# Geometric approaches to the problem of instability in Hamiltonian systems. An informal presentation

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**Abstract** We present (informally) some geometric structures that imply instability in Hamiltonian systems. We also present some finite calculations which can establish the presence of these structures in a given near integrable systems or in systems for which good numerical information is available. We also discuss some quantitative features of the diffusion mechanisms such as time of diffusion, Hausdorff dimension of diffusing orbits, etc.

## 1 Introduction

The goal of these lectures is to present an overview of some geometric programs to understand instability in Hamiltonian dynamical systems.

Roughly speaking, the problem of instability is to decide whether the effect of small time-dependent perturbations accumulates over time. Relatedly, to show that

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many orbits of a time-independent Hamiltonian system explore a large fraction of the energy surface.

Instability is a real problem arising in applications. For example, designers of accelerators or plasma confinement devices want to invent devices which are as devoid of instability as possible. Designers of space missions want to find orbits which can move freely over a wide region of space, but of course, they can only use the intertwining gravitational fields of the nearby celestial bodies. Chemists want to understand how reactions and reconfigurations of molecules take place. As is common with real problems, there are many mathematical formulations depending on the precise mathematical meaning attached to the vague words of the previous paragraph [1] and many techniques which have come to bear on these formulations. For example, the lectures of prof Cheng, Neishtadt, and Treschev in this volume present other points of view about the problem and will even present different treatments of the same mathematical model.

These lectures can only aim to present informally the ideas behind some of the methods that have been proposed. We do not aim to present all the hypothesis of the results and much less complete proofs. Even when we restrict to geometric methods, we cannot aim to present a complete survey. The subject is progressing very fast. We only hope that these lectures can present an entry point to a portion the literature and indicate what to look for while reading some papers. We just want to present several milestones of the programs and to give some indication of the arguments.

There are two basic steps in all the results presented here. In a first step, we will present several geometric facts that imply that there are orbits that move appreciable lengths. In a second step, we will present some finite calculations which can verify the existence of these objects in quasi-integrable systems or in systems of a special form. Hence, for some systems, deciding that instability happens can be established with a finite computation. This will have the conclusion that some types of diffusion or instability are generic in some sense in some class of systems.

*Remark 1.* It should be emphasized that there are different geometric mechanisms that lead to instability. These mechanisms involve different geometric objects, have different hypothesis and lead to orbits with different characteristics. Several different mechanisms can coexist in the same model. The existence of several mechanisms was documented in some of the heuristic literature. An early paper, which is still worth reading is [LT83].

*Remark 2.* Given the practical importance of the problem of instability, there is a very large numerical and heuristic literature. Even if not easy to read, this literature

[1] The previous paragraph contains several imprecise words such as *accumulate, many, explore, large, etc.*. There are several rigorous formulations of these ideas. Some of the authors of this paper remember a round table in [Sim99] which included Profs. Arnol'd, Gallavotti, Galgani, Herman, Moser, Simó, Sinai. The panel was asked the question to give a canonical definition of *diffusion* that was preferable to the other definitions then in use. The conclusion was that it was better that each paper contains a precise definition.

The reader is encouraged to compare the precise definitions of diffusion or Arnol'd diffusion used in each paper. See Remark3.

contains considerable insights and can suggest several theorems. As representative papers of the numerical literature — which we cannot discuss in more detail – we mention [Chi79, Ten82, LT83, ZZN$^+$89, Zas02, GLF05, FGL05, FLG06].

Perhaps the main insight from the numerical literature, is that resonances organize the diffusion (the so called *Arnold web*). This, indeed was one important motivation for several of the investigations reported here. On the other hand, we will discuss some mechanisms which do not quite fit in this paradigm. See Section 2, 7.

*Remark 3.* There are many precise mathematical formulations of what is meant by *diffusion* or *Arnol'd diffusion*. For some authors, the fact that there are whiskered tori as discussed in Section 2 is the key feature. We however take the presence large effects as the key feature. Many papers, for example [HM82] (which we will discuss more fully in Section 4) consider perturbations of size $\varepsilon$ of an integrable system and establish existence of whiskered tori with heteroclinic intersections. These chains of whiskered tori, however are rather short and lead only to changes in the action variable of order $\varepsilon^{1/2}$. We, on the other hand, prefer to emphasize the existence of changes of order 1 in the actions, even if they are not accomplished through whiskered tori. A careful discussion of these issues appears in [Moe96].

## *1.1 Two types of geometric programs*

With some simplification, there are two types of geometric programs that we will discuss.

**Programs based on invariant objects and their relations**

1. Find invariant objects (whiskered tori, normally hyperbolic invariant manifolds, periodic orbits, horseshoes, normally hyperbolic laminations, etc. as well as their stable and unstable manifolds).
2. Prove that if these objects satisfy some appropriate relations (e.g. there is a sequence of whiskered tori such that the unstable manifold of one torus intersects transversally the stable manifold of the next torus) then, there are orbits which move along the chain of invariant objects.

Incipient versions of programs of this type were already present in [Poi99]. The paper which has been more influential in the mathematical literature is [Arn64, Arn63]. The main invariant objects considered in [Arn64] were whiskered tori and their invariant manifolds. We will discuss this paper in some more detail in Section 2. Other early examples of instability were [Sit53, Ale68a, Ale68b, Ale69, Ale81], which were mainly based on hyperbolic and topological properties. The study of instability properties of oscillators was pioneered in [Lit66a, Lit66b, Lev92]. Other papers establishing instability in oscillators are [AO98, Ort97, LY97,

Ort04] The papers [Pus77a, Pus77b, Dou89, KPT95, Pus95] study instability in systems with collisions. The papers [Dou88, DLC83] construct examples of instability near elliptic points. The paper [CG98] revived geometric approaches and contained many useful techniques.

To study invariant objects, typically, one finds some representation of them as a function. The condition of invariance is then a functional equation, which is often studied by methods of functional analysis or just numerically or by asymptotic methods. Two very basic methods to study invariance equations are normal hyperbolicity or KAM theory. One often has to supplement them with some preliminary calculations based on averaging or on perturbative calculations.

**Programs based on finite orbits "with hooks"**

1. Find finite segments of pseudo-orbits such that one segment ends close to the beginning of the following segment.
2. Verify some extra properties of each of the segments.
3. Use these properties to show that there is an orbit that remains close to the whole segment of orbits.

We picturesquely describe the above situation as saying that the segments of orbits have *hooks* so that they can be chained together. The fact that one needs some extra properties of the segments is made clear by the existence of examples — e.g. rigid rotations of the torus — where the conclusions are false.

There are quite a number of mathematical results of this kind. The best known results of this type are, perhaps, the *shadowing* theorems for hyperbolic systems [Shu78]. The hook in this case, is hyperbolicity. For many applications, hyperbolicity is a hard hypothesis to verify – it is often even false! – so that there are many variants See, for example [Pal00, Pil99] and references there.

For us, the method which so far has proved to be more useful is the method based on *correctly aligned windows*. The basic idea is to use some kind of topological index of the segments of orbits so that one can show that there is an orbit in a neighborhood of the whole chain. One early example, is [Con68, Con78, Eas89]. We will discuss it in Section 5.

One should also mention the variational program started in the 30's using *broken geodesics* [Mor24, Hed32, Ban88]. The idea was that, if the segments are minimizers of a good variational principle, then, indeed, there are orbits that follow them. [2] Some early implementations of these ideas to the problem of diffusion appear in [Bes96]. More recent applications appear in [BBB03, BCV01]. These methods also have the advantage that they apply to PDE's [RS02, Ang87] Very deep variational methods that also involve global considerations appeared in [Mat93, Mañ97, CI99].

[2] The heuristic idea is that, in the space of segments, each of our minimizers is in the center of a ball whose boundary has more action. If we take the whole orbit, the phase space is the product of the phase space of the segments so that the approximate orbit is contained in a ball so that the boundary of a ball has more action.

*Remark 4.* Of course, there are relations between the methods. Even in [Arn64], the invariant objects were used to produce segments of orbits as well as some *obstruction* property which shows that there are orbits that follow the segment. In our discussion of applications of the method of correctly aligned windows, we will consider orbits suggested by the invariant objects [3]. Even the more global variational methods of [CY04b, CY04a] start by reducing the problem using the presence of a normally hyperbolic invariant manifold.

One can hope that in the near future there will be even more relations. In particular, the more local variational theories (broken geodesic methods) seem rather close to the geometric methods. One can find relations between variational methods and the windows method is [Moe05].

In these lectures, we will try to present different mechanisms as well as the verification of their presence in some quasi-integrable systems. For the geometric mechanisms we will present in these lectures, the verification of their presence in concrete systems, will involve a rather standard toolkit (averaging theory, the theory of normally hyperbolic manifolds – perturbation theory, $\lambda$-lemma —, KAM theory) and some less standard tools such as the scattering map (Section 3.2 ) and the correctly aligned windows (Section 5). We will omit most of the details, but refer to the literature. The only goal of these lectures is to present a road map to the programs and to indicate the significant mileposts to be reached. Some similar expositions are [DDLLS00, DLS03, dlL06]. The present one incorporates some progress since the previous exposition were written. Fortunately, the new developments have lead to more streamlined proofs.

## 2 Exposition of the Arnol'd example

This very explicit example was constructed in [Arn64]. It is, possibly, the best known example in the mathematical literature. Some more detailed expositions of several of the aspects appear in [AA67]. A very complete explanation of the model in [Arn64] and generalizations can be found in [FM01].

In the following paragraphs, we will present the result emphasizing some of the geometric aspects that will play a role in the following. We refer [FM01] for the technical details of many of the proofs. We will emphasize several geometric properties that will play in the future.

**Theorem 1.** *Consider a time-dependent system defined in the action-angle variables* $(I,\Phi) \in \mathbb{R}^2 \times \mathbb{T}^2$ *by:*

$$\begin{aligned} H(I,\Phi,t) =& \frac{1}{2}(I_1^2+I_2^2) + \varepsilon(\cos\Phi_1 - 1) \\ &+ \mu\varepsilon(\cos\Phi_1 - 1)\,(\sin\Phi_2 + \cos t)\ , \end{aligned} \tag{1}$$

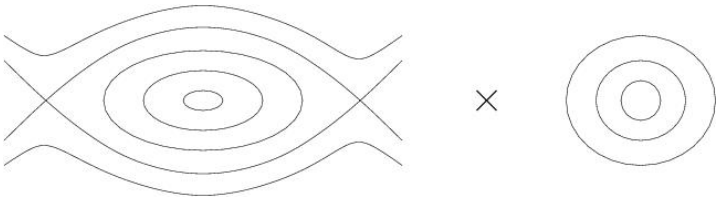
[3] Strictly speaking, the windowing method only needs that they are approximately invariant

*If* $0 < \mu \ll \varepsilon \ll 1$.
*Then, there exist orbits of the Hamilton's equation corresponding to* (1) *with*

$$|I(T) - I(0)| > 1 \ .$$

We point out that the Hamiltonian (1) satisfies the conditions of KAM and Nekhoroshev theorems (in spite of being partially degenerate). [Lla01, Nie07] so that the for $\varepsilon, \mu$ small, the orbits that satisfy the conclusion occupy a small measure (these orbits cannot be in KAM tori) and $T$ has to be very large (by Nekhoroshev's theorem). This gives an idea of the subtleness of the phenomenon.

The system (1) can be easily understood for $\varepsilon > 0, \mu = 0$ since it is a product of two simple systems (a rotator and a pendulum). We note, in particular that the



$\Phi_1, I_1$ plane $\times$ $\Phi_2, I_2$ plane.

**Fig. 1** Illustration of the dynamics of the time one map of the dynamics of (1) for $\varepsilon > 0, \mu = 0$

manifold $\Lambda$, obtained by fixing the pendulum variables to the hyperbolic fixed point, (i.e. $(I_1, \Phi_1) = (0,0)$) and letting the $(I_2, \Phi_2)$ vary is a normally hyperbolic manifold. Clearly, $\Lambda$ is topologically an annulus $\mathbb{R} \times \mathbb{T}^1$.

It will be important (for other mechanisms) to remark that the manifold $\Lambda$ is normally hyperbolic.

The main remark in [Arn64] is that the manifold $\Lambda$ is foliated by invariant tori (corresponding to fixing $I_2$). These tori are not normally hyperbolic (perturbations along the $I_2$ direction do not grow exponentially), but they are *whiskered tori* . That is, tori, whose normal directions contain stable directions (i.e. directions which contract exponentially fast in the future) and unstable directions (i.e. directions that contract exponentially fast in the past). The rates of contraction in the future and in the past are the contracting and expanding eigenvalues of the fixed point of the pendulum. It is easy to see that they are equal to $\lambda = \mp \varepsilon^{1/2}$.

It is shown, in general, that to whiskered tori, one can attach invariant stable (resp. unstable) manifolds consisting of the orbits which converge exponentially fast –with a rate similar to the rate of convergence of the linearization — in the future (resp. in the past). In the uncoupled case that we are considering now, the stable and unstable manifolds can be computed explicitly. The (un)stable manifolds are just the product of the tori and the separatrix of the pendulum. In particular, the stable and unstable manifolds of a torus agree.

Now, we consider $0 < \mu \ll \varepsilon$ and we will treat the term containing $\mu$ as a perturbation. In such a case, we can use the general theory of whiskered tori and their manifolds. The application of the general theory to (1) is rather simple because the example has been chosen carefully so that the perturbation and its gradient vanish on $\Lambda$. Hence, the family of tori, remains the same. It is part of the general theory that the tori keep being whiskered under the new dynamics and that they have (un)stable manifolds. Furthermore, the manifolds depend smoothly on $\mu$. The first order in the $\mu$ expansion can be computed easily by matching powers in formal expansions [4] and it is not difficult to show that the manifolds of nearby tori intersect transversally. In some ways the result is to be expected since the $\mu$ term, even if leaving $\Lambda$ invariant, is significant in the region occupied by the whiskers. It would be very easy to make perturbations with compact support intersecting the separatrices and which move them.

The construction so far, for any $\delta > 0$ allows to construct a $\delta$ pseudo-orbit that moves $I_2$ by 1. If we start in a torus $\tau$ with an irrational rotation, we wait for the appropriate moment, then, jump in its unstable manifold, in such a way that the orbit is also in the stable manifold of another torus $\tau'$. Once we are close enough to $\tau'$, we jump into a torus with an irrational rotation – such tori are dense –. Then, we can restart again.

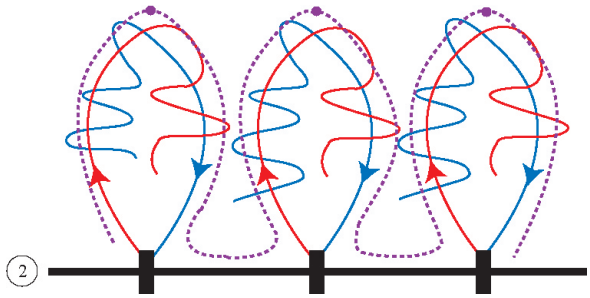



**Fig. 2** Illustration of some orbits in the dynamics of (1) for $0 < \mu \ll \varepsilon$. The 2 refers to the fact that $\Lambda$ is 2-dimensional.

