(H_j)\big|_{t=0}.$$

The next four steps complete the proof of Theorem 1.3 in the finite dimensional case.

Step 1. The relation $\tau^{-t}(H_X) = H_X - \sum_j (X_j/\beta_{\mathrm{eq}}) \int_0^t \tau^{-s}(\Phi_j)\mathrm{d}s$ yields that

$$\omega_X \circ \tau^t = \frac{1}{Z_X} \mathrm{e}^{-\beta_{\mathrm{eq}}\left(H_X - \sum_j (X_j/\beta_{\mathrm{eq}}) \int_0^t \tau^{-s}(\Phi_j)\mathrm{d}s\right)}. \tag{2.26}$$

Step 2. Step 1 and the Duhamel formula (see, for example, [BR2], pages 94-95) yield

$$\omega_X(\tau^t(A)) = \omega_X(A)\left(1 - \sum_j X_j \int_0^t \omega_X(\tau^{-s}(\Phi_j))\mathrm{d}s\right)$$

$$+ \sum_j \frac{X_j}{\beta_{\mathrm{eq}}} \int_0^t \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_X(A\sigma_X^{\mathrm{i}u}(\tau^{-s}(\Phi_j))) + O(|X|^2).$$

Step 3. If A is centered, then $\omega_X(A) = 0$ and $\omega_{X=0}(\tau^t(A)) = \omega_{X=0}(A) = 0$. Hence, Step 2 yields

$$\omega_X(\tau^t(A)) - \omega_{X=0}(\tau^t(A)) = \sum_j \frac{X_j}{\beta_{\mathrm{eq}}} \int_0^t \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_X(A\sigma_X^{\mathrm{i}u}(\tau^{-s}(\Phi_j))) + O(|X|^2), \tag{2.27}$$

Step 4. Since $\sigma_{X=0} = \tau$ (recall also that $\omega_{\mathrm{eq}} = \omega_{X=0}$),

$$\lim_{X\to 0} \int_0^t \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_X(A\sigma_X^{\mathrm{i}u}(\tau^{-s}(\Phi_j))) = \int_0^t \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_{\mathrm{eq}}(\tau^s(A)\tau^{\mathrm{i}u}(\Phi_j))).$$

and (2.27) yields

$$\partial_{X_j}\omega_X(\tau^t(A))\big|_{X=0} = \frac{1}{\beta_{\mathrm{eq}}} \int_0^t \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_{\mathrm{eq}}(\tau^s(A)\tau^{\mathrm{i}u}(\Phi_j)).$$

2.2 Proof of Theorem 1.3

Throughout this subsection we suppose that (G1) and (G5) hold. Under these assumptions each of the Steps 1-4 can be extended to the abstract system $\mathcal{S}+\mathcal{R}$.

We start with the Step 4. The following result was established in [JOP1] (Lemmas 3.3 and 3.4).

Lemma 2.1 (1) *The group τ preserves $\cap_j \mathrm{Dom}\,(\delta_j)$.*
(2) *For all $A \in \mathcal{O}$,*

$$\lim_{X\to 0} \omega_X(A) = \omega_{\mathrm{eq}}(A).$$

(3) *For all $A \in \mathcal{O}$ and $t \in \mathbb{R}$,*

$$\lim_{X\to 0} \sigma_X^t(A) = \tau^t(A).$$

We shall also need:

Lemma 2.2 *For all $A, B \in \mathcal{O}$ and $0 \leq u \leq \beta_{\mathrm{eq}}$,*

$$\lim_{X\to 0} \omega_X(A\sigma_X^{\mathrm{i}u}(B)) = \omega_{\mathrm{eq}}(A\tau^{\mathrm{i}u}(B)).$$

Proof. For $j = 1, 2, \ldots$ let

$$B_{jX} = \sqrt{\frac{j}{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-jt^2} \sigma_X^t(B)\mathrm{d}t.$$

By the properties of analytic approximations (see Section 2.5.3 in [BR1]),

$$\lim_{j\to\infty} \|B - B_{jX}\| = 0, \tag{2.28}$$

and

$$\sigma_X^{\mathrm{i}u}(B_{jX}) = \sqrt{\frac{j}{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-j(t-\mathrm{i}u)^2} \sigma_X^t(B)\mathrm{d}t. \tag{2.29}$$

We write $B_j = B_{jX=0}$. Relation (2.29) and Lemma 2.1 yield that

$$\begin{aligned} &\lim_{X\to 0} \sigma_X^{\mathrm{i}u}(B_{jX}) = \tau^{\mathrm{i}u}(B_j), \\ &\lim_{X\to 0} \omega_X(A\sigma_X^{\mathrm{i}u}(B_{jX})) = \omega_{\mathrm{eq}}(A\tau^{\mathrm{i}u}(B_j)). \end{aligned} \tag{2.30}$$

Since ω_X is a $(\sigma_X, \beta_{\mathrm{eq}})$-KMS state, the bound (1.1) implies that for all X,

$$|\omega_X(A\sigma_X^{\mathrm{i}u}(B)) - \omega_X(A\sigma_X^{\mathrm{i}u}(B_{jX}))| \leq \|A\|\|B - B_{jX}\|,$$

and so for all j,

$$\begin{aligned} |\omega_X(A\sigma_X^{\mathrm{i}u}(B)) - \omega_{\mathrm{eq}}(A\tau^{\mathrm{i}u}(B))\| &\leq \|A\|(\|B - B_{jX}\| + \|B - B_j\|) \\ &\quad + |\omega_X(A\sigma_X^{\mathrm{i}u}(B_{jX})) - \omega_{\mathrm{eq}}(A\tau^{\mathrm{i}u}(B_j))|. \end{aligned}$$

Relations (2.30) imply that for all j,

$$\limsup_{X\to 0} |\omega_X(A\sigma_X^{\mathrm{i}u}(B)) - \omega_{\mathrm{eq}}(A\tau^{\mathrm{i}u}(B))| \leq 2\|A\|\|B - B_j\|,$$

and (2.28) yields the statement. □

Lemma 2.2 and the bound

$$\|\omega_X(A\sigma_X^{\mathrm{i}u}(\tau^{-s}(\Phi_j)))\| \leq \|A\|\|\Phi_j\|,$$

yield the extension of the Step 4 to the abstract system $\mathcal{S}+\mathcal{R}$.

Proposition 2.3

$$\lim_{X\to 0}\int_0^t \mathrm{d}s\int_0^{\beta_{\rm eq}}\mathrm{d}u\,\omega_X(A\sigma_X^{\mathrm{i}u}(\tau^{-s}(\Phi_j)))=\int_0^t\mathrm{d}s\int_0^{\beta_{\rm eq}}\mathrm{d}u\,\omega_{\rm eq}(\tau^s(A)\tau^{\mathrm{i}u}(\Phi_j)).$$

We now turn to the Step 1. Let Γ_t be the unitary cocycle such that

$$\tau^t(A)=\Gamma_t\tau_0^t(A)\Gamma_t^*,$$

explicitly

$$\Gamma_t=\mathbb{1}+\sum_{n\geq 1}\mathrm{i}^n\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\tau_0^{t_n}(V)\cdots\tau_0^{t_1}(V),$$

see Proposition 5.4.1 in [BR2].

