Functional Integral Representation of the Pauli-Fierz Model with Spin 1/2

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Abstract

A Feynman-Kac-type formula for a Lévy and an infinite dimensional Gaussian random process associated with a quantized radiation field is derived. In particular, a functional integral representation of $e^{-tH_{\rm PF}}$ generated by the Pauli-Fierz Hamiltonian with spin 1/2 in non-relativistic quantum electrodynamics is constructed. When no external potential is applied $H_{\rm PF}$ turns translation invariant and it is decomposed as a direct integral $H_{\rm PF} = \int_{\mathbb{R}^3}^{\oplus} H_{\rm PF}(P)dP$. The functional integral representation of $e^{-tH_{\rm PF}(P)}$ is also given. Although all these Hamiltonians include spin, nevertheless the kernels obtained for the path measures are scalar rather than matrix expressions. As an application of the functional integral representations energy comparison inequalities are derived.

1 Introduction

Functional integration proved to be a useful approach in various applications to quantum field theory. For the case of a quantum particle linearly coupled to a scalar boson field, the so called Nelson model, it gives a tool to proving existence or absence of a ground state in Fock space [Spo98, LMS02a]. Furthermore, ground state properties can be derived in terms of path measure expectations [BHLMS02], and the question how the model Hamiltonian and its ground state behave under lifting the so called infrared and ultraviolet cutoffs can also be treated by the same method [LMS02b, GL07a, GL07b]. Another problem studied by this approach is that of the effective mass [BS05, Spo87]. Some of these results have been obtained by functional integration only, so sometimes it offers a complementary method rather than a mere alternative.

In contrast with Nelson's model, the Pauli-Fierz model describes a minimal coupling of a particle to the quantized radiation field. The spectrum of the Pauli-Fierz Hamiltonian has been extensively studied by a number of authors also using analytic methods. In particular, the bottom of the spectrum of the Pauli-Fierz Hamiltonian is contained in the absolutely continuous spectrum, no matter how small the coupling constant is. Nevertheless, a ground state exists for arbitrary values of the coupling constant without any infrared cutoff [BFS99, GLL01, LL03]. Functional integration is also useful in studying the spectrum of the Pauli-Fierz Hamiltonian which was addressed in the spinless case so far [BH07, Hir00a, Hir07, HL07].

The spinless Pauli-Fierz Hamiltonian is written as

$$\hat{H}_{\mathrm{PF}} := \frac{1}{2}(-i\nabla - e\mathcal{A})^2 + V + H_{\mathrm{rad}} \tag{1.1}$$

on $L^2(\mathbb{R}^3) \otimes L^2(\mathcal{Q})$, where the former is the particle state space and the latter is the state space of the quantum field, $\mathcal{A}$ stands for the vector potential, H_{rad} for the photon field, and V is an external potential acting on the electron. These objects will be explained in the following section in detail. The C_0-semigroup $e^{-t\hat{H}_{\mathrm{PF}}}$ is defined through spectral calculus. A functional integral representation of the semigroup $e^{-t\hat{H}_{\mathrm{PF}}}$ can be constructed on the space $C([0,\infty);\mathbb{R}^3) \times \mathcal{Q}_{\mathrm{E}}$, involving a process consisting of Brownian motion for the particle, and an infinite dimensional Ornstein-Uhlenbeck process on a function space $\mathcal{Q}_{\mathrm{E}}$ for the field [FFG97, Hab98, Hir97]. One immediate corollary for the functional integral representation is the diamagnetic inequality [AHS78, Hir00a]

$$\inf \sigma(-(1/2)\Delta + V + H_{\mathrm{rad}}) \le \inf \sigma(\hat{H}_{\mathrm{PF}}). \tag{1.2}$$

Using the fact that a path measure exists was also applied to proving self-adjointness of $\hat{H}_{\rm PF}$ for arbitrary values of the coupling constant e [Hir00b, Hir02]. Furthermore, whenever $\hat{H}_{\rm PF}$ has a ground state, the path measure can be used to prove its uniqueness [Hir00a] as an alternative to the methods making use of ergodic properties of the semigroup in [Gro72, GJ68]. Other applications for the study of the ground state include [BH07, HL07].

The path measure of the coupled Brownian motion and Ornstein-Uhlenbeck process can be written in terms of a mixture of two measures as the specific form of the coupling between particle and field allows an explicit calculation of the Gaussian part. The so obtained marginal over the particle is a Gibbs measure on Brownian paths with densities dependent on the twice iterated Itô integral of a pair potential function describing the effective field resulting from the Gaussian integration [Spo87, Hir00a, BH07, GL07a].

Previous applications of rigorous functional integration to quantum field theory covered, as far as we know, only cases when no spin was present in the model. In this paper our main concern is to study by means of a Feynman-Kac-type formula the Pauli-Fierz operator with spin 1/2. (1.1) is in this case replaced by

$$H_{\rm PF} := \frac{1}{2}\left(\sigma \cdot (-i\nabla - e\mathcal{A})\right)^2 + V + H_{\rm rad}, \tag{1.3}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices standing for the spin (see details in the next section). The random process of the particle modifies to a $3+1$ dimensional joint Wiener and jump process $(\xi_t)_{t\geq 0} = (B_t, \sigma_t)_{t\geq 0}$, where the effect of the spin appears in the process $\sigma_t = \sigma(-1)^{N_t}$ jumping between the two possible values of the spin variable σ, driven by a Poisson process $(N_t)_{t\geq 0}$. Our approach owes a debt to the ideas in [ALS83], where a path integral representation of a C_0-semigroup generated by Pauli operators in quantum mechanics was obtained by making use of an $\mathbb{R}^3 \times \mathbb{Z}_2$-valued process, with $\mathbb{Z}_2$ the additive group of order two. As we will see in the next subsection, the Pauli operator is of a similar form as $H_{\rm PF}$, in fact both operators describe minimal interactions. While in [ALS83] only a path integral representation of operators with non-vanishing off-diagonal elements was constructed, we improve on this here since this part of the spin interaction in general may have zeroes.

Another model considered in the present paper is the so called translation invariant Pauli-Fierz Hamiltonian which is the case of $H_{\rm PF}$ above with zero external potential V. Translation invariance yields to a fibred decomposition $H_{\rm PF} = \int_{\mathbb{R}^3}^{\oplus} H_{\rm PF}(P)dP$ with respect to total momentum $P^{\rm tot}$, where the fiber Hamiltonian is given by

$$H_{\rm PF}(P) := \frac{1}{2}\left(\sigma\cdot(P - P_{\rm f} - e\mathcal{A}(0))\right)^2 + H_{\rm rad}, \quad P \in \mathbb{R}^3. \tag{1.4}$$

Here $P_{\rm f}$ denotes the momentum operator of the field. While the translation invariant Hamiltonian has no point spectrum, $H_{\rm PF}(P)$ under some conditions does [Fro74, Che01]. In [Hir07] the functional integral representation of $e^{-t\hat{H}_{\rm PF}(P)}$ for the spinless fiber Hamiltonian is constructed, where

$$\hat{H}_{\rm PF}(P) := \frac{1}{2}\left(P - P_{\rm f} - e\mathcal{A}(0)\right)^2 + H_{\rm rad}, \quad P \in \mathbb{R}^3. \tag{1.5}$$

Then the uniqueness of the ground state of $\hat{H}_{\rm PF}(0)$ as well as the energy comparison inequality

$$\inf\sigma(\hat{H}_{\rm PF}(0)) \leq \inf\sigma(\hat{H}_{\rm PF}(P)) \tag{1.6}$$

are shown.

Our main purpose in this paper is to extend the results on the spinless Hamiltonians mentioned above to those with spin, i.e.,

(1) construct a functional integral representation of $e^{-tH_{\rm PF}}$ and $e^{-tH_{\rm PF}(P)}$ with *a scalar* kernel;

(2) derive some energy comparison inequalities for $H_{\rm PF}$ and $H_{\rm PF}(P)$.

We stress that $H_{\rm PF}$ and $H_{\rm PF}(P)$ include spin 1/2, nevertheless the kernels of their functional integrals obtained here are scalar. (1) is achieved in Theorems 4.11 and 5.2, and (2) in Corollaries 4.13 and 5.4 below.

Here is an outline of the key steps of proving (1) and (2). First we assume that the form factor $\hat{\varphi}$ is a sufficiently smooth function of compact support. Then we will see that there exists a Pauli operator $H^0_{\rm PF}(\phi)$, $\phi \in \mathcal{Q}$, on $L^2(\mathbb{R}^3\times\mathbb{Z}_2)$ which can be used to define

$$H^0_{\rm PF} := \int_{\mathcal{Q}}^{\oplus} H^0_{\rm PF}(\phi)d\mu(\phi). \tag{1.7}$$

As it will turn out, for *arbitrary* values of the coupling constant e,

$$H_{\rm PF} = H^0_{\rm PF} \dot{+} H_{\rm rad} \tag{1.8}$$

holds as an equality of self-adjoint operators ($\dot{+}$ denotes quadratic form sum). Although for weak couplings this results by the Kato-Rellich Theorem, it is non-trivial for arbitrary values of e. Thus it will suffice to construct a functional integral representation of the right hand side of (1.8). However, as was mentioned before, the off-diagonal part of $H^0_{\rm PF}(\phi)$ may have in general zeroes or a compact support. In order to prevent the off-diagonal part vanish we change $H^0_{\rm PF}(\phi)$ for $H^{0\varepsilon}_{\rm PF}(\phi)$ by adding a term controlled by a small parameter $\varepsilon > 0$. Then we work with

$$H^{\varepsilon}_{\rm PF} := H^{0\varepsilon}_{\rm PF} \dot{+} H_{\rm rad} \tag{1.9}$$

and obtain the original Hamiltonian by $\lim_{\varepsilon\to 0} e^{-tH^{\varepsilon}_{\rm PF}} = e^{-tH_{\rm PF}}$, where in fact $H^{0\varepsilon}_{\rm PF} := \int^{\oplus}_{\mathbb{R}^3} H^{0\varepsilon}_{\rm PF}(\phi) d\mu(\phi)$. In particular, instead of for the semigroup $e^{-tH_{\rm PF}}$, we construct the functional integral representation of $e^{-tH^{\varepsilon}_{\rm PF}}$. By the Trotter-Kato product formula we write

$$e^{-tH^{\varepsilon}_{\rm PF}} = \text{s--}\lim_{n\to\infty} \big(e^{-(t/n)H^{0\varepsilon}_{\rm PF}} e^{-(t/n)H_{\rm rad}}\big)^n \tag{1.10}$$

and derive the functional integral of the Pauli-operator $e^{-tH^{0\varepsilon}_{\rm PF}(\phi)}$ by using that the form factor $\hat{\varphi}$ is chosen to be bounded and sufficiently smooth, with non-zero off-diagonals. By making use of a hypercontractivity argument for second quantization and the Markov property of projections, we are able to construct the functional integral representation of $e^{-tH^{\varepsilon}_{\rm PF}}$. An approximation argument on $\hat{\varphi}$ leads us then to our main Theorem 4.11 for reasonable form factors.

The functional integral representation of $e^{-tH_{\rm PF}(P)}$ is further obtained by a combination of that of $e^{-tH_{\rm PF}}$ and [Hir07]. Since the functional integral kernels are scalar, we can estimate $|(F, e^{-tH_{\rm PF}}G)|$ and $|(F, e^{-tH_{\rm PF}(P)}G)|$ directly, and derive some energy comparison inequalities.

Our paper is organized as follows. In Section 2 we discuss the Fock space respectively Euclidean representations of the Pauli-Fierz Hamiltonian with spin 1/2 in detail. Section 3 is devoted to discussing Lévy processes and functional integral representations of Pauli operators. In Section 4 by using results of the previous section and hypercontractivity properties of second quantization we construct the functional integral representation of $e^{-tH_{\rm PF}}$ and derive comparison inequalities for ground state energies. In Section 5 we derive the functional integral of $e^{-tH_{\rm PF}(P)}$ and obtain energy inequalities for this case. In Section 6 we comment on the multiplicity of ground states of a model with spin. Section 7 is an appendix containing details on Poisson point processes and a related Itô formula adapted to our context.

2 Function space representation of the Pauli-Fierz model with spin

2.1 Pauli-Fierz model with spin $1/2$ in Fock space

We begin by defining the Pauli-Fierz Hamiltonian as a self-adjoint operator.

Fock space Let $h_{\rm b} := L^2(\mathbb{R}^3 \times \{-1,1\})$ be the Hilbert space of a single photon, where $\mathbb{R}^3 \times \{-1,1\} \ni (k,j)$ are its momentum and polarization, respectively. Denote n-fold symmetric tensor product by $\bigotimes_{\rm sym}^n$, with $\bigotimes_{\rm sym}^0 h_{\rm b} := \mathbb{C}$. The Fock space describing the full photon field is defined then as the Hilbert space $\mathcal{F} := \bigoplus_{n=0}^\infty \bigotimes_{\rm sym}^n h_{\rm b}$ with scalar product

$$(\Psi,\Phi)_{\mathcal{F}} := \sum_{n=0}^\infty (\Psi^{(n)},\Phi^{(n)})_{\otimes_{\rm sym}^n h_{\rm b}}, \tag{2.1}$$

and $\Psi = \bigoplus_{n=0}^\infty \Psi^{(n)}$, $\Phi = \bigoplus_{n=0}^\infty \Phi^{(n)}$. Alternatively, $\mathcal{F}$ can be identified as the set of ℓ_2-sequences $\{\Psi^{(n)}\}_{n=0}^\infty$ with $\Psi^{(n)} \in \bigotimes_{\rm sym}^n h_{\rm b}$. The vector $\Omega = \{1,0,0,...\} \in \mathcal{F}$ is called Fock vacuum. The finite particle subspace $\mathcal{F}_{\rm fin}$ is defined by

$$\mathcal{F}_{\rm fin} := \left\{ \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} \,|\, \exists M \in \mathbb{N} : \, \Psi^{(m)} = 0, \, \forall m \geq M \right\}.$$

Field operators With each $f \in h_{\rm b}$ a photon creation and annihilation operator is associated. The creation operator $a^\dagger(f) : \mathcal{F} \to \mathcal{F}$ is defined by

$$(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

where $S_n(f_1 \otimes \cdots \otimes f_n) = (1/n!) \sum_{\pi \in \Pi_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$ is the symmetrizer with respect to the permutation group Π_n of degree n. The domain of $a^\dagger(f)$ is maximally defined by

$$D(a^\dagger(f)) := \left\{ \{\Psi^{(n)}\}_{n=0}^\infty \,\middle|\, \sum_{n=1}^\infty n \|S_n(f \otimes \Psi^{(n-1)})\|^2 < \infty \right\}.$$

The annihilation operator $a(f)$ is introduced as the adjoint $a(f) = (a^\dagger(\bar{f}))^*$ of $a^\dagger(\bar{f})$ with respect to scalar product (2.1). $a^\dagger(f)$ and $a(f)$ are closable operators, their closed extensions will be denoted by the same symbols. Also, they leave $\mathcal{F}_{\rm fin}$ invariant and obey the canonical commutation relations on $\mathcal{F}_{\rm fin}$:

$$[a(f), a^\dagger(g)] = (\bar{f}, g)1, \quad [a(f), a(g)] = 0, \quad [a^\dagger(f), a^\dagger(g)] = 0.$$

Second quantization and free field Hamiltonian Although the free field Hamiltonian

$$H_{\rm rad}^{\rm F} = \sum_{j=\pm 1} \int |k| a^\dagger(k,j) a(k,j) dk$$

is usually given in terms of formal kernels of creation and annihilation operators, we define it as the infinitesimal generator of a one-parameter unitary group since this definition has advantages in studying functional integral representations. We use the label F for objects defined in Fock space. This unitary group is constructed through a functor Γ. Let $\mathcal{C}(X \to Y)$ denote the set of contraction operators from X to Y. Then $\Gamma : \mathcal{C}(h_{\rm b} \to h_{\rm b}) \to \mathcal{C}(\mathcal{F} \to \mathcal{F})$ is defined as

$$\Gamma(T) := \bigoplus_{n=0}^{\infty} [\otimes^n T]$$

for $T \in \mathcal{C}(h_{\rm b} \to h_{\rm b})$, where the tensor product for $n = 0$ is the identity operator. For a self-adjoint operator h on $h_{\rm b}$, $\Gamma(e^{ith})$, $t \in \mathbb{R}$, is a strongly continuous one-parameter unitary group on $\mathcal{F}$. Then by Stone's Theorem there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}$ such that $\Gamma(e^{ith}) = e^{itd\Gamma(h)}$, $t \in \mathbb{R}$. $d\Gamma(h)$ is called the second quantization of h. The second quantization of the identity operator, $N := d\Gamma(1)$ gives the photon number operator. Let $\omega_{\rm b}$ be the multiplication operator $f \mapsto \omega_{\rm b}(k) f(k,j) = |k| f(k,j)$, $k \in \mathbb{R}^3$, $j = \pm 1$ on $h_{\rm b}$. The operator $H_{\rm rad}^{\rm F} := d\Gamma(\omega_{\rm b})$ is then the free field Hamiltonian.

Polarization vectors Two vectors $e(k,+1)$ and $e(k,-1)$, $k \neq 0$, are polarization vectors whenever $e(k,-1), e(k,+1), k/|k|$ form a right-handed system in $\mathbb{R}^3$ with (1) $e(k,-1) \times e(k,+1) = k/|k|$, (2) $e(k,j) \cdot e(k,j') = \delta_{jj'}$, (3) $e(k,j) \cdot k/|k| = 0$. We have

$$\sum_{j=\pm 1} e_\mu(k,j) e_\nu(k,j) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2},$$

independently of the specific choice of these vectors. One can choose the polarization vectors at convenience since the Hamiltonians $H_{\rm PF}^{\rm F}$ defined below are unitary equivalent up to this choice [Sas06].

Quantized radiation field Note that $a^\sharp(f)$ is linear in f, where $a^\sharp = a, a^\dagger$, thus formally $a^\sharp(f) = \sum_{j=\pm 1} \int f(k,j) a^\sharp(k,j) dk$. The quantized radiation field with ultraviolet cutoff function (form factor) $\hat{\varphi}$ is defined through the vector potentials

$$A_\mu(x) := \frac{1}{\sqrt{2}} \sum_{j=\pm 1} \int e_\mu(k,j) \left(\frac{\hat{\varphi}(k)}{\sqrt{\omega_{\rm b}(k)}} a^\dagger(k,j) e^{-ik\cdot x} + \frac{\hat{\varphi}(-k)}{\sqrt{\omega_{\rm b}(k)}} a(k,j) e^{ik\cdot x} \right) dk.$$

Here $\hat{\varphi}$ is Fourier transform of φ. A standing assumption in this paper is

Assumption 2.1 *We take* $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$ *and* $\sqrt{\omega_{\rm b}}\hat{\varphi}$, $\hat{\varphi}/\omega_{\rm b} \in L^2(\mathbb{R}^3)$.

Under Assumption 2.1 $A_\mu(x)$ is a well-defined symmetric operator in $\mathcal{F}$. By $k \cdot e(k,j) = 0$, the Coulomb gauge condition

$$\sum_{\mu=1}^{3} [\partial_{x_\mu}, A_\mu(x)] = 0,$$

holds on $\mathcal{F}_{\rm fin}$. By the fact that $\sum_{n=0}^{\infty} \|A_\mu(x)^n \Phi\|/n! < \infty$ for $\Phi \in \mathcal{F}_{\rm fin}$, and Nelson's analytic vector theorem [RS75, Th.X.39] it follows that $A_\mu(x)\lceil_{\mathcal{F}_{\rm fin}}$ is essentially self-adjoint. We denote its closure $\overline{A_\mu(x)\lceil_{\mathcal{F}_{\rm fin}}}$ by the same symbol $A_\mu(x)$.

Electron state space and Schrödinger Hamiltonian The Hilbert space describing the electron is $L^2(\mathbb{R}^3; \mathbb{C}^2)$. Let $\sigma_1, \sigma_2, \sigma_3$ be the 2×2 Pauli matrices

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i \sum_{\gamma=1}^{3} \epsilon^{\alpha\beta\gamma} \sigma_\gamma$, where $\epsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor with $\epsilon^{123} = 1$. Then the electron Hamiltonian on $L^2(\mathbb{R}^3; \mathbb{C}^2)$ with external potential V is given by

$$H_{\rm p} = \frac{1}{2} \sum_{\mu=1}^{3} (\sigma_\mu(-i\nabla_\mu))^2 + V.$$

V acts as a multiplication operator and in some statements below it will be required to satisfy one or both of the following conditions:

Assumption 2.2 *Let* V *be*

(1) relatively bounded with respect to $(-1/2)\Delta$ *with a bound strictly less than 1;*

(2) $\sup_{x \in \mathbb{R}^3} \mathbb{E}^x \left[e^{-2\int_0^t V(B_s) ds} \right] < \infty$, *for all* $t \in (0, \infty)$.

(1) above is a usual ingredient for self-adjointness of Schrödinger operators. In (2) the expectation $\mathbb{E}^x$ is meant under Wiener measure for 3-dimensional Brownian motion $(B_s)_{s \geq 0}$ starting at x. It is in particular satisfied by Kato-class potentials which includes Coulomb potential.

Pauli-Fierz Hamiltonian The state space of the joint electron-field system is

$$\mathcal{H}^{\rm F} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}. \tag{2.2}$$

The non-interacting system is described by the total free Hamiltonian $H_{\rm p}\otimes 1+1\otimes H_{\rm rad}^{\rm F}$. To define the quantized radiation field A we identify $\mathcal{H}^{\rm F}$ with the set of $\mathbb{C}^2\otimes\mathcal{F}$-valued L^2 functions on $\mathbb{R}^3$, i.e., $\mathcal{H}^{\rm F}\cong\int_{\mathbb{R}^3}^{\oplus}(\mathbb{C}^2\otimes\mathcal{F})dx$. Then we have by definition $A_\mu=\int_{\mathbb{R}^3}^{\oplus}(1\otimes A_\mu(x))dx$. Hence $(A_\mu F)(x)=A_\mu(x)F(x)$ for $F(x)\in D(A_\mu(x))$ and A_μ is self-adjoint. Taking into account the minimal interaction $-i\nabla_\mu\mapsto -i\nabla_\mu-eA_\mu$, we obtain the Pauli-Fierz Hamiltonian

$$H_{\rm PF}^{\rm F}:=\frac{1}{2}\left(\sum_{\mu=1}^{3}\sigma_\mu(-i\nabla_\mu\otimes 1-eA_\mu)\right)^2+V\otimes 1+1\otimes H_{\rm rad}^{\rm F} \tag{2.3}$$

with coupling constant $e\in\mathbb{R}$, i.e.,

$$H_{\rm PF}^{\rm F}=-\frac{1}{2}(i\nabla-eA)^2+V+H_{\rm rad}^{\rm F}-\frac{e}{2}\sum_{\mu=1}^{3}\sigma_\mu B_\mu, \tag{2.4}$$

where we omit the tensor product for convenience and write

$$B_\mu(x)=-\frac{i}{\sqrt{2}}\sum_{j=\pm1}\int(k\times e(k,j))_\mu\frac{\hat{\varphi}(k)}{\sqrt{\omega_{\rm b}(k)}}\left(a^\dagger(k,j)e^{-ik\cdot x}-a(k,j)e^{ik\cdot x}\right)dk.$$

In fact, $B_\mu(x)=(\nabla\times A(x))_\mu$, however, we regard A and B as independent operators in this paper.

A first natural question is whether $H_{\rm PF}^{\rm F}$ is a self-adjoint operator.

Proposition 2.3 *Under Assumption 2.1 $H_{\rm PF}^{\rm F}$ is self-adjoint on $D(-\Delta)\cap D(H_{\rm rad}^{\rm F})$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $H_{\rm p}+H_{\rm rad}^{\rm F}$.*

PROOF: See [Hir00b, Hir02]. **qed**

A special case considered in this paper is the translation invariant Pauli-Fierz Hamiltonian obtained under $V=0$. Then

$$e^{itP_\mu^{\rm tot}}H_{\rm PF}^{\rm F}e^{-itP_\mu^{\rm tot}}=H_{\rm PF}^{\rm F},\quad t\in\mathbb{R},\ \mu=1,2,3,$$

where $P^{\rm tot}$ denotes the total electron-field momentum

$$P_\mu^{\rm tot}:=-i\nabla_\mu\otimes 1+1\otimes P_{{\rm f}\mu}^{\rm F}$$

and $P_{{\rm f}\mu}^{\rm F}=d\Gamma(k_\mu)$ is the momentum of the field. By translation invariance the Hilbert space $\mathcal{H}^{\rm F}$ and the Hamiltonian $H_{\rm PF}^{\rm F}$ can both be decomposed with respect to the spectrum of $P^{\rm tot}$ as $\int_{\mathbb{R}^3}^{\oplus} \mathcal{H}^{\rm F}(P)dP$ and $H_{\rm PF}^{\rm F} := \int_{\mathbb{R}^3}^{\oplus} K(P)dP$, with a self-adjoint operator $K(P)$ labeled by P on $\mathcal{H}^{\rm F}(P)$. It is seen that $K(P)$ and $\mathcal{H}^{\rm F}(P)$ are isomorphic with a self-adjoint operator resp. a Hilbert space. Define thus on $\mathbb{C}^2 \otimes \mathcal{F}$ the Pauli-Fierz operator at total momentum $P \in \mathbb{R}^3$ by

$$H_{\rm PF}^{\rm F}(P) := \frac{1}{2}(P - P_{\rm f}^{\rm F} - eA(0))^2 + H_{\rm rad}^{\rm F} - \frac{e}{2}\sum_{\mu=1}^{3} \sigma_\mu B_\mu(0). \tag{2.5}$$

Then we have

Proposition 2.4 *Under Assumption 2.1 $H_{\rm PF}^{\rm F}(P)$, $P \in \mathbb{R}^3$, is self-adjoint on the domain $D(H_{\rm rad}^{\rm F}) \bigcap_{\mu=1}^{3} D((P_{{\rm f}\mu}^{\rm F})^2)$, and essentially self-adjoint on any core of the self-adjoint operator $\frac{1}{2}\sum_{\mu=1}^{3}(P_{{\rm f}\mu}^{\rm F})^2 + H_{\rm rad}^{\rm F}$. Moreover, $\mathcal{H}^{\rm F} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes \mathcal{F} dP$ and $H_{\rm PF}^{\rm F} \cong \int_{\mathbb{R}^3}^{\oplus} H_{\rm PF}^{\rm F}(P)dP$ hold.*

Proof: See [Hir06, LMS06]. **qed**

Here is an incomplete list of results on the spectral properties on the Pauli-Fierz Hamiltonian. The existence of the ground state of $H_{\rm PF}$ is established in [BFS99, GLL01, LL03] and that of $H_{\rm PF}(P)$ in [Fro74, Che01, HaHe06]. The multiplicity of the ground state is estimated in [Hir00a, HS01, BFP05, Hir06], a spectral scattering theory and relaxation to ground states are studied in [Ara83a, Spo97, FGS01]. The perturbation of embedded eigenvalues is reduced to investigate resonances, which is done in [BFS98a, BFS98b]. Energy estimates are obtained [Fef96, FFG97, LL00] and the effective mass is studied in [Spo87, CH04, HS05, Che06, BCFS06, HI07]. Related works on particle systems interacting with quantum fields include [Ger00, BDG04, AGG04, LMS06, Sas06].

