

Symmetry Classification of quasi-linear PDE's Containing Arbitrary Functions

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Abstract

We consider the problem of performing the preliminary "symmetry classification" of a class of quasi-linear PDE's containing one or more arbitrary functions: we provide an easy condition involving these functions in order that nontrivial Lie point symmetries be admitted, and a "geometrical" characterization of the relevant system of equations determining these symmetries. Two detailed examples will elucidate the idea and the procedure: the first one concerns a nonlinear Laplace-type equation, the second a generalization of an equation (the Grad-Schlüter-Shafranov equation) which is used in magnetohydrodynamics.

Key words: symmetry classification; quasi-linear PDE's; symmetry determining equations; generalized Laplace equation; Grad-Schlüter-Shafranov equation

Running title: Symmetry Classification of quasi-linear PDE's

1 Introduction

The analysis of symmetry properties of differential equations is a well established and widely used tool both for studying general properties of the equations and for finding their solutions (see e.g. [1]-[7] and references therein); actually, the determination of Lie-point symmetries (we will consider only symmetries of this type) is by now an almost completely standard routine, thanks also to some suitably dedicated computer packages (see e.g. [8]-[10]).

The situation is however considerably different when one deals with the problem of performing the “symmetry classification” of an equation which contains one or more arbitrary functions, and one wants to discover how the symmetry properties depend on the choice of these functions: the discussion may be far from easy, as far as the symmetries may drastically change when the functions are changed: see e.g. [11]-[15] and references therein.

In this paper, we discuss the case of PDE’s of the form given below (eq.(1)) containing one or more arbitrary functions $F_\ell(u)$ of the unknown variable u , and we shall give an easy condition involving these functions in order that the equation admits nontrivial symmetries. This will provide a neat “geometrical” characterization of the relevant system of equations determining these symmetries, and a direct way to perform the complete symmetry classification of the given equation. Two examples will elucidate the idea and the procedure. The first one concerns a nonlinear Laplace-type equation; the second example deals with a generalization of an equation (the Grad-Schlüter-Shafranov equation) which is used in magnetohydrodynamics and plasma physics [16].

2 Preliminary results

For the sake of simplicity we consider only second order equations for the unknown function $u = u(x, y)$ of two independent variables x, y (but the extension to more general cases is completely straightforward), and we will deal with quasi-linear PDE’s of the following form

$$a_{11} u_{xx} + a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y = \sum_{\ell=1}^L \alpha_\ell F_\ell(u) \quad (1)$$

or, in a short-hand notation,

$$\mathcal{E}[u] = \alpha_\ell F_\ell(u)$$

where $a_{ij} = a_{ij}(x, y)$, $b_i = b_i(x, y)$, $\alpha_\ell = \alpha_\ell(x, y)$ are given (smooth) functions, and $F_\ell(u)$ are L arbitrary (smooth) functions of u (in the examples below we will deal with just one or two functions $F_\ell(u)$). It is understood that no linear relations exist between α_ℓ and between F_ℓ . We will exclude from our consideration the rather trivial case when $F_\ell(u)$ are linear functions of u , which usually can be more simply and conveniently discussed separately by means of direct calculations. In this paper we are looking only for Lie-point symmetries of (1); we will denote their Lie generator by

$$X = \xi(x, y, u) \frac{\partial}{\partial x} + \eta(x, y, u) \frac{\partial}{\partial y} + \varphi(x, y, u) \frac{\partial}{\partial u}.$$

Assume e.g. $a_{11} \neq 0$ (and put then $a_{11} = 1$).

Although essentially standard (see [1, 2, 7], and also [17] for some generalization), let us summarize for convenience and in order to fix notations, these first basic results, which can be easily obtained imposing the usual symmetry condition

$$X^{(2)}(\Delta) \Big|_{\Delta=0} = 0$$

where $\Delta = \mathcal{E}[u] - \alpha_\ell F_\ell$, and $X^{(2)}$ is the second prolongation of X .

