SUFFICIENCY CLASS FOR GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS IN $\mathbb{V}$

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Abstract. Let Ω be an open domain of class $\mathbb{C}^3$ contained in $\mathbb{R}^3$, let $(\mathbb{L}^2[\Omega])^3$ be the real Hilbert space of square integrable functions on Ω with values in $\mathbb{R}^3$, and let $\mathbf{D}[\Omega] = \{\mathbf{u} \in (\mathbb{C}_0^\infty[\Omega])^3 \mid \nabla \cdot \mathbf{u} = 0\}$. Let $\mathbb{H}[\Omega]$ be the completion of $\mathbf{D}$ with respect to the inner product of $(\mathbb{L}^2[\Omega])^3$ and let $\mathbb{V}[\Omega]$ be the completion of $\mathbf{D}[\Omega]$ with respect to the inner product of $\mathbb{H}^1[\Omega]$, the functions in $\mathbb{H}[\Omega]$ with weak derivatives in $(\mathbb{L}^2[\Omega])^3$. A well-known unsolved problem is the construction of a sufficient class of functions in $\mathbb{H}[\Omega]$ (respectively $\mathbb{V}[\Omega]$), which will allow global, in time, strong solutions to the three-dimensional Navier-Stokes equations. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. In this paper, we prove that, under appropriate conditions, there exists a number $\mathbf{u}_+$, depending only on the domain, the viscosity, the body forces and the eigenvalues of the Stokes operator, such that, for all functions in a dense set $\mathbb{D}$ contained in the closed ball $\mathbb{B}(\Omega)$ of radius $\mathbf{u}_+$ in $\mathbb{V}[\Omega]$, the Navier-Stokes equations have unique strong solutions in $\mathbb{C}^1((0,\infty), \mathbb{V}[\Omega])$.

Introduction

Let Ω be an open domain of class $\mathbb{C}^3$ contained in $\mathbb{R}^3$, let $(\mathbb{L}^2[\Omega])^3$ be the real Hilbert space of square integrable functions on Ω with values in $\mathbb{R}^3$, let $\mathbf{D}[\Omega]$ be

1991 *Mathematics Subject Classification.* Primary (35Q30) Secondary(47H20, 76DO3) .

Key words and phrases. Global (in time), 3D-Navier-Stokes Equations.

$\{\mathbf{u} \in (\mathbb{C}_0^\infty[\Omega])^3 \mid \nabla \cdot \mathbf{u} = 0\}$, let $\mathbb{H}[\Omega]$ be the completion of $\mathbf{D}[\Omega]$ with respect to the inner product of $(\mathbb{L}^2[\Omega])^3$, and let $\mathbb{V}[\Omega]$ be the completion of $\mathbf{D}[\Omega]$ with respect to the inner product of $\mathbb{H}^1[\Omega]$, the functions in $\mathbb{H}[\Omega]$ with weak derivatives in $(\mathbb{L}^2[\Omega])^3$. The global in time classical Navier-Stokes initial-value problem (for $\Omega \subset \mathbb{R}^3$, and all $T > 0$) is to find functions $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^3$, and $p : [0,T] \times \Omega \to \mathbb{R}$, such that

$$
\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T) \times \Omega, \\
&\nabla \cdot \mathbf{u} = 0 \text{ in } (0,T) \times \Omega, \\
&\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \text{ on } (0,T) \times \partial\Omega \ \text{ (in the distributional sense)}, \\
&\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega.
\end{aligned} \tag{1}
$$

The equations describe the time evolution of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ and the pressure p of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient ν in terms of a given initial velocity $\mathbf{u}_0(\mathbf{x})$ and given external body forces $\mathbf{f}(\mathbf{x}, t)$.

Purpose

Let $\mathbb{P}$ be the (Leray) orthogonal projection of $(\mathbb{L}^2[\Omega])^3$ onto $\mathbb{H}[\Omega]$ and define the Stokes operator by: $\mathbf{A}\mathbf{u} =: -\mathbb{P}\Delta \mathbf{u}$, for $\mathbf{u} \in D(\mathbf{A}) \subset \mathbb{H}^2[\Omega]$, the domain of $\mathbf{A}$. The purpose of this paper is to prove that there exists a number $\mathbf{u}_+$, depending only on $\mathbf{A}$, f, ν and Ω, such that, for all functions in $\mathbb{D} = D(\mathbf{A}^{3/2}) \cap \mathbb{B}(\Omega)$, where $D(\mathbf{A}^{3/2})$ is the domain of $\mathbf{A}^{3/2}$ and $\mathbb{B}(\Omega)$ is the closed ball of radius $\mathbf{u}_+$ in $\mathbb{V}(\Omega)$, the Navier-Stokes equations have unique strong solutions in $\mathbf{u} \in L^\infty_{\text{loc}}[[0,\infty); \mathbb{V}(\Omega)] \cap \mathbb{C}^1[(0,\infty); \mathbb{V}(\Omega)]$. We discuss this problem in $\mathbb{H}(\Omega)]$, in another paper.