2.2 Stochastic representation and spin variables in function space

2.2.1 Stochastic representation

In this section we prepare the necessary items for a $\mathcal{Q}$-space representation of $H_{\rm PF}^{\rm F}$ and explain how to accomodate spin in this framework.

To introduce a $\mathcal{Q}$-space representation, we define a bilinear form and construct the Gaussian random process with mean zero and covariance given in terms of this form. Define the field operator $A_\mu(\hat{f})$ by

$$A_\mu(\hat{f}) := \frac{1}{\sqrt{2}}\sum_{j=\pm 1}\int e_\mu(k,j)\left(\hat{f}(k)a^\dagger(k,j) + \hat{f}(-k)a(k,j)\right)dk$$

and the 3×3 matrix $D(k)$, $k \neq 0$, by

$$D(k) := \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2} \right)_{1 \le \mu,\nu \le 3}.$$

Consider the bilinear form $q_0 : \oplus^3 L^2(\mathbb{R}^3) \times \oplus^3 L^2(\mathbb{R}^3) \to \mathbb{C}$ given by the scalar product

$$q_0(f,g) := \sum_{\mu,\nu=1}^{3} (A_\mu(f)\Omega, A_\nu(g)\Omega)_{\mathcal{F}} = \frac{1}{2} \int_{\mathbb{R}^3} \overline{\hat{f}(k)} \cdot D(k) \hat{g}(k) dk.$$

Similarly to the representation of a Euclidean free field in terms of path integrals over the free Minkowski field in constructive quantum field theory [Sim74, Th.III.6], we introduce another bilinear form q_1 to define an additional Gaussian random process. Let $q_1 : \oplus^3 L^2(\mathbb{R}^{3+1}) \times \oplus^3 L^2(\mathbb{R}^{3+1}) \to \mathbb{C}$ be

$$q_1(F,G) := \frac{1}{2} \int_{\mathbb{R}^{3+1}} \overline{\hat{F}(k,k_0)} \cdot D(k) \hat{G}(k,k_0) dk dk_0.$$

Note that $D(k)$ is independent of k_0 in the definition of q_1. Use the label β for 0 or 1, let $\mathcal{S}(\mathbb{R}^{3+\beta})$ be the set of real-valued Schwartz test functions on $\mathbb{R}^{3+\beta}$ and put $\mathcal{S}_\beta := \oplus^3 \mathcal{S}(\mathbb{R}^{3+\beta})$. The properties (1) $\sum_{i,j=1}^n \bar{z}_i z_j \exp(-q_\beta(f_i - f_j, f_i - f_j)) \ge 0$ for arbitrary $z_i \in \mathbb{C}$ and $i = 1, ..., n$, $\forall n = 1, 2, ...$; (2) $\exp(-q_\beta(g,g))$ is strongly continuous in $g \in \oplus^3 L^2(\mathbb{R}^{3+\beta})$; (3) $\exp(-q_\beta(0,0)) = 1$ can be checked directly.

Let $\mathcal{Q}_\beta := \mathcal{S}'_\beta$, where $\mathcal{S}'_\beta$ is the dual space of $\mathcal{S}_\beta$, and denote the pairing between elements of $\mathcal{Q}_\beta$ and $\mathcal{S}_\beta$ by $\langle \phi, f \rangle_\beta \in \mathbb{R}$. By the three properties listed above and the Bochner-Minlos Theorem there exists a probability space $(\mathcal{Q}_\beta, \mathcal{B}_{\mathcal{Q}_\beta}, \mu_\beta)$ such that $\mathcal{B}_{\mathcal{Q}_\beta}$ is the smallest σ-field generated by $\{\langle \phi, f \rangle_\beta, f \in \mathcal{S}_\beta\}$ and $\langle \phi, f \rangle_\beta$ is a Gaussian random variable with mean zero and covariance given by

$$\int_{\mathcal{Q}_\beta} e^{i\langle \phi, f \rangle_\beta} d\mu_\beta(\phi) = e^{-q_\beta(f,f)}, \quad f \in \mathcal{S}_\beta. \tag{2.6}$$

Although $\langle \phi, \oplus_\mu^3 \delta_{\mu\nu} f \rangle_\beta$ is a $\mathcal{Q}$-representation of the quantized radiation field with the ultraviolet cutoff function $f \in \mathcal{S}(\mathbb{R}^3)$, we have to extend $f \in \mathcal{S}_\beta$ to a more general class since our cutoff is $(\hat{\varphi}/\sqrt{\omega})^\vee \in L^2(\mathbb{R}^3)$. This can be done in the following way. For any $f = f_{\rm Re} + i f_{\rm Im} \in \oplus^3 \mathcal{S}(\mathbb{R}^{3+\beta})$ we set $\langle \phi, f \rangle_\beta := \langle \phi, f_{\rm Re} \rangle_\beta + i \langle \phi, f_{\rm Im} \rangle_\beta$. Since $\mathcal{S}(\mathbb{R}^{3+\beta})$ is dense in $L^2(\mathbb{R}^{3+\beta})$ and the inequality

$$\int_{\mathcal{Q}_\beta} |\langle \phi, f \rangle_\beta|^2 d\mu_\beta(\phi) \le \|f\|^2_{\oplus^3 L^2(\mathbb{R}^{3+\beta})}$$

holds by (2.6), we can define $\langle\phi,f\rangle_\beta$ for $f\in\oplus^3L^2(\mathbb{R}^{3+\beta})$ by $\langle\phi,f\rangle_\beta=\text{s--}\lim_{n\to\infty}\langle\phi,f_n\rangle_\beta$ in $L^2(\mathcal{Q}_\beta)$, where $\{f_n\}_{n=1}^\infty\subset\oplus^3\mathcal{S}(\mathbb{R}^{3+\beta})$ is any sequence such that $\text{s--}\lim_{n\to\infty}f_n=f$ in $\oplus^3L^2(\mathbb{R}^{3+\beta})$. Thus we define the multiplication operator

$$\left(\mathcal{A}^\beta(f)F\right)(\phi):=\langle\phi,f\rangle_\beta F(\phi),\quad \phi\in\mathcal{Q}_\beta,$$

labeled by $f\in\oplus^3L^2(\mathbb{R}^{3+\beta})$ in $L^2(\mathcal{Q}_\beta)$, with domain

$$D(\mathcal{A}^\beta(f)):=\left\{F\in L^2(\mathcal{Q}_\beta)\Big|\int_{\mathcal{Q}_\beta}|\langle\phi,f\rangle_\beta F(\phi)|^2d\mu_\beta(\phi)<\infty\right\}.$$

Denote the identity function in $L^2(\mathcal{Q}_\beta)$ by $1_{\mathcal{Q}_\beta}$ and the function $\mathcal{A}^\beta(f)1_{\mathcal{Q}_\beta}$ by $\mathcal{A}^\beta(f)$ unless confusion may arise. It is known that $L^2(\mathcal{Q}_\beta)=\bigoplus_{n=0}^\infty L_n^2(\mathcal{Q}_\beta)$, with

$$L_n^2(\mathcal{Q}_\beta)=\overline{\text{L.H.}\{:\mathcal{A}^\beta(f_1)\cdots\mathcal{A}^\beta(f_n):|f_j\in\oplus^3L^2(\mathbb{R}^{3+\beta}),\,j=1,2,...,n\}}.$$

Here $L_0^2(\mathcal{Q}_\beta)=\{\alpha 1_{\mathcal{Q}_\beta}|\alpha\in\mathbb{C}\}$ and $:X:$ denotes Wick product recursively defined by

$$\begin{aligned}&:\mathcal{A}^\beta(f):=\mathcal{A}^\beta(f),\\&:\mathcal{A}^\beta(f)\mathcal{A}^\beta(f_1)\cdots\mathcal{A}^\beta(f_n):=\mathcal{A}^\beta(f):\mathcal{A}^\beta(f_1)\cdots\mathcal{A}^\beta(f_n):\\&\qquad\qquad-\sum_{j=1}^n q_\beta(f,f_j):\mathcal{A}^\beta(f_1)\cdots\widehat{\mathcal{A}^\beta(f_j)}\cdots\mathcal{A}^\beta(f_n):,\end{aligned}$$

where $\widehat{X}$ denotes removing X.

Next we define the second quantization $\Gamma_{\beta\beta'}$ in $\mathcal{Q}$-representation as the functor

$$\Gamma_{\beta\beta'}:\mathcal{C}\left(L^2(\mathbb{R}^{3+\beta})\to L^2(\mathbb{R}^{3+\beta'})\right)\;\to\;\mathcal{C}\left(L^2(\mathcal{Q}_\beta)\to L^2(\mathcal{Q}_{\beta'})\right).$$

With $T\in\mathcal{C}(L^2(\mathbb{R}^{3+\beta})\to L^2(\mathbb{R}^{3+\beta'}))$, $\Gamma_{\beta\beta'}(T)\in\mathcal{C}(L^2(\mathcal{Q}_\beta)\to L^2(\mathcal{Q}_{\beta'}))$ is defined by

$$\Gamma_{\beta\beta'}(T)1_{\mathcal{Q}_\beta}=1_{\mathcal{Q}_{\beta'}},\quad \Gamma_\beta(T):\mathcal{A}^\beta(f_1)\cdots\mathcal{A}^\beta(f_n):\;=\;:\mathcal{A}^{\beta'}(Tf_1)\cdots\mathcal{A}^{\beta'}(Tf_n):.$$

For notational simplicity we use Γ_β for $\Gamma_{\beta\beta}$. For each self-adjoint operator h in $L^2(\mathbb{R}^{3+\beta})$, $\Gamma_\beta(e^{ith})$ is a one-parameter unitary group. Then $\Gamma_\beta(e^{ith})=e^{itd\Gamma_\beta(h)}$, $t\in\mathbb{R}$, for the unique self-adjoint operator $d\Gamma_\beta(h)$ in $L^2(\mathcal{Q}_\beta)$. We write

$$\mathcal{Q}:=\mathcal{Q}_0,\quad \mathcal{Q}_{\rm E}:=\mathcal{Q}_1,\quad \mu:=\mu_0,\quad \mu_{\rm E}:=\mu_1,\quad \mathcal{A}:=\mathcal{A}^0,\quad \mathcal{A}^{\rm E}:=\mathcal{A}^1 \tag{2.7}$$

in what follows, using the label E for "Euclidean" objects to distinguish from Fock space objects. Thus it is seen that $\mathcal{F}$, $A_\mu(\hat{f})$ and $d\Gamma(h)$ are isomorphic to $L^2(\mathcal{Q})$, $\mathcal{A}(\oplus_{\nu=1}^3\delta_{\mu\nu}f)$ and $d\Gamma_0(\hat{h})$, respectively, where $\hat{h}={\rm F}h{\rm F}^{-1}$ and F denotes Fourier transform on $L^2(\mathbb{R}^3)$. That is, there exists a unitary operator $\mathbb{U}:\mathcal{F}\to L^2(\mathcal{Q})$ such that

(1) $\mathbb{U}\Omega = 1_{\mathcal{Q}}$,

(2) $\mathbb{U}A_\mu(\hat{f})\mathbb{U}^{-1} = \mathcal{A}(\oplus_{\nu=1}^3 \delta_{\mu\nu} f)$,

(3) $\mathbb{U}d\Gamma(h)\mathbb{U}^{-1} = d\Gamma_0(\hat{h})$.

The isomorphism $\mathcal{U} := 1 \otimes \mathbb{U} : \mathcal{H}^{\mathrm{F}} \to L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathcal{Q})$ maps $H_{\mathrm{PF}}^{\mathrm{F}}$ to a self-adjoint operator on $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathcal{Q})$. Let

$$\lambda := (\hat{\varphi}/\sqrt{\omega_{\mathrm{b}}})^\vee, \tag{2.8}$$

where $\check{f}$ denotes inverse Fourier transform of f. Set $\mathcal{A}_\mu(\lambda(\cdot - x)) := \mathcal{A}(\oplus_{\nu=1}^3 \delta_{\mu\nu}\lambda(\cdot - x))$ and $H_{\mathrm{rad}} := d\Gamma_0(\hat{\omega}_{\mathrm{b}})$ on $L^2(\mathcal{Q})$.

Finally we define H_{PF}, the main object in this paper by

$$H_{\mathrm{PF}} := \frac{1}{2}(-i\nabla - e\mathcal{A})^2 + V + H_{\mathrm{rad}} - \frac{e}{2}\sum_{\mu=1}^3 \sigma_\mu \mathcal{B}_\mu, \tag{2.9}$$

where $\mathcal{A}_\mu := \int_{\mathbb{R}^3}^{\oplus} \mathcal{A}_\mu(\lambda(\cdot - x))dx$ and $\mathcal{B}_\mu := \int_{\mathbb{R}^3}^{\oplus} \mathcal{B}_\mu(\lambda(\cdot - x))dx$, with

$$\mathcal{B}_\mu(\lambda(\cdot - x)) = \mathcal{A}(\oplus_{\nu=1}^3 \delta_{\nu\mu}(\nabla_x \times \lambda(\cdot - x))_\mu).$$

Here the self-adjoint operator H_{PF} is the $\mathcal{Q}$-representation of $H_{\mathrm{PF}}^{\mathrm{F}}$, obtained through the map $\mathcal{U}H_{\mathrm{PF}}^{\mathrm{F}}\mathcal{U}^{-1} = H_{\mathrm{PF}}$. In this representation A_μ and B_ν turn into the multiplication operators $\mathcal{A}_\mu$ and $\mathcal{B}_\nu$, respectively.

2.2.2 Spin variables in function space

In order to reduce (2.9) to a *scalar* operator, we introduce a two-valued variable σ. Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ and $[z]_2$ denote the equivalence class of $z \in \mathbb{Z}$. Use the affine map $x \mapsto 2x - 1$ to arrive at the conventional variables $\{-1, +1\} \cong \mathbb{Z}_2$. Addition modulo 2 gives $(+1) \oplus_{\mathbb{Z}_2} (+1) = +1$, $(+1) \oplus_{\mathbb{Z}_2} (-1) = -1$, $(-1) \oplus_{\mathbb{Z}_2} (-1) = +1$. Define

$$L^2(\mathbb{R}^3 \times \mathbb{Z}_2) := \left\{ f : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{C} \,|\, \|f\|_{L^2(\mathbb{R}^3 \times \mathbb{Z}_2)}^2 := \sum_{\sigma \in \mathbb{Z}_2} \|f(\cdot, \sigma)\|_{L^2(\mathbb{R}^3)}^2 < \infty \right\}.$$

The isomorphism between $L^2(\mathbb{R}^3; \mathbb{C}^2)$ and $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is given by

$$L^2(\mathbb{R}^3; \mathbb{C}^2) \ni \begin{bmatrix} u(x, +1) \\ u(x, -1) \end{bmatrix} \mapsto u(x, \sigma) \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2).$$

Let $F = \begin{bmatrix} F(+1) \\ F(-1) \end{bmatrix} \in \mathcal{H}^{\mathrm{F}}$ with $F(\pm 1) \in L^2(\mathbb{R}^3) \otimes L^2(\mathcal{Q})$. Then since

$$H_{\mathrm{PF}} = \frac{1}{2}(-i\nabla - e\mathcal{A})^2 + V + H_{\mathrm{rad}} - \frac{e}{2}\begin{bmatrix} \mathcal{B}_3 & \mathcal{B}_1 - i\mathcal{B}_2 \\ \mathcal{B}_1 + i\mathcal{B}_2 & -\mathcal{B}_3 \end{bmatrix},$$

our Hamiltonian can be regarded as the self-adjoint operator on

$$\mathcal{H} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathcal{Q}) \tag{2.10}$$

defined by

$$(H_{\mathrm{PF}}F)(\sigma) = \left(\frac{1}{2}(-i\nabla - e\mathcal{A})^2 + V + H_{\mathrm{rad}} + H_{\mathrm{d}}\right) F(\sigma) + H_{\mathrm{od}} F(-\sigma) \tag{2.11}$$

for $\sigma \in \mathbb{Z}_2$, where H_{d} and H_{od} denote the diagonal resp. off-diagonal parts of the spin interaction explicitly given by

$$\begin{aligned} H_{\mathrm{d}} &:= H_{\mathrm{d}}(x, \sigma) := -\frac{e}{2}\sigma\mathcal{B}_3(\lambda(\cdot - x)), \\ H_{\mathrm{od}} &:= H_{\mathrm{od}}(x, -\sigma) = -\frac{e}{2}\left(\mathcal{B}_1(\lambda(\cdot - x)) - i\sigma\mathcal{B}_2(\lambda(\cdot - x))\right). \end{aligned}$$

To investigate the translation invariant case let $P_{\mathrm{f}} := d\Gamma_0(-i\nabla)$. The translation invariant Pauli-Fierz Hamiltonian $H^{\mathrm{F}}_{\mathrm{PF}}(P)$ can also be mapped into a self-adjoint operator on $\ell_2(\mathbb{Z}_2) \otimes L^2(\mathcal{Q})$ defined by

$$(H_{\mathrm{PF}}(P)F)(\sigma) = \left(\frac{1}{2}(P - P_{\mathrm{f}} - e\mathcal{A}(0))^2 + H_{\mathrm{rad}} + H_{\mathrm{d}}(0)\right) F(\sigma) + H_{\mathrm{od}}(0)F(-\sigma), \tag{2.12}$$

where $F(\pm 1) \in L^2(\mathcal{Q})$, $\mathcal{A}_\mu(0) := \mathcal{A}_\mu(\lambda(\cdot - 0))$, $H_{\mathrm{d}}(0) = H_{\mathrm{d}}(0, \sigma)$ and $H_{\mathrm{od}}(0) = H_{\mathrm{od}}(0, -\sigma)$. In the following we will construct functional integral representations for (2.11) and (2.12).

3 A Feynman-Kac-type formula for jump processes

3.1 Pauli operators

In this section we consider the functional integral representation of the Pauli operator in the context of quantum mechanics. The spin will be described in terms of a $\mathbb{Z}_2$-valued Poisson point process. We start by reconsidering the path integral representation of the Pauli operator established in [ALS83]. We turn the results of De Angelis, Jona-Lasinio and Sirugue into precise statements and proofs, and add extensions and comments.

For a vector potential a we define the Pauli operator on $L^2(\mathbb{R}^3;\mathbb{C}^2)$ by

$$h(a,b) := \frac{1}{2}(-i\nabla - a)^2 + V - \frac{1}{2}\sum_{\mu=1}^{3}\sigma_\mu b_\mu. \tag{3.1}$$

Usually for Pauli operators $b = \nabla \times a$. However, for the remainder of this section we treat a and b as not necessarily dependent vectors. We require them to satisfy the following conditions:

Assumption 3.1 *Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be real valued with $a_\mu \in C^2_{\rm b}(\mathbb{R}^3)$ and $b_\nu \in L^\infty(\mathbb{R}^3)$, for $\mu, \nu = 1, 2, 3$.*

Under Assumptions 2.2 and 3.1 $h(a,b)$ is self-adjoint on $D(\Delta)$ and bounded from below, moreover it is essentially self-adjoint on any core of $-(1/2)\Delta$ as a consequence of the Kato-Rellich Theorem. In a similar manner to the previous section, $h(a,b)$ can also be reduced to the self-adjoint operator $\tilde{h}(a,b)$ on $L^2(\mathbb{R}^3\times\mathbb{Z}_2)$ to obtain

$$(\tilde{h}(a,b)f)(\sigma) := \left(\frac{1}{2}(-i\nabla - a)^2 + V - \frac{1}{2}\sigma b_3\right) f(\sigma) - \frac{1}{2}(b_1 - i\sigma b_2)f(-\sigma). \tag{3.2}$$

3.2 A $3+1$ dimensional jump process

In order to construct a Feynman-Kac formula for $e^{-t\tilde{h}(a,b)}$, in addition to the Brownian motion we need a Poisson point process to take the spin into account. For a summary of basic definitions and facts as well as notations we refer to the Appendix.

Let $(B_t)_{t\ge 0} = (B_t^\mu)_{t\ge 0,\, 1\le\mu\le 3}$ be three dimensional Brownian motion on $(W, \mathcal{B}_W, P_W^x)$ with the forward filtration $\mathcal{F}_t = \sigma(B_s, s \le t)$, $t \ge 0$, where $W = C([0,\infty);\mathbb{R}^3)$ and P_W^x is Wiener measure with $P_W^x(B_0 = x) = 1$. Let, moreover, $(S, \Sigma, P_{\rm P})$ be a probability space with a right-continuous increasing family of sub-σ-fields $(\Sigma_t)_{t\ge 0}$, and $\mathbb{E}_{\rm P}$ denote expectation with respect to $P_{\rm P}$. Fix a measurable space $(\mathcal{M}, B_\mathcal{M})$. Let $p : (0,\infty) \times S \to \mathcal{M}$ be a stationary (Σ_t)-Poisson point process, and $D(p) \subset (0,\infty)$ denote its domain. Note that $\#D(p)$ is finite for each $\tau \in S$. The intensity of p is given by $\Lambda(t,U) := \mathbb{E}_{\rm P}[N_p(t,U)] = tn(U)$ for some measure n on $\mathcal{M}$, where N_p denotes counting measure on $((0,\infty)\times\mathcal{M}, \mathcal{B}_{(0,\infty)} \times B_\mathcal{M})$ given by

$$N_p(t,U) := \#\left\{s \in D(p) \mid s \in (0,t],\, p(s) \in U\right\}, \quad t > 0,\ U \in B_\mathcal{M},$$

with $N_p[0,U] = 0$, and $\mathcal{B}_{(0,\infty)}$ is the Borel σ-field of $(0,\infty)$. Then

$$\mathbb{E}_{\rm P}[N_p(t,U) = N] = \frac{\Lambda(t)^N}{N!}e^{-\Lambda(t)}.$$

Assume that $n(\mathcal{M}) = 1$. Write

$$dN_t := \int_{\mathcal{M}} N_p(dtdm). \tag{3.3}$$

Hence

$$\int_0^{t+} f(s, N_s) dN_s = \sum_{\substack{r \in D(p) \\ 0 < r \leq t}} f(r, N_r). \tag{3.4}$$

Since $\#\{s \in D(p) \,|\, 0 < s \leq t\} < \infty$, for each $\tau \in S$ there exists $N = N(\tau) \in \mathbb{N}$ and $0 < s_1 = s_1(\tau), ..., s_N = s_N(\tau) \leq t$ such that

$$\int_0^{t+} f(s, N_s) dN_s = \sum_{j=1}^{N} f(s_j, N_{s_j}) = \sum_{j=1}^{N} f(s_j, j).$$

Since $\mathbb{E}_{\mathrm{P}}[N_t] = t$ and $\mathbb{E}_{\mathrm{P}}[N_t = N] = t^N e^{-t}/N!$, the expectation of (3.4) reduces to Lebesgue integral:

$$\mathbb{E}_{\mathrm{P}}\left[\int_0^{t+} f(s, N_s) dN_s\right] = \mathbb{E}_{\mathrm{P}}\left[\int_0^{t} f(s, N_s) ds\right] = \int_0^t \sum_{n=0}^{\infty} f(s, n) \frac{s^n}{n!} e^{-s} ds.$$

Write $(\Omega, \mathcal{B}_\Omega, P_\Omega) := (W \times S, \mathcal{B}_W \times \Sigma, P_W \otimes P_{\mathrm{P}})$ and $\omega := w \times \tau \in W \times S$. For $\omega = w \times \tau$, we put $B_t(\omega) := B_t(w)$ and $p(s, \omega) := p(s, \tau)$.

Definition 3.2 *The $\mathbb{Z}_2$-valued random process $\sigma_t : \mathbb{Z}_2 \times \Omega \to \mathbb{Z}_2$ is defined by*

$$\sigma_t := \sigma \oplus_{\mathbb{Z}_2} [N_t]_2 = \sigma(-1)^{N_t}, \quad \sigma \in \mathbb{Z}_2.$$

Here we have the paths $[N_t]_2$ with values $\pm 1 \in \mathbb{Z}_2$ corresponding to the equivalence classes. The electron and spin processes together give us finally the $(3+1)$-dimensional $\mathbb{R}^3 \times \mathbb{Z}_2$-valued random process

$$(\xi_t)_{t \geq 0} := (B_t, [N_t]_2)_{t \geq 0} = (B_t, \sigma_t)_{t \geq 0}$$

on $(\Omega, \mathcal{B}_\Omega, P_\Omega)$. Let $\Omega_t = \mathcal{F}_t \times \Sigma_t$, $t \geq 0$. For notational convenience, we write

$$\mathbb{E}^{x,\sigma}[f(\xi_\cdot)] := \int_\Omega f(x + B_\cdot, \sigma \oplus_{\mathbb{Z}_2} [N_\cdot]_2) dP_\Omega = \int_\Omega f(x + B_\cdot, \sigma_\cdot) dP_\Omega$$

as well as $\mathbb{E}_\Omega[f] = \int_\Omega f dP_\Omega$, $\mathbb{E}^x[f(B_\cdot)] = \int_W f(x + B_\cdot) dP_W^0 = \int_W f(B_\cdot) dP_W^x$, $\mathbb{E}^\sigma[g(\sigma_\cdot)] = \int_S g(\sigma_\cdot) dP_{\mathrm{P}}$, and $\sum_\sigma \int dx\, f(x, \sigma) := \sum_{\sigma \in \mathbb{Z}_2} \int_{\mathbb{R}^3} dx f(x, \sigma)$.