Lemma 2.4 $\Gamma_t\in\cap_j\mathrm{Dom}\,(\delta_j)$ *and*

$$\delta_j(\Gamma_t)\Gamma_t^*=\mathrm{i}\int_0^t\tau^s(\Phi_j)\mathrm{d}s. \tag{2.31}$$

Proof. Since $V\in\mathrm{Dom}\,(\delta_j)$, one easily shows that $\Gamma_t\in\mathrm{Dom}\,(\delta_j)$ and that

$$\delta_j(\Gamma_t)=\sum_{n\geq 1}\mathrm{i}^n\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\sum_k\tau_0^{t_n}(V)\cdots\tau_0^{t_k}(\delta_j(V))\cdots\tau^{t_1}(V).$$

This formula yields that the function $\mathbb{R}\ni t\mapsto\delta_j(\Gamma_t)\in\mathcal{O}$ is continuously differentiable and that

$$\frac{\mathrm{d}\delta_j(\Gamma_t)}{\mathrm{d}t}=\delta_j\left(\frac{\mathrm{d}\Gamma_t}{\mathrm{d}t}\right) \tag{2.32}$$

To prove relations (2.31), we recall that

$$\frac{\mathrm{d}\Gamma_t}{\mathrm{d}t}=\mathrm{i}\Gamma_t\tau_0^t(V),\qquad \frac{\mathrm{d}\Gamma_t^*}{\mathrm{d}t}=-\mathrm{i}\tau_0^t(V)\Gamma_t^*.$$

The first relation and (2.32) yield

$$\frac{\mathrm{d}\delta_j(\Gamma_t)}{\mathrm{d}t}=\mathrm{i}\delta_j(\Gamma_t)\tau_0^t(\Phi_j)+\mathrm{i}\Gamma_t\tau_0^t(\Phi_j).$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}\delta_j(\Gamma_t)}{\mathrm{d}t}\Gamma_t^*&=\mathrm{i}\delta_j(\Gamma_t)\tau_0^t(V)\Gamma_t^*+\mathrm{i}\Gamma_t\tau_0^t(\Phi_j)\Gamma_t^*\\&=-\delta_j(\Gamma_t)\frac{\mathrm{d}\Gamma_t^*}{\mathrm{d}t}+\mathrm{i}\tau^t(\Phi_j),\end{aligned}$$

and (2.31) follows. □

Set

$$P_{Xt}=-\sum_j\frac{X_j}{\beta_{\rm eq}}\int_0^t\tau^{-s}(\Phi_j)\mathrm{d}s. \tag{2.33}$$

Let t be fixed and let σ_{Xt} be the C^*-dynamics generated by

$$\delta_{Xt}=\delta_X+\mathrm{i}[P_{Xt},\,\cdot\,],$$

i.e. $\sigma_{Xt}^u=\mathrm{e}^{u\delta_{Xt}}$. The next proposition is the extension of the Step 1 to the abstract system $\mathcal{S}+\mathcal{R}$.

Proposition 2.5 *$\omega_X \circ \tau^t$ is a $(\sigma_{Xt}, \beta_{\rm eq})$-KMS state on $\mathcal{O}$.*

Proof. Let $A \in \cap_j \mathrm{Dom}\,(\delta_j)$. Relation $\Gamma_t^*\Gamma_t = \mathbb{1}$ and Part (1) of Lemma 2.1 yield

$$\begin{aligned}\delta_j(\tau^t(A)) &= \delta_j(\Gamma_t\tau_0^t(A)\Gamma_t^*)\\ &= \delta_j(\Gamma_t)\Gamma_t^*\tau^t(A) + \tau^t(\delta_j(A)) + \Gamma_t\tau_0^t(A)\delta_j(\Gamma_t^*),\end{aligned}$$

and

$$\Gamma_t\tau_0^t(A)\delta_j(\Gamma_t^*) = -\tau^t(A)\delta_j(\Gamma_t)\Gamma_t^*.$$

Hence,

$$\delta_j(\tau^t(A)) - \tau^t(\delta_j(A)) = [\delta_j(\Gamma_t)\Gamma_t^*, \tau^t(A)].$$

This identity and Lemma 2.4 yield

$$\tau^{-t}(\delta_j(\tau^t(A))) - \delta_j(A) = \mathrm{i}\int_0^t [\tau^{-s}(\Phi_j), A]\mathrm{d}s. \tag{2.34}$$

Since $\cap_j \mathrm{Dom}\,(\delta_j)$ is dense in $\mathcal{O}$, (2.34) implies that for all $u \in \mathbb{R}$,

$$\tau^{-t} \circ \sigma_X^u \circ \tau^t = \sigma_{Xt}^u. \tag{2.35}$$

Finally, since ω_X is a $(\sigma_X, \beta_{\rm eq})$-KMS state, (2.35) yields that $\omega_X \circ \tau^t$ is a $(\sigma_{Xt}, \beta_{\rm eq})$-KMS state. □

We now turn to the extension of the Step 2. Recall that $|X|_+ = \max|X_j|$

Proposition 2.6 *Let $A \in \mathcal{O}$ and t be fixed. Then there is a constant C such that if*

$$|X|_+ \le 1/(4|t|\sum_j \|\Phi_j\|), \tag{2.36}$$

then

$$\begin{aligned}\Big|\omega_X(\tau^t(A)) - \omega_X(A)\left(1 - \sum_j X_j\int_0^t \omega_X(\tau^{-s}(\Phi_j))\mathrm{d}s\right)\\ + \sum_j \frac{X_j}{\beta_{\rm eq}}\int_0^t \mathrm{d}s\int_0^{\beta_{\rm eq}}\mathrm{d}u\,\omega_X(A\sigma_X^{\mathrm{i}u}(\tau^{-s}(\Phi_j)))\Big| \le C|X|^2.\end{aligned}$$

Proof. Proposition 2.5 and Araki's theory of perturbation of KMS states (Theorem 5.44 Part (3) in [BR2]) yield that if $\|P_{Xt}\| < 1/2\beta_{\rm eq}$, then

$$\omega_X(\tau^t(A)) = \omega_X(A) - \int_0^{\beta_{\rm eq}}\mathrm{d}s\left[\omega_X(A\sigma_X^{\mathrm{i}u}(P_{Xt})) - \omega_X(A)\omega_X(P_{Xt})\right] + R,$$

where the remainder R can be estimated as

$$\|R\| \le \sum_{n=2}^{\infty}(2\beta_{\rm eq})^n\|P_{Xt}\|^n\|A\|. \tag{2.37}$$

The obvious estimate

$$\|P_{Xt}\| \le \frac{|t|}{\beta_{\rm eq}} \sum_j |X_j| \|\Phi_j\|,$$

combined with (2.36) and (2.37) implies

$$\|R\| \le 8\|A\| (2\beta_{\rm eq}|t| \sum_j \|\Phi_j\|)^2 |X|_+^2,$$

and the statement follows. □

As in the finite dimensional Part 3, the definition of a centered observable and Proposition 2.6 imply

Proposition 2.7 *Let $A \in \mathcal{O}$ be a centered observable and let t be given. Then*

$$\omega_X(\tau^t(A)) - \omega_{X=0}(\tau^t(A)) = \sum_j \frac{X_j}{\beta_{\rm eq}} \int_0^t {\rm d}s \int_0^{\beta_{\rm eq}} {\rm d}u\, \omega_X(A\sigma_X^{{\rm i}u}(\tau^{-s}(\Phi_j))) + O(|X|^2),$$

as $X \to 0$.