Preliminaries

In terms of notation and conventions, we follow Sell and You [SY]. In order to simplify our proofs, we always assume that all functions $\mathbf{u}, \mathbf{v}$ are in $D(\mathbf{A}^{3/2})$ and we let $c = \max\{c_i\}$, where c_i is one of nine positive constants that appear on pages 363-367 in [SY]. It will also be convenient to use the fact that the norms of $\mathbb{V}[\Omega]$ and $\mathbb{V}[\Omega]^{-1}$ are equivalent to their respective graph norms relative to $\mathbb{H}[\Omega]$. It is known that $\mathbf{A}$ is a positive linear operator with compact resolvent. It follows that the fractional powers $\mathbf{A}^{1/2}$ and $\mathbf{A}^{-1/2}$ are well defined. Moreover, it is also known (cf. [SY], [T1]) that the norms $\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]}$ and $\left\|\mathbf{A}^{-1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]}$ are equivalent to the corresponding norms induced by the Sobolev space $(H^1[\Omega])^3$, so that:

$$\|\mathbf{u}\|_{\mathbb{V}[\Omega]} \equiv \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]} \text{ and } \|\mathbf{u}\|_{\mathbb{V}[\Omega]^{-1}} \equiv \left\|\mathbf{A}^{-1/2}\mathbf{u}\right\|_{\mathbb{H}[\Omega]}. \tag{2}$$

In addition, it is known that $\mathbf{A}$ is an isomorphism from $D(\mathbf{A}) \xrightarrow{onto} \mathbb{H}[\Omega]$, and from $\mathbb{V}[\Omega] \xrightarrow{onto} \mathbb{V}[\Omega]^{-1}$. Furthermore, the embeddings $\mathbb{V}[\Omega] \to \mathbb{H}[\Omega] \to \mathbb{V}[\Omega]^{-1}$ are compact and the operator $\mathbf{A}^{-1}$ is a bounded compact map from $\mathbb{H}[\Omega]$ onto $D(\mathbf{A})$. Applying the Leray projection to equation (1), with $\mathbf{B}(\mathbf{u},\mathbf{u}) = \mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$, we can recast equation (1) in the standard form:

$$\begin{aligned} &\partial_t \mathbf{u} = -\nu\mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t) \text{ in } (0,T)\times\Omega, \\ &\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\partial\Omega, \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega, \end{aligned} \tag{3}$$

where we have used the fact that the orthogonal complement of $\mathbb{H}[\Omega]$ relative to $(\mathbb{L}^2[\Omega])^3$ is $\{\mathbf{v} : \mathbf{v} = \nabla q,\ q \in (H^1[\Omega])^3\}$ to eliminate the pressure term (see Galdi [GA] or [SY, T1]).

We will use the following inequalities from [SY], pages 363-367. (We use their numbering for easy reference.) Equation (61.8)

$$\left\|\mathbf{A}^{\alpha}\mathbf{u}\right\|_{\mathbb{H}}^{2} \geqslant \lambda_1^{2\alpha}\left\|\mathbf{u}\right\|_{\mathbb{H}}^{2}. \tag{4}$$

Equation (61.26)

$$\begin{aligned}&\left\|\mathbf{A}^{-1/2}\mathbf{B}(\mathbf{u},\mathbf{v})\right\|_{\mathbb{H}} \leqslant c_5\left\|\mathbf{u}\right\|_{\mathbb{H}}^{1/4}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}^{3/4}\left\|\mathbf{v}\right\|_{\mathbb{H}}^{1/4}\left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}}^{3/4}\\ &\leqslant c_5\lambda_1^{-1/4}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}} \leqslant c\lambda_1^{-1/4}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}},\\ &\Rightarrow \left\|\mathbf{A}^{-1/2}\mathbf{B}(\mathbf{u},\mathbf{u})\right\|_{\mathbb{H}} \leqslant c\lambda_1^{-1/4}\left\|\mathbf{u}\right\|_{\mathbb{V}}^{2}.\end{aligned} \tag{5}$$

