Strong Stability of KAM Tori
— from the view of Viscosity Solutions of H-J equations

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Abstract

In this paper, we prove the strong stability of Diophantine KAM tori in the view of viscosity solutions of Hamilton-Jacobi equations.

1 Introduction

The objective of this paper is to study the changes of the graphs of viscosity solutions of Hamilton-Jacobi equations

$$H(x,P+Du(x,P))=\overline{H}(P). \tag{1.1}$$

In (1.1), $H(x,p):\mathbb{R}^{2n}\to\mathbb{R}$ is a smooth Hamiltonian, strictly convex, i.e. $\frac{\partial^2 H}{\partial p^2}>\varsigma I>0$ uniformly, and superlinear growth in $p(\lim_{|p|\to\infty}\frac{H(x,p)}{\|p\|}=\infty)$, and $2\pi\mathbb{Z}^n$ periodic in x. Instead of studying a general Hamiltonian H as above, in this paper we will restrict us in the real analytic Lagrangian

$$L_0(x,\dot{x})=l_0(\dot{x})+\epsilon l_1(x,\dot{x}), \tag{1.2}$$

of which associated Hamiltonian is

$$H_0(x,p)=h_0(p)+\epsilon h_1(x,p), \tag{1.3}$$

where

$$\frac{\partial^2 l_0}{\partial \dot{x}^2}>0. \tag{1.4}$$

Except that, we also restrict us around the graph of a smooth viscosity solution, which is the so-called KAM torus. In [11], for the Lagrangian systems (1.2), Salamon and Zehnder proved that for any Diophantine frequency vector $\omega\in\mathbb{R}^n$, there exists an invariant torus Γ, which is in $T\mathbb{T}^n$, corresponds to it. Write $\tilde{\mathcal{L}}(\Gamma)=\mathcal{G}$, where $\tilde{\mathcal{L}}:\ T\mathbb{T}^n\to T^*\mathbb{T}^n$ is the

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Legendre transformation. In fact, $\mathcal{G}$ is a smooth graph of some viscosity solution. We can write $\mathcal{G}=\bigcup_{x\in\mathbb{T}^n}(x,P_0+Du(x,P_0))$, where $u(x,P_0)$ satisfies the Hamilton-Jacobi equation

$$H_0(x,P_0+Du(x,P_0))=\overline{H_0}(P_0). \tag{1.5}$$

From [5][1], we get many viscosity solutions for (1.5) for any P, where $\|P-P_0\|$ is small enough. Our problem is what the graphs of $\bigcup_{x\in\mathbb{T}^n}(x,P+Du(x,P))$ look like? What is the relationship between the graphs and the KAM torus $\mathcal{G}$? We will answer these problems in our theorems(see Theorem 1,2). In the following, we will give a heuristic description about our results. We remark that the notations in this section are independent of the following ones.

Suppose that ω satisfies

$$\begin{aligned}|\langle k,\omega\rangle|\geq\frac{\gamma}{|k|^\tau},\ k\neq 0,\\ \gamma>0,\ \tau\geq n-1.\end{aligned} \tag{1.6}$$

When $\|P-P_0\|$ small enough, we will have

$$\begin{aligned}\|Du(x,P)-Du(x,P_0)\|\leq C\|P-P_0\|^{\frac{1}{\tau+1}},\\ \|u(x,P)-u(x,P_0)\|\leq C\|P-P_0\|^{\frac{1}{\tau+1}}.\end{aligned}$$

Further, one gets

$$\|(P+Du(x,P))-(P_0+Du(x,P_0))\|\leq C\|P-P_0\|^{\frac{1}{\tau+1}}. \tag{1.7}$$

Definition 1.1 *For the H-J equation (1.1) and some $P_0\in R^n$, if the graph of $\bigcup_{x\in\mathbb{T}^n}(x,P_0+Du(x,P_0))$ corresponds to a KAM torus and this torus and the graphs of its nearby viscosity solutions satisfy*

$$\|(P+Du(x,P))-(P_0+Du(x,P_0))\|\leq C\|P-P_0\|^{\chi},\qquad 0<\chi\leq 1, \tag{1.8}$$

then we call this KAM torus strong stability, where $\|P-P_0\|$ is small enough and $x\in\Delta\subset\mathbb{T}^n$ and meas$(\{x\in\mathbb{T}^n\setminus\Delta\})=0$. χ is called the strong stability index.

Remark 1.1 *The definition for the strong stability of KAM torus is local. Therefore, the conditions of H, which are uniformly convex and superlinear growth in p, aren't necessary.*

From (1.7) and above discussions, we have known that the KAM torus $\mathcal{G}$ in $T^*\mathbb{T}^n$ is strong stability. We remark that the strong stability of KAM torus in the view of viscosity solutions of H-J equations has deep relationships with the stickiness of KAM torus(see [10], also see [9]) and the minimal property of the trajectories which lie in KAM tori(see [7]).

[1]Note here, we need neither that H_0 is superlinear in p, nor that H_0 is uniformly convex. We only need $\frac{\partial^2 h_0}{\partial p^2}>0$. The reason lies in that we only care about the dynamics of the small neighborhood of $\mathcal{G}$.

Let us close this introductory section with one important reference. In the Corollary 8.3 of [6], J. Mather has shown that if ω satisfies a Diophantine condition of order τ, then $\omega\to P_\omega(\xi)$ satisfies a Hölder condition of order $\frac{1}{2\tau}$ at ω, i.e. $|P_\omega(\xi)-P_\rho(\xi)|\le const.|\omega-\rho^*|^{\frac{1}{2\tau}}$, for $|\omega-\rho^*|\le 1$, where $P_\omega(\cdot)$ is Peierl's barrier. It is well-known that the barrier function can be represented by viscosity solutions. Our results about C^0 estimation partially generalize his result to high dimensional positive definite Hamiltonian systems.

2 Main Results

2.1 Theorem 1

We start from the Hamiltonian

$$\begin{aligned} H(x,p)=&\langle\omega,p\rangle+\frac{1}{2}\langle A(x)p,p\rangle+f(x,p)\\ =&N+R_1+R_2 \end{aligned} \tag{2.1}$$

where $N=\langle\omega,p\rangle$, $R_1=\frac{1}{2}\langle A(x)p,p\rangle$ and $R_2=f(x,p)=\mathcal{O}(p^3)$. H is assumed to be defined and real analytic in a neighbourhood of the origin, more precisely on a complex domain $D=D(R,\rho,\sigma)(\rho>0,\ \sigma>0)$ defined as follows: let us employ the Euclidean norm on complex numbers z and the max norm on complex vectors $\xi=(\xi_1,\cdots,\xi_n)$: $|z|=\{[Re(z)]^2+[Im(z)]^2\}^{\frac{1}{2}}$ and $|\xi|=\max\limits_{j=1,\cdots,n}|\xi_j|$. Denote by $\mathbb{T}^n+\sigma$ the complex σ-neighborhood of $\mathbb{T}^n$:

$$\mathbb{T}^n+\sigma=\{q\in\mathbb{C}^n/2\pi\mathbb{Z}^n||Im(q_i)|<\sigma,\ \forall j\}.$$

Similarly, for all $r<R$ denote by $B_r+\rho$ the ρ-neighborhood of B_r in $\mathbb{C}^n$:

$$B_r+\rho=\{p\in\mathbb{C}^n|\exists\ p'\in B_r \text{ such that } |p_j-p_j'|<\rho,\ \forall j\}.$$

For the combined complexified domain we write $D(r,\rho,\sigma)=(\mathbb{T}^n+\sigma)\times(B_r+\rho)$. A norm on the bounded complex-valued functions on $D(r,\rho,\sigma)$ is given by

$$\|F\|_{r,\rho,\sigma}=\sup_{(x,p)\in D(r,\rho,\sigma)}|F(x,p)|.$$

The Hamiltonian (2.1) is defined in $D(R,\rho,\sigma)$, where R will be chosen small enough. If $|p|\ge\Xi$, we define[2] $H=\frac{1}{2}\|p\|^2$, where Ξ will be chosen large enough. We extend H which defined in $\{(x,p)|x\in\mathbb{T}^n,\ |p|<R\}$ to $\{(x,p)|x\in\mathbb{T}^n,\ |p|\ge\Xi\}$ by a suitable smooth function and pertain the positive definiteness. We still write the new Hamiltonian by H for simplicity. Obviously, H is superlinear in p.