Unfortunately, this step does not allow to take the limit $\delta \to 0$ since the orbits change widely. If we make $\delta$ smaller, the orbits we constructed have to give more turns till the irrational rotation sets the phase exactly right for the jump.

[4] Of course, matching powers in formal expansions does not justify that the expansions exist. In this case, using the general theory of whiskered tori, we know that these expansions exist. Historically, power matching in cases similar to this one was routinely used many years before it was justified.

## 2.1 The obstruction property

The program of [Arn64] contains an extra step, the *obstruction property* – that constructs a true orbit shadowing some of the pseudo-orbit.

There is a substantial literature on the obstruction argument. We just call attention to the reader that part of the literature includes – sometimes without making it explicit – the assumption that one of the terms in the normal form of the torus vanishes. Some papers rely on normal forms to high order – hence only apply comfortably to $C^\infty$ or $C^\omega$ systems. Others assume that all the tori can be fit in a common system of coordinates. In some papers, the construction depends on the number of tori that the orbit has to explore. Therefore, increasing the number of tori changes substantially the orbit (the time the orbit has to spend in the neighborhood of each tori increases with the total number of tori to be visited). These constructions do not allow to pass to the limit and construct orbits which visit infinitely many tori. Of course, the diversity of arguments is just a reflection of the fact that there are many types of diffusing orbits each with different quantitative and qualitative properties. We cannot survey the rather extensive literature but just call attention on some points to watch for. We certainly hope somebody will write such a survey.

We also note that the obstruction argument is not the only way of constructing orbits which shadow the pseudo-orbits. In this lecture we will discuss the method of correctly aligned windows in other context, which is a topological method – applications to the shadowing of whiskered tori happen in [Rob02, GR04, CG03]. There are also variational methods [Bes96, BBB03, BCV01] for this step.

In practice, the step of constructing the shadowing orbits is what controls the time $T$ in the statement of the result. Many of the methods above lead to different estimates for $T$ and presumably to different orbits. This again reinforces the belief that diffusion is really a superposition of several mechanisms. Here, we will just present some simple argument – we follow closely [DLS00] – which makes more precise some of the ideas in the original papers [AA67] –. See also [FM01, FM03, FM00, Cre97]. The main ingredient is a somewhat sharp version of the $\lambda$-lemma – for example that in [FM00] and a point set topology argument. Since no normal forms to higher order are used the method has only modest differentiability requirements. It can also accommodate infinitely long chains. A more elaborate argument along similar lines, but also giving more control on the orbits appears in [DLS06c].

If $\{\mathscr{T}_i\}_{i=1}^\infty$ is a sequence of whiskered tori with irrational rotations and $\{\varepsilon_i\}_{i=1}^\infty$ a sequence of strictly positive numbers, we can find a point $P$ and an increasing sequence of numbers $T_i$ such that

$$\Psi_{T_i}(P) \in N_{\varepsilon_i}(\mathscr{T}_i)$$

where $N_{\varepsilon_i}(\mathscr{T}_i)$ is a neighborhood of size $\varepsilon_i$ of the torus $\mathscr{T}_i$. Here $\Psi_t$ represents the flow associated to the system.

To establish this result, note that if $x \in W^s_{\mathscr{T}_1}$, we can find a closed ball $B_1$, centered at $x$, and such that

$$\Psi_{T_1}(B_1) \subset N_{\varepsilon_1}(\mathscr{T}_1). \tag{2}$$

By the $\lambda$-lemma,

$$W^s_{\mathscr{T}_2} \cap B_1 \neq \emptyset.$$

Hence, there is a closed ball $B_2 \subset B_1$, centered at a point in $W^s_{\mathscr{T}_2}$ such that, besides satisfying (2):

$$\Psi_{T_2}(B_2) \subset N_{\varepsilon_2}(\mathscr{T}_2).$$

Proceeding by induction, we can find a sequence of closed balls

$$B_i \subset B_{i-1} \subset \cdots \subset B_1$$
$$\Psi_{T_j}(B_i) \subset N_{\varepsilon_j}(\mathscr{T}_j), \quad i \leq j.$$

Since the closed balls are compact, they have non-empty intersection and any point in the intersection satisfies the desired property.

This argument as presented above does not give estimates on the time needed to transfer. On the other hand, it gives several other information on the orbits. For example, the orbits never leave an $\varepsilon$ neighborhood of the segments of $W^{s,u}_{\mathscr{T}_i}$ so that we can be sure that the energy, or the actions, are described, up to errors of size $\varepsilon$ by the values along the sequences visited. For future purposes, it is important to point out that the argument only uses that the tori are whiskered and it does not use at all the way that the tori fit together. Later, in Section 4, we will apply this argument to sequences of tori which are not homotopic and that, therefore, cannot be fit in common system of coordinates.

## *2.2 Some final remarks on the example in [Arn64]*

The example (1) is remarkable for many reasons. Here, we just note that the diffusion happens in places where there are no resonances. Indeed, detecting the diffusion numerically in (1) is much harder than in other examples. It is somewhat ironic that much mathematical effort was spent proving instability in models for which the result is indeed very weak.

One feature of the example (1) which is important for the construction is that the second perturbation vanishes identically on a manifold. This is very non-generic and, indeed, it does not happen in many models of interest. [5]

We have done the first order expansion in $\mu$, assuming $\varepsilon > 0$ and fixed. The dependence on $\varepsilon$ of this theory is rather complicated. The first order term in the expansion in $\mu$ of the angle between the stable and the unstable manifoldso of a torus is of the order $\exp(-A\varepsilon^{-1/2})\mu\varepsilon$. The remainder, on the other hand, is not easy to

[5] One should remark, however, that it does happen in some models of interest. For example [dlLRR07] shows that perturbations which vanish on manifolds, happen naturally in some systems of physical interest such as billiards with moving boundaries and in oscillators provided that they have some symmetries and that an analysis very similar to that of [Arn64] leads to the existence of orbits of unbounded energy in these systems

bound better than $C\mu^2\varepsilon^2$. This is, of course, perfectly fine if $\mu \ll \exp(-A/2\varepsilon^{-1/2})$, but if $\mu = \varepsilon^p$, then, it could happen that the leading order of the perturbation in $\mu$ does not give the whole story.

As a consequence, the treatment above – based on just first order perturbation theory in $\mu$ can not establish the existence of instability in a whole ball in $\varepsilon, \mu$ or for $\mu = \varepsilon^p$.

# 3 Return to a normally hyperbolic manifold. The two dynamics approach

In the exposition of [Arn64] in the previous section, we have emphasized the normally hyperbolic manifold $\Lambda$ – which only appeared implicitly in [Arn64].

The reason is that the persistence of normally hyperbolic manifolds holds rather generally as was recognized in the 60's. [Sac65, Fen72, Fen74, HP70, HPS77, Pes04]. Of course, for examples other than the carefully chosen (1), one does not expect that the dynamics in the invariant manifold remains integrable. Indeed, as it is well known (we will present some ideas in Section4.3) the resonant tori break up under perturbation so that the foliation by invariant tori gets interrupted.

The general theory of normally hyperbolic invariant manifolds establishes not only the persistence of the normally hyperbolic invariant manifolds but also the existence of stable and unstable manifolds and the regularity of the dependence on parameters of these objects. A short summary of the theory of normally hyperbolic invariant manifolds can be found in Appendix A. Of course, this is no substitute for the references above.

The theory of dependence with respect to parameters, justifies the perturbation theory.

## *3.1 The basics of the mechanism of return to a normally hyperbolic invariant manifold*

The basic assumption is that the stable and unstable manifolds of $\Lambda$ intersect transversally. This means that there are orbits that leave the manifold but come back. We will refer to these orbits as homoclinic excursions. Note that a simple dimension counting — justified by the regularity given by the theory of normally hyperbolic invariant manifolds —- shows that the set of homoclinic excursions is, locally, a manifold of the same dimension as $\Lambda$. Hence, we expect that there is an open set $H_- \subset \Lambda$ such that all the points in $H_-$ can make an arbitrarily small jump and, go into the unstable manifold of $\Lambda$, perform an homoclinic excursion and come back to $\Lambda$. Since this homoclinic excursion moves the orbit far away from $\Lambda$ it is quite possible that it can be really affected by the perturbation and the action variables can change. In Section 3.2, we will describe some concrete descriptions of these sets.

When the system is conservative, one expects that some of the homoclinic excursions are *favorable* – e.g. the excursion gains energy or action – and others are *unfavorable* – the excursions looses energy or actions. Since there are rather explicit formulas – which we will explain in Sections 3.2 and 4.1, one expects that the points in $H_-$ which lead to favorable or unfavorable excursions are open sets separated by a codimension 1 manifold, which can be calculated as the zero set of a function (in the models discussed in Section 4 perturbative formulas for this function are rather standard).

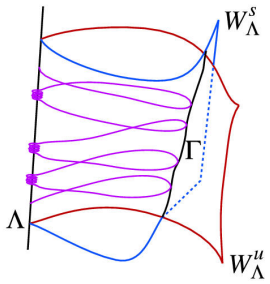



**Fig. 3** Illustration of orbits that gain energy by intertwining homoclinic excursions with staying around an invariant manifold

Note that $H_-$ and the separation between the favorable and unfavorable regions depend very strongly on the perturbation far away from $\Lambda$. Hence, we can expect that the dynamics on $\Lambda$ — which is unaffected by the perturbations away from the manifold — is completely unrelated to the separation between favorable and unfavorable excursions. Hence, unless this separation is invariant under the dynamics in $\Lambda$, one can stay around $\Lambda$ for a carefully chosen time and move into the favorable region. We emphasize that, explicit perturbative computations can give approximations to the manifold separating the favorable from the unfavorable excursions, so that a finite computation can establish that there are orbits in $\Lambda$ that move into the favorable region.

In this way, for many systems, one can construct pseudo orbits by interleaving orbits that follow a homoclinic excursion and orbits that remain in $\Lambda$ so that we go from the end of a homoclinic excursion to another favorable excursion. This can be compared to primitive sailing: When the wind is favorable, the boat moves. When the wind turns bad, it moves to the coast and anchors waiting till the wind becomes favorable again.

Of course, if one is interested in true orbits rather than on $\delta$ pseudo-orbits with $\delta$ arbitrarily small, one still needs an extra step – shadowing or obstruction. Some versions of these arguments are discussed in Section 2.1 and 5.

To make the above heuristic ideas rigorous, one uses: a) a tool to describe the homoclinic excursions, which allows explicit computations b) some explicit description of the dynamics on $\Lambda$, c) some tools to pass from the pseudo-orbits to the orbits.

Of course, the analysis of the dynamics restricted to $\Lambda$ is just the general problem of dynamical systems. The description of homoclinic intersections will be undertaken in Section 3.2.

We note that the scattering map is not the only way to discuss homoclinic excursions. The paper [Tre02a, Tre02b] introduce the *separatrix map*. We also call attention to [BK05].

## *3.2 The scattering map*

The scattering map is a particularly convenient way of describing the behavior on a homoclinic excursion. It was introduced explicitly in [DLS00] as a geometrically natural alternative to Melnikov theory so that issues of domain and monodromy could be discussed in detail. Related ideas for center manifolds were introduced in [Gar00]. A much more systematic theory of the scattering map was developed in [DLS06a].

An orbit is homoclinic if the future and the past converge exponentially fast to $\Lambda$. We adopt the same notation as in Appendix A.

We recall that the stable and unstable manifolds can be decomposed into stable manifolds of single points, namely: $W^s_\Lambda = \bigcup_{x\in\Lambda} W^s_x$, $W^u_\Lambda = \bigcup_{x\in\Lambda} W^u_x$. The above decompositions are are foliations because if $x, y \in \Lambda$, $x \neq y$, then $W^s_x \cap W^s_y = \emptyset$, $W^u_x \cap W^u_y = \emptyset$. We will refer to these foliations as $\mathscr{F}_{s,u}$ respectively.

Using the foliations $\mathscr{F}_{s,u}$ we can define the *"wave operators"* $\Omega_+$, $\Omega_-$

$$\Omega_\pm : W^{s,u}_\Lambda \longrightarrow \Lambda \tag{3}$$

defined by

$$x \in W^s_{\Omega_+(x)} \qquad x \in W^u_{\Omega_-(x)} \tag{4}$$

If there is a manifold $\Gamma \subset W^s_\Lambda \cap W^u_\Lambda$ such that $\Omega_-$ is a diffeomorphism from $\Gamma$ to its range $\Omega_-(\Gamma) \equiv H^\Gamma_-$, then we can define $(\Omega^\Gamma_-)^{-1} : H^\Gamma_- \to \Gamma$ and relatedly,

$$s^\Gamma = \Omega_+ \circ (\Omega^\Gamma_-)^{-1} \tag{5}$$

This set $H_-\Gamma$ is the set of initial points of trajectories having the property that an small push can make them go through $\Gamma$. This is a more precise version of the set $H_-$ wich we discussed in Section 3.1. The set $H^\Gamma_-$ specifies that the connections go through $\Gamma$.

The map $s^\Gamma : H_- \to H_+$, gives an encoding of the homoclinic excursions that pass through $\Gamma$. If we consider one such excursion, the orbit is asymptotically close to one orbit in $\Lambda$ in the past and to another orbit in $\Lambda$ in the future. The map $s^\Gamma$

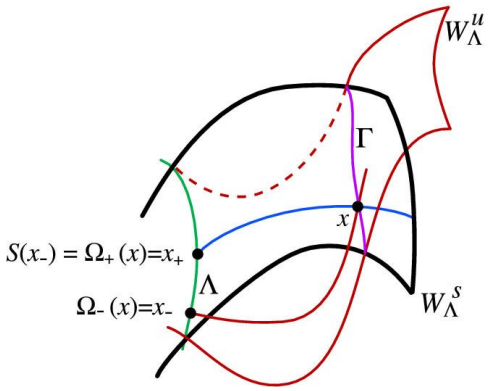



**Fig. 4** Illustration of the definition of the scattering map.

gives the future orbit as a function of the asymptotic orbit in the past. [6] Of course, the scattering map depends very strongly on the manifold $\Gamma$ we have chosen. Escaping from $\Lambda$ along different routes will, clearly, have very different effects and the scattering map will be very different. Some examples of celestial mechanics with explicit computations appear in [CDMR06].

Now, we discuss some natural hypothesis that imply that $\Omega_-^\Gamma$ is invertible from its range to $\Gamma$ and that this is maintained under perturbations and that there is good dependence with respect to parameters. Basically, we will reduce the definitions to transversality conditions so that the implicit function theorem gives the persistence and smooth dependence on parameters.

A natural set of conditions to define scattering map is that for all $x \in \Gamma$,

$$\begin{aligned} T_x W^s_\Lambda + T_x W^u_\Lambda &= T_x M \\ T_x W^s_\Lambda \cap T_x W^u_\Lambda &= T_x \Gamma \end{aligned} \tag{6}$$

$$\begin{aligned} T_x W^s_{\Omega_+ x} \oplus T_x \Gamma &= T_x W^s_\Lambda \\ T_x W^u_{\Omega_- x} \oplus T_x \Gamma &= T_x W^u_\Lambda \end{aligned} \tag{7}$$

The conditions in (6) mean that $W^s_\Lambda$, $W^u_\Lambda$ *"intersect transversally"* along $\Gamma$. The first condition in (7) means that $\Gamma$ is *"transversal to the foliation"* $\{W^s_x\}_{x\in\Lambda}$ inside $W^s_\Lambda$. The second equation in (7) means that $\Gamma$ satisfies an analogous property relative to the unstable foliation.