Lemma 1 *For any choice of the functions F_ℓ , the coefficients of the Lie-point symmetry operators admitted by a PDE of the form (1) satisfy the conditions $\xi_u = \eta_u = 0$; $\varphi_{uu} = 0$, i.e.*

$$\xi = \xi(x, y) \quad , \quad \eta = \eta(x, y) \quad , \quad \varphi = A(x, y) + uB(x, y) \quad (2)$$

and the unique determining equation which involves the functions $F_\ell(u)$ takes the form (with $F'_\ell = dF_\ell/du$, sum over $\ell = 1, \dots, L$)

$$p_\ell(x, y)F_\ell(u) + p_{L+\ell}(x, y)F'_\ell(u) + p_{2L+\ell}(x, y)uF'_\ell(u) + p_{3L+1}(x, y)u + p_{3L+2}(x, y) = 0 \quad (3)$$

where the coefficient functions $p_i(x, y)$ ($i = 1, \dots, 3L + 2$) are given by

$$p_\ell = B\alpha_\ell - (\xi_x + \eta_y)\alpha_\ell - a_{12}\xi_y\alpha_\ell - \xi\alpha_{\ell,x} - \eta\alpha_{\ell,y} \quad , \quad (4)$$

$$p_{L+\ell} = -\alpha_\ell A \quad , \quad p_{2L+1} = -\alpha_\ell B \quad , \quad p_{3L+1} = \mathcal{E}[B] \quad , \quad p_{3L+2} = \mathcal{E}[A]$$

with $\xi_x = \partial\xi/\partial x$, $\alpha_{\ell,x} = \partial\alpha_\ell/\partial x$ etc. ($\ell = 1, \dots, L$).

Considering now the determining equation (3), and observing that the p_i depend only on x, y and not on u , one immediately realizes that, if the $3L + 2$ functions f_i defined by

$$f \equiv (F_\ell, F'_\ell, uF'_\ell, u, 1) \quad (\ell = 1, \dots, L) \quad (5)$$

are *linearly independent*, then (3) can be satisfied if and only if

$$p_i = 0 \quad (i = 1, \dots, 3L + 2) . \quad (6)$$

Recalling now the definition of *kernel* of the full (or principal) symmetry groups [1] of eq. (1), i.e. the intersection of all symmetry groups admitted by (1) for any arbitrary choice of $F_\ell(u)$, we can then state the following property.

Lemma 2 *Conditions (6) characterize the kernel of the symmetry groups of equation (1).*

Indeed, conditions (6), together with the other determining equations (not involving F_ℓ), determine the functions ξ, η, A, B (i.e. the symmetries admitted by equation (1)), which are independent of the choice of the functions F_ℓ . These symmetries may be considered “trivial” in this context: for instance, if all coefficients a, b, α in (1) are independent of y , then such a symmetry operator is $\partial/\partial y$. Some not so obvious examples of symmetries of this type will be presented later.

Therefore, a first conclusion is that, in order to have “nontrivial” symmetries (i.e. really dependent on the choice of the functions F_ℓ), a necessary condition is the existence of some linear dependence among the functions (5).

Another relevant remark which will emerge from our discussion is the important role played also by the coefficient functions $\alpha_\ell(x, y)$ in the determination of the admitted symmetries.

3 Conditions for the existence of symmetries.

Consider the linear space generated by the $3L + 2$ functions f_i defined in (5), and, according to our above remarks, now assume that there are some linear relations among these functions. Then the f_i span a space with dimension $k < D = 3L + 2$; if this is the case, the D coefficients p_i are forced, according to (3), to belong to the *orthogonal* $(D - k)$ -dimensional subspace (with respect to the standard scalar product in \mathbf{R}^D), and the functions $p_i(x, y)$

turn out to be subjected to k linear conditions. For instance, if there is just one linear relationship between the f_i , say

$$\sum_{i=1}^D \lambda_i f_i = 0 \quad (7)$$

where not all the constants λ_i vanish, then $k = D - 1$ and the functions p_i span a 1-dimensional subspace and must satisfy $D - 1$ equations of the form (assuming that, e.g., $\lambda_D \neq 0$)

$$p_1 \lambda_D = \lambda_1 p_D, \quad p_2 \lambda_D = \lambda_2 p_D, \quad \dots, \quad p_{D-1} \lambda_D = \lambda_{D-1} p_D. \quad (8)$$

We can then state our main conclusion, which characterizes the crucial determining equation which contains the functions F_ℓ in the following “geometrical” form.