Suppose $A(x)(x\in\mathbb{T}^n)$ is a positive definite and symmetric matrix and satisfies

$$\lambda_2\|v\|^2\le\langle A(x)v,v\rangle\le\lambda_1\|v\|^2,\qquad v\in\mathbb{R}^n,$$

where $\lambda_2>0$. Further, we suppose (1.6). When $|p|<R$, the Hamilton equation of H is

$$\begin{cases}\dot{x}=\omega+A(x)p+\frac{\partial f}{\partial p}\\ \dot{p}=-\frac{1}{2}\langle\nabla A(x)p,p\rangle-\frac{\partial f}{\partial x},\end{cases} \tag{2.2}$$

[2] In this paper, define $\|p\|=(\sum\limits_{i=1}^n|p_i|^2)^{\frac{1}{2}}$.

where $\nabla A=(\frac{\partial A}{\partial x_1},\cdots,\frac{\partial A}{\partial x_n})^T$. When $p=0$, the above Hamiltonian equation admits a KAM torus with a Diophantine rotation number ω. It is well-known that the cell equation (1.1) admits viscosity solutions for any P. When $P=0$, it is clear that $H(x,0)=0$. This means that $u\equiv c$ is a smooth viscosity solution of (1.1) for $P=0$. From Lemma 5.3, we have the unique viscosity solution(mod constant) $u(x,0)=c$. Its graph $\bigcup_{x\in\mathbb{T}^n}(x,0)$ corresponds to the Diophantine KAM tori mentioned above[3].

Theorem 1 *For any $0<\delta\leq 1$, if $\|P\|\leq\min\{\epsilon_0,\eta_0,\eta_1\}$, then*

$$\|D_x u(x,P)\|\leq C\delta^{-1}\|P\|^{\frac{1}{\tau+1}},$$

for any $u(x,P)$ satisfying (1.1) and $x\in dom(Du(x,P))$[4].

Remark 2.1 *The constants ϵ_0, η_0 and η_1 will be explained in the following sections.*

Remark 2.2 *For $0<\delta\leq K_0$, the result is similar, where K_0 is any large constant.*

Define

$$\|u(x,P)-u(x,0)\|=\inf_c|u(x,P)-u(x,0)-c|,$$

then we have the following conclusion.

Corollary 1 *For any $0<\delta\leq 1$, if $\|P\|\leq\min\{\epsilon_0,\eta_0,\eta_1\}$, then*

$$\|u(x,P)-u(x,0)\|\leq C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}.$$

2.2 Theorem 2

In this subsection, we will give another important theorem. Consider the following Lagrangian systems

$$\frac{d}{dt}\frac{\partial L_0}{\partial\dot{x}}=\frac{\partial L_0}{\partial x}$$

with the Lagrangian

$$L_0(x,\dot{x})=l_0(\dot{x})+\epsilon l_1(x,\dot{x}), \tag{2.3}$$

where $\frac{\partial^2 l_0}{\partial\dot{x}^2}>0$. $L_0(x,v)$ is a real analytic function in the domain $|Imx|\leq 2\lambda_0 r_0$, $|Imv|\leq 2\lambda_0 r_0$ which is of period 2π in the $x-$variables. Let $\omega\in\mathbb{R}^n$ satisfying $\|\omega\|\leq M_0$, $|\langle k,\omega\rangle|\geq\frac{\gamma}{|k|^\tau}$, $0\neq k\in\mathbb{Z}^n$, for some constants $M_0\geq 1$, $\gamma_0>0$, $\tau\geq n-1$. $|\partial^\alpha L_0|_{2\lambda_0 r_0}\leq M_0$, $|\alpha|\leq 4$. From Theorem 1 in [11], we have the following conclusions: $\exists\delta^*=\delta^*(r_0,\ \tau,\ M_0,\ \lambda_0,\ n)>0$ and $c=c(r_0,\ \tau,\ M_0,\ \lambda_0,\ n)\geq 8M_0^3$ such that $c\delta^*\leq 1$. If $c_0\epsilon\leq\delta^*$, then there exits a real analytic torus diffeomorphism $x=f(\xi)$ mapping the strip $|Im\xi|\leq\frac{r_0}{2}$ into $|Imf(\xi)|\leq 2\lambda_0 r_0$, $|Im\mathcal{D}f(\xi)|\leq 2\lambda_0 r_0$ such that $f(\xi)$–ξ is of period 2π and $\mathcal{D}(L_0)_p(f,\mathcal{D}f)=$

[3]From Proposition 5.1 in the appendix, one gets $\|P+D_x u(x,P)\|\to 0$ when $\|P\|\to 0$. This means when $\|P\|$ is small, the Hamiltonian in the cell equation, which $u(x,P)$ satisfies, is the original Hamiltonian H.

[4]the notation dom(Du(x,P)) means the domain of definition of Du(x,P), i.e. the set of the points x where the derivative $D_x u(x,P)$ exists.

$(L_0)_x(f,\mathcal{D}f)$, where c_0 is a constant depending on L_0 and M and $\mathcal{D}=\sum_{j=1}^{n}\omega_j\frac{\partial}{\partial\xi_j}$. Moreover, the pair (L_0,f) is stable and satisfies the estimates

$$\begin{aligned} \|f-f_0\|_{\frac{r_0}{2}} &\le cc_0\epsilon r_0^{2\tau},\\ \|U-U_0\|_{\frac{r_0}{2}} &\le cc_0\epsilon r_0^{2\tau-1},\\ |U^T(L_0)_{pp}(f,\mathcal{D}f)U-a|_{\frac{r_0}{2}} &\le \frac{cc_0\epsilon}{4M_0^3}, \end{aligned} \tag{2.4}$$

where we denote $\|u\|_r=|u|_r+|\mathcal{D}u|_r+|\mathcal{D}^2u|_r$ and $U_0=\frac{\partial f_0}{\partial\xi}$ and $U=\frac{\partial f}{\partial\xi}$. For our conveniences, write the initial torus $\Gamma_0=\bigcup_{\xi\in\mathbb{T}^n}(f(\xi),Df\cdot\omega)$. For more concretely, please see [11].