If we have (6) for just one $x_0$, the implicit function theorem tells us that we can find a smooth manifold $\Gamma$ containing $x_0$ such that (6) holds for all $x \in \Gamma$. Since the

[6] This is remarkably similar to the definition of the scattering matrix in the time-dependent scattering theory in quantum mechanics. Indeed, there are many more analogies and we have chosen the notation to reflect them.

Other classical analogues of quantum scattering theory, somewhat different from those considered here, were considered in [Hun68, BT79, Thi83] and in a more general context in [Nel69].

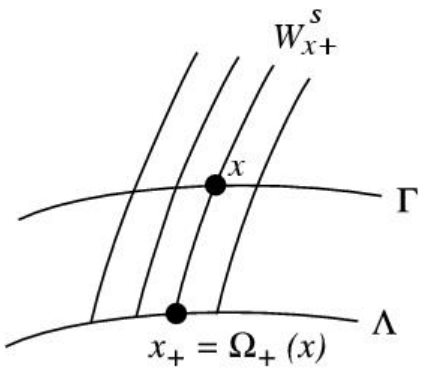



**Fig. 5** Illustration of the conditions in (7).

manifold $\Gamma$ is obtained applying the implicit function theorem, if both $W^s_\Lambda$, $W^u_\Lambda$, are $C^k$ manifolds in a neighborhood of $x$, then $\Gamma$ will also be a $C^k$ manifold.

Similarly, applying the the implicit function theorem, the regularity theory for the manifolds and their smooth dependence on parameters, discussed in Appendix A, we conclude that if $f_\varepsilon$ is a $C^1$ family and $f_0$ has a $\Lambda_0$, $\Gamma_0$ satisfying the normal hyperbolicity and transversality conditions, that there is a $C^1$ family of manifolds $\Lambda_\varepsilon$ which are normally hyperbolic and another family of manifolds $\Gamma_\varepsilon$ satisfying the properties. In the case that we can guarantee that $W^{s,u}_{\Lambda_\varepsilon}$ are $C^{\ell-1}$ families, we obtain that $\Gamma_\varepsilon$ is a $C^{\ell-1}$ family and we can also obtain smooth dependence on parameters for the $\Omega^{\Gamma_\varepsilon}_\pm$ and for the scattering map. [7]

The properties in (7) are very different. Even if the formulation of (7) does not require that the foliations $\mathscr{F}_{s,u}$ are smooth, they become more interesting when these foliations are $C^1$ foliations. In this case, the implicit function theorem tells that, when we move along $\Gamma$, we have to move across the foliation.

The implicit function theorem shows that, if the foliations $\mathscr{F}_{s,u}$ are $C^1$ – this is implied by properties of the hyperbolicity constants, so that it holds true in some $C^1$ open sets of examples – and (7) hold, then, $\Omega_\pm$ are locally invertible. Again, because this is just an application of the implicit function theorem and there is a good dependence on parameters, we obtain if (6), (7) are satisfied for a map, they will be satisfied – with a similar $\Lambda$, $\Gamma$ – for all the small $C^1$ perturbations. Furthermore, if we consider smooth families of maps, there will be smooth dependence on parameters.

*Remark 5.* One could argue heuritically that (7) could fail in a codimension 1 set of $\Gamma$ – transversality is a codimension 1 phenomenon –. Of course, this heuristic argument, could be false. Notably, the heuristic argument is false for the models

[7] The smooth dependence of a map in domains which are changing, should be understood in the sense that there is smooth family of maps from a fixed domain to the domains so that the composed map is smooth.

considered in [DLS00, DLS06c]. It however, applies to some examples considered in [DLS06b].

Nevertheless, as shown in [DLS06b], the existence of an open set is enough for the construction of orbits that move appreciable amounts. One can also note that one expects to have infinitely many $\Gamma$, each of which with a different scattering map. The argument does not require that all the excursions go through the same $\Gamma$, so that the set of points which cannot be moved by this argument should be empty in manu examples.

## *3.3 The scattering map and homoclinic intersections of submanifolds*

One important application of the scattering map is that it allows us to discuss transversal intersections of $W^u_{\Sigma_1}, W^s_{\Sigma_2}$ where $\Sigma_1, \Sigma_2 \subset \Lambda$ are invariant manifolds under the map $f$. One example is, of course, the whiskered tori inside the manifold $\Lambda$ that were discussed in Section 2. In Section 4 we will see other examples that are more challenging.

It was shown in [DLS00, DLS06b, DLS06c] that if, for some manifold $\Gamma$, satisfying (6) (7), we have [8]

$$s^\Gamma(\Sigma_1) \pitchfork^\Lambda \Sigma_2. \tag{8}$$

Then,

$$W^u_{\Sigma_1} \pitchfork W^s_{\Sigma_2}. \tag{9}$$

This result is useful because the hypothesis (8) is a hypothesis by calculations on the invariant manifold $\Lambda$. The conclusions is that the invariant manifolds are transverse in the full manifold $M$.

In the case that $\Sigma_1, \Sigma_2$ are invariant circles which are close together, the transversality of intersections is usually discussed using Melnikov theory. Notice, however that Melnikov theory – since it is based on first order calculations often done in a concrete coordinate system – requires that the manifolds $\Sigma_{1,2}$ are expressed in the same system of coordinates, in particular, they are homotopically equivalent. The above result, however, is coordinate independent. This is crucial for the applications in [DLS06b], discussed in Section 4, where $\Sigma_{1,2}$ are not topologically equivalent.

As we will see in Section 3.7 there are rather explicit – rapidly convergent – formulas for the perturbative computation of the scattering map. Therefore, the theory outlined above can give rather efficient ways of establishing intersections.

[8] We use the notation $\pitchfork^\Lambda$ to indicate that the manifolds intersect transversally as manifolds in $\Lambda$. In particular, when we use this symbol, we assume that the intersection is not empty.

### 3.4 Monodromy of the scattering map

Even if $\Omega_{\pm}^{\Gamma}$ are locally invertible, they could fail to be invertible in a domain which is large enough to include non-contractible closed loops. One interesting example was discussed already in [DLS00] and, in more detail in [DLS06c, DLS06a]. For example, when considering stable manifolds of a periodic orbit $\lambda$, the intersection manifold $\Gamma$ looks like a *helix*. That is, if we increase the phase of the intersection, then, eventually we go into a different homoclinic intersection of the time-1 map. This is a geometric counterpart of the fact that, in some calculation in first order perturbation theory of intersections of invariant manifolds – often called Melnikov theory – one has to add real variables to angle variables.

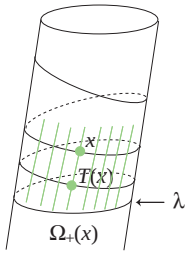



**Fig. 6** Illustration of the monodromy of the scattering map for the stable manifolds of periodic orbits.

### 3.5 Smoothness and smooth dependence on parameters

Note that the sufficient conditions (6), (7) that ensure the existence of the scattering map in a neighborhood are transversality conditions that are robust under perturbations. Hence, given a concrete system, they can be established with a finite precision calculation. Later, in Section 4.1 we will see how they can be verified by perturbative calculations from an integrable system. See [DLS06b, GL06a]. The conditions can also be verified numerically if one controls the precision of the calculations [CDMR06].

It follows from the general theory of dependence on parameters that, under the conditions (6), (7), and smoothness of the foliations $\mathscr{F}_{s,u}$ then, the scattering map is smooth jointly on the manifold and on parameters. [9]

## *3.6 Geometric properties of the scattering map*

So far, the discussion of the scattering map has only used normal hyperbolicity and regularity of the maps considered.

If the maps $f_\varepsilon$ have some geometric structure, the scattering map also inherits some geometric properties. Notably, if $f_\varepsilon$ is symplectic (resp. exact symplectic) and $\Lambda_0$ is a symplectic manifold (hence, exact symplectic if $f_\varepsilon$ is exact symplectic) then $s_\varepsilon$ is a symplectic (resp. exact symplectic) family of maps. This was proved in [DLS06a]. In the context of center manifolds it was proved in [Gar00].

There are two important consequences of the symplectic character.

- There are many techniques to discuss intersections of Lagrangian manifolds under symplectic mappings, see [Wei73, Wei79].
- There are very efficient perturbation theories for symplectic mappings. Historically this one of the reasons why Hamiltonian formalism was invented. We will discuss several versions of Hamiltonian perturbation theory here.

Taking advantage of both features at the same time, one gets a very efficient perturbative theory for the intersections of manifolds under the scattering map. In view of the results mentioned in Section 3.3, this is very useful to obtain transition tori.

In [DLS06a] it was proved that there is a unique smooth parameterization $k_\varepsilon(\Lambda_0) = \Lambda_\varepsilon$ such that $k_0$ is the immersion and that $k_\varepsilon^* \omega$ – the pull–back by $k_\varepsilon$ of the symplectic form $\omega$ – is independent of $\varepsilon$. This later condition is a natural normalization and it is shown in [DLS06a] that this natural normalization determines uniquely the deformation.

Then, denoting by $s_\varepsilon$ the scattering maps generated by a smooth family of manifolds $\Gamma_\varepsilon$ satisfying (6), (7), and invertibility of $\Omega_-^\Gamma$, we have that

$$\tilde{s}_\varepsilon \equiv k_\varepsilon^{-1} \circ s_\varepsilon \circ k_\varepsilon \tag{10}$$

is symplectic under $k_\varepsilon^* \omega \equiv k_0^* \omega$. Note that $\tilde{s}_\varepsilon : \Lambda_0 \to \Lambda_0$ can be thought of as the expression of $s_\varepsilon$ in the coordinates $k_\varepsilon$ mentioned above.

Furthermore, in [DLS06a], one can find explicit perturbative formulas for the canonical perturbation theory of $\tilde{s}_\varepsilon$. We will summarize them in Section 3.7.

[9] The discussion of smoothness with respect to parameters of the scattering map presents some technical annoyances such as that the domain of $s_\varepsilon$ is $\Lambda_\varepsilon$, which changes as $\varepsilon$ changes. An easy solution is to consider smooth (jointly with respect to the coordinates and the parameters) parameterizations $k_\varepsilon$ of the invariant manifold $\Lambda_\varepsilon$. That is $k_\varepsilon(\Lambda_0) = \Lambda_\varepsilon$. See Section 9.

### 3.7 Calculation of the scattering map

Given families of exact symplectic mappings there are very efficient ways of computing perturbation theories using the deformation method of singularity theory [LMM86].

If $g_\varepsilon$ is a family of exact symplectic mappings, it is natural to study instead the vector field $\mathscr{G}_\varepsilon$ generating the family.

$$\frac{d}{d\varepsilon} g_\varepsilon = \mathscr{G}_\varepsilon \circ g_\varepsilon. \tag{11}$$

The fact that $g_\varepsilon$ is exact symplectic for all $\varepsilon$ is equivalent to $g_0$ being exact symplectic and $\iota_{\mathscr{G}_\varepsilon}\omega = dG_\varepsilon$ (here $\iota_{\mathscr{G}_\varepsilon}\omega$ is the contraction of vectors and forms). Under enough regularity conditions, equation (11) admits a unique solution.

Hence, it is the same to work with $\mathscr{G}_\varepsilon$ or $G_\varepsilon$. The interesting thing is that the family of functions $G_\varepsilon$ satisfies much simpler equations. The reason is that the $\mathscr{G}_\varepsilon$ – and hence $G_\varepsilon$ can be thought as infinitesimal deformations and the only equations that one can form with infitesimal quantities are linear.

In the following, we will apply this idea to $g_\varepsilon$ being several of the families appearing in the problem. We will keep the convention of keeping the same letter for the objects corresponding to a family. We will use caligraphic for the vector field and capitals for the Hamiltonian.

In [DLS06a], it is shown that there are remarkably simple formulas for $\tilde{S}_\varepsilon$, the generator of the the map $\tilde{s}_\varepsilon$ – the expression of $s_\varepsilon$ in coordinates.

$$\begin{aligned}
\tilde{S}_\varepsilon &= \lim_{N_\pm\to+\infty} \sum_{j=0}^{N_- -1} F_\varepsilon \circ f_\varepsilon^{-j} \circ (\Omega_{\varepsilon-}^{\Gamma_\varepsilon})^{-1} \circ s_\varepsilon^{-1} \circ k_\varepsilon - F_\varepsilon \circ f_\varepsilon^{-j} \circ s_\varepsilon^{-1} \circ k_\varepsilon \\
&\quad + \sum_{j=1}^{N_+} F_\varepsilon \circ f_\varepsilon^{j} \circ (\Omega_{\varepsilon+}^{\Gamma_\varepsilon})^{-1} \circ k_\varepsilon - F_\varepsilon \circ f_\varepsilon^{j} \circ k_\varepsilon \\
&= \lim_{N_\pm\to+\infty} \sum_{j=0}^{N_- -1} F_\varepsilon \circ f_\varepsilon^{-j} \circ (\Omega_{\varepsilon-}^{\Gamma_\varepsilon})^{-1} \circ k_\varepsilon \circ s_\varepsilon^{-1} - F_\varepsilon \circ k_\varepsilon \circ r_\varepsilon^{-j} \circ s_\varepsilon^{-1} \\
&\quad + \sum_{j=1}^{N_+} F_\varepsilon \circ f_\varepsilon^{j} \circ (\Omega_{\varepsilon+}^{\Gamma_\varepsilon})^{-1} \circ k_\varepsilon - F_\varepsilon \circ k_\varepsilon \circ r_\varepsilon^{j}
\end{aligned} \tag{12}$$

Similarly, for Hamiltonian flows, we have

$$\begin{aligned}
S_\varepsilon &= \lim_{T_\pm\to\infty} \int_{-T_-}^{0} \frac{dH_\varepsilon}{d\varepsilon} \circ \Phi_{u,\varepsilon} \circ (\Omega_{\varepsilon-}^{\Gamma_\varepsilon})^{-1} \circ (s_\varepsilon)^{-1} \circ k_\varepsilon \\
&\quad - \frac{dH_\varepsilon}{d\varepsilon} \circ \Phi_{u,\varepsilon} \circ (s_\varepsilon)^{-1} \circ k_\varepsilon \\
&\quad + \int_0^{T_+} \frac{dH_\varepsilon}{d\varepsilon} \circ \Phi_{u,\varepsilon} \circ (\Omega_{\varepsilon+}^{\Gamma_\varepsilon})^{-1} \circ k_\varepsilon - \frac{dH_\varepsilon}{d\varepsilon} \circ \Phi_{u,\varepsilon} \circ k_\varepsilon
\end{aligned} \tag{13}$$

It is not difficult to see that the sums or the integrals converge uniformly.

The formulas (12) and (13) give the hamiltonian of the deformation as the integral of the generator of the perturbation over the homoclinic orbit minus the generator of the perturbation evaluated on the asymptotic orbits.

Note that, because of the exponential convergence of the homoclinic orbits and their asymptotic orbits, it is not difficult to see that the integrals in (12) and (13) converge exponentially fast. In [DLS06a] one can also find that derivatives up to an order (which is given by ratios of convergence exponents) also converge exponentially fast.

