EIGENVALUE ESTIMATES FOR RANDOM SCHRÖDINGER OPERATORS

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1. INTRODUCTION

In this paper we study the negative eigenvalues $\lambda_j(V)$ of the Schrödinger operator $-\Delta - V(x)$, $x \in \mathbb{R}^d$. If V decays as $|x| \to \infty$ in a certain integral sense, then the negative spectrum of the operator is discrete. The igenvalues $\lambda_j(V)$ can accumulate only to the point zero. Moreover, the rate of the accumulation is controlled by the relation

$$\sum_j |\lambda_j(V)|^\gamma \leq C \int |V(x)|^{d/2+\gamma}\, dx, \tag{1.1}$$

where $\gamma \geq 0$ for $d \geq 3$, $\gamma > 0$ for $d = 2$ and $\gamma \geq 1/2$ for $d = 1$.

The estimate (1.1) is called the classical Lieb-Thirring inequality. One needs to remark, that although for any $V \in L^{d/2+\gamma}$ the eigenvalue sum $\sum_j |\lambda_j|^\gamma$ converges for both V and $-V$, it follows from our results that converse need not be true. The sum $\sum_j |\lambda_j|^\gamma$ can converge even for potentials that are not functions of the class $L^{d/2+\gamma}$.

In the present paper we study the question: how typical is the situation when the right hand side of (1.1) is infinite, but nevertheless the series in the left hand side converges? For that purpose, we introduce a certain class of potentials that either decay slower than $L^{d/2+\gamma}$-functions or do not decay at all. Potentials in this class will depend on a parameter ω, which runs over a space with a probability measure, so that one can distinguish between typical and not typical ω. Instead of a decay of the potential, our theorems require random oscillations of $V = V_\omega$, which ensure that $\mathbb{E}[V_\omega(x)] = 0$ for all x. First, we establish the estimate

$$\mathbb{E}\big[\sum |\lambda_j(V_\omega)|^\gamma\big] \leq C \int \mathbb{E}\Big[|V_\omega(x)|^{d+2\gamma}\Big]\, dx, \qquad d \geq 3. \tag{1.2}$$

Then we move one step further and consider potentials with the property

$$\int \Big| \int |x-y|^{-(d-1)} \mathbb{E}\big[V_\omega(x)V_\omega(y)\big]\, dy \Big|^{d/2+\gamma} dx < \infty. \tag{1.3}$$

This condition holds not only if $V_\omega(x)$ decays at infinity, but it also holds when the frequency of random oscillations of V increases as $|x| \to \infty$.

We show that, even though potentials V_ω satisfying (1.3) do not necessary decay, the corresponding series $\sum_j |\lambda_j(V_\omega)|^\gamma$ for them might be still convergent.

The estimates obtained in the paper show that the probability to meet a $L^{d+2\gamma}$-potential for which the corresponding eigenvalue sum diverges is zero and that, for a typical V_ω, one has $\sum |\lambda_j(V_\omega)|^\gamma < \infty$. This illustrates the main difference between (1.2) and the classical Lieb-Thirring estimate (1.1) that holds for all potentials from $L^{d/2+\gamma}$, even for the worst ones.

Relation (1.2) holds for $d \geq 3$. A close result holds in the case $d = 2$ for potentials $|V_\omega(x)| \leq C(1+|x|)^{-s}$. The case $d = 1$ is essentially different from other dimensions.

The solution of the problem studied in this paper relies heavily on the classical Lieb-Thirring estimates. The important role of these estimates in the theory of Schrödinger operators is illustrated by the large number of references we decided to give in the corresponding section (see [1]-[3], [7]-[15] and [18]).

In the 4th section , we give some examples of applications of the estimate (1.2) to the theory of the absolutely continuous spectrum of $H_\omega = -\Delta - V_\omega$. These examples are based on the relation between the negative and the positive part of the spectrum.

2. PRELIMINARIES. THE BIRMAN-SCHWINGER PRINCIPLE

1. Throughout the paper we denote the probability space by Ω. All random variables f in our considerations are functions on Ω;

$$\mathbb{E}[f] = \int_\Omega f(\omega)d\omega.$$

2. For any self adjoint operator T and $s > 0$ we define

$$n_+(s,T) = \mathrm{rank} E_T(s,+\infty),$$

where $E_T(\cdot)$ denotes the spectral measure of T. Recall the following relation (see [5])

$$n_+(s+t, T+S) \leq n_+(s,T) + n_+(t,S); \tag{2.1}$$

The next statement is known as the Birman-Schwinger principle.