From the above, it is easy to check that the Lagrangian equation

$$\frac{d}{dt}\frac{\partial L_1}{\partial\dot\xi}=\frac{\partial L_1}{\partial\xi} \tag{2.5}$$

has the solution $(\xi_0+\omega t,\omega)$, $\xi_0\in\mathbb{T}^n$, where $L_1(\xi,\dot\xi)=L_0(f(\xi),\frac{\partial f}{\partial\xi}\dot\xi)$. Write $\bigcup_{\xi\in\mathbb{T}^n}(\xi,\omega)=\Gamma$. From [7], $\exists$ a closed 1-form η, $[\eta]=P_0$, such that

$$(L_1-\eta)|_\Gamma=0,\ (L_1-\eta)|_{\notin\Gamma}>0. \tag{2.6}$$

Write $L_2=L_1-\eta$. From (2.6), we have

$$L_2=(\dot\xi-\omega)^T\frac{\partial^2L_2}{\partial\xi^2}(\xi,\omega)(\dot\xi-\omega)+\mathcal{O}(\dot\xi-\omega)^3.$$

Write $\frac{\partial^2L_2}{\partial\xi^2}(\xi,\omega)=\frac{1}{2}A^{-1}(\xi)$. Clearly, $A^{-1}(\xi)>0$. Therefore,

$$L_2=\frac{1}{2}(\dot\xi-\omega)^TA^{-1}(\xi)(\dot\xi-\omega)+\mathcal{O}(\dot\xi-\omega)^3.$$

And its associated Hamiltonian is

$$H_2(\xi,p)=\langle\omega,p\rangle+\frac{1}{2}\langle A(\xi)p,p\rangle+\mathcal{O}(p^3).$$

Since $L_0(x,v)$ is real analytic, it is easy to see that there exist $r'>0$ and $\sigma'>0$ and $H_2(\xi,p)$ is real analytic in $(\mathbb{T}^n+\sigma')\times(B_{r'}(0)+\rho')$, where

$$B_{r'}(0)+\rho'=\{p\in\mathbb{C}^n|\exists\ p'\in B_{r'}(0)\text{ such that }|p_j-p_j'|\le\rho',\ \forall j\}.$$

Clearly, we can choose r', σ' and ρ' small enough such that Theorem 1 can been used. Therefore, for any $u(\xi,P)$ satisfying the equation $H_2(\xi,P+Du(\xi,P))=\overline{H}_2(P)$, we have

$$\|Du(\xi,P)\|\le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}, \tag{2.7}$$

for $\forall\xi\in Dom(Du,P)$ and $\forall\delta\in(0,1]$ and for $\|P\|$ small enough. From Corollary 1, we also get that for $\forall\delta\in(0,1]$ and $\|P\|$ small enough,

$$\|u(\xi,P)-u(\xi,0)\|\le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}, \tag{2.8}$$

where $\xi\in\mathbb{T}^n$. Write $\eta=P_0d\xi+df_1,\ f_1\in C^\omega(\mathbb{T}^n)$. Therefore[5], $L_1=L_2+P_0d\xi+df_1$ and $L_1-\langle P,\dot\xi\rangle=L_2-\langle P-P_0-f_1',\dot\xi\rangle$. Further, one has

$$H_2(\xi,P-P_0+D(v-f_1))=\overline{H}_2(P-P_0). \tag{2.9}$$

From (2.7) and (2.8), we have for $\|P-P_0\|$ small enough,

$$\begin{aligned}\|D(v-f_1)(P-P_0)-D(v-f_1)(0)\|\le C\delta^{-1}\|P-P_0\|^{\frac{1}{\tau+1}},\\ \|v(\xi,P-P_0)-v(\xi,0)\|\le C\delta^{-1}\|P-P_0\|^{\frac{1}{\tau+1}},\end{aligned}$$

where $Dv(0)=Df_1$. Note $v(\xi,P-P_0)$ is viscosity solution corresponding with $L_1-\langle P,\dot\xi\rangle$. We will denote $v_1(\xi,P)=v(\xi,P-P_0)$. Therefore, for $\|P-P_0\|$ small, we have

$$\begin{aligned}\|Dv_1(\xi,P)-Dv_1(\xi,P_0)\|\le C\delta^{-1}\|P-P_0\|^{\frac{1}{\tau+1}},\\ \|v_1(\xi,P)-v_1(\xi,P_0)\|\le C\delta^{-1}\|P-P_0\|^{\frac{1}{\tau+1}},\end{aligned} \tag{2.10}$$

where $Dv_1(\xi,P_0)=Df_1$ and $\mathcal{G}=\bigcup_{\xi\in\mathbb{T}^n}(\xi,P_0+Dv_1(\xi,P_0)=\bigcup_{\xi\in\mathbb{T}^n}(\xi,P_0+Df_1)$ is the smooth torus. From (2.6) and Lemma 5.3, we get

$$\tilde{\mathcal{L}}(\Gamma)=\mathcal{G}, \tag{2.11}$$

$\tilde{\mathcal{L}}:\ T\mathbb{T}^n\to T^*\mathbb{T}^n$ is the Legendre transformation. Further, we obtain

$$Df\cdot\frac{\partial L_0}{\partial\dot q}(f(\xi),Df\cdot\omega)=P_0+Dv_1(\xi,P_0). \tag{2.12}$$

Lemma 2.1 *$v_1(\xi,P)$ is the viscosity solution of $L_1-\langle P,\dot\xi\rangle$ and satisfies (2.10). From the real analytic torus diffeomorphism $\xi=f^{-1}(x)$, we have the Lagrangian*

$$L_1(f^{-1}(x),Df^{-1}(x)\dot x)-\langle P,Df^{-1}(x)\dot x\rangle=L_0(x,\dot x)-\langle P,Df^{-1}(x)\dot x\rangle.$$

Write $\eta_{P_1}(\dot x)=\langle P,Df^{-1}\dot x\rangle$, then η_{P_1} is a closed 1-form and $\eta_{P_1}=P_1dx+df_2$, where $[\eta_{P_1}]=C(0)P=P_1$, $C(0)=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}Df^{-1}(x)dx$ and $f_2(x)\in C^\omega(x)$. $v_2(x,P_1)=v_1(f^{-1}(x),P)$ is the viscosity solution of $L_0-\eta_{P_1}$. $v_2(x,P_1)$ satisfies

$$H_0(x,P_1+D(v_2+f_2))=\overline{H}_0(P_1). \tag{2.13}$$

For $\|P-P_0\|$ small enough and $\forall\delta\in(0,1]$, we have

$$\begin{aligned}\|Dv_2(x,P_1)-Dv_2(x,P_1^0)\|\le C\delta^{-1}\|P-P_0\|^{\frac{1}{\tau+1}}\le C\delta^{-1}\|P_1-P_1^0\|^{\frac{1}{\tau+1}},\ x\in\Lambda,\\ \|v_2(x,P_1)-v_2(x,P_1^0)\|\le C\delta^{-1}\|P_1-P_1^0\|^{\frac{1}{\tau+1}},\ x\in\mathbb{T}^n,\end{aligned}$$

where $P_1^0=C(0)P_0$ and $\Lambda=\{x\in\mathbb{T}^n|x\in Dom(Dv_2,P_1)\cap f^{-1}(x)\in Dom(Dv_1(\xi),P)\}$.

Proof. As Proposition 4.4.8 in [3], we need prove two points:

(1). $v_2(x,P_1)\prec L_1(f^{-1}(x),Df^{-1}(x)\dot x)-\langle P,Df^{-1}(x)\dot x\rangle+\overline{H}_0(P_1)$.

[5] In fact, we need extend H_2, L_2 as before. The same for L_1 and etc.. We skip the steps for simplicity.

