NOTES ON N. BOURBAKI'S

Integration

Notes by S. K. Berberian

Notes on text through Ch.V, §5, No. 4 (10-19-08)

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Foreword

When I first studied the original French fascicles of N. Bourbaki's *Intégration* (in the 1970's) I made a handwritten working translation. Whenever I reached an assertion that I didn't "see" after a few seconds of reflection (which was often), I opened a pair of braces { } and worked out an explanation to myself before proceeding further. This document is a transcription of those explanations, indexed by the page and line numbers of the text in the published translation to which they pertain (N. Bourbaki, *Integration*, Vols. I, II, Springer–Verlag, 2004). On restudying the work while preparing the new translation (starting in 1998), I often consulted the notes, occasionally finding my explanations inadequate; the notes in their present form aspire to repair the inadequacies.

All of the above is rather personal; what justification can there be for putting the notes out in front of everybody? Some of the gaps I thought I saw proved to be trivial (*after* I saw the light); others took several days to work out—but there might have been shortcuts that I didn't find. My motivation was high; I was determined to acquire the prerequisites for reading the author's *Théories spectrales* and J. Dixmier's books on operator algebras, so I ground at the obstacle until it yielded. In answer to the question posed above: My hope is that the reader of *Integration*, who comes upon a sticking point and does not have the leisure to grind it away, will find in these notes an explanation that will enable him to keep going without loss of momentum.

Paragraph 3 of the *mode d'emploi* ("To the Reader") makes it plain that the typical reader of *Integration* comes to the task well-equipped. We can infer that the "gaps" are there on purpose; the missing details were doubtless present in an initial draft of a proof, but (if we may be permitted to read the author's mind) they have been deliberately pruned away to highlight the main outline of the proof while leaving enough details to enable the determined reader to fill in the gaps. This is marvelous exercise for the research muscles, surely one of the author's didactic aims. That is good news; the reader can be reassured that the gaps are well thought out and the chances good that a moderate amount of effort will see him through and render these notes superfluous.

Errata for *Integration* (INT)

III.52, ℓ. −3. For I read L.

IV.17, ℓ. 2. After "belong to $\mathscr{A}$," insert "are $\leqslant f$,".

IV.30, ℓ. −2. Delete inverted comma at the end of the line.

IV.40, ℓ. −15. For "step function" as translation of *fonction en escalier*, see the Note for this line (on p. IV.x26 below) and especially the Note for IV.66, ℓ. −15, −14 (on p. IV.x88).

IV.49, Footnote. The period at the end should immediately follow the parenthesis.

IV.51, Footnote. Vis-a-vis the translation of *fonction étagée*, see the Notes mentioned above in the entry for IV.40, ℓ. −15.

IV.73, ℓ. −8. For "Cor." read "Cor. 1".

IV.84, ℓ. −13. For $|\mathbf{y}-\mathbf{z}| \leqslant \delta$ read $|\mathbf{y}-\mathbf{z}| < \delta$. (See the Note for this line.)

IV.88, Running head. For §6 read §5.

IV.108, ℓ. 20. For TVS read EVT. {See the Note for ℓ. 20–24 on p. IV.x248 below.}

IV.113, ℓ. 1. The argument needed for (16_n) is more complicated. {See the Note for this line on p. IV.x265 below.}

IV.115, ℓ. −5. In the formula (19), for $f(x)$ read $\overline{f}(x)$.

IV.117, ℓ. 21. For $h(x) = 1$ read, say, $h(z) = 1$. (The letter x is reserved for an element of M in the statement of the Corollary, and y occurs later in the proof as another element of M.)

V.12, ℓ. −9. For "No. 3" read "No. 4".

V.46, ℓ. −10. For (g_α) read (g_n).

V.49, ℓ. 9. For "Lemma 1" read "Lemma 2".

V.77, ℓ. 1 and −9. Lines incompletely printed ('washed out').

VI.10, ℓ. 9. For "q is gauge" read "q its gauge".

VI.14, ℓ. −9. For "subset" read "subsets".

Extended "asides"

III.16, ℓ. 13–15. The discrete measure on $X = \mathbf{N}$ with unit mass at every point. {III.x13–x18}

IV.66, ℓ. $-15, -14$. "Fonction étagée" *vs.* "fonction en escalier" ("step function" *vs.* "interval step function"). {IV.x88, x89}. See also IV.40, ℓ. -15. {IV.x26, x27}

IV.79, ℓ. $3, 4$. Conditions c') and c'') for No. 10, Prop. 15. {IV.x127}

IV.79, ℓ. 7. Condition d'). {IV.x129}

IV.80, ℓ. -17 to -14. Measurability of functions defined on a measurable subset. {IV.x133–x146}

IV.85, ℓ. 19–27 and the "echo" that follows it. Measure as a set function, 'comparative anatomy' of the Bourbaki and Halmos formulations of measure on a locally compact space. {IV.x170–x175 and IV.x176–179}

V.10, ℓ. $13, 14$. 'Bilinearity' of $(\mathbf{f}, \mu) \mapsto \int \mathbf{f}\, d\mu$. {V.x15–x21, especially the *Theorem* (V.x17) and its *Corollary 3* (V.x19)}

V.17, ℓ. -6 to -4. Solution of Exer. 7 for Ch. V, §3. {V.x33–x36}

V.17, ℓ. -4. Solution of Exer. 8 for Ch. V, §3. {V.x38, x39}

V.25, ℓ. 8–11. The analog of "Lebesgue's theorem" (Ch. IV, §3, No. 7, Th. 6) for essentially integrable functions is worked out in this Note. {V.x68, *Remark*}

V.27, ℓ. $-14, -13$. Revisiting the notation $\mu = \int \varepsilon_x\, d\mu(x)$ of Ch. III, §3, No. 1, *Example* 2, in the context of diffusions. {V.x73–x75, *An example*}

V.29, ℓ. -11 to -9. A handy 'scholium' for bounded diffusions. {V.x80}

V.30, ℓ. 6–28. Composition of diffusions when viewed as mappings $\mu \mapsto \mu\Lambda$ ($\mu \in$ domain of Λ), and the inclusion $\Lambda \circ \mathrm{H} \subset \Lambda\mathrm{H}$. {V.x82–x85}

V.38, ℓ. -10 to -3 and **V.39,** ℓ. 22–24. Adjustments to the definition of local integrability. {V.x106–x109 and V.x110, *Proposition*}

V.39, ℓ. $26, 27$ and **V.39,** ℓ. -12 to -10. Review of the locally convex topology generated by a set of semi-norms. {V.x111–x113 and V.x113, x114}

V.46, ℓ. $16, 17$. A useful criterion for measurability of a numerical function ($\mathbf{f}$ is measurable $\Leftrightarrow \mathbf{f}\varphi_{\mathrm{K}}$ is measurable for every compact set K). {V.x138, *Lemma*}

V.47, ℓ. 14–16. If $\mathbf{f}$ and $\mathbf{g}$ are measurable numerical functions $\geqslant 0$, then $\mathbf{fg}$ is measurable. {V.x145, *Lemma*}

Loose ends: some questions posed in the Notes

III.23, ℓ. 16–19. A question about a direct limit of induced topologies. {III.x28, ℓ. −6 to −4}.

III.54, ℓ. 11–13. Wanted: A simpler proof of the associative law for the product of measures. {III.x135, ℓ. 8–10}

IV.85, ℓ. 19–27. Is $\nu = \rho$ always? {IV.x170, ℓ. $-2, -1$}

V.32, ℓ. 18–20. A question about the definition of vague measurability of mappings $\Lambda : t \mapsto \lambda_t$. {V.x90, ℓ. $-8, -7$}

V.45, ℓ. $10, 11$. A question about moderated measures posed here is answered in an addendum to the Note for V.48, ℓ. $2, 3$.

CHAPTER I

Inequalities of convexity

I.2, ℓ. 15–16.

"…therefore there exists a $\lambda_0 > 0$ such that the relation $\lambda \geqslant \lambda_0$ implies $\lambda\mathbf{t} \in \mathrm{K}$;"

For example, set $\lambda_0 = \varphi(t_1, t_2, \ldots, t_n)^{-1}$ and cite the positive homogeneity of φ: if $\lambda \geqslant \lambda_0$, then $\varphi(\lambda t_1, \ldots, \lambda t_n) = \lambda\lambda_0^{-1} \geqslant 1$, whence $\lambda\mathbf{t} \in \mathrm{K}$.

I.2, ℓ. 17.

"…the relations $t_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ imply $\alpha_{\iota 1}t_1 + \cdots + \alpha_{\iota n}t_n \geqslant 0$"

Suppose first that $t_i > 0$ for all i, and choose λ_0 as above. Then, for all $\lambda \geqslant \lambda_0$, $\lambda\mathbf{t} \in \mathrm{K}$ and therefore, for each ι,

$$\alpha_{\iota 1}(\lambda t_1) + \cdots + \alpha_{\iota n}(\lambda t_n) \geqslant \beta_\iota\,,$$

that is,

$$\alpha_{\iota 1}t_1 + \cdots + \alpha_{\iota n}t_n \geqslant \lambda^{-1}\beta_\iota\,,$$

and letting $\lambda \to +\infty$ yields $\alpha_{\iota 1}t_1 + \cdots + \alpha_{\iota n}t_n \geqslant 0$.

If now $\mathbf{t} = (t_i)$ with $t_i \geqslant 0$ for $i = 1, \ldots, n$, then, for each ι, the foregoing yields $\alpha_{\iota 1}(t_1 + \varepsilon) + \cdots + \alpha_{\iota n}(t_n + \varepsilon) \geqslant 0$ for every $\varepsilon > 0$, whence $\alpha_{\iota 1}t_1 + \cdots + \alpha_{\iota n}t_n \geqslant 0$.

I.2, ℓ. 18–19.

"…it is then clear that K is also the intersection of the half-spaces $t_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ and the U_ι such that $\beta_\iota \geqslant 0$;"

If $\beta_\iota < 0$, the inequality (2) is trivially satisfied when the t_i are $\geqslant 0$.

I.2, ℓ. 20.

"…since the origin does not belong to K"

Writing $\mathbf{0} = (0, \ldots, 0)$, $\varphi(\mathbf{0}) = \varphi(2 \cdot \mathbf{0}) = 2\varphi(\mathbf{0})$ implies $\varphi(\mathbf{0}) = 0$ by finiteness, whence $\mathbf{0} \notin \mathrm{K}$.

I.2, ℓ. 20–21.

"…there exists at least one index ι such that $\beta_\iota > 0$."

If one had $\beta_\iota = 0$ for all ι, then $\mathbf{0} = (0, \dots, 0)$ would satisfy all of the inequalities defining K, that is, it would belong to all of the half-spaces whose intersection is K, contrary to $\mathbf{0} \notin \mathrm{K}$.

I.2, ℓ. −11 to −9.

"Now let C be the convex cone in $\mathbf{R}^{n+1}$ defined by the relations $t_i \geqslant 0$ $(1 \leqslant i \leqslant n+1)$, $t_{n+1} \leqslant \varphi(t_1, t_2, \dots, t_n)$ (the closure of the convex cone generated in $\mathbf{R}^{n+1}$ by the convex set $\mathrm{K} \times \{1\}$)"

It is clear from the continuity of φ that C is closed. Let D be the convex cone generated by the convex set $\mathrm{K} \times \{1\}$, namely,

$$\begin{aligned} \mathrm{D} &= \bigcup_{\alpha > 0} \alpha(\mathrm{K} \times \{1\}) \\ &= \{(\alpha t_1, \dots, \alpha t_n, \alpha) : \ \alpha > 0 \ \& \ (t_1, \dots, t_n) \in \mathrm{K}\}. \end{aligned}$$

We are to show that $\mathrm{C} = \overline{\mathrm{D}}$.

If $\alpha > 0$ and $(t_1, \dots, t_n) \in \mathrm{K}$ then

$$\varphi(\alpha t_1, \dots, \alpha t_n) = \alpha \varphi(t_1, \dots, t_n) \geqslant \alpha \cdot 1 = \alpha,$$

whence $(\alpha t_1, \dots, \alpha t_n, \alpha) \in \mathrm{C}$. Thus $\mathrm{D} \subset \mathrm{C}$ and, since C is closed, $\overline{\mathrm{D}} \subset \mathrm{C}$.

To prove the reverse inclusion $\mathrm{C} \subset \overline{\mathrm{D}}$, suppose $(t_1, \dots, t_n, t_{n+1}) \in \mathrm{C}$, that is, $t_i \geqslant 0$ for $1 \leqslant i \leqslant n+1$ and $t_{n+1} \leqslant \varphi(t_1, \dots, t_n)$.

case 1: $t_{n+1} > 0$.

Then $1 \leqslant \varphi(t_{n+1}^{-1} \cdot t_1, \dots, t_{n+1}^{-1} \cdot t_n)$, thus $t_{n+1}^{-1}(t_1, \dots, t_n) \in \mathrm{K}$, $(t_1, \dots, t_n) \in t_{n+1}\mathrm{K}$, and so $(t_1, \dots, t_n, t_{n+1}) \in t_{n+1}(\mathrm{K} \times \{1\}) \subset \mathrm{D} \subset \overline{\mathrm{D}}$.

case 2: $t_{n+1} = 0$.

Thus $0 = t_{n+1} \leqslant \varphi(t_1, \dots, t_n)$. We consider two sub-cases.

If $\varphi(t_1, \dots, t_n) > 0$, then for $\gamma_0 = \varphi(t_1, \dots, t_n)^{-1}$ we have

$$\gamma_0 \varphi(t_1, \dots, t_n) = 1,$$

whence $\gamma \varphi(t_1, \dots, t_n) \geqslant 1$ for all $\gamma \geqslant \gamma_0$; thus, for all $\gamma \geqslant \gamma_0$, we have $\varphi(\gamma t_1, \dots, \gamma t_n) \geqslant 1$, so that $(\gamma t_1, \dots, \gamma t_n) \in \mathrm{K}$ and therefore $(\gamma t_n, \dots, \gamma t_n, 1) \in \mathrm{K} \times 1$, whence $(t_1, \dots, t_n, \gamma^{-1}) \in \gamma^{-1}(\mathrm{K} \times \{1\}) \subset \mathrm{D}$; letting $\gamma \to +\infty$ one sees that $(t_1, \dots, t_n, 0) \in \overline{\mathrm{D}}$.

Finally, suppose $\varphi(t_1, \dots, t_n) = 0$. Let $t_i' > t_i$ for $i = 1, \dots, n$. In particular, $t_i' > 0$ for $i = 1, \dots, n$, hence $\varphi(t_1', \dots, t_n') > 0 = t_{n+1}$, thus $(t_1', \dots, t_n', 0) \in \overline{\mathrm{D}}$ by the preceding sub-case; letting $t_i' \to t_i$ yields $(t_1, \dots, t_n, 0) \in \overline{\mathrm{D}}$ as desired.