Lemma 2.1. *Let V be a real valued function defined on the space $\mathbb{R}^d$. Let $N(\lambda, V)$ be the number of eigenvalues of $-\Delta - V$ below $\lambda < 0$. Then*

$$N(\lambda,V) = n_+(1, (-\Delta-\lambda)^{-1/2} V (-\Delta-\lambda)^{-1/2}).$$

Combining this lemma with (2.1) we obtain

Corollary 2.1. *For any* $\epsilon \in (0,1)$

$$N(\lambda, V_1+V_2) \le N(\lambda, \epsilon^{-1}V_1) + N(\lambda, (1-\epsilon)^{-1}V_2). \tag{2.2}$$

We would like to remark, that since

$$\sum_j |\lambda_j(V)|^\gamma = \int_0^\infty \gamma s^{\gamma-1} N(-s,V)\, ds, \tag{2.3}$$

we can always represent the Lieb-Thirring sum as the integral

$$\sum_j |\lambda_j(V)|^\gamma = \gamma \int_0^\infty s^{\gamma-1} n_+(1, (-\Delta+s)^{-1/2} V (-\Delta+s)^{-1/2})\, ds.$$

3. Estimates for the expectation of the Lieb-Thirring sum

1. Let ω_n be independent bounded identically distributed random variables with $\mathbb{E}[\omega_n]=0$ and $\mathbb{E}[\omega_n^2]=1$. Let χ_n be the characteristic functions of disjoint measurable sets $\Delta_n \subset \mathbb{R}^d$ and let $n \in \mathbb{R}^d$ be fixed points in Δ_n. Consider the potential

$$V_\omega := \sum_n v_n \omega_n \chi_n$$

where v_n are fixed real coefficients. We introduce the operator

$$H_\omega = -\Delta - V_\omega$$

and study the negative eigenvalues $\lambda_j(V_\omega)$ of H_ω. For simplicity, assume that the diameters of Δ_n are bounded:

$$\sup_n \sup_{x,y\in\Delta_n} |x-y| < \infty. \tag{3.1}$$

Denote

$$\tau_n = \sup_x \int_{\Delta_n} \frac{1}{|x-y|^{d-1}} dy, \qquad |\Delta_n| = \int_{\Delta_n} dx.$$

The following statement is the main result of the paper. It includes the case when the diameter of Δ_n tends to 0 as $|n| \to \infty$.

Theorem 3.1. *Let* $d \ge 3$*. Assume that sizes of the sets* Δ_n *are uniformly bounded so that they satisfy* (3.1)*. Then for any* $\gamma \ge 0$

$$\mathbb{E}\Big[\sum_j |\lambda_j(V_\omega)|^\gamma\Big] \le C \sum_n |v_n|^{d+2\gamma} \tau_n^{d/2+\gamma} |\Delta_n|. \tag{3.2}$$

In particular, the number of eigenvalues $\lambda_j(V_\omega)$ *is almost surely finite if the series in the right hand side of* (3.2) *converges for* $\gamma = 0$.

Remarks. 1. If $\Delta_n = [0,1)^d + n$ with $n \in \mathbb{Z}^d$, then relation (3.2) can be written in the following form

$$\mathbb{E}\Big(\sum_j |\lambda_j(V_\omega)|^\gamma\Big) \le C \int \mathbb{E}\Big(|V_\omega(x)|^{d+2\gamma}\Big)dx.$$

2. One can replace functions χ_n in the theorem by any collection of functions whose absolute values are not bigger than χ_n

3. Let us note that a non-random potential V (that is V_ω with all $\omega_n = 1$) has to be in L^q with $q = d/2 + \gamma$ in order to guarantee that $\sum_j |\lambda_j(V)|^\gamma < \infty$. In the case $\Delta_n = [0,1)^d + n$ with $n \in \mathbb{Z}^d$, our theorem allows potentials from L^{2q}. That means they can decay twice slower. We gain nicer properties of the discrete spectrum because of random oscillations of ω_n.