Proposition 1 *Eq. (1) admits nontrivial symmetries only if the $D = 3L + 2$ functions (5) are linearly dependent. If this is the case, the D functions $p_i(x, y)$ given by (4) appearing in the determining equation (3) span the subspace orthogonal (with respect to the standard scalar product in \mathbf{R}^D) to the k -dimensional ($k < D$) subspace spanned by the functions (5). The admitted symmetries are completely determined by imposing this orthogonality condition to the coefficient functions $p_i(x, y)$, together with the other determining equations not involving the functions $F_\ell(u)$.*

Before considering explicit examples, let us remark that the complete symmetry classification must be accompanied by the determination of the equivalence group [1], i.e. the group of the transformations which leave invariant the differential structure of the PDE. Standard calculations show easily that, for any fixed choice of the functions a_{ij} , b_i , α_ℓ in equation (1), the equivalence group includes in particular, expectedly, the scalings $u \rightarrow cu$, $F_\ell \rightarrow cF_\ell$ and the translation $u \rightarrow u + c_0$ ($c, c_0 = \text{const.}$). Other transformations involving also the variables x, y can appear for particular choices of the functions a_{ij} , b_i , α_ℓ . The transformations belonging to the equivalence group will play an important role in performing the complete symmetry classification of our equations.

4 First example: a generalized Laplace equation

To illustrate the main idea and the procedure, and also to clarify some details, we are now going to examine some examples, which can be noteworthy.

thy also for their different and interesting peculiarities. We start considering the simplest case, where the r.h.s. of (1) contains only one term $\alpha(x, y)F(u)$ (then $D = 5$).

First of all, let us remark that, independent of the form of the equation (1), not all linear combinations among the f_i defined in (5) can be chosen arbitrarily. For instance, no relation exists between u and 1, and also, having excluded the trivial case of linear $F(u)$, between F , u , 1. On the other hand, the necessary linear dependence between the functions f_i implies immediately that the presence of some symmetry is possible only if $F(u)$ is an exponential or a power of u . It can also happen that more than one linear relation holds: e.g., if $F = u^2/2 + u$, then both relations

$$F' - u - 1 = 0 \quad \text{and} \quad 2F - uF' - u = 0 \quad (9)$$

hold simultaneously, i.e. $k = 3$. In this case, the orthogonality condition expressed by Proposition 1 becomes

$$p_1 + 2p_4 - 2p_5 = 0, \quad p_2 + p_5 = 0, \quad p_3 - p_4 + p_5 = 0. \quad (10)$$

As an explicit example, let us consider the following generalization of the classical nonlinear Laplace equation

$$\Delta \equiv \nabla^2 u - \alpha(x, y)F(u) = 0 \quad (11)$$

where $\alpha(x, y)$ is a given function. In this case, the determining equations not containing F imply in particular

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x, \quad B = \text{const} \quad (12)$$

whereas the crucial determining equation (3) involving F is

$$p_1 F + p_2 F' + p_3 u F' + p_4 u + p_5 = 0 \quad (13)$$

with the coefficient functions $p_i(x, y)$ given by

$$\begin{aligned} p_1 &= B\alpha - (\xi_x + \eta_y)\alpha - (\xi\alpha_x + \eta\alpha_y), \quad p_2 = -\alpha A, \\ p_3 &= -\alpha B, \quad p_4 = \nabla^2 B = 0, \quad p_5 = \nabla^2 A. \end{aligned} \quad (14)$$

Let us first discuss the kernel group. The conditions $p_i = 0$ ($i = 1, \dots, 5$) characterizing the transformations in the kernel group (see Lemma 2) give

now $A = B = \varphi = 0$ and the condition $p_1 = 0$, which now reads $\alpha(\xi_x + \eta_y) + (\xi\alpha_x + \eta\alpha_y) = 0$. Introducing a harmonic function $\Phi = \Phi(x, y)$ such that

$$\Phi_x = \xi \quad \Phi_y = -\eta$$

this condition can be more conveniently transformed into an equation for the single unknown Φ :

$$\alpha(\Phi_{xx} - \Phi_{yy}) + \alpha_x\Phi_x - \alpha_y\Phi_y = 0. \quad (15)$$