(2). for each $x\in\mathbb{T}^n$, there exists a

$$(v_2(x,P_1),\ L_1(f^{-1}(x),Df^{-1}(x)\dot{x})-\langle P,Df^{-1}(x)\dot{x}\rangle,\ \overline{H}_0(P_1))-\text{calibrated curve}$$

$\gamma^x:(-\infty,0]\to\mathbb{T}^n$ such that $\gamma^x(0)=x$, and $\forall t$,

$$\begin{aligned}v_1(f^{-1}(x),P)-v_1(f^{-1}(\gamma(-t)),P)=&\int_{-t}^{0}L_1(f^{-1}(\gamma(s)),Df^{-1}\dot{\gamma}(s))\\&-\langle P,Df^{-1}\dot{\gamma}(s)\rangle ds+\overline{H}_0(P_1)t.\end{aligned}\tag{2.14}$$

We only prove the second point and the first one is similar. For any $x\in\mathbb{T}^n$, there exists only one $\xi\in\mathbb{T}^n$ such that $\xi=f^{-1}(x)$. Since $v_1(\xi,P)$ is the viscosity solution of $L_1-\langle P,\dot{\xi}\rangle$, we have the following: $\exists$ a minimizing extremal curve $\tilde{\gamma}^\xi:(-\infty,0]\to\mathbb{T}^n$ with $\tilde{\gamma}^\xi(0)=\xi$ and such that $\forall t\in(-\infty,0]$,

$$v_1(\xi,P)-v_1(\tilde{\gamma}(-t),P)=\int_{-t}^{0}L_1(\tilde{\gamma}(s),\dot{\tilde{\gamma}}(s))-\langle P,\dot{\tilde{\gamma}}(s)\rangle ds+\overline{H}_1(P)t.\tag{2.15}$$

We define $\gamma^x(s)=f(\tilde{\gamma}^\xi(s))$ and $\overline{H}_0(P_1)=\overline{H}_1(P)$. Clearly, (2.14) holds from (2.15). As (2.9), one has (2.13). If we choose $x\in\Lambda=Dom(Dv_2,P_1)\cap f^{-1}(x)\in Dom(Dv_1(\xi),P)$, then,

$$Dv_2(x,P_1)=\frac{\partial f^{-1}(x)}{\partial x}Dv_1(f^{-1}(x),P),\tag{2.16}$$

and

$$Dv_2(x,P_1^0)=\frac{\partial f^{-1}(x)}{\partial x}Dv_1(f^{-1}(x),P_0).\tag{2.17}$$

Therefore, from (2.4), (2.16) and (2.17), one gets

$$\begin{aligned}\|Dv_2(x,P_1)-Dv_2(x,P_1^0)\|\le&\, C\|Dv_1(f^{-1}(x),P)-Dv_1(f^{-1}(x),P_0)\|\\\le&\, C\delta^{-1}\|P-P_0\|^{\frac{1}{\tau+1}}\\\le&\, C\delta^{-1}\|P_1-P_1^0\|^{\frac{1}{\tau+1}}.\end{aligned}$$

The rest are obvious. ∎

If denote $\mathcal{G}_0=\bigcup_{x\in\mathbb{T}^n}(x,P_1^0+D(f_2+v_2(x,P_1^0)))$, from (2.12), it is easy to check that

$$\tilde{\mathcal{L}}(\Gamma_0)=\mathcal{G}_0.$$

From all above in this subsection, we have the following theorem.

Theorem 2 *The Lagrangian*

$$L_0(x,\dot{x})=l_0(\dot{x})+\epsilon l_1(x,\dot{x})$$

with associated Hamiltonian H_0, where $\frac{\partial^2 l_0}{\partial\dot{x}^2}>0$. $L_0(x,v)$ is a real analytic function in the domain $|Imx|\le 2\lambda_0 r_0$, $|Imv|\le 2\lambda_0 r_0$ which is of period 2π in the $x-$variables. Let

$\omega\in\mathbb{R}^n$ satisfying $\|\omega\|\le M_0$, $|\langle k,\omega\rangle|\ge\frac{\gamma}{|k|^\tau}$, $0\ne k\in\mathbb{Z}^n$, for some constants $M_0\ge 1$, $\gamma>0$, $\tau\ge n-1$. $|\partial^\alpha L_0|_{2\lambda_0 r_0}\le M_0$, $|\alpha|\le 4$. If $c_0\epsilon\le\delta^*$, then $\exists$ a real analytic smooth torus

$$\Gamma_0=\bigcup_{\xi\in\mathbb{T}^n}(f(\xi),Df\cdot\omega)=\tilde{\mathcal{L}}^{-1}(\bigcup_{x\in\mathbb{T}^n}(x,P_1^0+D(f_2+v_2(x,P_1^0)))),$$

where $v_2(x,P_1)$ satisfies

$$H_0(x,P_1+D(f_2+v_2(x,P_1)))=\overline{H}_0(P_1).$$

If $\|P_1-P_1^0\|$ small and for any $\delta\in(0,1]$, we have

$$\|Dv_2(x,P_1)-Dv_2(x,P_1^0)\|\le C\delta^{-1}\|P_1-P_1^0\|^{\frac{1}{\tau+1}},\ x\in\Lambda,$$
$$\|v_2(x,P_1)-v_2(x,P_1^0)\|\le C\delta^{-1}\|P_1-P_1^0\|^{\frac{1}{\tau+1}},\ x\in\mathbb{T}^n.$$

Remark 2.3 *$meas(\{x\in\mathbb{T}^n\setminus\Lambda\})=0$.*

3 Three Preparing Lemmata

In this section and the following ones, we will prove Theorem 1. Before that, we will give three preparing lemmata.

3.1 Lemma 3.1

Lemma 3.1 *If $\|P\|\le\bar\delta$, there exists at least one point $x_0\in{}^6\mathcal{M}_P$ satisfying*

$$\|P+D_xu(x_0,P)\|\le C_0\|P\|, \tag{3.1}$$

where $\bar\delta$ depends on n, λ_1, λ_2 and f and C_0 depends on λ_1, λ_2.

Before we prove Lemma 3.1, we first prove the following proposition.

Proposition 3.1 *If $\|P\|\le\delta_0$, at least there exists one point $x_0\in\mathcal{M}_P$ such that*

$$\|v(x)-\omega\|\le C_1\|P\|, \tag{3.2}$$

where δ_0 depends on n, λ_1 and f and C_1 depends on λ_1.

Proof. The corresponding Lagrangian with the hamiltonian (2.1) is

$$L=\frac{1}{2}(\dot x-\omega)^TA^{-1}(x)(\dot x-\omega)+\mathcal{O}((\dot x-\omega)^3). \tag{3.3}$$

Our aim is to prove at least one point $(x,v)\in\tilde{\mathcal{M}}_P$ satisfying (3.2), where the corresponding Lagrangian is $L-\langle P,\dot x\rangle$. Suppose $|\mathcal{O}((\dot x-\omega)^3)|\le C\|\dot x-\omega\|^3$. Since $Mané's$ set is upper semi-continuous(see [8]), for $|P|\le\delta_0$, we have

$$L\ge\frac{1}{4}\lambda_1^{-1}\|\dot x-\omega\|^2$$

[6] In this paper, we will admit the notations in [3] without further explanations. For more details, please refer to [3].

where δ_0 depend on λ_1 and $\mathcal{O}((\dot{x}-\omega)^3)$ and $(x,\dot{x})\in\tilde{\mathcal{M}}_P$. Therefore,

$$\begin{aligned}\bar{L}=L-\langle P,\dot{x}\rangle &\geq \frac{1}{4}\lambda_1^{-1}\|\dot{x}-\omega\|^2-\langle P,\dot{x}\rangle\\ &=\frac{1}{4}\lambda_1^{-1}\|\dot{x}-\omega-2\lambda_1 P\|^2-\lambda_1\|P\|^2-\langle P,\omega\rangle.\end{aligned}$$

In the following we will discuss from the contrary. If $\|v(x)-\omega\|\geq A\|P\|$ and $A>4\lambda_1$, then

$$\begin{aligned}\|\dot{x}-\omega-2\lambda_1 P\| &\geq \|\dot{x}-\omega\|-2\lambda_1\|P\|\\ &\geq \frac{A}{2}\|P\|.\end{aligned}$$

Therefore,

$$\int \bar{L}d\mu > -\langle P,\omega\rangle = \int \bar{L}d\mu_0,$$

where μ_0 denote the Borel probability measure, invariant by the Euler-Lagrange flow, supporting on the KAM torus $\{(x,\omega)|x\in\mathbb{T}^n\}$. It contradicts with the definition of the minimizing measure μ.