Propositions 2.3 and 2.7 yield Theorem 1.3.

Remark. The density matrix (2.26) or the corresponding infinite dimensional expression (2.33) are the starting point of Zubarev construction of NESS. In some sense, they provide a way to map thermodynamical perturbations into mechanical ones.

3 Proof of Theorem 1.11

The proof of Theorem 1.11 is based on techniques and estimates of [JP1, JP3]. We recall the ingredients we need. Throughout this section we assume that (A1)-(A3) hold. The GNS-representation of the algebra $\mathcal{O}$ associated to the product state $\omega_X^{(0)}$ can be explicitly computed [AW]. We will describe it in the glued form of [JP3]. Denote by $e_\pm$ the eigenvectors of σ_z associated to the eigenvalues ± 1. Set $\mathcal{H}_\mathcal{S} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and define a unit vector in $\mathcal{H}_\mathcal{S}$ by

$$\Omega_\mathcal{S} = \frac{1}{\sqrt{2}}(e_- \otimes e_- + e_+ \otimes e_+).$$

Let $\pi_\mathcal{S} : \mathcal{O}_\mathcal{S} \to \mathcal{B}(\mathcal{H}_\mathcal{S})$ be given by

$$\pi_\mathcal{S}(A) = A \otimes I.$$

The triple $(\mathcal{H}_\mathcal{S}, \pi_\mathcal{S}, \Omega_\mathcal{S})$ is the GNS representation of $\mathcal{O}_\mathcal{S}$ associated to $\omega_\mathcal{S}$. We set

$$\mathcal{L}_\mathcal{S} = H_\mathcal{S} \otimes I - I \otimes H_\mathcal{S}.$$

Let $\mathcal{F}_j$ be the anti-symmetric Fock space over $\tilde{\mathfrak{h}}_j = L^2(\mathbb{R}, {\rm d}s; \mathfrak{H}_j)$ and Ω_j the vacuum vector in $\mathcal{F}_j$. We denote by $\tilde{a}_j$, $\tilde{a}_j^*$ the annihilation and creation operators and by N_j the number operator on $\mathcal{F}_j$. Let $\mathcal{L}_j = {\rm d}\Gamma(s)$ be the second quantization of the operator of multiplication by s on $\tilde{\mathfrak{h}}_j$. To any $f_j \in \mathfrak{h}_j$ we associate $\tilde{f}_j \in \tilde{\mathfrak{h}}_j$ by (1.13). For $X \in \mathbb{R}^M$ we set

$$\tilde{f}_{jX}(s) = \left({\rm e}^{(X_j - \beta_{\rm eq})s} + 1\right)^{-1/2} \tilde{f}_j(s).$$

Finally, we define a map $\pi_{jX} : \mathcal{O}_j \to \mathcal{B}(\mathcal{F}_j)$ by

$$\pi_{jX}(\varphi_j(f_j)) = \tilde{\varphi}_j(\tilde{f}_{jX}) = \frac{1}{\sqrt{2}}\left(\tilde{a}_j(\tilde{f}_{jX}) + \tilde{a}_j^*(\tilde{f}_{jX})\right).$$

The map π_{jX} uniquely extends to a representation of $\mathcal{O}_j$ on the Hilbert space $\mathcal{F}_j$.

We set

$$\mathcal{H}_{\mathcal{R}} = \otimes_{j=1}^{M}\mathcal{F}_j, \qquad \pi_{\mathcal{R}X} = \otimes_{j=1}^{M}\pi_{jX}, \qquad \Omega_{\mathcal{R}} = \otimes_{j=1}^{M}\Omega_j.$$

The triple $(\mathcal{H}_{\mathcal{R}}, \pi_{\mathcal{R}X}, \Omega_{\mathcal{R}})$ is the GNS representation of the algebra $\mathcal{O}_{\mathcal{R}}$ associated to the state $\omega_{\mathcal{R}X}$. Let

$$\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}, \qquad \pi_X = \pi_{\mathcal{S}} \otimes \pi_{\mathcal{R}X}, \qquad \Omega = \Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{R}}.$$

The triple $(\mathcal{H}, \pi_X, \Omega)$ is the GNS-representation of the algebra $\mathcal{O}$ associated to the state $\omega_X^{(0)}$. Note that $\mathcal{H}$ and Ω do not depend on X.

The spectral theory of NESS is based on a particular non-selfadjoint operator acting on $\mathcal{H}$, the adjoint of the so-called C-Liouvillean. This operator is defined as follows. Let $\mathcal{L}_{\mathcal{R}} = \sum_j \mathcal{L}_j$ and

$$\mathcal{L}_0 = \mathcal{L}_{\mathcal{S}} + \mathcal{L}_{\mathcal{R}}.$$

Let

$$V_{jX} = \pi_X(V_j) = \sigma_x \otimes I \otimes \tilde{\varphi}_j(\tilde{\alpha}_{jX}),$$

$$W_{jX} = I \otimes \sigma_x \otimes \frac{1}{\sqrt{2}}(-I)^{N_j}\left(\tilde{a}_j^*(\mathrm{e}^{(X_j-\beta_{\mathrm{eq}})s}\tilde{\alpha}_{jX}) - \tilde{a}_j(\tilde{\alpha}_{jX})\right),$$

and

$$V_X = \sum_j V_{jX}, \qquad W_X = \sum_j W_{jX}.$$

The adjoint of the C-Liouvillean associated to the triple $(\mathcal{O}, \tau_\lambda, \omega_X^{(0)})$ is

$$\mathcal{L}_{\lambda X} = \mathcal{L}_0 + \lambda(V_X + W_X).$$

This operator is closed on $\mathrm{Dom}\,(\mathcal{L}_0)$ and generates a quasi-bounded strongly continuous group $\mathrm{e}^{\mathrm{i}t\mathcal{L}_{\lambda X}}$ on $\mathcal{H}$. The operator $\mathcal{L}_{\lambda X}$ is characterized by the following two properties:

(i) For any $A \in \mathcal{O}$ and any $t \in \mathbb{R}$, $\pi_X(\tau_\lambda^t(A)) = \mathrm{e}^{\mathrm{i}t\mathcal{L}_{\lambda X}}\pi_X(A)\mathrm{e}^{-\mathrm{i}t\mathcal{L}_{\lambda X}}$.

(ii) $\mathcal{L}_{\lambda X}^*\Omega = 0$.

Thus, for $A, B \in \mathcal{O}$ we have

$$\omega_X^{(0)}(\tau_\lambda^t(A)B) = (\pi_X(A^*)\Omega, \mathrm{e}^{-\mathrm{i}t\mathcal{L}_{\lambda X}}\pi_X(B)\Omega), \tag{3.38}$$

and hence the function

$$z \mapsto \int_0^\infty \omega_X^{(0)}(\tau_\lambda^t(A)B)\,\mathrm{e}^{\mathrm{i}tz}\mathrm{d}t = \mathrm{i}(\pi_X(A^*)\Omega, (z-\mathcal{L}_{\lambda X})^{-1}\pi_X(B)\Omega),$$

is analytic in the upper half-plane. The basic strategy of [JP3] is to show that for appropriate A, B this function has a meromorphic continuation to a larger half-plane and that the behavior of $t \mapsto \omega_X^{(0)}(\tau_\lambda^t(A)B)$ as $t \to \infty$ is controlled by the poles of this continuation (the resonances) via the inverse Laplace transform.