Solutions of this equation clearly depend on the choice of the function $\alpha(x, y)$. For instance, if $\alpha = \text{const.}$, it gives $\xi_x + \eta_y = 0$, which, together with (12), implies that the symmetries in the kernel group are, as expected, only translations and rotations of the variables x, y . If $\alpha = \exp(2x)$ then $\Phi = \exp(-x)(c_1 \cos y + c_2 \sin y) + c_3 y + c_4$ and the kernel contains, apart from the translation generated by $\partial/\partial y$, the transformations generated by

$$X_1 = \exp(-x) \left(\cos y \frac{\partial}{\partial x} - \sin y \frac{\partial}{\partial y} \right) \quad , \quad X_2 = \exp(-x) \left(\sin y \frac{\partial}{\partial x} + \cos y \frac{\partial}{\partial y} \right).$$

With $\alpha = x^r$, one has that if $r \neq -2$ then the kernel contains only the translation generator $\partial/\partial y$, whereas if $r = -2$ it also contains the transformations generated by the two operators

$$X_1 = 2xy \frac{\partial}{\partial x} - (x^2 - y^2) \frac{\partial}{\partial y} \quad \text{and} \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

The first transformation describes the kernel group even if $\alpha = x^{-2}\beta(y/(x^2 + y^2))$, where β is an arbitrary function.

It can be remarked, incidentally, that if we reverse the argument for a moment, one has that: given *any* harmonic function Φ (and then any couple of harmonic conjugate functions ξ, η), there are some $\alpha(x, y)$ which solve equation (15), and then, with these functions α , the kernel group contains precisely the symmetry generated by the operator $X = \xi(\partial/\partial x) + \eta(\partial/\partial y)$.

Let us now finally perform the symmetry classification of eq. (11). As already remarked, its equivalence group may contain, in addition to the transformations listed at the end of Section 3, and depending on the specific choice of the function α , other transformations possibly involving also x, y . As we shall see, however, these are not relevant for the symmetry classification of equation (11).

According to our procedure, it is immediately seen that just *one* linear relation between the five f_i can exist. For instance, in the case $F = u^2/2 + u$

mentioned above, admitting the two linear relations (9), conditions (10) would lead to $p_i = 0$, i.e. only the kernel symmetries. We then assume the existence of a single linear relation:

$$\lambda_1 F + \lambda_2 F' + \lambda_3 u F' + \lambda_4 u + \lambda_5 = 0 \quad (16)$$

with not all λ_i equal to zero. Observing that $p_4 = 0$, one has from (8) that $\lambda_4 = 0$. We now distinguish the cases $\lambda_3 \neq 0$ and $\lambda_3 = 0$.

Let $\lambda_3 \neq 0$. It is not restrictive to put $\lambda_3 = 1$, and (up to a translation of u) $\lambda_2 = 0$, which implies $\lambda_5 p_2 = 0$. If $\lambda_5 \neq 0$, then $p_2 = 0$ would imply $A = 0$ and also $p_5 = 0 = p_3 \lambda_5$; now, if $p_3 = 0$ it remains only $p_1 \neq 0$, and from $\lambda_5 p_1 = \lambda_1 p_5 = 0$ one concludes that $\lambda_5 = 0$. Therefore, we get $\lambda_1 F + u F' = 0$, i.e.

$$F(u) = u^m \quad \text{with} \quad m = -\lambda_1 \quad (m \neq 0, 1) .$$

Using now (8), which become $p_1 + m p_3 = 0$, $p_2 = p_4 = p_5 = 0$, we get

$$A = 0 \quad \text{and} \quad \alpha B(m - 1) + (\xi_x + \eta_y)\alpha + (\xi\alpha_x + \eta\alpha_y) = 0 .$$

The last equation relates the symmetry coefficients ξ , η , B with the specific form of the function $\alpha(x, y)$. If for instance $\alpha = x^r$, then $B(m - 1) + \xi_x + \eta_y = 0$, but B must be $\neq 0$, otherwise also $p_3 = p_1 = 0$, i.e. the kernel group. Therefore, ξ must be proportional to x and equation (11) admits the symmetry operator

$$X = (m - 1) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - (r + 2) u \frac{\partial}{\partial u}$$

(and obviously the translation of the variable y , and also the translation of x and the rotations of x, y in the case $r = 0$, i.e. if $\alpha = \text{const.}$).