Proof of Lemma 3.1:

Proof. From [3], if $(x,v)\in\tilde{\mathcal{M}}_P$, one has

$$P+D_x u(x,P)=\frac{\partial L}{\partial v}(x,v).$$

From (3.3), it is easy to get

$$\frac{\partial L}{\partial \dot{x}}=A^{-1}(\dot{x}-\omega)+\mathcal{O}((\dot{x}-\omega)^2).$$

Then, for $\|P\|\leq\delta_1$, one obtains

$$\|\frac{\partial L}{\partial \dot{x}}\|\leq\lambda_2^{-1}\|\dot{x}-\omega\|+C\|\dot{x}-\omega\|^2\leq 2\lambda_2^{-1}\|\dot{x}-\omega\|,$$

where δ_1 depends on λ_2 and $\mathcal{O}((\dot{x}-\omega)^3)$. Therefore, if choose $x_0\in\mathcal{M}_P$ satisfying (3.2), then

$$\begin{aligned}\|P+D_x u(x_0,P)\| &\leq \|\frac{\partial L}{\partial \dot{x}}\|\\ &\leq 2\lambda_2^{-1}\|\dot{x}_0-\omega\|\\ &\leq C_0\|P\|,\end{aligned}$$

for $\|P\|\leq\bar{\delta}=\min\{\delta_0,\delta_1\}$.

3.2 Lemma 3.2

We introduce the theorem from [10] in the following:

Theorem 3 *Consider a Hamiltonian of the form*

$$H(x,p)=\langle\omega,p\rangle+F(x,p), \tag{3.4}$$

where ω satisfies (1.6). F is a bounded real-analytic function on $D(R,\rho,\sigma)$ for some positive constants R,ρ and $\sigma<1$. Furthermore, assume that F, regarded as a function of p, is order $|p|^2$, so that there exists $E'\geq 0$ such that $|F(x,p)|<E'\frac{|p|^2}{R^2}$ for all $(q,p)\in D(R,\rho,\sigma)$. Assume $\rho<4R$, and choose

$$a\in(0,\beta),\ \beta=\frac{1-\frac{\rho}{4R}}{1+\frac{\rho}{4R}}.$$

Then for all $r\leq R$,

$$|p(0)|\leq ar\Rightarrow|p(t)|\leq r\qquad\text{for all } |t|\leq T,$$

where

$$\begin{aligned}&T=\Gamma(\frac{R}{r})e^{(\frac{k_3R}{r})^\alpha},\ \Gamma=(\beta-a)(1+\frac{\rho}{4R})^2\frac{\sigma R}{2k_1},\ k_1=eE,\\&E=E'(1+\frac{\rho}{R})^2,\ k_3=k(1+\frac{\rho}{4R}),\ \alpha=\frac{1}{\tau+2},\\&k=\frac{\gamma\rho}{D_{\tau,n}E}(\frac{\sigma}{8\kappa_\tau})^{\frac{1}{\alpha}},\ D_{\tau,n}=\frac{\sqrt{(2\tau)!}}{2^{\tau-n-1}},\ \kappa_\tau=\frac{1}{4}(\frac{e^2}{e-1}a_\tau+eb_\tau)^{\frac{1}{\tau+2}},\\&a_\tau=2^{\tau+3}+2^{3\tau+7},\ b_\tau=2^{\tau+3}+2^{2\tau+5}+2^{3\tau+7}.\end{aligned}$$

Remark 3.1 *If choose $\rho=2R$ and $a=\frac{1}{6}$ in the above theorem, then for all $r\leq R$, $|p(0)|\leq\frac{1}{6}r\Rightarrow|p(t)|\leq r$ for all*

$$|t|\leq T=\frac{c\sigma R^2}{E'r}exp[(\frac{C_{\tau,n}\gamma\rho R}{E'r})^\alpha\sigma],$$

where c is an absolute constant and $C_{\tau,n}$ is a general constant depending on n,τ.

From Remark 3.1, we have the following proposition:

Proposition 3.2 *If $\epsilon\leq R$, then for any initial value point $(x_0,\ p_0)(|p_0|\leq\frac{1}{6}\epsilon)$, its solution curve $(x(t),\ p(t))$ under (2.2) satisfies $|p(t)|\leq\epsilon$ for*

$$|t|\leq\frac{c\sigma R^2}{E'\epsilon}exp[(\frac{C_{\tau,n}\gamma\rho R}{E'\epsilon})^\alpha\sigma],$$

where c is an absolute constant and $C_{\tau,n}$ is a general constant depending on n,τ.

Lemma 3.2 *(a). For any $x\in\mathcal{M}_P$, then*

$$(x,v(x))=\tilde{\mathcal{L}}^{-1}(x,D_xu(x,P))\in\tilde{\mathcal{M}}_P,$$

where $u(x,P)$ is the viscosity solution of (1.1) and $\tilde{\mathcal{L}}\colon T\mathbb{T}^n\to T^*\mathbb{T}^n$ is the global Legendre transformation associated to $\tilde{L}=L-\langle P,\dot{x}\rangle$. Denote $w(s)=(x(s),v(s))=\phi^s_{\tilde{L}}(x,v(x))$, then

$w(s)\in \tilde{M}_P$, where $\phi_{\tilde{L}}$ is the Lagrangian flow of $\tilde{L}$. The corresponding curve in $T^*\mathbb{T}^n$ is denoted by $l(s)=(x(s),\tilde{p}(s))=\phi^s_{\tilde{H}}(x,D_xu(x,P))$, where $\tilde{H}(x,\tilde{p})=H(x,P+\tilde{p})$, H as (2.1).

(b). For $t_1\in\mathbb{R}$, $D_xu(x_1,P)+P=\frac{\partial L}{\partial v}(x_1,v(x_1))$, where $(x_1,v(x_1))=(x(t_1),v(t_1))=w(t_1)$.

(c). Denote the curve by $l(s)$ of which the initial point is $(x_0,D_xu(x_0,P))$, where x_0 and $u(x,P)$ satisfy (3.1). Denote $\pi: T^*\mathbb{T}^n\to\mathbb{T}^n$ and $x(s)=\pi(l(s))$. The obit of $x(s)$ will ergodize $\mathbb{T}^n$ to within $C_6\|P\|^{\frac{1}{\tau+1}}$, if $\|P\|\le\epsilon_0$ where ϵ_0 depends on R, n, τ, γ, σ, λ_1, λ_2, ρ, f and C_6 depends on f, n, τ, γ, λ_1, λ_2. More concretely, for $\forall\theta\in\mathbb{T}^n$, $\exists 0\le t_0\le \frac{C_{n,\tau}}{\gamma\|P\|^{\frac{\tau}{\tau+1}}}$ and $x(-t_0)$ such that

$$\|x(-t_0)-\theta\|\le C_6\|P\|^{\frac{1}{\tau+1}},$$

where $C_{n,\tau}$ is a general constant depending on n and τ.