Let $p_j = \mathrm{i}\partial_s$ be the generator of the group of translations on $\tilde{\mathfrak{h}}_j$, $P_j = \mathrm{d}\Gamma(p_j)$ its second quantization. Let $U_j(\theta) = \mathrm{e}^{-\mathrm{i}\theta P_j} = \Gamma(\mathrm{e}^{-\mathrm{i}\theta p_j})$, $\theta \in \mathbb{R}$, be the second quantization of this group and

$$V_X(\theta) = \sum_j U_j(\theta)V_{jX}U_j(-\theta) = \sum_j \sigma_x \otimes I \otimes \tilde{\varphi}_j(\mathrm{e}^{-\mathrm{i}\theta p_j}\tilde{\alpha}_{jX}),$$

$$W_X(\theta) = \sum_j U_j(\theta)W_{jX}U_j(-\theta) = \sum_j I \otimes \sigma_x \otimes \frac{1}{\sqrt{2}}(-I)^{N_j}\left(\tilde{a}_j^*(\mathrm{e}^{-\mathrm{i}\theta p_j}\tilde{\alpha}_{jX}) - \tilde{a}_j(\mathrm{e}^{-\mathrm{i}\theta p_j}(\mathrm{e}^{(X_j-\beta_{\mathrm{eq}})s}\tilde{\alpha}_{jX}))\right).$$

Lemma 3.1 *There exist $\epsilon > 0$ and $\delta' > 0$ such that the maps*

$$(X,\theta) \mapsto V_X(\theta), \qquad (X,\theta) \mapsto W_X(\theta),$$

extend to analytic operator-valued functions on $D_\epsilon \times I(\delta')$ satisfying

$$\sup_{X\in D_\epsilon, \theta\in I(\delta')} (\|V_X(\theta)\| + \|W_X(\theta)\|) < \infty. \tag{3.39}$$

In particular, one has

$$\sup_{X\in D_\epsilon, |t|\le 1} \left\|\mathrm{e}^{\mathrm{i}t\mathcal{L}_{\lambda X}}\right\| < \infty. \tag{3.40}$$

Proof. The proof of the first part of this result is the same as the proof of Lemma 4.1 and Proposition 4.4 (iii) in [JP1]. The only additional fact needed is that for some $\epsilon > 0$ and $\mu > 0$ the function

$$\mathbb{R} \times \mathbb{R} \ni (x,s) \mapsto w(x,s) = (\mathrm{e}^{-xs}+1)^{-1/2},$$

has an analytic continuation to the region $O = \{z : |z-\beta_{\mathrm{eq}}| < \epsilon\} \times I(\mu)$ such that

$$\sup_{(z,\theta)\in O} |w(z,\theta)| < \infty.$$

Since $\mathcal{L}_0$ is self-adjoint, the bound (3.40) is a simple consequence of (3.39). □

Let $N = \sum_j N_j$. For $X \in D_\epsilon$ and $\theta \in I(\delta')$ we set

$$\mathcal{L}_0(\theta) = \mathcal{L}_0 + \theta N,$$

$$\mathcal{L}_{\lambda X}(\theta) = \mathcal{L}_0(\theta) + \lambda(V_X(\theta) + W_X(\theta)).$$

The family of operators $\mathcal{L}_{\lambda X}(\theta)$, $X \in D_\epsilon, \theta \in I(\delta')$, is a complex deformation of the family of operators $\mathcal{L}_{\lambda X}$, $X \in \mathbb{I}_\epsilon$. Note that $\mathcal{L}_{0X}(\theta) = \mathcal{L}_0(\theta)$ is a normal operator which does not depend on X. The spectrum of $\mathcal{L}_0(\theta)$ consists of two simple eigenvalues ± 2, a doubly degenerate eigenvalue 0 and a sequence of lines $\{x + \mathrm{i}n\mathrm{Im}\,\theta \,|\, x \in \mathbb{R}, n \ge 1\}$. The next lemma is a consequence of Lemma 3.1 and regular perturbation theory and is deduced in the same way as the corresponding results in [JP1, JP3].

Proposition 3.2 *There exist $\Lambda > 0$, $\epsilon > 0$ and $0 < \mu < \delta'$ such that for $|\lambda| < \Lambda$, $-\mu < \mathrm{Im}\,\theta < -3\mu/4$ and $X \in D_\epsilon$, the spectrum of $\mathcal{L}_{\lambda X}(\theta)$ is contained in the set*

$$\{z \,|\, \mathrm{Im}\, z > -\mu/8\} \cup \{z \,|\, \mathrm{Im}\, z < -\mu/2\}.$$

The spectrum inside the half-plane $\{z \,|\, \mathrm{Im}\, z > -\mu/8\}$ is discrete and, for $\lambda \neq 0$, consists of four simple eigenvalues $E_{j\lambda X}$ which do not depend on θ and are bounded analytic functions of $(\lambda, X) \in \{\lambda \,|\, |\lambda| < \Lambda\} \times D_\epsilon$. Moreover, $E_{0\lambda X} = 0$ and $\mathrm{Im}\, E_{j\lambda X} < 0$ for $j = 1,2,3$, $X \in D_\epsilon$, and $0 < |\lambda| < \Lambda$. The corresponding eigenprojections $P_{j\lambda X}(\theta)$ are bounded analytic functions of the variables (λ, X, θ).

With regard to the results of [JP1, JP3], the only part of Proposition 3.2 that requires a comment are the relations $E_{0\lambda X} = 0$ and $\mathrm{Im}\, E_{j\lambda X} < 0$ for $j = 1,2,3$, which hold for $X \in D_\epsilon$ and $0 < |\lambda| < \Lambda$. Regular perturbation theory and an explicit Fermi Golden Rule computation yield that the eigenvalues $E_{j\lambda X}$, $j = 2,3$, which are respectively near ± 2, satisfy

$$E_{2\lambda X} = -2 + \frac{\lambda^2}{2} \sum_j \left(-\mathrm{i}\pi \|\alpha_j(2)\|^2_{\mathfrak{H}_j} - \mathrm{PV}\int_{\mathbb{R}} \frac{\|\tilde{\alpha}_j(s)\|^2_{\mathfrak{H}_j}}{s-2} \mathrm{d}s \right) + \lambda^4 R_2(\lambda, X),$$

$$E_{3\lambda X} = 2 + \frac{\lambda^2}{2} \sum_j \left(-\mathrm{i}\pi \|\alpha_j(2)\|^2_{\mathfrak{H}_j} + \mathrm{PV}\int_{\mathbb{R}} \frac{\|\tilde{\alpha}_j(s)\|^2_{\mathfrak{H}_j}}{s-2} \mathrm{d}s \right) + \lambda^4 R_3(\lambda, X),$$

where PV stands for Cauchy's principal value and the functions $R_j(\lambda, X)$, $j = 2, 3$, are bounded and analytic for $X \in D_\epsilon$ and $|\lambda| < \Lambda$. Clearly, by choosing Λ small enough, we have that $\operatorname{Im} E_{j\lambda X} < 0$ for $j = 2, 3$, $X \in D_\epsilon$, and $0 < |\lambda| < \Lambda$. The eigenvalues $E_{j\lambda X}$, $j = 0, 1$, which are near 0, are the eigenvalues of a 2×2-matrix $\Sigma_{\lambda X}$ which has the form

$$\Sigma_{\lambda X} = \lambda^2 \Sigma_2(X) + \lambda^4 R(\lambda, X),$$

where the matrix-valued function $R(\lambda, X)$ is analytic and bounded for $X \in D_\epsilon$ and $|\lambda| < \Lambda$ and

$$\Sigma_2(X) = -\mathrm{i}\pi \sum_j \|\alpha_j(2)\|^2_{\mathfrak{H}_j} \frac{1}{2\cosh\beta_j} \begin{bmatrix} \mathrm{e}^{\beta_j} & -\mathrm{e}^{-\beta_j} \\ -\mathrm{e}^{\beta_j} & \mathrm{e}^{-\beta_j} \end{bmatrix}, \qquad \beta_j = \beta_{\mathrm{eq}} - X_j. \tag{3.41}$$