I.2, ℓ. -9 to -7.

"...it is immediate that C is also defined by the relations $t_i \geqslant 0$ $(1 \leqslant i \leqslant n+1)$ and

$$(3) \qquad \beta_\iota t_{n+1} \leqslant \alpha_{\iota 1} t_1 + \cdots + \alpha_{\iota n} t_n \qquad (\iota \in \mathrm{I},\ \beta_\iota \geqslant 0)\,."$$

Write E for the set of $(t_1, \ldots, t_n, t_{n+1})$ defined by these relations. It is clear that E is a closed convex cone; we are to show that $\mathrm{E} = \mathrm{C}$, that is, that $\mathrm{E} = \overline{\mathrm{D}}$ (where D is defined as in the preceding remarks).

To show that $\overline{\mathrm{D}} \subset \mathrm{E}$, it will suffice to show that $\mathrm{K} \times \{1\} \subset \mathrm{E}$ (because $\overline{\mathrm{D}}$ is the closed convex cone generated by $\mathrm{K} \times \{1\}$). If $(t_1, \ldots, t_n) \in \mathrm{K}$ then $t_i \geqslant 0$ for $i = 1, \ldots, n$ and by (2) we have $\beta_\iota \leqslant \alpha_{\iota 1} t_1 + \cdots + \alpha_{\iota n} t_n$ for all $\iota \in \mathrm{I}$, thus $(t_1, \ldots, t_n, 1)$ satisfies (3). Thus $\mathrm{K} \times \{1\} \subset \mathrm{E}$, and so $\overline{\mathrm{D}} \subset \mathrm{E}$.

Conversely, suppose $t_1, \ldots, t_n, t_{n+1}$ are $\geqslant 0$ and satisfy (3); we are to show that $(t_1, \ldots, t_n, t_{n+1}) \in \overline{\mathrm{D}}$. We consider two cases, according as $t_{n+1} > 0$ or $t_{n+1} = 0$:

If $t_{n+1} > 0$ then (3) yields

$$\beta_\iota \leqslant \alpha_{\iota 1}\Big(\frac{1}{t_{n+1}} \cdot t_1\Big) + \cdots + \alpha_{\iota n}\Big(\frac{1}{t_{n+1}} \cdot t_n\Big) \quad \text{for all } \iota \in \mathrm{I}\,,$$

thus $\dfrac{1}{t_{n+1}}(t_1, \ldots, t_n) \in \mathrm{K}$, so that $(t_1, \ldots, t_n) \in t_{n+1}\mathrm{K}$, whence

$$(t_1, \ldots, t_n, t_{n+1}) \in (t_{n+1}\mathrm{K} \times \{t_{n+1}\}) = t_{n+1}(\mathrm{K} \times \{1\}) \subset \mathrm{D}\,.$$

If $t_{n+1} = 0$ then, since $\varphi(t_1, \ldots, t_n) \geqslant 0$ by property 1° and the continuity of φ, we have $t_{n+1} = 0 \leqslant \varphi(t_1, \ldots, t_n)$, and the argument used in "case 2" of the preceding remarks yields $(t_1, \ldots, t_n, 0) \in \overline{\mathrm{D}}$.

I.2, ℓ. -6 to -4.

"For every $x \in \mathrm{X}$, we therefore have

$$(4) \qquad \beta_\iota \varphi\big(f_1(x), \ldots, f_n(x)\big) \leqslant \alpha_{\iota 1} f_1(x) + \cdots + \alpha_{\iota n} f_n(x)$$

for all $\iota \in \mathrm{I}$."

Set $t_i = f_i(x)$ for $i = 1, \ldots, n$ and set $t_{n+1} = \varphi(t_1, \ldots, t_n)$. Obviously

$$t_i \geqslant 0 \ \ (1 \leqslant i \leqslant n) \quad \text{and} \quad t_{n+1} \leqslant \varphi(t_1, \ldots, t_n)\,,$$

thus $(t_1, \ldots, t_n, t_{n+1}) \in \mathrm{C}$, where C is the convex cone defined earlier in the proof. But $\mathrm{C} = \overline{\mathrm{D}} = \mathrm{E}$ (where D and E are defined as in the comments on ℓ. $-10, -9, -8$), thus $(t_1, \ldots, t_n, t_{n+1})$ satisfies (3), which is precisely the assertion (4).

I.2, ℓ. $-3, -2$.

" $M(\varphi(f_1, f_2, \ldots, f_n))$ is finite and

$$\beta_\iota M\big(\varphi(f_1, f_2, \ldots, f_n)\big) \leqslant \alpha_{\iota 1} M(f_1) + \alpha_{\iota 2} M(f_2) + \cdots + \alpha_{\iota n} M(f_n)\text{"}$$

We are assuming $\beta_\iota > 0$ (such an ι exists, as noted earlier in the proof). Write $f = \varphi(f_1, \ldots, f_n)$ for the function on X defined by $x \mapsto \varphi\big(f_1(x_1), \ldots, f_n(x)\big)$. Since (4) holds for all $x \in \mathrm{X}$, we have

$$\beta_\iota f \leqslant \alpha_{\iota 1} f_1 + \cdots + \alpha_{\iota n} f_n \,,$$

and since $\beta_\iota > 0$ we have, by the properties of M,

$$\beta_\iota M(f) = M(\beta_\iota f) \leqslant M(\alpha_{\iota 1} f_1 + \cdots + \alpha_{\iota n} f_n) \leqslant \alpha_{\iota 1} M(f_1) + \cdots + \alpha_{\iota n} M(f_n)\,,$$

whence $M(f) < +\infty$.

I.3, Proposition 2.

Note that the condition $\alpha + \beta = 1$ assures that the function $\varphi(t_1, t_2) = t_1^\alpha t_2^\beta$ is positively homogeneous.

I.4, ℓ. $10, 11$.

"…if g is a function belonging to $\mathscr{F}^p(\mathrm{X}, M)$ and if $|f| \leqslant |g|$, then f also belongs to $\mathscr{F}^p(\mathrm{X}, M)$;"

Since $|f|^p \leqslant |g|^p$, one has $M(|f|^p) \leqslant M(|g|^p)$, therefore $M(|f|^p)^{1/p} \leqslant M(|g|^p)^{1/p} < +\infty$, thus $f \in \mathscr{F}(\mathrm{X}, M)$.

I.4, ℓ. 12.

"…the sum of two functions in $\mathscr{F}^p(\mathrm{X}, M)$ also belongs to this set;"

For any finite numerical function f on X, write $\mathrm{N}_p(f) = \big(M(|f|^p)\big)^{1/p}$. By Minkowski's inequality (Prop. 3), for any numerical functions f, g on X one has $\mathrm{N}_p(|f| + |g|) \leqslant \mathrm{N}_p(f) + \mathrm{N}_p(g)$, and it follows from $|f + g| \leqslant |f| + |g|$ that

$$\mathrm{N}_p(f + g) \leqslant \mathrm{N}_p(|f| + |g|) \leqslant \mathrm{N}_p(f) + \mathrm{N}_p(g)\,.$$

Therefore if f and g belong to $\mathscr{F}^p(\mathrm{X}, M)$, that is, if $\mathrm{N}_p(f)$ and $\mathrm{N}_p(g)$ are finite, then $\mathrm{N}_p(f + g)$ is also finite, thus $f + g \in \mathscr{F}^p(\mathrm{X}, M)$.

I.5, ℓ. 7.

"…the corollary is proved."

Explicitly, if $0 < r < p$ one sees that $\mathrm{N}_r(f) \leqslant \mathrm{N}_p(f)$ by choosing q so that $\dfrac{1}{q} = \dfrac{1}{r} - \dfrac{1}{p}$ and citing (9) with $g = 1$.

I.5, ℓ. 8–12.

PROPOSITION 5. — *For every finite numerical function* f *defined on* X, *the set* I *of values of* $1/p$ $(p > 0)$ *such that* $\mathrm{N}_p(f)$ *is finite is either empty or is an interval*; *if* I *is not reduced to a point, then the mapping* $\alpha \mapsto \log \mathrm{N}_{1/\alpha}(f)$ *is either convex on* I *or is equal to* $-\infty$ *on the interior of* I.

Rearrangement of the proof. Fix a finite numerical function f on X and define $\mathrm{J} = \{p > 0 : \ \mathrm{N}_p(f) < +\infty\}$; thus, in the notation of the statement of the proposition, $\mathrm{I} = \{1/p : \ p \in \mathrm{J}\}$.

claim 1: Either $\mathrm{J} = \varnothing$ or J is an interval (possibly degenerate, i.e., reduced to a point).

In other words, the assertion is that J is a convex subset of $]0, +\infty[$.

Suppose $r, s \in \mathrm{J}$ and $0 < \alpha < 1$; writing $p = \alpha r + (1-\alpha)s$, we are to show that $p \in \mathrm{J}$, i.e., that $M(|f|^p) < +\infty$. Since $r, s \in \mathrm{J}$ we have $M(|f|^r) < +\infty$ and $M(|f|^s) < +\infty$; by Hölder's inequality (Prop. 2),

$$M\big((|f|^r)^\alpha (|f|^s)^{1-\alpha}\big) \leqslant \big(M(|f|^r)\big)^\alpha \big(M(|f|^s)\big)^{1-\alpha},$$

that is,

$$M(|f|^p) \leqslant \big(M(|f|^r)\big)^\alpha \big(M(|f|^s)\big)^{1-\alpha},$$

which can also be written

$$(*) \qquad \big(\mathrm{N}_p(f)\big)^p \leqslant \big(\mathrm{N}_r(f)\big)^{r\alpha} \big(\mathrm{N}_s(f)\big)^{s(1-\alpha)},$$

whence $\mathrm{N}_p(f) < +\infty$. It follows that either $\mathrm{I} = \varnothing$ or I is an interval (possibly degenerate).

claim 2: If J is a nondegenerate interval and if $\mathrm{N}_r(f) = 0$ for *some* interior point r of J, then $\mathrm{N}_p(f) = 0$ for *every* interior point p of J.

Suppose, for example, that $p > r$. Choose $s \in \mathrm{J}$ such that $r < p < s$ and write $p = \lambda r + (1-\lambda)s$ with $0 < \lambda < 1$ (namely, $\lambda = \dfrac{s-p}{s-r}$). Since $\mathrm{N}_r(f) = 0$ and $\mathrm{N}_s(f)$ is finite, it is clear from (*) that $\mathrm{N}_p(f) = 0$.

Assume henceforth that J (hence also I) is a nondegenerate interval, and that there exists a $p \in \mathring{\mathrm{J}}$ with $\mathrm{N}_p(f) > 0$ (hence, by the foregoing, that $\mathrm{N}_p(f) > 0$ for all interior points p of J). Define $\varphi : \mathring{\mathrm{I}} \to \mathbf{R}$ as follows: if $\alpha \in \mathring{\mathrm{I}}$ then $1/\alpha \in \mathring{\mathrm{J}}$, hence $0 < \mathrm{N}_{1/\alpha}(f) < +\infty$; we define

$$\varphi(\alpha) = \log \mathrm{N}_{1/\alpha}(f) \qquad (\alpha \in \mathring{\mathrm{I}}).$$

The proof will be completed by showing:

claim 3: φ is convex.

Suppose $r, s \in \mathring{\mathrm{J}}$; thus $1/r, 1/s$ are typical elements of $\mathring{\mathrm{I}}$. Given $0 < t < 1$, define $p > 0$ by the formula

$$\frac{1}{p} = t \cdot \frac{1}{r} + (1-t) \cdot \frac{1}{s}\,, \tag{†}$$

so that $1 = tp/r + (1-t)p/s$; thus $1/p \in \mathring{\mathrm{I}}$ and the problem is to show that

$$\varphi\big(\frac{1}{p}\big) \leqslant t\,\varphi\big(\frac{1}{r}\big) + (1-t)\varphi\big(\frac{1}{s}\big)\,,$$

i.e., that

$$\log \mathrm{N}_p(f) \leqslant t\,\log \mathrm{N}_r(f) + (1-t)\log \mathrm{N}_s(f)\,,$$

equivalenty, that

$$\mathrm{N}_p(f) \leqslant \big(\mathrm{N}_r(f)\big)^t \big(\mathrm{N}_s(f)\big)^{1-t},$$

i.e., that

$$\big(M(|f|^p)\big)^{1/p} \leqslant \big(M(|f|^r)\big)^{t/r} \big(M|f|^s\big)^{(1-t)/s},$$

i.e., that $M(|f|^p) \leqslant \big(M(|f|^r)\big)^{tp/r} \big(M(|f|^s)\big)^{(1-t)p/s}$.

Setting $\lambda = tp/r$, we have, by (†),

$$1 - \lambda = 1 - \frac{tp}{r} = \frac{(1-t)p}{s} > 0\,,$$

thus $0 < \lambda < 1$ and our problem is to show that

$$M(|f|^p) \leqslant \big(M(|f|^r)\big)^{\lambda} \big(M(|f|^s)\big)^{1-\lambda}. \tag{**}$$

Note that

$$r\lambda + s(1-\lambda) = tp + (1-t)p = p\,;$$

since $M(|f|^r) < +\infty$ and $M(|f|^s) < +\infty$, it follows from Hölder's inequality (Prop. 2) that

$$\begin{aligned} M(|f|^p) &= M(|f|^{r\lambda + s(1-\lambda)}) = M\big((|f|^r)^{\lambda} \cdot (|f|^s)^{1-\lambda}\big) \\ &\leqslant \big(M(|f|^r)\big)^{\lambda} \big(M(|f|^s)\big)^{1-\lambda}, \end{aligned}$$

which is the desired relation (**).

{Note: In verifying (**), r and/or s can be endpoints of J; it is for taking $\log$ that one restricts to the interior.}

CHAPTER II

Riesz spaces

§1. RIESZ SPACES AND FULLY LATTICE-ORDERED SPACES

II.2, ℓ. $-2, -1$.

"...whence, in particular,

$$\sup(x,y) = x + (y-x)^+ = \tfrac{1}{2}(x+y+|x-y|)\,. \text{''} \tag{6}$$

Setting $z = -x$ in (5) yields $\sup(0, y-x) = -x + \sup(x,y)$, that is, $\sup(x,y) = x + \sup(0, y-x) = x + (y-x)^+$. But

$$(y-x)^+ = \frac{1}{2}(|y-x| + y - x) = \frac{1}{2}(|x-y| + y - x)\,,$$

thus $x + (y-x)^+ = \dfrac{1}{2}(2x + |x-y| + y - x)$, whence (6).

II.3, ℓ. 6–8.

"If A and B are two subsets of E each of which has a supremum, then $\mathrm{A}+\mathrm{B}$ also has a supremum and

$$\sup(\mathrm{A}+\mathrm{B}) = \sup \mathrm{A} + \sup \mathrm{B}\,. \text{''} \tag{10}$$

Let $a = \sup \mathrm{A}$, $b = \sup \mathrm{B}$. For all $x \in \mathrm{A}$ and $y \in \mathrm{B}$, one has $x \leqslant a$ and $y \leqslant b$, hence $x + y \leqslant a + b$, thus $\mathrm{A}+\mathrm{B}$ admits $a+b$ as an upper bound.