Let now $\lambda_3 = 0$. Then necessarily $\lambda_2 \neq 0$, and one can put $\lambda_2 = 1$ and also $\lambda_1 = 1$ (possibly up to a scaling of u). Assume first $\lambda_5 = 0$; therefore, from (16),

$$F(u) = \exp(-u)$$

and the conditions (8),(14) for the functions p_i become now

$$p_3 = -\alpha B = 0, \quad p_5 = \nabla^2 A = 0, \quad \text{and} \quad p_1 = p_2 \quad \text{i.e.}$$

$$(\xi_x + \eta_y)\alpha + (\xi\alpha_x + \eta\alpha_y) = \alpha A .$$

As before, we can consider some examples. If $\alpha = \text{const.}$, the last equation shows $A = \xi_x + \eta_y$ and then the most general symmetry of the equation $\nabla^2 u = \exp(-u)$ is

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + (\xi_x + \eta_y) \frac{\partial}{\partial u}$$

where ξ, η are arbitrary harmonic conjugate functions: this is the well known case of the classical Liouville equation (in its “elliptic” form; for a full discussion of the symmetry properties and other related features of the Liouville-type equations, see [18]-[23]). If instead for instance $\alpha = y^r$ then the admitted symmetry operators are

$$X_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + (2+r) \frac{\partial}{\partial u} \quad , \quad X_2 = (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} + (2r+4)x \frac{\partial}{\partial u}$$

and the translation $\partial/\partial x$. Assume now $\lambda_5 \neq 0$, then

$$F(u) = \exp(-u) - \lambda_5 .$$

Introducing the transformation $u \rightarrow u + \tilde{u}$, where $\tilde{u} = \tilde{u}(x, y)$ satisfies the equation $\nabla^2 \tilde{u} + \lambda_5 \alpha = 0$, one obtains the new equation

$$\nabla^2 u - \tilde{\alpha} \exp(-u) = 0$$

where $\tilde{\alpha}(x, y) := \alpha \exp(-\tilde{u})$, which has precisely the same form as the equation considered before. Without repeating details, it can be interesting to provide just one illustrative example. Let

$$\nabla^2 u - k x^{-2} (\exp(-u) + 1) = 0$$

where $k = \text{const.}$ It is easy to see that if $k = 2$ this equation admits the symmetries

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \left(\xi_x + \eta_y - \frac{2}{x} \xi \right) \frac{\partial}{\partial u}$$

where ξ, η are arbitrary harmonic conjugate functions; if instead $k \neq 2$, the admitted symmetries are only those in the kernel group.

The above results concerning eq. (11) can be stated in a complete form as follows.

Proposition 2 *Given a function $\alpha = \alpha(x, y)$, consider this equation for the harmonic function $\Phi = \Phi(x, y)$*

$$\alpha(\Phi_{xx} - \Phi_{yy}) + \alpha_x \Phi_x - \alpha_y \Phi_y - \alpha C = 0 . \quad (17)$$

Let $\xi = \Phi_x$, $\eta = -\Phi_y$. Assume first $C = 0$: for any solution Φ of (17), the kernel group of the generalized Laplace equation (11) is generated by the symmetry operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} .$$

If $F(u) = u^m$, for any solution Φ of (17) with $C = \text{const.} \neq 0$, eq. (11) admits the symmetry operator

$$X = (m - 1) \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) - Cu \frac{\partial}{\partial u} .$$

If $F(u) = \exp(-u)$, for any solution Φ of (17) with $C = C(x, y)$ any harmonic function, eq. (11) admits the symmetry operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + C(x, y) \frac{\partial}{\partial u} .$$

In particular, if $\alpha = \text{const.}$ then $C = \xi_x + \eta_y$ where ξ, η are arbitrary harmonic conjugate functions, and the case of the standard “elliptic” Liouville equation is recovered. If finally $F(u) = \exp(-u) + c$ ($c = \text{const.}$), the above case is recovered by means of the transformation $u \rightarrow u + \tilde{u}$, where $\tilde{u} = \tilde{u}(x, y)$ satisfies the equation $\nabla^2 \tilde{u} - c\alpha = 0$. This completes the symmetry classification of the PDE (11), apart from the transformations in the equivalence group.