The eigenvalues of $\Sigma_2(X)$ are 0 and $-\mathrm{i}\pi \sum_j \|\alpha_j(2)\|^2_{\mathfrak{H}_j}$, and we conclude that for Λ small enough the eigenvalues $E_{0\lambda X}$ and $E_{1\lambda X}$ are analytic functions, that $E_{0\lambda X} \neq E_{1\lambda X}$ for $\lambda \neq 0$, and that $\operatorname{Im} E_{1\lambda X} < 0$ for $X \in D_\epsilon$, $0 < |\lambda| < \Lambda$. By construction of the C-Liouvillean, $E_{0\lambda X} = 0$ for X real. Hence, by analyticity, $E_{0\lambda X} = 0$ for $X \in D_\epsilon$ and $|\lambda| < \Lambda$.

The next technical result we need is:

Proposition 3.3 *There exist $\Lambda > 0$, $\epsilon > 0$, and $\mu > 0$ such that for all $|\lambda| < \Lambda$, all θ in the strip $-\mu < \operatorname{Im}\theta < -3\mu/4$ and all $\Psi \in \mathcal{H}$, the functions defined by*

$$F_+(z) = \sup_{X \in D_\epsilon} \|(z - \mathcal{L}_{\lambda X}(\theta))^{-1}\Psi\|, \quad F_-(z) = \sup_{X \in D_\epsilon} \|(\overline{z} - \mathcal{L}_{\lambda X}(\theta)^*)^{-1}\Psi\|,$$

satisfy

$$\int_{\mathbb{R}} |F_\pm(x \pm \mathrm{i}\mu)|^2 \mathrm{d}x \leq \frac{16\pi}{\mu} \|\Psi\|^2, \tag{3.42}$$

and

$$\lim_{|x| \to \infty} F_\pm(x + \mathrm{i}\eta) = 0. \tag{3.43}$$

for all $|\eta| \leq \mu/4$.

Proof. We only deal with $F_+(z)$, the other case is similar. We start with Λ, ϵ, and μ as in Proposition 3.2 and set

$$Q_\mu = (\mathbb{R} + \mathrm{i}\mu/4) \cup (\mathbb{R} - \mathrm{i}\mu/4) \cup \{z \in \mathbb{C} \,|\, |\operatorname{Re} z| \geq 2 + \mu/4, |\operatorname{Im} z| \leq \mu/4\}.$$

Since $\mathcal{L}_0(\theta)$ is normal and $\mathrm{dist}(Q_\mu, \sigma(\mathcal{L}_0(\theta))) \geq \mu/4$ for $\operatorname{Im}\theta \leq -3\mu/4$, the spectral theorem yields that

$$\sup_{z \in Q_\mu, \operatorname{Im}\theta \leq -3\mu/4} \|(z - \mathcal{L}_0(\theta))^{-1}\| \leq \frac{4}{\mu}. \tag{3.44}$$

The estimate

$$\int_{\mathbb{R}} \|(x \pm \mathrm{i}\mu/4 - \mathcal{L}_0(\theta))^{-1}\Psi\|^2 \mathrm{d}x \leq \frac{4\pi\|\Psi\|^2}{\mu}, \tag{3.45}$$

holds for all $\Psi \in \mathcal{H}$, and the dominated convergence theorem yields

$$\lim_{|z| \to \infty, z \in Q_\mu} \|(z - \mathcal{L}_0(\theta))^{-1}\Psi\| = 0. \tag{3.46}$$

We further impose that Λ and μ satisfy

$$\sup_{X \in D_\epsilon, -\mu < \operatorname{Im}\theta < 0} \|V_X(\theta) + W_X(\theta)\| \leq \frac{\mu}{8\Lambda}.$$

The resolvent identity yields

$$(z-\mathcal{L}_{\lambda X}(\theta))^{-1}=G(z,\lambda,X,\theta)(z-\mathcal{L}_0(\theta))^{-1},$$

where

$$G=G(z,\lambda,X,\theta)=\left(I-\lambda(z-\mathcal{L}_0(\theta))^{-1}(V_X(\theta)+W_X(\theta))\right)^{-1}.$$

The estimate (3.44) yields

$$\sup\|G\|\leq 2,$$

where the supremum is taken over $z\in Q_\mu$, $|\lambda|<\Lambda$, $X\in D_\epsilon$, and θ in the strip $-\mu<\operatorname{Im}\theta<-3\mu/4$. Hence, for $z\in Q_\mu$,

$$\sup_{X\in D_\epsilon}\|(z-\mathcal{L}_{\lambda X}(\theta))^{-1}\Psi\|\leq 2\,\|(z-\mathcal{L}_0(\theta))^{-1}\Psi\|,$$

and (3.45), (3.46) yield (3.42), (3.43). □

Assumption (A2) ensures that there is $\epsilon>0$ such that the operators

$$V(X,u)=\sum_{j=1}^{M}\sigma_x\otimes I\otimes\frac{1}{\sqrt{2}}\left(\tilde{a}_j^*(\mathrm{e}^{-u(1-X_j/\beta_{\rm eq})s}\tilde{\alpha}_{jX})+\tilde{a}_j(\mathrm{e}^{u(1-X_j/\beta_{\rm eq})s}\tilde{\alpha}_{jX})\right),$$

acting on $\mathcal{H}$ are well-defined continuous functions of $(X,u)\in\mathbb{I}_\epsilon\times[0,\beta_{\rm eq}]$ satisfying

$$\sup_{(X,u)\in\mathbb{I}_\epsilon\times[0,\beta_{\rm eq}]}\|V(x,u)\|<\infty.$$

If we set

$$\mathcal{G}_{\lambda X}=\mathbb{1}+\sum_{n\geq 1}(-\beta_{\rm eq})^n\int_{0\leq t_n\leq\cdots\leq t_1\leq 1}(\lambda V(X,\beta_{\rm eq}t_n)+\pi_X(H_{\mathcal{S}}))\cdots(\lambda V(X,\beta_{\rm eq}t_1)+\pi_X(H_{\mathcal{S}}))\mathrm{d}t_1\cdots\mathrm{d}t_n,$$

then the Araki perturbation theory [Ar, BR2, DJP] yields that the reference state $\omega_{\lambda X}$ can be written as

$$\omega_{\lambda X}(A)=\frac{(\Omega,\pi_X(A)\mathcal{G}_{\lambda X}\Omega)}{(\Omega,\mathcal{G}_{\lambda X}\Omega)}. \tag{3.47}$$

Consider the unitary group

$$U(\theta)=\mathrm{e}^{-\mathrm{i}\theta\sum_j P_j},$$

on $\mathcal{H}$.