Proof of Theorem 3.1. We represent the function V_ω in the form (see [4])

$$V_\omega = \operatorname{div} Q_\omega, \quad Q_\omega = \nabla\Big(\Delta^{-1} V_\omega\Big).$$

Put it differently, we introduce the vector potential

$$Q_\omega = c_d \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} V_\omega(y)\, dy, \tag{3.3}$$

where c_d is so chosen that the right hand side of (3.3) is the convolution of V_ω with the kernel of the operator $\nabla\Delta^{-1}$. We will return shortly to the question of convergence of this integral and show, that under conditions of the Theorem 3.1, Q_ω is in the space $L^{d+2\gamma}$. Note that the idea to introduce Q appeared in [4] where the author studied the absolutely continuous spectrum of a random Schrödinger operator. Since $V_\omega = \operatorname{div} Q_\omega$, we obtain that the operator

$$-\Delta - 2V_\omega + 4Q_\omega^2 = (\nabla + 2Q_\omega)^*(\nabla + 2Q_\omega) \ge 0$$

is positive. Now, since

$$V = (V - 2Q^2) + 2Q^2,$$

we obtain from Corollary 2.1 and formula (2.3), that

$$\sum |\lambda_j(V)|^\gamma \le \sum |\lambda_j(2V - 4Q^2)|^\gamma + \sum |\lambda_j(4Q^2)|^\gamma. \tag{3.4}$$

Since the operator $-\Delta - 2\operatorname{div} Q_0 + 4Q_0^2$ is positive, the first sum in the right hand side of (3.4) equals zero. Thus

$$\sum |\lambda_j(V_\omega)|^\gamma \le \sum |\lambda_j(4Q_\omega^2)|^\gamma. \tag{3.5}$$

Now formulas (3.5) and (1.1) lead to the following important intermediate result:

$$\sum_j |\lambda_j(V_\omega)|^\gamma \le C \int |Q_\omega|^{d+2\gamma} dx. \tag{3.6}$$

Theorem 3.2. *Let $d \ge 3$ and let $Q_\omega = \nabla\Delta^{-1}V_\omega$. Assume that sizes of the sets Δ_n are uniformly bounded in the sense of* (3.1). *Then for any integer number $p \ge 1$*

$$\int_{\mathbb{R}^d} \mathbb{E}\Big[|Q_\omega|^{2p}\Big] dx \le C \sum_n |v_n|^{2p} \tau_n^p |\Delta_n|. \tag{3.7}$$

Proof. We represent Q in the form of a sum

$$Q = Q_1 + Q_2,$$

where

$$Q_1 = c_d \int \frac{x-y}{|x-y|^d} \chi(x-y) V(y) dy$$

and χ is the characteristic function of the unit ball $\{x : \ |x| < 1\}$. We will establish the estimates

$$\int_{\mathbb{R}^d} \mathbb{E}\Big[|Q_1|^{2p}\Big] dx \le C \sum_n |v_n|^{2p} \tau_n^p |\Delta_n|, \tag{3.8}$$

and

$$\int_{\mathbb{R}^d} \mathbb{E}\Big[|Q_2|^{2p}\Big] dx \le C \sum_n |v_n|^{2p} \tau_n^p |\Delta_n| \tag{3.9}$$

separately. Let us prove the estimate (3.8) for Q_1 first.

Since $\mathbb{E}[\omega_n] = 0$, we obtain that

$$\begin{aligned}
\mathbb{E}[Q_1^{2p}(x)] &\le c_d \sum_{m_1+\cdots+m_k=2p} \prod_j \frac{2p!}{m_1!\dots m_k!} \sum_n \Big(\int_{\Delta_n} \frac{v_n \chi(x-y)\, dy}{|x-y|^{d-1}}\Big)^{m_j} \\
&\le C_1 \sum_{m_1+\cdots+m_k=2p} \prod_j \sum_n |v_n|^{m_j} \tau_n^{m_j/2} \Big(\int_{\Delta_n} \frac{\chi(x-y)\, dy}{|x-y|^{d-1}}\Big)^{m_j/2} \\
&\le C_2 \sum_{m_1+\cdots+m_k=2p} \prod_j \sum_n |v_n|^{m_j} \tau_n^{m_j/2} \int_{\Delta_n} \frac{\chi(x-y)\, dy}{|x-y|^{d-1}}
\end{aligned}$$

simply because all $m_j \ge 2$ and Δ_n are uniformly bounded.