Assuming c is any upper bound for $\mathrm{A}+\mathrm{B}$, we need only show that $a + b \leqslant c$. If $y \in \mathrm{B}$ is fixed then, for every $x \in \mathrm{A}$, $x + y \leqslant c$, $x \leqslant c - y$, whence $a \leqslant c - y$. Thus $y \leqslant c - a$ for all $y \in \mathrm{B}$, whence $b \leqslant c - a$.

II.7, ℓ. 6–7.

"However, if A is infinite, a subset of $\mathscr{B}(\mathrm{A})$ may be *bounded above in* $\mathbf{R}^{\mathrm{A}}$ *without being bounded above in* $\mathscr{B}(\mathrm{A})$"

For example, let A be the set of positive integers, x the function on A such that $x(k) = k$ for all k, and, for each positive integer n, let x_n be the function on A such that $x_n(n) = n$ and $x_n(k) = 0$ for all $k \neq n$. Then $x_n \leqslant x$ for all n, but the sequence (x_n) has no bounded majorant.

II.8, ℓ. −7.

"Corollary. — *Let* a *be an element of a fully lattice-ordered space* E, B_a *the band generated by* a, B'_a *the band of elements alien to* a. *For every element* $x \geqslant 0$ *of* E, *the component of* x *in* B_a (for the decomposition of E as the ordered direct sum of B_a and B'_a) *is equal to* $\sup_{n\in\mathbf{N}} \big(\inf(n|a|, x)\big)$.

This follows from Proposition 6, applied to $\mathrm{M} = \{a\}$, and Proposition 5."

Adopt the notations of Prop. 6 and its proof. Thus (when $\mathrm{M} = \{a\}$)

$$\mathrm{M}_1 = \{t :\ 0 \leqslant t \leqslant n|a| \quad \text{for some } n \in \mathbf{N}\};$$

then M_1 has the properties of the set A of Prop. 5, and M_2 is the set M of Prop. 5. Let $x \in \mathrm{E}$, $x \geqslant 0$. By Prop. 5, one can write $x = y + z$ with $y \geqslant 0$, $z \geqslant 0$ and

$$\begin{aligned} y &= \sup\{v :\ v \in \mathrm{M}_1 \text{ and } v \leqslant x\} \\ &= \sup\{v :\ 0 \leqslant v \leqslant n|a| \quad \text{for some } n \in \mathbf{N} \text{ and } v \leqslant x\} \\ &= \sup\{v :\ 0 \leqslant v \leqslant \inf(n|a|, x) \quad \text{for some } n \in \mathbf{N}\} \\ &= \sup\{v :\ v = \inf(n|a|, x) \quad \text{for some } n \in \mathbf{N}\} \\ &= \sup_{n\in\mathbf{N}} \inf(n|a|, x), \end{aligned}$$

and where $z \in \mathrm{M}'_2 = \mathrm{M}' = \mathrm{M}''' = \mathrm{B}'_a$. Thus, as noted in the proof of Prop. 6, $x = y + z$ is the representation of x in the direct sum $\mathrm{E} = \mathrm{B}_a \oplus \mathrm{B}'_a$. In particular, the component of x in B_a (namely y) is given by the desired formula.

§2. LINEAR FORMS ON A RIESZ SPACE

II.10, ℓ. −5.

"Theorem 1. — 1° *In order that a linear form* L *on a Riesz space* E *be relatively bounded, it is necessary and sufficient that it be the difference of two positive linear forms.*

2° *The ordered vector space* Ω *of relatively bounded linear forms on* E *is a Riesz space that is fully lattice-ordered.*"

The proof can perhaps be clarified by separating out a part of it as a lemma:

Lemma 0. — *Let* L *be a relatively bounded linear form on the Riesz space* E. *Let* $\mathrm{P} = \{x \in \mathrm{E} :\ x \geqslant 0\}$ *and define* $M : \mathrm{P} \to [0, +\infty[$ *by the formula*

$$M(x) = \sup_{0\leqslant y\leqslant x} L(y) \quad \textit{for all } x \in \mathrm{P}.$$

Then M *may be extended to a positive linear form on* E *that is* $\geqslant L$.

Proof. Note first that the indicated supremum is finite (because L is relatively bounded). Moreover, $M(x) \geqslant 0$ for every $x \geqslant 0$; for, $y = 0$ satisfies $0 \leqslant y \leqslant x$ and so $0 = L(0) \leqslant M(x)$. Also, $M(x) \geqslant L(x)$ for every $x \geqslant 0$ because $y = x$ satisfies $0 \leqslant y \leqslant x$. Thus if we show that M can be extended to a linear form on E, it will be a positive linear form $\geqslant L$; in view of Prop. 3, it will suffice to show that M is additive.

Suppose $x, x' \in \mathrm{P}$ and define

$$\alpha = M(x) = \sup_{0 \leqslant y \leqslant x} L(y)$$

$$\beta = M(x') = \sup_{0 \leqslant y' \leqslant x'} L(y')$$

$$\gamma = M(x + x') = \sup_{0 \leqslant z \leqslant x + x'} L(z)\,;$$

we are to show that $\alpha + \beta = \gamma$.

To see that $\alpha + \beta \leqslant \gamma$, let $0 \leqslant y \leqslant x$ and $0 \leqslant y' \leqslant x'$. Then $0 \leqslant y + y' \leqslant x + x'$, so

$$L(y) + L(y') = L(y + y') \leqslant \gamma\,;$$

varying y and y' independently, one concludes that $\alpha + \beta \leqslant \gamma$. In detail, for each y', $L(y) \leqslant \gamma - L(y')$ for all y, whence

$$\alpha = \sup_{0 \leqslant y \leqslant x} L(y) \leqslant \gamma - L(y')\,;$$

thus $L(y') \leqslant \gamma - \alpha$ for all y', whence $\beta \leqslant \gamma - \alpha$.

To see that $\gamma \leqslant \alpha + \beta$, suppose $0 \leqslant z \leqslant x + x'$. By the decomposition theorem (A, VI, §1, No. 10, Th. 1) one can write $z = y + y'$ with $0 \leqslant y \leqslant x$ and $0 \leqslant y' \leqslant x'$; explicitly, writing $z' = (x + x') - z \geqslant 0$, one has $z + z' = x + x'$ and the first part of the proof of the cited theorem (*loc. cit.*) shows that the elements $y = \sup(0, z - x')$ and $y' = z - y$ meet the requirements. Then $L(z) = L(y) + L(y') \leqslant \alpha + \beta$; varying z, $\gamma \leqslant \alpha + \beta$. This completes the proof of the lemma.

Proof of 1°. "Necessity": Suppose L is relatively bounded. By the above Lemma 0, there exists a positive linear form M on E such that $M \geqslant L$, and the formula $L = M - (M - L)$ exhibits L as the difference of two positive linear forms on E.

"Sufficiency": Suppose $L = U - V$, where U and V are positive linear forms on E. Let $x \in \mathrm{E}$, $x \geqslant 0$. If $|y| \leqslant x$, i.e., $\sup(y, -y) \leqslant x$, i.e., $-x \leqslant y \leqslant x$, then $-U(x) \leqslant U(y) \leqslant U(x)$, i.e., $|U(y)| \leqslant U(x)$, and similarly $|V(y)| \leqslant V(x)$, whence

$$|L(y)| = |U(y) - V(y)| \leqslant |U(y)| + |V(y)| \leqslant U(x) + V(x)\,,$$

thus the set $\{|L(y)| : |y| \leqslant x\}$ is bounded. In other words, L is relatively bounded.

Proof of 2°. Write Ω for the set of linear forms characterized in 1°. Let $L \in \Omega$ and let M be the positive linear form constructed in Lemma 0, such that

$$M(x) = \sup_{0 \leqslant y \leqslant x} L(y) \quad \text{for all } x \geqslant 0 .$$

We assert that in the ordered vector space Ω (with the positive linear forms serving as the convex cone of positive elements), the elements L and 0 have a supremum, namely M. For, on the one hand, $M \geqslant 0$ and $M \geqslant L$. On the other hand, if $N \in \Omega$ satisfies $N \geqslant 0$ and $N \geqslant L$, then for every $x \geqslant 0$ one has

$$0 \leqslant y \leqslant x \quad \Rightarrow \quad N(x) \geqslant N(y) \geqslant L(y) ,$$

whence

$$N(x) \geqslant \sup_{0 \leqslant y \leqslant x} L(y) = M(x) ,$$

thus $N \geqslant M$ and the assertion is proved, authorizing us to write $M = \sup(L, 0)$. It follows that Ω is a Riesz space; for, if L, M are any two elements of Ω, the linear form $\sup(L - M, 0) + M$ clearly serves as a supremum for L and M. It follows from the foregoing calculations that, for every $x \geqslant 0$,

$$(1) \qquad L^+(x) = \big(\sup(L, 0)\big)(x) = \sup_{0 \leqslant y \leqslant x} L(y) .$$

It remains only to show that Ω is *fully* lattice-ordered. In view of §1, Prop. 1 (on p. II.4) it suffices to show that if $\mathscr{H}$ is any nonempty subset of Ω that is bounded above in Ω and is directed upward by $\leqslant$, then $\mathscr{H}$ admits a supremum. By the Lemma on p. II.12, we know that $\mathscr{H}$ has a supremum M in the algebraic dual E^* of E, such that

$$M(x) = \sup_{L \in \mathscr{H}} L(x) \quad \text{for all } x \geqslant 0 .$$

In particular, for any $L \in \mathscr{H}$, the relations $M \geqslant L$ and $M = L + (M - L)$ show that M is the sum of two elements of Ω, hence $M \in \Omega$.

II.12, formulas (3).

"From the formula (1), one deduces immediately that if L and M are two relatively bounded linear forms on E then, for every $x \geqslant 0$,

$$(3) \qquad \begin{cases} \sup(L, M)(x) = \displaystyle\sup_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} \big(L(y) + M(z)\big) \\ \inf(L, M)(x) = \displaystyle\inf_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} \big(L(y) + M(z)\big) ." \end{cases}$$

Write $N = \sup(L, M)$. Citing formula (6) of §1, No. 1, one has

$$N = (L - M)^{+} + M\,,$$

thus, for every $x \geqslant 0$,

$$\begin{aligned} N(x) &= (L - M)^{+}(x) + M(x) \\ &= \sup_{0 \leqslant y \leqslant x} [L(y) - M(y)] + M(x) \\ &= \sup_{0 \leqslant y \leqslant x} [L(y) - M(y) + M(x)] \\ &= \sup_{0 \leqslant y \leqslant x} [L(y) + M(x - y)] \\ &= \sup_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [L(y) + M(z)]\,. \end{aligned}$$

Since $\inf(L, M) = L + M - \sup(L, M)$ (formula (8) of §1, No. 1), for all $x \geqslant 0$ one has

$$\begin{aligned} \big(\inf(L, M)\big)(x) &= L(x) + M(x) - \sup_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [L(y) + M(z)] \\ &= L(x) + M(x) + \inf_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [-L(y) - M(z)] \\ &= \inf_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [L(x) + M(x) - L(y) - M(z)] \\ &= \inf_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [L(x - y) + M(x - z)] \\ &= \inf_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [L(z) + M(y)] \\ &= \inf_{y \geqslant 0,\, z \geqslant 0,\, y + z = x} [L(y) + M(z)] \end{aligned}$$

(the latter equality, by the symmetry of the roles of y and z).

II.12, ℓ. -7.

"...if $x = y + z$, $y \geqslant 0$ and $z \geqslant 0$, then $-x \leqslant y - z \leqslant x$"

For, $y = x - z \leqslant (x - z) + 2z = x + z$ and $z = x - y \leqslant (x - y) + 2y = x + y$, thus $y - z \leqslant x$ and $-x \leqslant y - z$, whence the assertion (equivalently, $|y - z| \leqslant x$).

II.12, ℓ. -6.

"the relation $|u| \leqslant x$ implies $L(u) \leqslant |L|(|u|) \leqslant |L|(x)$."

The inequality $L(|u|) \leqslant |L|(x)$ is immediate from $|u| \leqslant x$ and the positivity of $|L|$. To see that $L(u) \leqslant |L|(|u|)$, set $y = u^{+}$, $z = u^{-}$ and $x = y + z = u^{+} + u^{-} = |u|$; then $y - z = u^{+} - u^{-} = u$ and, by the formula displayed in ℓ. -8, $L(u) = L(y - z) \leqslant |L|(x) = |L|(|u|)$.

II.12, ℓ. -4.

"(4) $$|L|(x) = \sup_{|y| \leqslant x} L(y) \qquad \text{for } x \geqslant 0\text{"}$$

Let $x \geqslant 0$ and let $s = \sup_{|y| \leqslant x} L(y)$. If $|u| \leqslant x$ then $L(u) \leqslant |L|(x)$ by ℓ. -6, hence $s \leqslant |L|(x)$.

On the other hand, if $x = y + z$ with $y \geqslant 0$ and $z \geqslant 0$, then (as noted in connection with ℓ. -7) the element $u = y - z$ satisfies $|u| \leqslant x$, hence $L(y-z) = L(u) \leqslant s$; varying y and z (subject to $y \geqslant 0$, $z \geqslant 0$, $y+z = x$) it follows from the formula displayed in ℓ. -8 that $|L|(x) \leqslant s$.

II.14, ℓ. 5.

"Then every continuous linear form $x' \in \mathrm{E}'$ is *relatively bounded*"

Recall that the convex cone P is proper (TVS, II, §2, No. 5, Prop. 13). One has

$$\begin{aligned}\{y:\ |y| \leqslant x\} &= \{y:\ -x \leqslant y \leqslant x\} \\ &= \{y:\ x - y \in \mathrm{P} \ \text{ and } \ x + y \in \mathrm{P}\} \\ &= \{y:\ y - x \in -\mathrm{P} \ \text{ and } \ x + y \in \mathrm{P}\} \\ &= (x - \mathrm{P}) \cap (-x + \mathrm{P});\end{aligned}$$

this set is homeomorphic (via translation by x) to the set $(2x - \mathrm{P}) \cap \mathrm{P}$, which is compact for $\sigma(\mathrm{E}, \mathrm{E}')$ (TVS, II, §6, No. 8, Cor. 2 of Prop. 11).

Since $x' \in \mathrm{E}'$ is continuous for $\sigma(\mathrm{E}, \mathrm{E}')$, it follows that the set $x'(\{y:\ |y| \leqslant x\})$ is compact, hence x' is bounded on $\{y:\ |y| \leqslant x\}$; thus x' is relatively bounded (Def. 2).

II.14, ℓ. 11–12.

