Symplectic Non-Squeezing Theorems, Quantization of Integrable Systems, and Quantum Uncertainty

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Abstract

The ground energy level of an oscillator cannot be zero because of Heisenberg's uncertainty principle. We use methods from symplectic topology (Gromov's non-squeezing theorem, and the existence of symplectic capacities) to analyze and extend this heuristic observation to Liouville-integrable systems, and to propose a topological quantization scheme for such systems, thus extending previous results of ours.

1 Introduction

The fact that the ground energy level of a harmonic oscillator is different from zero is heuristically justified in the physical literature by the following observation: since Heisenberg's uncertainty relation $\Delta p \Delta x \geq \frac{1}{2}\hbar$ prevent us from assigning simultaneously a precise value to both position and momentum, the oscillator cannot be at rest. To show that the lowest energy has the value$\frac{1}{2}\hbar\omega$ predicted by quantum mechanics one then argues as follows: since we cannot distinguish the origin from a phase plane trajectory whose all points lie inside the double hyperbola $px < \frac{1}{2}\hbar$, we must require that at least one point (x, p) of that trajectory is such that $|px| \geq \frac{1}{2}\hbar$; multiplying both sides of the trivial inequality

$$\frac{p^2}{m\omega} + m\omega x^2 \geq 2|px| \geq \hbar$$

by $\omega/2$ we then get

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \geq \frac{1}{2}\hbar\omega$$

which is the correct lower bound for the quantum energy. The argument above can also be reversed: since the lowest energy of an oscillator with frequency ω and mass m is $\frac{1}{2}\hbar\omega$, the minimal phase space trajectory will be the ellipse

$$\frac{p^2}{m\hbar\omega} + \frac{x^2}{(\hbar/m\omega)} = 1;$$

that ellipse encloses an area equal to $\frac{1}{2}h$, which is a topological, or geometrical, version of the uncertainty principle. Everything in the discussion above immediately extends to the n-dimensional oscillator with phase space coordinates $(x_1, ..., x_n; p_1, ..., p_n)$ by using each of the uncertainty relations $\Delta p_j \Delta x_j \geq \frac{1}{2}\hbar$, and one not only recovers the correct ground energy level, but one also finds that conversely, the projection of the motion on any plane of conjugate variables x_j, p_j will always enclose a surface having an area at least equal to $\frac{1}{2}h$.

These heuristic observations leading to *exact* results suggest that there might be a precise relation between the uncertainty principle and the ground energy level in more general cases. The aim of this paper is to show that one can in fact use with benefit recent advances in symplectic topology (Gromov's [9] surprising "non-squeezing theorem" and the "symplectic capacities" of Ekeland and Hofer [4]), to both extend and make rigorous the considerations above, not only for quadratic Hamiltonians, but also for Liouville-integrable Hamiltonian systems.

It is not taking too great risks to conjecture that these new symplectic methods –which were still unknown to mathematicians only two decades ago– will play in the future a fundamental role in physics, both classical and quantum. In [5] and [6] we already discussed EBK quantization from the perspective of symplectic capacities; our argument however relied on an *ad hoc* physical assumption: part of the present paper makes these results mathematically rigorous.

This paper is to a great extent self-contained; symplectic non-squeezing results are, for the time being, not widely known by physicists (and perhaps not even fully appreciated outside specialized mathematical circles): we have therefore devoted Section 2 of this paper to an (elementary) review of these recent advances; we prove, in passing, a linear version of Gromov's theorem (Proposition 1) using a (probably) new approach. EBK quantization of Lagrangian manifolds is also discussed in some detail, and a precise definition of the Maslov index is given. For a technical mathematical study of non-squeezing and general Lagrangian manifold we refer to our previous paper [8]. We also note that Dragoman [3] has used the related notion of quantum blob we have introduced in [7] to propose an axiomatic construction of quantum mechanics in phase space.

NOTATION. We will use the following notation in this paper. The phase space $\mathbb{R}^{2n}_z \equiv \mathbb{R}^n_x \times \mathbb{R}^n_p$ is equipped with the standard symplectic form

$$\sigma(z, z') = p \cdot x' - p' \cdot x$$

($z = (x, p)$, $z' = (x', p')$); in differential notation:

$$dp \wedge dx = dp_1 \wedge dx_1 + \cdots + dp_n \wedge dx_n$$

where $x = (x_1, ..., x_n)$, $p = (p_1, ..., p_n)$. We will call each pair (x_j, p_j) a pair of *conjugate coordinates*. The symplectic group of $(\mathbb{R}^{2n}_z, \sigma)$ is denoted by $Sp(n)$: it is the group of all linear automorphisms of $\mathbb{R}^{2n}_z$ such that $\sigma(Sz, Sz') = \sigma(z, z')$ for all (z, z'). We will call a diffeomorphism $f : \mathbb{D} \subset \mathbb{R}^{2n}_z \longrightarrow \mathbb{R}^{2n}_z$ a *symplectomorphism* if the Jacobian matrix $Df(z)$ is in $Sp(n)$ for every $z \in \mathbb{D}$. The Lagrangian Grassmannian of $(\mathbb{R}^{2n}_z, \sigma)$ is denoted by $\Lambda(n)$; it is the manifold of all Lagrangian planes in $(\mathbb{R}^{2n}_z, \sigma)$, i.e. of the n-dimensional subspaces of $\mathbb{R}^{2n}_z$ on which σ is identically zero. A Lagrangian manifold is a manifold whose tangent spaces are Lagrangian planes.

A solution $t \longmapsto z(t) = (x(t), p(t))$ of the Hamilton equations

$$\dot{x}(t) = \partial_p H(z(t)) \quad , \quad \dot{p}(t) = -\partial_x H(z(t))$$

for $H \in C^\infty(\mathbb{R}^{2n}_z, \mathbb{R})$ will be called indifferently "solution curve" or "motion".

We will denote by $B(\bar{z}', R)$ the Euclidean phase space ball $|z - \bar{z}'| \leq R$ and by $S^1_j(\bar{z}, r)$ the circle in the conjugate plane x_j, p_j plane with radius r and centered at $\bar{z}$. The phase space cylinder $S^1_j(\bar{z}, r) \times \mathbb{R}^{2n}_z$ based on the x_j, p_j plane is denoted by $Z_j(\bar{z}, r)$. We will write $B(0, R) = B(R)$, $S^1_j(0, r) = S^1_j(0, r)$, and $Z_j(0, r) = Z_j(r)$.

2 Symplectic Non-Squeezing Theorems

The determinant of a symplectic matrix is equal to one; it follows that symplectomorphisms are volume preserving: this is essentially the message of Liouville's theorem on conservation of phase space volume by Hamiltonian flows; it is however not a characteristic of Hamiltonian systems: Liouville's theorem holds for the flow of any divergence-free vector field. What really singles out symplectomorphisms among all volume-preserving diffeomorphisms is the following "non-squeezing property" proved by Gromov [9] in 1985. One way of expressing Gromov's theorem is to say that for every symplectomorphism f defined in a neighbourhood of $B(\bar{z}', R)$, the area of the orthogonal projection of $f(B(\bar{z}', R))$ on any of the conjugate planes x_j, p_j will have an area which is at least equal to that of the projection of $B(\bar{z}', R)$ itself on that plane, that is πR^2. It follows from this statement that there exists no symplectomorphism f such that $f(B(\bar{z}', R)) \subset Z_j(\bar{z}, r)$ if $R > r$. (That there exist such symplectomorphisms if $R \leq r$ is obvious: translations in $\mathbb{R}_z^{2n}$ are trivially symplectic).

All known proofs of Gromov's theorem rely on rather complicated mathematical methods (e.g. the theory of pseudo-holomorphic curves). Here is however a proof in the affine case (an affine symplectomorphism is the compose of an element of $Sp(n)$ and of a translation in $\mathbb{R}_z^{2n}$). Since phase space translations trivially satisfy Gromov's theorem, we may, without loss of generality, reduce the proof to the case $\bar{z} = \bar{z}' = 0$. We are in fact going to show that for every $S \in Sp(n)$ the area of the orthogonal projection of $S(B(R))$ (where $B(R) = B(0, R)$) on any of the conjugate planes x_j, p_j is $\geq \pi R^2$.

Proposition 1 *Let $S \in Sp(n)$. (1) The area of the intersection of $S(B(R))$ with any of the conjugate planes x_j, p_j is equal to πR^2; (2) The area of the orthogonal projection of $S(B(R))$ on any of the conjugate planes x_j, p_j is at least πR^2.*

Proof. The second statement follows from the first since the orthogonal projection of $S(B(R))$ on the x_j, p_j plane contains the intersection of $S(B(R))$ with that plane. Let us prove (1). The area of the plane surface $\Gamma = S(B(R)) \cap \mathbb{R}^2_{x_j, p_j}$ is

$$A(\Gamma) = \oint_\gamma p_j dx_j = \oint_\gamma p dx$$

where γ is the (positively) oriented boundary of Γ and pdx the Liouville form $p_1 dx_1 + \cdots + p_n dx_n$. Since S is linear the set $\Gamma' = S^{-1}(\Gamma)$ is a surface lying in a plane passing through the origin, and its boundary $\gamma' = S^{-1}(\gamma)$ is hence a big circle of the sphere $|z| = R$. Using Stoke's theorem together with the fact that the symplectic form $dp \wedge dx$ is preserved by S we have