Now by the Hölder inequality for sequence spaces l^p,

$$\sum_n |v_n|^{m_j} \tau_n^{m_j/2} \int_{\Delta_n} \frac{\chi(x-y)\, dy}{|x-y|^{d-1}} \le$$

$$\le C_3\Big(\sum_n |v_n|^{2p}\tau_n^p\int_{\Delta_n}\frac{\chi(x-y)\,dy}{|x-y|^{d-1}}\Big)^{m_j/2p}\Big(\int_{\mathbb{R}^d}\frac{\chi(x-y)\,dy}{|x-y|^{d-1}}\Big)^{1-m_j/2p}$$

Consequently,

$$\mathbb{E}[Q_1^{2p}(x)]\le C_4\sum_n |v_n|^{2p}\tau_n^p\int_{\Delta_n}\frac{\chi(x-y)\,dy}{|x-y|^{d-1}}$$

Integrating this inequality with respect to x we obtain (3.8).

Similarly we obtain estimate (3.9) for Q_2 Since $\mathbb{E}[\omega_n]=0$, we obtain that

$$\mathbb{E}[Q_2^{2p}(x)]\le C_5\sum_{m_1+\dots+m_k=2p}\prod_j\frac{2p!}{m_1!\dots m_k!}\sum_n\Big(\int_{\Delta_n}\frac{v_n\,dy}{(1+|x-y|)^{d-1}}\Big)^{m_j}$$

Applying the Hölder inequality for L^p-functions, we get

$$\sum_n\Big(\int_{\Delta_n}\frac{v_n\,dy}{(1+|x-y|)^{d-1}}\Big)^{m_j}\le\sum_n|v_n|^{m_j}\Delta_n^{m_j/2}\Big(\int_{\Delta_n}\frac{dy}{(1+|x-y|)^{2(d-1)}}\Big)^{m_j/2}\le$$

$$\le C_6\sum_n|v_n|^{m_j}\Delta_n^{m_j/2}\int_{\Delta_n}\frac{dy}{(1+|x-y|)^{2(d-1)}}$$

simply because all $m_j\ge 2$ and Δ_n are uniformly bounded.

Now applying the Hölder inequality for sequences, we derive

$$\sum_n|v_n|^{m_j}\Delta_n^{m_j/2}\int_{\Delta_n}\frac{dy}{(1+|x-y|)^{2(d-1)}}\le$$

$$\le\Big(\sum_n|v_n|^{2p}\Delta_n^p\int_{\Delta_n}\frac{dy}{(1+|x-y|)^{2(d-1)}}\Big)^{m_j/2p}\Big(\int_{\mathbb{R}^d}\frac{dy}{(1+|x-y|)^{2(d-1)}}\Big)^{1-m_j/2p}$$

Consequently,

$$\mathbb{E}[Q_2^{2p}(x)]\le C_7\sum_n|v_n|^{2p}\Delta_n^p\int_{\Delta_n}\frac{dy}{(1+|x-y|)^{2(d-1)}}$$

Integrating this inequality with respect to x and estimating Δ_n by τ_n we obtain (3.9). Thus the statement of the theorem follows from the triangle inequality

$$\Big(\int\mathbb{E}[Q^{2p}(x)]\,dx\Big)^{1/2p}\le\Big(\int\mathbb{E}[Q_1^{2p}(x)]\,dx\Big)^{1/2p}+\Big(\int\mathbb{E}[Q_2^{2p}(x)]\,dx\Big)^{1/2p}.$$

The proof is completed. □

Estimate (3.7) is proven only for integer p. It follows for arbitrary $p\ge 1$ by interpolation arguments. Indeed, for every integer $p\ge 1$, consider the mapping

$$\mathfrak{T}:\{q_n\}\mapsto Q_\omega$$

where $q_n = v_n \tau_n$. If Δ_n are fixed, this mapping is linear and continuous from the space with the norm $(\sum q_n^{2p} \Delta_n)^{1/2p}$ to the space $L^{2p}(\Omega \times \mathbb{R}^d)$. Interpolation of $\mathfrak{T}$ leads to estimate (3.7) for arbitrary $p \geq 1$.

