New Constants Arising in Non-linear Unimodal Maps

Milton M. Chowdhury
mchowdhur@lycos.com
The University of Manchester Institute of Science and Technology

Abstract

We give new constants that arise in non linear unimodal maps. We discuss the arithmetic character of Feigenbaum's constant and related constants arising in mathematical physics.

Keywords

Non linear map, period doubling.

1. On The Logistic Map

Theorem 1a

Define $u(x)$ by $u(x) = 1 \ if \ x < 0, u(x) = 0 \ if \ \ x \geq 0$. The logistic map [2] which is $x_{n+1} = \lambda x_n(1 - x_n)$, $\lambda \in R$ we give new results for $\lambda = 4$ this is the full logistic map. The full logistic map has the non-recursive representation $x_n = \frac{(1-\cos(2^n \arccos(1-2x_0)))}{2}$ [2]. It can be shown that if $x_{n+1} = 4x_n(1 - x_n)$, and $x_n = \frac{(1-\cos(2^n \arccos(1-2x_0)))}{2}$ then $x_0 = \frac{1}{2} - \frac{1}{2}\beta_0$ and $x_n = \frac{1}{2} - \frac{1}{2}\beta_n$ with $\beta_{n+1} = 2\beta_n^2 - 1$, $-1 \leq \beta_0 \leq 1$, $n \geq 0$, with $\beta_0 = \delta$, $-1 \leq \delta \leq 1$ we have

$$\sum_{n=0}^{\infty} \frac{u(\beta_n)}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{u(1-2x_n)}{2^{n+1}} = \theta$$

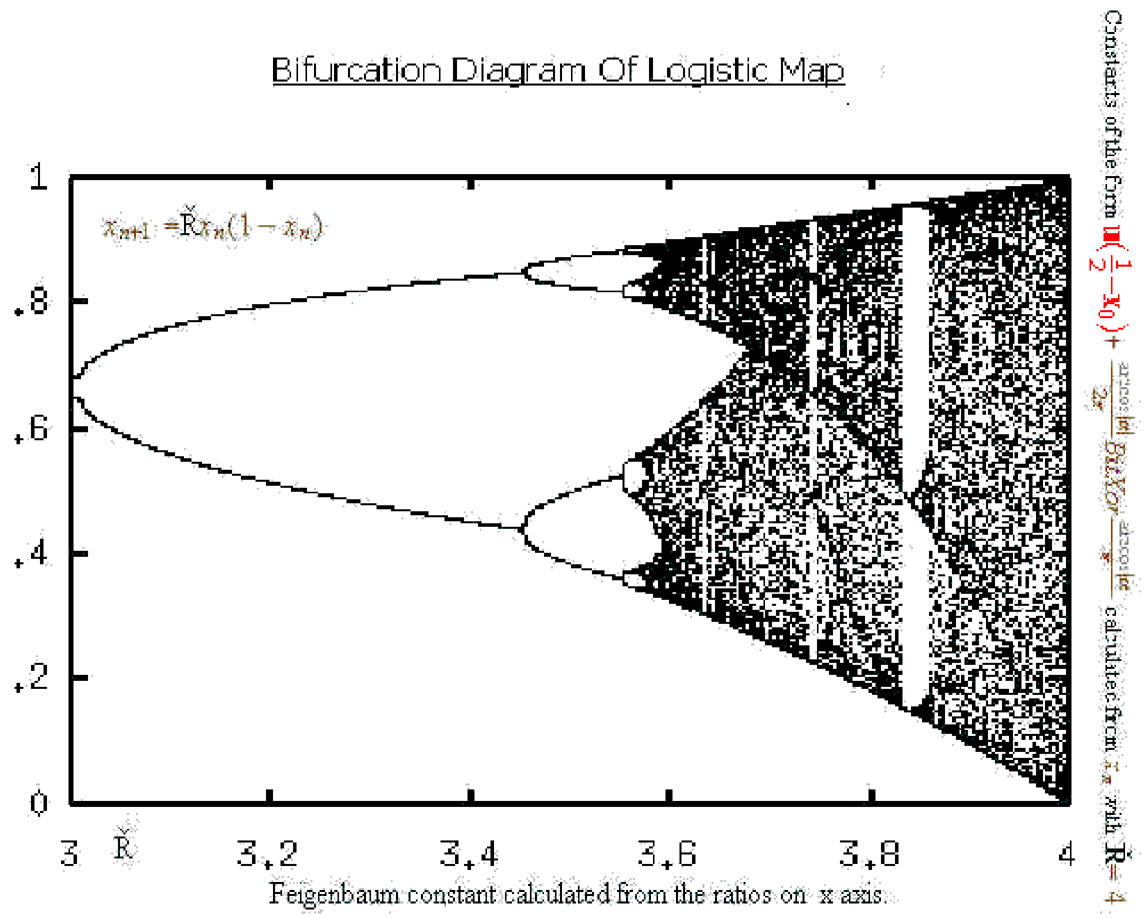
$$\theta = \frac{\arccos(\delta)}{2\pi} BitXor \frac{\arccos(\delta)}{\pi} \text{ if } 0 < \delta \leq 1$$