5 An example with two arbitrary functions.

We now consider the case of a PDE of the form (1) with two arbitrary functions $F_\ell(u)$, i.e. $L = 2$, $D = 8$. To avoid excessive generality, let us restrict our study to a PDE of the following form

$$u_{xx} + u_{yy} + \frac{a}{x} u_x = \alpha(x, y) F_1(u) + F_2(u) \quad (18)$$

here $b_1 = a/x$, $a \neq 0$ is a constant, $\alpha_1 = \alpha(x, y)$ a given function and $\alpha_2 = 1$. The choice of this equation is motivated and suggested by the theory of plasma physics: it is indeed a generalization of the Grad-Schlüter-Shafranov equation (see [16]), which is obtained putting in (18) $a = -1$, $\alpha = x^2$, and describes the magnetohydrodynamic force balance in a magnetically confined toroidal plasma. In this context, u is the so-called magnetic flux variable, x is a radial variable (then $x \geq 0$), while the two arbitrary functions

$F_1(u)$, $F_2(u)$ are flux functions related to the plasma pressure and current density profiles.

The determining equations not involving $F_\ell(u)$ give in this case

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x, \quad \text{and} \quad B = \frac{-a\xi}{2x} + b, \quad b = \text{const} .$$

First of all, the kernel group is immediately seen to be trivial (apart obviously from the translation generated by $\partial/\partial y$, in the case where α depends only on x ; the possible presence of this symmetry will be tacitly understood in the following). Indeed, from $p_i = 0$ (see Lemma 2), one has $A = B = 0$, then the above equation implies $\xi = (2b/a)x$, and the condition $p_2 = 0$ with $\alpha_2 = 1$ gives finally $\xi = 0$.

Let us now start assuming that there are exactly *two* linear relations involving the functions F_1 and F_2 separately:

$$\begin{aligned} \lambda_1 F_1 + \lambda_3 F_1' + \lambda_5 u F_1' + \lambda_7 u + \lambda_8 &= 0 \\ \lambda_2 F_2 + \lambda_4 F_2' + \lambda_6 u F_2' + \mu_7 u + \mu_8 &= 0 \end{aligned} \quad (19)$$

with not all λ_i, μ_i equal to zero.

Let $\lambda_5 \lambda_6 \neq 0$, and put $\lambda_5 = \lambda_6 = 1$. According to Proposition 1, the symmetry coefficients p_i , given by (4), satisfy then the six linear conditions

$$\begin{aligned} p_1 = \lambda_1 p_5, \quad p_2 = \lambda_2 p_6, \quad p_3 = \lambda_3 p_5, \quad p_4 = \lambda_4 p_6, \\ p_7 = \lambda_7 p_5 + \mu_7 p_6, \quad p_8 = \lambda_8 p_5 + \mu_8 p_6 \end{aligned} \quad (20)$$

With $\lambda_5 \neq 0$, we can put $\lambda_3 = 0$, up to a translation of u . Conditions (20) and the expression of the coefficients p_i give $p_3 = 0$, $A = 0$, then $p_4 = p_8 = 0$ and therefore also $\lambda_4 = \lambda_8 = \mu_8 = 0$ (indeed, $p_5 p_6 \neq 0$, otherwise all $p_i = 0$); condition $p_2 = \lambda_2 p_6$ implies that $\xi(x, y)$ must satisfy an equation of the form

$$\xi_x = k_0 \frac{\xi}{x} + k_1 \quad (k_0, k_1 = \text{const})$$

which admits harmonic solution only of the form $\xi = cx$, $c = \text{const}$. On the other hand, $\lambda_1 = p_1/p_5$, $\lambda_2 = p_2/p_6$ imply that α is forced to satisfy

$$\frac{x\alpha_x + y\alpha_y}{\alpha} = r = \text{const} . \quad (21)$$

This means that if α does not satisfy this condition, no symmetry is allowed; we then assume for α the form

$$\alpha(x, y) = x^r \beta(y/x)$$

where β is arbitrary. Notice that, with α of this form, a new transformation is included in the equivalence group, namely the scaling $x \rightarrow cx$, $y \rightarrow cy$, $F_1 \rightarrow c^{2-r}F_1$, $F_2 \rightarrow c^{-2}F_2$. We also deduce $B = \text{const.} \neq 0$, $p_7 = 0$, which implies in turn $\lambda_7 = \mu_7 = 0$. Then we are left with