"(the latter being compatible with the ordered vector space structure of E)"

The point is that P, which is assumed to be closed for the original topology of E (TVS, II, §2, No. 7, axiom (TO)), is also closed for $\sigma(\mathrm{E}, \mathrm{E}')$ on account of its convexity (TVS, II, §5, No. 3, Cor. 1 of Prop. 4); thus $\sigma(\mathrm{E}, \mathrm{E}')$ is also compatible with the ordered vector space structure of E.

II.14, ℓ. -6.

"By translation, we can suppose that $\mathrm{H} \subset \mathrm{P}$"

Choose any $x_0 \in \mathrm{H}$, drop down to $\mathrm{H}_0 = \{x \in \mathrm{H}:\ x \geqslant x_0\}$, and consider the translate $-x_0 + \mathrm{H}_0 \subset \mathrm{P}$.

II.14, ℓ. $-5, -4$.

"...or again that every continuous linear form $x' \in \mathrm{E}'$ has a limit with respect to $\mathfrak{F}$"

Assume that this condition is satisfied. Then, given any $x' \in \mathrm{E}'$ and any $\varepsilon > 0$, there exists an $\mathrm{F}_0 \in \mathfrak{F}$ such that $|x'(x) - x'(y)| \leqslant \varepsilon$ for all $x, y \in \mathrm{F}_0$. Since $\mathfrak{F}$ is closed under finite intersections, it follows that if $x'_1, \ldots, x'_n \in \mathrm{E}'$ and $\varepsilon > 0$, there exists a set $\mathrm{F}_0 \in \mathfrak{F}$ such that $|x'_i(x) - x'_i(y)| \leqslant \varepsilon$ for $i = 1, \ldots, n$ and $x, y \in \mathrm{F}_0$, that is, writing

$$\mathrm{U} = \{(x, y) \in \mathrm{H} \times \mathrm{H} : \ |x'_i(x) - x'_i(y)| \leqslant \varepsilon \quad \text{for } i = 1, \ldots, n\},$$

we have $\mathrm{F}_0 \times \mathrm{F}_0 \subset \mathrm{U}$, i.e., F_0 is small of order U. Since such sets U are basic entourages for the uniform structure on H, it follows that $\mathfrak{F}$ is Cauchy for that uniform structure.

II.14, ℓ. -4 to -2.

"But this follows at once from the monotone limit theorem when x' is a *positive* linear form"

For, $x'(\mathrm{H})$ is an increasing directed set in $\mathbf{R}$ that is bounded above, hence x' has a limit with respect to $\mathfrak{F}$, namely $\lim_{\mathfrak{F}} x' = \sup x'(\mathrm{H})$.

CHAPTER III

Measures on locally compact spaces

§1. MEASURES ON A LOCALLY COMPACT SPACE

III.1, ℓ. $-6, -5$.

"(namely, the subspace of continuous mappings of K into E that are zero on the boundary of K)."

Recall that the boundary of a subset A of a topological space X is the set $\mathrm{B} = \overline{\mathrm{A}} \cap \overline{\mathrm{X} - \mathrm{A}}$ (which is also the boundary of $\mathrm{X} - \mathrm{A}$). Note that the union of A with its boundary is the closure of A: $\mathrm{A} \cup \mathrm{B} = \overline{\mathrm{A}}$; for,

$$\mathrm{A} \cup \mathrm{B} = \mathrm{A} \cup (\overline{\mathrm{A}} \cap \overline{\mathrm{X} - \mathrm{A}}) = (\mathrm{A} \cup \overline{\mathrm{A}}) \cap (\mathrm{A} \cup \overline{\mathrm{X} - \mathrm{A}}) = \overline{\mathrm{A}} \cap \mathrm{X} = \overline{\mathrm{A}}\,.$$

Incidentally, $\overline{\mathrm{X} - \mathrm{A}} = \complement \mathring{\mathrm{A}}$, where $\mathring{\mathrm{A}}$ is the interior of A (GT, I, §1, No. 6, formulas (2)). And when A is a closed set, A contains its boundary, since $\mathrm{A} = \overline{\mathrm{A}} = \mathrm{A} \cup \mathrm{B}$.

Suppose now that A is a closed subset of X and $f : \mathrm{A} \to \mathrm{Y}$ is a continuous mapping of A into a topological space Y, such that the restriction of f to its boundary B is constant, say $f(\mathrm{B}) = \{y_0\}$ for some $y_0 \in \mathrm{Y}$. Define $g : \mathrm{X} \to \mathrm{Y}$ to be the function equal to f on A and to y_0 on $\mathrm{X} - \mathrm{A}$. Since B is also the boundary of $\mathrm{X} - \mathrm{A}$, g has the constant value y_0 on the set $\overline{\mathrm{X} - \mathrm{A}} = (\mathrm{X} - \mathrm{A}) \cup \mathrm{B}$. Thus, g is continuous on each of the closed sets A and $\overline{\mathrm{X} - \mathrm{A}}$, whose union is X, consequently g is continuous on X (GT, I, §3, No. 2, Prop. 4).

In particular, if K is a compact subset of the locally compact space X, E is a topological vector space over **R** or **C**, and $f : \mathrm{K} \to \mathrm{E}$ is a continuous function that is zero on the boundary of K, then the extension g of f to X defined by setting $g(x) = 0$ for all $x \in \mathrm{X} - \mathrm{K}$ is continuous, and its support, being a closed subset of K, is compact. Conversely, if K is a compact subset of X and $g \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$, that is, g is continuous on X with support contained in K, i.e., $\overline{\{x : g(x) \neq 0\}} \subset \mathrm{K}$, equivalently g is equal to 0 on $\mathrm{X} - \mathrm{K}$, then g is also equal to 0 on $\overline{\mathrm{X} - \mathrm{K}}$, hence on the boundary $K \cap \overline{X - K}$ of K, thus the restriction $f = g|K$ is a continuous function on K equal to 0 on the boundary of K.

III.1, ℓ. $-5, -4$.

"When $\mathscr{C}(K;E)$ is equipped with the topology of uniform convergence in K, $\mathscr{K}(X,K;E)$ is a *closed* subspace of $\mathscr{C}(K;E)$."

More generally, suppose (g_α) is a directed family in $\mathscr{K}(X,K;E)$ that converges pointwise to a function $g \in \mathscr{C}(K;E)$. Then (assuming E is Hausdorff) g is equal to 0 on the boundary of K, hence belongs to $\mathscr{K}(X,K;E)$. For, if there existed a point x of the boundary of K such that $g(x) \neq 0$, one could choose a neighborhood V of 0 in E such that $g(x) \notin V$; but there exists an α_0 such that $\alpha \geqslant \alpha_0 \Rightarrow g(x) - g_\alpha(x) \in V$, a contradiction since $g_\alpha(x) = 0$.

III.1, ℓ. -2.

"...if the topology of E is defined by the semi-norms p_n"

Recall that a *Fréchet space* is a complete, metrizable locally convex space, and that the topology of a metrizable locally convex space can be defined by a countable set of semi-norms (TVS, II, §4, No. 1, comments following the Corollary of Prop. 1).

III.2, ℓ. $7, 8$.

"If E is *locally convex*, one can therefore define on $\mathscr{K}(X;E)$ the *direct limit* of the locally convex topologies of the $\mathscr{K}(X,K;E)$"

Recall (TVS, II, §4, No. 4, Prop. 5) that this topology (call it $\mathscr{T}$) is the finest locally convex topology on $\mathscr{K}(X;E)$ that renders continuous the canonical injections

$$u_K : \mathscr{K}(X,K;E) \to \mathscr{K}(X;E),$$

where, for each compact subset K of X, the space $\mathscr{K}(X,K;E)$ bears the topology of uniform convergence in K; and, for a linear mapping $u : \mathscr{K}(X;E) \to G$ (G a locally convex space), the following conditions are equivalent:

(a) u is continuous for the topology $\mathscr{T}$;

(b) for every compact subset K of X, the linear mapping $u \circ u_K : \mathscr{K}(X,K;E) \to G$ is continuous; that is,

(b$'$) for every compact subset K of X and for every directed family (f_α) in $\mathscr{K}(X,K;E)$ that converges uniformly to a function $f \in \mathscr{K}(X,K;E)$, one has $u(f_\alpha) \to u(f)$ in G.

The mappings u_K are continuous when $\mathscr{K}(X,K;E)$ is equipped with the topology of uniform convergence in K and $\mathscr{K}(X;E)$ is equipped with the topology $\mathscr{T}_u$ of uniform convergence in X, consequently $\mathscr{T}$ is finer than $\mathscr{T}_u$ (TVS, *loc. cit.*, *Example*. CAUTION: In the cited *Example*, TVS p. II.29, in lines 4 and 5 read "Denote by $\mathscr{T}_{\mathrm{K}}$ the topology induced on E_{K} by the topology $\mathscr{T}_u$ of *uniform convergence* on X"; the sentence is correct in EVT, p. II.31).

SCHOLIUM: The direct limit topology on $\mathscr{K}(\mathrm{X};\mathrm{E})$ is finer than the topology of uniform convergence.

III.2, ℓ. −5.

"...hence belongs to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$."

Here $f \in \mathscr{K}(\mathrm{X},\mathrm{K}';\mathrm{E})$ and there exists a family (f_α) in $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ such that $f_\alpha \to f$ uniformly on K', hence uniformly on $\mathrm{K}' - \mathrm{K}$, hence pointwise on $\mathrm{K}' - \mathrm{K}$. Since the f_α are 0 on $\mathrm{X} - \mathrm{K}$, it follows that $f = 0$ on $\mathrm{K}' - \mathrm{K}$; but also $f = 0$ on $\mathrm{X} - \mathrm{K}'$, hence $f = 0$ on $\mathrm{X} - \mathrm{K} = (\mathrm{X} - \mathrm{K}') \cup (\mathrm{K}' - \mathrm{K})$, that is, $f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$.

III.2, ℓ. −4, −3.

"(ii) The criterion for continuity in a direct limit (TVS, II, §4, No. 4, Prop. 5) shows at once that the mapping $f \mapsto (\mathrm{pr}_i \circ f)$ is continuous"

It suffices to show that for each i, $f \mapsto \mathrm{pr}_i \circ f$ is a continuous mapping $\mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E}_i)$. Fix a compact set $\mathrm{K} \subset \mathrm{X}$; by the cited result in TVS, it is enough to show that the composite mapping

$$\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E}_i)$$

defined by $f \mapsto f \mapsto \mathrm{pr}_i \circ f$ is continuous. Indeed, if $f_\alpha, f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ and $f_\alpha \to f$ uniformly on K, then $\mathrm{pr}_i \circ f_\alpha \to \mathrm{pr}_i \circ f$ uniformly on K (regard $\mathrm{E}_i \subset \mathrm{E}$ in the canonical way); thus $\mathrm{pr}_i \circ f_\alpha$, $\mathrm{pr}_i \circ f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}_i)$ and $\mathrm{pr}_i \circ f_\alpha \to \mathrm{pr}_i \circ f$ in the space $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}_i)$ (equipped with the topology of uniform convergence on K) hence in the space $\mathscr{K}(\mathrm{X};\mathrm{E}_i)$ (by (i)).

III.2, ℓ. −2.

"...the same is true of the inverse mapping"

If $(f_i)_{1 \leqslant i \leqslant n} \in \prod_{1 \leqslant i \leqslant n} \mathscr{K}(\mathrm{X};\mathrm{E}_i)$, for each i define a function $f_i' \in \mathscr{K}(\mathrm{X};\mathrm{E})$ to be the mapping such that $\mathrm{pr}_i \circ f_i' = f_i$ and $\mathrm{pr}_j \circ f_i' = 0$ when $j \neq i$. Since each of the mappings $f_i \mapsto f_i'$ $(1 \leqslant i \leqslant n)$ is obviously continuous, so is the mapping $\Phi : \prod_{1 \leqslant i \leqslant n} \mathscr{K}(\mathrm{X};\mathrm{E}_i) \to \mathscr{K}(\mathrm{X};\mathrm{E})$ defined by $\Phi\big((f_i)\big) = \sum_{1 \leqslant i \leqslant n} f_i'$, and $\mathrm{pr}_j\big(\Phi\big((f_i)\big)\big) = f_j$ $(1 \leqslant j \leqslant n)$ shows that Φ is the mapping inverse to the mapping $f \mapsto (\mathrm{pr}_i \circ f)$ of $\mathscr{K}(\mathrm{X};\mathrm{E})$ into $\prod_{1 \leqslant i \leqslant n} \mathscr{K}(\mathrm{X};\mathrm{E}_i)$ defined above.

III.3, ℓ. 8–9.

"...it is immediate that the mapping $f_\lambda \mapsto f''_\lambda$ of $\mathscr{K}(\mathrm{X}_\lambda;\mathrm{E})$ into $\mathscr{K}(\mathrm{X};\mathrm{E})$ is continuous."

For each compact $\mathrm{K}_\lambda \subset \mathrm{X}_\lambda$, the composite mapping

$$\mathscr{K}(\mathrm{X}_\lambda,\mathrm{K}_\lambda;\mathrm{E}) \to \mathscr{K}(\mathrm{X}_\lambda;\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E})$$

defined by $f_\lambda \mapsto f_\lambda \mapsto f''_\lambda$ is obviously continuous.

III.3, ℓ. 9–11.

"The assertion (iii) follows from these remarks and the criterion for continuity in direct limits (TVS, II, §4, No. 4, Prop. 5)."

Define $\varphi : \mathscr{K}(\mathrm{X};\mathrm{E}) \to \bigoplus_{\lambda\in\mathrm{L}} \mathscr{K}(\mathrm{X}_\lambda;\mathrm{E})$ as follows. Let $f \in \mathscr{K}(\mathrm{X};\mathrm{E})$ and let K be the support of f. Since the X_λ form an open covering of X, K intersects X_λ for at most finitely many λ, therefore $f|\mathrm{X}_\lambda = 0$ for all but finitely many λ; thus the family $(f|\mathrm{X}_\lambda)_{\lambda\in\mathrm{L}}$ qualifies for membership in $\bigoplus_{\lambda\in\mathrm{L}} \mathscr{K}(\mathrm{X}_\lambda;\mathrm{E})$ and we may define $\varphi(f) = (f|\mathrm{X}_\lambda)_{\lambda\in\mathrm{L}}$. Conversely, if $(g_\lambda)_{\lambda\in\mathrm{L}} \in \bigoplus_{\lambda\in\mathrm{L}} \mathscr{K}(\mathrm{X}_\lambda;\mathrm{E})$, in particular $g_\lambda = 0$ for all but finitely many λ, it is clear that the function f on X defined by $f|\mathrm{X}_\lambda = g_\lambda$ for all λ is continuous and has compact support (equal to the union of the supports of the nonzero g_λ's), and that $\varphi(f) = (g_\lambda)_{\lambda\in\mathrm{L}}$. Clearly φ is a vector space isomorphism.