$$\theta = \frac{1}{2} + \frac{\arccos(|\delta|)}{2\pi} BitXor \frac{\arccos(|\delta|)}{\pi} \text{ if } -1 \leq \delta < 0$$

$$\theta = \frac{1}{4} = u(0) + \frac{1}{4} \text{ if } \delta = 0$$

We can show constants of the form, ($\alpha = \delta$), $\frac{u(\frac{1}{2}-x_0)}{2} + \frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ are irrational we sketch the proof, it is known $\cos(p\pi) = q$ when p & q are rational $p = 0, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}$ and 1 and $q = 1, \frac{1}{2}, 0, -\frac{1}{2}$ and -1 respectively are the only possible values see [4].

We immediately see that $\frac{\arccos\ p}{\pi} = q$ is irrational for nearly all and infinitely many p. The above determines a (finite) total of all the periodic orbits that have rational initial values. It can be shown that $aBitXor\frac{a}{2}$ is rational if a is rational. The bifurcation diagram below shows the relation of Feigenbaum's constant [3] and constants of the form $\frac{u(\frac{1}{2}-x_0)}{2} + \frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$.



2. Generalisations

The series $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\sin 2^n) = \frac{1}{2\pi}$ and $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\tan 2^n) = \frac{1}{\pi}$ are given in [5], [6]. Now define $a(n, x_k)$ below recursively, $k \in N$. x is understood to be x_k for some k. $a(n, x_k) = a_n$ with initial value x_k we use similiar definitions for $b(n, x_k) = b_n$ etc.

$$
\begin{aligned}
a(n, x_k) &= \sin(2^n \arcsin(a_0)) \\
&= a_0 \quad = x_k \text{ if } n = 0,\ 0 < x_k < 1 \\
&= 2a_0\sqrt{1 - a_0^2} \text{ if } n = 1 \\
&= 2a_{n-1}(1 - 2a_{n-2}) \text{ if } n \geq 2
\end{aligned}
$$

this recursive definition and the similar ones that follow can be derived using the double angle formulae for tan, sine and cos etc. Then $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(a(n, x_k)) \ = \frac{\arcsin(x_k)}{2\pi}$ see [5]. Define b_n by

$$
\begin{aligned}
b(n, x_k) &= \cos(2^n \arccos(b_0)) \\
&= x_k = b_0,\ 0 < x_k < 1 \text{ if } n = 0 \\
&= 2b_{n-1}^2 - 1 \text{ if } n \geq 1
\end{aligned}
$$

then it can be shown (for example a proof based on theorem 1) that

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(b(n,x_k)) = \frac{\arccos(x_k)}{\pi} BitXor \frac{\arccos(x_k)}{2\pi}$$