$$\lambda_1 F_1 + u F_1' = 0 \quad , \quad \lambda_2 F_2 + u F_2' = 0$$

giving (thanks to some scalings – all these transformations belong indeed to the equivalence group) $F_1 = u^{-\lambda_1}$, $F_2 = u^{-\lambda_2}$ where

$$-\lambda_1 = 1 - \frac{c}{B}(2+r) = 1 + \frac{2+r}{q} \quad , \quad -\lambda_2 = 1 + \frac{2}{q} \quad \text{with} \quad B = -cq$$

with admitted symmetry generated by

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - q u \frac{\partial}{\partial u} .$$

Let now $\lambda_5 = \lambda_6 = 0$. Then necessarily $\lambda_3 \lambda_4 \neq 0$. According to Proposition 1, the orthogonality condition now reads (with $\lambda_3 = \lambda_4 = 1$)

$$p_1 = \lambda_1 p_3 \quad , \quad p_2 = \lambda_2 p_4 \quad , \quad p_5 = p_6 = 0 \quad , \quad p_7 = \lambda_7 p_3 + \mu_7 p_4 \quad , \quad p_8 = \lambda_8 p_3 + \mu_8 p_4 .$$

In this case, one has immediately $B = 0$, then $p_7 = 0$ and $\lambda_7 = \mu_7 = 0$, and again $\xi = cx$. From $p_2 = p_4$ one has $2\xi_x = A = \text{const.}$, which gives $p_8 = \lambda_8 = \mu_8 = 0$. Up to a scaling of u , one can choose $\lambda_2 = 1$, the equations for F_1, F_2 are then

$$\lambda_1 F_1 + F_1' = 0 \quad , \quad F_2 + F_2' = 0$$

giving $F_1 = \exp(-\lambda_1 u)$, $F_2 = \exp(-u)$, and finally from $\lambda_1 = p_1/p_3$ one deduces the same condition (21) as before for the function $\alpha(x, y)$, and $\lambda_1 = 1 + (r/2)$.

The conclusion will be stated in complete form in the following Proposition 3. Indeed, with the same, and even simpler, arguments used in the two above cases, it is an easy task to see that no other possibilities are left to the PDE (18) of admitting symmetries.

Proposition 3 *The kernel group of equation (18) is trivial (apart from the translation $y \rightarrow y + c$ if α depends only on x). Except for this, a necessary condition in order that equation (18) may admit symmetries is that the function α has the form $\alpha(x, y) = x^r \beta(y/x)$, where β is an arbitrary function.*

With α of this form, equation (18) admits a symmetry only with the following choices for the functions $F_1(u), F_2(u)$ (up to transition to equivalent functions via equivalence group):

$$a) \quad F_1(u) = u^{1+(r+2)/q} \quad , \quad F_2(u) = u^{1+(2/q)}$$

for all $q \neq 0$, with admitted symmetry operator

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - q u \frac{\partial}{\partial u}$$

and

$$b) \quad F_1(u) = \exp\left(-\left(1 + \frac{r}{2}\right)u\right) \quad , \quad F_2(u) = \exp(-u)$$

with symmetry operator

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2 \frac{\partial}{\partial u} .$$

As stated at the beginning of Section 2, we have excluded from our analysis the case where the functions F_1 and F_2 are linear functions of u . This case indeed, in general of minor interest, can be more conveniently considered by means of a separate and direct calculation. Specifically, if the r.h.s. of equation (18) has the form $\tilde{\alpha}(x, y)u + \tilde{\beta}(x, y)$, the admitted symmetry is, not surprisingly, generated by

$$X = (cx + \Psi(x, y)) \frac{\partial}{\partial u}$$

where c is a constant and $\Psi(x, y)$ is any solution of the PDE

$$\mathcal{E}[\Psi] = \tilde{\alpha}\Psi - c\tilde{\beta} .$$

In the particular case of the Grad-Schlüter-Shafranov equation of plasma physics [16], the above results were already presented in Ref. [24], but without any details in the calculations and without any reference to the procedure, which is instead one of the main purposes of the present paper. In the same reference [24] one can also find some physical comment on the symmetry properties of the above equation.

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