Now the statement of Theorem 3.1 follows from (3.6) and (3.7) for $p = d/2 + \gamma$.

2. In the two dimensional case Theorem 3.1 holds in a somewhat weaker form. We assume that the potential V admits the estimate:

$$|V_\omega(x)| \leq C(1 + |x|)^{-s}, \quad s > 0. \tag{3.10}$$

Note that then $V_\omega \in L^{d+2\gamma}$ for $s > \frac{d}{d+2\gamma}$. Therefore a natural version of Theorem 3.1 for potentials (3.10) is the following statement, which we formulate only for $d = 2$.

Theorem 3.3. *Let $d = 2$, $\Delta_n = [0,1)^d + n$, $n \in \mathbb{Z}^2$ and $s > \frac{1}{1+\gamma}$. Then*

$$\mathbb{E}\Big[\sum_j |\lambda_j(V_\omega)|^\gamma\Big] \leq C\Big(\sup_n (1 + |n|)^s |v_n|\Big)^{2+2\gamma}, \qquad \gamma > 0.$$

For $\gamma = 0$ the condition $s > 1$ in (3.10) *implies that the number of negative eigenvalues $\lambda_j(V_\omega)$ is finite with probability 1.*

We allow ourself to omit the proof of Theorem 3.3, since it differs very little from the proof of Theorem 3.1.

Finally, consider the case $d = 1$. Note that estimate (1.2) with $\gamma = 0$ imlies finiteness of the number of eigenvalues below zero. It means that the operators with potentials (3.10) have finite negative spectrum for $s > 1$. The same is true in $d = 2$. The situation changes for $d = 1$. It turns out that the number of eigenvalues $\lambda_j(V)$ of the operator with a potential satisfying estimate (3.10) with $s > 1$ can be infinite. Nevertheless, one can prove the following result.

Theorem 3.4. *Let $d = 1$ and $\Delta_n = [n, n+1)$, $n \in \mathbb{Z}$. Then the condition that*

$$|v_n| \leq C(1 + |n|)^{-3/2-\epsilon}, \qquad \epsilon > 0,$$

implies that the number of negative eigenvalues of the operator $-d^2/dx^2 - V_\omega$ is finite with probability 1.

This result is sharp in the power scale (see Theorem 5.3).

4. Consequences of the main theorem

In this section we give some examples of applications of Theorem 3.1 to the problems, where instead of negative eigenvalues one studies the positive spectrum. We shall say that the absolutely continuous spectrum of the operator $H_\omega = -\Delta + V_\omega$ is essentially supported by $\mathbb{R}_+$ if the spectral projection

$E_{H_\omega}(\delta)$ is different from zero, as soon as the Lebesgue measure of the set $\delta \subset \mathbb{R}_+ = (0,\infty)$ is positive. In other words,

$$E_{H_\omega}(\delta) = 0, \quad \delta \subset \mathbb{R}_+, \quad \text{implies } |\delta| = 0.$$

It is known that the singular spectrum of a self-adjoint operator on a separable Hilbert space is concentrated on the set of zero Lebesgue measure. Therefore the property of the spectral projections, mentioned above, holds only for operators, whose absolutely continuous spectrum fills the positive real line.

Our first theorem in this section is based on the connection between the properties of the absolutely continuous spectrum and behavior of the negative eigenvalues of H_ω.

Theorem 4.1. *Let $d \geq 3$. Assume that $\{v_n\} \in l^\infty$ and*

$$\sum_n |v_n|^{d+1} \tau_n^{(d+1)/2} |\Delta_n| < \infty. \tag{4.1}$$

Then the absolutely continuous spectrum of the operator H_ω is essentially supported by $(0,\infty)$ with prbability one.