3.3 Generator and a Feynman-Kac formula for ξ_t

Next we compute the generator of the process ξ_t and derive a version of the Feynman-Kac formula.

Let $\sigma_{\rm F}$ be the fermionic harmonic oscillator defined by

$$\sigma_{\rm F} := \frac{1}{2}(\sigma_3 + i\sigma_2)(\sigma_3 - i\sigma_2) - \frac{1}{2}. \tag{3.5}$$

Note that $\sigma_{\rm F} = -\sigma_1$. A direct computation yields

$$(f, e^{-t(-(1/2)\Delta+\epsilon\sigma_{\rm F})}g) = \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}[\bar{f}(\xi_0)g(\xi_t)\epsilon^{N_t}]. \tag{3.6}$$

Thus the generator of ξ_t is given by

$$-\frac{1}{2}\Delta + \sigma_{\rm F}$$

and by making use of the two-valued variable σ,

$$\left((-\frac{1}{2}\Delta + \epsilon\sigma_{\rm F})f\right)(\sigma) = \frac{1}{2}\Delta f(\sigma) - \epsilon f(-\sigma)$$

follows.

Proposition 3.3 [**De Angelis, Jona-Lasinio, Sirugue**] *Suppose*

$$\int_0^t ds \int_{\mathbb{R}^3} (2\pi s)^{-3/2} \left|\log \frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2}\right| e^{-|y-x|^2/(2s)} dy < \infty \tag{3.7}$$

for all $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. *Then*

$$\left(e^{-t\tilde{h}(a,b)}g\right)(\sigma, x) = e^t \mathbb{E}^{x,\sigma}[e^{Z_t} g(\xi_t)]. \tag{3.8}$$

Here

$$\begin{aligned} Z_t &= -i\sum_{\mu=1}^3 \int_0^t a_\mu(B_s)\circ dB_s^\mu - \int_0^t V(B_s)ds \\ &\qquad - \int_0^t \left(-\frac{1}{2}\right)\sigma_s b_3(B_s)ds + \int_0^{t+} W(B_s, -\sigma_{s-})dN_s, \end{aligned}$$

$\int_0^t a_\mu(B_s)\circ dB_s^\mu$ *denoting Stratonovich integral and*

$$W(x, -\sigma) := \log\left(\frac{1}{2}(b_1(x) - i\sigma b_2(x))\right).$$

Remark 3.4 We will prove Proposition 3.3 by making use of the Itô formula. In order that Itô's formula applies, however, the integrand in $\int_0^{t+}\dots dN_s$ must be predictable with respect to the given filtration. σ_s is, though, right continuous in s for each $\omega\in\Omega$, so we define $\sigma_{s-}=\lim_{\epsilon\uparrow 0}\sigma_{s-\epsilon}$. Then σ_{s-} is left continuous and $W(B_s,-\sigma_{s-})$ is predictable, i.e., $W(B_s,-\sigma_{s-})$ is Ω_s measurable and left continuous in s for each $\omega\in\Omega$. This allows then an application of Itô's formula to $\int_0^{t+}W(B_s,-\sigma_{s-})dN_s$, for more details see the Appendix.

Before turning to the proof of Proposition 3.3, we consider a simplified model. Let $U(\cdot,\sigma)$ and $W(\cdot,-\sigma)$ be multiplication operators on $L^2(\mathbb{R}^3\times\mathbb{Z}_2)$. Define the operator $K:L^2(\mathbb{R}^3\times\mathbb{Z}_2)\to L^2(\mathbb{R}^3\times\mathbb{Z}_2)$ by

$$(Kf)(x,\sigma):=U(x,\sigma)f(x,\sigma)-e^{W(x,-\sigma)}f(x,-\sigma). \tag{3.9}$$

First we construct a functional integral for e^{-tK}.

Proposition 3.5 *Let $U(x,\sigma)$ and $W(x,-\sigma)$ be continuous bounded functions in $x\in\mathbb{R}^3$, for each $\sigma=\pm1$, such that $\overline{U(x,\sigma)}=U(x,\sigma)$, $\overline{W(x,-\sigma)}=W(x,+\sigma)$. Then K is self-adjoint and*

$$(e^{-tK}g)(x,\sigma)=e^t\mathbb{E}^{x,\sigma}\left[g(x,\sigma_t)e^{-\int_0^t U(x,\sigma_s)ds+\int_0^{t+}W(x,-\sigma_{s-})dN_s}\right]. \tag{3.10}$$

Proof: The proof of the self-adjointness of K is trivial. Write

$$K_tg(x,\sigma):=\mathbb{E}^{x,\sigma}\left[g(x,\sigma_t)e^{-\int_0^t U(x,\sigma_s)ds+\int_0^{t+}W(x,-\sigma_{s-})dN_s}\right].$$

Note that for each $(x,\omega)\in\mathbb{R}^3\times\Omega$,

$$\left|\int_0^{t+}W(x,-\sigma_{s-})dN_s\right|\le M\int_0^t dN_s=MN_t, \tag{3.11}$$

where $M=\sup_{x\in\mathbb{R}^3,\sigma\in\mathbb{Z}_2}|W(x,-\sigma)|$. Then

$$\|K_tg\|\le\|g\|\,e^{tM'}\,\mathbb{E}^{x,\sigma}[e^{MN_t}]=\|g\|\,e^{tM'}\,e^{t(e^M-1)},$$

where $M'=\sup_{x\in\mathbb{R}^3,\sigma\in\mathbb{Z}_2}\mathbb{E}^{x,\sigma}[e^{-\int_0^t U(x,\sigma_s)ds}]$, and K_t is bounded. For each $(x,\omega)\in\mathbb{R}^3\times\Omega$ it is seen that $\int_0^{t+}W(x,-\sigma_{s-})dN_s$ is continuous in a neighborhood of $t=0$, since $\#\{0<s<\epsilon\,|\,s\in D(p)\}=0$ for sufficiently small $\epsilon>0$, and then

$$\int_0^{t+}W(x,-\sigma_{s-})dN_s=\sum_{\substack{s\in D(p)\\0<s\le t}}W\left(x,-\sigma(-1)^{N_s}\right)=0$$

for small enough t. Hence for $g \in C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_2)$,

$$\lim_{t\to 0} \|g - K_t g\|^2$$
$$\le \lim_{t\to 0} \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}\left[|g(x,\sigma) - g(x,\sigma_t) e^{-\int_0^t U(x,\sigma_s)ds + \int_0^{t+} W(x,-\sigma_{s-})dN_s} |^2 \right] = 0$$

by dominated convergence. Since $C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_2)$ is dense in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, it follows that K_t is strongly continuous at $t = 0$. Also, K_t has the following semigroup property. Since N_s is a Markov process, for each $(x,\sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2$, we have

$$\begin{aligned}
&(K_s K_t g)(x,\sigma) \\
&= e^{-\int_0^s U(x,\sigma_r)dr + \int_0^{s+} W(x,-\sigma_{r-})dN_r} \mathbb{E}^{B_s,\sigma_s}\left[e^{-\int_0^t U(x,\sigma_l)dl + \int_0^{t+} W(x,-\sigma_{l-})dN_l} g(x,\sigma_t) \right] \\
&= \mathbb{E}^{x,\sigma}\Bigg[e^{-\int_0^s U(x,\sigma_r)dr + \int_0^{s+} W(x,-\sigma_{r-})dN_r} \\
&\qquad \times \mathbb{E}^{x,\sigma}\left[e^{-\int_s^{s+t} U(x,\sigma_l)dl + \int_s^{(s+t)+} W(x,-\sigma_{l-})dN_l}\, g(x,\sigma_{s+t}) |\, \Omega_s \right]\Bigg] \\
&= \mathbb{E}^{x,\sigma}\left[e^{-\int_0^s U(x,\sigma_r)dr + \int_0^{s+} W(x,-\sigma_{r-})dN_r} e^{-\int_s^{s+t} U(x,\sigma_l)dl + \int_s^{(s+t)+} W(x,-\sigma_{l-})dN_l} g(x,\sigma_{s+t}) \right] \\
&= (K_{s+t} g)(x,\sigma).
\end{aligned}$$

K_t is thus a C_0-semigroup, hence the Hille-Yoshida Theorem says that there is a closed operator h in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ such that $K_t = e^{-th}$, $t \ge 0$. We show that $h = K + 1$.

By Itô's formula, see Proposition 7.7 below, we have $d\sigma_t = \int_0^{t+}(-2\sigma_{s-})dN_s$ and $dg(x,\sigma_t) = \int_0^{t+} \left(g(x,-\sigma_{s-}) - g(x,\sigma_{s-}) \right) dN_s$. Let

$$Y_t := -\int_0^t U(x,\sigma_s)ds + \int_0^{t+} W(x,-\sigma_{s-})dN_s.$$

Then it follows that

$$de^{Y_t} = -\int_0^t e^{Y_s} U(x,\sigma_s)ds + \int_0^{t+} e^{Y_{s-}} (e^{W(x,-\sigma_{s-})} - 1)dN_s.$$

By using the product rule we get

$$\begin{aligned}
&d\left(e^{Y_t} g(x,\sigma_t) \right) \\
&= -\int_0^t g(x,\sigma_s) e^{Y_s} U(x,\sigma_s)ds + \int_0^{t+} g(x,\sigma_{s-}) e^{Y_{s-}} (e^{W(x,-\sigma_{s-})} - 1)dN_s \\
&\quad + \int_0^{t+} e^{Y_{s-}} (g(x,-\sigma_{s-}) - g(x,\sigma_{s-})dN_s \\
&\quad + \int_0^{t+} (g(x,-\sigma_{s-}) - g(x,\sigma_{s-})) e^{Y_{s-}} (e^{W(x,-\sigma_{s-})} - 1)dN_s \\
&= -\int_0^t g(x,\sigma_s) e^{Y_s} U(x,\sigma_s)ds + \int_0^{t+} e^{Y_{s-}} \left(g(x,-\sigma_{s-}) e^{W(x,-\sigma_{s-})} - g(x,\sigma_{s-}) \right) dN_s.
\end{aligned}$$

Therefore

$$\mathbb{E}^{x,\sigma}\left[e^{Y_t}g(x,\sigma_t)-e^{Y_0}g(x,\sigma(0))\right]=\int_0^t \mathbb{E}^{x,\sigma}[G(s)]ds, \tag{3.12}$$

where $G(s)=G(x,\sigma,s)$ is defined by

$$G(s):=\begin{cases} -e^{Y_s}g(x,\sigma_s)U(x,\sigma_s)+e^{Y_{s-}}(g(x,-\sigma_{s-})e^{W(x,-\sigma_{s-})}-g(x,\sigma_{s-})), & s>0,\\ -g(x,\sigma)U(x,\sigma)+g(x,-\sigma)e^{W(x,-\sigma)}-g(x,\sigma), & s=0.\end{cases}$$

Thus for each $(x,\omega)\in\mathbb{R}^3\times\Omega$, $G(s)$ is continuous in s at $s=0$ and is bounded as $|G(s)|\le e^{MN_s}M'|g(x,\sigma)|$, with constants M and M'. Dominated convergence gives then

$$\lim_{s\to 0+}\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}[G(s)]=\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}[G(0)].$$

Hence

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}(f,(K_tg-g))\\
&=\lim_{t\to 0}\frac{1}{t}\sum_\sigma\int dx\,\overline{f(x,\sigma)}\mathbb{E}^{x,\sigma}[e^{Y_t}g(x,\sigma_t)-e^{Y_0}g(x,\sigma)]\\
&=\lim_{t\to 0}\frac{1}{t}\int_0^t ds\sum_\sigma\int dx\,\overline{f(x,\sigma)}\mathbb{E}^{x,\sigma}[G(s)]\\
&=\sum_\sigma\int dx\,\overline{f(x,\sigma)}\mathbb{E}^{x,\sigma}[G(0)]\\
&=\sum_\sigma\int dx\,\overline{f(x,\sigma)}\left(-U(x,\sigma)g(x,\sigma)+g(x,-\sigma)e^{W(x,-\sigma)}-g(x,\sigma)\right)\\
&=(f,-(K+1)g).
\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^3\times\mathbb{Z}_2)$ is a core of K, $h=K+1$ follows. **qed**

Proof of Proposition 3.3: We put $U(x,\sigma)=-(1/2)\sigma b_3(x)$ and $W(x,-\sigma)=\log[(1/2)(b_1(x)-i\sigma b_2(x))]$. Recall that

$$Z_t=-i\sum_{\mu=1}^3\int_0^t a_\mu(B_s)\circ dB_s^\mu-\int_0^t U(B_s,\sigma_s)ds+\int_0^{t+}W(B_s,-\sigma_{s-})dN_s.$$

$W(B_s,-\sigma_{s-})$ is predictable and first we have to check that $|\int_0^{t+}W(B_s,-\sigma_{s-})dN_s|$ is finite for almost every $\omega\in\Omega$ in order to apply Itô's formula. Indeed,

$$\begin{aligned}
&\left|\mathbb{E}^{x,\sigma}\left[\int_0^{t+}W(B_s,-\sigma_{s-})dN_s\right]\right|\\
&\quad\le \mathbb{E}^{x,\sigma}\left[\int_0^t\left|\log\left(\frac{1}{2}\sqrt{b_1(B_s)^2+b_2(B_s)^2}\right)\right|dN_s\right]\\
&\quad=2\int_0^t ds\int_{\mathbb{R}^3}(2\pi s)^{-3/2}e^{-|y-x|^2/(2s)}\left|\log\left(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}\right)\right|dy
\end{aligned}$$

is finite by the assumption, hence $|\int_0^{t+}W(B_s,-\sigma_{s-})dN_s|<\infty$, for almost every $\omega\in\Omega$.

Define $S_t: L^2(\mathbb{R}^3\times\mathbb{Z}_2)\to L^2(\mathbb{R}^3\times\mathbb{Z}_2)$ by

$$S_tg(x,\sigma)=\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}e^{Z_t}g(\sigma_t,B_t)\right].$$

It can be seen that

$$\|S_tg\|\le V_M^{1/2}e^{M't}e^{(M-1)t/2}\|g\|,$$

where $M'=\sup_{x\in\mathbb{R}^3}|b_3(x)/2|$, $M=\sup_{x\in\mathbb{R}^3}(b_1^2(x)+b_2^2(x))/4$ and

$$V_M:=\sup_{x\in\mathbb{R}^3}\mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}], \tag{3.13}$$

which is finite by Assumption 2.2. Thus S_t is bounded. Since Z_t is continuous at $t=0$ for each $\omega\in\Omega$, dominated convergence yields

$$\|S_tg-g\|\le\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}[|g(x,\sigma)-g(B_t,\sigma_t)e^{Z_t}|]\to 0$$

as $t\to 0$. The semigroup property of S_t follows from the Markov property of the process (B_t,N_t), which is shown in a similar way as that of K_t in Proposition 3.5. Thus S_t is a C_0-semigroup. Denote the generator of S_t by the closed operator h. We will see below that $S_t=e^{-th}=e^{-t(h(a,b)+1)}$. From Proposition 7.7 it follows that

$$\begin{aligned} dg(B_t,\sigma_t) &= \sum_{\mu=1}^3\int_0^t\partial_{x_\mu}g(B_s,\sigma_s)dB_s^\mu+\frac12\int_0^t\Delta_xg(B_s,\sigma_s)ds\\ &\quad+\int_0^{t+}\left(g(B_s,-\sigma_{s-})-g(B_s,\sigma_{s-})\right)dN_s,\end{aligned}$$

and

$$\begin{aligned} de^{Z_t} &= \sum_{\mu=1}^3\int_0^t e^{Z_s}(-ia_\mu(B_s))\circ dB_s^\mu+\int_0^t e^{Z_s}(-V(B_s))ds\\ &\quad+\frac12\int_0^t e^{Z_s}(-ia(B_s))^2ds+\int_0^t e^{Z_s}(-U(B_s,\sigma_s))ds\\ &\quad+\int_0^{t+}\left(e^{Z_{s-}+W(B_s,-\sigma_{s-})}-e^{Z_{s-}}\right)dN_s.\end{aligned}$$

By the product rule and the two identities above we have

$$\begin{aligned} d(e^{Z_t}g(B_t,\sigma_t)) &= \int_0^t e^{Z_s}\left[\frac12\Delta_xg(B_s,\sigma_s)+(-ia(B_s))\cdot(\nabla_xg)(B_s,\sigma_s)\right.\\ &\quad\left.+\left(\frac12(-ia(B_s))^2-V(B_s)-U(B_s,\sigma_s)\right)g(B_s,\sigma_s)\right]ds\end{aligned}$$

$$+\sum_{\mu=1}^{3}\int_0^t e^{Z_s}\Big(\partial_{x_\mu} g(B_s,\sigma_s)+(-ia_\mu(B_s))g(B_s,\sigma_s)\Big)\cdot dB_s^\mu$$
$$+\int_0^{t+} e^{Z_{s-}}\Big[\ (g(B_s,-\sigma_{s-})-g(B_s,\sigma_{s-}))$$
$$+(g(B_s,-\sigma_{s-})-g(B_s,\sigma_{s-}))(e^{W(B_s,-\sigma_{s-})}-1)$$
$$+g(B_s,\sigma_{-s})(e^{W(B_s,-\sigma_{s-})}-1)\Big]\,dN_s.$$

Take expectation on both sides above. The martingale part vanishes and by (7.3) we we obtain that

$$\mathbb{E}^{x,\sigma}[e^{Z_t}g(B_t,\sigma_t)-g(x,\sigma)]=\int_0^t\mathbb{E}^{x,\sigma}[G(s)]ds,$$

where

$$\begin{aligned}G(s) \ := \ & e^{Z_s}\Big[\frac{1}{2}\Delta_x g(B_s,\sigma_s)+(-ia(B_s))\cdot(\nabla_x g)(B_s,\sigma_s)\\ &+\Big(\frac{1}{2}(-ia(B_s))^2-V(B_s)-U(B_s,\sigma_s)\Big)g(B_s,\sigma_s)\Big]\\ &+e^{Z_{s-}}\Big((g(B_s,-\sigma_{s-})e^{W(B_s,-\sigma_{s-})}-g(B_s,\sigma_{s-})\Big),\end{aligned}$$

with $s>0$, and

$$\begin{aligned}G(0) := &\Big\{\frac{1}{2}\Delta_x-ia(x)\cdot\nabla_x+\frac{1}{2}(-ia(x))^2-V(x)-U(x,\sigma)-1\Big\}g(x,\sigma)\\ &+e^{W(x,-\sigma)}g(x,-\sigma)\\ =&-(h(a,b)+1)g(x,\sigma).\end{aligned}$$

We see that $G(s)$ is continuous at $s=0$, for each $\omega\in\Omega$, whence

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(f,(S_t-1)g) \ &= \ \lim_{t\to 0}\frac{1}{t}\int_0^t ds\sum_\sigma\int dx\,\overline{f(x,\sigma)}\mathbb{E}^{x,\sigma}[G(s)]\\ &= \ \sum_\sigma\int dx\,\bar{f}(x,\sigma)\mathbb{E}^{x,\sigma}[G(0)]\\ &= \ (f,-(h(a,b)+1)g).\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^3\times\mathbb{Z}_2)$ is a core of $h(a,b)$, (3.8) follows. **qed**

Note that (3.7) is a sufficient condition making sure that

$$\int_0^{t+}|W(B_s,-\sigma_{s-})|dN_s<\infty,\quad \text{a.e. } \omega\in\Omega. \tag{3.14}$$

When, however, $b_1(x) - i\sigma b_2(x)$ vanishes for some (x, σ), (3.14) is not clear. This case is relevant and Proposition 3.3 must be improved since we have to construct the path integral representation of $e^{-t\tilde{h}(a,b)}$ in which the off-diagonal part $b_1 - i\sigma b_2$ of $\tilde{h}(a,b)$ has zeroes or a compact support. Since the generator of ξ_t is $-(1/2)\Delta + \sigma_{\rm F}$, as was seen above, this then becomes singular. Take $\epsilon \to 0$ on both sides of

$$(f, e^{-t(-(1/2)\Delta + \epsilon\sigma_{\rm F})} g) = \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}[\bar{f}(\xi_0) g(\xi_t) \epsilon^{N_t}]. \tag{3.15}$$

Then the right hand side of (3.15) converges to $\sum_\sigma \int dx\, \mathbb{E}^x[\bar{f}(x,\sigma) g(B_t,\sigma)]$, see Remark 3.7 below. The off-diagonal part of $h(a,b)$, however, in general may have zeroes. For instance, a_μ for all $\mu = 1,2,3$ have compact support, and so does the off-diagonal part in the case of $b = \nabla \times a$. Therefore, in order to avoid that the diagonal part vanishes, we introduce

$$\begin{aligned} \tilde{h}^\varepsilon(a,b) f(\sigma) &:= \left(\frac{1}{2}(-i\nabla - a)^2 + V - \frac{1}{2}\sigma b_3\right) f(\sigma) \\ &\quad + \left(-\frac{1}{2}(b_1 - i\sigma b_2) + \varepsilon \psi_\varepsilon \left(-\frac{1}{2}(b_1 - i\sigma b_2)\right)\right) f(-\sigma), \end{aligned} \tag{3.16}$$

where ψ_ε is the indicator function

$$\psi_\varepsilon(x) := \begin{cases} 1, & |x| < \varepsilon/2, \\ 0, & |x| \ge \varepsilon/2. \end{cases} \tag{3.17}$$

We define $\psi_\varepsilon(K)$ for a self-adjoint operator K by the spectral theorem. In particular, the identity

$$\psi_\varepsilon(K) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{\psi}_\varepsilon(k) e^{ikK} dk$$

holds. Thus $| -\frac{1}{2}(b_1 - i\sigma b_2) + \varepsilon\psi_\varepsilon(-\frac{1}{2}(b_1 - i\sigma b_2))| > \varepsilon/2$, which does not vanish for any $\varepsilon > 0$.

Proposition 3.6 *We have*

$$\left(e^{-t\tilde{h}^\varepsilon(a,b)} g\right)(\sigma, x) = e^t \mathbb{E}^{x,\sigma}[e^{Z_t^\varepsilon} g(\xi_t)], \tag{3.18}$$

and

$$\left(e^{-t\tilde{h}(a,b)} g\right)(\sigma, x) = \lim_{\varepsilon \to 0} e^t \mathbb{E}^{x,\sigma}[e^{Z_t^\varepsilon} g(\xi_t)], \tag{3.19}$$

where

$$\begin{aligned} Z_t^\varepsilon &= -i \sum_{\mu=1}^3 \int_0^t a_\mu(B_s) \circ dB_s^\mu - \int_0^t V(B_s) ds \\ &\quad - \int_0^t \left(-\frac{1}{2}\right) \sigma_s b_3(B_s) ds + \int_0^{t+} W_\varepsilon(B_s, -\sigma_{s-}) dN_s, \end{aligned}$$

and

$$W^\varepsilon(x,-\sigma) := \log\left(\frac{1}{2}(b_1(x)-i\sigma b_2(x)) + \varepsilon\psi_\varepsilon\left(\frac{1}{2}(b_1(x)-i\sigma b_2(x))\right)\right).$$

PROOF: (3.18) is derived as in Proposition 3.3. Since $e^{-t\tilde{h}^\varepsilon(a,b)}$ converges strongly to $e^{-t\tilde{h}(a,b)}$ as $\varepsilon \to 0$, (3.19) follows. **qed**

Remark 3.7 We have the following cases.

(1) Let the measure of

$$\mathcal{O}_\varepsilon = \left\{(x,\sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2 \mid |(1/2)(b_1(x)-i\sigma b_2(x))| < \varepsilon/2\right\}$$

be zero for some $\varepsilon > 0$. Then Proposition 3.3 stays valid.

(2) In case when the off-diagonal part identically vanishes, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}^{x,\sigma}[e^{Z_t^\varepsilon} g(\xi_t)] \\
&= \lim_{\varepsilon\to 0} e^t \mathbb{E}^{x,\sigma}[e^{-i\sum_{\mu=1}^3\int_0^t a_\mu(B_s)\circ dB_s^\mu - \int_0^t V(B_s)ds - \int_0^t(-\frac{1}{2})\sigma_s b_3(B_s)ds}\varepsilon^{N_t} g(\xi_t)] \\
&= \mathbb{E}^x[e^{-i\sum_{\mu=1}^3\int_0^t a_\mu(B_s)\circ dB_s^\mu - \int_0^t V(B_s)ds - \int_0^t(-\frac{1}{2})\sigma_s b_3(B_s)ds} g(B_t,\sigma)] \\
&= e^{-t\left(\frac{1}{2}(-i\nabla-a)^2+V-\frac{1}{2}\sigma_3 b_3\right)} g(x,\sigma).
\end{aligned}$$

Here we used that as $\varepsilon \to 0$ the functions on $K_t := \{\omega \in \Omega \,|\, N_t(\omega) \geq 1\}$ vanish and those on $K_t^c := \{\omega \in \Omega \,|\, N_t(\omega) = 0\}$ stay different from zero. Note that for $\omega \in K_t^c$, $N_s(\omega) = 0$ whenever $0 \leq s \leq t$, as N_t is counting measure. Clearly, then the right hand side in the expression above describes the diagonal Hamiltonian.