Proof. (a) is easy. and (b) is clear from Theorem 4.8.3 in [3]. We mainly prove (c). Note

$$\|P+D_xu(x_0,P)\|\le C_0\|P\|. \tag{3.5}$$

Then $(x(t),p(t))=(x(t),P+\tilde{p}(t))$ satisfies (2.2). From (3.5), we have $|P+D_xu(x_0,P)|\le C_0'|P|$, where C_0' depends on λ_1, λ_2 and n. Denote $|P|=\frac{\epsilon}{6C_0'}$. If P satisfies

$$|P|\le\min\{\frac{R}{6C_0},\ \bar{\delta}\} \tag{3.6}$$

then for initial value point $(x_0,\ P+D_xu(x_0,P))$, its solution curve $(x(t),p(t))$ under (2.2) satisfies

$$|p(t)|\le 6C_0|P|, \tag{3.7}$$

$$|t|\le\frac{c\sigma R^2}{E'C_0|P|}exp[(\frac{C_{\tau,n}\gamma\rho R}{E'C_0|P|})^{\alpha}\sigma], \tag{3.8}$$

where c is an absolute constant and $C_{\tau,n}$ is a general constant depending on n,τ. From

$$\dot{x}=\omega+A(x)p+\frac{\partial f}{\partial p}(x,p),$$

we have

$$x(t)-x_0-\omega t=\int_0^t(A(x)p+\frac{\partial f}{\partial p}(x,p))ds. \tag{3.9}$$

Denote $|\frac{\partial f}{\partial p}|\le C_2|p|^2$. From (3.7) and (3.9), if

$$|t|\le\frac{c\sigma R^2}{E'C_0|P|}exp[(\frac{C_{\tau,n}\gamma\rho R}{E'C_0|P|})^{\alpha}\sigma],$$

then

$$|x(t)-x_0-\omega t|\le C_3|t||P|. \tag{3.10}$$

From [1](also see [2], [4]), one obtains for $\forall\theta\in\mathbb{T}^n$, $\forall r_1^M$(r_1^M will be chosen in the following), $\exists t_0$ satisfies

$$0\leq t_0\leq\frac{C_{n,\tau}}{\gamma r_1^{M\tau}}$$

such that

$$|\theta-(-\omega t_0+x_0)|\leq r_1^M. \tag{3.11}$$

From (3.10) and (3.11), if

$$0\leq t_0\leq\frac{C(n,\tau)}{\gamma r_1^{M\tau}}\leq\frac{c\sigma R^2}{E'C_0|P|}exp[(\frac{C_{\tau,n}\gamma\rho R}{E'C_0|P|})^\alpha\sigma], \tag{3.12}$$

we will have for $\forall\theta\in\mathbb{T}^n$, $\exists 0\leq t_0\leq\frac{C(n,\tau)}{\gamma r_1^{M\tau}}$ such that

$$\begin{aligned}|x(-t_0)-\theta|&\leq|x(-t_0)-(-\omega t_0+x_0)|+|\theta-(-\omega t_0+x_0)|\\&\leq C_3t_0|P|+r_1^M\\&\leq C_3\frac{C(n,\tau)}{\gamma r_1^{M\tau}}|P|+r_1^M\\&=\frac{C_4}{r_1^{M\tau}}|P|+r_1^M.\end{aligned}$$

If we choose $r_1^M=|P|^{\frac{1}{\tau+1}}$, then

$$|x(-t_0)-\theta|\leq C_5|P|^{\frac{1}{\tau+1}}.$$

Note (3.6) and (3.12), if $|P|\leq\epsilon_0$, then

$$\frac{C(n,\tau)}{\gamma|P|^{\frac{\tau}{\tau+1}}}\leq\frac{c\sigma R^2}{E'C_0|P|}exp[(\frac{C_{\tau,n}\gamma\rho R}{E'C_0|P|})^\alpha\sigma]$$

holds naturally, where ϵ_0 depends on R, n, τ, γ, σ, λ_1, λ_2, ρ, f and C_5 depends on f, n, τ, γ, λ_1, λ_2. Obviously, if $\|P\|\leq\epsilon_0$, then $|P|\leq\epsilon_0$. For any $\theta\in\mathbb{T}^n$,

$$\exists 0\leq t_0\leq\frac{C_{n,\tau}}{\gamma\|P\|^{\frac{\tau}{\tau+1}}}\text{ and }x(-t_0)\text{ such that} \tag{3.13}$$

$$\|x(-t_0)-\theta\|\leq C_6\|P\|^{\frac{1}{\tau+1}}, \tag{3.14}$$

where C_6 depends on f, n, τ, γ, λ_1, λ_2. ■

3.3 Lemma 3.3

The following lemma is almost a direct corollary of Theorem 4.7.5 of [3]. As [3], for $\delta>0$ is given and any map $u \prec L-\langle P,\dot{x}\rangle+\overline{\tilde{H}}(P)$, we define the set $\mathcal{A}_{\delta,u}$ formed by the $x\in\mathbb{T}^n$ for which there exists a (continuous) piecewise C^1 curve $\gamma:[-\delta,\delta]\to\mathbb{T}^n$ with $\gamma(0)=x$ and

$$u(\gamma(\delta))-u(\gamma(-\delta))=\int_{-\delta}^{\delta}L(\gamma(s),\dot{\gamma(s)})-\langle P,\dot{\gamma}(s)\rangle ds+2\overline{\tilde{H}}(P)\delta.$$

Lemma 3.3 *For any $u(x,P)$ satisfying (1.1), we define*

$$Graph(Du(x,P))=\{(x,D_xu(x,P))|x\in dom(Du(x,P))\}.$$

Then $(\phi_{\tilde{H}}^{-\delta})^*Graph(Du(x,P))\subset\mathcal{A}_{\delta,u}$. It is a Lipschitzian graph with Lipschitzian constant depending on a fixed $\delta>0$ on each subset with diameter $\le\eta$, where $\phi_{\tilde{H}}$ is the Hamiltonian flow of $\tilde{H}(x,p)=H(x,P+p)$ in $T^*\mathbb{T}^n$ and η doesn't depend on δ.

Proof. Since $u(x,P)$ is a viscosity solution of (1.1), obviously, we have

$$T_t^-u+\overline{\tilde{H}}(P)t=u,\ t\ge 0,$$

where the relative Lagrangian of T_t^- is $L-\langle P,\dot{x}\rangle$. From Proposition 4.4.8 of [3], for any $x\in dom(Du(x,P))$, $\exists$ a $(u,L-\langle P,\dot{x}\rangle,\overline{H}(P))$-calibrated curve $\gamma_-^x:(-\infty,0]\to\mathbb{T}^n$, such that $\gamma_-^x(0)=x$. This means that for any $t\ge 0$,

$$u(x)-u(\gamma_-^x(-t))=\int_{-t}^{0}(L(\gamma_-^x(s),\dot{\gamma}_-^x(s))-\langle P,\dot{\gamma}_-^x(s)\rangle)ds+\overline{\tilde{H}}(P)t. \tag{3.15}$$

Since u has a derivative at x, we have $P+D_xu=\frac{\partial L}{\partial v}(x,\dot{\gamma}_-^x(0))$ and $\phi_{-\delta}^*(x,D_xu)=D_{\gamma_-^x(-\delta)}u$, where $\phi_{\tilde{H}}$ is the Hamiltonian flow of $\tilde{H}(x,p)=H(x,P+p)$. In order to apply Theorem 4.7.5 of [3], we choose $t=2\delta$ in (3.15). In the following, we will give the Lipschitzian constant. From (5.3), we know that when $\|P\|\le\eta_0$, $\|D_xu(x,P)\|\le 1$. Write

$$D_1=\tilde{\mathcal{L}}^{-1}D_2,$$

where $D_2=\{(x,D_xu(x,P))\in T^*\mathbb{T}^n|\forall x\in\mathbb{T}^n,\ \forall u(x,P)$ satisfying (1.1), $\|P\|\le\eta_0\}$. It is clear that there exists compact sets D_3 and D_4 such that $D_2\subset D_3$ and $D_1\subset D_4$. It is clear that

$$\sup_{x\in D_4}|\frac{\partial^2L}{\partial x^2}|,\ \sup_{x\in D_4}|\frac{\partial^2L}{\partial x\partial v}|,\ \sup_{x\in D_4}|\frac{\partial^2L}{\partial v^2}|\le\frac{1}{2}K.$$