To see that φ is continuous, suppose $\mathrm{K} \subset \mathrm{X}$ compact and consider the composite mapping

$$\varphi_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E}) \to \bigoplus_{\lambda\in\mathrm{L}} \mathscr{K}(\mathrm{X}_\lambda;\mathrm{E})$$

defined by $f \mapsto f \mapsto \bigoplus_{\lambda\in\mathrm{L}} (f|\mathrm{X}_\lambda)$. By the preceding discussion, K intersects X_λ only for λ in some finite subset of L, thus the the above composite has range contained in the subspace $\bigoplus_{\lambda\in\mathrm{H}} \mathscr{K}(\mathrm{X}_\lambda;\mathrm{E})$ for some finite subset H of L, and it defines a bicontinuous isomorphism of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ onto that subspace; in particular, φ_{K} is continuous for every compact $\mathrm{K} \subset \mathrm{X}$, whence the continuity of φ.

To see that φ^{-1} is continuous, it is enough, by the definition of topological direct sum, to note that for each $\lambda \in \mathrm{L}$, the composite

$$\mathscr{K}(\mathrm{X}_\lambda;\mathrm{E}) \to \bigoplus_{\mu\in\mathrm{L}} \mathscr{K}(\mathrm{X}_\mu;\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E})$$

of the canonical injection and φ^{-1} is continuous; indeed, this composite is precisely the continuous mapping $f_\lambda \mapsto f''_\lambda$ $(f_\lambda \in \mathscr{K}(X_\lambda; E)$.

III.4, ℓ. 5.

"1° it is closed;"

Note that since $\mathscr{K}(X, K; E)$ is closed in $\mathscr{K}(X; E)$, a subset of $\mathscr{K}(X, K; E)$ is closed in $\mathscr{K}(X, K; E)$ if and only if it is closed in $\mathscr{K}(X; E)$.

III.4, ℓ. 5–6.

"2° it is equicontinuous;"

Note that a subset H of $\mathscr{K}(X, K; E)$ is equicontinuous on K if and only if it is equicontinuous on X, since every function in H is zero on $X - K$.

III.4, Proof of the COROLLARY..

"It suffices, by virtue of Proposition 2, (ii), to note that for every compact subset K of X, $\mathscr{K}(X, K; E)$ is a closed subspace of $\mathscr{C}(K; E)$, which is quasi-complete since every bounded subset of $\mathscr{C}(K; E)$ consists of functions taking values in a same bounded subset of E."

The proof is a "Knight's tour" of the prerequisites; let us review some fundamental definitions.

A subset B of a topological vector space E is *bounded* if, for every neighborhood V of 0 in E, there exists a nonzero scalar λ such that $A \subset \lambda V$ (TVS, III, §1, No. 2, comments following Def. 3). In testing for boundedness, one can obviously restrict attention to V's belonging to a fundamental system of neighborhoods of 0.

A locally convex topological vector space F is said to be *quasi-complete* if every closed bounded subset B of F is complete for the uniform structure induced on B by that of F.

Suppose H is a closed bounded subset of $\mathscr{K}(X; E)$; we are to show that H is complete. By Prop. 2, (ii), there exists a compact subset K of X such that $H \subset \mathscr{K}(X, K; E)$. As noted in No. 1, $\mathscr{K}(X, K; E)$ may be identified with a closed subspace of $\mathscr{C}(K; E)$, namely, the subspace $\mathscr{C}_0(K; E)$ consisting of the functions $f = g|K$, where $g \in \mathscr{K}(X, K; E)$ and $g = 0$ on the boundary $K \cap \overline{X - K}$ of K in X; the mapping $g \mapsto g|K$ is a topological vector space isomorphism of $\mathscr{K}(X, K; E)$ onto $\mathscr{C}_0(K; E)$.

Let us digress to describe the bounded subsets of $\mathscr{C}(K; E)$, where $\mathscr{C}(K; E)$ is equipped with the topology of uniform convergence on K (GT, X, §1, No. 1, Def. 1). In detail, a fundamental system of entourages for the uniform structure on E is formed by the sets

$$W = \{(a, b) \in E \times E : \ a - b \in V\},$$

where V runs over any fundamental system of neighborhoods of 0 in E (TVS, I, §1, No. 4 and GT, III, §3, No. 1, Def. 1). A fundamental system of entourages for the uniform structure on $\mathscr{C}(\mathrm{K};\mathrm{E})$ is formed by the sets

$$\mathscr{W} = \{(f,g) \in \mathscr{C}(\mathrm{K};\mathrm{E}) : \quad \big(f(x), g(x)\big) \in \mathrm{W} \quad \text{for all } x \in \mathrm{K}\},$$

where W runs over any fundamental system of entourages for the uniform structure on E (GT, X, §1, No. 1, Def. 1). Thus a fundamental system of entourages for the uniform structure on $\mathscr{C}(\mathrm{K};\mathrm{E})$ is formed by the sets

$$\mathscr{W} = \{(f,g) \in \mathscr{C}(\mathrm{K};\mathrm{E}) : \quad f(x) - g(x) \in \mathrm{V} \ \text{ for all } x \in \mathrm{K}\},$$

where V runs over any fundamental system of neighborhoods of 0 in E. In particular, a fundamental system of neighborhoods of a function $g \in \mathscr{C}(\mathrm{K};\mathrm{E})$ is formed by the sets

$$\{f \in \mathscr{C}(\mathrm{K};\mathrm{E}) : \quad f(x) - g(x) \in \mathrm{V} \ \text{ for all } x \in \mathrm{K}\},$$

where V runs over a fundamental system of neighborhoods of 0 in E (GT, II, §1, No. 2, Prop. 1); and so a fundamental system of neighborhoods of the zero function $0 \in \mathscr{C}(\mathrm{K};\mathrm{E})$ is formed by the sets

$$\begin{aligned} \mathscr{N}_{\mathrm{V}} &= \{f \in \mathscr{C}(\mathrm{K};\mathrm{E}) : \quad f(x) - 0 \in \mathrm{V} \quad \text{for all } x \in \mathrm{K}\} \\ &= \{f \in \mathscr{C}(\mathrm{K};\mathrm{E}) : \quad f(\mathrm{K}) \subset \mathrm{V}\}, \end{aligned}$$

where V runs over a fundamental system of neighborhoods of 0 in E.

The bounded subsets of $\mathscr{C}(\mathrm{K};\mathrm{E})$ may now be described as follows: in order that a subset B of $\mathscr{C}(\mathrm{K};\mathrm{E})$ be bounded, it is necessary and sufficient that for every neighborhood V of 0 in E, there exist a scalar $\lambda \neq 0$ such that $\mathrm{B} \subset \lambda\mathscr{N}_{\mathrm{V}}$, this inclusion being equivalent to each of the following conditions:

$$\begin{aligned} \lambda^{-1}\mathrm{B} &\subset \mathscr{N}_{\mathrm{V}} \\ \lambda^{-1}\mathrm{B} &\subset \{f \in \mathscr{C}(\mathrm{K};\mathrm{E}) : \quad f(\mathrm{K}) \subset \mathrm{V}\} \\ \lambda^{-1}u(\mathrm{K}) &\subset \mathrm{V} \quad \text{for all } u \in \mathrm{B} \\ u(\mathrm{K}) &\subset \lambda\mathrm{V} \quad \text{for all } u \in \mathrm{B} \\ \bigcup_{u \in \mathrm{B}} u(\mathrm{K}) &\subset \lambda\mathrm{V}. \end{aligned}$$

In brief, a subset B of $\mathscr{C}(\mathrm{K};\mathrm{E})$ is bounded if and only if the set $\bigcup_{u \in \mathrm{B}} u(\mathrm{K})$ is bounded in E.

Let us return to the closed bounded subset H of $\mathscr{K}(\mathrm{X};\mathrm{E})$ and let K be a compact subset of X such that $\mathrm{H} \subset \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$. Let $\mathrm{H}_0 = \{g|\mathrm{K} : \ g \in \mathrm{H}\}$; thus $\mathrm{H}_0 \subset \mathscr{C}_0(\mathrm{K};\mathrm{E}) \subset \mathscr{C}(\mathrm{K};\mathrm{E})$. Since the mapping

$g \mapsto g\big|\mathrm{K}$ is an isomorphism of topological vector spaces, it is clear that H_0 is a closed and bounded subset of $\mathscr{C}_0(\mathrm{K};\mathrm{E})$ and hence of $\mathscr{C}(\mathrm{K};\mathrm{E})$. As shown in the preceding paragraph, the set $\bigcup_{f\in \mathrm{H}_0} f(\mathrm{K})$ is bounded in E, hence (TVS, III, §1, No. 2, Prop. 1, *d*)) so is its closure

$$\mathrm{A} = \overline{\bigcup_{f\in \mathrm{H}_0} f(\mathrm{K})},$$

and since E is quasi-complete, A is complete (for the induced uniform structure). Thus H_0 may be regarded as a subset of the space $\mathscr{C}(\mathrm{K};\mathrm{A})$ equipped with the topology of uniform convergence. Since A is complete, so is $\mathscr{C}(\mathrm{K};\mathrm{A})$ (GT, X, §1, No. 6, Cor. 1 of Th. 2). Regarding $\mathscr{C}(\mathrm{K};\mathrm{A})$ as a uniform subspace of $\mathscr{C}(\mathrm{K};\mathrm{E})$, we have

$$\mathrm{H}_0 \subset \mathscr{C}(\mathrm{K};\mathrm{A}) \subset \mathscr{C}(\mathrm{K};\mathrm{E}),$$

where H_0 is closed in $\mathscr{C}(\mathrm{K};\mathrm{E})$, hence in $\mathscr{C}(\mathrm{K};\mathrm{A})$, and since $\mathscr{C}(\mathrm{K};\mathrm{A})$ is complete we conclude that H_0 is complete (GT, II, §3, No. 4, Prop. 8), therefore so is H, and we have proved that $\mathscr{K}(\mathrm{X};\mathrm{E})$ is quasi-complete.

III.4, ℓ. −12, −11.

"(GT, IX, §4, No. 3, Prop. 3)"

The cited proposition exploits Urysohn's theorem (GT, IX, §4, No. 1, Th. 1) via the normality of a compact space (*loc. cit.*, Prop. 1). The corresponding result in the bound French edition is TG, IX, §4, No. 3, Th. 3. The difference is due to the fact that the translation in GT is based on an earlier French edition of Chapter IX.

III.6, ℓ. −14.

"... whence our assertion."

Let us review the topology of compact convergence on $\mathscr{C}(\mathrm{X};\mathrm{E})$ (cf., GT, X, §1, No. 3, Example III), that is, the topology of uniform convergence in the compact subsets of X. (See also the notes for III.4, ℓ. 5–6 and III.39, ℓ. 8–11.)

Let (N_α) be a fundamental system of neighborhoods of 0 in E. The sets

$$\mathrm{V}_\alpha = \{(\mathbf{x},\mathbf{y}) \in \mathrm{E}\times\mathrm{E} : \ \mathbf{x}-\mathbf{y} \in \mathrm{N}_\alpha\}$$

then form a fundamental system of entourages for the uniformity on E (TVS, I, §1, No. 4 and GT, III, §3, No. 1, remark following Def. 1). A fundamental system of entourages for a uniformity on $\mathscr{C} = \mathscr{C}(\mathrm{K};\mathrm{E})$ is then given by the sets

$$\begin{aligned}\mathrm{W}_{\mathrm{K},\alpha} &= \{(\mathbf{f},\mathbf{g}) \in \mathscr{C}\times\mathscr{C} : \ \big(\mathbf{f}(x),\mathbf{g}(x)\big) \in \mathrm{V}_\alpha \ \text{ for all } x\in\mathrm{K}\}\\ &= \{(\mathbf{f},\mathbf{g}) \in \mathscr{C}\times\mathscr{C} : \ \mathbf{f}(x)-\mathbf{g}(x) \in \mathrm{N}_\alpha \ \text{ for all } x\in\mathrm{K}\}\end{aligned}$$

where K runs over the set of all compact subsets of X. The deduced topology is called the *topology of compact convergence* and will be denoted τ_{cc}; it has as fundamental system of neighborhoods of 0 the sets

$$\mathscr{N}_{\mathrm{K},\alpha} = \{\mathbf{f} \in \mathscr{C} : \ \mathbf{f}(\mathrm{K}) \subset \mathrm{N}_\alpha\}$$

(notably, given indices $\alpha_1, \ldots, \alpha_n$ and compact sets $\mathrm{K}_1, \ldots, \mathrm{K}_n$, if α is chosen so that $\mathrm{V}_{\alpha_1} \cap \cdots \cap \mathrm{V}_{\alpha_n} \supset \mathrm{V}_\alpha$, then $\mathscr{N}_{\mathrm{K}_1,\alpha_1} \cap \cdots \cap \mathscr{N}_{\mathrm{K}_n,\alpha_n} \supset \mathscr{N}_{\mathrm{K}_1 \cup \cdots \cup \mathrm{K}_n,\alpha}$); and, for fixed $\mathbf{f} \in \mathscr{C}$, the sets

$$\mathscr{V}_{\mathrm{K},\alpha}(\mathbf{f}) = \mathbf{f} + \mathscr{N}_{\mathrm{K},\alpha} = \{\mathbf{g} \in \mathscr{C} : \ \mathbf{g} - \mathbf{f} \in \mathscr{N}_{\mathrm{K},\alpha}\} = \{\mathbf{g} \in \mathscr{C} : \ (\mathbf{g} - \mathbf{f})(\mathrm{K}) \subset \mathrm{N}_\alpha\}$$

form a fundamental system of neighborhoods of $\mathbf{f}$ for τ_{cc}. One writes $\mathscr{C}_c(\mathrm{X};\mathrm{E})$ for $\mathscr{C}(\mathrm{X};\mathrm{E})$ equipped with the topology τ_{cc}.

Given $\mathbf{f} \in \mathscr{C}_c(\mathrm{X};\mathrm{E})$, we are to show that $\mathbf{f}$ belongs to the closure of $\mathscr{K}(\mathrm{X};\mathrm{E})$; it suffices to show that every neighborhood $\mathscr{V}_{\mathrm{K},\alpha}(\mathbf{f})$ contains an element of $\mathscr{K}(\mathrm{X};\mathrm{E})$. Indeed, with $h \in \mathscr{K}(\mathrm{X};\mathbf{R})$ chosen as in the proof, one has $h\mathbf{f} = \mathbf{f}$ on K, therefore

$$(h\mathbf{f} - \mathbf{f})(\mathrm{K}) = \{0\} \subset \mathrm{N}_\alpha,$$

thus $h\mathbf{f} \in \mathscr{V}_{\mathrm{K},\alpha}(\mathbf{f})$.

III.6, ℓ. −7.