(3) Since the diagonal part $-(1/2)\sigma b_3(x)$ acts as an external potential up to the sign $\sigma = \pm$, heuristically we have the integral $\int_0^t(-1/2)\sigma_s b_3(B_s)ds$ in Z_t. This explains why $\int_0^t \log[(1/2)(b_1(B_s) - i\sigma_s b_2(B_s))]dN_s$ appears in Z_t. Consider $T_tF(x,\sigma) := \mathbb{E}^{x,\sigma}[F(B_t,\sigma_t)e^{\int_0^t W(B_s,-\sigma_{s-})dN_s}]$. Take, for simplicity, that W has no zeroes. Compute the generator $-K$ of T_t by Itô's formula for Lévy processes to obtain

$$\begin{aligned}
d\left(e^{\int_0^{t+} W(B_s,-\sigma_s)dN_s}\right) &= \left(e^{\int_0^{t+} W(B_s,-\sigma_{s-})dN_s + W(B_t,-\sigma_t)} - e^{\int_0^{t+} W(B_s,-\sigma_{s-})dN_s}\right) dN_t \\
&= e^{\int_0^{t+} W(B_s,-\sigma_{s-})dN_s}(e^{W(B_t,-\sigma_t)} - 1)dN_t. \qquad (3.20)
\end{aligned}$$

On the other hand, we have

$$d\left(e^{-\int_0^t V(B_s)ds}\right) = e^{-\int_0^t V(B_s)ds}(-V(B_t))dt. \tag{3.21}$$

From this we obtain that $e^{-t(-(1/2)\Delta+V)}f(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)ds}f(B_t)]$. Comparing (3.20) and (3.21), it is seen that Itô's formula gives the differential for continuous processes and the difference for discontinuous ones. From (3.20) it follows that the generator K of T_t is given by

$$Kf(\sigma) = (-\frac{1}{2}\Delta - e^{W(x,-\sigma)} + 1)f(-\sigma).$$

Thus $e^{-tK}F(x,\sigma) = e^t\mathbb{E}^\sigma[F(x,\sigma_t)e^{\int_0^t W(x,-\sigma_{s-})dN_s}]$ giving rise to the special form of the off-diagonal part.

4 Functional integral representation of $e^{-tH_{\mathrm{PF}}}$

4.1 Hypercontractivity and Markov property

In this section we discuss hypercontractivity and turn to the functional integral representation of $e^{-tH_{\mathrm{PF}}}$. Also, we derive a comparison inequality for ground state energies.

Let $\|F\|_p = \left(\int_{\mathcal{Q}_\beta} |F(\phi)|^p d\mu_\beta(\phi)\right)^{1/p}$ be L^p-norm on $(\mathcal{Q}_\beta, \mu_\beta)$ and $(\cdot,\cdot)_2$ the scalar product on $L^2(\mathcal{Q}_\beta)$. As explained in Section 2, $\Gamma_\beta(T)$ for $\|T\| \le 1$ is a contraction on $L^2(\mathcal{Q}_\beta)$. It has also the strong property of *hypercontractivity*, i.e., for a bounded operator $K : L(\mathbb{R}^{3+\beta}) \to L^2(\mathbb{R}^{3+\beta'})$ such that $\|K\| < 1$, $\Gamma_{\beta\beta'}(K)$ is a bounded operator from $L^2(\mathcal{Q}_\beta)$ to $L^4(\mathcal{Q}_\beta)$. Nelson proved the sharper result below.

Proposition 4.1 *Let $1 \le q \le p$ and $\|T\|^2 \le (q-1)(p-1)^{-1} \le 1$. Then $\Gamma_\beta(T)$ is a contraction operator from $L^q(\mathcal{Q}_\beta)$ to $L^p(\mathcal{Q}_\beta)$, i.e., for $\Phi \in L^q(\mathcal{Q}_\beta)$, $\Gamma_\beta(T)\Phi \in L^p(\mathcal{Q}_\beta)$ and $\|\Gamma_\beta(T)\Phi\|_p \le \|\Phi\|_q$.*

PROOF: See [Nel73]. **qed**

We factorize $e^{-tH_{\mathrm{rad}}}$ as is usually done. Let $j_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^{3+1})$, $t \ge 0$, be defined by

$$\widehat{j_t f}(k,k_0) := \frac{e^{-itk_0}}{\sqrt{\pi}}\sqrt{\frac{\omega_{\mathrm{b}}(k)}{\omega_{\mathrm{b}}(k)^2 + |k_0|^2}}\hat{f}(k), \quad (k,k_0) \in \mathbb{R}^3 \times \mathbb{R}.$$

The range of j_t, $a \le t \le b$, defines the σ-field $\Sigma_{[a,b]}$ of $\mathcal{Q}_{\rm E}$, and the projection $E_{[a,b]}$ to the set of $\Sigma_{[a,b]}$-measurable functions can be represented as the second quantization of a contraction operator. By using the Markov property of the family of projections $E_{[\cdots]}$ and hypercontractivity of $E_{[a,b]}E_{[c,d]}$ with $[a,b]\cap[c,d]=\emptyset$, it can be shown that $\int_{\mathcal{Q}_{\rm E}}|J_aF||J_bG||\Phi|d\mu_{\rm E}<\infty$ for $F,G\in L^2(\mathcal{Q})$ and $\Phi\in L^1(\mathcal{Q}_{\rm E})$. We will prove this for the *massless* case in Corollary 4.4.

The isometry j_t preserves realness and $j_t^*j_s=e^{-|t-s|\omega_{\rm b}(-i\nabla)}$, $s,t\in\mathbb{R}$, follows. Define

$$J_t:=\Gamma_{01}(j_t),\quad J_t:L^2(\mathcal{Q})\to L^2(\mathcal{Q}_{\rm E}).$$

Hence $J_t^*J_s=e^{-|t-s|H_{\rm rad}}$ on $L^2(\mathcal{Q})$. The operator $e_t:=j_tj_t^*$ is the projection from $L^2_{\rm real}(\mathbb{R}^{3+1})$ to ${\rm Ran} j_t$. Define

$$U_{[a,b]}:=\overline{{\rm L.H.}\{f\in L^2_{\rm real}(\mathbb{R}^{3+1})\mid f\in {\rm Ran}j_t\ \text{ for some } t\in[a,b]\}}$$

and let $e_{[a,b]}:L^2_{\rm real}(\mathbb{R}^{3+1})\to U_{[a,b]}$ denote orthogonal projection. Define the projections on $L^2(\mathcal{Q}_{\rm E})$ by $E_t:=J_tJ_t^*=\Gamma_1(e_t)$ and $E_{[a,b]}:=\Gamma_1(e_{[a,b]})$. Let $\Sigma_{[a,b]}$ be the minimal σ-field generated by $\{\mathcal{A}^{\rm E}(f)\in L^2(\mathcal{Q}_{\rm E})\mid f\in U_{[a,b]}\}$ and denote the set of $\Sigma_{[a,b]}$-measurable functions in $L^2(\mathcal{Q}_{\rm E})$ by $\mathcal{E}_{[a,b]}$. The projection $E_{[a,b]}$ has the properties below:

Lemma 4.2 *Let $a\le b\le t\le c\le d$. Then (1) $e_ae_be_c=e_ae_c$, (2) $e_{[a,b]}e_te_{[c,d]}=e_{[a,b]}e_{[c,d]}$, (3) ${\rm Ran}E_{[a,b]}=\mathcal{E}_{[a,b]}$, (4) $E_{[a,b]}E_tE_{[c,d]}=E_{[a,b]}E_{[c,d]}$.*

PROOF: See [Sim74, Hir97]. **qed**

Lemma 4.2 implies that $E_{[a,b]}$ is the projection from $L^2(\mathcal{Q}_{\rm E})$ onto $\mathcal{E}_{[a,b]}$. The fact that $E_{[a,b]}E_tE_{[c,d]}=E_{[a,b]}E_{[c,d]}$ is called *Markov property* of the family E_s. Let $\omega_{{\rm b},m}=\sqrt{|k|^2+m^2}$ with $m\ge 0$. Define $j_t^{(m)}$, $J_t^{(m)}$, $e_{[a,b]}^{(m)}$, $e_t^{(m)}$, $E_{[a,b]}^{(m)}$, $E_t^{(m)}$ and $\mathcal{E}_{[a,b]}^{(m)}$ by j_t, J_t, $e_{[a,b]}$, e_t, $E_{[a,b]}$, E_t and $\mathcal{E}_{[a,b]}$ with $\omega_{\rm b}$ replaced by $\omega_{{\rm b},m}$, respectively. Then Lemma 4.2 stays true for $e_{[a,b]}$ and $E_{[a,b]}$ replaced by $e_{[a,b]}^{(m)}$ and $E_{[a,b]}^{(m)}$, respectively. Note that $\Gamma_{01}(e^{-t\omega_{{\rm b},m}})$, $m>0$, is hypercontractive but it fails to be so for $m=0$.

Lemma 4.3 *Let $a\le b<t<c\le d$, $F\in\mathcal{E}_{[a,b]}^{(m)}$ and $G\in\mathcal{E}_{[c,d]}^{(m)}$. Take $1\le r<\infty$, $1<p$, $1<q$, $r<p$ and $r<q$. Suppose that $e^{-2m(c-b)}\le(p/r-1)(q/r-1)\le 1$ and $F\in L^p(\mathcal{Q}_{\rm E})$ and $G\in L^q(\mathcal{Q}_{\rm E})$. Then $FG\in L^r(\mathcal{Q}_{\rm E})$ and $\|FG\|_r\le\|F\|_p\|G\|_q$. In particular, for r such that*

$$r\in[1,\frac{2}{1+e^{-m(c-b)}}]\cup[\frac{2}{1-e^{-m(c-b)}},\infty),$$

we have $\|FG\|_r\le\|F\|_2\|G\|_2$.

PROOF: Let $F_N = \begin{cases} F, & |F| < N, \\ 0, & |F| \geq N, \end{cases}$ and $G_N = \begin{cases} G, & |G| < N, \\ 0, & |G| \geq N. \end{cases}$ Then $|F_N|^r \in \mathcal{E}_{[a,b]}^{(m)}$, $|G_N|^r \in \mathcal{E}_{[c,d]}^{(m)}$, and it follows that

$$\int_{\mathcal{Q}_{\mathrm{E}}} |F_N|^r |G_N|^r d\mu_{\mathrm{E}} = \left(E_{[a,b]}^{(m)} |F_N|^r, E_{[c,d]}^{(m)} |G_N|^r \right)_2 = \left(|F_N|^r, \Gamma_1(e_{[a,b]}^{(m)} e_{[c,d]}^{(m)}) |G_N|^r \right)_2.$$

Note that $T_e := e_{[a,b]}^{(m)} e_{[c,d]}^{(m)}$ satisfies

$$\begin{aligned} \|T_e\|^2 &= \|e_{[a,b]}^{(m)} e_b^{(m)} e_c^{(m)} e_{[c,d]}^{(m)}\|^2 \leq \|j_b^{(m)*} j_c^{(m)}\|^2 \\ &= \|e^{-|c-b|\omega_{\mathrm{b},m}}\|^2 \leq e^{-2m(c-b)} \leq (p/r-1)(q/r-1). \end{aligned}$$

Thus by Hölder inequality,

$$\|F_N G_N\|_r^r \leq \||F_N|^r\|_{q/r} \|\Gamma_1(T_e)|G_N|^r\|_s, \tag{4.1}$$

where $1 = \dfrac{1}{s} + \dfrac{r}{q}$. Since $\|T_e\|^2 \leq (p/r-1)(q/r-1) = (p/r-1)(s-1)^{-1} \leq 1$, by Proposition 4.1 it is seen that $\|\Gamma_1(T_e)|G_N|^r\|_s \leq \||G_N|^r\|_{p/r}$. Together with (4.1) this yields

$$\|F_N G_N\|_r \leq \|F_N\|_q \|G_N\|_p \leq \|F\|_q \|G\|_p. \tag{4.2}$$

Taking the limit $N \to \infty$ on both sides of (4.2), by monotone convergence the lemma follows. **qed**

An immediate consequence is

Corollary 4.4 *Let $\Phi \in L^1(\mathcal{Q}_{\mathrm{E}})$ and $F, G \in L^2(\mathcal{Q}_{\mathrm{E}})$. Then, for $a \neq b$, $(J_a F)\Phi(J_b G) \in L^1(\mathcal{Q}_{\mathrm{E}})$ and*

$$\int_{\mathcal{Q}_{\mathrm{E}}} |(J_a F)\Phi(J_b G)| d\mu_{\mathrm{E}} \leq \|\Phi\|_1 \|F\|_2 \|G\|_2. \tag{4.3}$$

PROOF: Let $a < b$, and $r^{(m)} = \dfrac{2}{1 - e^{-m(b-a)}}$ and $s^{(m)} > 1$ be such that $\dfrac{1}{r^{(m)}} + \dfrac{1}{s^{(m)}} = 1$, i.e., $s^{(m)} = r^{(m)}/(r^{(m)} - 1)$. Without loss of generality we can assume that Φ is a real-valued function. Truncate Φ as

$$\Phi_N := \begin{cases} N, & \Phi > N, \\ \Phi, & |\Phi| \leq N, \\ -N, & \Phi < -N. \end{cases}$$

By Lemma 4.3

$$\begin{aligned} |(J_a^{(m)} F, \Phi_N J_b^{(m)} G)_2| &\leq \int_{\mathcal{Q}_{\mathrm{E}}} |(J_a^{(m)} F)||\Phi_N||(J_b^{(m)} G)| d\mu_{\mathrm{E}} \\ &\leq \|\Phi_N\|_{s^{(m)}} \|(J_a^{(m)} F)(J_b^{(m)} G)\|_{r^{(m)}} \\ &= \|\Phi_N\|_{s^{(m)}} \|J_a^{(m)} F\|_2 \|J_b^{(m)} G\|_2 \\ &= \|\Phi_N\|_{s^{(m)}} \|F\|_2 \|G\|_2. \end{aligned}$$

Since $\text{s–lim}_{m\to 0} J_t^{(m)} = J_t$ in $L^2(\mathcal{Q}_{\rm E})$ by $\text{s–lim}_{m\to 0} j_t^{(m)} = j_t$ in $L^2(\mathbb{R}^{3+1})$, and Φ_N is a bounded multiplication operator, we have

$$(|J_aF|, |\Phi_N||J_bG|)_2 \le \|\Phi_N\|_1\|F\|_2\|G\|_2 \le \|\Phi\|_1\|F\|_2\|G\|_2. \tag{4.4}$$

Since $|\Phi_N| \uparrow |\Phi|$ as $N \to \infty$, by monotone convergence $|J_aF||\Phi||J_bG| \in L^1(\mathcal{Q}_{\rm E})$ and (4.3) follow. This completes the proof. **qed**

4.2 Functional integral

As explained in Section 1, a key idea of constructing a functional integral representation of $e^{-tH_{\rm PF}}$ is to use the identity

$$\mathcal{H} = \int_{\mathcal{Q}}^{\oplus} L^2(\mathbb{R}^3\times\mathbb{Z}_2)d\mu(\phi). \tag{4.5}$$

We define the Pauli operator $H^0_{\rm PF}(\phi)$ in (4.7) for each fiber $\phi \in \mathcal{Q}$ and set

$$K_{\rm PF} := H_{\rm rad} \dotplus \int_{\mathcal{Q}}^{\oplus} H^0_{\rm PF}(\phi)d\mu(\phi), \tag{4.6}$$

where $\dotplus$ denotes quadratic form sum. It is seen that $H_{\rm PF} = K_{\rm PF}$ as a self-adjoint operator. Using the path integral representation of Pauli operators discussed in Section 3, we can construct the functional integral representation of $e^{-tH^0_{\rm PF}(\phi)}$ for each $\phi \in \mathcal{Q}$. From this the path integral representation of $e^{-tH_{\rm PF}}$ can be derived through the identity $H_{\rm PF} = K_{\rm PF}$ and the Trotter product formula for quadratic form sums [KM78].

Define the Pauli operator $H^0_{\rm PF}(\phi)$ on $L^2(\mathbb{R}^3\times\mathbb{Z}_2)$ by

$$(H^0_{\rm PF}(\phi)f)(\sigma) := \left(\frac{1}{2}(-i\nabla - e\mathcal{A}(\phi))^2 + V + H_{\rm d}(\phi)\right) f(\sigma) + H_{\rm od}(\phi)f(-\sigma), \tag{4.7}$$

where

$$\begin{aligned} H_{\rm d}(\phi) &= H_{\rm d}(x,\sigma,\phi) = -\frac{e}{2}\sigma\mathcal{B}_3(\phi),\\ H_{\rm od}(\phi) &= H_{\rm od}(x,-\sigma,\phi) = -\frac{e}{2}(\mathcal{B}_1(\phi) - i\sigma\mathcal{B}_2(\phi)). \end{aligned}$$

To avoid that the off-diagonal part $H_{\rm od}(\phi)$ vanishes, we introduce $H^{0\varepsilon}_{\rm PF}(\phi)$ in the similar manner as in $\tilde{h}^\varepsilon(a,b)$ above by

$$\begin{aligned} (H^{0\varepsilon}_{\rm PF}(\phi)f)(\sigma) := &\left(\frac{1}{2}(-i\nabla - e\mathcal{A}(\phi))^2 + V + H_{\rm d}(\phi)\right) f(\sigma) \\ &+ (H_{\rm od}(\phi) + \varepsilon\psi_\varepsilon(H_{\rm od}(\phi)))\, f(-\sigma), \end{aligned} \tag{4.8}$$

where ψ_ε is the indicator function given by (3.17). Since $|H_{\rm d}(\phi)+\varepsilon\psi_\varepsilon(H_{\rm d}(\phi))|\geq\varepsilon/2$ for all $(x,\sigma)\in\mathbb{R}^3\times\mathbb{Z}_2$, we can define

$$W_\phi^\varepsilon(x,-\sigma):=\log\left[-H_{\rm od}(x,-\sigma,\phi)-\varepsilon\psi_\varepsilon(H_{\rm od}(x,-\sigma,\phi))\right].$$

Lemma 4.5 *Assume that $\lambda\in C_0^\infty(\mathbb{R}^3)$. Then for each $\phi\in\mathcal{Q}$, $H_{\rm PF}^{0\varepsilon}(\phi)$ is self-adjoint on $D(-\Delta)\otimes\mathbb{Z}_2$ and for $g\in L^2(\mathbb{R}^3\times\mathbb{Z}_2)$,*

$$(e^{-tH_{\rm PF}^{0\varepsilon}(\phi)}g)(x,\sigma)=\mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_s)ds}e^{Z_t(\phi,\varepsilon)}g(\xi_t)],$$

where

$$\begin{aligned}Z_t(\phi,\varepsilon) &= -i\sum_{\mu=1}^3\int_0^t\mathcal{A}_\mu(\lambda(\cdot-B_s),\phi)dB_s^\mu\\ &\quad -\int_0^t H_{\rm d}(B_s,\sigma_s,\phi)ds+\int_0^{t+}W_\phi^\varepsilon(B_s,-\sigma_{s-})dN_s.\end{aligned}$$

Proof: Since $\lambda\in C_0^\infty(\mathbb{R}^3)$, we have

$$\mathcal{A}_\mu(\phi)=\mathcal{A}_\mu(\lambda(\cdot-x),\phi):=\langle\phi,\oplus_{\nu=1}^3\delta_{\mu\nu}\lambda(\cdot-x)\rangle_0\in C_{\rm b}^\infty(\mathbb{R}_x^3),\quad \phi\in\mathcal{Q}.$$

Then $H_{\rm PF}^{0\varepsilon}(\phi)$ is the Pauli operator with a sufficiently smooth bounded vector potential $\mathcal{A}(\phi)$, and the off-diagonal part is perturbed by the bounded operator $\varepsilon\psi_\varepsilon(H_{\rm od}(\phi))$. Hence it is self-adjoint on $D(-\Delta)\otimes\mathbb{Z}_2$ and the functional integral representation follows by Proposition 3.3. **qed**

Next we define the operator $K_{\rm PF}^\varepsilon$ on $\mathcal{H}$ through $H_{\rm PF}^{0\varepsilon}(\phi)$ and the constant fiber direct integral representation (4.5) of $\mathcal{H}$. Assume that $\lambda\in C_0^\infty(\mathbb{R}^3)$. Define the self-adjoint operator $H_{\rm PF}^{0\varepsilon}$ on $\mathcal{H}$ by

$$H_{\rm PF}^{0\varepsilon}:=\int_{\mathcal{Q}}^{\oplus}H_{\rm PF}^{0\varepsilon}(\phi)d\mu(\phi),$$

that is, $(H_{\rm PF}^{0\varepsilon}F)(\phi)=H_{\rm PF}^{0\varepsilon}(\phi)F(\phi)$ with domain

$$D(H_{\rm PF}^{0\varepsilon})=\left\{F\in\mathcal{H}\,|\,\int_{\mathcal{Q}}\|(H_{\rm PF}^{0\varepsilon}F)(\phi)\|_{L^2(\mathbb{R}^3\times\mathbb{Z}_2)}^2d\mu(\phi)<\infty\right\}.$$

Set

$$K_{\rm PF}^\varepsilon:=H_{\rm PF}^{0\varepsilon}\dotplus H_{\rm rad}. \tag{4.9}$$

Let $L_{\rm fin}^2(\mathcal{Q}):=\bigcup_{m=0}^\infty\{\bigoplus_{n=0}^m L_n^2(\mathcal{Q})\bigoplus_{n=m+1}^\infty\{0\}\}$ and define the dense subspace

$$\mathcal{H}_0:=C_0^\infty(\mathbb{R}^3\times\mathbb{Z}_2)\,\hat{\otimes}\,L_{\rm fin}^2(\mathcal{Q}), \tag{4.10}$$

where $\hat{\otimes}$ denotes algebraic tensor product. Also, define

$$H_{\mathrm{PF}}^{\varepsilon} := H_{\mathrm{PF}} - \frac{e}{2}\begin{bmatrix} 0 & \varepsilon\psi_\varepsilon(\mathcal{B}_1 - i\mathcal{B}_2) \\ \varepsilon\psi_\varepsilon(\mathcal{B}_1 + i\mathcal{B}_2) & 0 \end{bmatrix}. \tag{4.11}$$

Lemma 4.6 *Let $\lambda \in C_0^\infty(\mathbb{R}^3)$. Then*

$$(F, e^{-tH_{\mathrm{PF}}}G) = \lim_{\varepsilon\to 0}(F, e^{-tK_{\mathrm{PF}}^{\varepsilon}}G). \tag{4.12}$$

Proof: It is seen that $K_{\mathrm{PF}}^{\varepsilon} = H_{\mathrm{PF}}^{\varepsilon}$ on $\mathcal{H}_0$, implying that $K_{\mathrm{PF}}^{\varepsilon} = H_{\mathrm{PF}}^{\varepsilon}$ as a self-adjoint operator since $\mathcal{H}_0$ is a core of $H_{\mathrm{PF}}^{\varepsilon}$ [Hir00b, Hir02]. Moreover, $H_{\mathrm{PF}}^{\varepsilon} \to H_{\mathrm{PF}}$ on $\mathcal{H}_0$ as $\varepsilon \to 0$ and $\mathcal{H}_0$ is a common core of the sequence $\{H_{\mathrm{PF}}^{\varepsilon}\}_{\varepsilon\geq 0}$. Thus $\mathrm{s}\text{–}\lim_{\varepsilon\to 0} e^{-tH_{\mathrm{PF}}^{\varepsilon}} = e^{-tH_{\mathrm{PF}}}$, whence (4.12) follows. **qed**

By (4.12) it suffices to construct a functional integral representation for the expressions at its right hand side and then use a limiting procedure. Set

$$H_{\mathrm{d}}^{\mathrm{E}}(x, \sigma, s) = -\frac{e}{2}\sigma\mathcal{B}_3^{\mathrm{E}}(j_s\lambda(\cdot - x)),$$
$$H_{\mathrm{od}}^{\mathrm{E}}(x, -\sigma, s) = -\frac{e}{2}(\mathcal{B}_1^{\mathrm{E}}(j_s\lambda(\cdot - x)) - i\sigma\mathcal{B}_2^{\mathrm{E}}(j_s\lambda(\cdot - x))).$$

Lemma 4.7 *As a bounded multiplication operator on $L^2(\mathcal{Q})$, for each $(x, \sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2$*

$$J_s\psi_\varepsilon(H_{\mathrm{od}}(x, -\sigma))J_s^* = E_s\psi_\varepsilon(H_{\mathrm{od}}^{\mathrm{E}}(x, -\sigma, s))E_s. \tag{4.13}$$

Proof: Note that $\psi_\varepsilon(H_{\mathrm{od}}(x, -\sigma))$ is a function of the Gaussian random variable $\Phi := H_{\mathrm{od}}(x, -\sigma) = \mathcal{B}_1(x) - i\sigma\mathcal{B}_2(x)$ of mean zero and covariance

$$\rho := \int_{\mathcal{Q}} \Phi^2 d\mu = \int_{\mathcal{Q}} (\mathcal{B}_1(x)^2 + \mathcal{B}_2(x)^2)d\mu = \frac{1}{2}\int \frac{|\hat{\varphi}(k)|^2}{\omega_{\mathrm{b}}(k)}|k|^2\left(2 - \frac{|k_1|^2 + |k_2|^2}{|k|^2}\right)dk, \tag{4.14}$$

since

$$\sum_{j=\pm 1}(k \times e(k, j))_\mu(k \times e(k, j))_\nu = |k|^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right).$$

In general, for a given function $g \in L^2(\mathbb{R})$, $g(\Phi)$ is approximated by

$$g_n(\Phi) = (2\pi)^{-1/2}\int_{\mathbb{R}} \hat{g}_n(k)e^{ik\Phi}dk \tag{4.15}$$

in $L^2(\mathcal{Q})$, where $g_n \in \mathcal{S}(\mathbb{R})$ is such that $g_n \to g$ as $n \to \infty$ in $L^2(\mathbb{R})$. This follows from

$$\|g(\Phi) - g_n(\Phi)\|_2^2 \leq (2\pi\rho)^{-1/2}\int_{\mathbb{R}} |g(x) - g_n(x)|^2 dx. \tag{4.16}$$

For the vector

$$F=\int f(k_1,...,k_n)e^{-i\sum_{j=1}^n\langle\phi,h_j\rangle_0}dk_1\cdots dk_n$$

with $f\in\mathcal{S}(\mathbb{R}^n)$ and $h_j\in\oplus^3L^2(\mathbb{R}^3)$, we have $\lim_{n\to\infty}g_n(\Phi)F=g(\Phi)F$ strongly by (4.16). Since the set of vectors of form F are dense in $L^2(\mathcal{Q})$, as bounded multiplication operators $g_n(\Phi)$ strongly converge to $g(\Phi)$ as $n\to\infty$. Thus there is a sequence $\{\psi_\varepsilon^n(\Phi)\}_{n=1}^\infty$ such that

$$\psi_\varepsilon^n(\Phi)=(2\pi)^{-1/2}\int_{\mathbb{R}}\hat{\psi}_\varepsilon^n(k)e^{ik\Phi}dk \tag{4.17}$$

with $\hat{\psi}_\varepsilon^n\in\mathcal{S}(\mathbb{R})$ and $\lim_{n\to\infty}\psi_\varepsilon^n(\Phi)=\psi_\varepsilon(\Phi)$ in strong sense. By (4.17)

$$\begin{aligned}J_s\psi_\varepsilon^n(H_{\rm od}(x,-\sigma))J_s^*&=(2\pi)^{-1/2}\int_{\mathbb{R}}\hat{\psi}_\varepsilon^n(k)J_se^{ik\Phi}J_s^*dk\\&=(2\pi)^{-1/2}\int_{\mathbb{R}}\hat{\psi}_\varepsilon^n(k)E_se^{ik\Phi_s}E_sdk=E_s\psi_\varepsilon^n(H_{\rm od}^{\rm E}(x,-\sigma,s))E_s,\end{aligned}$$

where $\Phi(s)=\mathcal{B}_1^{\rm E}(j_s\lambda(\cdot-x))-i\sigma\mathcal{B}_2^{\rm E}(j_s\lambda(\cdot-x))$, and $\psi_\varepsilon^n(H_{\rm od}^{\rm E}(x,-\sigma,s))$ converges strongly to $\psi_\varepsilon(H_{\rm od}^{\rm E}(x,-\sigma,s))$ with $n\to\infty$ as a bounded multiplication operator on $L^2(\mathcal{Q}_{\rm E})$, yielding (4.13). **qed**

The next statement is our key lemma.