Proposition 3.4 *There exist $\epsilon>0$ and $\mu>0$ such that:*
(1) *The function*

$$\mathbb{I}_\epsilon\times\mathbb{R}\ni(X,\theta)\mapsto U(\theta)\mathcal{G}_{\lambda X}\Omega\in\mathcal{H},$$

extends to a bounded analytic $\mathcal{H}$-valued function in the region $D_\epsilon\times I(\mu)$ for all $\lambda\in\mathbb{R}$. We denote this analytic extension by $\Omega_{\lambda X\theta}$.
(2) *For all $A\in\tilde{\mathcal{O}}$ the function*

$$\mathbb{I}_\epsilon\times\mathbb{R}\ni(X,\theta)\mapsto U(\theta)\pi_X(A)\Omega\in\mathcal{H},$$

extend to bounded analytic $\mathcal{H}$-valued functions in the region $D_\epsilon\times I(\mu)$. We denote this analytic extensions by $\Psi_{AX\theta}$.

Proof. We sketch the proof of (1). The proof of (2) is similar and simpler.

For $(X, u, \theta) \in \mathbb{I}_\epsilon \times [0, \beta_{\rm eq}] \times \mathbb{R}$ we set

$$\begin{aligned} V_\theta(X,u) &= U(\theta)V(X,u)U(\theta)^* \\ &= \sum_{j=1}^{M} \sigma_x \otimes I \otimes \frac{1}{\sqrt{2}} \left(\tilde{a}_j^*(\mathrm{e}^{-\mathrm{i}\theta p_j}\mathrm{e}^{-u(1-X_j/\beta_{\rm eq})s}\tilde{\alpha}_{jX}) + \tilde{a}_j(\mathrm{e}^{-\mathrm{i}\theta p_j}\mathrm{e}^{u(1-X_j/\beta_{\rm eq})s}\tilde{\alpha}_{jX}) \right). \end{aligned}$$

Since $U(\theta)\Omega = \Omega$, we can write $U(\theta)\mathcal{G}_{\lambda X}\Omega = \mathcal{G}_{\lambda X \theta}\Omega$ where $\mathcal{G}_{\lambda X\theta}$ is obtained by replacing $V(X,u)$ by $V_\theta(X,u)$ in the definition of $\mathcal{G}_{\lambda X}$. It is easy to see for any $\epsilon > 0$, $\mu > 0$ and $\rho > 0$ the entire analytic function $g(u,z,s) = \mathrm{e}^{u(1-z/\beta_{\rm eq})s}$ satisfies

$$\sup_{|u|<(1+\rho)\beta_{\rm eq}, |z|<\epsilon, |\mathrm{Im}\, s|<\mu} \left| \frac{g(u,z,s)}{\cosh(ls)} \right| < \infty,$$

where $l = (1+\rho)(\epsilon + \beta_{\rm eq})$. Let $\kappa > \beta_{\rm eq}$ be as in Assumption (A2). Choose ρ and ϵ such that $l < \kappa$. Since by (A2) one has $\cosh(\kappa s)\tilde{\alpha}_{jX} \in H_j^2(\delta)$, it follows that $V_\theta(X,u)$ has a bounded analytic extension to the set

$$\{(X,u,\theta) \,|\, X \in D_\epsilon, u \in \mathbb{C}, |u| < (1+\rho)\beta_{\rm eq}, |\mathrm{Im}\, \theta| < \mu\}.$$

This yields the statement. □

Proof of Theorem 1.11. We choose $\Lambda > 0$, $\epsilon > 0$, and $\mu > 0$ sufficiently small so that the statements in Propositions 3.2, 3.3 and 3.4 hold. Combining (3.38) and (3.47) we can write

$$\omega_{\lambda X}(\tau_\lambda^t(A)) = \frac{(\pi_X(A^*)\Omega, \mathrm{e}^{-\mathrm{i}t\mathcal{L}_{\lambda X}}\mathcal{G}_{\lambda X}\Omega)}{(\Omega, \mathcal{G}_{\lambda X}\Omega)}. \tag{3.48}$$

Since for $X \in I_\epsilon$

$$(\Omega, \mathcal{G}_{\lambda X}\Omega) = \|\mathrm{e}^{-\beta_{\rm eq}(\sum_j(1-X_j/\beta_{\rm eq})\mathcal{L}_j + \pi_X(\lambda V + H_{\mathcal{S}}))/2}\Omega\|^2 > 0,$$

by Proposition 3.4 (and by possibly taking ϵ smaller), the function $X \mapsto (\Omega, \mathcal{G}_{\lambda X}\Omega)$ extends to an analytic function in the region D_ϵ such that

$$\inf_{X \in D_\epsilon} |(\Omega, \mathcal{G}_{\lambda X}\Omega)| > 0.$$

Thus, it suffices to consider the numerator in (3.48). For $\mathrm{Im}\, z > 0$ we set

$$D_X(z) = \mathrm{i}(\pi_X(A^*)\Omega, (z - \mathcal{L}_{\lambda X})^{-1}\mathcal{G}_{\lambda X}\Omega).$$

For $|\lambda| < \Lambda$, $X \in \mathbb{I}_\epsilon$ and $-\mu < \mathrm{Im}\, \theta < -3\mu/4$ one has

$$D_X(z) = \mathrm{i}(\Psi_{A^* X \overline{\theta}}, (z - \mathcal{L}_{\lambda X}(\theta))^{-1}\Omega_{\lambda X \theta}),$$

which, by Proposition 3.2, has a meromorphic extension to the half-plane $\{\mathrm{Im}\, z > -\mu/2\}$. For $\alpha > 0$ denote by Γ_α the boundary of the rectangle with vertices $\pm\alpha \pm \mathrm{i}\mu/4$. For large enough α one has

$$I_X(t) = \oint_{\Gamma_\alpha} \mathrm{e}^{-\mathrm{i}tz} D_X(z) \frac{\mathrm{d}z}{2\pi\mathrm{i}} = \mathrm{i}\sum_{j=0}^{3} (\Psi_{A^* X\overline{\theta}}, P_{j\lambda X}(\theta)\Omega_{\lambda X\theta})\mathrm{e}^{-\mathrm{i}tE_{j\lambda X}}.$$