…the second assertion is an obvious consequence of the first"

For definiteness, let us consider the case of real scalars (the same argument works in the complex case). The identification in question associates with $\mathbf{f} = \sum_{k=1}^{n} \varphi_k \otimes \mathbf{y}_k$ (where $\varphi_k \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{R})$ and $\mathbf{y}_k \in \mathrm{E}$) the function

$$x \mapsto \sum_{k=1}^{n} \varphi_k(x)\mathbf{y}_k \qquad (x \in \mathrm{X}),$$

clearly an element of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$. By definition,

$$\text{(i)} \qquad \mathscr{K}(\mathrm{X};\mathrm{E}) = \bigcup_{\mathrm{K}} \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}),$$

where K runs over the set of all compact subsets of X; citing (i) for $\mathrm{E} = \mathbf{R}$, we infer that

$$\text{(ii)} \qquad \mathscr{K}(\mathrm{X};\mathbf{R}) \otimes \mathrm{E} = \bigcup_{\mathrm{K}} \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{R}) \otimes \mathrm{E}.$$

The left side of (ii), viewed as a set of (continuous) functions $X \to E$, is clearly contained in the left side of (i).

Recall (No. 1) that the topology on $\mathscr{K}(X, K; E)$, for K compact in X, is the topology of uniform convergence in X, equivalently in K (it is described in detail in the notes for III.39, ℓ. 8–11), and the topology on $\mathscr{K}(X; E)$ is the direct limit of the topologies on the $\mathscr{K}(X, K; E)$ as K varies over the set of all compact subsets of X. The first assertion of the proposition is that the closure of $\mathscr{K}(X, K; \mathbf{R}) \otimes E$ in $\mathscr{K}(X, K; E)$ is equal to $\mathscr{K}(X, K; E)$. Since $\mathscr{K}(X, K; E)$ is a closed topological subspace of $\mathscr{K}(X; E)$ (No. 1, Prop. 1), $\mathscr{K}(X, K; E)$ is also the closure of $\mathscr{K}(X, K; \mathbf{R}) \otimes E$ in $\mathscr{K}(X; E)$ (GT, I, §3, No. 1, Cor. of Prop. 1); it then follows from (ii) that the closure of $\mathscr{K}(X; \mathbf{R}) \otimes E$ in $\mathscr{K}(X; E)$ contains every term on the right side of (i), hence is equal to $\mathscr{K}(X; E)$.

III.6, ℓ. $-2, -1$.

"...this proves the proposition, by the definition of the topology of $\mathscr{K}(X, K; E)$."

The formulation of the topology of (hence convergence in) $\mathscr{K}(X, K; E)$ in terms of continuous semi-norms on E is worked out in the note for III.39, ℓ. 8–11; explicitly, the argument given here shows that, in the notations of (7) and (8) of that note, given any $f \in \mathscr{K}(X, K; E)$ and any neighborhood of 0 in $\mathscr{K}(X, K; E)$ of the form

$$U_\alpha = \{\mathbf{f} \in \mathscr{K}(X, K; E) : \quad q_\alpha\big(\mathbf{f}(x)\big) \leqslant 1 \quad \text{for all } x \in X\},$$

with q_α a continuous semi-norm on E, there exists an element $\sum_{j=1}^{n} \varphi_j \otimes f(x_j)$ of $\mathscr{K}(X, K; \mathbf{R}) \otimes E$ (interpreted as a function on X) such that

$$f - \sum_{j=1}^{n} \varphi_j \otimes f(x_j) \in U_\alpha,$$

whence the second assertion of the proposition.

III.10, ℓ. 13.

"...whence our assertion (TVS, II, §4, No. 4, Prop. 5)."

Write $u : \mathscr{K}(X; \mathbf{C}) \to \mathscr{K}(X; \mathbf{C})$, $u(f) = gf$. The foregoing shows that for each compact set $K \subset X$, $f \mapsto gf$ is a continuous linear mapping of $\mathscr{K}(X, K; \mathbf{C})$ into itself. Then for each K, the restricted mapping

$$u\big|\mathscr{K}(X, K; \mathbf{C}) : \mathscr{K}(X, K; \mathbf{C}) \to \mathscr{K}(X; \mathbf{C})$$

is the composite

$$\mathscr{K}(X, K; \mathbf{C}) \to \mathscr{K}(X, K; \mathbf{C}) \to \mathscr{K}(X; \mathbf{C})$$

of two continuous mappings ($f \mapsto gf \mapsto gf$), hence is continuous. Therefore u is continuous by the cited proposition (with $\mathrm{G} = \mathscr{K}(\mathrm{X};\mathbf{C})$).

More generally, suppose E, F are locally convex spaces that are the direct limits of families (with same index set) of subspaces $\mathrm{E}_\alpha, \mathrm{F}_\alpha$ of E, F, respectively. If $u : \mathrm{E} \to \mathrm{F}$ is a linear mapping such that for every α, $u(\mathrm{E}_\alpha) \subset \mathrm{F}_\alpha$ and $u|\mathrm{E}_\alpha : \mathrm{E}_\alpha \to \mathrm{F}$ is continuous, then u is continuous; for, $u|\mathrm{E}_\alpha : \mathrm{E}_\alpha \to \mathrm{F}$ is the composite $\mathrm{E}_\alpha \to \mathrm{F}_\alpha \to \mathrm{F}$ of two continuous mappings ($x \mapsto u(x) \mapsto u(x)$).

III.11, ℓ. 4–6.

"...moreover, the mapping $(f_1, f_2) \mapsto f_1 + if_2$ is an *isomorphism* of the product topological vector space $\mathscr{K}(\mathrm{X};\mathbf{R}) \times \mathscr{K}(\mathrm{X};\mathbf{R})$ onto the real topological vector space $\mathscr{K}(\mathrm{X};\mathbf{C})$ (No. 1, Prop. 1)."

By the cited proposition, one has an isomorphism of real topological vector spaces

$$\Phi : \mathscr{K}(\mathrm{X};\mathbf{R}^2) \to \mathscr{K}(\mathrm{X};\mathbf{R}) \times \mathscr{K}(\mathrm{X};\mathbf{R})$$

defined by $\Phi(g) = (\mathrm{pr}_1 \circ g, \mathrm{pr}_2 \circ g)$, where $\mathrm{pr}_1(a_1, a_2) = a_1$, $\mathrm{pr}_2(a_1, a_2) = a_2$ for $(a_1, a_2) \in \mathbf{R}^2$; the inverse mapping is $\Phi^{-1}(f_1, f_2) = g$, where $g(x) = \big(f_1(x), f_2(x)\big)$. Since the mapping $\theta : \mathbf{R}^2 \to \mathbf{C}$ defined by $\theta(a_1, a_2) = a_1 + ia_2$ is an isomorphism of real topological vector spaces, the mapping

$$\Psi : \mathscr{K}(\mathrm{X};\mathbf{R}^2) \to \mathscr{K}(\mathrm{X};\mathbf{C})$$

defined by $\Psi(g) = \theta \circ g$ is also an isomorphism of real topological vector spaces. Composing, we have an isomorphism

$$\Psi \circ \Phi^{-1} : \mathscr{K}(\mathrm{X};\mathbf{R}) \times \mathscr{K}(\mathrm{X};\mathbf{R}) \to \mathscr{K}(\mathrm{X};\mathbf{C}),$$

namely

$$\begin{aligned}(\Psi \circ \Phi^{-1})(f_1, f_2) &= \Psi\big(\Phi^{-1}(f_1, f_2)\big) \\ &= \Psi(g) \quad (\text{where } g(x) = (f_1(x), f_2(x))) \\ &= \theta \circ g,\end{aligned}$$

where $(\theta \circ g)(x) = \theta(g(x)) = \theta\big((f_1(x), f_2(x)\big) = f_1(x) + if_2(x)$, thus $(\Psi \circ \Phi^{-1})(f_1, f_2) = f_1 + if_2$ is the desired isomorphism.

III.14, ℓ. 1–3.

"Moreover, we can suppose ζ so chosen that

$$|\mu(g_1) + \zeta\mu(g_2)| = |\mu(g_1)| + |\mu(g_2)|;$$

since $|\mu(g_i)|$ is arbitrarily close to $L(f_i)$ $(i = 1, 2)\ldots$"

First choose g_i so that $|\mu(g_i)|$ is near $L(f_i)$, *then* choose ζ so as to rotate $\mu(g_2)$ to point in the same direction as $\mu(g_1)$ in the complex plane.

III.14, ℓ. 9–10.

"…which proves the continuity of g_i at the points where $f_1(x) + f_2(x) = 0$ $(i = 1, 2)$, since at these points we have also $g(x) = 0$."

Some preliminaries are in order. Given $g \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ with $|g| \leqslant f_1 + f_2$, let $\mathrm{U} = \{x \in \mathrm{X} : f_1(x) + f_2(x) > 0\}$; since the f_i are $\geqslant 0$, $\mathrm{X} - \mathrm{U} = \{x \in \mathrm{X} : f_1(x) = f_2(x) = 0\}$. Define $g_i : \mathrm{X} \to \mathbf{C}$ $(i = 1, 2)$ by the formulas

$$g_i = \begin{cases} \dfrac{g f_i}{f_1 + f_2} & \text{on } \mathrm{U} \\ 0 & \text{on } \mathrm{X} - \mathrm{U}. \end{cases}$$

The continuity of g_i at the points of U is obvious. Note that $g_1 + g_2 = g$; this is clear at the points of U, whereas if $x \in \mathrm{X} - \mathrm{U}$ then $g_1(x) = g_2(x) = 0$ by definition, and $g(x) = 0$ follows from $|g(x)| \leqslant f_1(x) + f_2(x) = 0$. Also, $|g_i| \leqslant g$; for, if $x \in \mathrm{X} - \mathrm{U}$ then $g_i(x) = g(x) = 0$, whereas if $x \in \mathrm{U}$ then

$$|g_i(x)| = |g(x)| \cdot \frac{f_i(x)}{f_1(x) + f_2(x)} \leqslant |g(x)|.$$

And $|g_i| \leqslant f_i$; for, if $x \in \mathrm{X} - \mathrm{U}$ then $f_i(x) = g_i(x) = 0$, whereas if $x \in \mathrm{U}$ then

$$|g_i(x)| = f_i(x) \cdot \frac{|g(x)|}{f_1(x) + f_2(x)} \leqslant f_i(x)$$

because $|g| \leqslant f_1 + f_2$. End of preliminaries.

Consider the covering of X by the closed sets $\mathrm{X} - \mathrm{U}$ and $\overline{\mathrm{U}}$. Since $g_i \big| \mathrm{X} - \mathrm{U} = 0$ is continuous, to prove that g_i is continuous on X we need only show that $g_i \big| \overline{\mathrm{U}}$ is continuous (GT, I, §3, No. 2, Prop. 4).

Given $x_0 \in \overline{\mathrm{U}}$ and a directed family (x_α) in $\overline{\mathrm{U}}$ such that $x_\alpha \to x_0$, we are to show that $g_i(x_\alpha) \to g_i(x_0)$ for $i = 1, 2$. If $x_0 \in \mathrm{U}$ then $x_\alpha \in \mathrm{U}$ from some index onward, whence $g_i(x_\alpha) \to g_i(x_0)$ by the continuity of g_i on U. Thus we can suppose that $x_0 \in \overline{\mathrm{U}} - \mathrm{U} = \overline{\mathrm{U}} \cap (\mathrm{X} - \mathrm{U})$. Since $x_0 \in \mathrm{X} - \mathrm{U}$ we have $g_i(x_0) = 0$; also $f_1(x_0) = f_2(x_0) = 0$. By the continuity of f_i, we know that $f_i(x_\alpha) \to f_i(x_0) = 0$; but $|g_i(x_\alpha)| \leqslant f_i(x_\alpha) \to 0$, whence $g_i(x_\alpha) \to 0 = g_i(x_0)$.

III.14, ℓ. 16.

"When μ is a *real* measure, it follows from formula (9) that $|\mu| \leqslant L$"

Let $f \in \mathscr{K}_+(\mathrm{X})$. If $g \in \mathscr{K}(\mathrm{X}; \mathbf{R})$ and $|g| \leqslant f$ then $\int g\, d\mu \in \mathbf{R}$ and

$$\int g\, d\mu \leqslant \left| \int g\, d\mu \right| \leqslant L(f)$$

by the definition of L; taking the supremum over all such g, $\int f\,d|\mu| \leqslant L(f)$ by (9); thus $|\mu|(f) \leqslant L(f)$ for all $f \in \mathscr{K}_+(\mathrm{X})$, that is, $|\mu| \leqslant L$ as positive linear forms on $\mathscr{K}(\mathrm{X};\mathbf{R})$.

III.14, ℓ. −7, −6.

"consequently, for every function $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$,

$$\Big|\int g\,d\mu\Big| \leqslant \int |g|\,d|\mu|\,.\text{''} \tag{13}$$

Putting $f = |g|$ in (12),

$$|\mu|(|g|) = \sup_{|h|\leqslant|g|,\,h\in\mathscr{K}(\mathrm{X};\mathbf{C})} |\mu(h)| \geqslant |\mu(g)|$$

(because $|g| \leqslant |g|$).

III.15, ℓ. 4, 5.

"therefore

$$|\overline{\mu}| = |\mu|\,.\text{''} \tag{16}$$

For all $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with $|g| \leqslant f \in \mathscr{K}_+(\mathrm{X})$, $|\overline{\mu}(g)| = \big|\overline{\mu(\overline{g})}\big| = |\mu(\overline{g})|$; taking supremum over all such g, one has $|\overline{\mu}|(f) = |\mu|(f)$ for all $f \in \mathscr{K}_+(\mathrm{X})$, hence $|\overline{\mu}| = |\mu|$ on $\mathscr{K}(\mathrm{X};\mathbf{C})$.

III.15, ℓ. 14, 15.

"if V is a *dense* linear subspace of $\mathscr{K}(\mathrm{X};\mathbf{C})$,"

Caution: Recall that $\mathscr{K}(\mathrm{X};\mathbf{C})$ (without express mention to the contrary) bears the direct limit topology $\mathscr{T}$, which is finer than the topology $\mathscr{T}_u$ of uniform convergence in X, thus a linear subspace dense in the sense of $\mathscr{T}$ is dense for $\mathscr{T}_u$; but the converse may fail (TVS, II, §4, No. 4, *Example* on page TVS II.29).

III.15, ℓ. 16–18.

"...every linear form on V that is continuous for the topology induced by that of $\mathscr{K}(\mathrm{X};\mathbf{C})$ may be extended (in only one way) to a measure on X."

Since $\mathscr{K}(\mathrm{X};\mathbf{C})$ is locally convex for the direct limit topology (§1, No. 1), the Hahn–Banach theorem is applicable (TVS, II, §8, No. 3, Cor. 2 of Th. 1).

III.15, ℓ. −2, −1.

"It then suffices to apply Th. 1 of No. 5 and Prop. 1 of TVS, II, §3, No. 1."