$$A(\Gamma) = \int_\Gamma dp \wedge dx = \int_{S^{-1}(\Gamma)} dp' \wedge dx'$$

so it is sufficient to show that

$$\int_{S^{-1}(\Gamma)} dp' \wedge dx' = \pi R^2. \tag{1}$$

A new application of Stoke's formula yields

$$\int_{S^{-1}(\Gamma)} dp' \wedge dx' = \oint_{\gamma'} p' dx';$$

parametrizing γ' by

$$\begin{gathered} x'_j(t) = x'_j \cos t + p'_j \sin t \\ p'_j(t) = -x'_j \sin t + p'_j \cos t \\ \sum_{j=1}^{n} (x'^2_j + p'^2_j) = R^2 \end{gathered}$$

$(1 \leq j \leq n \ , \ 0 \leq t \leq 1)$ we have

$$\oint_{\gamma'} p' dx' = \pi \sum_{j=1}^{n} (x'^2_j + p'^2_j) = \pi R^2$$

proving (1) and hence the proposition. ■

Remark 2 *It would certainly be interesting to extend the proof above along the same lines to the case of arbitrary symplectomorphisms. This might perhaps be achieved by exploiting the fact that the image of a conjugate plane by any symplectomorphism is a two-dimensional symplectic manifold.*

Gromov's theorem is equivalent to the existence of *symplectic capacities.* A symplectic capacity (for short: *capacity*) on $\mathbb{R}^{2n}_z$ is the assignment $c : \Omega \longmapsto c(\Omega)$ to every subset Ω of $\mathbb{R}^{2n}_z$ of a number ≥ 0, or $+\infty$, satisfying the following axioms:

- $\Omega \subset \Omega' \Longrightarrow c(\Omega) \leq c(\Omega')$ for all $\Omega, \Omega' \subset \mathbb{R}^{2n}_z$;
- $c(\lambda\Omega) = \lambda^2 c(\Omega)$ for all $\lambda \in \mathbb{R}$;
- $c(f(\Omega)) = c(\Omega)$ for every symplectomorphism $f : \mathbb{R}^{2n}_z \longrightarrow \mathbb{R}^{2n}_z$;
- $c(B(R)) = \pi R^2 = c(Z_j(R))$.

The first and fourth axioms obviously imply the following very useful property:

$$B(R) \subset \Omega \subset Z_j(R) \Longrightarrow c(\Omega) = \pi R^2 \tag{2}$$

for every symplectic capacity c.

In the case $n = 1$ (the phase plane), the usual notion of area is a symplectic capacity (for measurable sets); in higher dimensions volume is however never a capacity (the second axiom would be violated); it seems that there is no useful relation between volumes and capacities for $n > 1$: property (2) shows that sets with very different sizes and volumes (even infinite) can have the same capacity.

The existence of symplectic capacities is actually equivalent to Gromov's theorem. This can be seen by introducing the lower and upper "Gromov capacities" c_{G} and c^{G}. They are defined as follows: $c_{\mathrm{G}}(\Omega) = \pi R^2$ where R is the supremum of the radii of all balls that can be sent in Ω using symplectomorphisms; $c^{\mathrm{G}}(\Omega)$ is the infimum of the radii of all cylinders $Z_j(R)$ into which Ω can be sent using symplectomorphisms. The first three axioms above are trivially satisfied by c_{G} and c^{G}; the fourth axiom is a restatement of Gromov's theorem. One moreover easily checks that c_{G} and c^{G} are lower and upper bounds for all capacities: we have

$$c_{\mathrm{G}}(\Omega) \leq c(\Omega) \leq c^{\mathrm{G}}(\Omega) \tag{3}$$

for all $\Omega \subset \mathbb{R}^{2n}$ and every symplectic capacity c.

Although there is at this time no general formula allowing the calculation of the capacities of arbitrary sets, there are some partial results. Here are two that will be used in this paper.

Ellipsoids. Let Q be a positive definite quadratic form on $\mathbb{R}_z^{2n}$. There exists a linear symplectomorphism S and a unique n-tuplet $(R_1, ..., R_n)$ of numbers > 0 (the "symplectic spectrum of Q") such that

$$Q(Sz) = \sum_{j=1}^{n} \frac{1}{R_j^2}(x_j^2 + p_j^2)$$

All the capacities of the ellipsoid $Q(z) \leq 1$ are equal and are given by the formula

$$c(B(R_1, ..., R_n)) = \pi \inf_{1 \leq j \leq n} R_j^2 \tag{4}$$

(see e.g. Hofer–Zehnder [11] for a proof).