The effect of the homoclinic excursions on slowly changing variables can be computed using more conventional methods – we will present some of these computations in Section 4.1 –.

One novelty of the geometric theory presented in this section is that it allows computation of the effect of the homoclinic excursions not only on the slow variables, but also on the fast variables.

Notice also that, we can compute the intersection between objects of different topologies very simply. This extends many calculations usually done using Melnikov theory. It suffices to apply (8). Note that the present theory only involves convergent integrals. This was somewhat controversial in the so-called Melnikov theory. See [Rob88]. [10]

The Hamiltonian theory is particularly effective when the manifolds $\Sigma$ are level sets of a function. We will see some examples in Section 4.6.

## 4 The large gap model

The model is basically a rotor coupled to one or several penduli and subject to a periodic perturbation.

This model was introduced in [HM82], but it appears naturally as a model of the motion near a multiplicity 1 resonance. A fuller treatment of multiplicity 1 resonances appears in [DLS07].

One could consider that it is a version of the example (1) when we set $\varepsilon = 1$ (hence rename as $\varepsilon$ the parameter $\mu$ in (1)) but we allow the perturbing term to be a general one. In the paper [GL06b] it was remarked that the fact that the pendulum variables have only 1 degree of freedom can be easily removed and one could consider many penduli. Hence, the geometric treatment can be easily generalized to the case that the hyperbolic variables have several components.

Hence, we consider the model

[10] Unfortunately, many references in Melnikov theory still invoke the use of Melnikov functions given by integrals of quasi-periodic functions. The textbook explanation is that these integrals converge along subsequences. Unfortunately, the resulting limit – and hence the predictions of these theories – depend on the sequence taken, so that the textbook explanation cannot be true. The real explanation is that these references forgot to take into account some important effect. In many cases, it is the change of the target manifold.

$$H_\varepsilon(p_1,\dots,p_n,q_1,\dots,q_n,I,\phi,t) = \sum_{i=1}^{n} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i)\right) + h_0(I) + \varepsilon h(p_1,\dots,p_n,q_1,\dots,q_n,I,\phi,t;\varepsilon), \tag{14}$$

where $(p_i,q_i)$, $(I,\phi)$ are symplectically conjugate. We will assume that $V_i'(0)=0$, $V_i''(0)>0$. This means that $V_i$ has a non-degenerate local minimum – that we set at 0. We will also assume that the pendulum $P_i$ has a homoclinic orbit to 0. This is implied by the fact that there is no other critical point $p$ with $V_i(p)=0$. Both conditions are implied by $V_i$ being a Morse function.

The version of (14) considered in [HM82, DLS06b, GL06a] consider only the case $n=1$, but, as we will see, the complications introduced by several variables is not too important. A full treatment of (14) for general $n$ appears in [GL06b]. We will explain it in Section4.1.

One extra assumption in [DLS06b] – which we will maintain in the discussion in this section – is that the perturbation term $h$ was a trigonometric polynomial in the angle variables. This assumption simplifies the calculations since there is only a finite number of resonances to be studied. It allows us to emphasize the geometric objects appearing at each resonance. When $h$ is not a polynomial, for each value of $\varepsilon>0$ it suffices to study a finite number of resonances, but the number of resonances to be considered is $\varepsilon^{-\alpha}$. One needs to do some rather explicit quantitative estimates on the resonances. The assumption that the perturbation is a polynomial has been removed by very different methods. The paper [DH06] contains a very deep study of resonances taking into account the effect of the size of the Fourier coefficients on the size of the resonant region. The paper [GL06b] considers very large windows, much larger than the resonance zones and uses the method of correctly aligned windows to conclude existence of diffusion **without having to analyze** what happens in the region of resonance. This leads to less conditions than the analysis in [DLS06b, GL06a]. Also, the method in [GL06b] leads to optimal estimates on the time.

The analysis of (14) we will present starts by noting that $\Lambda_0 = \{p_i=0, q_i=0\}$ is a normally hyperbolic invariant manifold for the time-1 map. Applying the theory of normally hyperbolic manifolds, we conclude that, for $\varepsilon$ small enough, it persists. In contrast with the example (1), the motion on the invariant manifold will not remain integrable. Indeed, the foliation of KAM tori will present gaps of size $\approx \varepsilon^{1/2}$. In the rest of the section, we will describe how to construct orbits that indeed jump over the resonance zone. [11]

[11] The paper [HM82] showed only that there were heteroclinic intersections between some whiskered tori. The length of the heteroclinic chains constructed in [HM82] goes to 0 as $\varepsilon \to 0$. This was the meaning of *Arnol'd diffusion* adopted in that paper. It is very interesting to compare the Melnikov theory developed there with the based on the scattering map.

## 4.1 Generation of intersections. Melnikov theory for normally hyperbolic manifolds

In the model (14), even if the manifold $\Lambda_0$ is normally hyperbolic, its stable and unstable manifolds coincide.

In this section, we want to argue that, under some non-degeneracy conditions on $h$ which we will make explicit, for $0 < |\varepsilon|$, there is a manifold $\Gamma_\varepsilon$ satisfying the conditions (6), (7). Furthermore, one can define the scattering map in a patch which is rather large and uniform with respect to $\varepsilon$.

The fact that there is a $\Gamma_\varepsilon$ which depends smoothly on parameters and, in particular, can be continued through $\varepsilon = 0$ is well known to experts and we present the ideas of a simple proof later. See also [GL06b]. These are sometimes called *primary intersections* of the stable and unstable manifolds, to distinguish them from other intersections which do not have a limit as $\varepsilon \to 0$. See [Mos73, p. 99 ff.]. Subsequent steps of the construction of diffusing orbits could use any of these intersections for which the next non-degeneracy assumptions can be verified. The calculations we will develop here will work just as well for any of the primary intersections. The use of the secondary intersections deserves more study.

Very elegant geometric theories of intersections of stable and unstable manifolds can be found in [LMS03]. In these lectures, we will follow [GL06b] and present a very simpleminded calculation for the model using coordinates. The paper [GL06b] contains significantly more details than those presented here.

We call attention that the calculation here does not assume that the variables $I, \phi$ in (14) are one-dimensional. This will play a role in Section 7.

A key observation is that, by the theory of normally hyperbolic manifolds, we already know that $\Lambda_\varepsilon$, $W^{s,u}_{\Lambda_\varepsilon}$ depend smoothly on parameters. We just need to compute explicitly what are the derivatives of these objects. The non-degeneracy conditions alluded above are just that the first order in $\varepsilon$ calculation predicts an intersection satisfying (6), (7). If the first order perturbation predicts a transversal intersection, the implicit function theorem allows us to conclude that indeed there is an intersection, and that the formal calculation gives the leading order.

For this calculation, the fundamental theorem of calculus will play an important role, hence it is better to consider flows rather than time-1 maps. To make it autonomous, we will just add a variable $t$. We will use the notation $\tilde{\Lambda}$ to refer to the invariant manifold in these coordinates.

For each of the penduli, we choose a homoclinic orbit $x_i$ and consider the unperturbed homoclinic manifold $\{(x_1(\tau_1), x_2(\tau_2), \ldots, x_n(\tau_n))\}$. [12] The variables $\tau_i$ are variables parameterizing the separatrix of the $i$ pendulum.

[12] Note that, in general, each of the penduli will have 2 homoclinic orbits to the critical point (one going in one direction and the other going in the opposite direction). So that, there will be $2^n$ homoclinic manifolds with parameterizations similar to the ones considered in the text. Since the conditions we will considering be are sufficient conditions for existence of unstable orbits, having many orbits at our disposal makes it more likely that we have instability.

We note that in a neighborhood of the homoclinic manifold – excluding a neighborhood of the critical points – we can extend the variables $\tau_i$. The variables $\tau_i$ and $P_i$ constitute a good system of coordinates in this neighborhood.

Again, appealing to the smoothness of the dependence of the stable manifolds on parameters, we know that the perturbed manifolds can be written as the graph of a function that gives the $P_i$ as a function of $\tau, I, \phi, t$. Furthermore, this function will depend smoothly on parameters. Our only goal, then, is to compute the first order expansion of this function, knowing already that such an expansion exists.

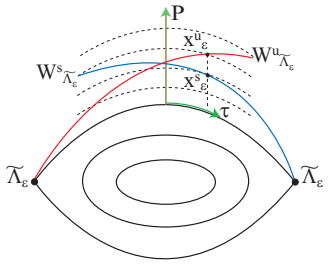



**Fig. 7** Illustration of the system of coordinates in a neighborhood of the homoclinic manifold

We will denote the time evolution of a point by $\Psi^s_\varepsilon$. Remember that, to make the system autonomous, we consider $t$ as a variable, which takes values on a circle. We will denote the invariant manifolds in the extended phase space as $\tilde\Lambda$.

Let $x$ be a point in $W^s_{\tilde\Lambda_\varepsilon}$, by the fundamental theorem of calculus, we have, for any $T$,

$$\begin{aligned} P_i(x) - P_i(\Omega^\varepsilon_+ x) =& P_i(\Psi^T_\varepsilon(x)) - P_i(\Psi^T_\varepsilon(\Omega^\varepsilon_+ x) \\ & - \int_0^T \frac{d}{ds}\left[P_i(\Psi^s_\varepsilon(x)) - P_i(\Psi^s_\varepsilon(\Omega^\varepsilon_+ x))\right] ds \end{aligned}$$

and, taking limits $T \to \infty$, we obtain

$$P_i(x) - P_i(\Omega^\varepsilon_+ x) = -\int_0^\infty \frac{d}{ds}\left[P_i(\Psi^s_\varepsilon(x)) - P_i(\Psi^s_\varepsilon(\Omega^\varepsilon_+ x))\right] ds \tag{15}$$

Now, recalling that we are only computing up to order $\varepsilon$, we can simplify significantly the formula.

We note that because $P_i$ has a critical point at 0, we have $P_i(\Omega^\varepsilon_+ x) = O(\varepsilon^2)$, We also note that

$$\begin{aligned}\frac{d}{ds}\left[P_i(\Psi^s_\varepsilon(x)) - P_i(\Psi^s_\varepsilon(\Omega^\varepsilon_+ x)\right] &= \varepsilon\left(\{P_i,h\}\circ\Psi^s_\varepsilon(x)) - \{P_i,h\}\circ\Psi^s_\varepsilon(\Omega^\varepsilon_+(x))\right)\\ &= O(\varepsilon)\end{aligned}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket.

Notice also that the integrand in (15) is converging exponentially fast to zero. Hence, we have:

$$P_i(x) = -\varepsilon\int_0^{c|\log(\varepsilon)|} ds, \left[\{P_i,h\}(\Psi^s_\varepsilon(x)) - \{P_i,h\}(\Psi^s_\varepsilon(\Omega^\varepsilon_+ x))\right] + O(\varepsilon^2)$$

Since the integral is over a finite interval, we observe that, if $|s| \le c|\ln(\varepsilon)|$, then

$$|\Psi^s_\varepsilon(x) - \Psi^s_0(x)| \le c|\ln(\varepsilon)|\varepsilon$$

Also, using the smooth dependence of the stable and unstable foliations, we obtain that

$$|\Psi^s_\varepsilon(\Omega^x_+) - \Psi^s_0(\Omega^0_+ x)| \le c|\ln(\varepsilon)|\varepsilon$$

Hence, we can transform the integral into

$$P_i(x) = -\varepsilon\int_0^{c|\log(\varepsilon)|} ds, \left[\{P_i,h\}(\Psi^s_0(x)) - \{P_i,h\}(\Psi^s_0(\Omega^0_+ x))\right] + O(\varepsilon^2|\ln(\varepsilon)|)$$

*Remark 6.* The above calculation identifies the derivative of the manifold with respect to $\varepsilon$ when we consider the $C^0$ topology of functions.

In the case that we know that the derivative in $C^r$ sense exists, the previous expression has to be the derivative in the $C^r$ sense too.

In [GL06b], one can find justification of the slightly stronger result that the integrals above converge uniformly in $C^r$ – provided that the Hamiltonians are uniformly $C^{r+2}$.

A very similar formula – reversing the time – can be obtained for an expression of the unstable manifold as a graph. Subtracting them, we obtain an expression for the first order expansion of the separation $\Delta$ of the $P_i$ coordinates of the manifolds as a function of the $\tau_i, I, \phi, t$

$$\Delta_i(\underline{\tau}, I, \phi, t; \varepsilon) = \varepsilon\Delta^0_i(\underline{\tau}, I, \phi, t) + O(\varepsilon^2)$$

where the $O(\varepsilon^2)$ can be understood in the sense that the $C^1$ norm is bounded by $C\varepsilon^2$.

The implicit function theorem shows that if we find a zero of $\Delta^0_i = 0$ which is non-degenerate (i,e, rank$D_{\underline{\tau}}\Delta^0 = n$) then we can find $\tau^*(\varepsilon, I, \phi, t)$ such that $\Delta(\tau^*(\varepsilon, I, \phi, t), I, \phi, t; \varepsilon) = 0$. Hence, substituting in the variables $P$ we can onbtain a parameterization of the intersection.

A more detailed analysis shows that the expressions of $Delta_i$ are derivaties of a potential function with some periodicities [DR97]. Hence they have to have zeros. The assumption that these zeros are non-degenerate is a mild non-degeneracy assumption that can be verified in practical problems. It also holds generically. The case $n = 1$ is studied in great detail in [DLS06b]. In [GL06b] one can find an study of how to produce several of these solutions for $n > 1$.

## *4.2 Computation of the scattering map*

The calculation of the scattering map in this case can be done as a particular case of the general theory of Section 3.2.

Notice that the formulas (12) are given in terms of limit of the intersection as $\varepsilon \to 0$, which we computed in the previous section using the easy part of the Melnikov theory.

The calculation in [DLS06b], was done by a different method since at the time that [DLS06b] was written, the authors were not aware of the symplectic theory of the scattering map.

The method of [DLS06b] was more elementary. Only the effect of the scattering map in one of the coordinates was computed. This was done using the fact that one of the coordinates in the invariant manifold – namely the energy – has a slow variation, so that in the calculation of the change of energy along a homoclinic excursion, one can use – up to the accuracy needed – just the fundamental theorem of calculus integrating over the unperturbed trajectory. The calculation can be done in very similar way to the calculation done in Section 4.1. [13] The fact that in [DLS06b] one only got control on one of the variables made the calculation of subsequent properties more complicated than what is nowadays possible using geometric theory. See [DLS07]. On the other hand, the calculation based on estimating the change of energy is natural for the purposes of the study of the intersection with KAM tori – which are given as level sets of the averaged energy.

For the purpose of this exposition, we will just mention that, for the model considered, once we settle on one primary homoclinic intersection, the scattering map can be computed as an explicit perturbation series with well controlled remainders. As in all the steps of this strategy, the calculations required can be done by very different methods. Te more modern methods, taking more advantage of geometric cancellations seem more efficient even if the older methods can compute some features faster.

The conclusions is that – under conditions which can be checked explicitly and which, in particular, hold generically – the domain of definition of the scattering map contains a set which is independent of $\varepsilon$ as $\varepsilon \to 0$. We call attention to the

[13] The actual calculation done in [DLS06b] uses not the energy – which is easily seen to be an slow variable – but rather a linear approximation to the energy –. This makes only higher order differences. This linear approximation had been used customarily in the literature. At the time that [DLS06b] was written, it was important to make contact with the previous literature.

fact that the formulas for the scattering map depend heavily on the behavior of the perturbation along the whole homoclinic excursion.