Proof. This theorem follows from the main result of [16] that says that the condition

$$\sum_j \sqrt{|\lambda_j(V)|} + \sum_j \sqrt{|\lambda_j(-V)|} < \infty$$

implies that the absolutely continuous spectrum of $-\Delta - V$ is essentially supported by $(0,\infty)$. □

Without any doubt, this result can not be considered as a trivial consequence of the classical scattering theory, because the potentials satisfying (4.1) do not have to decay faster than the Coulomb potential. On the other hand the presence of the absolutely continuous spectrum is expected in the case when

$$\int \frac{V^2(x)}{(1+|x|)^{d-1}} dx < \infty \tag{4.2}$$

or, which is almost the same, at least under the condition

$$V \in L^{2q}(\mathbb{R}^d), \qquad \text{for some} \quad q < d. \tag{4.3}$$

The statement that the absolutely continuous spectrum of $-\Delta - V(x)$ fills the positive real line under the condition (4.2) is called B. Simon's conjecture. There is no proof of this conjecture in the full extent. However, given that $V = V_\omega$ is random ($\Delta_n = [0,1)^d + n,\ n \in \mathbb{Z}^d$) and $\mathbb{E}(V_\omega) = 0$, this statement can be considered to be proven (see[4]) under a certain convention. Namely, instead of (4.2) one has to impose the condition

$$|V(x)| \leq \frac{C}{(1+|x|)^s}, \qquad s > 1/2.$$

In this sense, any integral condition of type (4.3) or (4.1) are again meaningful, because they do not assume that the decay of V as $|x| \to \infty$ is uniform with respect to the direction.

In the next theorem we say that a random potential of class (4.3) can be slightly perturbed so that the spectrum of the Schrödinger operator will gain nicer properties.

We shall say that a real valued potential W belongs to the class of fast decaying potentials $\mathfrak{A}$ if

$$\int_{\mathbb{R}^d} \frac{|W(x)|dx}{(1+|x|)^{d-1}} < \infty.$$

Theorem 4.2. *Let $d \geq 3$ and let $\{v_n\} \in l^\infty$. Assume that*

$$\sum_n |v_n|^{2q} \tau_n^q |\Delta_n| < \infty$$

for some $1 < q < d$. Then for almost every $\omega \in \Omega$ there is a potential $W_\omega \in \mathfrak{A}$ such that the absolutely continuous spectrum of the operator

$$H_\omega + W_\omega = -\Delta + V_\omega + W_\omega$$

is essentially supported by $(0, \infty)$.

Proof. As the matter of fact,

$$W_\omega = Q_\omega^2,$$

where $Q_\omega = \nabla \Delta^{-1} V_\omega$. According to estimate (3.7),

$$Q \in L^{2q}.$$

Consequently, $W \in L^q$ with $q < d$. Therefore,

$$\int_{\mathbb{R}^d} \frac{|W(x)|dx}{(1+|x|)^{d-1}} \leq \Big(\int |W|^q dx\Big)^{1/q} \Big(\int \frac{dx}{(1+|x|)^{q(d-1)/(q-1)}}\Big)^{1-1/q} < \infty$$

it remains to refer to [17] where it is proven that if both operators $H_+ = -\Delta + V + W$ and $H_- = -\Delta - V + W$ are positive, then the absolutely continuous spectra of operators $H_\pm$ are essentially supported by $(0, \infty)$. Positivity of the operators $H_\pm$ follows in its turn from the relations $W = Q^2$ and $V = \operatorname{div} Q$. $\square$

In the case of the normal lattice $\Delta_n = [0,1)^d + n$ with $n \in \mathbb{Z}^d$, this result can be improved. Namely, as it was shown by Denissov [4], H_ω has a.c.

spectrum allover the positive real line under conditions that are similar to the ones in Theorem 4.2.

5. CONDITIONS THAT GUARANTEE THE PRESENCE OF INFINITELY MANY EIGENVALUES IN LOW DIMENSIONS.

Inequality (3.2) for $\gamma = 0$ guarantees that the number of eigenvalues below zero is finite. Let us discuss the converse question. Namely, under what conditions on V the operator $-\Delta + V_\omega$ has infinitely many negative eigenvalues. Here we shall consider the case of the standard lattice $\Delta_n = [0,1)^d + n,\ n \in \mathbb{Z}^d$ in dimensions $d = 1, 2$ and shall be interested only in potentials V_ω satisfying the condition

$$|V_\omega| \leq \frac{C}{(1+|x|)^s}, \qquad s > 0.$$

For simplicity of calculations we shall assume that the random variables ω_n are normally distributed. This means that the density of distribution for ω_n is a function of the form

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\bigl(-\frac{t^2}{2\sigma^2}\bigr).$$

Consider first the two-dimensional case, when χ_n are the characteristic functions of the squares $\Delta_n = [0,1)^2 + n,\ n \in \mathbb{Z}^2$. We intend to construct a potential V_ω that decays as $|x|^{-1+\epsilon}$ as $|x| \to \infty$, but nevertheless the operator $-\Delta - V_\omega$ has infinitely many eigenvalues below zero.