Lemma 4.8 *Let $\lambda\in C_0^\infty(\mathbb{R}^3)$, $F\in\mathcal{E}_{[a,b]}$ and $s\notin[a,b]$. Then*

$$(F,J_se^{-tH_{\rm PF}^{0\varepsilon}}J_s^*G)=e^t\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^tV(B_r)dr}\int_{\mathcal{Q}_{\rm E}}\overline{F(\xi_0)}e^{X_t(\varepsilon,s)}E_sG(\xi_t)d\mu_{\rm E}\right].\tag{4.18}$$

Here

$$\begin{aligned}X_t(\varepsilon,s)&=-ie\sum_{\mu=1}^3\int_0^t\mathcal{A}_\mu^{\rm E}(j_s\lambda(\cdot-B_r))dB_r^\mu\\&\quad-\int_0^tH_{\rm d}^{\rm E}(B_r,\sigma_r,s)dr+\int_0^{t+}W^\varepsilon(B_r,-\sigma_{r-},s)dN_r,\end{aligned}\tag{4.19}$$

and

$$W^\varepsilon(x,-\sigma,s):=\log\left(-H_{\rm od}^{\rm E}(x,-\sigma,s)-\varepsilon\psi_\varepsilon(H_{\rm od}^{\rm E}(x,-\sigma,s))\right)\tag{4.20}$$

Proof: First notice that the right hand side of (4.18) is bounded. By Corollary 4.4, $F(x,\sigma)=J_lJ_l^*F(x,\sigma)$ for some $l\in[a,b]$ and $E_sG(B_t,\sigma_t)=J_sJ_s^*G(B_t,\sigma_t)$. We obtain

$$|\text{r.h.s. }(4.18)|\ \le\ \mathbb{E}_\Omega\left[e^{-\int_0^tV(B_r)dr}\sum_\sigma\int dx\,\|F(x,\sigma)\|_2\|G(B_t+x,\sigma_t)\|_2\|e^{X_t(\varepsilon,s)}\|_1\right].\tag{4.21}$$

We will prove in Lemma 4.9 below that there exists a random variable $c=c(\omega)$ such that

(1) $\|e^{X_t(\varepsilon,s)}\|_1^2 \le c$, a.e. $\omega\in\Omega$,

(2) c is independent of $(x,\sigma)\in\mathbb{R}^3\times\mathbb{Z}_2$,

(3) c is independent of B_t^μ, $\mu=1,2,3$,

(4) $\mathbb{E}_\Omega[c^{1/2}]<\infty$.

By (4.21),

$$\begin{aligned}
&|\text{r.h.s. } (4.18)|\\
&\le \mathbb{E}_\Omega\left[\left(\sum_\sigma\int dx\,\|G(B_t+x,\sigma_t)\|_2^2\right)^{1/2}\left(\sum_\sigma\int dx\,\|F(x,\sigma)\|_2^2 e^{-2\int_0^t V(B_r+x)dr}c\right)^{1/2}\right]\\
&\le \|G\|_{\mathcal{H}}\,\mathbb{E}_\Omega\left[c^{1/2}\left(\sum_\sigma\int dx\,\|F(x,\sigma)\|_2^2 e^{-2\int_0^t V(B_r+x)dr}\right)^{1/2}\right]\\
&\le \|G\|_{\mathcal{H}}\,\mathbb{E}_\Omega[c^{1/2}]\,\mathbb{E}_\Omega\left[\left(\sum_\sigma\int dx\,\|F(x,\sigma)\|_2^2 e^{-2\int_0^t V(B_r+x)dr}\right)^{1/2}\right]\\
&\le \|G\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} V_M^{1/2}\mathbb{E}_\Omega[c^{1/2}]<\infty, \qquad (4.22)
\end{aligned}$$

where we used (1) above in the second line, (2) in the third line, (3) in the fourth line, Assumption 2.2 and (4) in the fifth line, where V_M is defined in (3.13).

Next we prove (4.18). By Lemma 4.5 we have

$$\begin{aligned}
&(J_s^*F, e^{-tH_{\mathrm{PF}}^{0\varepsilon}}J_s^*G)\\
&\quad=\int_{\mathcal{Q}} d\mu(\phi)((J_s^*F)(\phi), e^{-tH_{\mathrm{PF}}^{0\varepsilon}(\phi)}(J_s^*G)(\phi))_{L^2(\mathbb{R}^3;\mathbb{C}^2)}\\
&\quad=\int_{\mathcal{Q}} d\mu(\phi)\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_r)dr}\overline{(J_s^*F)(\phi,\xi_0)}e^{Z_t(\phi,\varepsilon)}(J_s^*G)(\phi,\xi_t)\right]\\
&\quad=\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_r)dr}\int_{\mathcal{Q}} d\mu(\phi)\overline{(J_s^*F)(\phi,\xi_0)}e^{Z_t(\phi,\varepsilon)}(J_s^*G)(\phi,\xi_t)\right].
\end{aligned}$$

Here we used Fubini's Theorem in the fourth line. Put

$$Z_t(\varepsilon)=-ie\sum_{\mu=1}^3\int_0^t \mathcal{A}_\mu(\lambda(\cdot-B_s))dB_s^\mu-\int_0^t H_{\mathrm{d}}(B_s,\sigma_s)ds+\int_0^{t+}W^\varepsilon(B_s,-\sigma_{s-})dN_s,$$

with $W^\varepsilon(x,-\sigma):=\log\left(-H_{\mathrm{od}}(x,-\sigma)-\varepsilon\psi_\varepsilon(H_{\mathrm{od}}(x,-\sigma))\right)$. Pick $F,G\in\mathcal{H}_0$. Given that $J_s^*F\in L^2(\mathcal{Q}_{\mathrm{E}})$ and $e^{Z_t(\varepsilon)}J_s^*G(B_t,\sigma_t)\in L^2(\mathcal{Q}_{\mathrm{E}})$, we rewrite as

$$(J_s^*F, e^{-tH_{\mathrm{PF}}^{0\varepsilon}}J_s^*G)=\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_r)dr}(F(\xi_0), J_s e^{Z_t(\varepsilon)}J_s^*G(\xi_t))_{L^2(\mathcal{Q}_{\mathrm{E}})}\right].$$

The kernel $J_s e^{Z_t(\varepsilon)} J_s^*$ is computed as follows. Divide it up into

$$J_s e^{Z_t(\varepsilon)} J_s^* = \underbrace{J_s e^{-ie\sum_{\mu=1}^3 \int_0^t \mathcal{A}_\mu(\lambda(\cdot - B_r))dB_r^\mu} J_s^*}_{:=\mathrm{I}} \underbrace{J_s e^{-\int_0^t H_\mathrm{d}(B_r,\sigma_r)dr} J_s^*}_{:=\mathrm{II}} \times \underbrace{J_s e^{\int_0^{t+} W^\varepsilon(B_r,-\sigma_{r-})dN_r} J_s^*}_{:=\mathrm{III}}. \tag{4.23}$$

We compute the three factors I, II, III separately. First, by [Hir97]

$$J_s \exp\left(-ie\sum_{\mu=1}^3 \int_0^t \mathcal{A}_\mu(\lambda(\cdot - B_r))dB_r^\mu\right) J_s^* = E_s \exp\left(-ie\sum_{\mu=1}^3 \int_0^t \mathcal{A}_\mu^\mathrm{E}(j_s\lambda(\cdot - B_r))dB_r^\mu\right) E_s.$$

Secondly, for $\omega \in \Omega$, there exist $N = N(\omega) \in \mathbb{N}$ and $s_1 = s_1(\omega), ..., s_N = s_N(\omega) \in (0,\infty)$ such that on $\mathcal{H}_0$

$$\begin{aligned}
&J_s \exp\left(\int_0^{t+} W^\varepsilon(B_r, -\sigma_{r-})dN_r\right) J_s^* \\
&= J_s \prod_{i=1}^N (-H_\mathrm{od}(B_{s_i}, -\sigma_{s_i -}) - \varepsilon\psi_\varepsilon(-H_\mathrm{od}(B_{s_i}, -\sigma_{s_i -})))\, J_s^* \\
&= E_s \prod_{i=1}^N \left(-H_\mathrm{od}^\mathrm{E}(B_{s_i}, -\sigma_{s_i -}, s) - \varepsilon\psi_\varepsilon(-H_\mathrm{od}^\mathrm{E}(B_{s_i}, -\sigma_{s_i -}, s))\right) E_s \\
&= E_s \exp\left(\int_0^{t+} W^\varepsilon(B_r, -\sigma_{r-}, s)dN_r\right) E_s,
\end{aligned}$$

where we used that $J_s\mathcal{A}(f_1)\cdots\mathcal{A}(f_n)J_s^* = E_s\mathcal{A}^\mathrm{E}(j_sf_1)\cdots\mathcal{A}^\mathrm{E}(j_sf_n)E_s$ as multiplication operators, and that $J_s\psi_\varepsilon(H_\mathrm{od}(B_{s_i}, -\sigma_{s_i -}))J_s^* = E_s\psi_\varepsilon(H_\mathrm{od}^\mathrm{E}(B_{s_i}, -\sigma_{s_i -}, s))E_s$ by Lemma 4.7. Finally, it can be seen that, similarly to III, factor II is computed on $\mathcal{H}_0$ as

$$\begin{aligned}
J_s \exp\left(-\int_0^t H_\mathrm{d}(B_r,\sigma_r)dr\right) J_s^* &= \lim_{n\to\infty} J_s \prod_{i=0}^n \exp\left(H_\mathrm{d}(B_{it/n}, \sigma_{it/n})\frac{t}{n}\right) J_s^* \\
= \lim_{n\to\infty} \prod_{i=0}^n E_s \exp\left(H_\mathrm{d}^\mathrm{E}(B_{it/n}, \sigma_{it/n}, s)\frac{t}{n}\right) E_s &= \exp\left(-\int_0^t H_\mathrm{d}^\mathrm{E}(B_r,\sigma_r,s)dr\right) E_s.
\end{aligned}$$

Putting all this together we get

$$(F, J_s e^{-tH_\mathrm{PF}^{0\varepsilon}} J_s^* G) = \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_r)dr} \int_{\mathcal{Q}_\mathrm{E}} d\mu_\mathrm{E} \overline{F(\xi_0)} e^{X_t(\varepsilon,s)} E_s G(\xi_t)\right] \tag{4.24}$$

for $F, G \in \mathcal{H}_0$. By a limiting argument and the bound (4.22) it is seen that (4.24) extends for $F, G \in \mathcal{H}$, completing the proof. **qed**

Lemma 4.9 *There exists a random variable $c = c(\omega)$ satisfying (1)-(4) in the proof of Lemma 4.8.*

Proof: Note that

$$\|e^{X_t(\varepsilon,s)}\|_1^2 \le \|e^{-\int_0^t H_{\rm d}^{\rm E}(B_r,\sigma_r,s)dr}\|_2^2 \, \|e^{\int_0^t |W^\varepsilon(B_r,-\sigma_{r-},s)|dN_r}\|_2^2.$$

We estimate the right-hand side of this expression. Since

$$\int_0^t H_{\rm d}^{\rm E}(B_r,\sigma_r,s)dr = \mathcal{B}_3^{\rm E}\left(-\frac{e}{2}\int_0^t \sigma_r j_s\lambda(\cdot - B_r)dr\right)$$

and $\mathcal{B}_j^{\rm E}(f)$ is a Gaussian random variable with mean zero and covariance

$$\int_{\mathcal{Q}_{\rm E}} \mathcal{B}_\mu^{\rm E}(f)\mathcal{B}_\nu^{\rm E}(g)d\mu_{\rm E} = \frac{1}{2}\int \overline{\hat{f}(k,k_0)}\hat{g}(k,k_0)|k|^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right)dkdk_0, \tag{4.25}$$

we have

$$\begin{aligned}\left\|e^{-\int_0^t H_{\rm d}^{\rm E}(B_r,\sigma_r,s)dr}\right\|_2^2 &= \left(1_{\mathcal{Q}_{\rm E}}, e^{-2\int_0^t H_{\rm d}^{\rm E}(B_r,\sigma_r,s)dr}1_{\mathcal{Q}_{\rm E}}\right)\\ &= \exp\left(4\frac{1}{2}\left(\frac{e}{2}\right)^2\frac{1}{2}\int_0^t dr\int_0^t dl\sigma_r\sigma_l\int_{\mathbb{R}^3}\frac{|\hat{\varphi}(k)|^2}{\omega_{\rm b}(k)}e^{-ik\cdot(B_r-B_l)}(|k_1|^2+|k_2|^2)dk\right)\\ &\le \exp\left(\left(\frac{e}{2}\right)^2 t^2\int_{\mathbb{R}^3}\frac{|\hat{\varphi}(k)|^2}{\omega_{\rm b}(k)}|k|^2dk\right) := c_1 < \infty. \end{aligned}\tag{4.26}$$

c_1 is thus independent of $(x,\sigma)\in\mathbb{R}^3\times\mathbb{Z}_2$. Next consider $\|e^{\int_0^t |W_\varepsilon(B_r,-\sigma_{r-},s)|dN_r}\|_2^2$. Set $\mathcal{B}_j^{\rm E}(t) := \mathcal{B}_j^{\rm E}(j_s\lambda(\cdot - B_t))$ for notational convenience. For each $\omega\in\Omega$, there exists $N = N(\omega)\in\mathbb{N}$ and $s_1 = s_1(\omega),...,s_N = s_N(\omega)\in(0,\infty)$ such that

$$\begin{aligned}&\left\|e^{\int_0^t |W^\varepsilon(B_r,-\sigma_{r-},s)|dN_r}\right\|_2^2 \\ &\le \left(1_{\mathcal{Q}_{\rm E}}, \exp\left(2\int_0^t \log\left[\frac{|e|}{\sqrt{2}}\sqrt{\mathcal{B}_1^{\rm E}(r)^2+\mathcal{B}_2^{\rm E}(r)^2+\varepsilon^2}\right]dN_r\right)1_{\mathcal{Q}_{\rm E}}\right)_2\\ &= \left(1_{\mathcal{Q}_{\rm E}}, \exp\left(2\sum_{i=1}^N \log\left[\frac{|e|}{\sqrt{2}}\sqrt{\mathcal{B}_1^{\rm E}(s_i)^2+\mathcal{B}_2^{\rm E}(s_i)^2+\varepsilon^2}\right]\right)1_{\mathcal{Q}_{\rm E}}\right)_2\\ &= \left(\frac{|e|}{\sqrt{2}}\right)^{2N}\left(1_{\mathcal{Q}_{\rm E}}, \prod_{i=1}^N\left(\mathcal{B}_1^{\rm E}(s_i)^2+\mathcal{B}_2^{\rm E}(s_i)^2+\varepsilon^2\right)1_{\mathcal{Q}_{\rm E}}\right)_2\\ &= \left(\frac{|e|}{\sqrt{2}}\right)^{2N}\sum_{m=0}^N \varepsilon^{2(N-m)}\sum_{{\rm comb}_m}(1_{\mathcal{Q}_{\rm E}}, \underbrace{(\mathcal{B}_\#^{\rm E})^2\cdots(\mathcal{B}_\#^{\rm E})^2}_{m\text{-fold}}1_{\mathcal{Q}_{\rm E}})_2 \end{aligned}\tag{4.27}$$

$$
\begin{aligned}
&= \left(\frac{|e|}{\sqrt{2}}\right)^{2N} \sum_{m=0}^{N} \varepsilon^{2(N-m)} \sum_{\mathrm{comb}_m} \| \underbrace{\mathcal{B}_{\#}^{\mathrm{E}} \cdots \mathcal{B}_{\#}^{\mathrm{E}}}_{m\text{-fold}} 1_{\mathcal{Q}_{\mathrm{E}}} \|_2^2 \\
&\leq \left(\frac{|e|}{\sqrt{2}}\right)^{2N} \sum_{m=0}^{N} \varepsilon^{2(N-m)} 2^m \, (\sqrt{2})^{2m} \, m! \, \|\sqrt{|k|}\hat{\varphi}\|^{2m} \\
&= \left(\frac{|e|}{\sqrt{2}}\right)^{2N} \sum_{m=0}^{N} \varepsilon^{2(N-m)} \, m! \, 2^{2m} \, \|\sqrt{|k|}\hat{\varphi}\|^{2m} \ := \ c_2(\omega),
\end{aligned}
\tag{4.28}
$$

where $\sum_{\mathrm{comb}_m}$ denotes summation over the 2^m terms in the expansion of the product $\prod_{i=1}^m (\mathcal{B}_1^{\mathrm{E}}(s_i)^2 + \mathcal{B}_2^{\mathrm{E}}(s_i)^2)$, $\mathcal{B}_{\#}^{\mathrm{E}}$ denotes one of $\mathcal{B}_{\mu}^{\mathrm{E}}(s_i)$, $\mu = 1, 2$, $i = 1, ..., N$, and we used that $|a+ib+\varepsilon| \leq \sqrt{2}\sqrt{a^2+b^2+\varepsilon^2}$, $a, b, \varepsilon \in \mathbb{R}$, in the first line, and the basic inequality $\|\mathcal{B}_{\mu}^{\mathrm{E}}(s_i)\Psi\|_2 \ \leq \ \sqrt{2}\|\sqrt{|k|}\hat{\varphi}\|\|N_{\mathrm{b}}^{1/2}\Psi\|_2$ in the sixth. Note that $c_2(\omega)$ is independent of $(x, \sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2$ and B_t^{μ}. Set

$$
c(\omega) = c_1 c_2(\omega). \tag{4.29}
$$

Then

$$
\mathbb{E}_{\Omega}[c^{1/2}] \ \leq \ e^{\frac{1}{2}(|e|/2)^2 t^2 \|\sqrt{|k|}\hat{\varphi}\|^2} \sum_{N=0}^{\infty} \left(\frac{|e|}{\sqrt{2}}\right)^{N} \sum_{m=0}^{N} \frac{\varepsilon^{N-m} \sqrt{m!} \, 2^m \, \|\sqrt{|k|}\hat{\varphi}\|^m}{N!} e^{-t} < \infty. \tag{4.30}
$$

This completes the proof of claims (1)-(4) above. **qed**

Next we define the $L^2(\mathbb{R}^{3+1})$-valued stochastic integral $\int_0^t j_s \lambda(\cdot - B_s) dB_s^{\mu}$ by a limiting procedure. Let $\Delta_n(s)$ be the step function on the interval $[0, t]$ given by

$$
\Delta_n(s) := \sum_{i=1}^{n} \frac{t(i-1)}{n} 1_{(t(i-1)/n, ti/n]}(s). \tag{4.31}
$$

Define the sequence of the $L^2(\mathbb{R}^{3+1})$-valued random variable $\xi_n^{\mu} : \Omega \to L^2(\mathbb{R}^{3+1})$ by

$$
\xi_n^{\mu} := \int_0^t j_{\Delta_n(s)} \lambda(\cdot - B_s) dB_s^{\mu}, \quad \mu = 1, 2, 3.
$$

This sequence converges, which is guaranteed by

$$
\begin{aligned}
\mathbb{E}_{\Omega}[\|\xi_n^{\mu} - \xi_m^{\mu}\|^2] &= \mathbb{E}_{\Omega}\left[\int_0^t \|j_{\Delta_n(s)}\lambda(\cdot - B_s) - j_{\Delta_m(s)}\lambda(\cdot - B_s)\|^2 ds\right] \\
&= 2\mathbb{E}^{x,\sigma}\left[\int_0^t \left(\|\lambda\|^2 - (\lambda(\cdot - B_s), e^{-|\Delta_n(s) - \Delta_m(s)|\omega_{\mathrm{b}}}\lambda(\cdot - B_s))\right) ds\right] \to 0
\end{aligned}
$$

as $n, m \to \infty$.

Definition 4.10 *We define*

$$\int_0^t j_s\lambda(\cdot - B_s)dB_s^\mu := \text{s–}\lim_{n\to\infty}\xi_n^\mu, \quad \mu = 1,2,3,$$

and set

$$\int_0^t \mathcal{A}_\mu^{\rm E}(j_s\lambda(\cdot - B_s))dB_s^\mu := \mathcal{A}_\mu^{\rm E}\left(\int_0^t j_s\lambda(\cdot - B_s)dB_s^\mu\right).$$

Now we are in the position to state the main theorem of this section.

Theorem 4.11 *For every $t \geq 0$ and all $F, G \in \mathcal{H}$*

$$(F, e^{-tH_{\rm PF}^\varepsilon}G) = e^t\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}} d\mu_{\rm E}\overline{J_0F(\xi_0)}e^{X_t(\varepsilon)}J_tG(\xi_t)\right] \tag{4.32}$$

and

$$(F, e^{-tH_{\rm PF}}G) = \lim_{\varepsilon\to 0} e^t\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}} d\mu_{\rm E}\overline{J_0F(\xi_0)}e^{X_t(\varepsilon)}J_tG(\xi_t)\right]. \tag{4.33}$$

Here

$$X_t(\varepsilon) = -ie\sum_{\mu=1}^3\int_0^t \mathcal{A}_\mu^{\rm E}(j_s\lambda(\cdot - B_s))dB_s^\mu$$
$$-\int_0^t H_{\rm d}^{\rm E}(B_s,\sigma_s,s)ds + \int_0^{t+}\log\left(-H_{\rm od}^{\rm E}(B_s,-\sigma_{s-},s) - \varepsilon\psi_\varepsilon(H_{\rm od}^{\rm E}(B_s,-\sigma_{s-},s))\right)dN_s.$$

Proof: Notice that $\mathcal{B}_\mu^{\rm E}(j_sf)$, $f \in L^2(\mathbb{R}^3)$, $s \in \mathbb{R}$, $\mu = 1,2,3$, is a Gaussian random variable with mean zero and covariance

$$\int_{\mathcal{Q}_{\rm E}} \mathcal{B}_\mu^{\rm E}(j_sf)\mathcal{B}_\nu^{\rm E}(j_tg)d\mu_{\rm E} = \frac{1}{2}\int_{\mathbb{R}^3}\overline{\hat{f}(k)}\hat{g}(k)|k|^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right)e^{-|t-s|\omega_{\rm b}(k)}dk.$$

Then similarly to (4.22) we obtain $|\text{r.h.s.}(4.32)| \leq \|F\|_\mathcal{H}\|G\|_\mathcal{H}V_M^{1/2}\mathbb{E}^{x,\sigma}[c^{1/2}] < C$, where c is given by (4.29) and C is a constant independent of ε. Since $e^{-tH_{\rm PF}^\varepsilon} \to e^{-tH_{\rm PF}}$ strongly as $\varepsilon \to 0$, (4.33) follows from (4.32).