Solid Lagrangian tori. A solid Lagrangian torus is a product

$$\mathbb{D}^n(R_1, ..., R_n) = D_1^2(R_1) \times \cdots \times D_n^2(R_n)$$

where $D_j^2(R_j)$ is the disk $x_j^2+p_j^2 \leq R_j^2$ lying in the x_j, p_j plane. To calculate $c(\mathbb{D}^n(R_1,...,R_n))$ we proceed as follows: let $B(R_1,...,R_n)$ be the ellipsoid defined by

$$\sum_{j=1}^{n} \frac{1}{R_j^2}(x_j^2+p_j^2) \leq 1.$$

We have inclusions

$$B(R_1,...,R_n) \subset \mathbb{D}^n(R_1,...,R_n) \subset Z_j(R_j)$$

for every $j = 1, 2, ..., n$ and hence

$$c(B(R_1,...,R_n)) \leq c(\mathbb{D}^n(R_1,...,R_n)) \leq c(Z_j(R_j)).$$

In view of (4) and the equality $c(Z_j(R_j)) = \pi R_j^2$ we get, using the third and fourth axioms for symplectic capacities,

$$\pi \inf_{1\leq j\leq n} R_j^2 \leq c(\mathbb{D}^n(R_1,...,R_n)) \leq \pi R_j^2$$

for $j = 1, 2, ..., n$; choosing in particular j such that $R_j^2 = \inf_{1\leq j\leq n} R_j^2$ it follows that all the capacities of the solid torus are equal and are given by

$$c(\mathbb{D}^n(R_1,...,R_n)) = \pi \inf_{1\leq j\leq n} R_j^2. \tag{5}$$

3 Quadratic Hamiltonians

Let H be a positive definite quadratic form in the position and momentum variables, that is

$$H(z) = \frac{1}{2}Rz \cdot z = \frac{1}{2}z^T Rz$$

where R is a real symmetric $2n \times 2n$ matrix with > 0 eigenvalues (it is the Hessian matrix of H). Let $S \in Sp(n)$ be such that

$$H(Sz) = \sum_{j=1}^{n} \frac{1}{R_j}(p_j^2 + x_j^2)$$

where $R_1, ..., R_n > 0$; setting $\omega_j = \sqrt{2/R_j}$, the compose $H \circ S$ can be written in the familiar form

$$H(Sz) = \sum_{j=1}^{n} \frac{\omega_j}{2}(p_j^2 + x_j^2);$$

notice that the frequencies ω_j are uniquely determined by H and are thus independent of the choice of S. Solving Hamilton's equations for $H \circ S$ with initial datum (x, p) at time $t = 0$ yields

$$\begin{aligned} x_j(t) &= x_j \cos \omega_j t + p_j \sin \omega_j t \\ p_j(t) &= -x_j \sin \omega_j t + p_j \cos \omega_j t \end{aligned}$$

$= (x_j t = x_j \cos \omega_j t + p_j \sin \omega_j t, p_j t = -x_j \sin \omega_j t + p_j \cos \omega_j t)$ for $1 \leq j \leq n$ and hence the motion winds around a torus

$$\mathbb{T}^n = S_1^1(R_1) \times \cdots \times S_n^1(R_n) \quad , \quad R_j = \sqrt{x_j^2 + p_j^2}. \tag{6}$$

We know from standard quantum mechanics that the exact quantized energy levels of H are given by the formula

$$E_{N_1,\ldots,N_n} = \sum_{j=1}^{n} (N_j + \tfrac{1}{2})\hbar\omega_j$$

where the N_j are integers ≥ 0; in particular the ground energy level is

$$E_0 = \sum_{j=1}^{n} \tfrac{1}{2}\hbar\omega_j. \tag{7}$$

In quantum mechanics this property is usually restated by saying that "a sum of harmonic oscillators is in its ground energy level if and only if each of its components is". Let us discuss formula (7) from a semiclassical perspective. The fact that each individual oscillator with Hamiltonian

$$H_j(x_j, p_j) = \frac{\omega_j}{2}(p_j^2 + x_j^2)$$

has ground energy $\frac{1}{2}\hbar\omega_j$ means that the corresponding semiclassical motion takes place on the circle

$$\frac{\omega_j}{2}(p_j^2 + x_j^2) = \tfrac{1}{2}\hbar\omega_j$$

with radius $R_j = \sqrt{\hbar}$. It follows that the motion determined by the complete Hamiltonian $H = H_1 + \cdots + H_n$ is carried by the torus

$$\mathbb{T}^n(\sqrt{\hbar}) = S_1^1(\sqrt{\hbar}) \times \cdots \times S_n^1(\sqrt{\hbar}).$$