### *4.3 The averaging method. Resonant averaging*

The averaging method for nearly integrable systems goes back at least to [LP66]. Modern expositions are [LM88, AKN88, DG96]. An introduction for practitioners is [Car81]. See also [Mey91].

The basic idea is very simple. Given a quasi-integrable system, one tries to make changes of variables that reduce the perturbed system to another integrable system up to high powers in the perturbation parameter. This is accomplished by solving recursively cohomology equations.

There are many contexts and variations which make the literature extensive, even if there is only one guiding principle. For example one can consider autonomous perturbations or periodic perturbations, maps, flows etc. There are different possible meanings of "as simple as possible". One difference that leads to several variants is the fact that one can parameterize perturbations in different ways (generating functions, several types of Lie Series, deformation method, etc.) A systematic comparison of differences between these perturbation theories was undertaken in [LMM86].

In the present problem, we consider periodic perturbations of integrable flows with one degree of freedom. To make comparisons with the literature easier, it will be convenient to make the system autonomous and symplectic by adding an extra variable $A$ symplectically conjugated to $t$

$$H_\varepsilon(I,\phi,t,A) = H^0(I) + A + \varepsilon H^1(I,\phi,t) + \varepsilon^2 H^2(I,\phi,t) + \cdots \tag{16}$$

where, of course, $H(I,\phi,t+1,A) = H(I,\phi,t,A)$, so that $t$ can be considered as an angle variable. The $A$ is added to keep the symplectic structure. Notice that it does not enter into the evolution of the other variables.

Again, for the sake of expediency in this presentation, we will omit considerations of issues of differentiability, estimates of reminders etc. We refer to [DLS06b, Section 8], but the averaging method is covered in many other references, including some of the lectures in this volume.

For simplicity also, we will assume that all the terms in the expansion in $\varepsilon$ are trigonometric polynomials with the same set of indices. That is,

$$H^i(I,\phi,t) = \sum_{k,l \in \mathcal{N}_i \subset \mathbb{Z}^2} H^i_{k,l}(I) \exp(k\phi + lt). \tag{17}$$

Note that in the Appendix A, we show that this assumption for the case that we are interested in, follows from the assumption that the $h$ in (14) is a trigonometric polynomial. The general theory of averaging does not require this assumption, but it involves several analysis consideration, which we prefer to avoid in an exposition.

We try to find a time periodic family of symplectic changes of variables $k_\varepsilon(I,\phi,t) = (I,\phi) + O(\varepsilon)$ in such a way that $H_\varepsilon(k_\varepsilon(I,\phi,t),t)$ is as simple as possible.

One possible way to try to generate the $k_\varepsilon$'s is to write them as the time-1 solutions of a differential equation

$$\frac{d}{ds}k_\varepsilon^s = \varepsilon J\nabla K_\varepsilon \circ k_\varepsilon^s, \quad k_\varepsilon^0 = \mathrm{Id},$$

where $J$ is the symplectic matrix. In this case, we consider the evolution in the $p,q,A,I,\phi,t$ variables and the $\varepsilon$ is just a parameter (this is not what we did in the section on deformation method). The gradient $\nabla$ refers to the $p,q,A,I,\phi,t$ variables. The function $K_\varepsilon$ is called the Hamiltonian. This way of parameterizing changes of variables is one of the variants of Lie transforms, [Car81, Mey91]. We will assume that $K_\varepsilon = \varepsilon K^1 + \varepsilon^2 K^2 + \cdots$,

It is well known from Hamiltonian mechanics [Arn89, AM78, Car81, Mey91] that

$$H_\varepsilon \circ k_\varepsilon = H^0 + \varepsilon(H^1 + \{H^0 + A, K^1\}) + O(\varepsilon^2)$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket in the variables $I,\phi,A,t$.

Therefore, our goal is to find $K^1$ in such a way that

$$R^1 \equiv H^1 + \{H^0 + A, K^1\} \tag{18}$$

is somewhat simple (we will make precise what "simple" means in our case). Since $R^1$ is the dominant term in $H_\varepsilon \circ k_\varepsilon$, one can hope that the dynamics expressed in the new coordinates is simple.

In terms of Fourier coefficients, (18) is equivalent to

$$R^1_{k,l}(I) = H^1_{k,l}(I) + i(k\omega(I) + l)K^1_{k,l}(I), \tag{19}$$

where $\omega(I) = \frac{\partial}{\partial I}H_0(I)$. The assumptions include that $H_0$ is twist. That is that $\omega(I)$ is monotonic, so that for each $k,l$ there is one and only one $p_{k,l}$ such that $k\omega(I_{k,l}) + l = 0$. Of course, $I_{nk,nl} = I_{n,k}$. The points $I_{k,l}$ are called resonances.

Because of the assumption that the perturbation is a polynomial, we have to consider $k,l$ ranging only over the finite set $\mathscr{N} \subset \mathbb{Z}^2$.

We see that (19) has very different character depending on whether $(k\omega(I)+l) = 0$ or not. If $(k\omega(I)+l) = 0$, we have to set $R^1_{j,k}(I) = H^1_{j,k}(I)$ but we can choose $K^i_{k,l}(I)$ arbitrary. Since we want that our solutions are differentiable, we have to make sure that the choices are made in a differentiable way. A particularly simple way – used in [DLS06b] to make these choices is to take a fixed $C^\infty$ cut-off function $\Psi$ and a fixed number $L$ so that denoting $\Psi_L(t) = \Psi(t/L)$, we take the choice

$$\begin{aligned} R^1(I,\phi,t) &= \sum_{k,l\in\mathscr{N}} \Psi_L(I - I_{k,l}) H^1_{k,l}(I_{k,l}) \exp(i(k\phi + lt)), \\ K^1(I,\phi,t) &= \sum_{k,l\in\mathscr{N}} (1 - \Psi_L(I - I_{k,l}))/i(\omega(I)k + l) H^1_{k,l}(I) \exp(i(k\phi + lt). \end{aligned} \tag{20}$$

If we choose conveniently $L$ – we are considering only a finite number of resonances – we can ensure that the intervals $[-2L+I_{k,l}, 2L+I_{k,l}]$ do not intersect for different resonances.

So, we can divide the phase space into two regions:

• One *"non-resonant region"* where – in the appropriate coordinates – the system is integrable up to an error of order $\varepsilon^2$.

• A finite number of *"resonant regions"*. Each of the resonant regions can be labeled by a frequency $l/k$ expressed in an irreducible fraction. In one of these resonant regions, in the appropriate coordinates, the Hamiltonian is: [14]

$$\begin{aligned} H_0(I)+A+\varepsilon\sum_{n\in\natural} H^1_{nk,nl}(I_{k,l})\exp(in(k\phi+lt))+O(\varepsilon^2) \\ = \qquad H_0(I)+A+\varepsilon V(k\phi+lt)+O(\varepsilon^2). \end{aligned} \tag{21}$$

The dynamics of the Hamiltonian (21) are easy to understand. If we introduce the variables $\tilde{\phi} = k\phi + lt$, $\tilde{I} = I - I_{k,l}$ – this change of variables is not symplectic, but it just multiplies the symplectic structure by a constant, so that the equations of motion – up to a constant change in time are also given by a Hamiltonian. Note also that in this change of variables, the period of the angle variables is changed. Hence, in the new variables, the hamiltonian is:

$$\alpha H_{k,l}(\tilde{I}) + A + \varepsilon V_{k,l}(\tilde{\phi}) + O(\varepsilon^2) \tag{22}$$

Since at the resonance the variable $\tilde{\phi}$ has frequency 0, we have that

$$H_{k,l}(\tilde{I}) = \alpha_{k,l} I^2 + O(\tilde{I}^3)$$

Furthermore, $\alpha$ will not be zero since it will be close to the second derivative of the unperturbed Hamiltonian, which we assumed is strictly positive (twist condition).

Note that the dynamics of (22) is very similar to the dynamics of a pendulum with a potential of size $\varepsilon$. In this case, the variable $A$ does not play any role at all. There will be homoclinic orbits to the maximum of the potential. These orbits will be given by the conservation of energy and the form of the kinetic energy as

$$\tilde{I} = \pm\varepsilon^{1/2}\sqrt{\alpha^{-1}(\max V - V(\tilde{\phi}))} + O(\varepsilon). \tag{23}$$

Inside these curves, the system does a rotation.

If the maximum is non-degenerate – another hypothesis which is easy to verify in practice and which holds for generic $V$ – we see that the orbits described in (23) are orbits that start and end in a critical point, which is hyperbolic. They are at the same time the stable and the unstable manifolds of this hyperbolic fixed point.

Note that these orbits are very different from the KAM tori. This is the reason why the KAM foliation gets interrupted by gaps of order $\varepsilon^{1/2}$.

[14] Again, we ignore regularity issues. It is not hard to show that if we assume that the function $H^1$ is $C^r$, then, $K^1$, $R^1$ are $C^{r-2}$ so that the error term in (21) can be considered in the $C^{r-2}$ norm. Again, we refer to [DLS06b].

It is important to remark that the stable and unstable manifolds of these periodic points have Lyapunov exponents $O(\varepsilon^{1/2})$. This is much smaller than the Lyapunov exponents in the transverse directions, which are independent of $\varepsilon$. Hence, when we talk about the stable manifolds restricted to $\Lambda$ this is not the same as the $W^s$ in the sense of the theory of normally hyperbolic invariant manifolds , which requires convergence at an exponential rate of order 1.

The dynamics of the averaged system – we will see that many of these features are preserved in the full system – consists of the foliation of – more or less horizontal – curves given by the orbits of the integrable system interrupted by a group of *eyes* or *islands*. At a resonance of type $k,l$ we obtain $k$ eyes. The amplitude of these eyes is $O(\varepsilon^{1/2})$.

*Remark 7.* The above classification ignores some stripes of width $O(\varepsilon)$ near the separation of the regions. The conclusions remain valid if we realize that the separation between the zones – the choice of $L$ – was a choice we made. We can repeat the same analysis with an slightly different $L$ and see that the ambiguous zones are different in the two procedures. So that by doing the analysis twice with slightly different $L$ one establishes the conclusions above for all the phase space.

*Remark 8.* The choice of separation between the resonances zones is rather wasteful (even if it makes the estimates and the concepts easier). We assign the same width to all the resonances even if it is clear that the real width will decrease with $\varepsilon$. (In particular, we expect that the optimal size would be close to $\varepsilon^{1/2}$). Furthermore, if the original Hamiltonian is several times differentiable, then, its Fourier coefficients will decrease at least like a power of $k,l$. Hence the $V_{k,l}$ will become smaller with $k,l$. Hence, if for a fixed $\varepsilon$ we decide to consider only resonant regions of size $\varepsilon^B$, we only need to consider a finite number of resonances – which will grow as $\varepsilon \to 0$ if $B > 1/2$.

Considerations of these type were known heuristically since at least [Chi79]. A rigorous implementation appears in [DH06]. The paper [DH06] includes also considerations of repeated averaging – discussed in the next section – and a very detailed analysis of the motion in each resonance with error terms.

## *4.4 Repeated averaging*

The method of averaging can be applied several times. Indeed, in celestial mechanics it has been common for centuries to do at least two steps of averaging.

In the region that was marked as integrable in the first step, after we perform the change of variables, we are left with a quasi-integrable system. The perturbation parameter is $\varepsilon^2$. We can restart the procedure and get again some regions where the system can be made integrable up to $O(\varepsilon^2)$ and new resonant regions in which the dynamics has *eyes*, which will now be of size $\varepsilon$ rather than $\varepsilon^{1/2}$.

In the resonant regions, nothing much happens except that the resonant potential $V_{k,l}$ gets deformed.

In the case that the perturbation is a trigonometric polynomial, the number of resonances we get at each step is finite and given a number of steps, we can get an $L$ which works for all cases.

The result of applying averaging twice is depicted crudely in Figure 8 [15] For future analysis, the only important thing is that near resonances, we encounter separatrices well approximated by other tori and that, outside the resonances the system is very approximately integrable.

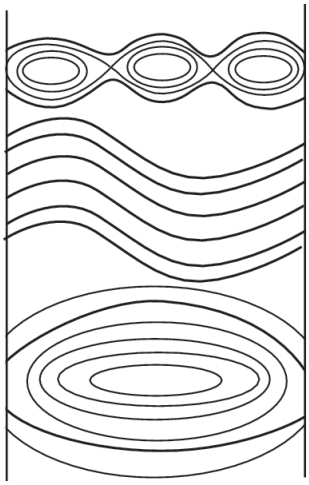


**Fig. 8** Schematic description of the predictions for the dynamics by the averaging method.

## *4.5 Invariant objects generated by resonances: Secondary tori, Lower dimensional tori*

The resonant averaging described above, gives very accurate predictions of the dynamics.

The difference between the perturbed system expressed in a system of coordinates and the true system – in a smooth norm – is smaller than $C_N\varepsilon^N$. The constants $C_N$ grow very fast.

This can be taken advantage off in two different ways:

A) If some perturbation theories apply, we can conclude that some of the invariant objects for the integrable system, persist for the true system.

[15] We have ignored, for example, the fact that inside the big islands of size $\varepsilon^{1/2}$ there are other *baby islands* of size $\varepsilon$ going around.

B) We have good control of some long orbits that, using some conditions can be glued together or shadowed.

This can be applied to the two types of geometric programs mentioned in Section1.1.

In this section, we will be concerned mainly with point A) and will produce invariant objects. We will come to point B) in Section 6.

If we consider the averaged system, we see that near resonances of order $j$, we obtain hyperbolic orbits, whose Lyapunov exponents are $C\varepsilon^{j/2} + O(\varepsilon^{(j+1)/2})$ and such that the angle between the stable and unstable directions are $C\varepsilon^{j/2} + O(\varepsilon^{(j+1)/2})$. Then, applying the implicit function theorem if $N > j$, we get that there are periodic orbits that persist. [16] More importantly for our later applications we obtain that the stable and unstable manifolds are very similar to those of the integrable system. The results are depicted in Figure 9.

We also can show that some of the quasi-periodic orbits with sufficiently large Diophantine constant persist. It is important to note that, one can get invariant tori of two types. One is tori which *"go across"*. These are the *"primary tori"* which are continuous deformations of the tori that were present in the unperturbed system. The tori inside the eyes of the resonance are of a completely different type. These are the *"secondary tori"* which were not present in the unperturbed system, but rather were created by the resonances. Note that as $\varepsilon \to 0$, the eyes become flatter and the limit of the tori is just a segment of periodic points. The tori merge with the stable and unstable manifolds. So that at the limit $\varepsilon = 0$ there is change of the topology.

One point which is important is that there are invariant tori very close to the resonances both from the inside and from the outside. These problems had been considered in [Neĭ84, Her83] under slightly different hypothesis. The method used in [DLS06b] was, mainly, to study in detail the expansion of the action-angle coordinates in a neighborhood of the separatrix. Using the – more or less explicit – formulas one can find in textbooks, it is possible to show that the $C^r$ norm of the change to action angle variables can be bounded by $d^{-rA}$ where $A$ is an explicit number. As it turns out the twist constant does not degenerate – the frequency is singular, but in the good direction that the twist becomes infinite. On the other hand, remember that the error of the averaging method was less than $C_N\varepsilon^N$. It follows that one can apply the KAM theorem at a distance $\varepsilon^{N/B}$. So that, one can get KAM tori – both rotating or librating – faster that a power of $\varepsilon$. The power is arbitrarily large assuming that the system is differentiable enough.