Now we turn to proving (4.32). Take $\lambda = (\hat{\varphi}/\sqrt{\omega_{\rm b}})^\vee \in C_0^\infty(\mathbb{R}^3)$. Then by (4.22) $\mathbb{E}^{x,\sigma}[e^{-\int_0^t V(B_r)dr}e^{X_t(\varepsilon,s)}G(\xi_t)] \in \mathcal{H}$ for $G \in \mathcal{H}$, and

$$\left\|\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_r)dr}e^{X_t(\varepsilon,s)}G(\xi_t)\right]_\mathcal{H}\right\| \leq V_M^{1/2}\,\mathbb{E}^{x,\sigma}[c^{1/2}]\,\|G\|_\mathcal{H}.$$

Remember that $X_t(\varepsilon,s)$ was defined in (4.19) and V_M in (3.13). Define the bounded operator

$$(S_{t,s}^\varepsilon G)(x,\sigma) := e^t\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_u)du}e^{X_t(\varepsilon,s)}G(\xi_t)\right], \quad \mathcal{H}\to\mathcal{H}.$$

Set

$$X_{S,T}(\varepsilon,s)=-ie\sum_{\mu=1}^{3}\int_S^T\mathcal{A}_\mu(j_s\lambda(\cdot-B_l))dB_l^\mu \\ -\int_S^T H_{\rm d}(B_l,\sigma_l,s)dl+\int_S^{T+}W^\varepsilon(B_l,-\sigma_{l-},s)dN_l.$$

By making use of the Markov property of ξ_t we get

$$\begin{aligned}&(S^\varepsilon_{t,r}S^\varepsilon_{s,l}G)(x,\sigma)\\ &=e^{s+t}\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_u)du}e^{X_{0,t}(\varepsilon,r)}\mathbb{E}^{B_t,\sigma_t}\left[e^{-\int_0^s V(B_u)du}e^{X_{0,s}(\varepsilon,l)}G(\xi_s)\right]\right]\\ &=e^{s+t}\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_u)du}e^{X_{0,t}(\varepsilon,r)}\mathbb{E}^{x,\sigma}\left[e^{-\int_s^{s+t} V(B_u)du}e^{X_{t,s+t}(\varepsilon,l)}G(B_{s+t},\sigma_{s+t})\,|\,\Omega_t\right]\right]\\ &=e^{s+t}\mathbb{E}^{x,\sigma}\left[e^{-\int_0^{s+t} V(B_u)du}e^{X_{0,t}(\varepsilon,r)+X_{t,s+t}(\varepsilon,l)}G(B_{s+t},\sigma_{s+t})\right].\end{aligned}\tag{4.34}$$

Note that for $s_1\le\cdots\le s_n$,

$$\exp\Big(X_{0,t_1}(\varepsilon,s_1)+X_{t_1,t_1+t_2}(\varepsilon,s_2)+\cdots+X_{t_1+\cdots+t_{n-1},t_1+\cdots+t_n}(\varepsilon,s_n)\Big)\in E_{[s_1,s_n]}L^2(\mathcal{Q}_{\rm E}).\tag{4.35}$$

For operators T_j, $j=1,...,N$, write $\prod_{i=1}^n T_i:=T_1T_2\cdots T_n$. By using the identity $H^\varepsilon_{\rm PF}=H_{\rm rad}\dot{+}\int_{\mathcal{Q}}^{\oplus}H^0_{\rm PF}(\phi)d\mu(\phi)$, we have

$$\begin{aligned}(F,e^{-tH^\varepsilon_{\rm PF}}G)&=\Big(F,e^{-t(H^{0\varepsilon}_{\rm PF}\dot{+}H_{\rm rad})}G\Big)\\ &=\lim_{n\to\infty}\Big(F,\Big(e^{-(t/n)H^{0\varepsilon}_{\rm PF}}e^{-(t/n)H_{\rm rad}}\Big)^nG\Big)\\ &=\lim_{n\to\infty}\left(J_0F,\left(\prod_{i=0}^{n-1}J_{it/n}e^{-(t/n)H^{0\varepsilon}_{\rm PF}}J^*_{it/n}\right)J_tG\right)\\ &=\lim_{n\to\infty}\left(J_0F,\left(\prod_{i=0}^{n-1}E_{it/n}S^\varepsilon_{t/n,it/n}E_{it/n}\right)J_tG\right)\\ &=\lim_{n\to\infty}\left(J_0F,\left(\prod_{i=0}^{n-1}S^\varepsilon_{t/n,it/n}\right)J_tG\right)\\ &=e^t\lim_{n\to\infty}\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_r)dr}\int_{\mathcal{Q}_{\rm E}}d\mu_{\rm E}\overline{J_0F(x,\sigma)}e^{X^n_t(\varepsilon)}J_tG(\xi_t)\right],\end{aligned}\tag{4.36}$$

where we applied the Trotter-Kato product formula [KM78] to the quadratic form sum in the second line, the equality $J_s^*J_t=e^{-|t-s|H_{\rm rad}}$ in the third, Lemma 4.8 in the fourth, (4.35) and the Markov property of the family of projections $E_{[\dots]}$ in the fifth, and (4.34) in the sixth line. Moreover $X_t^n(\varepsilon)=Y_t^n(1)+Y_t^n(2)+Y_t^n(3,\varepsilon)$, with

$$\begin{aligned}
Y_t^n(1) &:= -ie\sum_{\mu=1}^3\sum_{i=1}^n\int_{t(i-1)/n}^{ti/n}\mathcal{A}^{\rm E}(j_{t(i-1)/n}\lambda(\cdot-B_s))dB_s^\mu\\
&= -ie\mathcal{A}^{\rm E}\left(\oplus_{\mu=1}^3\int_0^t j_{\Delta_n(s)}\lambda(\cdot-B_s)dB_s^\mu\right),\\
Y_t^n(2) &:= -\sum_{i=1}^n\int_{t(i-1)/n}^{ti/n}H_{\rm d}^{\rm E}(B_s,\sigma_s,t(i-1)/n)ds=-\int_0^t H_{\rm d}^{\rm E}(B_s,\sigma_s,\Delta_n(s))ds,\\
Y_t^n(3,\varepsilon) &:= \sum_{i=1}^n\int_{t(i-1)/n}^{ti/n+}W^\varepsilon(B_s,-\sigma_{s-},t(i-1)/n)dN_s=\int_0^t W^\varepsilon(B_s,-\sigma_{s-},\Delta_n(s))dN_s,
\end{aligned}$$

and with $W^\varepsilon(x,-\sigma,r)$ defined in (4.20) and step function $\Delta_n(s)$ given by (4.31). Furthermore, put

$$\begin{aligned}
Y_t(1) &:= -ie\mathcal{A}^{\rm E}\left(\oplus_{\mu=1}^3\int_0^t j_s\lambda(\cdot-B_s)dB_s^\mu\right),\\
Y_t(2) &:= -\int_0^t H_{\rm d}^{\rm E}(B_s,\sigma_s,s)ds,\\
Y_t(3,\varepsilon) &:= \int_0^{t+}W^\varepsilon(B_s,-\sigma_{s-},s)dN_s.
\end{aligned}$$

Then $X_t(\varepsilon)=Y_t(1)+Y_t(2)+Y_t(3,\varepsilon)$. We claim that

$$\text{r.h.s. (4.36)}=e^t\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}}d\mu_{\rm E}\overline{J_0F(\xi_0)}e^{X_t(\varepsilon)}J_tG(\xi_t)\right]. \tag{4.37}$$

Note that

$$\begin{aligned}
&\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}}|J_0F(\xi_0)|\,|J_tG(\xi_t)|\,|e^{X_t^n(\varepsilon)}-e^{X_t(\varepsilon)}|d\mu_{\rm E}\right]\\
&\le \|G\|_{\mathcal{H}}\,\mathbb{E}^{x,\sigma}\left[\left(\sum_\sigma\int dx\,e^{-2\int_0^t V(B_s)ds}\|F(x,\sigma)\|_2^2\,\|e^{X_t^n(\varepsilon)}-e^{X_t(\varepsilon)}\|_1^2\right)^{1/2}\right]
\end{aligned} \tag{4.38}$$

and

$$\|e^{X_t^n(\varepsilon)}\|_1^2\le\left(1_{\mathcal{Q}_{\rm E}},|e^{Y_t^n(2)}|^2 1_{\mathcal{Q}_{\rm E}}\right)\left(1_{\mathcal{Q}_{\rm E}},|e^{Y_t^n(3,\varepsilon)}|^2 1_{\mathcal{Q}_{\rm E}}\right).$$

We continue by estimating the right-hand side above. It readily follows that

$$\begin{aligned}
&\left(1_{\mathcal{Q}_{\rm E}},e^{2Y_t^n(2)}1_{\mathcal{Q}_{\rm E}}\right)\\
&=\exp\left(\frac{e^2}{4}\int_0^t ds\int_0^t dr\sigma_s\sigma_r\int_{\mathbb{R}^3}\frac{|\hat\varphi(k)|^2}{\omega_{\rm b}(k)}e^{-ik(B_s-B_r)}(|k_1|^2+|k_2|^2)e^{-|\Delta_n(s)-\Delta_n(r)|\omega_{\rm b}(k)}dk\right)\\
&\le\exp\left(\frac{e^2}{4}t^2\int_{\mathbb{R}^3}|\hat\varphi(k)|^2|k|dk\right)=c_1,
\end{aligned} \tag{4.39}$$

and the estimate of $\left\|e^{\int_0^t W^\varepsilon(B_s,-\sigma_{s-},\Delta_n(s))dN_s}\right\|_2^2$ goes as that of $\left\|e^{\int_0^t W^\varepsilon(B_r,-\sigma_{r-},s)dN_r}\right\|_2^2$ explained in (4.28), with $\mathcal{B}_\mu^{\rm E}(j_{s_i}\lambda(\cdot - B_{s_i}))$ replaced by $\mathcal{B}_\mu^{\rm E}(j_{\Delta_n(s_i)}\lambda(\cdot - B_{s_i}))$. Then, for each $\omega \in \Omega$, $\left\|e^{\int_0^t W^\varepsilon(B_s,-\sigma_{s-},\Delta_n(s))dN_s}\right\|_2^2 \le c_2(\omega)$, with $c_2(\omega)$ given in (4.28). Thus we conclude that $\|e^{X_t^n(\varepsilon)}\|_1^2 < c(\omega)$, where $c(\omega) = c_1 c_2(\omega)$ and $\mathbb{E}^{x,\sigma}[c^{1/2}] < \infty$. Similarly, $\|e^{X_t(\varepsilon)}\|_1 < C(\omega)$ and $\mathbb{E}^{x,\sigma}[C^{1/2}] < \infty$ follows for a random variable $C(\omega)$. Note that both c and C are independent of $(x,\sigma) \in \mathbb{R}^3 \times \mathbb{Z}_2$, B_t^μ and n. Thus by (4.38) and dominated convergence, it suffices to show that for almost every $\omega \in \Omega$, $e^{X_t^n(\varepsilon)} \to e^{X_t(\varepsilon)}$ as $n \to \infty$ in $L^1(\mathcal{Q}_{\rm E})$. We have

$$
\begin{aligned}
e^{X_t^n(\varepsilon)} - e^{X_t(\varepsilon)} &= \underbrace{e^{Y_t^n(1)}e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)}}_{:=\rm I} \\
&\quad + \underbrace{e^{Y_t(1)}e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t(2)}e^{Y_t^n(3,\varepsilon)}}_{:=\rm II} \\
&\quad + \underbrace{e^{Y_t(1)}e^{Y_t(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t(2)}e^{Y_t(3)}}_{:=\rm III}. \qquad (4.40)
\end{aligned}
$$

We estimate I, II and III. Notice that

$$
\|{\rm I}\|_1 \le \|e^{Y_t^n(1)} - e^{Y_t(1)}\|_2 \, \|e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)}\|_2, \qquad (4.41)
$$

By a minor modification of (4.26) and (4.28) it is seen that there is $N = N(\omega)$ such that

$$
\begin{aligned}
\|e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)}\|_2^2 &\le \||e^{Y_t^n(2)}|^2\|_2 \||e^{Y_t^n(3,\varepsilon)}|^2\|_2 \qquad (4.42) \\
&\le \underbrace{e^{4(e/2)^2 t^2 \|\sqrt{|k|}\hat\varphi\|^2} \left(\frac{|e|}{\sqrt{2}}\right)^{4N} \sum_{m=0}^{2N} \varepsilon^{2N-m}\, m!\, 2^{2m}\, \|\sqrt{|k|}\hat\varphi\|^{2m}}_{:=c_3}.
\end{aligned}
$$

By the expression of $Y_t(1)$ in Definition 4.10

$$
\left(e^{Y_t^n(1)}, e^{Y_t(1)}\right)_2 = \exp\left(-\frac{e^2}{2} q_1(\varrho_1^n, \varrho_1^n)\right),
$$

with $\varrho_1^n = \oplus_{\mu=1}^3 \int_0^t (j_{\Delta_n(s)}\lambda(\cdot - B_s) - j_s\lambda(\cdot - B_s))dB_s^\mu$. Moreover,

$$
\begin{aligned}
\mathbb{E}^{x,\sigma}\left[q_1(\varrho_1^n, \varrho_1^n)\right] &\le \frac{3}{2}\mathbb{E}^{x,\sigma}\left[\int_0^t \|j_{\Delta_n(s)}\lambda(\cdot - B_s) - j_s\lambda(\cdot - B_s)\|^2 ds\right] \\
&\le \frac{3}{2}\mathbb{E}^{x,\sigma}\left[\int_0^t \left(2\|\lambda\|^2 - 2\Re(\lambda(\cdot - B_s), e^{-|\Delta_n(s)-s|\omega_{\rm b}}\lambda(\cdot - B_s))\right) ds\right] \to 0
\end{aligned}
$$

as $n \to 0$. This implies that there exists a subsequence m such that for almost every $\omega \in \Omega$, $\lim_{m\to\infty}(e^{Y_t^m(1)}, e^{Y_t(1)})_2 = 1$ and thus $\|e^{Y_t^m(1)} - e^{Y_t(1)}\|_2 \to 0$. We relabel this subsequence by n. Then

$$\lim_{n\to\infty} \|\mathrm{I}\|_1 = 0 \tag{4.43}$$

follows by (4.41) for almost every $\omega \in \Omega$.

Next we estimate II. Since $|e^{Y_t(1)}| = 1$, we have

$$\|\mathrm{II}\|_1 \le \|e^{Y_t^n(2)} - e^{Y_t(2)}\|_2 \, \|e^{Y_t^n(3,\varepsilon)}\|_2$$

and $\|e^{Y_t^n(3,\varepsilon)}\|_2 \le c_3(\omega)$, see (4.42). A direct computation yields

$$\begin{aligned}
&\|e^{Y_t^n(2)}\|_2^2 \\
&= \exp\left(\left(\frac{e}{2}\right)^2 \int_0^t ds \int_0^t ds \sigma_s\sigma_r \int dk \frac{|\hat{\varphi}(k)|^2}{\omega_{\rm b}(k)} e^{-ik(B_s-B_r)}(|k_1|^2+|k_2|^2)e^{-|\Delta_n(s)-\Delta_n(r)|\omega_{\rm b}(k)}\right) \\
&\quad \to \exp\left(\left(\frac{e}{2}\right)^2 \int_0^t ds \int_0^t dr \sigma_s\sigma_r \int dk \frac{|\hat{\varphi}(k)|^2}{\omega_{\rm b}(k)} e^{-ik(B_s-B_r)}(|k_1|^2+|k_2|^2)e^{-|s-r|\omega_{\rm b}(k)}\right) \\
&\quad = \|e^{Y_t(2)}\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
&(e^{Y_t^n(2)}, e^{Y_t(2)})_2 \\
&\quad = \exp\left(\frac{1}{4}\left(\frac{e}{2}\right)^2 \int_0^t ds \int_0^t dr \sigma_s\sigma_r \int dk \frac{|\hat{\varphi}(k)|^2}{\omega_{\rm b}(k)} e^{-ik\cdot(B_s-B_r)}(|k_1|^2+|k_2|^2)\right. \\
&\qquad\qquad \left.\times \left(e^{-|s-r|\omega_{\rm b}(k)} + e^{-|s-\Delta_n(r)|\omega_{\rm b}(k)} + e^{-|r-\Delta_n(s)|\omega_{\rm b}(k)} + e^{-|\Delta_n(s)-\Delta_n(r)|\omega_{\rm b}(k)}\right)\right) \\
&\quad \to \exp\left(\left(\frac{e}{2}\right)^2 \int_0^t ds \int_0^t dr \sigma_s\sigma_r \int dk \frac{|\hat{\varphi}(k)|^2}{\omega_{\rm b}(k)} e^{-ik\cdot(B_s-B_r)}(|k_1|^2+|k_2|^2)e^{-|s-r|\omega_{\rm b}(k)}\right) \\
&\quad = \|e^{Y_t(2)}\|_2^2
\end{aligned}$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} \|\mathrm{II}\|_1^2 \le \lim_{n\to\infty} (\|e^{Y_t^n(2)}\|_2^2 - 2\Re(e^{Y_t^n(2)}, e^{Y_t(2)})_2 + \|e^{Y_t(2)}\|_2^2)c_3^2 = 0 \tag{4.44}$$

is obtained.

Finally, we deal with III. Since

$$\|e^{Y_t(1)}e^{Y_t(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t(2)}e^{Y_t(3,\varepsilon)}\|_1 \le \|e^{Y_t(2)}\|_2 \, \|e^{Y_t^n(3,\varepsilon)} - e^{Y_t(3,\varepsilon)}\|_2$$

and $\|e^{Y_t(2)}\|_2^2 \le e^{4(e/2)t^2\|\sqrt{|k|}\hat{\varphi}\|^2}$, it is enough to show that $e^{Y_t^n(3,\varepsilon)} \to e^{Y_t(3,\varepsilon)}$ in $L^2(\mathcal{Q}_{\rm E})$. By the definition of $Y_t^n(3,\varepsilon)$ we have

$$e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^n \exp\left(\int_{t(i-1)/n}^{ti/n+} W^\varepsilon(B_s, -\sigma_{s-}, t(i-1)/n) dN_s\right).$$

For each $\omega \in \Omega$ there exists $N = N(\omega) \in \mathbb{N}$ such that $D(p) = \{s_1, ..., s_N\}$, where p is the point process defining the counting measure N_t, see (3.3). For sufficiently large n the number of s_k contained in the interval $(t(i-1)/n, ti/n]$ is at most one. Then by taking n large enough and putting $(n(s_i), n(s_i) + t/n]$ for the interval containing s_i, $i = 1, ..., N$, we get

$$e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^N \left(-H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i)) - \varepsilon\psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i))\right). \tag{4.45}$$

Clearly, $n(s_i) \to s_i$ as $n \to \infty$. We want to show that

$$\lim_{m\to\infty} \text{r.h.s. (4.45)} = \prod_{i=1}^N \left(-H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i) - \varepsilon\psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i))\right). \tag{4.46}$$

Since $H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i))$ converges strongly to $H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)$ as $n \to \infty$ in $L^2(\mathcal{Q}_{\rm E})$, we have by Lemma 4.12 below that in $L^2(\mathcal{Q}_{\rm E})$

$$\lim_{n\to 0} \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i))) = \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)). \tag{4.47}$$

Set $I(n,i) := \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i)))$, $I(\infty, i) := \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i))$, $A(n,i) := H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i))$ and $A(\infty, i) := H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)$. Since these are commutative as operators, the right hand side of (4.45) can be expanded as a finite sum of functions of the form $C(n) := \prod_k I(n, \#) \prod_{N-k} A(n, \#)$, where $\#$ stands for one of $1, ..., N$. It suffices to show that each $C(n)$ converges to $C(\infty)$ as $n \to \infty$ in $L^2(\mathcal{Q}_{\rm E})$, where $C(\infty)$ is $C(n)$ with $n(s_i)$ replaced by s_i, $i = 1, ..., N$. Take, for example $C_0(n) := I(n,1) \cdots I(n,k) A(n, k+1) \cdots A(n, N)$. Then

$$\begin{aligned} &C_0(n) - C_0(\infty) = \\ &\quad I(n,1)\cdots I(n,k)\left(A(n,k+1)\cdots A(n,N) - A(\infty,k+1)\cdots A(\infty,N)\right) \\ &\quad + \left(I(n,1)\cdots I(n,k) - I(\infty,1)\cdots I(\infty,k)\right) A(\infty,k+1)\cdots A(\infty,N). \end{aligned} \tag{4.48}$$

Since $I(n,i)$ is uniformly bounded in n, the first term at the right hand side of (4.48) goes to zero as $n \to \infty$ in $L^2(\mathcal{Q}_{\rm E})$. The second term can be estimated in this way.

First note that

$$\begin{aligned}&\| (I(n,i) - I(\infty,i))\, A(\infty,k+1)\cdots A(\infty,N)\|_2^2 = \\ &\quad \Big(A(\infty,k+1)^2\cdots A(\infty,N)^2, I(n,i)-I(\infty,i)\Big)_2 .\end{aligned}$$

Since $\lim_{n\to\infty}\|(I(n,i)-I(\infty,i))^2\| = \lim_{n\to\infty}\|I(n,i)-I(\infty,i)\| = 0$ by (4.47), the second term of the right hand side of (4.48) also converges to zero. Then $C_0(n)\to C_0(\infty)$ as $n\to\infty$ in $L^2(\mathcal{Q}_{\rm E})$ follows, and hence (4.46). Since the right-hand side of (4.46) equals $e^{Y_t(3,\varepsilon)}$, it is seen that $\lim_{n\to\infty}\|e^{Y_t^n(3,\varepsilon)}-e^{Y_t(3,\varepsilon)}\|_2 = 0$, and

$$\lim_{n\to\infty}\|{\rm III}\|_1 = 0. \tag{4.49}$$

A combination of (4.43), (4.44) and (4.49) implies (4.37), and thus (4.32).

Now we extend (4.33) to form factors for which $\sqrt{\omega_{\rm b}}\hat\varphi$, $\hat\varphi/\sqrt{\omega_{\rm b}}\in L^2(\mathbb{R}^3)$, through a limiting argument. Let $\hat\varphi_m\in C_0^\infty(\mathbb{R}^3)$ satisfy $\hat\varphi_m/\sqrt{\omega_{\rm b}}\to\hat\varphi/\sqrt{\omega_{\rm b}}$ and $\sqrt{\omega_{\rm b}}\hat\varphi_m\to\sqrt{\omega_{\rm b}}\hat\varphi$ strongly in $L^2(\mathbb{R}^3)$ as $m\to\infty$.For each $\hat\varphi_m$, (4.33) holds. Let $H^\varepsilon_{\rm PF}(m)$ be $H^\varepsilon_{\rm PF}$ with $\hat\varphi$ replaced by $\hat\varphi_m$. Thus $H^\varepsilon_{\rm PF}(m)\to H^\varepsilon_{\rm PF}$ as $m\to\infty$ on the common core $\mathcal{H}_0$. Then $e^{-tH^\varepsilon_{\rm PF}(m)}\to e^{-tH^\varepsilon_{\rm PF}}$ strongly in $\mathcal{H}$ as $m\to\infty$. Define $X_t^{(m)}(\varepsilon)$, $Y_t^{(m)}(1)$, $Y_t^{(m)}(2)$ and $Y_t^{(m)}(3,\varepsilon)$ by $X_t(\varepsilon)$, $Y_t(1)$, $Y_t(2)$ and $Y_t(3,\varepsilon)$ with $\hat\varphi$ replaced by $\hat\varphi_m$, respectively. It is enough to see that $e^{X_t^{(m)}(\varepsilon)}\to e^{X_t(\varepsilon)}$ in $L^1(\mathcal{Q}_{\rm E})$. We divide $e^{X_t^{(m)}(\varepsilon)}-e^{X_t(\varepsilon)}$ in the same way as (4.40) with $Y_t^n(j)$ replaced by $Y_t^{(m)}(i)$. Then it suffices to show that $e^{Y_t^{(m)}(i)}\to e^{Y_t(i)}$ strongly in $L^2(\mathcal{Q}_{\rm E})$, for almost every $\omega\in\Omega$ as $m\to\infty$. First, we have

$$(e^{Y_t^{(m)}(1)}, e^{Y_t(1)})_2 = \exp\left(-\frac{e^2}{2}q_1(\varrho_2^m,\varrho_2^m)\right),$$

where $\varrho_2^m = \oplus_{\mu=1}^3\int_0^t (j_s\lambda_m(\cdot - B_s) - j_s\lambda(\cdot - B_s))dB_s^\mu$ and $\lambda_m = (\hat\varphi_m/\sqrt{\omega_{\rm b}})^\vee$. Furthermore,

$$\begin{aligned}\mathbb{E}^{x,\sigma}[q_1(\varrho_2^m,\varrho_2^m)] &\le \frac{3}{2}\mathbb{E}^{x,\sigma}\left[\int_0^t \|j_s\lambda_m(\cdot - B_s) - j_s\lambda(\cdot - B_s)\|^2 ds\right]\\ &\le \frac{3}{2}\|\hat\varphi_m/\sqrt{\omega_{\rm b}} - \hat\varphi/\sqrt{\omega_{\rm b}}\|\to 0\end{aligned}$$

as $m\to\infty$. Then there is a subsequence l such that $(e^{Y_t^{(l)}(1)}, e^{Y_t(1)})_2\to 1$ as $l\to\infty$ for almost every $\omega\in\Omega$, and hence

$$\lim_{l\to\infty}\|e^{Y_t^{(l)}(1)} - e^{Y_t(1)}\|_2 = 0. \tag{4.50}$$

We relabel l as m again. Secondly, we have

$$\|e^{Y_t^{(m)}(2)}\|_2^2$$
$$= \exp\left(\left(\frac{e}{2}\right)^2 \int_0^t ds \int_0^t dr \sigma_s \sigma_r \int dk \frac{|\hat{\varphi}_m(k)|^2}{\omega_{\rm b}(k)} e^{-ik\cdot(B_s-B_r)}(|k_1|^2+|k_2|^2)e^{-|s-r|\omega_{\rm b}(k)}\right),$$

$$(e^{Y_t^{(m)}(2)}, e^{Y_t(2)})_2$$
$$= \exp\left(\frac{1}{4}\left(\frac{e}{2}\right)^2 \int_0^t ds \int_0^t dr \sigma_s \sigma_r \int_{\mathbb{R}^3} dk \frac{|\hat{\varphi}(k)+\hat{\varphi}_m(k)|^2}{\omega_{\rm b}(k)} e^{-ik\cdot(B_s-B_r)}\right.$$
$$\left. \times (|k_1|^2+|k_2|^2)e^{-|s-r|\omega_{\rm b}(k)}\right).$$

From here

$$\lim_{m\to\infty} \|e^{Y_t^{(m)}(2)} - e^{Y_t(2)}\|_2^2 = \lim_{m\to\infty}\left(\|e^{Y_t^{(m)}(2)}\|_2^2 - 2\Re(e^{Y_t^{(m)}(1)}, e^{Y_t(1)})_2 + \|e^{Y_t(2)}\|_2^2\right) = 0 \tag{4.51}$$

follows. Finally we see that for each $\omega\in\Omega$, $e^{Y_t^{(m)}(3,\varepsilon)}1_{\mathcal{Q}_{\rm E}} \to e^{Y_t(3,\varepsilon)}1_{\mathcal{Q}_{\rm E}}$ as $m\to\infty$ in $L^2(\mathcal{Q}_{\rm E})$. There exists $N=N(\omega)\in\mathbb{N}$, $s_1=s_1(\omega),...,s_N(\omega)\in(0,\infty)$ such that

$$e^{Y_t^{(m)}(3,\varepsilon)} = \prod_{i=1}^N \left(-H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m) - \varepsilon\psi_\varepsilon\left(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m)\right)\right),$$

where $H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m)$ is defined by $H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)$ with $\hat{\varphi}$ replaced by $\hat{\varphi}_m$. Since $H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m)$ converges strongly to $H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)$ as $m\to 0$ in $L^2(\mathcal{Q}_{\rm E})$, by Lemma 4.12 we obtain

$$\lim_{m\to 0}\psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m)) = \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)) \tag{4.52}$$

in $L^2(\mathcal{Q}_{\rm E})$. Similarly to the proof of $\lim_{n\to\infty} e^{Y_t^n(3,\varepsilon)} = e^{Y_t(3,\varepsilon)}$, we argue that

$$\lim_{m\to\infty} \|e^{Y_t^{(m)}(3,\varepsilon)} - e^{Y_t(3,\varepsilon)}\|_2 = 0. \tag{4.53}$$

From (4.50), (4.51) and (4.53) we finally obtain (4.37), completing the proof. **qed**

It remains to show (4.47) and (4.52).