We can use equation (61.21):

$$\left\|\mathbf{B}(\mathbf{u},\mathbf{v})\right\|_{\mathbb{H}} \leqslant c_1\left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}^{1/4}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}^{3/4}\left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}^{1/4}\left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}}^{3/4}$$

and the fact that $\lambda_1^{-3/4} \leq \lambda_1^{-1/4}$, along with equation (61.8) to get that:

$$\begin{aligned}&\left|\langle\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{V}}\right| = \left|\langle\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{A}\mathbf{w}\rangle_{\mathbb{H}}\right|\\ &\leqslant c_1\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}^{3/4}\left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}^{1/4}\left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{H}}^{3/4}\left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}^{1/4}\left\|\mathbf{A}\mathbf{w}\right\|_{\mathbb{H}}\\ &\leqslant c_1\lambda_1^{-3/4}\left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}\left\|\mathbf{A}\mathbf{w}\right\|_{\mathbb{H}} \leqslant c\lambda_1^{-1/4}\left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}\left\|\mathbf{A}\mathbf{w}\right\|_{\mathbb{H}}.\end{aligned} \tag{6}$$

Using the fact that

$$\begin{aligned}&\left\langle\left[\mathbf{B}(\mathbf{u},\mathbf{u})-\mathbf{B}(\mathbf{v},\mathbf{v})\right],\mathbf{u}-\mathbf{v}\right\rangle_{\mathbb{V}}\\ &= \tfrac{1}{2}\left\langle\left[\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{u})+\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{v})\right],\mathbf{A}(\mathbf{u}-\mathbf{v})\right\rangle_{\mathbb{H}},\end{aligned} \tag{7}$$

we have from equation (6) that:

$$\begin{aligned}&\left\langle\left[\mathbf{B}(\mathbf{u},\mathbf{u})-\mathbf{B}(\mathbf{v},\mathbf{v})\right],\mathbf{u}-\mathbf{v}\right\rangle_{\mathbb{V}}\\ &\leqslant c\lambda_1^{-1/4}\left\|\mathbf{A}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}^{2}\left\{\left\|\mathbf{A}\mathbf{u}\right\|_{\mathbb{H}}+\left\|\mathbf{A}\mathbf{v}\right\|_{\mathbb{H}}\right\}.\end{aligned} \tag{8}$$

Definition 1. *We say that the operator* $\mathbf{J}(\cdot,t)$ *is (for each* t*)*

(1) *0-Dissipative if* $\langle\mathbf{J}(\mathbf{u},t),\mathbf{u}\rangle_{\mathbb{V}} \leq 0$.

(2) *Dissipative if* $\langle \mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{v},t), \mathbf{u}-\mathbf{v}\rangle_{\mathbb{V}} \leq 0$.

(3) *Strongly dissipative if there exists a constant* $\alpha > 0$ *such that*

$$\langle \mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{v},t), \mathbf{u}-\mathbf{v}\rangle_{\mathbb{V}} \leq -\alpha \left\|\mathbf{u}-\mathbf{v}\right\|_{\mathbb{V}}^2.$$

Theorem 2 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35, page 887], while Theorem 3 is from Miyadera [M, p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]) .

Theorem 2. *Let* $\mathbb{B}[\Omega]$ *be a closed, bounded, convex subset of* $\mathbb{V}[\Omega]$*. If* $\mathbf{J}(\cdot,t)$: $\mathbb{B}[\Omega] \to \mathbb{V}[\Omega]$ *is closed and strongly dissipative for each fixed* $t \geq 0$*, then, for each* $\mathbf{b} \in \mathbb{B}[\Omega]$*, there is a* $\mathbf{u} \in \mathbb{B}[\Omega]$ *with* $\mathbf{J}(\mathbf{u},t) = \mathbf{b}$ *(the range Ran[*$\mathbf{J}(\cdot,t)$*]* $\supset \mathbb{B}[\Omega]$*).*

Theorem 3. *Let* $\{\ \mathcal{A}(t), t \in I = [0,\infty)\}$ *be a family of operators defined on* $\mathbb{V}[\Omega]$ *with domains* $D(\mathcal{A}(t)) = D$ *independent of* t*. We assume that* $\mathbb{D} = D \cap \mathbb{B}[\Omega]$ *is a closed convex set (in an appropriate topology):*

(1) *The operator* $\mathcal{A}(t)$ *is the generator of a contraction semigroup for each* $t \in I$*.*

(2) *The function* $\mathcal{A}(t)\mathbf{u}$ *is continuous in both variables on* $I \times \mathbb{D}$*.*

Then, for every $\mathbf{u}_0 \in \mathbb{D}$*, the problem* $\partial_t \mathbf{u}(t,\mathbf{x}) = \mathcal{A}(t)\mathbf{u}(t,\mathbf{x})$*,* $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$*, has a unique solution* $\mathbf{u}(t,\mathbf{x}) \in \mathbb{C}^1(I;\mathbb{D})$*.*