It follows from formula (5) of Section 2 that every capacity of the solid torus

$$\mathbb{D}^n(\sqrt{\hbar}) = D_1^2(\sqrt{\hbar}) \times \cdots \times D_n^2(\sqrt{\hbar})$$

is equal to

$$c(\mathbb{D}^n(\sqrt{\hbar})) = \pi(\sqrt{\hbar})^2 = \tfrac{1}{2}h. \tag{8}$$

Remark 3 *In the light of Gromov's theorem (8) can be viewed as a topological form of the uncertainty principle: choosing* $c = c^G$*, (8) shows that* $\mathbb{D}^n(\sqrt{\hbar})$ *cannot be squeezed inside a cylinder* $Z_j(R)$ *with radius* $R < \sqrt{\hbar}$*using symplectomorphisms.*

Suppose now, conversely, that the motion takes place on a torus $\mathbb{T}^n(R_1, ..., R_n)$ and that the capacity of the corresponding solid torus is

$$c(\mathbb{D}^n(R_1, ..., R_n)) = \tfrac{1}{2}h. \tag{9}$$

Using again formula (5) we get

$$c(\mathbb{D}^n(R_1, ..., R_n)) = \pi R^2 = \tfrac{1}{2}h$$

where $R = \inf_{1 \leq j \leq n} R_j$. It follows that we have $R_j^2 \geq \hbar$ for $j = 1, 2, ..., n$; the energy of the motion being given by

$$E = \sum_{j=1}^{n} \frac{\omega_j}{2}(p_j^2 + x_j^2) = \sum_{j=1}^{n} \frac{\omega_j}{2} R_j^2$$

it follows that the assumption (9) implies that $E \geq E_0$ where E_0 is the ground energy level (7), and thus implies the correct lower bound for the energy.

Summarizing:

- A necessary condition for the motion of a positive definite quadratic Hamiltonian to be quantized is that it lies on a torus $\mathbb{T}^n$ such that the corresponding solid torus $\mathbb{D}^n$ has symplectic capacity $c(\mathbb{D}^n)$ at least equal to $\frac{1}{2}h$, that is half the quantum of action.

- The condition that the motion is carried by a torus $\mathbb{T}^n$ such that $c(\mathbb{D}^n)$ is not sufficient to conclude that this motion is quantized; its energy is however bounded from below by the ground energy level.

Let us extend this discussion to a class of more general Hamiltonian systems.

4 Liouville-Integrable Hamiltonian Systems

Let H be a Hamiltonian function on $\mathbb{R}_z^{2n}$; we assume that there exists a symplectomorphism $f : (x, p) \longmapsto (\phi, I)$ of $\mathbb{R}_z^{2n}$ (not necessarily globally defined) such that $K = H \circ f^{-1}$ only depends on the variables $I = (I_1, ..., I_n)$:

$$H(x, p) = K(I).$$

The Hamilton equations for K (and hence for H) are immediately solved, and one finds that

$$\phi_j(t) = \omega_j(I(0))t + \phi_j(0) \quad , \quad I_j(t) = I_j(0) \ \text{ for } \ 1 \leq j \leq n; \tag{10}$$

the frequencies ω_j are the derivatives of K:

$$\omega_j(I) = \partial_{I_j} K(I) \quad , \quad 1 \leq j \leq n. \tag{11}$$

Such a situation typically occurs when the Hamiltonian system associated to H is Liouville integrable, that is when: (1) there exist n independent constants of the motion $F_1 = H, F_2, ..., F_n$ so that the set

$$\mathbb{V}^n = \{z : F_1(z) = f_1, ..., F_n(z) = f_n\};$$

is a n-dimensional manifold for almost all values $f_1, ..., f_n$ of $F_1, ..., F_n$; (2) these constants of the motion are in involution:

$$\{F_i, F_j\} = \partial_x F_i \cdot \partial_p F_j - \partial_x F_j \cdot \partial_p F_i = 0$$

hence the manifold $\mathbb{V}^n$ is Lagrangian; each motion takes place on such a manifold. In fact, the manifolds $\mathbb{V}^n$ can be parametrized the n-parameter $I = (I_1, ..., I_n)$ consisting of the "action variables", at least in some open subset $U \subset \mathbb{R}^n$. If the manifolds $\mathbb{V}^n$ are compact and connected there exists a symplectomorphism $f : (x, p) \longmapsto (\phi, I)$, defined in a neighbourhood of $\mathbb{V}^n$, such that $f(\mathbb{V}^n)$ is the torus

$$\mathbb{T}^n(R_1, ..., R_n) = S_1^1(R_1) \times \cdots \times S_n^1(R_n)$$

(the variables $\phi = (\phi_1, ..., \phi_n)$ are here cyclic).