Denote by S_α the part of the above contour integral corresponding to the two vertical sides of Γ_α. It follows from the dominated convergence theorem and Proposition 3.3 that $\lim_{\alpha\to\infty} S_\alpha = 0$. Since by Proposition 3.3 the function $x \mapsto D_X(x + \mathrm{i}\mu/4)$ is in $L^2(\mathbb{R}, \mathrm{d}x)$ it follows from the Plancherel theorem that there exists a sequence α_n such that

$$\lim_n \int_{-\alpha_n}^{\alpha_n} \mathrm{e}^{-\mathrm{i}t(x+\mathrm{i}\mu/4)} D_X(x + \mathrm{i}\mu/4) \frac{\mathrm{d}x}{2\pi} = (\pi_X(A^*)\Omega, \mathrm{e}^{-\mathrm{i}t\mathcal{L}_{\lambda X}}\mathcal{G}_{\lambda X}\Omega),$$

for Lebesgue almost all $t > 0$. Integration by parts and (3.43) yield that for $t > 0$

$$\lim_n \int_{-\alpha_n}^{\alpha_n} \mathrm{e}^{-\mathrm{i}t(x-\mathrm{i}\mu/4)} D_X(x - \mathrm{i}\mu/4) \frac{\mathrm{d}x}{2\pi} = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t(x-\mathrm{i}\mu/4)} D'_X(x - \mathrm{i}\mu/4) \frac{\mathrm{d}x}{2\pi \mathrm{i}t},$$

where $D'_X(z)$ denotes the derivative of $D_X(z)$ with respect to z. Combining these facts we obtain the identity

$$\begin{aligned}(\pi_X(\Phi_j)\Omega, \mathrm{e}^{-\mathrm{i}t\mathcal{L}_{\lambda X}} \mathcal{G}_{\lambda X}\Omega) = &\sum_{j=0}^{3} (\Psi_{A^* X\overline{\theta}}, P_{j\lambda X}(\theta)\Omega_{\lambda X\theta}) \mathrm{e}^{-\mathrm{i}tE_{j\lambda X}} \\ &- \frac{\mathrm{e}^{-\mu t/4}}{2\pi t} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} (\Psi_{A^* X\overline{\theta}}, (x - \mathrm{i}\mu/4 - \mathcal{L}_{\lambda X}(\theta))^{-2} \Omega_{\lambda X\theta}) \mathrm{d}x,\end{aligned} \tag{3.49}$$

which holds for Lebesgue for almost all $t > 0$. By Proposition 3.3 the integrand on the right hand side of (3.49) is in $L^1(\mathbb{R}, \mathrm{d}x)$. Hence, both side of this identity are continuous functions of t and (3.49) holds for all $t > 0$. By Propositions 3.2 and 3.4 both terms on the right hand side of (3.49) have analytic extensions to $X \in D_\epsilon$ which are bounded uniformly in X and $t \geq 1$. The bound (3.40) and Proposition 3.4 yield that

$$\sup_{X \in D_\epsilon, t \in [0,1]} \left| (\Psi_{A^* X0}, \mathrm{e}^{-\mathrm{i}t\mathcal{L}_{\lambda X}} \Omega_{\lambda X0}) \right| < \infty,$$

and the result follows. □

4 Proof of Theorem 1.13

In Part (1) of Theorem 1.12 we have established that for given λ and $A \in \tilde{\mathcal{O}}$, the function $X \mapsto \omega_{\lambda X+}(A)$ is analytic near zero. In fact, a stronger result holds.

Theorem 4.1 *Assume that (A1)-(A3) hold and let $A \in \tilde{\mathcal{O}}$. Then there is $\Lambda > 0$ and $\epsilon > 0$ such that the maps*

$$(\lambda, X) \mapsto \omega_{\lambda X+}(A),$$

extend to analytic functions on $\{\lambda \,|\, |\lambda| < \Lambda\} \times D_\epsilon$.

Proof. By the construction of the NESS $\omega_{\lambda X+}$,

$$\omega_{\lambda X+}(A) = (\Omega, P_{0\lambda X}(\theta) U(\theta) \pi_X(A)\Omega),$$

where $-\mu < \mathrm{Im}\,\theta < -3\mu/4$ and $P_{0\lambda X}(\theta)$ and μ are as in Proposition 3.2. The analyticity of $P_{0\lambda X}(\theta)$ and Part (2) of Proposition 3.4 yield the statement. □

Theorem 4.1 yields that the function $\lambda \mapsto L_{\lambda kj}$ is analytic near zero. To compute the leading term in its power expansion we argue as follows.

By the relation (1.25) established in [JP3, JP4],

$$\omega_{\lambda X+}(\Phi_k) = \lambda^2 \omega_{\mathcal{S}X+}(\overline{\Phi}_{kX}) + O(\lambda^3),$$

where the remainder is uniform in X. Hence, (1.17) holds and

$$L_{kj}^{(2)} = \partial_{X_j} \omega_{\mathcal{S}X+}(\overline{\Phi}_{kX})\big|_{X=0}.$$

Let $\mathcal{D} \subset \mathcal{O}_{\mathcal{S}}$ be the set of observables which are diagonal in the eigenbasis $\{e_+, e_-\}$ of $H_{\mathcal{S}}$. The generators K_X and K_{X_k} preserve $\mathcal{D}$. The vector space $\mathcal{D}$ is naturally identified with $\mathbb{C}^2$. After this identification, $K_X = \mathrm{i}\Sigma_2(X)^*$, where $\Sigma_2(X)$ is given by (3.41), and

$$K_{X_k} = -\frac{\pi\|\alpha_k(2)\|^2_{\mathfrak{H}_k}}{2\cosh\beta_k}\begin{bmatrix} \mathrm{e}^{\beta_k} & -\mathrm{e}^{\beta_k} \\ -\mathrm{e}^{-\beta_k} & \mathrm{e}^{-\beta_k}\end{bmatrix}, \qquad \beta_k = \beta_{\mathrm{eq}} - X_k.$$

These relations between the generators K_X, K_{X_k} and the Fermi Golden Rule for the resonances of the C-Liouvillean are quite general—for the proofs and additional information we refer the reader to [DJ1]. Hence,

$$\overline{\Phi}_{kX} = K_{X_k}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{\pi\|\alpha_k(2)\|^2_{\mathfrak{H}_k}}{\cosh\beta_k}\begin{bmatrix} \mathrm{e}^{\beta_k} \\ -\mathrm{e}^{-\beta_k}\end{bmatrix}.$$

The density matrix describing $\omega_{\mathcal{S}X+}$ (which we denote by the same letter) is also diagonal in the basis $\{e_+, e_-\}$ and the vector in $\mathbb{C}^2$ associated to its diagonal elements is the eigenvector of $\Sigma_2(X)$ corresponding to the eigenvalue 0. Hence,

$$\omega_{\mathcal{S}X+} = \left(\sum_i \|\alpha_i(2)\|^2_{\mathfrak{H}_i}\right)^{-1}\begin{bmatrix} \displaystyle\sum_i \frac{\|\alpha_i(2)\|^2_{\mathfrak{H}_i}\mathrm{e}^{-\beta_i}}{2\cosh\beta_i} \\ \displaystyle\sum_i \frac{\|\alpha_i(2)\|^2_{\mathfrak{H}_i}\mathrm{e}^{\beta_i}}{2\cosh\beta_i}\end{bmatrix},$$

and we get

$$\omega_{\mathcal{S}X+}(\overline{\Phi}_{kX}) = \pi\left(\sum_i \|\alpha_i(2)\|^2_{\mathfrak{H}_i}\right)^{-1}\frac{\|\alpha_k(2)\|^2_{\mathfrak{H}_k}}{\cosh\beta_k}\sum_i \|\alpha_i(2)\|^2_{\mathfrak{H}_i}\frac{\sinh(\beta_i-\beta_k)}{\cosh\beta_i}. \tag{4.50}$$

It follows that for $j \neq k$,

$$L^{(2)}_{kj} = \partial_{X_j}\omega_{\mathcal{S}X+}(\overline{\Phi}_{kX})\big|_{X=0} = -\frac{\pi}{(\cosh\beta_{\mathrm{eq}})^2}\frac{\|\alpha_k(2)\|^2_{\mathfrak{H}_k}\|\alpha_j(2)\|^2_{\mathfrak{H}_j}}{\sum_i\|\alpha_i(2)\|^2_{\mathfrak{H}_i}}.$$

Since $\sum_k \omega_{\mathcal{S}X+}(\overline{\Phi}_{kX}) = 0$ we can conclude that $L^{(2)}_{jj} = -\sum_{k\neq j} L^{(2)}_{kj}$.