M-Dissipative Conditions

We assume that $\mathbf{f}(t) \in L^\infty[[0,\infty);\mathbb{V}(\Omega)]$ and is Lipschitz continuous in t, with $\left\|\mathbf{f}(t) - \mathbf{f}(\tau)\right\|_{\mathbb{V}} \leq d\left|t-\tau\right|^\theta,\ d > 0,\ 0 < \theta < 1$. We can rewrite equation (3) in the form:

$$\begin{aligned} &\partial_t \mathbf{u} = \nu \mathbf{A}\mathbf{J}(\mathbf{u},t) \text{ in } (0,T)\times\Omega, \\ &\mathbf{J}(\mathbf{u},t) = -\mathbf{u} - \nu^{-1}\mathbf{A}^{-1}\mathbf{B}(\mathbf{u},\mathbf{u}) + \nu^{-1}\mathbf{A}^{-1}\mathbb{P}\mathbf{f}(t). \end{aligned} \tag{9}$$

Approach

We begin with a study of the operator $\mathbf{J}(\cdot,t)$, for fixed t, and seek conditions depending on $\mathbf{A}$, ν, Ω and $\mathbf{f}(t)$ which guarantee that $\mathbf{J}(\cdot,t)$ is m-dissipative for each t. Clearly $\mathbf{J}(\cdot,t) : D(\mathbf{A}) \xrightarrow{onto} D(\mathbf{A})$ and, since $\nu\mathbf{A} = \nu\mathbb{P}[-\Delta]$ is a closed positive (m-accretive) operator, so that $-\mathbf{A}$ generates a linear contraction semigroup, we expect that $\nu\mathbf{A}\mathbf{J}(\cdot,t)$ will be m-dissipative for each t.

Theorem 4. *For $t \in I = [0,\infty)$ and, for each fixed $\mathbf{u}$, $\mathbf{J}(\mathbf{u},t)$ is Lipschitz continuous, with $\|\mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},\tau)\|_{\mathbb{V}} \le d'\,|t-\tau|^{\theta}$, where $d' = d\nu^{-1}(\lambda_1)^{-1}$, d is the Lipschitz constant for the function $\mathbf{f}(t)$ and λ_1 is the first eigenvalue of $\mathbf{A}$.*

Proof. For fixed $\mathbf{u}$,

$$\begin{aligned} &\|\mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},\tau)\|_{\mathbb{V}} = \nu^{-1}\left\|\mathbf{A}^{-1}[\mathbb{P}\mathbf{f}(t) - \mathbb{P}\mathbf{f}(\tau)]\right\|_{\mathbb{V}} \\ &\quad \le d\nu^{-1}(\lambda_1)^{-1}\,|t-\tau|^{\theta} = d'\,|t-\tau|^{\theta}. \end{aligned}$$

We have used the fact that $\mathbf{A}$ is unbounded, and every function $\mathbf{h}(t) \in \mathbb{V}(\Omega)$ has an expansion in terms of the eigenfunctions of $\mathbf{A}$, so that $\mathbf{A}^{-1}\,\mathbf{h}(t) = \sum_{k=1}^{\infty}\lambda_k^{-1}h_k(t)\mathbf{e}^k(\mathbf{x})$, and, from here, it is easy to see that $\left\|\mathbf{A}^{-1}\,\mathbf{h}(t)\right\|_{\mathbb{V}} \le \lambda_1^{-1}\,\|\,\mathbf{h}(t)\|_{\mathbb{V}}$. (It is well known that the eigenvalues of $\mathbf{A}$ are positive and increasing (see Temam [T2]).) □

Main Results

Theorem 5. *Let $f = \sup_{t\in\mathbf{R}^+} \|\mathbb{P}\mathbf{f}(t)\|_{\mathbb{H}} < \infty$, then there exists a positive constant $\mathbf{u}_+$, depending only on f, $\mathbf{A}$, ν and Ω, such that for all $\mathbf{u}$, with $\|\mathbf{u}\|_{\mathbb{V}} \le \mathbf{u}_+$, $\mathbf{J}(\cdot, t)$ is strongly dissipative.*

Proof. The proof of our first assertion has two parts. First, we require that the nonlinear operator $\mathbf{J}(\cdot, t)$ be 0-dissipative, which gives us an upper bound $\mathbf{u}_+$, in terms of the norm (e.g., $\|\mathbf{u}\|_{\mathbb{V}} \leqslant \mathbf{u}_+$). We then use this part to show that $\mathbf{J}(\cdot, t)$ is strongly dissipative on the closed ball, $\mathbb{B} = \{\mathbf{u} \in \mathbb{V}(\Omega) : \|\mathbf{u}\|_{\mathbb{V}} \leqslant \mathbf{u}_+\}$.