The passage to semiclassical mechanics consists in imposing selection rules on the Lagrangian manifolds $\mathbb{V}^n$; these rules are the EBK (Einstein–Brillouin–Keller) quantum conditions:

$$\frac{1}{2\pi\hbar} \oint_\gamma p dx - \frac{1}{4} m(\gamma) \ \text{ is an integer} \tag{12}$$
$$\text{for all one-cycles } \gamma \text{ on } \mathbb{V}^n$$

(see e.g. Arnol'd, Leray [12], Maslov [13], Maslov–Fedoriuk [14]; the conditions (12) are sometimes also called the Bohr–Sommerfeld–Maslov conditions in the literature). The integer $m(\gamma)$ appearing in (12) is the Maslov index of γ; its vocation is to "count" the number of caustics of $\mathbb{V}^n$ traversed by γ (a caustic of $\mathbb{V}^n$ is a point of $\mathbb{V}^n$ which does not have any neighbourhood diffeomorphic to an open subset of the position space $\mathbb{R}^n_x$). The Maslov index is calculated as follows (Arnol'd [1], Leray [12], Souriau [16]). Parametrize γ by $t \in [0,1]$ and set $\ell(t) = T_{\gamma(t)}\mathbb{V}^n$ (the tangent plane to $\mathbb{V}^n$ at $\gamma(t)$. The mapping $t \longmapsto \ell(t)$, $0 \leq t \leq 1$, is a loop γ_Λ in the Lagrangian Grassmannian $\Lambda(n)$. Identifying $\Lambda(n)$ with the manifold $W(n)$ of all symmetric unitary matrices of order n (see Souriau [16]; also Guillemin–Sternberg [10]), the loop γ_Λ is identified with a loop $\gamma_W : t \longmapsto w(t)$, $0 \leq t \leq 1$, in $W(n)$. The Maslov index of γ is by definition the integer

$$m(\gamma) = \frac{1}{2\pi i}\int_0^1 \frac{d(\det w(t))}{\det w(t)}.$$

One shows that $m(\gamma)$ only depends on the homotopy class of γ in $\mathbb{V}^n$. An important property of the Maslov index is the following:

$$\mathbb{V}^n \text{ oriented } \Longrightarrow m(\gamma) \text{ is even} \tag{13}$$

(Souriau [17]).

The semiclassical values of the energy are obtained from the EBK condition as follows: let $I = (I_1, ..., I_n)$ be the action variables corresponding to the basic one-cycles $\gamma^1, ..., \gamma^n$ on $\mathbb{V}^n$. These are defined as follows: let $\bar{\gamma}^1, ..., \bar{\gamma}^n$ be the loops in $\mathbb{T}^n(R_1, ..., R_n)$ defined, for $0 \leq t \leq 2\pi$, by

$$\begin{aligned}
\bar{\gamma}^1(t) &= R_1(\cos t, 0, ..., 0; \sin t, 0, ..., 0) \\
\bar{\gamma}^2(t) &= R_2(0, \cos t, ..., 0; 0, \sin t, ..., 0) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.. \\
\bar{\gamma}^n(t) &= R_n(0, ..., 0, \cos t; 0, ..., 0, \sin t).
\end{aligned}$$

The basic one-cycles $\gamma^1, ..., \gamma^1$ of $\mathbb{V}^n$ are then just

$$\gamma^1 = f^{-1}(\bar{\gamma}^1), ..., \gamma^n = f^{-1}(\bar{\gamma}^n).$$

The action variables being given by

$$I_j = \frac{1}{2\pi}\oint_{\bar{\gamma}^j} I d\phi = \frac{1}{2\pi}\oint_{\gamma^j} pdx \quad , \quad 1 \leq j \leq n$$

the EBK quantization conditions (12) imply that we must have

$$I_j = (N_j + \tfrac{1}{4}m(\gamma^j))\hbar \quad \text{for} \quad 1 \leq j \leq n \tag{14}$$

each N_j being an integer ≥ 0. Writing $H(x,p) = K(I)$ the semiclassical energy levels are then given by the formula

$$E_{N_1,...,N_n} = K((N_1 + \tfrac{1}{4}m(\gamma^1))\hbar, ..., (N_n + \tfrac{1}{4}m(\gamma^1))\hbar) \tag{15}$$

where $N_1, ..., N_n$ range over all *non-negative* integers; they correspond to the physical "quantum states" labeled by the sequence $(N_1, ..., N_n)$.

(We do not discuss here the ambiguity that might arise in the calculation of the energy because of the non-uniqueness of the angle action coordinates; that ambiguity actually disappears if one requires that the system under consideration is non-degenerate, that is $\partial^2 K(I) \neq 0$.)

Let us state and prove a result which generalizes to the Liouville integrable case the discussion of quadratic Hamiltonians we did in the last Section.