By the cited proposition (see TVS II.21), the property of V just derived assures that every positive linear form μ_0 on V (for the relative order) may be extended to a positive linear form on $\mathscr{K}(\mathrm{X};\mathbf{R})$, that is, to a positive measure μ on X (No. 5, Th. 1); if, in addition, V is dense (for the direct limit topology), then the extension is unique by the remarks at the beginning of the present subsection.

III.16, ℓ. 6–8.

"Definition 3. — *A measure on a locally compact space* X *is said to be bounded if it is continuous on* $\mathscr{K}(\mathrm{X};\mathbf{C})$ *for the topology of uniform convergence*."

Note that a linear form on $\mathscr{K}(\mathrm{X};\mathbf{C})$ continuous for the topology of uniform convergence $\mathscr{T}_u$ (i.e., the norm topology) is automatically continuous for the direct limit topology $\mathscr{T}$ (because $\mathscr{T}_u \subset \mathscr{T}$), hence is a measure.

III.16, ℓ. 13–15.

"To say that μ is a bounded measure thus signifies that μ belongs to the dual of the space $\mathscr{K}(\mathrm{X};\mathbf{C})$ *normed* by $\|f\|$; we shall denote this dual by $\mathscr{M}^1(\mathrm{X};\mathbf{C})$ (or simply $\mathscr{M}^1(\mathrm{X})$ when no confusion can result)."

The case $\mathrm{X} = \mathbf{N}$. The direct limit topology on $\mathscr{K}(\mathrm{X};\mathbf{C})$ remains a slippery concept to grasp; it is instructive to trace through the concepts on an atypical but simple example: $\mathrm{X} = \mathbf{N} = \{0,1,2,3,\ldots\}$ with the discrete topology.

(1) $\mathscr{F}(\mathrm{X})$, the set of all functions $f : \mathbf{N} \to \mathbf{C}$, may be identified with the vector space (s) (in the notation of Banach's book) of all sequences $(c_k)_{k\geqslant 0}$ of complex numbers with the termwise linear operations (or, when viewed as a product space $\mathbf{C}^{\aleph_0}$, with the coordinatewise linear operations).

(2) $\mathscr{C}(\mathrm{X}) = \mathscr{F}(\mathrm{X})$, as all functions from a discrete space to a topological space ($\mathbf{C}$ in this instance) are continuous.

(3) $\mathscr{C}^b(\mathrm{X})$: This is the linear subspace of $\mathscr{F}(\mathrm{X})$ consisting of all bounded complex functions on X; it may be identified with the space (m) (in the notation of Banach's book) of bounded sequences of complex numbers.

(4) $\mathscr{K}(\mathrm{X})$: As the compact subsets K of $\mathrm{X} = \mathbf{N}$ are its finite subsets, $\mathscr{K}(\mathrm{X})$ is the set of all complex functions on X with finite support; it may be identified with the linear subspace of (m) consisting of all complex sequences that terminate in zeros. In particular, $\mathscr{K}(\mathrm{X},\mathrm{K})$ is finite-dimensional, $\mathscr{K}(\mathrm{X})$ being the union of its linear subspaces $\mathscr{K}(\mathrm{X},\mathrm{K}_n)$, where $\mathrm{K}_n = \{0,1,2,\ldots,n\}$ $(n = 0,1,2,\ldots)$.

(5) *The topology on* $\mathscr{K}(\mathrm{X},\mathrm{K})$, K *compact* (i.e., finite).

The topology of uniform convergence on $\mathscr{K}(\mathrm{X},\mathrm{K})$ is simply the topology of pointwise convergence; since $\mathscr{K}(\mathrm{X},\mathrm{K})$ is finite-dimensional, this is the unique topology on $\mathscr{K}(\mathrm{X},\mathrm{K})$ that makes it a Hausdorff topological vector space (TVS, I, §2, No. 3, Th. 2). One can define a semi-norm p_{K} on $\mathscr{K}(\mathrm{X})$ by

$$p_{\mathrm{K}}(f) = \sup_{k\in\mathrm{K}} |f(k)| = \max_{k\in\mathrm{K}} |f(k)|$$

(also denoted $\|f\|_{\mathrm{K}}$); the restriction of p_{K} to $\mathscr{K}(\mathrm{X},\mathrm{K})$ is a *norm* that generates the topology just described.

(6) *The topology on* $\mathscr{K}(\mathrm{X})$ *of uniform convergence in the compact subsets of* X.

Since the compact sets are finite, this is just the topology of pointwise convergence.

(7) *The topology on* $\mathscr{K}(\mathrm{X})$ *of uniform convergence in* X.

This is the topology on $\mathscr{K}(\mathrm{X})$ derived from the norm

$$\|f\| = \sup_{k\in\mathbf{N}} |f(k)| = \max_{k\in\mathbf{N}} |f(k)| = \sup\{p_{\mathrm{K}}(f) :\ \mathrm{K}\subset\mathbf{N}\text{ finite}\};$$

it is obviously finer than the topology of pointwise convergence. The sequence $f_n = n\varphi_n$, where φ_n is the characteristic function of $\{n\}$, converges pointwise to 0, but $\|f_n\| = n$; thus the topology of uniform convergence in X is strictly finer than the topology of uniform convergence in the compact subsets of X.

The normed space $\big(\mathscr{K}(\mathrm{X}), \|\cdot\|\big)$ is incomplete, i.e., is not a Banach space: for example, if f_n is the function defined by $f_n(k) = \dfrac{1}{k+1}$ for $k = 0, 1, \ldots, n$ and $f_n(k) = 0$ for $k > n$, then $\|f_m - f_n\| \to 0$ as $m, n \to \infty$; but the sequence (f_n) is not convergent in $\mathscr{K}(\mathrm{X})$, indeed $f_n \to f \in \mathscr{F}(\mathrm{X})$ pointwise implies that $f(k) = \dfrac{1}{k+1}$ for all $k \in \mathbf{N}$, whence $f \notin \mathscr{K}(\mathrm{X})$.

(8) *The direct limit topology* $\mathscr{T}$ *on* $\mathscr{K}(\mathrm{X})$.

The key observation: If $\mathscr{S}$ is a locally convex (Hausdorff) topology on $\mathscr{K}(\mathrm{X})$, then $\mathscr{S}$ is coarser than $\mathscr{T}$ (i.e., $\mathscr{S} \subset \mathscr{T}$, that is, the open sets for $\mathscr{S}$ are open for $\mathscr{T}$). For, for each finite subset K of X, the insertion mapping (or "canonical injection")

$$u_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}), \quad u_{\mathrm{K}}(f) = f$$

(where $\mathscr{K}(\mathrm{X},\mathrm{K})$ is equipped with its unique compatible Hausdorff topology, and $\mathscr{K}(\mathrm{X})$ is equipped with $\mathscr{S}$) is continuous because $\mathscr{K}(\mathrm{X},\mathrm{K})$ is finite-dimensional (TVS, I, §2, No. 3, Cor. 2 of Th. 2), whence $\mathscr{S} \subset \mathscr{T}$ by the definition of $\mathscr{T}$. On the other hand, there exists on $\mathscr{K}(\mathrm{X})$ a finest locally convex topology $\mathscr{T}_\omega$ (TVS, II, §4, No. 2, item 2) and *loc. cit.*, §8, No. 2, fifth paragraph). Putting $\mathscr{S} = \mathscr{T}_\omega$ in the foregoing, $\mathscr{T}_\omega \subset \mathscr{T}$ and so

$\mathscr{T} = \mathscr{T}_\omega$ by the maximality of $\mathscr{T}_\omega$. Thus, $\mathscr{T}$ is the topological vector space topology on $\mathscr{K}(\mathrm{X})$ for which the set of all absorbent, balanced, convex sets in $\mathscr{K}(\mathrm{X})$ forms a fundamental system of neighborhoods of 0 (TVS, *loc. cit.*). *A fortiori*, every closed (for $\mathscr{T}$), absorbent, balanced convex set in $\mathscr{K}(\mathrm{X})$ is a neighborhood of 0, consequently $\mathscr{K}(\mathrm{X})$ is barreled (TVS, III, §4, No. 1, Def. 2). Since discrete spaces are paracompact (GT, I, §9, No. 10), $\mathscr{K}(\mathrm{X})$ is quasi-complete (for $\mathscr{T}$) (III, §1, No. 1, Cor. of Prop. 2); indeed, a bounded (in the topological vector space sense) subset of $\mathscr{K}(\mathrm{X})$ is contained in the finite-dimensional space $\mathscr{K}(\mathrm{X},\mathrm{K})$ for some finite $\mathrm{K} \subset \mathrm{X}$ (*loc. cit.*, Prop. 2), thus a closed and bounded subset of $\mathscr{K}(\mathrm{X})$ is compact (hence complete) by the Weierstrass–Bolzano theorem (for $\mathbf{C}^n$).

(9) $\mathscr{M}(\mathrm{X}) = \mathscr{K}(\mathrm{X})^*$ (*the algebraic dual of* $\mathscr{K}(\mathrm{X})$).

Every linear mapping of $\mathscr{K}(\mathrm{X})$ into a locally convex space is continuous (for the topology $\mathscr{T}$). For, suppose $v : \mathscr{K}(\mathrm{X}) \to \mathrm{G}$ is a linear mapping, G a locally convex space. For every finite subset K of X, the composite mapping $v \circ u_{\mathrm{K}}$,

$$\mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}) \to \mathrm{G}$$

is continuous by the finite-dimensionality of $\mathscr{K}(\mathrm{X},\mathrm{K})$, whence the continuity of v (TVS, II, §4, No. 4, Prop. 5). In particular, every linear form on $\mathscr{K}(\mathrm{X})$ is continuous (for $\mathscr{T}$), i.e., is a measure on X. Since a linear form continuous for the topology $\mathscr{T}_u$ of uniform convergence must be bounded, one infers that $\mathscr{T} \supset \mathscr{T}_u$ properly.

{Since $\mathscr{T} = \mathscr{T}_\omega$, an alternative proof is available: If E is a vector space equipped with its finest locally convex topology, then every linear mapping of E into a locally convex space G is continuous; for, the inverse image of an absorbent balanced convex subset of G has the same properties, hence is a neighborhood of 0 in E, and G has a fundamental system of neighborhoods of 0 (automatically absorbent) that are balanced and convex (TVS, II, §4, No. 1, p. TVS II.23, ℓ. -9 to -7, and *loc. cit.*, §8, No. 2, p. TVS II.62).}

(10) *The vector space* $\mathscr{M}(\mathrm{X}) = \mathscr{K}(\mathrm{X})^*$ *may be identified with the space* (s) *of all complex sequences.*

For every $n \in \mathbf{N}$ let $\varphi_n : \mathbf{N} \to \mathbf{C}$ be the characteristic function of $\{n\}$: $\varphi_n(k) = \delta_{nk}$ $(k = 0, 1, 2, \ldots)$. Since every $f \in \mathscr{K}(\mathrm{X})$ has finite support, it is clear that the φ_n form a basis of $\mathscr{K}(\mathrm{X})$; so if $\mu \in \mathscr{M}(\mathrm{X})$, μ is characterized by its values $c_n = \mu(\varphi_n)$ $(n \in \mathbf{N})$, thus $\mu \mapsto \big(\mu(\varphi_n)\big)_{n\geqslant 0}$ is a (clearly injective) linear mapping of $\mathscr{M}(\mathrm{X})$ into the space (s) of all complex sequences. This mapping is bijective; for, if $c = (c_n) \in (s)$, the linear form $\mu : \mathscr{K}(\mathrm{X})) \to \mathbf{C}$ defined by $\mu(f) = \sum_{k=0}^{\infty} c_k f(k)$ (actually a finite sum) satisfies $\mu(\varphi_n) = c_n$ for all $n \in \mathbf{N}$, so that $\mu \mapsto c$ under the mapping.

(11) *The vague topology on* $\mathscr{M}(\mathrm{X})$.

In $\mathscr{M}(\mathrm{X})$, "$\mu_\alpha \to \mu$ vaguely" means that $\mu_\alpha(f) \to \mu(f)$ in $\mathbf{C}$ for every $f \in \mathscr{K}(\mathrm{X})$; equivalently, $\mu_\alpha(\varphi_n) \to \mu(\varphi_n)$ for every $n \in \mathbf{N}$. Thus, in the identification $\mathscr{M}(\mathrm{X}) \to (s)$, the vague topology on $\mathscr{M}(\mathrm{X})$ corresponds to the topology of coordinatewise convergence in (s), viewed as a product space $\mathbf{C}^{\aleph_0}$. The latter topology is metrizable (Banach, p. 10), by the metric

$$d\big((a_n),(b_n)\big) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{|a_n - b_n|}{1 + |a_n - b_n|}\,,$$

for which (s) is complete. The corresponding metric D on $\mathscr{M}(\mathrm{X})$ is

$$D(\mu,\nu) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{|\mu(\varphi_n) - \nu(\varphi_n)|}{1 + |\mu(\varphi_n) - \nu(\varphi_n)|}$$

(φ_n the characteristic function of $\{n\}$). The metric D is invariant in the sense that $D(\mu+\rho,\nu+\rho) = D(\mu,\nu)$; consequently the additive group of $\mathscr{M}(\mathrm{X})$ is a complete topological group for the vague topology (cf. Prop. 7 on p. III.8 above, with $\mathfrak{S}$ the set of all singletons $\{\varphi_n\}$, $n \in \mathbf{N}$). In other words, $\mathscr{M}(\mathrm{X})$ is a complete topological vector space; being locally convex and metrizable, it is a Fréchet space (TVS, II, §4, No. 1, p. TVS II.24), hence is barreled (TVS, III, §4, No. 1, Cor. of Prop. 2).

(12) *The space* $\mathscr{M}^1(\mathrm{X})$ *of bounded measures on* $\mathrm{X} = \mathbf{N}$.

Let μ be a bounded measure on X, that is, a linear form on $\mathscr{K}(\mathrm{X})$ that is continuous for the topology of uniform convergence in X, i.e., for the norm

$$\|f\| = \sup_{n\in\mathbf{N}} |f(n)| = \max_{n\in\mathbf{N}} |f(n)|\,.$$

An equivalent condition on μ is that it be a "bounded linear form" on the normed space $\big(\mathscr{K}(\mathrm{X}), \|\cdot\|\big)$ in the sense that there exists a constant $M \geqslant 0$ such that

$$|\mu(f)| \leqslant M\|f\| \ \text{ for all } f \in \mathscr{K}(\mathrm{X})\,.$$

The smallest possible value of M (the infimum of all such M) is the number $\sup\limits_{f\in\mathscr{K}(\mathrm{X}),\,\|f\|\leqslant 1} |\mu(f)|$, which is denoted $\|\mu\|$.