Part 1) From equation (7), we have

$$\langle \mathbf{J}(\mathbf{u}, t), \mathbf{u}\rangle_{\mathbb{V}} = -\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbb{V}} - \nu^{-1}\left\langle \mathbf{A}^{-1}\mathbf{B}(\mathbf{u}, \mathbf{u}) + \mathbf{A}^{-1}\mathbb{P}\mathbf{f}(t), \mathbf{u}\right\rangle_{\mathbb{V}}$$

$$\leqslant -\|\mathbf{u}\|_{\mathbb{V}}^2 + \nu^{-1}\left\|\mathbf{A}^{-1}\mathbf{B}(\mathbf{u}, \mathbf{u})\right\|_{\mathbb{V}} \|\mathbf{u}\|_{\mathbb{V}} + \nu^{-1}\left\|\mathbf{A}^{-1}\mathbb{P}\mathbf{f}(t)\right\|_{\mathbb{V}} \|\mathbf{u}\|_{\mathbb{V}}$$

$$= -\|\mathbf{u}\|_{\mathbb{V}}^2 + \nu^{-1}\left\|\mathbf{A}^{-1/2}\mathbf{B}(\mathbf{u}, \mathbf{u})\right\|_{\mathbb{H}} \|\mathbf{u}\|_{\mathbb{V}} + \nu^{-1}\left\|\mathbf{A}^{-1/2}\mathbb{P}\mathbf{f}(t)\right\|_{\mathbb{H}} \|\mathbf{u}\|_{\mathbb{V}}$$

Using $\left\|\mathbf{A}^{-1/2}\mathbf{B}(\mathbf{u}, \mathbf{u})\right\|_{\mathbb{H}} \leqslant c\lambda_1^{-1/4}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{H}}^2$ and $\left\|\mathbf{A}^{-1/2}\mathbb{P}\mathbf{f}(t)\right\|_{\mathbb{H}} \leqslant \lambda_1^{-1/2} f$, we have that

$$\langle \mathbf{J}(\mathbf{u}, t), \mathbf{u}\rangle_{\mathbb{V}} \leqslant -\|\mathbf{u}\|_{\mathbb{V}}^2 + \nu^{-1}c\lambda_1^{-1/4}\|\mathbf{u}\|_{\mathbb{V}}^2 \|\mathbf{u}\|_{\mathbb{V}} + \nu^{-1}\lambda_1^{-1/2} f \|\mathbf{u}\|_{\mathbb{V}}$$

$$= -\|\mathbf{u}\|_{\mathbb{V}}^2 + \nu^{-1}c\lambda_1^{-1/4}\|\mathbf{u}\|_{\mathbb{V}}^3 + \nu^{-1}\lambda_1^{-1/2} f \|\mathbf{u}\|_{\mathbb{V}} \leqslant 0$$

$$\Rightarrow$$

$$\|\mathbf{u}\|_{\mathbb{V}}\left\{\nu^{-1}c\lambda_1^{-1/4}\|\mathbf{u}\|_{\mathbb{V}}^2 - \|\mathbf{u}\|_{\mathbb{V}} + \nu^{-1}\lambda_1^{-1/2} f\right\} \leqslant 0.$$

Since $\|\mathbf{u}\|_{\mathbb{V}} > 0$, we can solve to get that:

$$\mathbf{u}_{\pm} = \tfrac{1}{2}\nu\lambda_1^{1/4}c^{-1}\left\{1 \pm \sqrt{1 - [4cf \big/ \lambda_1^{3/4}\nu^2]}\right\} = \tfrac{1}{2}\nu\lambda_1^{1/4}c^{-1}\left\{1 \pm \sqrt{1-\gamma}\right\}$$

Since we want real distinct solutions, we must require that

$$\gamma = 4cf \Big/ \lambda_1^{3/4}\nu^2 < 1 \Rightarrow \lambda_1^{3/4}\nu^2 > 4cf \ \Rightarrow \ \nu > 2\lambda_1^{-3/8}(cf)^{1/2}.$$

It follows that if $\mathbb{P}\mathbf{f} \neq \mathbf{0}$, then $\mathbf{u}_- < \mathbf{u}_+$, and our requirement that $\mathbf{J}$ is 0-dissipative implies that

$$\|\mathbf{u}\|_{\mathbb{V}} - \mathbf{u}_+ \leqslant 0, \ \ \|\mathbf{u}\|_{\mathbb{V}} - \mathbf{u}_- \geqslant 0.$$

This means that, whenever $\mathbf{u}_- \leqslant \|\mathbf{u}\|_{\mathbb{V}} \leqslant \mathbf{u}_+, \ \langle \mathbf{J}(\mathbf{u},t), \mathbf{u}\rangle_{\mathbb{V}} \leqslant 0$. (It is clear that when $\mathbb{P}\mathbf{f}(t) = \mathbf{0}, \ \mathbf{u}_- = \mathbf{0}$, and $\mathbf{u}_+ = \nu\lambda_1^{1/4}c^{-1}$.)