Theorem 4 *Assume that the Lagrangian manifold $\mathbb{V}^n$ is compact and connected. (1) If $\mathbb{V}^n$ satisfies the EBK condition (12), then for every symplectic capacity c we have*

$$c(\bar{\mathbb{V}}^n) \geq \tfrac{1}{2}h \tag{16}$$

where $\bar{\mathbb{V}}^n$ is defined by $f(\bar{\mathbb{V}}^n) = \mathbb{D}^n$ (f the mapping $(x,p) \longmapsto (\phi, I)$) and $\mathbb{D}^n$ is the "solid torus" corresponding to $\mathbb{T}^n$. (2) If conversely $\bar{\mathbb{V}}^n$ satisfies (16), and the frequencies ω_j are everywhere > 0 then the energy E of the motion carried by $\mathbb{V}^n$ is such that

$$E \geq E_0 = K(\tfrac{1}{2}\hbar, ..., \tfrac{1}{2}\hbar); \tag{17}$$

In view of (20) we have $m(\gamma^j) \geq 2$ for every basic one-cycle γ^j on $\mathbb{V}^n$ and hence

$$E_{N_1,...,N_n} \geq E_0 = K(\tfrac{1}{2}\hbar, ..., \tfrac{1}{2}\hbar); \tag{18}$$

the number E_0 is a lower bound for the quantized energy levels $E_{N_1,...,N_n}$ given by (15).

Proof. (1) Since capacities are symplectic invariants, we may assume without restricting the generality of the argument that $\mathbb{V}^n$ is the torus $\mathbb{T}^n = \mathbb{T}^n(R_1, ..., R_n)$ itself. Since

$$I_j = \frac{1}{2\pi}\oint_{\gamma^j} pdx = \frac{1}{2\pi}\oint_{\bar{\gamma}^j} I_j d\phi_j = \frac{1}{2}R_j^2$$

the quantization conditions (14) are equivalent to the conditions

$$R_j^2 = (2N_j + \tfrac{1}{2}m(\gamma^j))\hbar.$$

As a manifold $\mathbb{V}^n$ (and hence $\mathbb{T}^n$) has dimension n; we must thus have $R_j > 0$ for every j, and this implies that $m(\gamma^j) > 0$ for every basic one-cycle γ^j. It follows that

$$\inf_{1\leq j\leq n} R_j^2 \geq \tfrac{1}{2} \inf_{1\leq j\leq n} m(\gamma^j))\hbar > 0. \tag{19}$$

We next observe that the torus $\mathbb{T}^n = \mathbb{T}^n(R_1, ..., R_n)$ is an oriented manifold (because it is a product of circles, which are oriented manifolds). It follows that $\mathbb{V}^n = f^{-1}(\mathbb{T}^n)$ is also oriented (symplectomorphisms are orientation preserving). Souriau's theorem (13) thus implies that the Maslov index $m(\gamma^j)$ of every basic one-cycle on $\mathbb{V}^n$ is even, and hence

$$\inf_{1\leq j\leq n} m(\gamma^j) \geq 2. \tag{20}$$

It follows from the inequalities (19) and (20) that we have

$$c(\mathbb{D}^n(R_1, ..., R_n)) = \pi \inf_{1\leq j\leq n} R_j^2 \geq \tfrac{1}{2}h$$

as was to be proven. (2) Assume that conversely

$$c(\bar{\mathbb{V}}^n) = c(\mathbb{D}^n) \geq \tfrac{1}{2}h.$$

The motion thus takes place on a torus $\mathbb{T}^n = \mathbb{T}^n(R_1, ..., R_n)$ such

$$\pi \inf_{1\leq j\leq n} R_j^2 \geq \tfrac{1}{2}h$$

and we thus have

$$I_j = \frac{1}{2\pi} \oint_{\bar{\gamma}^j} I_j d\phi_j = \frac{1}{2} R_j^2 \geq \frac{1}{2}\hbar.$$

The assumption $\omega_j(I) = \partial_{I_j} K(I) > 0$ implies that K is an increasing function of the variables $I = (I_1, ..., I_n)$ and we thus have

$$E = K(I_1, ..., I_n) \geq K(\tfrac{1}{2}\hbar, ..., \tfrac{1}{2}\hbar).$$

In view of (20) we have $m(\gamma^j) \geq 2$ for every basic one-cycle γ^j on $\mathbb{V}^n$ and hence

$$E_{N_1,...,N_n} \geq K(\tfrac{1}{2}\hbar, ..., \tfrac{1}{2}\hbar)$$

which ends the proof of the Theorem. ■

Remark 5 *If we impose the EBK conditions on the tori* $\mathbb{T}^n$ *themselves, and not on the Lagrangian manifolds* $\mathbb{V}^n$ *then the number* E_0 *in (17) effectively coincides with the ground energy level; but in the general case we have* $E_0 < E_{N_1,...,N_n}$*. This is due to the fact that the Maslov index is not a symplectic invariant and that* $\mathbb{V}^n$ *can have more caustics than* $\mathbb{T}^n$ *(see the discussion above following the definition of the Maslov index).*