Define the Plouffe recursion [6] with c_n by

$$\begin{aligned} c(n,x_k) &= \tan(2^n \arctan(c_0)) \\ &= c_0 = x_k \text{ if } n = 0 \\ &= \frac{2c_{n-1}^2}{1-c_{n-1}^2} \text{ if } n \geq 1,\ |c_k| \neq 1 \\ &= -\infty \text{ if } n \geq 1, |c_k| = 1 \end{aligned}$$

we consider $0 < x_k < 1$, $\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(c(n,x_k)) = \frac{\arctan(x_k)}{\pi}$ see [5]. Let $d(n,x), e(n,x), f(n,x)$ be the analogous recursions for sec, csc and cot respectively which can obtained by using the double angle formula so for example for $d(n,x)$ we have

$$\begin{aligned} d(n,x_k) &= x_k = d_0 = 0 < x_k < 1 \text{ if } n = 0 \\ &= \frac{1}{-1+\frac{2}{d_{n-1}^2}} \text{ if } n \geq 1 \end{aligned}$$

then it can be shown that (which is a new result)

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(d(n,x_k)) = \frac{\operatorname{arcsec}(x_k)}{\pi} BitXor \frac{\operatorname{arcsec}(x_k)}{2\pi}$$

Define

$$\sum_{\forall n} \overset{\wedge}{BitXor}\ f(v_n) = f(v_1) BitXor \frac{f(v_1)}{2} ... f(v_n) BitXor \frac{f(v_n)}{2}$$

Theorem 2

It can be shown that

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(\prod_{\forall A} a(n,x_A)\prod_{\forall B} b(n,x_B)\prod_{\forall C} c(n,x_C)\prod_{\forall D}(d(n,x_D)\prod_{\forall E} e(n,x_E)\prod_{\forall F} f(n,x_F)) =$$

$$(\sum_{\forall A} BitXor \frac{1}{\pi}\arcsin(x_A)) BitXor (\sum_{\forall B} \overset{\wedge}{BitXor}\ \frac{1}{\pi}\arccos(x_B)) BitXor (\sum_{\forall C} BitXor$$

$$\frac{1}{\pi}\arctan(x_C)) BitXor (\sum_{\forall D} \overset{\wedge}{BitXor}\ \frac{1}{\pi}\operatorname{arcsec}(x_D)) BitXor (\sum_{\forall E} BitXor \frac{1}{\pi}\operatorname{arccsc}(x_E))$$

$$BitXor (\sum_{\forall F} BitXor \frac{1}{\pi}\operatorname{arccot}(x_F))$$

It is known that the logistic has invariant measure $\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ if x_0 is not an element of a set of measure zero [1].

Theorem 1b

For $x_0 = \frac{1-\alpha}{2}$ and $0 < |\alpha| \leq 1$ in the logistic map then except when α is not an element of the set of measure zero above there are exist infinitely many numbers $i)$ and $ii)$ defined by $i)$ $\frac{\arccos|\alpha|}{\pi}$, $ii)$ $\frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ that satisfy the following conditions 1, 2 & 3 (we call these collectively condition A) in a base 2^f (for f≥ 1), 1) simply normal, 2) normal & 3) digit dense.