Part 2): Now, for any $\mathbf{u},\mathbf{v} \in \mathbb{V}(\Omega)$ with $\max(\ \|\mathbf{u}\|_{\mathbb{V}}, \|\mathbf{v}\|_{\mathbb{V}}) \leq \mathbf{u}_+$, we have that

$$\langle \mathbf{J}(\mathbf{u},t) - \mathbf{J}(\mathbf{v},t), \mathbf{u}-\mathbf{v}\rangle_{\mathbb{V}} = -\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2 + \tfrac{1}{2}\nu^{-1}\left\langle \mathbf{A}^{-1}\left\{\mathbf{B}[(\mathbf{u}-\mathbf{v}),\mathbf{u}] + \mathbf{B}[(\mathbf{u}-\mathbf{v}),\mathbf{v}]\right\}, (\mathbf{u}-\mathbf{v})\right\rangle_{\mathbb{V}}$$

$$\leqslant -\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2 + \tfrac{1}{2}\nu^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}\left(\left\|\mathbf{A}^{-1}\mathbf{B}[(\mathbf{u}-\mathbf{v}),\mathbf{u}]\right\|_{\mathbb{V}} + \left\|\mathbf{A}^{-1}\mathbf{B}[(\mathbf{u}-\mathbf{v}),\mathbf{v}]\right\|_{\mathbb{V}}\right)$$

$$\leqslant -\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2 + \tfrac{1}{2}c(\nu\lambda_1^{1/4})^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2\left(\|\mathbf{u}\|_{\mathbb{V}} + \|\mathbf{v}\|_{\mathbb{V}}\right)$$

$$\leqslant -\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2 + c(\nu\lambda_1^{1/4})^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2(\mathbf{u}_+)$$

$$= -\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2 + c(\nu\lambda_1^{1/4})^{-1}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2\left(\tfrac{1}{2}\nu\lambda_1^{1/4}c^{-1}\left\{1+\sqrt{1-\gamma}\right\}\right)$$

$$= -\tfrac{1}{2}\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2\left\{1-\sqrt{1-\gamma}\right\}$$

$$= -\alpha\|\mathbf{u}-\mathbf{v}\|_{\mathbb{V}}^2, \ \alpha = \tfrac{1}{2}\left\{1-\sqrt{1-\gamma}\right\}.$$

It follows that $\mathbf{J}(\mathbf{x},t)$ is strongly dissipative. □

Let $\mathbb{B}(\Omega) = \{\mathbf{u} \in \mathbb{V}(\Omega) : \|\mathbf{u}\|_{\mathbb{V}} \leqslant \mathbf{u}_+\}$, $\mathbb{B}_+(\Omega) = \left\{\mathbf{u} \in \mathbb{V}(\Omega) : \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{V}} \leqslant \mathbf{u}_+\right\}$ and $\mathbb{B}_{++}(\Omega) = \{\mathbf{u} \in \mathbb{V}(\Omega) : \|\mathbf{A}\mathbf{u}\|_{\mathbb{V}} \leqslant \mathbf{u}_+\}$. We now show that $Ran(I - \beta\nu\mathbf{A}\mathbf{J}) \supset \mathbb{B}(\Omega), \ \beta > 0$.

Theorem 6. *The operator $\mathcal{A}(t) = \nu \mathbf{AJ}(\cdot, t)$ is closed, dissipative and jointly continuous in $\mathbf{u}$ and t. Furthermore, for each $t \in \mathbf{R}^+$ and $\beta > 0$, $Ran[I - \beta\mathcal{A}(t)] \supset \mathbb{B}[\Omega]$, so that $\mathcal{A}(t)$ is m-dissipative on $\mathbb{B}_{++}$.*

Proof. Since $\mathbf{J}(\cdot, t)$ is strongly dissipative and closed on $\mathbb{V}[\Omega]$, it follows from Theorem 6 that $Ran[\mathbf{J}(\cdot, t)] \supset \mathbb{B}[\Omega]$.