Lemma 4.12 *We have*

$$\lim_{n\to\infty}\psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, n(s_i))) = \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)) \tag{4.54}$$
$$\lim_{m\to 0}\psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m)) = \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)) \tag{4.55}$$

strongly in $L^2(\mathcal{Q}_{\rm E})$.

PROOF: We show (4.55), the proof of (4.54) is similar. Put $\eta_m = H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i, m)$ and $\eta = H_{\rm od}^{\rm E}(B_{s_i}, -\sigma_{s_i-}, s_i)$. Let $g_n \in \mathcal{S}(\mathbb{R})$ be such that $g_n \to \psi_\varepsilon$ as $n \to \infty$ in $L^2(\mathbb{R})$. We have

$$\|\psi_\varepsilon(\eta) - \psi_\varepsilon(\eta_m)\| \le \|\psi_\varepsilon(\eta) - g_n(\eta)\| + \|g_n(\eta) - g_n(\eta_m)\| + \|g_n(\eta_m) - \psi_\varepsilon(\eta_m)\|.$$

It is readily seen that

$$\|\psi_\varepsilon(\eta) - g_n(\eta)\|^2 = \int |\psi_\varepsilon(x) - g_n(x)|^2 e^{-|x|^2/(2\rho)}(2\pi\rho)^{-1/2}dx \tag{4.56}$$

and

$$\|g_n(\eta_m) - \psi_\varepsilon(\eta_m)\|^2 = \int |\psi_\varepsilon(x) - g_n(x)|^2 e^{-|x|^2/(2\rho_m)}(2\pi\rho_m)^{-1/2}dx, \tag{4.57}$$

where ρ is given by (4.14) and ρ_m is obtained by replacing $\hat{\varphi}$ by $\hat{\varphi}_m$. Since $\rho_m \to \rho$ as $m \to 0$, the left hand sides of (4.56) and (4.57) are bounded by $C\|\psi_\varepsilon - g_n\|^2$ with some constant C independent of m. Consequently, they both converge to zero uniformly in m. We also see that

$$\|g_n(\eta) - g_n(\eta_m)\| \le (2\pi)^{-1/2}\int_{\mathbb{R}} |\hat{g}_n(k)|\|e^{ix\eta} - e^{ix\eta_m}\|dx. \tag{4.58}$$

Since $\|e^{ix\eta} - e^{ix\eta_m}\| \to 0$ as $m \to 0$ for each n, the left hand side of (4.58) converges to zero as $m \to 0$. This gives the lemma. **qed**

4.3 Energy comparison inequality

Write

$$\inf\sigma(H_{\rm PF}) = E(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$$

for the bottom of the spectrum of $H_{\rm PF}$. Then for the spinless Pauli-Fierz Hamiltonian $\hat{H}_{\rm PF}$ we have $\inf\sigma(\hat{H}_{\rm PF}) = E(\mathcal{A}, 0, 0, 0)$ and the diamagnetic inequality $E(0,0,0,0) \le E(\mathcal{A}, 0, 0, 0)$ is well-known [AHS78, Hir97]. In this subsection we extend this inequality to the case of the Hamiltonian with spin.

Define

$$H_{\rm PF}^{\perp} := H_{\rm p} + H_{\rm rad} - \begin{bmatrix} \frac{e}{2}\mathcal{B}_3 & \frac{|e|}{2}\sqrt{\mathcal{B}_1^2 + \mathcal{B}_2^2} \\ \frac{|e|}{2}\sqrt{\mathcal{B}_1^2 + \mathcal{B}_2^2} & -\frac{e}{2}\mathcal{B}_3 \end{bmatrix}. \tag{4.59}$$

Furthermore, to avoid zeroes of the off-diagonal part occur we also define

$$H_{\rm PF}^{\perp\varepsilon} := H_{\rm PF}^{\perp} - \begin{bmatrix} 0 & \varepsilon\psi_\varepsilon\left(\frac{|e|}{2}\sqrt{\mathcal{B}_1^2 + \mathcal{B}_2^2}\right) \\ \varepsilon\psi_\varepsilon\left(\frac{|e|}{2}\sqrt{\mathcal{B}_1^2 + \mathcal{B}_2^2}\right) & 0 \end{bmatrix}. \tag{4.60}$$

Since the spin interaction is infinitesimally small with respect to the free Hamiltonian $H_{\rm p}+H_{\rm rad}$, $H_{\rm PF}^\perp$ and $H_{\rm PF}^{\perp\varepsilon}$ are self-adjoint on $D(-\Delta)\cap D(H_{\rm rad})$ and bounded from below. Note that $|H_{\rm od}|=\frac{|e|}{2}\sqrt{\mathcal{B}_1^2+\mathcal{B}_2^2}$ and $\psi_\varepsilon(H_{\rm od})=\psi_\varepsilon(|H_{\rm od}|)=\psi_\varepsilon(\frac{|e|}{2}\sqrt{\mathcal{B}_1^2+\mathcal{B}_2^2})$. The functional integral representation of $e^{-tH_{\rm PF}^\perp}$ is given by

$$
\begin{aligned}
(F,e^{-tH_{\rm PF}^\perp}G)&=\lim_{\varepsilon\to 0}(F,e^{-tH_{\rm PF}^{\perp\varepsilon}}G)\\
&=\lim_{\varepsilon\to 0}\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}}d\mu_{\rm E}\overline{J_0F(\xi_0)}e^{X_t^\perp(\varepsilon)}J_tG(\xi_t)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
X_t^\perp(\varepsilon)=&-\int_0^t H_{\rm d}(B_s,\sigma_s,s)ds\\
&+\int_0^{t+}\log\left[|H_{\rm od}^{\rm E}(B_s,-\sigma_{s-},s)|+\varepsilon\psi_\varepsilon(|H_{\rm od}^{\rm E}(B_s,-\sigma_{s-},s)|)\right]dN_s.
\end{aligned}
$$

Corollary 4.13 *For all $t\geq 0$ and $F,G\in\mathcal{H}$ we have*

$$
|(F,e^{-tH_{\rm PF}}G)|\ \leq\ \Big(|F|,e^{-tH_{\rm PF}^\perp}|G|\Big) \tag{4.61}
$$

and

$$
\max\left\{\begin{array}{l}E(0,\sqrt{\mathcal{B}_1^2+\mathcal{B}_2^2},0,\mathcal{B}_3)\\ E(0,\sqrt{\mathcal{B}_3^2+\mathcal{B}_1^2},0,\mathcal{B}_2)\\ E(0,\sqrt{\mathcal{B}_2^2+\mathcal{B}_3^2},0,\mathcal{B}_1)\end{array}\right\}\leq E(\mathcal{A},\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3). \tag{4.62}
$$

PROOF: Since $H_{\rm PF}^\perp$ is unitary equivalent with the Hamiltonian obtained on replacing e by $-e$, we may assume that $e>0$ without loss of generality. By the functional integral representation of $e^{-tH_{\rm PF}}$ we have

$$
\begin{aligned}
|(F,e^{-tH_{\rm PF}}G)|&=\lim_{\varepsilon\to 0}|(F,e^{-tH_{\rm PF}^\varepsilon}G)|\\
&\leq\lim_{\varepsilon\to 0}\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}}d\mu_{\rm E}|J_0F(\xi_0)||J_tG(\xi_t)|e^{X_t^\perp(\varepsilon)}\right]\\
&\leq\lim_{\varepsilon\to 0}\sum_\sigma\int dx\,\mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}\int_{\mathcal{Q}_{\rm E}}d\mu_{\rm E}(J_0|F(\xi_0)|)(J_t|G(\xi_t)|)e^{X_t^\perp(\varepsilon)}\right],\\
&=\lim_{\varepsilon\to 0}(|F|,e^{-tH_{\rm PF}^{\perp\varepsilon}}|G|)=(|F|,e^{-tH_{\rm PF}^\perp}|G|),
\end{aligned}
$$

where we used $|e^{X_t(\varepsilon)}|\leq e^{X_t^\perp(\varepsilon)}$ and the fact that $|J_tG|\leq J_t|G|$ as J_t is positivity preserving. Thus (4.61) follows. From this, $E(0,\sqrt{\mathcal{B}_1^2+\mathcal{B}_2^2},0,\mathcal{B}_3)\leq E(\mathcal{A},\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3)$ is obtained. Since $E(\mathcal{A},\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3)=E(\mathcal{A},\mathcal{B}_3,\mathcal{B}_1,\mathcal{B}_2)=E(\mathcal{A},\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_1)$ by symmetry, (4.62) follows. **qed**

5 Translation invariant Hamiltonians

In this section we assume that $V = 0$. In the previous section we derived the functional integral representation of $e^{-tH_{\rm PF}}$ and $e^{-tH_{\rm PF}^\varepsilon}$. By using them we can construct the functional integral representation of the translation invariant Hamiltonian

$$H_{\rm PF}(P) = \frac{1}{2}(P - P_{\rm f} - e\mathcal{A}(0))^2 + H_{\rm rad} - \frac{e}{2}\sum_{\mu=1}^{3}\sigma_\mu \mathcal{B}_\mu(0).$$

Before going to do this, we show translation invariance of the operator $H_{\rm PF}^\varepsilon$ defined in (4.11).

Lemma 5.1 $H_{\rm PF}^\varepsilon$ *is translation invariant and it follows that*

$$H_{\rm PF}^\varepsilon = \int_{\mathbb{R}^3}^{\oplus} H_{\rm PF}^\varepsilon(P)dP,$$

where

$$H_{\rm PF}^\varepsilon(P) = H_{\rm PF}(P) - \frac{e}{2}\begin{bmatrix} 0 & \varepsilon\psi_\varepsilon(\mathcal{B}_1(0) - i\mathcal{B}_2(0)) \\ \varepsilon\psi_\varepsilon(\mathcal{B}_1(0) + i\mathcal{B}_2(0)) & 0 \end{bmatrix}. \tag{5.1}$$

PROOF: Let $\Phi = \Phi(x) = \mathcal{B}_1(\lambda(\cdot - x) - i\mathcal{B}_2(\lambda(\cdot - x))$. Note that

$$H_{\rm PF}^\varepsilon = H_{\rm PF} - \frac{e}{2}\begin{bmatrix} 0 & \varepsilon\psi_\varepsilon(\Phi) \\ \varepsilon\psi_\varepsilon(\bar\Phi) & 0 \end{bmatrix},$$

where $\bar\Phi$ denotes the complex conjugate of Φ. The term $H_{\rm PF}$ is translation invariant, therefore we only show that so is $\psi_\varepsilon(\Phi)$. We already know that there exists $\psi_\varepsilon^n \in \mathcal{S}(\mathbb{R})$ such that $\psi_\varepsilon^n(\Phi) \to \psi_\varepsilon(\Phi)$ strongly as a bounded multiplication operator when $n \to \infty$, where $\psi_\varepsilon^n(\Phi) = (2\pi)^{-1/2}\int_{\mathbb{R}} \hat\psi_\varepsilon^n(k)e^{ik\Phi}dk$. Thus ψ_ε^n is translation invariant, since Φ is. Hence $\psi_\varepsilon(\Phi)$ is also a translation invariant bounded multiplication operator. The proof for $\psi_\varepsilon(\bar\Phi)$ is similar.

Furthermore, $H_{\rm PF} + \psi_\varepsilon^n(\Phi)$ is decomposed as

$$H_{\rm PF} + \begin{bmatrix} 0 & \psi_\varepsilon^n(\Phi) \\ \psi_\varepsilon^n(\bar\Phi) & 0 \end{bmatrix} = \int_{\mathbb{R}^3}^{\oplus}\left(H_{\rm PF}(P) + \begin{bmatrix} 0 & \varepsilon\psi_\varepsilon^n(\Phi(0)) \\ \varepsilon\psi_\varepsilon^n(\bar\Phi(0)) & 0 \end{bmatrix}\right)dP.$$

Since $\psi_\varepsilon^n(\Phi(0))$ and $\psi_\varepsilon^n(\bar\Phi(0))$ converge strongly to $\psi_\varepsilon(\Phi(0))$ and $\psi_\varepsilon(\bar\Phi(0))$, respectively, (5.1) follows. **qed**

Theorem 5.2 *For $t \geq 0$ and $\Phi, \Psi \in \mathbb{Z}_2 \otimes L^2(\mathcal{Q})$ we have*

$$(\Phi, e^{-tH^\varepsilon_{\rm PF}(P)}\Psi) = e^t \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}^{0,\sigma}\left[e^{iP\cdot B_t}\int_{\mathcal{Q}_{\rm E}} d\mu_{\rm E}\overline{J_0\Phi(\sigma)}e^{X_t(\varepsilon)}J_t e^{-iP_{\rm f}\cdot B_t}\Psi(\sigma_t)\right] \quad (5.2)$$

and

$$(\Phi, e^{-tH_{\rm PF}(P)}\Psi) = \lim_{\varepsilon\to 0} e^t \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}^{0,\sigma}\left[e^{iP\cdot B_t}\int_{\mathcal{Q}_{\rm E}} \overline{J_0\Phi(\sigma)}e^{X_t(\varepsilon)}J_t e^{-iP_{\rm f}\cdot B_t}\Psi(\sigma_t)d\mu_{\rm E}\right]. \quad (5.3)$$

Proof: It suffices to show (5.2). The idea of proof is similar to that of Theorem 3.3 in [Hir06]. Set $F_s(\sigma) = \rho_s \otimes \Phi(\sigma)$ and $G_r(\sigma) = \rho_r \otimes \Psi(\sigma)$, where $\rho_s(x) = (2\pi s)^{-3/2}\exp(-|x|^2/(2s))$, $s > 0$, is the heat kernel, and $\Phi(\sigma), \Psi(\sigma) \in L^2_{\rm fin}(\mathcal{Q})$. We have by Lemma 5.1, for $\xi \in \mathbb{R}^3$,

$$(F_s, e^{-tH^\varepsilon_{\rm PF}}e^{-i\xi\cdot P^{\rm tot}}G_r)_{\mathcal{H}} = \int_{\mathbb{R}^3} dP((UF_s)(P), e^{-tH^\varepsilon_{\rm PF}(P)}e^{-i\xi\cdot P}(UG_r)(P))_{\mathbb{Z}_2\otimes\mathcal{F}},$$

where the unitary operator $U : \mathcal{H} \to \mathcal{H}$ is defined by

$$(UF_s)(P) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-ix\cdot P}e^{ix\cdot P_{\rm f}}\rho_s(x)\Psi(\sigma)dx.$$

Hence we have

$$\lim_{s\to 0}(F_s, e^{-tH^\varepsilon_{\rm PF}}e^{-i\xi\cdot P^{\rm tot}}G_r)_{\mathcal{H}} = (2\pi)^{-3/2}\int_{\mathbb{R}^3} dP(\Psi, e^{-tH^\varepsilon_{\rm PF}(P)}e^{-i\xi\cdot P}(UG_r)(P))_{\mathbb{Z}_2\otimes\mathcal{F}}. \quad (5.4)$$

On the other hand, we have through the functional integral representation (4.33),

$$(F_s, e^{-tH^\varepsilon_{\rm PF}}e^{-i\xi\cdot P^{\rm tot}}G_r)_{\mathcal{H}} = \int \rho_s(x)\Upsilon(x)dx,$$

where

$$\Upsilon(x) = \sum_\sigma \mathbb{E}^{x,\sigma}\left[\rho_r(B_t-\xi)\int_{\mathcal{Q}_{\rm E}} \overline{J_0\Psi(\sigma)}e^{X_t(\varepsilon)}J_t e^{-i\xi\cdot P_{\rm f}}\Phi(\sigma_t)d\mu_{\rm E}\right].$$

In Lemma 5.3 below we show that Υ is bounded and is continuous at $x = 0$. Thus further we obtain that

$$\lim_{s\to 0}\int \rho_s(x)\Upsilon(x)dx = \Upsilon(0) = \sum_\sigma \mathbb{E}^{0,\sigma}\left[\rho_r(B_t-\xi)\int_{\mathcal{Q}_{\rm E}} \overline{J_0\Psi(\sigma)}e^{X_t(\varepsilon)}J_t e^{-i\xi\cdot P_{\rm f}}\Phi(\sigma_t)d\mu_{\rm E}\right].$$

Hence, together with (5.4) we have

$$\begin{aligned}&(2\pi)^{-3/2}\int_{\mathbb{R}^3} dPe^{-i\xi\cdot P}(\Psi, e^{-tH^\varepsilon_{\rm PF}(P)}(UG_r)(P))_{\mathbb{Z}_2\otimes\mathcal{F}}\\ &\quad= \sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^{0,\sigma}[\rho_r(B_t-\xi)\overline{J_0\Psi(\sigma)}e^{X_t(\varepsilon)}J_t e^{-i\xi\cdot P_{\rm f}}\Phi(\sigma_t)]. \quad (5.5)\end{aligned}$$

Since $(\Psi, e^{-tH^{\varepsilon}_{\rm PF}(\cdot)}(UG_r)(\cdot))_{\mathbb{Z}_2\otimes\mathcal{F}} \in L^2(\mathbb{R}^3)$, by taking inverse Fourier transform on both sides of (5.5) we arrive at

$$\begin{aligned}&\Big(\Psi, e^{-tH^{\varepsilon}_{\rm PF}(P)}(UG_r)(P)\Big)_{\mathbb{Z}_2\otimes\mathcal{F}} \\ &\quad = (2\pi)^{-3/2}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^{0,\sigma}\left[\int_{\mathbb{R}^3} d\xi e^{i\xi\cdot P}\rho_r(B_t-\xi)\int_{\mathcal{Q}_{\rm E}}\overline{J_0\Psi(\sigma)}e^{X_t(\varepsilon)}J_t e^{-i\xi\cdot P_{\rm f}}\Phi(\sigma_t)d\mu_{\rm E}\right]\end{aligned} \tag{5.6}$$

for almost every $P \in \mathbb{R}^3$. Since both sides of (5.6) are continuous in P, the equality holds for all $P \in \mathbb{R}^3$. Taking $r \to 0$ on both sides of (5.6), we get the desired result. **qed**

We conclude by showing the lemma used above.

Lemma 5.3 Υ *is bounded and is continuous at* $x = 0$.

Proof: The boundedness is trivial, we proceed to show the continuity. We have

$$|\Upsilon(x)-\Upsilon(0)| \le \sum_\sigma \mathbb{E}^{0,\sigma}\left[\|\Psi(\sigma)\|_2\|\Phi(\sigma_t)\|_2\|e^{Z^x_t(\varepsilon)}-e^{Z^0_t(\varepsilon)}\|_1\right], \tag{5.7}$$

with

$$\begin{aligned} Z^x_t(\varepsilon) = & \underbrace{-ie\sum_{\mu=1}^3\int_0^t \mathcal{A}^{\rm E}_\mu(j_s\lambda(\cdot-B_s-x))dB^\mu_s}_{:=Z^x_t(1)} \underbrace{-\int_0^t H_{\rm d}(B_s+x,\sigma_s,s)ds}_{:=Z^x_t(2)} \\ & \underbrace{+\int_0^{t+}\log\left[-H_{\rm od}(B_s+x,-\sigma_{s-},s)-\varepsilon\chi_\varepsilon(B_s+x,\sigma_{s-},s)\right]dN_s}_{:=Z^x_t(3,\varepsilon)}. \end{aligned}$$

By (5.7) it is enough to show that

$$\lim_{x\to 0}\mathbb{E}^{0,\sigma}[\|e^{Z^x_t(\varepsilon)}-e^{Z^0_t(\varepsilon)}\|_1] = 0, \tag{5.8}$$

similarly to the proof of Theorem 4.11. We estimate $\mathrm{I}, \mathrm{II}, \mathrm{III}$ below:

$$\begin{aligned} e^{Z^x_t(\varepsilon)}-e^{Z^0_t(\varepsilon)} &= \underbrace{e^{Z^x_t(1)}e^{Z^x_t(2)}e^{Z^x_t(3,\varepsilon)}-e^{Z^0_t(1)}e^{Z^x_t(2)}e^{Z^x_t(3,\varepsilon)}}_{:=\mathrm{I}} \\ &\quad + \underbrace{e^{Z^0_t(1)}e^{Z^x_t(2)}e^{Z^x_t(3,\varepsilon)}-e^{Z^0_t(0)}e^{Z^0_t(2)}e^{Z^x_t(3,\varepsilon)}}_{:=\mathrm{II}} \\ &\quad + \underbrace{e^{Z^0_t(1)}e^{Z^0_t(2)}e^{Z^x_t(3,\varepsilon)}-e^{Z^0_t(1)}e^{Z^0_t(2)}e^{Z^0_t(3,\varepsilon)}}_{:=\mathrm{III}}. \end{aligned} \tag{5.9}$$

We have $\|e^{Z_t^x(2)}e^{Z_t^x(3,\varepsilon)}\|_2 \le e^{4(e/2)^2t^2\|\sqrt{|k|}\hat\varphi\|^2}c_3(\omega) := c_4(\omega)$, where $c_3(\omega)$ is given in (4.42), and

$$\|e^{Z_t^x(1)}-e^{Z_t^0(1)}\|_2^2 = 2-2\Re(e^{Z_t^x(1)},e^{Z_t^0(1)}) = 2-2\exp\left(-\frac{e^2}{2}q_1(\varrho_3^x,\varrho_3^x)\right),$$

where $\varrho_3^x = \oplus_{\mu=1}^3\int_0^t j_s(\lambda(\cdot-B_s-x)-\lambda(\cdot-B_s))dB_s^\mu$. Moreover,

$$\mathbb{E}^{0,\sigma}[q_1(\varrho_3^x,\varrho_3^x)] \le \frac{3}{2}\,\mathbb{E}^{0,\sigma}\left[\int_0^t \|\lambda(\cdot-B_s-x)-\lambda(\cdot-B_s)\|^2 ds\right] \to 0$$

as $x\to 0$. Thus

$$\begin{aligned}
\lim_{x\to 0}\mathbb{E}^{0,\sigma}\|I\|_1 &\le \lim_{x\to 0}\mathbb{E}^{0,\sigma}\|e^{Z_t^x(1)}-e^{Z_t^0(1)}\|_2\|e^{Z_t^x(2)}e^{Z_t^x(3,\varepsilon)}\|_2\\
&\le \lim_{x\to 0}\mathbb{E}^{0,\sigma}\|e^{Z_t^x(1)}-e^{Z_t^0(1)}\|_2\mathbb{E}^{0,\sigma}[c_4^{1/2}]\\
&\le \lim_{x\to 0}\mathbb{E}^{0,\sigma}[1-e^{-(e^2/2)q_1(\varphi(x),\varphi(x))}]\mathbb{E}^{0,\sigma}[c_4^{1/2}]\\
&\le \lim_{x\to 0}\mathbb{E}^{0,\sigma}[(e^2/2)q_1(\varphi(x),\varphi(x))]\mathbb{E}^{0,\sigma}[c_4^{1/2}] = 0.
\end{aligned}$$

Next we estimate II. We have

$$\begin{aligned}
&(e^{Z_t^x(2)},e^{Z_t^0(2)})_2\\
&= \exp\left(\frac{e^2}{2}\int_0^t ds\int_0^t dr\sigma_s\sigma_r\int dk\frac{|\hat\varphi(k)|^2}{\omega_{\rm b}(k)}e^{-ik(B_s-B_r-x)}(|k_1|^2+|k_2|^2)e^{-|s-r|\omega_{\rm b}(k)}\right)\\
&\to \|e^{Z_t^0(2)}\|_2^2
\end{aligned}$$

as $x\to 0$. Then from $\|e^{Z_t^x(2)}-e^{Z_t^0(2)}\|_2^2 = 2\|e^{Z_t^0(2)}\|_2^2 - 2\Re(e^{Z_t^x(2)},e^{Z_t^0(2)}) \to 0$ follows

$$\lim_{x\to 0}\|\mathrm{II}\|_1^2 \le c_3\lim_{x\to 0}\|e^{Z_t^x(2)}-e^{Z_t^0(2)}\|_2^2 = 0$$

for almost every $\omega\in\Omega$. Finally we estimate III. For each $\omega\in\Omega$, there exist $N = N(\omega)\in\mathbb{N}$ and $s_1 = s_1(\omega),...,s_N(\omega)\in(0,\infty)$ such that

$$e^{Z_t^x(3,\varepsilon)} = \prod_{i=1}^N\left(-H_{\rm od}^{\rm E}(x+B_{s_i},-\sigma_{s_i-},s_i)-\varepsilon\psi_\varepsilon\left(H_{\rm od}^{\rm E}(x+B_{s_i},-\sigma_{s_i-},s_i)\right)\right).$$