The paper [DLS06b, Section 8] contained other considerations on properties of the KAM tori as graphs and how the set of KAM tori close to the invariant circle can be interpolated with the others where the averaging method is different. The problem is somewhat difficult because depending on how does one relate the $\varepsilon$ to the distance to the separatrix, and to the fixed point, the expression of the KAM tori has different leading expressions.

[16] There are many versions of this argument on persistence of periodic orbits. The basic idea goes back at least to Poincaré and Birkhoff.

It seems possible that using more the geometric methods developed after [DLS06b] was written, many of these technical calculations can be eliminated or improved in many ways. A significant extension of the results can be found in [DH06]. Another line of argument that seems promising is the use of KAM theory without action angle coordinates – the singularity of the action angle variables and the different expressions in different regions is one of the source of problems – [dlLGJV05, FS07] so that one can prove directly the persistence of the orbits in the level sets of the averaged energy. We hope to come back to this.

In summary, it is possible to show that one can get persistence of many of the orbits predicted by the averaging method. For our purposes, it is enough to claim that we get an scaffolding of orbits which are much closer that $\varepsilon$ – the size of the effect of the scattering map.

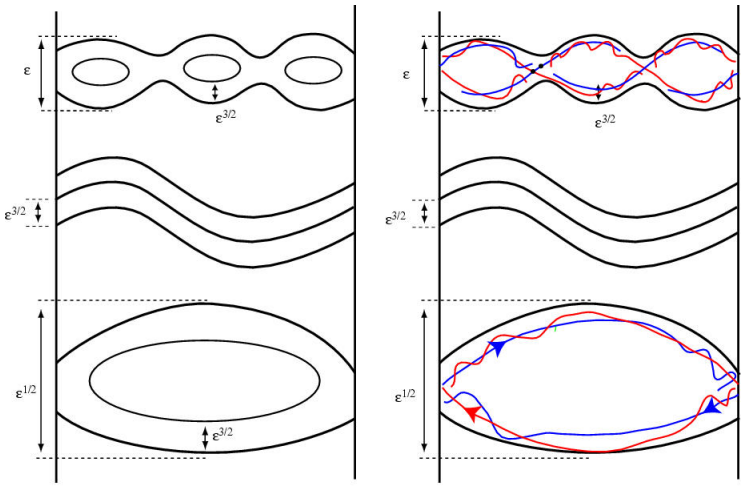



**Fig. 9** Illustration of an scaffolding of invariant objects in $\Lambda$. These invariant objects are $\varepsilon^{3/2}$ dense in the manifold.

## *4.6 Heteroclinic intersections between the invariant objects generated by resonances*

Now we want to argue that the objects discussed in the previous sections possess heteroclinic intersections. Since these objects have different topologies and very different characteristics, it is useful to use the scattering map and the argument discussed in Section 3.3.

To establish this intersection, we just compute the image of these invariant objects under the scattering map and check whether one can verify (8).

Given that we have computed rather explicitly the leading expansions of the scattering map and the leading expansions of the invariant objects, it is possible to compute the angles of intersections of manifolds. If these angles are not zero in the leading approximation, then, the implicit function theorem will establish that the true invariant manifolds satisfy (8).

The effect of the scattering map on the invariant objects is depicted schematically in Figure 10

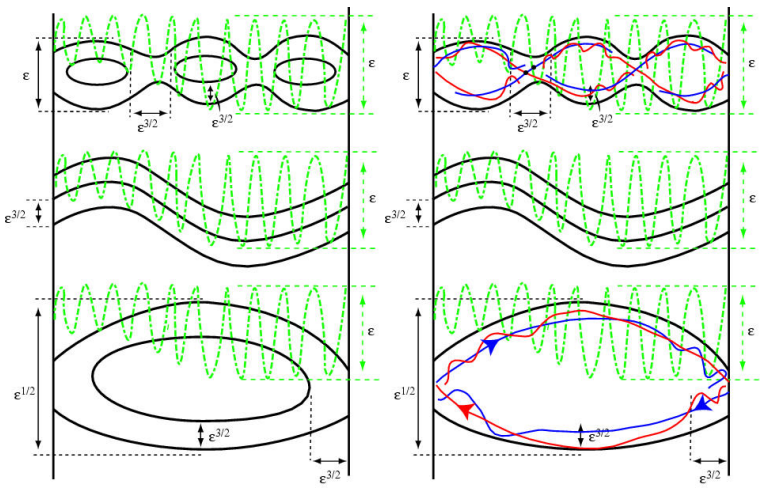



**Fig. 10** Effect of the scattering map on the invariant objects found in Figure 4.5
.

Therefore, the above calculation gives – rather explicit – expressions so that, if they do not vanish, then indeed we can obtain heteroclinic excursions between a primary torus below the resonance, to a secondary torus inside the resonance, and then to another torus above the resonance.

The non-vanishing of these explicit expressions giving the angles is a non-degeneracy assumption on the perturbation.

It is intuitively clear that the conditions hold rather generically. Basically, they are a comparison of two effects: the deformation of the invariant objects in $\Lambda$ and the effect of the scattering map. We note that the first effect, is very much affected by the behavior of the perturbation near $\Lambda$, but not by the behavior of the perturbation near $\Gamma$. The scattering map has the opposite properties. Hence, if by some miracle, the angles happened to be zero, some perturbation near $\Gamma$ could destroy this coincidence.

*Remark 9.* The calculation of the scattering map in [DLS06b] was based on traditional methods of perturbations of slow variables. This had the consequence that only the energy component of the scattering map could be computed.

The use of the symplectic properties, which was developed in [DLS06a] and explained in Section3.7, simplifies and extends the calculation. Note also that we mentioned that the invariant objects are very close to the level sets of a function $\Psi_\varepsilon$. Since the scattering map is a symplectic map close to the identity, the images of the level sets of $\Psi_\varepsilon$ will be level sets of the function $\Psi_\varepsilon + \varepsilon\{\Psi_\varepsilon, S_0\} + \text{h.o.t.}$. See [DH06].

## 5 The method of correctly aligned windows

The method of correctly aligned windows is a way of proving that given segments of orbits – with some extra conditions – one can get an orbit that tracks them. Since we never have to consider more than finite orbits, in principle, we do not need the existence of invariant objects. On the other hand, considerations about times become relevant. This is the reason why one gets explicit estimates on diffusion time.

The method has its origins in [Eas78, EM79, Eas89]. The version we will discuss comes from [ZG04, GZ04].

One can think of a window, as a topological version of a rectangle with some marked sides. Windows are correctly aligned when the image of one *stretches across the other*.

A window in a $n$-dimensional manifold $M$ is a compact subset $W$ of $M$ together with a $C^0$-coordinate system $(x,y) : U \to \mathbb{R}^u \times \mathbb{R}^s$ defined in neighborhood $U$ of $W$, where $u+s=n$, such that the homeomorphic image of $W$ through this coordinate system is the rectangle $[0,1]^u \times [0,1]^s$. The subset $W^-$ of $W$ that corresponds through the coordinates $(x,y)$ to $\partial[0,1]^u \times [0,1]^s$ is called the 'exit set' and the subset $W^+$ of $W$ that corresponds through the local coordinates $(x,y)$ to $[0,1]^u \times \partial[0,1]^s$ is called the 'entry set' of $W$. Here $\partial$ denotes the topological boundary of a set. If we want to specify the dimension $u$ of the unstable-like direction and the dimension $s$ of the stable-like direction of a window $W$, we refer to $W$ as an $(u,s)$-window. We will assume that $u > 0$.

Let $W_1$, $W_2$ be two $(u,s)$-windows in $M$, and let $(x_1,y_1) : U_1 \to \mathbb{R}^n$ and $(x_2,y_2) : U_2 \to \mathbb{R}^n$ be the corresponding coordinates systems. Let $f$ be a continuous map on $M$; we will denote its expression $(x_2,y_2) = f(x_1,y_1)$ in local coordinates also by $f$. Assume $f(U_1) \subseteq U_2$. We say that $W_1$ is correctly aligned with $W_2$ under $f$ provided that the following conditions are satisfied:

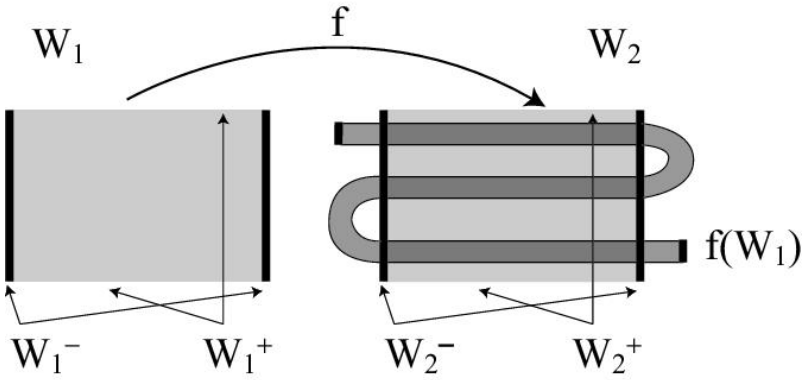
(i) $f(\partial[0,1]^u \times [0,1]^s) \cap [0,1]^u \times [0,1]^s = \emptyset, \quad f([0,1]^u \times [0,1]^s) \cap ([0,1]^u \times \partial[0,1]^s) = \emptyset$.

(ii)there exists a point $y_0 \in [0,1]^s$ such that

(ii.a)
$$f([0,1]^u \times \{y_0\}) \subseteq \text{int}\,([0,1]^u \times [0,1]^s \cup (\mathbb{R}^u \setminus (0,1)^u) \times \mathbb{R}^s)\,,$$

(ii.b) The map $A_{y_0} : \mathbb{R}^u \to \mathbb{R}^u$ defined by $A_{y_0}(x) = \pi_1\,(f(x,y_0))$ satisfies

$$A_{y_0}(\partial[0,1]^u) \subseteq \mathbb{R}^u \setminus [0,1]^u,$$
$$\deg(A_{y_0}, 0) \neq 0.$$




The main result is that "One can see through correctly aligned windows". See [ZG04, GZ04].

Let $W_i$ be a collection of $(u,s)$-windows in $M$, where $i \in \mathbb{Z}$ or $i \in \{0,\dots,d-1\}$, with $d > 0$ (in the latter case, for convenience, we let $W_i := W_{(i \bmod d)}$ for all $i \in \mathbb{Z}$). Let $f_i$ be a collection of continuous maps on $M$. If $W_i$ is correctly aligned with $W_{i+1}$, for all $i$, then there exists a point $p \in W_0$ such that

$$f_i \circ \dots \circ f_0(p) \in W_{i+1},$$

Moreover, if $W_{i+k} = W_i$ for some $k > 0$ and all $i$, then the point $p$ can be chosen so that

$$f_{k-1} \circ \dots \circ f_0(p) = p.$$

If one takes very small windows, the behavior of the windows is determined by the derivative of the orbit. If the orbit is hyperbolic, by choosing the rectangles as products of balls along the stable direction and the unstable direction with the unstable being the exit direction, we can get the correct alignment. Then the result that one can see through chains of correctly aligned windows becomes the standard shadowing result. On the other hand, the method is more flexible since we can choose the sizes of the windows and the time we take to put them along the orbits. This has some advantages for non-uniformly hyperbolic systems. See the proof of the non-uniformly hyperbolic closing lemma in [Pol93].

On the other hand, the windows do not need to be small. As we will see in the next section, one can take advantage of large scale effects to get the alignment of

windows. Notably, when one has some twist – shear – that causes some stretching, this can be used in place of the stretching caused by the hyperbolicity. It is also important to notice that, to check whether windows are well aligned or not, one can just study what happens on the boundary.

In our applications the time of diffusion can be computed by the time that it takes the windows to stretch.

An important technical tool [GL06a] is that, for systems that are close to product of systems, one can construct product windows and verify the alignment checking conditions on each of the factors.

## 6 The large gap model: The method of correctly aligned windows

The method of correctly aligned windows has been applied to the large gap model in [GL06a, GL06b].

The construction of windows adapted to the problem of diffusion basically requires to choose the parameters of a sequence of windows (the length of the sides, the center in a good coordinate system) and choose the times taken to go from one to the next. Then, one has to verify that all the steps match. In practice this amounts to choosing two dozen of parameters and verifying about a dozen of trivial inequalities.

Even if verifying the validity of the choices is not very hard, coming up with the good choices requires a good understanding of the behavior of the model. We now discuss some of the reasons behind the choices.

We have already discussed the pseudo-orbits that appear. We go from the intersection to the manifold, rotate around and then escape back again.

It is important to note that even the unperturbed system is not hyperbolic. The vector along the separatrices of the pendulum contracts both in the future and in the past. So that, these vectors in the intersection of the stable and unstable subspace and the forward Lyapunov exponent is different from the backward Lyapunov exponent. [17]

The construction of windows, however, can take advantage of the fact that there are some direction with good hyperbolicity for a long stretch ( $O(|\ln(\varepsilon)|)$ ) of time while the orbit moves from $\Gamma$ to $\Lambda$ or back. The fact that one can control the behavior in the hyperbolic directions is possible because of the transversal intersection. (On the other hand, the windowing method, being a topological method could work with much weaker assumptions. [GR04].)

The treatment of the center directions is much more interesting. Of course, the windows that start close to $\Lambda$, go to $\Gamma$ and come back to $\Lambda$ are very well described by the scattering map. One does not have any hyperbolicity in these directions, but on the other hand, the twist does distort the windows and one can use this distortion to construct windows that are correctly aligned. This is very similar to the *torsion-hyperbolicity* mechanism.

[17] The equality of these two exponents was called *regularity* by Lyapunov and plays a very important role. See [BP01].

In the paper, [GL06a] the windows were taken very thin in the action variables, but they were taken of order 1 in the angle. This allowed to avoid discussions of *ergodization times* and produced rather concrete estimates on the time. In [GL06b] the windows are chosen in a scale $O(1/|\ln(\varepsilon)|)$. This, of course, goes to zero, but it is much larger than the scales of the resonance. The orbits also do not come too close to the manifold. This has the effect that the method does not need to analyze what happens in the resonances. This method also leads to times of order $O(\varepsilon|\ln(\varepsilon)|)$ that – up to, perhaps, a constant – match the upper bounds obtained in [BBB03]. Similar results appear in [Tre04].

## 7 The large gap model in higher dimensions

Some of the analysis in Section 4 can be adapted to higher dimensional models. See [DLS07].

We consider the same model as in (14), but now $I$, $\phi$ are higher dimensional variables. Again, for simplicity, for the moment, we assume that the perturbation $h$ is a trigonometric polynomial.

The averaging method described in Section4.3 can be carried out pretty much the same way. The only difference is that now, that the resonances $\omega(I)\cdot k = n$ are codimension 1 manifolds. If the number of degrees of freedom is more than 1, there will be intersections of these resonant surfaces. The intersection of two independent resonances are called *multiple resonances*. The multiplicity of the resonance – not to be confused with the order – is the dimension of the module of vectors $k, n$ for which there is resonance relation. The order is the power of $\varepsilon$ of the terms that cannot be eliminated.