To show that $\mathcal{A}(t) = \nu \mathbf{AJ}(\cdot, t)$ is dissipative, first note that for $\mathbf{u}, \mathbf{v} \in \mathbb{B}_+$, and using equation (8), we have

$$\begin{aligned}&\tfrac{1}{2}\left|\left\langle \mathbf{A}^{1/2}\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{v}), \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\rangle_{\mathbb{H}} + \left\langle \mathbf{A}^{1/2}\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{u}), \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\rangle_{\mathbb{H}}\right| \\ &\leqslant \tfrac{1}{2}c\lambda_1^{-1/4}\left\|\mathbf{A}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}^2\left(\left\|\mathbf{Au}\right\|_{\mathbb{H}} + \left\|\mathbf{Av}\right\|_{\mathbb{H}}\right).\end{aligned}$$

Using this result, we have that

$$\begin{aligned}&\langle \mathcal{A}(t)\mathbf{u} - \mathcal{A}(t)\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle_{\mathbb{V}} = -\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{V}}^2 \\ &-\tfrac{1}{2}\left\langle \mathbf{A}^{1/2}\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{u}) + \mathbf{A}^{1/2}\mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{v}), \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\rangle_{\mathbb{H}} \\ &\leqslant -\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{V}}^2 + \tfrac{1}{2}\lambda_1^{-1/4}c\left\|\mathbf{A}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{H}}^2\left(\left\|\mathbf{Au}\right\|_{\mathbb{H}} + \left\|\mathbf{Av}\right\|_{\mathbb{H}}\right) \\ &= \nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{V}}^2\left[-1 + \tfrac{1}{2}c\nu^{-1}\lambda_1^{-1/4}\left(\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{\mathbb{V}} + \left\|\mathbf{A}^{1/2}\mathbf{v}\right\|_{\mathbb{V}}\right)\right] \\ &\leqslant \tfrac{1}{2}\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{V}}^2\left[-1 + c\nu^{-1}\lambda_1^{-1/4}\mathbf{u}_+\right] \\ &= \tfrac{1}{2}\nu\left\|\mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v})\right\|_{\mathbb{V}}^2\left[-1 + \sqrt{1-\gamma}\right] < 0.\end{aligned}$$

It follows that $\mathcal{A}(t)$ is dissipative. Since $-\mathbf{A}$ is m-dissipative for $\beta > 0$, $Ran(I + \beta\mathbf{A}) = \mathbb{V}(\Omega)$. As $\mathbf{J}$ is strongly dissipative, closed, with $Ran[\mathbf{J}] \supset \mathbb{B}[\Omega]$, and $\mathbf{J}(\cdot, t) : D(\mathbf{A}) \xrightarrow{onto} D(\mathbf{A})$, $\mathcal{A}(t)$ is maximal dissipative, and also closed, so that $Ran[I - \beta\mathcal{A}(t)] \supset \mathbb{B}[\Omega]$. It follows that $\mathcal{A}(t)$ is m-dissipative on $\mathbb{B}_+[\Omega]$ for each $t \in \mathbf{R}^+$ (since $\mathbb{V}[\Omega]$ is a Hilbert space). To see that $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables, let $\mathbf{u}_n, \mathbf{u} \in \mathbb{B}_{++}$, $\left\|\mathbf{Au}_n - \mathbf{Au}\right\|_{\mathbb{V}} \to 0$, with $t_n, t \in I$ and $t_n \to t$. Then (see equation (6))

$$\|\mathcal{A}(t_n)\mathbf{u}_n - \mathcal{A}(t)\mathbf{u}\|_{\mathbb{V}} \leqslant \|\mathcal{A}(t_n)\mathbf{u} - \mathcal{A}(t)\mathbf{u}\|_{\mathbb{V}} + \|\mathcal{A}(t_n)\mathbf{u}_n - \mathcal{A}(t_n)\mathbf{u}\|_{\mathbb{V}}$$

$$= \|[\mathbb{P}\mathbf{f}(t_n) - \mathbb{P}\mathbf{f}(t)]\|_{\mathbb{V}} + \left\|\nu\mathbf{A}(\mathbf{u}_n - \mathbf{u}) - \tfrac{1}{2}[\mathbf{B}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}) + \mathbf{B}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n)]\right\|_{\mathbb{V}}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u} - \mathbf{v})\|_{\mathbb{V}} + \tfrac{1}{2}\left\|\mathbf{A}^{1/2}\mathbf{B}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}) + \mathbf{A}^{1/2}\mathbf{B}(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n)\right\|_{\mathbb{H}}$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{V}} + \tfrac{1}{2}\left\|\mathbf{A}^{1/2}(\mathbf{u}_n - \mathbf{u})\right\|_{\mathbb{H}}^{1/4}\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}^{3/4}[\|\mathbf{A}\mathbf{u}\|_{\mathbb{H}} + \|\mathbf{A}\mathbf{u}_n\|_{\mathbb{H}}]$$

$$\leqslant d\,|t_n - t|^{\theta} + \nu\,\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{V}} + \left\|\mathbf{A}^{1/2}(\mathbf{u}_n - \mathbf{u})\right\|_{\mathbb{H}}^{1/4}\|\mathbf{A}(\mathbf{u}_n - \mathbf{u})\|_{\mathbb{H}}^{3/4}\mathbf{u}_+.$$