Proof

Consider when x_0 is not an element of the set of measure zero above. Observe that from the above probability distribution $u(\frac{1}{2} - x_n)$ behaves similar to a binary valued uniformly distributed random variable. From the invariant measure we see that $\int_0^{1/2} \rho(x) = \int_{1/2}^1 \rho(x) = \frac{1}{2}$ it follows that ii) and i) are simply normal in base two with x_0. Normality of a number in base b is equivalent to the digits being generated by a fair b sided die hence it follows that $ii)$ are normal in base 2.
The logistic map has chaotic dense orbits with x_0 it follows that then $ii)$ are digit dense and the above result also follows from the invariant measure of the logistic map.
Say $ii)$ has the binary expansion $y_0y_1....y_my_{m+1}...$ and $i)$ derived from the above expansion has the expansion $z_0z_1....z_m....$ then the possibilities for z_m are 1 if $y_{m+1} \neq y_m$ and 0 otherwise hence the probability of z_m being 0 or 1 is $\frac{1}{2}$ and so it follows $i)$ are normal and simply normal in base 2.
From the chaotic dense orbits of the logistic map it follows that there can be any given sequence bits in $ii)$ and this implies $i)$ are digit dense in base 2.
For $k \in \Re$ the digits of k_{2^f}, $f \geq 0$ can be computed from the digits of k_2 it follows that the above results are true in base 2^f.
For example condition A is true for $\frac{1}{\pi}{}_{2^f}$ and $\frac{1}{2\pi}BitXor\frac{1}{\pi}{}_{2^f}$ iff $x_0 = \cos 1$ is not an element of the set of measure zero, and $x_0 = \cos 1$ is not an element of the set of measure zero see [1] also this is a consequence of Theorem 7 in [7]. By a similar proof it can be shown that condition A is true for $\frac{\arccos|\alpha|}{2^Z\pi}BitXor\frac{\arccos|\alpha|}{\pi}{}_{2^f}$, $\frac{\text{arcsec}(\gamma)}{\pi}{}_{2^f}$ and $\frac{\text{arcsec}(\gamma)}{\pi}BitXor\frac{\text{arcsec}(\gamma)}{2^Z\pi}{}_{2^f}$

for γ not an element of a set of measure zero with $Z \geq 0$ and show chaotic properties with maps associated with the above constants. We can use the above result to construct sets for x_0 so that invariant measure of the logistic map does not apply or does apply for example we can select $\alpha = \cos(c\pi)$ for some non-normal number c or normal number c.

Chaotic Orbits of the Logistic Map

$\frac{\arccos\, p}{\pi}$ is irrational for infinitely many p (see above) for example take $p = \cos(f\pi)$ for some irrational $0 < f < 1$ and another value of p can be selected by choosing another two different irrationals (because $aBitXor\frac{a}{2}$ is a two to one function for a$\in \Re$) for the value of f repeating the above shows that there an infinite number of orbits that are not asymptotically periodic. It is easy to construct a countable set of infinite irrationals for f. We take the orbit so there is at there are 2 different symbols in the itinerary. By topological conjugacy the above constructed non asymptotically orbits mean there are infinite non asymptotic orbits for the tent map and such an itenary orbit for the tent map cannot be a sequence of k zeroes. Let L,T,C be the logistic, tent and conjugacy maps respectively. Consider an orbit of T and the corresponding orbit in L. By the using the conjugacy map we have

$$\frac{1}{k}\ln T^{'}(x_k)...T^{'}(x_2)T^{'}(x_1) = \frac{1}{k}(\ln|C^{'}(x_1)| + \ln|C^{'}(x_{k+1})| + \sum_{i=1}^{k}\ln|G^{'}(C(x_i))|).$$

One way to show the Lyanpunov exponent is positive is to observe the for a not asymptotically periodic orbit of the tent map that does not have the k consecutive same symbols in its itinerary the orbit never enters the intervals [0,2^{-k}] and [1-2^{-k-1},1] up to the iteration x_k. Hence this gives the bounds

$$\frac{\pi}{2}\sin(\frac{\pi}{2^{k+1}}) \leq |C^{'}(x_k)| \leq \frac{\pi}{2} \Rightarrow$$
$$(\ln\frac{\pi}{2} + \ln\sin(\frac{\pi}{2^{k+1}}))/k \leq (\ln|C^{'}(x_k)|)/k \leq (\ln\frac{\pi}{2})/k \Rightarrow \lim_{k\to\infty}(\ln|C^{'}(x_k)|)/k = 0 \Rightarrow$$
$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\ln|T^{'}(x_i)| = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\ln|G^{'}(C(x_i))|$$

(it is known that all chaotic orbits of T has Lyanpunov exponent $\ln 2$) then the non asymptotic orbits we have considered for L has Lyanpunov exponent $\ln 2$ and hence are chaotic. Hence there are infinite number of chaotic orbits for L.
We conjecture that the above results can be used to prove that Feigenbaum constant is simply normal.

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