Theorem 5.1. *Let $d = 2$ For any $\epsilon > 0$ there exist coefficients v_n satisfying the estimate*

$$|v_n| \leq (1+|n|)^{-1+\epsilon},$$

such that the operator $-\Delta - V_\omega$ with the potential

$$V_\omega = \sum_n v_n \omega_n \chi_n$$

has infinite number of negative eigenvalues.

Proof. In order to construct $V = V_\omega$ we introduce characteristic functions θ_m of the spherical layers

$$\{\, x \in \mathbb{R}^2 : \ 2\cdot 4^n \leq |x| < 3\cdot 4^n \,\}.$$

The potential V_ω in our example will be the function

$$V_\omega = \sum_{m=1}^{\infty} \theta_m \sum_n \omega_n (1+|n|)^{-1+\epsilon} \chi_n(x)$$

Note that the quantity

$$\xi_m = \int \theta_m V \, dx$$

is a normally distributed random variable having the variance

$$\sigma_m^2 = c \int_{\mathbb{R}^2} \theta_m(x) \sum_n (1+|n|)^{-2+2\epsilon} \chi_n(x) \, dx \to \infty, \text{ as } m \to \infty.$$

Consequently, the probability, that

$$\frac{\xi_m}{\sigma_m} > s, \quad \text{equals} \quad \frac{1}{\sqrt{2\pi}} \int_s^\infty \exp(-t^2/2) \, dt.$$

Consider now functions

$$\phi_n(x) = \phi\big(\frac{x}{4^n}\big), \qquad \text{where} \quad \phi \in H^1(\mathbb{R}^2)$$

equals 1 on the layer $\{x \in \mathbb{R}^2 : \ 2 < |x| < 3\}$ and supported in the set $\{x \in \mathbb{R}^2 : \ 1 < |x| < 4\}$. We note that the supports of the functions ϕ_n are disjoint. Besides that,

$$\int |\nabla \phi_n|^2 \, dx = \int |\nabla \phi|^2 \, dx = const.$$

Consequently, the probability that

$$\int |\nabla \phi_n|^2 \, dx - \int V_\omega(x) |\phi_n|^2 \, dx < 0 \tag{5.1}$$

tends to $1/2$ as $|n| \to \infty$. In other words, (5.1) holds approximately for ”one half” of the indices n. Since the number of indices in question is infinite, the inequality (5.1) holds for infinitely many n, which means that the operator $H_\omega = -\Delta - V_\omega$ has infinitely many negative eigenvalues. □

As a matter of fact, one can prove

Theorem 5.2. *Let $d = 2$. There exist coefficients v_n satisfying the estimate*

$$|v_n| \le (1+|n|)^{-1},$$

such that the operator $-\Delta - V_\omega$ with the potential

$$V_\omega = \sum_n v_n \omega_n \chi_n$$

has infinite number of negative eigenvalues.

The same arguments work in the one-dimensional case

Theorem 5.3. *Let $d = 1$ and let χ_n be the characteristic functions of the intervals $[n, n+1)$. For any $\epsilon > 0$ there exist coefficients v_n satisfying the estimate*

$$|v_n| \leq (1 + |n|)^{-3/2+\epsilon},$$

such that the operator $-\Delta - V_\omega$ with the potential

$$V_\omega = \sum_n v_n \omega_n \chi_n$$

has infinite number of negative eigenvalues.

On the other hand, the number of negative eigenvalues of the Schrödinger operator with the potential

$$V_\omega = C \sum_n (1 + |n|)^{-3/2-\epsilon} \omega_n \chi_n$$

is finite. That shows that the corresponding result is sharp.

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