Since $H_{\rm od}^{\rm E}(x+B_{s_i},-\sigma_{s_i-},s_i)$ converges strongly to $H_{\rm od}^{\rm E}(B_{s_i},-\sigma_{s_i-},s_i)$ as $x\to 0$ in $L^2(\mathcal{Q}_{\rm E})$, we see that $\lim_{x\to 0}\psi_\varepsilon(H_{\rm od}^{\rm E}(x+B_{s_i},-\sigma_{s_i-},s_i)) = \psi_\varepsilon(H_{\rm od}^{\rm E}(B_{s_i},-\sigma_{s_i-},s_i))$ in $L^2(\mathcal{Q}_{\rm E})$. This can be proven in the same way as Lemma 4.12. Hence

$$\begin{aligned}
&\lim_{x\to 0}\prod_{i=1}^N\left(-H_{\rm od}^{\rm E}(x+B_{s_i},-\sigma_{s_i-},s_i)-\varepsilon\psi_\varepsilon\left(H_{\rm od}^{\rm E}(x+B_{s_i},-\sigma_{s_i-},s_i)\right)\right)\\
&=\prod_{i=1}^N\left(-H_{\rm od}^{\rm E}(B_{s_i},-\sigma_{s_i-},s_i)-\varepsilon\psi_\varepsilon\left(H_{\rm od}^{\rm E}(B_{s_i},-\sigma_{s_i-},s_i)\right)\right) \qquad (5.10)
\end{aligned}$$

follows. Thus we obtain $\lim_{x\to 0}\|e^{Z_t^x(3,\varepsilon)}-e^{Z_t^0(3,\varepsilon)}\|_2=0$ as well as $\lim_{x\to 0}\|\mathrm{III}\|_1\leq \lim_{x\to 0}\|e^{Z_t^x(3,\varepsilon)}-e^{Z_t^0(3,\varepsilon)}\|_2\|e^{Z_t^0(2)}\|_2=0$ for almost every $\omega\in\Omega$, proving (5.8). **qed**

From (5.3), we can derive energy inequalities in a similar manner to Corollary 4.13. Write

$$\inf\sigma(H_{\rm PF}(P))=E(P,\mathcal{A},\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3),$$

and define

$$H_{\rm PF}^{\perp}(P)=\frac{1}{2}(P-P_{\rm f})^2+H_{\rm rad}-\begin{bmatrix}\frac{e}{2}\mathcal{B}_3(0) & \frac{|e|}{2}\sqrt{\mathcal{B}_1(0)^2+\mathcal{B}_2(0)^2}\\ \frac{|e|}{2}\sqrt{\mathcal{B}_1(0)^2+\mathcal{B}_2(0)^2} & -\frac{e}{2}\mathcal{B}_3(0)\end{bmatrix}.$$

Corollary 5.4 *For $t\geq 0$*

$$|(\Phi,e^{-tH_{\rm PF}(P)}\Psi)|\leq\Big(|\Phi|,e^{-tH_{\rm PF}^{\perp}(0)}|\Psi|\Big) \tag{5.11}$$

and

$$\max\left\{\begin{array}{l}E(0,0,\sqrt{\mathcal{B}_1^2+\mathcal{B}_2^2},0,\mathcal{B}_3)\\ E(0,0,\sqrt{\mathcal{B}_3^2+\mathcal{B}_1^2},0,\mathcal{B}_2)\\ E(0,0,\sqrt{\mathcal{B}_2^2+\mathcal{B}_3^2},0,\mathcal{B}_1)\end{array}\right\}\leq E(P,\mathcal{A},\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3). \tag{5.12}$$

PROOF: Clearly, $|e^{-iP_{\rm f}\cdot B_t}\Psi|\leq e^{-iP_{\rm f}\cdot B_t}|\Psi|$. Therefore

$$\begin{aligned}|(\Phi,e^{-tH_{\rm PF}(P)}\Psi)| &\leq e^t\lim_{\varepsilon\to 0}\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}^{x,\sigma}\left[\int_{\mathcal{Q}_{\rm E}}(J_0|\Phi(\sigma)|)e^{X_t^{\perp}(\varepsilon)}(J_te^{-iP_{\rm f}\cdot B_t}|\Phi(\sigma_t)|)\right]d\mu_{\rm E}\\ &= \text{r.h.s. (5.11)}.\end{aligned}$$

(5.12) is immediate from (5.11). **qed**

6 Concluding remarks

It is known that $H_{\rm PF}$ has degenerate ground states for weak enough couplings [HS01, Hir06]. In this subsection we comment on the breaking of ground state degeneracy of a toy model by using the functional integral obtained in Theorem 4.11.

Consider the self-adjoint operator on $\mathcal{H}$ with the spin interaction replaced by the fermion harmonic oscillator (3.5) in $H_{\rm PF}$:

$$H(\epsilon)=\frac{1}{2}(-i\nabla-e\mathcal{A})^2+V+H_{\rm rad}+\epsilon\sigma_{\rm F}.$$

Whenever $\epsilon = 0$, the ground state of $H(0)$ is degenerate at any coupling. In this case

$$\begin{aligned}(F, e^{-tH(0)}G) &= e^t \lim_{\varepsilon\to 0} \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}\left[e^{-\int_0^t V(B_s)ds}(J_0F(\xi_0), e^{-iA}\varepsilon^{N_t}J_tG(\xi_t))\right] \\ &= e^t \sum_\sigma \int dx\, \mathbb{E}^{\sigma}\left[e^{-\int_0^t V(B_s)ds}(J_0F(x,\sigma), e^{-iA}J_tG(B_t,\sigma))\right].\end{aligned}$$

We show, however, that the ground state of $H(\epsilon)$ becomes unique for arbitrary values of coupling constants as soon as $\epsilon \neq 0$. Since the fermion harmonic oscillator $\sigma_{\rm F}$ is identical to $-\sigma_1$, the off-diagonal part of $H(\epsilon)$ is the non-zero constant $-\epsilon$. Then we have the functional integral representation of $e^{-tH(\epsilon)}$ with the exponent $X_t(0)$ in (4.33) replaced by

$$-ieA + \int_0^t \log \epsilon dN_s,$$

where $A = \mathcal{A}^{\rm E}(\oplus_{\mu=1}^3 \int_0^t j_s\lambda(\cdot - B_s)dB_s^\mu)$. Thus

$$(F, e^{-tH(\epsilon)}G) = e^t \sum_\sigma \int dx\, \mathbb{E}^{x,\sigma}[\epsilon^{N_t} e^{-\int_0^t V(B_s)ds}(J_0F(\xi_0), e^{-ieA}J_tG(\xi_t))].$$

Take the unitary operator $\theta = e^{-i(\pi/2)N}$. In [Hir00a] it was seen that $T_t := J_0^*\theta^{-1}e^{-iA}\theta J_t$ is positivity improving. This implies

Corollary 6.1 $\theta^{-1}e^{H(\epsilon)}\theta$ *is positivity improving for* $\epsilon > 0$ *and, in particular, the ground state of* $H(\epsilon)$, $\epsilon \neq 0$, *is unique whenever it exists.*

PROOF: Note that $H(\epsilon)$ and $H(-\epsilon)$ are isomorphic, therefore we only take $\epsilon > 0$. By a direct computation and the definition of T_t, we have

$$\begin{aligned}&(F, \theta^{-1}e^{-tH(\epsilon)}\theta G) \\ &\quad = e^t \sum_\sigma \int dx\, \mathbb{E}^{x}\left[e^{-\int_0^t V(B_s)ds} \times \right.\\ &\qquad \left. \times \left((F(x,\sigma), T_tG(B_t,\sigma))\cosh \epsilon t + (F(x,\sigma), T_tG(B_t,-\sigma))\sinh \epsilon t\right)\right].\end{aligned}$$

Then for non-zero $0 \le F, G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2 \times \mathcal{Q})$ we see that the right-hand side above is strictly positive, i.e., $(F, \theta^{-1}e^{-tH(\epsilon)}G) > 0$. This means that $e^{-tH(\epsilon)}$ is positivity improving. The uniqueness of the ground state follows by an application of the Perron-Frobenius theorem [GJ68, Gro72]. **qed**

The translation invariant version of the model is given by

$$H(\epsilon, P) := \frac{1}{2}(P - P_{\rm f} - e\mathcal{A}(0))^2 + H_{\rm rad} + \epsilon\sigma_{\rm F}.$$

The ground state of $H(0,P)$ is degenerate, whenever it exists, however in this case too the degeneracy is broken. By Theorem 5.2, the functional integral representation of $e^{-tH(\epsilon,P)}$ is given by

$$(\Psi, e^{-tH(\epsilon,P)}\Phi) = e^t \sum_{\sigma\in\mathbb{Z}_2} \mathbb{E}^{0,\sigma}\left[\epsilon^{N_t} e^{iP\cdot B_t}(J_0\Phi(\sigma), e^{-iA}J_t e^{-iP_{\rm f}\cdot B_t}\Psi(\sigma_t))\right]. \tag{6.1}$$

If $P=0$, the phase $e^{iP\cdot B_t}$ vanishes. Then, since $e^{-iP_{\rm f}\cdot B_t}$ is positivity preserving in $\mathcal{Q}$-representation, similarly to Corollary 6.1 we see that for $P=0$ and $\epsilon>0$, $\theta^{-1}e^{-tH(\epsilon,0)}\theta$ is positivity improving. This yields

Corollary 6.2 *Let $P=0$ and $\epsilon\neq 0$. Then $\theta^{-1}e^{-tH(\epsilon,0)}\theta$ is positivity improving and the ground state of $H(\epsilon,0)$ is unique, whenever it exists.*

Remark 6.3 The spin-boson model is defined by

$$H_{\rm SB} = \sigma_1\otimes 1 + 1\otimes H_{\rm f} + \alpha\sigma_3\otimes\phi(f),\ \alpha\in\mathbb{R},$$

on $\mathbb{C}^2\otimes\mathcal{F}(L^2(\mathbb{R}^3))$, where $H_{\rm f}$ is the free field Hamiltonian of $\mathcal{F}(L^2(\mathbb{R}^3))$ and $\phi(f)$ is the field operator labeled by $f\in L^2(\mathbb{R}^3)$. We can also construct the functional integral representation of $e^{-tH_{\rm SB}}$ by making use of the $\mathbb{Z}_2$-valued jump process σ_t. The functional integral can then be used to prove uniqueness of the ground state whenever it exists [Spo89, Hik99, Hik01, HH07].

7 Appendix: Itô formula for Lévy processes

In this appendix we recall and discuss some basic facts on Poisson processes and related Itô formulas to make this paper sufficiently self-contained. A general reference on this subject is [IW81, DV07].

Let $(S,\Sigma,P_{\rm P})$ be a complete probability space with a right-continuous increasing family of sub-σ-fields $(\Sigma_t)_{t\geq 0}$, where each Σ_t contains all $P_{\rm P}$-null sets. Also, let $(\mathcal{X},\mathcal{B}_{\mathcal{X}})$ be a measurable space and ϖ the set of $\mathbb{Z}_+\cup\{\infty\}$-valued measures on $(\mathcal{X},\mathcal{B}_{\mathcal{X}})$. Denote by $\mathcal{B}_\varpi$ be the smallest σ-field on ϖ such that $\varpi\ni\mu\mapsto\mu(B)$, $B\in\mathcal{B}_{\mathcal{X}}$, are measurable.

We define a class of measure-valued random variables.

Definition 7.1 *The $(\varpi,\mathcal{B}_\varpi)$-valued random variable N on $(S,\Sigma,P_{\rm P})$ is a* Poisson random measure *on $(\mathcal{X},\mathcal{B}_{\mathcal{X}})$ whenever the conditions below are satisfied:*

(1) $P(N(A)=n) = e^{-\Lambda(A)}\Lambda(A)^n/n!$, $A\in\mathcal{B}_{\mathcal{X}}$, where $\Lambda(A) := \mathbb{E}_{\rm P}[N(A)]$,

(2) if $A_1, ..., A_n \in \mathcal{B}_{\mathcal{X}}$ *are pairwise disjoint, then* $N(A_1), ..., N(A_n)$ *are independent.*

$\Lambda(A)$ is called the *intensity* of $N(A)$, and $\mathbb{E}_{\mathrm{P}}[e^{-\alpha N(A)}] = e^{\Lambda(A)(e^{-\alpha}-1)}$ holds.

Fix a measurable space $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$. By an $\mathcal{M}$-valued point function p we mean a map $p : D(p) \to \mathcal{M}$, where the domain $D(p)$ is a countable subset of $(0, \infty)$. Define the counting measure $N_p(dtdm)$ on the measure space $((0,\infty) \times \mathcal{M}, \mathcal{B}_{(0,\infty)} \times \mathcal{B}_{\mathcal{M}})$ by

$$N_p(t, U) := N_p((0,t] \times U) = \#\{s \in D(p) \,|\, s \in (0,t], p(s) \in U\}, \quad t > 0,\ U \in \mathcal{B}_{\mathcal{M}},$$

where $\mathcal{B}_{(0,\infty)}$ is the Borel σ-field on $(0,\infty)$. Let $\Pi(\mathcal{M})$ denote the set of all point functions on $\mathcal{M}$, and $\mathcal{B}_{\Pi(\mathcal{M})}$ be the smallest σ-field on $\Pi(\mathcal{M})$ with respect to which $p \longmapsto N_p(t, U)$, $t > 0$, $U \in \mathcal{B}_{\mathcal{M}}$, are measurable.

Definition 7.2 *A* $(\Pi(\mathcal{M}), \mathcal{B}_{\Pi(\mathcal{M})})$*-valued random variable* p *on* $(S, \Sigma, P_{\mathrm{P}})$ *is called an* $\mathcal{M}$*-valued point process on* $(S, \Sigma, P_{\mathrm{P}})$*.*

The point process p is called a *stationary point process* if and only if $p(\cdot)$ and $p(s + \cdot)$ have the same law for all $s \geq 0$, with $D(p(s + \cdot)) = \{t \in (0,\infty) \,|\, s + t \in D(p)\}$.

Definition 7.3 *An* $\mathcal{M}$*-valued point process* p *on* $(S, \Sigma, P_{\mathrm{P}})$ *is called a* Poisson point process *if and only if the counting measure* $N_p(dtdm)$ *is a Poisson random measure on* $((0,\infty) \times \mathcal{M}, \mathcal{B}_{(0,\infty)} \times \mathcal{B}_{\mathcal{M}})$*.*

It is known that a Poisson point process p is stationary if and only if its intensity measure is of the form

$$\mathbb{E}_{\mathrm{P}}[N_p(dtdm)] = dtn(dm) \tag{7.1}$$

for some measure n on $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$. An $\mathcal{M}$-valued point process p on $(S, \Sigma, P_{\mathrm{P}})$ is called (Σ_t)*-adapted* if for every $t > 0$ and $U \in \mathcal{B}_{\mathcal{M}}$, $N_p(t, U)$ is Σ_t measurable for all $t > 0$. It is called σ*-finite* if there exists $U_n \in \mathcal{B}_{\mathcal{M}}$, $n = 1, 2, ...$, such that $U_n \uparrow \mathcal{M}$ and $\mathbb{E}_{\mathrm{P}}[N_p(t, U_n)] < \infty$, for all $t > 0$ and $n = 1, 2, ...$ Let p be a (Σ_t)-adapted, σ-finite point process. When $\mathbb{E}_{\mathrm{P}}[N_p(t, U)] < \infty$, $\forall t > 0$, there exists a natural integrable increasing process $(\hat{N}_p(t, U))_{t \geq 0}$ on $(S, \Sigma, P_{\mathrm{P}})$ such that

$$N_p(t, U) - \hat{N}_p(t, U) := \tilde{N}_p(t, U)$$

is a martingale. $\hat{N}_p(t, U)$ is called the *compensator* of point process p.

Definition 7.4 *An $\mathcal{M}$-valued point process p on $(S, \Sigma, P_{\rm P})$ is called a (Σ_t)-*Poisson point process *if it is an (Σ_t)-adapted, σ-finite Poisson point process such that the increments*

$$\{N_p(t+h, U) - N_p(t, U) :\ h > 0, U \in \mathcal{B}_{\mathcal{M}}\}$$

are independent of Σ_t.

Let p be a (Σ_t)-Poisson point process. Then if $t \mapsto \mathbb{E}_{\rm P}[N_p(t, U)]$ is continuous, it holds that $\hat{N}_p(t, U) = \mathbb{E}_{\rm P}[N_p(t, U)]$. In particular, a stationary (Σ_t)-Poisson point process has the compensator $\hat{N}_p(t, U) = tn(U)$, where n is that of (7.1), and for a disjoint family of U_i in Σ, $i = 1, ..., N$,

$$\mathbb{E}_{\rm P}\left[e^{-\sum_{i=1}^N \alpha_i N_p((s,t]\times U_i)}\right] = \exp\left((t-s)\sum_{i=1}^N (e^{-\alpha_i} - 1) n(U_i)\right).$$

Fix a stationary (Σ_t)-Poisson point process p on $(S, \Sigma, P_{\rm P})$ with values in $\mathcal{M}$. In Section 3 we set $(\Omega, \mathcal{B}_\Omega, P_\Omega) := (W \times S, \mathcal{B}_W \times \Sigma, P_W^0 \otimes P)$ and $\omega := w \times \tau \in W \times S = \Omega$. Let Π be the smallest σ-field on $[0, \infty) \times \mathcal{M} \times \Omega$ such that all g having the properties below are measurable:

(1) for each $t > 0$, $(m, \omega) \mapsto g(t, m, \omega)$ is $\mathcal{B}_{\mathcal{M}} \times \Omega_t$ measurable,

(2) for each (m, ω), $t \mapsto g(t, m, \omega)$ is left continuous.

Definition 7.5 *A Π-measurable function $h : [0, \infty) \times \mathcal{M} \times \Omega \to \mathbb{R}$ is called (Ω_t)-*predictable *and their set is denoted by $\Omega_{\rm pred}$.*

Write

$$\mathbb{F} := \left\{ f \in \Omega_{\rm pred} \,|\, \int_0^{t+} \int_{\mathcal{M}} |f(s, m, \omega)| N_p(dsdm) < \infty \ \text{ for } t > 0, \text{ a.e. } \omega \right\},$$
$$\mathbb{F}^2 := \left\{ f \in \Omega_{\rm pred} \,|\, \mathbb{E}_\Omega\left[\int_0^t \int_{\mathcal{M}} |f(s, m, \omega)|^2 \hat{N}_p(dsdm)\right] < \infty \ \text{ for } t > 0 \right\}$$

and

$$\mathbb{F}^{2,\rm loc} := \left\{ f \in \Omega_{\rm pred} \,|\, \exists\, \tau_n \ \ (\Omega_t)\text{−stopping times} :\ \tau_n \uparrow \infty \ \text{ and } 1_{[0,\tau_n]}(t) f(t, m, \omega) \in \mathbb{F}^2 \right\}.$$

Let $f^i(t, \omega)$ and $g^i(s, \omega)$ be adapted with respect to (Ω_t), $\mathbb{E}_\Omega[\int_0^t |f^i(s, \cdot)|^2 ds] < \infty$ and $g^i(\cdot, \omega) \in L^1_{\rm loc}(\mathbb{R})$ for a.e. $\omega \in \Omega$. Furthermore, take $h_1^i \in \mathbb{F}$ and $h_2^i \in \mathbb{F}^{2,\rm loc}$. Define the semi-martingale $X_t = (X_t^1, ..., X_t^d)$ on $(\Omega, \mathcal{B}_\Omega, P_\Omega)$ by

$$X_t^i = \int_0^t f^i(s,\omega)dB_s^i + \int_0^t g^i(s,\omega)ds \qquad (7.2)$$
$$+ \int_0^{t+}\int_{\mathcal{M}} h_1^i(s,m,\omega)N_p(dsdm) + \int_0^{t+}\int_{\mathcal{M}} h_2^i(s,m,\omega)\tilde{N}_p(dsdm).$$

Here $\tilde{N}_p(dsdm) = N_p(dsdm) - dsn(dm)$.

Proposition 7.6 *Let $F \in C^2(\mathbb{R}^d)$ and $X_t = (X_t^1, ..., X_t^d)$ be given by (7.2). Suppose $h_1^i \in \mathbb{F}$, $h_2^j \in \mathbb{F}^{2,loc}$, and $h_1^i h_2^j = 0$ for $i, j = 1, ..., d$. Then $F(X_t)$ is a semimartingale and the following Itô formula holds:*

$$dF(X_t) = \sum_{i=1}^d \sum_{\mu=1}^3 \int_0^t \frac{\partial F(X_s)}{\partial x_i} f_\mu^i(s,\omega)dB_s^\mu$$
$$+\sum_{i=1}^d \int_0^t \frac{\partial F(X_s)}{\partial x_i} g^i(s,\omega)ds + \frac{1}{2}\sum_{i=1}^d \int_0^t \frac{\partial^2 F(X_s)}{\partial x_i^2}(f^i(s,\omega))^2 ds$$
$$+\int_0^{t+}\int_{\mathcal{M}} (F(X_{s-} + h_1(s,m,\omega)) - F(X_{s-})\, N_p(dsdm)$$
$$+\int_0^{t+}\int_{\mathcal{M}} (F(X_{s-} + h_2(s,m,\omega)) - F(X_{s-}))\, \tilde{N}_p(dsdm)$$
$$+\int_0^t \int_{\mathcal{M}} \left(F(X_s + h_2(s,m,\omega)) - F(X_s) - \sum_{i=1}^d h_2^i(s,m,\omega)\frac{\partial F(X_s)}{\partial x_i}\right) \hat{N}_p(dsdm),$$

where $\hat{N}_p(dsdm) = dsn(dm)$.

Proof: See, e.g., [IW81, Theorem 5.1]. **qed**

Write (7.2) as $dX^i = f^i dB^i + g^i dt + \int_{\mathcal{M}} h_1^i dN + \int_{\mathcal{M}} h_2^i d\tilde{N}$ in concise notation. Let $d = 1$, $B_t^1 = B_t$ and

$$dZ = u_Z dt + v_Z dB + \int_{\mathcal{M}} f_Z dN + \int_X g_Z d\tilde{N},$$
$$dY = u_Y dt + v_Y dB + \int_{\mathcal{M}} f_Y dN + \int_X g_Y d\tilde{N}$$

with $f_Z g_Z = 0$, $f_Z g_Y = 0$, $f_Y g_Y = 0$ and $f_Y g_Z = 0$. Then by Proposition 7.6 we have the product rule

$$d(ZY) = Z_s u_Y ds + Z_s v_Y dB_s + \int_{\mathcal{M}} Z_{s-} f_Y N_p(dsdm) + \int_{\mathcal{M}} Z_{s-} g_Y \tilde{N}_p(dsdm)$$
$$+Y_s u_Z ds + Y(s) v_Z dB_s + \int_{\mathcal{M}} Y_{s-} f_Z N_p(dsdm) + \int_{\mathcal{M}} Y(s-) g_Z \tilde{N}_p(dsdm)$$
$$+v_Z v_Y ds + \int_{\mathcal{M}} (f_Z f_Y + g_Z g_Y) N_p(dsdm).$$

This formula is written as $d(ZY) = dZ \cdot Y + Z \cdot dY + dZ \cdot dY$ in the concise notation.

Suppose $n(\mathcal{M}) = 1$ and set $N_t := N_p((0,t] \times \mathcal{M})$ and $dN_t := \int_{\mathcal{M}} N_p(dtdm)$ as mentioned in Section 3.2. Then the compensator of p is given by $\hat{N}_p(t, \mathcal{M}) = t$ and $\mathbb{E}_\Omega[e^{-\alpha N_t}] = e^{t(e^{-\alpha}-1)}$. Moreover,

$$\mathbb{E}_\Omega\left[\int_0^{t+}\int_{\mathcal{M}} f(s,\omega,m)N_p(dsdm)\right] = \mathbb{E}_\Omega\left[\int_0^t\int_{\mathcal{M}} f(s,\omega,m)dsn(dm)\right].$$

Hence we have for $f = f(s,\omega)$ independent of $m \in \mathcal{M}$,

$$\mathbb{E}_\Omega\left[\int_0^{t+} f(s,\omega)dN_s\right] = \mathbb{E}_\Omega\left[\int_0^t f(s,\omega)ds\right]. \tag{7.3}$$

Furthermore, Proposition 7.6 gives

Proposition 7.7 *Suppose $h^i \in \mathbb{F}$, $i = 1, ..., d$, are independent of $m \in \mathcal{M}$. Let $dX^i = f^i_\mu dB^\mu + g^i dt + h^i dN$, $i = 1, ..., d$, and $F \in C^2(\mathbb{R}^d)$. Then*

$$\begin{aligned} dF(X_t) &= \sum_{i=1}^d\sum_{\mu=1}^3\int_0^t \frac{\partial F(X_s)}{\partial x_i} f^i_\mu(s,\omega)dB^\mu_s \\ &\quad + \sum_{i=1}^d \int_0^t \frac{\partial F(X_s)}{\partial x_i} g^i(s,\omega)ds + \frac{1}{2}\sum_{i=1}^d\int_0^t \frac{\partial^2 F(X_s)}{\partial x_i^2}(f^i(s,\omega))^2 ds \\ &\quad + \int_0^{t+} \left(F(X_{s-} + h(s,\omega)) - F(X_{s-})\right) dN_s. \end{aligned}$$

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