Finally, we remark that the formula (4.50) yields that

$$\overline{\mathrm{Ep}} = \frac{\pi}{2}\left(\sum_i \|\alpha_i(2)\|^2_{\mathfrak{H}_i}\right)^{-1}\sum_{k,j}\frac{\|\alpha_k(2)\|^2_{\mathfrak{H}_k}\|\alpha_j(2)\|^2_{\mathfrak{H}_j}}{\cosh\beta_k\cosh\beta_j}(\beta_k-\beta_j)\sinh(\beta_k-\beta_j). \tag{4.51}$$

Clearly, $\overline{\mathrm{Ep}} > 0$ whenever β_j's are not all equal.

5 Proofs of Theorems 1.14 and 1.15.

In this section we use the notational conventions of Subsection 1.5.

Proof of Theorem 1.14. The only part that requires a proof is (3). We only sketch the argument. Let $\vec{\beta}_0 = (\beta_{10}, \ldots, \beta_{M0})$ be a given point and $O_\epsilon = \{\vec{\beta} \in \mathbb{C}^M \mid |\vec{\beta} - \vec{\beta}_0| < \epsilon\}$. Arguing as in the proof of Lemma 3.1 one shows that there exists $\epsilon > 0$ and $\delta' > 0$ such that such that the maps

$$(\vec{\beta}, \theta) \mapsto V_{\vec{\beta}}(\theta), \qquad (\vec{\beta}, \theta) \mapsto W_{\vec{\beta}}(\theta),$$

extend to analytic operator-valued functions on $O_\epsilon \times I(\delta')$ satisfying

$$\sup_{\vec{\beta}\in O_\epsilon, \theta\in I(\delta')}\Big(\|V_{\vec{\beta}}(\theta)\| + \|W_{\vec{\beta}}(\theta)\|\Big) < \infty.$$

This implies that Proposition 3.2 holds with D_ϵ replaced with O_ϵ (of course, the index X is also replaced by $\vec{\beta}$). Note that Λ depends on the ϵ. Complementing the construction in [JP3] with arguments used in the proof of Proposition 3.4 one easily shows that there exists a norm-dense $*$-algebra $\mathcal{O}_0$ of $\mathcal{O}$ such that:
(a) $\mathcal{O}_0$ does not depend on the choice of $\vec{\beta}_0$;
(b) $\Phi_j \in \mathcal{O}_0$;
(c) for all $A \in \mathcal{O}_0$ the functions

$$(\vec{\beta}, \theta) \mapsto U(\theta)\pi_{\vec{\beta}}(A)\Omega \in \mathcal{H},$$

extend to bounded analytic $\mathcal{H}$-valued functions in the region $O_\epsilon \times I(\mu)$. The representation

$$\omega_{\lambda\vec{\beta}+}(A) = (\Omega, P_{0\lambda\vec{\beta}}(\theta)U(\theta)\pi_{\vec{\beta}}(A)\Omega),$$

where $-\mu < \operatorname{Im}\theta < -3\mu/4$ and $P_{0\lambda\vec{\beta}}(\theta)$ and μ are as in the analog of Proposition 3.2, yields the following statement: For any given $\vec{\beta}_0 \in \mathbb{I}_{\gamma_1\gamma_2}$ there exists Λ and ϵ such that the function

$$(\lambda, \vec{\beta}) \mapsto \omega_{\lambda\vec{\beta}+}(A),$$

extends to an analytic functions on $\{\lambda \,|\, |\lambda| < \Lambda\} \times O_\epsilon$ for all $A \in \mathcal{O}_0$. This fact and the compactness of $I_{\gamma_1\gamma_2}$ yield the statement. $\Box$

Proof of Theorem 1.15. By Remark 1 after Theorem 1.15, it suffices to establish Part (1). By Remark 2, it suffices to show that there exists $\delta > 0$ and $\Lambda > 0$ such that for $0 < |\lambda| < \Lambda$

$$\mathrm{Ep}(\omega_{\lambda\vec{\beta}+}) > 0,$$

for $\vec{\beta} \in \mathbb{I}_{\gamma_1\gamma_2}$ satisfying $0 < \sum_{i,j} |\beta_i - \beta_j| < \delta$.

Let $\vec{\beta}_0 = (\beta_0, \dots, \beta_0)$ be a given point on the diagonal of $\mathbb{I}_{\gamma_1\gamma_2}$. We set

$$O_\delta = \{\vec{\beta} \in \mathbb{C}^M \mid \sum_j |\beta_j - \beta_0| < \delta\},$$

and $\mathbb{I}_\delta = O_\delta \cap \mathbb{R}^M$. One can choose Λ and δ such that $(\lambda, \vec{\beta}) \mapsto \mathrm{Ep}(\omega_{\lambda\vec{\beta}+})$ is an analytic function on $\{|\lambda| < \Lambda\} \times O_\delta$. We set

$$Y_{\vec{\beta}} = (\beta_2 - \beta_1, \dots, \beta_M - \beta_1).$$

Setting $\beta_1 = \beta_{\mathrm{eq}}$ one deduces from the formula (4.51) and the Taylor series for $\mathrm{Ep}(\omega_{\lambda\vec{\beta}+})$ (use that $\mathrm{Ep}(\omega_{\lambda\vec{\beta}+})$ and $\partial_{\beta_i}\mathrm{Ep}(\omega_{\lambda\vec{\beta}+})$ vanish when all β_j are equal) that there exists $(M-1)\times(M-1)$-matrix valued functions $A(\vec{\beta})$ and $B(\lambda, \vec{\beta})$ such that:
(a) $A(\vec{\beta})$ is analytic for $\vec{\beta} \in O_\delta$ and strictly positive for $\vec{\beta}$ real;
(b) $B(\lambda, \vec{\beta})$ is analytic and bounded on $\{|\lambda| < \Lambda\} \times O_\epsilon$;
(c)

$$\mathrm{Ep}(\omega_{\lambda\vec{\beta}+}) = \lambda^2 (Y_{\vec{\beta}}, A(\vec{\beta})Y_{\vec{\beta}}) + \lambda^3 (Y_{\vec{\beta}}, B(\lambda, \vec{\beta})Y_{\vec{\beta}}).$$

By choosing Λ small enough we can ensure that for all $\vec{\beta} \in \mathbb{I}_\delta$ and $|\lambda| < \Lambda$,

$$(Y_{\vec{\beta}}, A(\vec{\beta})Y_{\vec{\beta}}) > |\lambda (Y_{\vec{\beta}}, B(\lambda, \vec{\beta}))Y_{\vec{\beta}})|.$$

This yields that $\mathrm{Ep}(\omega_{\lambda\vec{\beta}+}) > 0$ for $0 < |\lambda| < \Lambda$ and $\vec{\beta} \in \mathbb{I}_\delta$ satisfying $Y_{\vec{\beta}} \neq 0$. This local result combined with an obvious compactness argument yields the statement. $\Box$

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