The mathematical analysis of multiple resonances and their role in diffusion remains a very interesting problem. Very important progress has been done in [Hal97, Hal99].

Nevertheless in [DLS07] it is argued that there exist diffusing orbits – under the assumption that $h$ is polynomial – plus some non-degeneracy assumptions.

The key observation is that, under a twist condition, the multiple resonances can be contoured. (Since they happen on sets of codimension 2 or higher, there are paths that go around them).

The analysis of resonances of order 1 in higher dimensional systems is very similar to the analysis carried in Section 4 [18]

The upshot is that, under explicit non-degeneracy conditions, for any path in the space of actions that crosses only multiplicity one resonances, for $0 < \varepsilon$ small enough there orbits whose actions evolve along the path – up to errors that go to zero with $\varepsilon$.

[18] The scattering map does not require any change, but the persistence of tori of lower dimension becomes more complicated (one has to use KAM theory rather than the implicit function theorem). Also the secondary tori require some extra considerations.

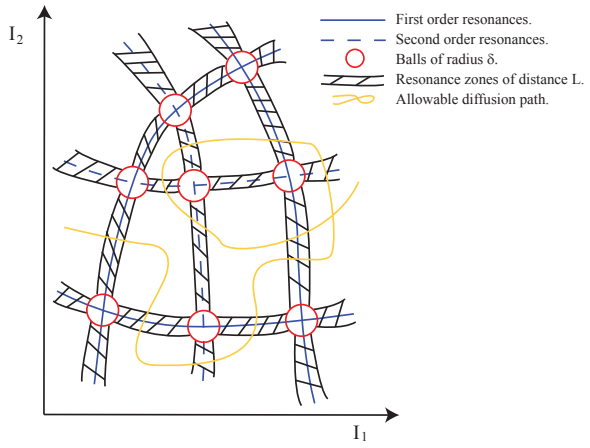



**Fig. 11** Illustration of the paths of diffusion avoiding higher order resonances

## 8 Instability caused by normally hyperbolic laminations

One of the standard heuristics in the numerical studies is that of *modulational diffusion* [Chi79, TLL80]. It is often described as saying that *A degree of freedom becomes chaotic and drives another one*.

Mathematically, one can formulate this as perturbing a system which is the product of a system with some hyperbolic behavior, and another system which is integrable: $F_\varepsilon = F_h \times F_i + O(\varepsilon)$, where $F_h(\Lambda) = \Lambda$ and $\Lambda$ is a hyperbolic set, and $F_i : M \mapsto M$ is an integrable map.

In the mathematical literature, some rigorous results have been obtained. The paper [MS02] constructed a specific system of this type. The paper [Moe02] used topological methods in two dimensions. Closely related to this paper is [EMR01].

One systematic way to make sense of the above [Lla04, dlL06] is to observe that the set $\cup_{x\in\Lambda}\{x\}\times M$ is a normally hyperbolic lamination for $F_0$. See Appendix A.3. The laminae are $\{x\}\times M$ are permuted by the map and the normal directions are hyperbolic.

It was shown in [HPS77, Ch. 15] that these structures persist under perturbations in the sense that one can get slightly deformed collections of laminae which are also permuted under the map $F_\varepsilon$. Of course, the dynamics on these laminae is not integrable anymore. The dynamics on the integrable parts is a random composition of maps, which one can consider as uncoupled as in [MS02].

## 8.1 Models with two time scales: geodesic flows, billiards with moving boundaries, Littlewood problems

The above mechanism is particularly effective in systems that have two time scales.

One important system is the model of a geodesic flow perturbed by a periodic or quasi-periodic potential considered by other methods in [Mat96, BT99, DLS00, DLS06c].

This dynamical system is defined on the cotangent bundle $T^*M$ of a compact manifold $M$. It has the form:

$$\dot{p} = -\nabla V(q, \omega t), \quad \dot{q} = p, \tag{24}$$

where the potential $V : M \times \mathbb{T}^d$ and $\omega \in \mathbb{R}^d$ is a non-resonant vector. When $d = 1$, the potential depends periodically on time.

We note that the system satisfies some scaling properties. Setting $p = \varepsilon\tilde{p}$, $q = \tilde{q}$, $t = \varepsilon^{-1}\tilde{t}$ and denoting by $'$ the derivative with respect to $\tilde{t}$, the system, above becomes

$$\tilde{p}' = -\varepsilon^2 \nabla V(\tilde{q}, \varepsilon\omega t) \quad \tilde{q}' = \tilde{q} \tag{25}$$

So that, for high energy, the potential can be considered as a slow and weak perturbation.

We will assume that the unperturbed geodesic flow has a horseshoe in the unit energy surface. Using the above scaling, we obtain that, considering the system for all the energies it possesses an invariant lamination. By the theory of persistence of normally hyperbolic invariant laminations, we obtain that this structure just gets deformed.

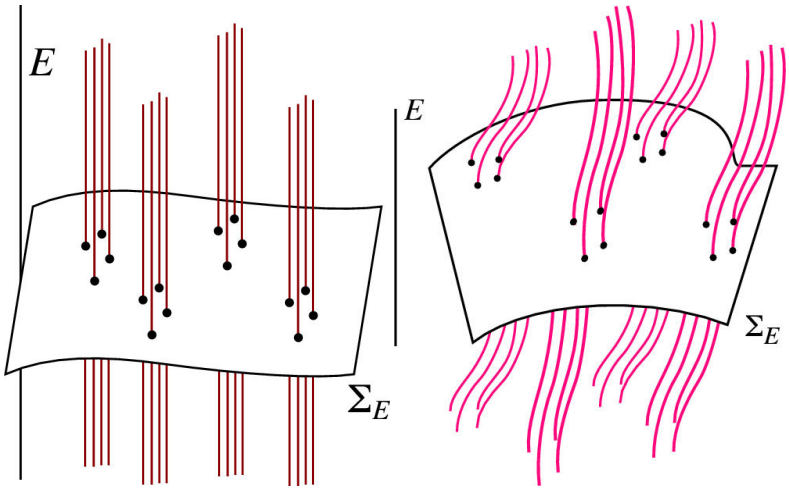



**Fig. 12** Invariant normally hyperbolic laminations associated to the geodesic flow and the periodic geodesic flow.

If $\gamma_1,\dots,\gamma_N$ are periodic orbits in the horseshoe, we denote $|\gamma_i|$ the period and define:

$$G_i(t) = \frac{1}{|\gamma_i|}\int_0^{|\gamma_i|} \frac{\partial}{\partial t} V(\gamma_i(s),t)\,ds$$

This has the meaning of the gain of energy per unit time for orbits that stay in a close proximity to the periodic orbit. Note that $\int_0^1 G_i(t)\,dt = 0$.

Recall that, in the horseshoe, we have a symbolic dynamics for the hyperbolic orbits. That is, if we fix neighborhoods of these orbits, we can move from one to the other in arbitrary order. Each of the steps can be accomplished in a fixed time.

By the persistence of the normally hyperbolic laminations, the same property persists when we consider the perturbation by the potential. So, we can switch from a neighborhood of an orbit to another one in a fixed time for the geodesic flow. For the potential, this is a slow time.

In the periodic case, $d = 1$, we assume without loss of generality that $V(q,t+1) = V(q,t)$, $\omega = 1$. If we assume that there exist $0 = a_0 < a_1 < \cdots < a_N = 1$ in such a way that

$$A \equiv \sum_{i=1}^{N} \int_{a_{i-1}}^{a_i} G_i(t)\,dt > 0 \tag{26}$$

then, we can construct orbits whose energy as function of time is larger than $At - B$. The idea is very simple. We stay close to $\gamma_1$ during the macroscopic times $[a_0,a_1]$. Using the symbolic dynamics, we can move to $\gamma_2$, etc. Hence, during a cycle, we have gained roughly $A$.

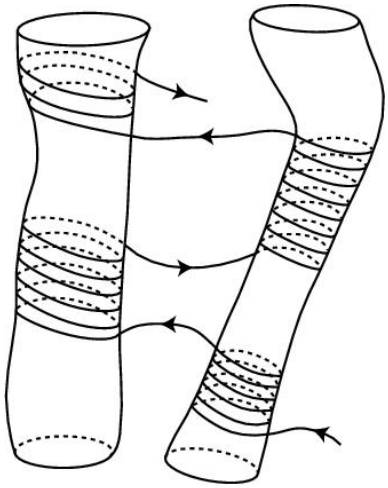


**Fig. 13** Illustration of the mechanisms of gain of energy based in locally hyperbolic manifolds

In the quasi-periodic case, we just need to assume that it is possible to write $\mathbb{T}^d = \cup_{i=1}^N O_i$ where $O_i$ are sets with smooth boundary transversal to the rotation, which only overlap in the boundary, and such that $A \equiv \sum_{i=1}^N \int_{O_i} G_i(\tau)\, d\tau > 0$.

If we look at the symbolic dynamics, we see that the space of sequences that lead to linear gain in energy has positive Hausdorff dimension. Then, using that the conjugacy given by the stability, we obtain that, when the (26) are satisfied, the orbits with energy growing linearly are of positive Hausdorff dimension.

It is shown in [Lla04] that if the metric is of negative curvature and, in case that it has dimension $\geq 3$, that it satisfies some pinching conditions, then, the only $C^3$ potentials for which it is impossible to find orbits satisfying the hypothesis of the above result are the potentials of the form $V(q,t) = V_1(q) + V_2(t)$.

Very similar analysis applies to other systems which have two scales.

One example is what we call the *Littlewood models* in higher dimensions.

$$H(p,q,t) = \frac{1}{2}p^2 + V_n(q) + V_m(q,t) \tag{27}$$

where $p,q \in \mathbb{R}^d$, $d \geq 2$, $V_n$, $V_m$ are homogeneous of degree $n,m$ respectively, $n > m$, $n > 2$,$V_n > 0$, $V_m$ periodic or quasi-periodic in $t$. The fact that different terms have different homogeneities makes the geometric analysis similar to that of the geodesic flows.

In the case $d = 1$, [Lit66a, Lit66b] constructed examples of potentials – which are not polynomials and with not very smooth dependence on time – with orbits with unbounded energy. Unfortunately, the papers contain a serious error. The papers [LL91, LZ95] showed that for terms which are like polynomials, and with smooth quasiperiodic perturbations the orbits stay bounded. An excellent survey of the history of these models and simplification of the results is [Lev92].

When the number of degrees of freedom is greater or equal than 2, a very similar analysis to the one carried out above for geodesic flows applies. We note that if we scale, $p = \varepsilon^{m/2}\tilde{p}, q = \varepsilon\tilde{q}, t = \varepsilon^{-m}$ we get that the system (27) can be rewritten as:

$$H(\tilde{p},\tilde{q},\tilde{t}) = \frac{1}{2}(\tilde{p})^2 + V_m(\tilde{q}) + \varepsilon^{2m-n}V_n(q,\varepsilon^m t)$$

so that the low degree polynomial can be considered as a small and slow perturbation and an analysis very similar to the one carried above for the geodesic flow applies. The only difference is that one gets that the orbits grow like a power. This is optimal due to a calculation in [LZ95].

One interesting example, which does not fit in the above theory proposed as a challenge by M. Levi is the system defined by a Hamiltonian

$$\frac{1}{2}p^2 + q_1^6 + q_1^4 + \eta q_1^2 q_2^2 + q_1 f(t)$$

This is a challenging model because for large energy, the dominant term is the one degree of freedom system for which the theorem of [LZ95] applies.

Another model which has scaling behavior is the billiard with moving boundaries. A higher dimensional model of the Fermi acceleration.

For all these systems, when they are sufficiently chaotic, it seems possible to derive – heuristically – stochastic models for the growth of energy. These stochastic models can be analyzed rigorously and the final results compared satisfactorily with numerical simulations. [DdlL06]. Even if parts of a stochastic theory of diffusion can be made rigorous, deriving a fully rigorous stochastic theory of diffusion remains a very challenging problem.

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## A: Normally hyperbolic manifolds

In this section, we recall some results in the literature on normally hyperbolic manifolds. Good references are [Fen72, Fen74, Fen77, HPS77, Pes04].

For simplicity, we will discuss only the case of diffeomorphisms. The case of flows is very similar. For many of the applications (persistence of invariant man-

ifolds, regularity) the case of flows follows from the case of diffeomorphism by taking time-1 maps.

Let $M$ be a smooth $d$-dimensional manifold, $f : M \to M$ a $C^r$ diffeomorphism, $r \geq 1$.

**Definition 1.** Let $\Lambda \subset M$ be a $C^1$ submanifold invariant under $f$, $f(\Lambda) = \Lambda$. We say that $\Lambda$ is a normally hyperbolic invariant manifold if there exist a constant $C > 0$, rates $0 < \lambda < \mu^{-1} < 1$ and a splitting for every $x \in \Lambda$

$$T_x M = E_x^s \oplus E_x^u \oplus T_x \Lambda$$

in such a way that

$$\begin{aligned} v \in E_x^s &\Leftrightarrow |Df^n(x)v| \leq C\lambda^n |v| \qquad n \geq 0 \\ v \in E_x^u &\Leftrightarrow |Df^n(x)v| \leq C\lambda^{|n|} |v| \qquad n \leq 0 \\ v \in T_x\Lambda &\Leftrightarrow |Df^n(x)v| \leq C\mu^{|n|} |v| \qquad n \in \mathbb{Z} \end{aligned} \tag{28}$$

In this exposition, we will assume that $\Lambda$ is compact and, without loss of generality, connected.

*Remark 10.* The set up can be weakened in several directions which appear in applications.

For example, as remarked in [HPS77], instead of assuming that $\Lambda$ is compact, it suffices to assume that $f$ is $C^r$ in a neighborhood of $\Lambda$ with all the derivatives of order up to $r$ uniformly bounded. The non-compact case involves some complications such as study of extension operators. These considerations become much more important in the extension of the theory to infinite dimensional Banach spaces, which we will also not consider [BLZ98]. In these infinite dimensional cases, the standard arguments often give one or two derivatives less in the conclusions than the finite dimensional compact arguments.

We also note that some parts of the theory are also true for manifolds with boundary such that $f(\Lambda) \subset \Lambda, d(f(\partial\Lambda), \partial\Lambda) > 0$ (inflowing) or $f(\Lambda) \subset \Lambda, d(f(\partial\Lambda), \partial\Lambda) > 0$ (outflowing). Note that the definition of stable (resp. unstable) directions in (28) requires serious changes in the outflowing (resp. inflowing) cases. An adaptation of the theory to the inflowing and outflowing cases is done in [Fen72]. Note that, even if these definitions become possible, the resulting objects may lack some of the properties of the more standard definitions. For example, the stable spaces are not unique in the inflowing case, so that issues of regularity are more delicate, even if well understood in the literature.

In some applications to instability, one often gets systems with two time scales, so that the hyperbolicity degenerates. Therefore it is useful to keep explicit track of how $C, \lambda, \mu$, the parameters affecting the quality of the hyperbolicity in (28) enter in the hypothesis of the theorems. See [Fen79].

A self-contained detailed treatment of a case that involves several of these complications can be found in Appendix A of [DLS06c].