From Theorem 4 we easily deduce the following form of the uncertainty principle:

Corollary 6 *Let* $\mathbb{V}^n$ *be a compact and connected Lagrangian manifold associated to a Liouville-integrable Hamiltonian system. If* $\mathbb{V}^n$ *satisfies the EBK condition (12) then the following property holds: Let* Λ_j *be a connected subset in the* x_j, p_j *plane bounded by a simple curve* λ_j*. If* Λ_j *contains the projection of* $\mathbb{V}^n$ *then we have*

$$\text{Area}(\Lambda_j)) \geq \frac{1}{2}h. \tag{21}$$

Proof. Recall from Theorem 4 that if $\mathbb{V}^n$ is quantized then $c(\bar{\mathbb{V}}^n) \geq \frac{1}{2}h$. Let us first assume that λ_j is a circle $S_j^1(R)$. Then $\bar{\mathbb{V}}^n \subset Z_j(R)$ and hence

$$\tfrac{1}{2}h \leq c(\bar{\mathbb{V}}^n) \leq c(Z_j(R)) = \pi R^2$$

proving the claim in that case. If λ_j is not a circle, choose an area-preserving diffeomorphism f of the x_j, p_j plane taking Λ_j into a circle $S_j^1(R)$. The phase space transformation F taking (x_j, p_j) into $f(x_j, p_j)$ and leaving all other coordinates unchanged is symplectic, and the projection of $F(\bar{\mathbb{V}}^n)$ lies inside $S_j^1(R)$. We have

$$c(\bar{\mathbb{V}}^n) = c(F(\bar{\mathbb{V}}^n)) \leq c(Z_j(R)) = \text{Area}(\Lambda_j)$$

hence again $\text{Area}(\Lambda_j) \geq \frac{1}{2}h$. ■

Remark 7 *This result can also be deduced from Gromov's theorem: if* $c(\bar{\mathbb{V}}^n) \geq \frac{1}{2}h$ *for every symplectic capacity c then, in particular,* $c_{\mathrm{G}}(\mathbb{V}^n) \geq \frac{1}{2}h$ *(*c_{G} *the Gromov capacity defined in Section 2). The largest ball* $B(R)$ *that can be squeezed inside* $\mathbb{V}^n$ *by a symplectomorphism* f *has therefore radius* $\sqrt{\hbar}$*, and the area of the orthogonal projection of* $f(B(\sqrt{\hbar}))$ *on any* x_j, p_j *plane is at least* $\frac{1}{2}h$*.*

Remark 8 *The inequality (21) is symplectically invariant in the sense that it remains true if we replace* $\bar{\mathbb{V}}^n$ *by* $g(\bar{\mathbb{V}}^n)$*,* g *any symplectomorphism: this follows from the symplectic invariance of symplectic capacities.*

5 Discussion and Conclusion

We have been able to relate the Heisenberg uncertainty principle to the existence of a non-zero ground energy level for integrable systems with compact Lagrangian tori. The results we have obtained are however not sharp, in the sense that we have not been able to recover the exact ground energy from the minimum uncertainty, but only a lower bound for that energy. A possible way to refine and generalize these results would perhaps be to use the powerful tool of "Hofer displacement energy" (see Hofer–Zehnder [11], Polterovich [15]).

In [5, 7] we have shown that a classical uncertainty principle, formally identical with the quantum uncertainty principle, can be derived for all linear Hamiltonian systems; it would perhaps be interesting to extend these results to more general Hamiltonians (integrable or not), and to study the *classical* implications of this principle from the point of view of the methods outlined in this paper: perhaps the existence of these classical uncertainty relations could be used with some profit in the study of non-integrable (chaotic) Hamiltonian systems.

Gromov's theorem, and its implications, shows that Hamiltonian mechanics is "aerial" in nature; symplectic capacities are symplectic invariants that have the physical dimension of an area, that is of *action*. They certainly deserve to be further studied within the contexts of both classical and quantum mechanics. A possible application of the notion of symplectic capacity might be a global characterization of adiabatic invariance (and of the method of "adiabatic switching" in semiclassical mechanics). One might envisage that in multi-dimensional Hamiltonian systems the best candidate for adiabatic conservation is not the action of periodic orbits, but rather the capacity of some sets (for instance that of Lagrangian solid tori in the integrable case, or that of the set bounded by the energy shell in the ergodic case). We hope to come back to these important topics in forthcoming work.

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