From Theorem 4.7.5 and Proposition 4.7.1 of [3], the Lipschitzian constant is no more than $\frac{K}{\delta}$. ∎

4 Proofs of Theorem 1 and Corollary 1

Proof. For any $x \in dom(Du(x,P))$, from the proof of Lemma 3.3, there exists a calibrated curve γ^x such that $\phi^*_{-\delta}(x, D_x u) = (\gamma^x(-\delta), D_{\gamma^x(-\delta)} u)$. Denote $x_1 = \gamma^x(-\delta)$. For $\|P\| \le \epsilon_0$, from Lemma 3.2 and (3.14), there exists $x(-t_0) = x_2 \in \mathcal{M}_P$ such that

$$\|x_1 - x_2\| \le C_6 \|P\|^{\frac{1}{\tau+1}}. \tag{4.1}$$

Moreover, from (3.7), (3.8), (3.13) and (3.14), one gets $|p(-t_0)| \le 6C_0|P|$ and $|p(-t_0+\delta)| \le 6C_0|P|$. Then $\|\tilde{p}(-t_0)\| \le (6C_0+1)\sqrt{n}\|P\|$ and $\|\tilde{p}(-t_0-\delta)\| \le (6C_0+1)\sqrt{n}\|P\|$. Clearly, for x_0, there exists a calibrated curve γ_1 such that $\gamma_1(0) = x_0$. Write $\gamma_1(-t_0) = x_2$ and $\gamma_1(-t_0+\delta) = x_3$. From Lemma 3.2, we have

$$P + D_x u(x_3, P) = \frac{\partial L}{\partial v}(x_3, v(x_3)).$$

It is clear $\tilde{p}(-t_0+\delta) = D_x u(x_3, P)$. It results in

$$\|D_x u(x_3, P)\| \le (6C_0+1)\sqrt{n}\|P\| = C_7\|P\|. \tag{4.2}$$

From the proof of Lemma 3.3, we know that for $\|P\| \le \eta_0$, $\|D_x u(x,P)\| \le 1$. Therefore, there exits a compact set $D_5 \subset T^*\mathbb{T}^n$ such that

$$\{\phi^{-\delta}_{\tilde{H}}(x, D_x u(x,P)) | \forall x \in dom(Du(x,P)),\ \forall u(x,P) \text{ satisfying (1.1)}\} \subset D_5.$$

Denote d the flat metric in $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$. Clearly, for $\|P\| \le \eta_1$, one has

$$\|x_1 - x_2\| \le C_6\|P\|^{\frac{1}{\tau+1}} \le \eta.$$

Then for $\|P\| \le \min\{\epsilon_0, \eta_0, \eta_1\}$, one gets

$$\begin{aligned}
& d(\phi^{\delta}_{\tilde{H}}(\phi^{-\delta}_{\tilde{H}}(x, D_x u(x,P))), \phi^{\delta}_{\tilde{H}}(\phi^{-\delta}_{\tilde{H}}(x_3, D_x u(x_3,P))) \\
& \le \sup_{D_5} |D\phi^{\delta}_{\tilde{H}}| d(\phi^{-\delta}_{\tilde{H}}(x, D_x u(x,P)), \phi^{-\delta}_{\tilde{H}}(x_3, D_x u(x_3,P))) \\
\text{(from a compact discussion)} \quad & \le C d((x_1, D_{x_1} u(x_1,P)), (x_2, D_{x_2} u(x_2,P))) \\
\text{(from Lemma 3.3)} \quad & \le C\delta^{-1}\|x_1 - x_2\| \\
\text{(from (4.1))} \quad & \le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}.
\end{aligned}$$

Therefore,

$$\|D_x u(x,P) - D_x u(x_3,P)\| \le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}.$$

Combining with (4.2), we get

$$\|D_x u(x,P)\| \le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}.$$

∎

The following is the proof of Corollary 1.

Proof.

$$\|u(x,P)-u(x,0)\| \le |u(x,P)-\min_{x\in\mathbb{T}^n} u(x,P)| \\ = |u(x,P)-u(y,P)|.$$

From the proof of Lemma 5.1 and Theorem 1, if $0<\delta\le 1$ and $\|P\|\le \min\{\epsilon_0,\eta_0,\eta_1\}$, then

$$|u(x,P)-u(y,P)| \le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}\|x-y\| \\ \le C\delta^{-1}\|P\|^{\frac{1}{\tau+1}}.$$

∎

5 Appendix

Lemma 5.1 *u_n is a viscosity solution of*

$$H(x,P_n+Du_n(x,P_n)) = \overline{H}(P_n), \tag{5.1}$$

$P_n\in D$, where D is a compact set in $\mathbb{R}^n$, then $\exists$ K_2' depending only on H and D, such that

$$|u_n(x,P_n)-u_n(y,P_n)| \le K_2'\|x-y\|,$$

where $x,\ y\in\mathbb{T}^n$.

Proof. From $P_n\in D$, $\exists$ K_1' depending only on D and H, such that $|\overline{H}(P_n)|\le K_1'$. Since H is superlinear and (5.1), it is clear that the set $\{(x,p):\ |H(x,p)|\le K_1'\}$ is compact. It follows that $\exists K_2'$ depending only on H and D, such that $|Du_n(x,P_n)|\le K_2'$, where $x\in W_{P_n}=dom(Du_n,P_n)$. For each P_n, $u_n(x,P_n)$ is Lipschitz function. So, $\mathbb{T}^n\setminus W_{P_n}$ is negligible for Lebsgue measure.

Given two point $x,\ y\in\mathbb{T}^n$, by Fubini theorem, there exist two sequences of points $x_k,\ y_k\in\mathbb{T}^n$ such that $x_k\to x,\ y_k\to y$, and the affine segment $\Gamma_k:\ \gamma_k(t)=x_k+\frac{y_k-x_k}{|y_k-x_k|}t$ intersect W_{P_n} in a set of full linear measure in Γ_k. We have

$$|u_n(y_k,P_n)-u_n(x_k,P_n)| \le \int_0^{\|y_k-x_k\|} |\langle D_xu_n(x,P_n),\dot{\gamma}_k\rangle|dt \\ \le \int_0^{\|y_k-x_k\|} |\|D_xu_n(x,P_n)\|\|\dot{\gamma}_k\|dt \\ \le K_2'\|y_k-x_k\|.$$

Since $u_n(x,P_n)$ is continuous and $x_k\to x,\ y_k\to y$, we have

$$|u_n(x,P_n)-u_n(y,P_n)| \le K_2'\|x-y\|.$$

Lemma 5.2 *Write $[u_m](x) = u_m(x) - \min_{x\in\mathbb{T}^n} u_m(x)$. If u_n is a viscosity solution of $H(x, P_n + Du_n(x, P_n)) = \overline{H}(P_n)$, where $P_n \in D$ and D is a compact set in $\mathbb{R}^n$, then there exits a sequence of $[u_n]$ and $u_0 \in C^0(M, \mathbb{R})$ such that $[u_n] \to u_0$ in the C^0 topology uniformly on M.*

Proof. Write $u_n(x_0^n, P_n) = \min_{x\in\mathbb{T}^n} u_n(x, P_n)$. From Lemma 5.1, for $\forall n$, we get

$$|u_n(x, P_n) - u_n(x_0^n, P_n)| \le K_2' \|x - x_0^n\| \le K_3',$$

where K_3' depends on H and D. This means that $|[u_n]| \le K_3'$, for any n. Similarly, from Lemma 5.1, for any $x,\ y \in \mathbb{T}^n$, we have

$$\begin{aligned}|[u_n](x, P_n) - [u_n](y, P_n)| &= |u_n(x, P_n) - u_n(y, P_n)| \\ &\le K_2' \|x - y\|.\end{aligned}$$

From Ascoli's Theorem, we get the conclusion.