It follows that $\mathcal{A}(t)\mathbf{u}$ is continuous in both variables. □

Since $\mathbb{D} = \mathbb{B}_{++}$ is the closure of $D(\mathbf{A}^{3/2}) \cap \mathbb{B}[\Omega]$ equipped with the restriction of the graph norm of $\mathbf{A}^{3/2}$ induced on $D(\mathbf{A}^{3/2})$, it follows that $\mathbb{D}$ is a closed, bounded, convex set. We now have:

Theorem 7. *For each $T \in \mathbf{R}^+$, $t \in (0,T)$ and $\mathbf{u}_0 \in \mathbb{D} \subset \mathbb{B}[\Omega]$, the global in time Navier-Stokes initial-value problem in $\Omega \subset \mathbb{R}^3$:*

$$\begin{aligned} &\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \ \text{in } (0,T)\times\Omega, \\ &\nabla\cdot\mathbf{u} = 0 \ \text{in } (0,T)\times\Omega, \\ &\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \ \text{on } (0,T)\times\partial\Omega, \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \ \text{in } \Omega. \end{aligned} \tag{10}$$

has a unique strong solution $\mathbf{u}(t,\mathbf{x})$, which is in $L^2_{loc}[[0,\infty);\mathbb{H}^2(\Omega)]$ and in $L^\infty_{loc}[[0,\infty);\mathbb{V}(\Omega)] \cap \mathbb{C}^1[(0,\infty);\mathbb{V}(\Omega)]$.

Proof. Theorem 6 allows us to conclude that when $\mathbf{u}_0 \in \mathbb{D}$, the initial value problem is solved and the solution $\mathbf{u}(t,\mathbf{x})$ is in $\mathbb{C}^1[(0,\infty);\mathbb{D}(\Omega)]$. Since $\mathbb{D} \subset \mathbb{H}^2[\Omega]$, it follows that $\mathbf{u}(t,\mathbf{x})$ is also in $\mathbb{V}(\Omega)$, for each $t>0$. It is now clear that for any $T>0$,

$$\int_0^T \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{H}[\Omega]}^2\, dt < \infty, \text{ and } \sup_{0<t<T} \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{V}[\Omega]}^2 < \infty.$$

This gives our conclusion. □

Discussion

It is clear from our results that the stationary problem also has a unique solution in $\mathbb{B}_+[\Omega]$. It is also known that, if $\mathbf{u}_0 \in \mathbb{V}$ and $\mathbf{f}(t)$ is $L^\infty[(0,\infty),\mathbb{H}]$, then there is a time $T>0$ such that a weak solution with this data is uniquely determined on any subinterval of $[0,T)$ (see Sell and You, page 396, [SY]). Thus, we also have that:

Corollary 8. *For each $t \in \mathbf{R}^+$ and $\mathbf{u}_0 \in \mathbb{D}$ the Navier-Stokes initial-value problem in $\Omega \subset \mathbb{R}^3$:*

$$\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) \ \text{in } (0,T)\times\Omega,\\
&\nabla\cdot\mathbf{u} = 0 \ \text{in } (0,T)\times\Omega,\\
&\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \ \text{on } (0,T)\times\partial\Omega,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \ \text{in } \Omega.
\end{aligned} \tag{11}$$

has a unique weak solution $\mathbf{u}(t,\mathbf{x})$ which is in $L^2_{loc}[[0,\infty);\mathbb{H}^2(\Omega)]$ and in $L^\infty_{loc}[[0,\infty);\mathbb{V}(\Omega)] \cap \mathbb{C}^1[(0,\infty);\mathbb{H}(\Omega)]$.

Since we require that our initial data be in $\mathbb{H}^{3/2}[\Omega]$, the conditions for the Leray-Hopf weak solutions are not satisfied. However, it was an open question as to whether these solutions developed singularities, even if $\mathbf{u}_0 \in \mathbb{C}_0^\infty[\Omega]$ (see Giga [G], and references therein). The above Corollary shows that it suffices that $\mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^2(\Omega)]$ to insure that the solutions develop no singularities.

Acknowledgements. *Our interest in this problem was stimulated during a joint research project with Professor George Sell which began in 1989. Over the last eighteen years, we have benefited from his friendship, generous sharing of knowledge, encouragement and (constructive) criticism.*

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