It follows from (28) that $E^s_x$, $E^u_x$ depend continuously on $x$. In particular, the dimension of $E^s_x$, $E^u_x$ are independent of $x$. In fact, using the invariant section theorem [HP70] or some direct arguments [Fen74, Fen77] they are $C^{\ell-1}$,

$$\ell < \min\left(r, \frac{|\log\lambda|}{\log\mu}\right). \tag{29}$$

Indeed, using some variants of these arguments, it is possible to show that the invariant manifold $\Lambda$ is $C^\ell$ – even if the hypothesis of the definition only require it is $C^1$. In general, one cannot improve on these regularities. [Mos69] contains explicit examples – even trigonometric polynomials – where the regularity claimed above is sharp, and [HW99] shows that this regularity is indeed sharp for generic examples. Hence, in general, one cannot expect that the normally hyperbolic invariant manifolds are $C^\infty$ even if $f$ is a polynomial. One can however have uniform lower bounds for all the $C^r$ maps which are in a $C^1$ neighborhood. The regularity of overflowing (resp. inflowing) manifolds is even more problematic since the stable (resp. unstable) bundles are not uniquely defined, hence the hyperbolicity constants do not have a unique value.

Given a normally hyperbolic invariant manifold $\Lambda$ we define

$$\begin{aligned} W^s_\Lambda &= \{y \in M \mid d(f^n(y),\Lambda) \le C_y\lambda^n,\ n \ge 0\} \\ W^u_\Lambda &= \{y \in M \mid d(f^n(y),\Lambda) \le C_y\lambda^{|n|},\ n \le 0\} \end{aligned}$$

Furthermore, for each $x \in \Lambda$, we define

$$\begin{aligned} W^s_x &= \{y \in M \mid d(f^n(x),f^n(y)) \le C_{x,y}\lambda^n,\ n \ge 0\} \\ W^u_x &= \{y \in M \mid d(f^n(x),f^n(y)) \le C_{x,y}\lambda^{|n|},\ n \le 0\} \end{aligned}$$

and we note that $E^s_x = T_x W^s_x$ and $E^u_x = T_x W^u_x$. It is a fact that

$$\begin{aligned} W^s_\Lambda &= \bigcup_{x\in\Lambda} W^s_x \\ W^u_\Lambda &= \bigcup_{x\in\Lambda} W^u_x \end{aligned} \tag{30}$$

Moreover, $x \neq \tilde{x} \Rightarrow W^s_x \cap W^s_{\tilde{x}} = \emptyset$, $W^u_x \cap W^u_{\tilde{x}} = \emptyset$.

The decomposition (30) can expressed geometrically saying that $\{W^s_x\}_{x\in\Lambda}$, $\{W^u_x\}_{x\in\Lambda}$ are a foliation of $W^s_\Lambda$, $W^u_\Lambda$, respectively. We will refer to these foliations as $\mathscr{F}_s$, $\mathscr{F}_u$.

Dynamically, the above statement means that, when the orbit of a point is approaching $\Lambda$, it approaches the orbit of a single point. This, as well as the uniqueness can be established easily by noting that, for two points in $\Lambda$, we have $d(f^n(x),f^n(x)) \ge C\mu^{-n}$. Since $\lambda\mu < 1$, we can see that if we fix $y$ there can only be one $x$ such that $d(f^n(x),f^n(y)) \le C\lambda^n$.

We recall that in these circumstances we have that

1. $\Lambda$ is a $C^\ell$ manifold with $\ell$ given in (29).

2. $W^s_\Lambda$, $W^u_\Lambda$ are $C^{\ell-1}$ manifolds
3. $W^s_x$, $W^u_x$ are $C^r$ manifolds
4. The maps $x \mapsto W^s_x,\ W^u_x$ are $C^{\ell-1-j}$, when $W^s_x,\ W^u_x$ are given the $C^j$ topologies in compact sets.
5. When $x \in \Lambda$, we have

$$T_x W^{s,u}_\Lambda = E^{s,u}_x \qquad T_x W^{s,u}_x = E^{s,u}_x$$

6. As a consequence of the above, using the implicit function theorem, we have:
   Denote by $W^{s,\delta}_\Lambda$ a $\delta$-neighborhood of $\Lambda$ in $W^s_\Lambda$ and by $W^{s,\delta}_x$ a $\delta$ neighborhood of $x$ in $W^s_x$.
   Then, for sufficiently small $\delta$, there is a $C^{\ell-1}$ diffeomorphism $h^s$ from $W^{s,\delta}_\Lambda$ to a neighborhood of the zero section in $E^s$. Furthermore, $h^s(W^s_x) \subset E^s_x$.

Note, that, even if $W^s_x$ are as smooth as the map, the dependence of the point on the base point has only some finite regularity that depends on the regularity exponents entering in (28).

The manifold $W^s_\Lambda$ is invariant. That is $f(W^s_\Lambda) = W^s_\Lambda$. Analogously, of course, the unstable manifolds.

On the other hand, the manifolds $W^s_x$ are not invariant. They, however satisfy a covariance property

$$f(W^s_x) = W^s_{f(x)} \tag{31}$$

The local behavior in a neighborhood of a normally hyperbolic invariant manifold is described very precisely by the following theorem in [HPS77, PS70], who show who show that if $\Lambda$ is a normally hyperbolic invariant manifold , then there is a homeomorphism $h$ from a neighborhood of the zero section in $T\Lambda$ to a neighborhood in $\Lambda$ in such a way that if $x \in \Lambda$, $\eta \in T_x M$ and $|\eta|$ is sufficiently small, we have

$$f \circ h(x,\eta) = h(f(x), Df(x)\eta) \tag{32}$$

The homeomorphism $h$ is, of course, highly non-unique. Note that, in the case that $\Lambda$ is just a point, the theorem reduces to the celebrated Hartman-Grobman theorem. Indeed the proof of the references above, after some clever reductions, becomes the Hartman-Grobman theorem in infinite dimensions.

An important consequence of the linearization theorem is that if $\Lambda$ is a normally hyperbolic invariant manifold , then, for any sufficiently small open neighborhood $U$ of $\Lambda$ we have

$$\Lambda = \bigcup_{n\in\mathbb{Z}} f^n(U)$$

Of course, if $\Lambda \subset V \subset U$, then $\Lambda = \bigcup_{n\in\mathbb{Z}} f^n(V)$.

The homeomorphism $h$ solving (32) is not unique and there are really terrible choices.[19] Nevertheless, there are choices which are continuous and indeed Hölder in some of the variables. We also have that, $W^{s,u\mathrm{loc}}_x = h(x, E^{s,u}_x \cup B_\delta)$.

[19] The lovers of pathologies can amuse themselves using the axiom of choice – Argh!! – to produce $h$ solving (32) which are not measurable.

The linearization (32) is a generalization of Hartman-Grobman theorem. Under appropriate non-resonance conditions on the possible rates of growth of the vectors on $T_xM|_{x\in\Lambda}$ it is possible to obtain more precise linearizations [Rob71, KP90, BK94]. In contrast with the Sternberg Linearization theorem, the non-resonance conditions can fail in $C^1$ open sets of diffeomorphisms. When the conditions for the linearization apply, then one can obtain very good estimates for the orbits that "fly by" the invariant manifold. In particular, one can get very detailed information about the separatrix map. Note that the time that one can spend in a "fly by" is unbounded, so that linearization gives information over trajectories that go over a long time.

## *A.1 Persistence and dependence on parameters*

One of the most important results of the theory of normally hyperbolic invariant manifolds is that they persist under perturbations and that they depend smoothly under parameters.

Persistence means, roughly, that if a map $f$ has an invariant manifold $\Lambda_f$ and $g$ is sufficiently $C^1$ close to $f$, then $g$ also has an invariant manifold $\Lambda_g$.

In these cases, the results on dependence on parameters and can be obtained very economically from the results on persistence by considering an extended system.

Let $f(x,\varepsilon) : M \times \Sigma \to M$ is a family of maps ($\varepsilon$ is the parameter). ¡– We will also use $f_\varepsilon = f(\cdot,\varepsilon)$. We consider $\tilde{f} = f \times \mathrm{Id}$ and $\tilde{f}_0 = f_0 \times \mathrm{Id}$.

We note that if $\Lambda_0$ is a normally hyperbolic invariant manifold for $f_0$, then $\Lambda_0 \times \Sigma$ is a normally hyperbolic invariant manifold for $\tilde{f}_0$. Furthermore, the hyperbolicity for $\tilde{f}_0$ admits the same constants in (28) than $f_0$. Hence, if $\tilde{f}$ is $C^1$ close to $\tilde{f}_0$, the persistence result implies that we can find a manifold $\tilde{\Lambda}$ that is invariant for $\tilde{f}$. Because $\tilde{f}$ is the identity in the $\varepsilon$ variable, we have that $\tilde{\Lambda}$ has to have the form $\bigcup_\varepsilon \Lambda_\varepsilon \times \{\varepsilon\}$, where $\Lambda_\varepsilon$ is invariant under $f_\varepsilon$.

Another important result in the theory of persistence of invariant manifolds is that the change in the hyperbolicity constants can be controlled by the $C^1$ distance of the maps. This is important since the regularity of the foliations $\mathscr{F}_{s,u}$ can be bounded uniformly in sets which are the intersection of $C^1$ open sets and $C^r$. For example in Section 3.2 it was convenient to assume that the foliations $\mathscr{F}_{s,u}$ are $C^1$. The previous remark implies that this assumption holds in some open sets, characterized by ratios in the contractions exponents.

A very efficient way of describing the results of persistence and smooth dependence on parameters is to use a parameterization method.

We write $\Lambda_\varepsilon$ as $k_\varepsilon(\Lambda_0)$ where $k : \Lambda_0 \times \Sigma \to M$ The fact that $\Lambda_\varepsilon$ is invariant is equivalent to

$$f_\varepsilon \circ k_\varepsilon = k_\varepsilon \circ r_\varepsilon \tag{33}$$

where $r_\varepsilon : \Lambda_0 \to \Lambda_0$ is a representation of dynamics of $f_\varepsilon$ restricted to the invariant manifold.

The result that $\Lambda_\varepsilon$ is $C^\ell$, means that $k_\varepsilon$ can be chosen to be $C^\ell$ in $\Lambda_0 \times \Sigma$. Hence, $\frac{\partial^j}{\partial \varepsilon^j} k_\varepsilon(x)$ is $C^{\ell-j}$. So that the map $\varepsilon - \Lambda_\varepsilon$ is $C^{\ell-j}$ when the manifolds are given the $C^j$ topology.

Even if in this presentations we have argued that the standard theory of normally hyperbolic invariant manifolds implies the existence of solutions of (33), it is possible consider (33) as an equation for $k_\varepsilon, r_\varepsilon$ and show that there are solutions. This is an alternative approach to the theory of existence of normally hyperbolic invariant manifolds developed in [HdlL07]. This has several advantages from the point of view of numerical computation. See [HdlL06c, HdlL06b, HdlL06a] for some simpler cases.

Notice that (33) is a geometrically natural equation. We also note that – since all geometrically natural equations are invariant under the choice of a system of coordinates in $\Lambda_0$ – if $k_\varepsilon, r_\varepsilon$ is a solution of (33) and $h_\varepsilon : \Lambda_0 \to \Lambda_0$ is a diffeomorphism we have that $\tilde{k}_\varepsilon = k_\varepsilon \circ h_\varepsilon$, $\tilde{r}_\varepsilon = h_\varepsilon^{-1} \circ r_\varepsilon \circ h_\varepsilon$ is also a solution of (33). This lack of uniqueness can be chosen to impose some supplementary conditions. For example, in [DLS06a] it is shown that if $f_\varepsilon$ preserve a symplectic form $\omega$, there is one and only one $k_\varepsilon$ such that $k_\varepsilon^* \omega = k_0^* \omega \equiv \omega|_{\Lambda_0}$. (This choice also has other geometric properties, we refer to the [DLS06a].

*Remark 11.* There is a large literature on formal perturbation theories based on *"expanding to first order"* and solving the resulting equations. This, in general, is not a correct procedure, but in the case that we know that there is a derivative, it is easy to show that this derivative satisfies a functional equation (which is the equation considered by the formal expansion). If the solution of this equation is unique, then, the solution of this equation will be the derivative.

## *A.2 The $\lambda$-lemma and the exchange lemma*

The simplest version of the $\lambda$ lemma states that if there is manifold $\Sigma$ which intersects transversally $W^s_x$, then, for large $n$, $f^n(\Sigma)$ will have a patch which is exponentially close – in a smooth topology – to $W^u(U_n)$ where $U_n \subset \Lambda$ is an open set around $f^n(x)$.

The sizes of the $U_n$ may decrease exponentially – but the rate is bounded by $\mu^{-n}$ –

## *A.3 Normally hyperbolic laminations*

This is a very interesting concept developed in [HPS77, Ch. 15]. See the results in Section 8.

In the simplest formulation, a lamination is a closed set of manifolds which do not intersect. $\{\Lambda_\sigma\}_{\sigma \in \Sigma}$.

A lamination is invariant if $f(\Lambda_\sigma) \subset \Lambda_{\Phi(\sigma)}$. A lamination is normally hyperbolic if, for $x \in \Lambda_\sigma$ we can find decompositions $T_xM = T_x\Lambda_\sigma \oplus E^s_x \oplus E^u_x$ satisfying estimates similar to those in (28).

The result of [HPS77, Ch. 15] is that this situation is stable under perturbations. Some improvements were developed in [Lla02]. Namely, that we can find another lamination $\Lambda^g_\sigma$ and a map $h_\sigma : \Lambda^f_\sigma \to \Lambda^g_\sigma$ in such a way that $g \circ h_\sigma = h_\sigma \circ f$.

A heuristic point of view which is useful is that one can consider the laminae as points, so that the above result is just the structural stability. As shown in [HPS77], there are also shadowing theorems and many other results analogue to the results for hyperbolic sets.

In a way similar to the stability of normally hyperbolic invariant manifolds , it is convenient to describe the stability of invariant manifolds using a parameterization method.

If $F_0(L_\sigma) = L_{f(sigma)}$, satisfying the hypothesis of normal hyperbolicity, we can try to find $h^\varepsilon_\sigma : L_\sigma \to M$ and $r^\varepsilon_\sigma : L_\sigma \to L_{f(\sigma)}$ that $F_\varepsilon \circ h^\varepsilon_\sigma = h^\varepsilon_\sigma \circ r^\varepsilon_\sigma$. Clearly $L^\varepsilon_\sigma = h^\varepsilon(L_\sigma)$ satisfy the invariance properties of laminations, $F_\varepsilon(L^\varepsilon_\sigma) = L^\varepsilon_{f(\sigma)}$.

The $h^\varepsilon_\sigma$ $r^\varepsilon_\sigma$ are parameterizations of the new laminae in terms of the old and the $r^\varepsilon_\sigma$ are expressions of the dynamics.

It follows from the results in [HPS77] that, for fixed $\sigma$ the $h^\varepsilon_\sigma(x)$, $r^\varepsilon_\sigma(x)$, are $C^\ell$ on $(\varepsilon, x)$, where $\ell$ depends on the exponents.

One small improvement from the results of [HPS77] that is found in [Lla04] is the observation that, the mappings $\sigma \mapsto h^\varepsilon_\sigma, r^\varepsilon_\sigma$ are Hölder when the $h, r$ are given a $C^\ell$ topology.