Lemma 5.3 *M is a compact and connected manifold. L is a Tonelli Lagrangian. Denote by $H : T^*M \to \mathbb{R}$ its associated Hamiltonian. If $\mathcal{A}_0 = \mathcal{M}_0 = M$ and u is a viscosity solution of $H(x, Du) = c$, then u is unique(mod a constant).*

Proof. From Section 5.2 in [3], we have the following:

$$\begin{aligned}\mathcal{I}_{(u_-,u_+)} &= \{x \in M | u_-(x) = u_+(x)\} \\ \tilde{\mathcal{I}}_{(u_-,u_+)} &= \{(x, v) | x \in \mathcal{I}_{(u_-,u_+)}, D_x u_- = D_x u_+ = \frac{\partial L}{\partial v}(x, v)\} \\ \tilde{\mathcal{A}}_0 &= \cap \tilde{\mathcal{I}}_{(u_-,u_+)},\ \mathcal{A}_0 = \pi(\tilde{\mathcal{A}}_0).\end{aligned} \tag{5.2}$$

From (5.2) and Theorem 5.2.8 in [3], we know that if v is another viscosity solution(also weak KAM solution) of $H(x, Du) = c$, then $D_x u = D_x v$, for any $x \in M$. The conclusion is clear.

Proposition 5.1 *When $P \to 0$, then*

$$\|P + D_x u(x, P)\| \to 0, \tag{5.3}$$

for any $x \in dom(Du(x, P))$ and any $u(x, P)$ satisfying (1.1).

Proof. Suppose it is not true, then we have the following: $\exists \epsilon_0 > 0$, $\forall \delta_n = \frac{1}{n} \to 0$, $\exists \|P_n\| \le \delta_n$, $\exists u_n(x, P_n)$ and $x_n \in dom(Du_n(x, P_n))$ such that

$$\|P_n + D_x u_n(x_n, P_n)\| \ge \epsilon_0.$$

Since u_n is a viscosity solution(weak KAM solution) and satisfies $H(x, P_n + Du) = \overline{H}(P_n)$, from Proposition 4.4.8 of [3], we have the following: for $x_n \in \mathbb{T}^n$, $\exists (u_n, L - \langle P_n, \dot{x}\rangle, \overline{H}(P_n))$-calibrated curve γ_n such that for $\forall t > 0$,

$$u_n(\gamma_n(0)) - u_n(\gamma_n(-t)) = \int_{-t}^{0} [L - \langle P_n, \dot{x}\rangle] ds + \overline{H}(P_n)t, \tag{5.4}$$

where $\gamma_n(0)=x_n$. We also have

$$P_n+D_xu_n(x_n,P_n)=\frac{\partial L}{\partial v}(\gamma_n(0),\dot\gamma_n(0)). \tag{5.5}$$

Since $\|P_n\|\le 1$, it is easy to see that $\exists K_1>0$, such that

$$\|P_n+D_xu_n(x_n,P_n)\|\le K_1. \tag{5.6}$$

Otherwise, we will have $\|P_{n_i}+D_xu_{n_i}(x_{n_i},P_{n_i})\|\to\infty$. Then $H(x_{n_i},P_{n_i}+D_xu_{n_i}(x_{n_i},P_{n_i}))=\overline{H}(P_{n_i})\to\infty$. But it is impossible since $\overline{H}(P_{n_i})$ is bounded.

From (5.5) and (5.6), one gets that $\exists K_2>0$ such that

$$\|(\gamma_n(0),\dot\gamma_n(0))\|\le K_2. \tag{5.7}$$

Therefore, there exists a sequence denoted still by $(\gamma_n(0),\dot\gamma_n(0))$ and $(x_0,\dot x_0)\in T\mathbb{T}^n$ such that

$$(\gamma_n(0),\dot\gamma_n(0))\to(x_0,\dot x_0). \tag{5.8}$$

Denote

$$(\gamma_0(s),\dot\gamma_0(s))=(\phi^L_{-t})_*(x_0,\dot x_0),$$

where $t>0$ and ϕ^L_t is the Lagrangian flow in $T\mathbb{T}^n$. From (5.8), it is clear that

$$\lim_{n\to\infty}(\phi^L_{-t})_*(\gamma_n(0),\dot\gamma_n(0))=(\phi^L_{-t})_*(x_0,\dot x_0)=(\gamma_0(t),\dot\gamma_0(-t)). \tag{5.9}$$

This means for any $t\in[0,\infty)$, we have

$$(\gamma_n(-t),\dot\gamma_n(-t))\to(\gamma_0(-t),\dot\gamma_0(-t)). \tag{5.10}$$

From Lemma 5.2 in the appendix, we know that there exists a sequence $[u_n]$ and $u\in C^0(M,\mathbb{R})$ such that $[u_n]\to u$ in the C^0 topology uniformly on M. Since $\|P_n\|\le\delta_n\to 0$, one obtains $P_n\to 0$ when $n\to 0$. Further, we have

$$\overline{H}(P_n)\to\overline{H}(0)=0. \tag{5.11}$$

From (5.4), (5.10) and (5.11), when $n\to 0$, we have for any $t>0$,

$$u(\gamma_0(0))-u(\gamma_0(-t))=\int_{-t}^0 L(\gamma_0(s),\dot\gamma_0(s))ds. \tag{5.12}$$

Write $H_n(x,p)=H(x,P_n+p)-\overline{H}(P_n)$, from Theorem 8.1.1 of [3], one has that u is a viscosity solution of $H(x,d_xu)=0$. From Theorem 7.6.1 of [3], we have

$$u\prec L. \tag{5.13}$$

From (5.12) and (5.13), it is obvious that $\gamma_0(s)$ is $(u,L,0)$-calibrated. From Proposition 4.4.10 of [3], we have the following: there exists a Borel probability measure μ on $T\mathbb{T}^n$, invariant by ϕ^L_t, carried by the α-limit set of the orbit of $(x_0,\dot x_0)$, and such that $\int Ld\mu=0$.

From Corollary 4.8.4 of [3], we have $\overline{supp\mu} \subset \{(x,\omega)|x\in\mathbb{T}^n\}$. Therefore, $\exists t_1>0$ large enough and $\epsilon_1>0$ small enough, such that

$$\|\dot{\gamma}_0(-t_1)-\omega\|=\epsilon_1>0. \tag{5.14}$$

But

$$L(\gamma_0(-t_1),\dot{\gamma}_0(-t_1))\geq\frac{1}{4}\lambda_1^{-1}\|\dot{\gamma}_0(-t_1)-\omega\|^2=\frac{1}{4}\lambda_1^{-1}\epsilon_1^2>0. \tag{5.15}$$

From Lemma 5.3, we know that u is unique(mod constant). But from $H(x,du)=0$, it is clear that $u=c$ is the smooth solution. Then (5.12) is changed into the following: for any $t>0$, $\int_{-t}^{0}L(\gamma_0(s),\dot{\gamma}_0(s))ds=0$. Then for any $t>0$, we have $L(\gamma_0(-t),\dot{\gamma}_0(-t))=0$. This contradicts with (5.15). ■

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