Explicitly, $\|\mu\| = \sum\limits_{n=0}^{\infty} |\mu(\varphi_n)|$, where φ_n is the characteristic function of $\{n\}$. The details are as follows. If $f \in \mathscr{K}(\mathrm{X})$, say $f = \sum\limits_{n=0}^{\infty} a_n\varphi_n$ (a finite sum), then $\mu(f) = \sum\limits_{n=0}^{\infty} a_n\mu(\varphi_n)$, thus

$$\left|\sum_{n=0}^{\infty} a_n \mu(\varphi_n)\right| = |\mu(f)| \leqslant \|\mu\| \cdot \|f\| = \|\mu\| \cdot \sup_{n\in\mathbf{N}} |a_n| .$$

Writing $c_n = \mu(\varphi_n)$ for all n, we have

$$\left|\sum_{n=0}^{\infty} a_n c_n\right| \leqslant \|\mu\| \cdot \sup_{n\in\mathbf{N}} |a_n|$$

for all complex sequences (a_n) that terminate in zeros. For every $n \in \mathbf{N}$ choose a complex number a_n such that $|a_n| = 1$ and $a_n c_n = |c_n|$. By the foregoing, applied to the sequence $a_0, a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots$,

$$\sum_{k=0}^{n} |c_k| = \left|\sum_{k=0}^{n} a_k c_k\right| \leqslant \|\mu\| \cdot 1$$

for every $n \in \mathbf{N}$, thus the series $\sum\limits_{n=0}^{\infty} c_k$ is absolutely convergent with $\sum\limits_{n=0}^{\infty} |c_n| \leqslant \|\mu\|$. On the other hand, for every finite sequence $a_0, a_1, \ldots, a_n$ of complex numbers, we have

$$\left|\sum_{k=0}^{n} a_k c_k\right| \leqslant \sum_{k=0}^{n} |a_k| \cdot |c_k| \leqslant \left(\sum_{k=0}^{n} |c_k|\right) \cdot \max_{0\leqslant k\leqslant n} |a_k| \leqslant \left(\sum_{k=0}^{\infty} |c_k|\right) \cdot \max_{0\leqslant k\leqslant n} |a_k|;$$

it follows that for every $f \in \mathscr{K}(\mathrm{X})$,

$$|\mu(f)| \leqslant \left(\sum_{n=0}^{\infty} |c_n|\right) \|f\| ,$$

whence $\|\mu\| \leqslant \sum\limits_{n=0}^{\infty} |c_n|$.

The set of all such measures, denoted $\mathscr{M}^1(\mathrm{X})$, is a linear subspace of $\mathscr{M}(\mathrm{X}) = \mathscr{K}(\mathrm{X})^*$; equipped with the norm just defined, it is the dual space of the normed space $\big(\mathscr{K}(\mathrm{X}), \|\cdot\|\big)$ and is a Banach space (TVS, III, §3, No. 8, Cor. 2 of Prop. 12).

The foregoing associates, to each bounded measure μ on X, the sequence $\big(\mu(\varphi_n)\big)$ satisfying $\sum\limits_{n=0}^{\infty} |\mu(\varphi_n)| < +\infty$. The set of all complex sequences $c = (c_n)_{n\geqslant 0}$ satisfying $\sum\limits_{n=0}^{\infty} |c_n| < +\infty$ is a linear subspace of (s)

and is a Banach space for the norm $\|c\| = \sum_{n=0}^{\infty} |c_n|$; it is denoted $(\ell^{(1)})$ (Banach, p. 12). The correspondence $\mu \mapsto \big(\mu(\varphi_n)\big)$ is an isometric linear mapping of $\mathscr{M}^1(\mathrm{X})$ onto $(\ell^{(1)})$. The space $(\ell^{(1)})$ may be regarded as the space of functions integrable with respect to the measure on $\mathrm{X} = \mathbf{N}$ that assigns measure 1 to every set $\{n\}$, namely, the linear form $\lambda : \mathscr{K}(\mathrm{X}) \to \mathbf{C}$ defined by $\lambda(f) = \sum_{n=0}^{\infty} f(n)$ (a finite sum) for every $f \in \mathscr{K}(\mathrm{X})$.

Identifying $\mathscr{K}(\mathrm{X})$ with the space (c_{00}) of sequences that terminate in 0's, its completion for the sup-norm $\|\cdot\|$ may be identified with the space (c_0) of sequences that converge to 0; the Banach space (c_0) has the same dual as its dense linear subspace (c_{00}), thus

$$(c_0)' \cong (c_{00})' \cong \big(\mathscr{K}(\mathrm{X}),\ \|\cdot\|\big)' = \mathscr{M}^1(\mathrm{X}) \cong (\ell^{(1)})$$

(cf. Hewitt–Stromberg (7.13) and (14.25), or Banach pp. 66, 67). Banach spaces whose dual is isometrically isomorphic to the space of functions integrable for some measure are called L^1-*predual spaces* (cf. the book of H.E. Lacey, *The isometric theory of classical Banach spaces*, Grundlehren math. Wiss. Bd. 208, Springer–Verlag, New York, 1974).

III.17, ℓ. 9–13.

"Proposition 10. — *For every measure* μ *on* X,

$$\|\mu\| = \sup_{0 \leqslant f \leqslant 1,\ f \in \mathscr{K}(\mathrm{X};\mathbf{R})} |\mu|(f)\,. \tag{22}$$

"For, taking into account the formula (12) that defines the absolute value of a measure, the second member of (22) may be written

$$\sup_{0 \leqslant f \leqslant 1,\ f \in \mathscr{K}(\mathrm{X};\mathbf{R})} \left(\sup_{|g| \leqslant f,\ g \in \mathscr{K}(\mathrm{X};\mathbf{C})} |\mu(g)| \right) = \sup_{\|g\| \leqslant 1,\ g \in \mathscr{K}(\mathrm{X};\mathbf{C})} |\mu(g)|\,."$$

Write α for the expression on the left (where, in the right side of (22), $|\mu|(f)$ has been replaced by its formula given by (12)), β for the expression on the right (which is the definition of $\|\mu\|$).

Proof that $\alpha \leqslant \beta$: Suppose $f \in \mathscr{K}(\mathrm{X};\mathbf{R})$, $0 \leqslant f \leqslant 1$. If $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and $|g| \leqslant f$, then $\|g\| \leqslant \|f\| \leqslant 1$, hence $|\mu(g)| \leqslant \beta$; varying g—and then f—yields $\alpha \leqslant \beta$.

Proof that $\beta \leqslant \alpha$: Suppose $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$, $\|g\| \leqslant 1$; we are to show that $|\mu(g)| \leqslant \alpha$. Say $g \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$. Choose $f \in \mathscr{K}_+(\mathrm{X})$ with $0 \leqslant f \leqslant 1$ and $f = 1$ on K. Evidently $|g| \leqslant f$, whence, citing (12), $|\mu(g)| \leqslant |\mu|(f) \leqslant \alpha$.

III.18, ℓ. 1–4.

"Corollary 3. — *For every real measure* μ *on a locally compact space* X,

$$\|\mu\| = \sup_{\|f\|\leqslant 1,\ f\in\mathscr{K}(\mathrm{X};\mathbf{R})} |\mu(f)|\,. \tag{24}$$

"It suffices to make use of the formula (22) and the expression (9) for $|\mu|(f)$ when μ is a real measure and $f \in \mathscr{K}_+(\mathrm{X})$."

Citing (22), then (9),

$$\|\mu\| = \sup_{f\in\mathscr{K}(\mathrm{X};\mathbf{R}),\,0\leqslant f\leqslant 1} |\mu|(f) = \sup_{f\in\mathscr{K}(\mathrm{X};\mathbf{R}),\,0\leqslant f\leqslant 1} \left(\sup_{g\in\mathscr{K}(\mathrm{X};\mathbf{R}),\,|g|\leqslant f} \mu(g) \right);$$

the g's involved on the right side of this equality are precisely the functions $g \in \mathscr{K}(\mathrm{X};\mathbf{R})$ such that $\|g\| \leqslant 1$, thus the right side may be rewritten as $\sup_{g\in\mathscr{K}(\mathrm{X};\mathbf{R}),\,\|g\|\leqslant 1} \mu(g)$, which equals the right side of (24) (where the absolute value signs are superfluous since $-f$ satisfies the same conditions as f).

III.18, ℓ. 7, 8.

"The canonical injection $\mathscr{M}^1(\mathrm{X},\mathbf{R}) \to \mathscr{M}^1(\mathrm{X};\mathbf{C})$ is an *isometry* by virtue of (24)."

Let μ be a bounded real measure on X. The left side of (24) is $\|\mu\|$ as calculated in $\mathscr{M}^1(\mathrm{X};\mathbf{C})$; the right side is the norm of μ regarded as an element of $\mathscr{M}^1(\mathrm{X},\mathbf{R})$.

III.18, ℓ. −3 to −1.

"Let X be a locally compact space. On the space $\mathscr{M}(\mathrm{X};\mathbf{C})$, one can consider the topology of *pointwise* convergence in $\mathscr{K}(\mathrm{X};\mathbf{C})$, which we shall call the *vague topology* on $\mathscr{M}(\mathrm{X};\mathbf{C})$."

Since $\mathscr{M}(\mathrm{X};\mathbf{C})$ is by definition the dual space of the locally convex space $\mathscr{K}(\mathrm{X};\mathbf{C})$ (equipped with the direct limit topology), the vague topology is the "weak topology" $\sigma\big(\mathscr{M}(\mathrm{X};\mathbf{C}),\mathscr{K}(\mathrm{X};\mathbf{C})\big)$ (TVS, II, §6, No. 2, Def. 2).

The space $\mathscr{M}(\mathrm{X};\mathbf{C})$, equipped with the vague topology, may also be described as the locally convex space $\mathscr{L}_{\mathfrak{S}}\big(\mathscr{K}(\mathrm{X};\mathbf{C});\mathbf{C}\big)$, where $\mathscr{K}(\mathrm{X};\mathbf{C})$ bears the direct limit topology and $\mathfrak{S}$ is the set of all 1-element subsets $\{f\}$ of $\mathscr{K}(\mathrm{X};\mathbf{C})$, or, equivalently, the set of all finite subsets of $\mathscr{K}(\mathrm{X};\mathbf{C})$ (TVS, III, §3, No. 1, p. TVS III.13, ℓ. −3 to −1; recall that every finite subset of a topological vector space is bounded).

III.19, ℓ. 3–6.

"To say that a filter $\mathfrak{F}$ on $\mathscr{M}(\mathrm{X};\mathbf{C})$ *converges vaguely* to a measure μ_0 signifies that

$$\mu_0(f) = \lim_{\mu,\,\mathfrak{F}} \mu(f)$$

for every function $f \in \mathscr{K}(\mathrm{X};\mathbf{R})$."

This is a property of initial topologies (GT, I, §7, No. 6, Prop. 10).

Incidentally, a filter $\mathfrak{F}$ on $\mathscr{M}(\mathrm{X};\mathbf{C})$ is vaguely Cauchy (i.e., Cauchy for the uniformity associated with the vague topology) if and only if for every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ (or every $f \in \mathscr{K}(\mathrm{X};\mathbf{R})$), the filter base

$$\mathfrak{F}(f) = \{\mathrm{V}(f) :\ \mathrm{V} \in \mathfrak{F}\}$$

(where $\mathrm{V}(f) = \{\mu(f) :\ \mu \in \mathrm{V}\}$) is Cauchy (i.e., convergent) in $\mathbf{C}$; this is a property of initial uniform structures (GT, II, §3, No. 1, Prop. 4).

III.19, ℓ. 6–7.

"For every function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, the mapping $\mu \mapsto \mu(f)$ is a *vaguely continuous* linear form on the space $\mathscr{M}(\mathrm{X};\mathbf{C})$."

And these are the *only* vaguely continuous linear forms on $\mathscr{M}(\mathrm{X};\mathbf{C})$ (TVS, II, §6, No. 2, Prop. 3).

III.20, ℓ. 13.

"It is clear that d) implies a)."

Suppose H satisfies *d*). Given any function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, we assert that the set $\mathrm{H}(f) = \{\mu(f) :\ \mu \in \mathrm{H}\}$ is bounded in $\mathbf{C}$; for, there exists a compact set K such that $f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$, and choosing $\mathrm{M_K}$ as in *d*), we have in particular $|\mu(f)| \leqslant \mathrm{M_K}\|f\|$ for all $\mu \in \mathrm{H}$.

Thus the implication *d*) $\Rightarrow$ *a*) is a consequence of the following proposition: a subset H of $\mathscr{M}(\mathrm{X};\mathbf{C})$ is vaguely bounded (i.e., bounded for the vague topology) if and only if, for every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, the set $\mathrm{H}(f) = \{\mu(f) :\ \mu \in \mathrm{H}\}$ is bounded in $\mathbf{C}$. The details are as follows.

Suppose H is vaguely bounded and let $f \in \mathrm{H}$. Since the set

$$\mathrm{V} = \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) :\ |\mu(f)| \leqslant 1\}$$

is a vague neighborhood of 0, there exists a scalar $a \neq 0$ such that $\mathrm{H} \subset a\mathrm{V}$ (TVS, III, §1, No. 2, remarks following Def. 3), thus the set $\{|\mu(f)| :\ \mu \in \mathrm{H}\}$ is bounded above by $|a|$.

Conversely, suppose H has the stated property. Given a vague neighborhood V of 0 in $\mathscr{M}(\mathrm{X};\mathbf{C})$, we seek a scalar $a \neq 0$ such that $\mathrm{H} \subset a\mathrm{V}$. We can suppose that

$$\mathrm{V} = \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) :\ |\mu(f_i)| \leqslant \varepsilon_i \ \text{ for } \ i = 1, \dots, n\}$$

where $f_i \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and $\varepsilon_i > 0$ $(i = 1, \ldots, n)$, since such sets V form a fundamental system of neighborhoods of 0 for the vague topology. By assumption, for each i, the set

$$\mathrm{S}_i = \{\mu(f_i) : \ \mu \in \mathrm{H}\}$$

is bounded in $\mathbf{C}$, hence so is the set $\mathrm{S} = \mathrm{S}_1 \cup \ldots \cup \mathrm{S}_n$, say $|\mu(f_i)| \leqslant M < +\infty$ for all $\mu \in \mathrm{H}$ and for $i = 1, \ldots, n$. If $a > 0$ is so chosen that $a^{-1}M \leqslant \varepsilon_i$ for $i = 1, \ldots, n$, then, for all $\mu \in \mathrm{H}$ and all $i = 1, \ldots, n$,

$$a^{-1}|\mu(f_i)| \leqslant a^{-1}M \leqslant \varepsilon_i$$

thus $a^{-1}\mu \in \mathrm{V}$ for all $\mu \in \mathrm{H}$, that is, $\mathrm{H} \subset a\mathrm{V}$.

III.20, ℓ. 13–15.

"Finally, if H is equicontinuous then the set of restrictions of the measures $\mu \in \mathrm{H}$ to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is also equicontinuous, ..."

The point is that the relative topology on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ induced by the direct limit topology of $\mathscr{K}(\mathrm{X};\mathbf{C})$ is equal to the norm topology on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$, by No. 1, Prop. 1, (i).

III.20, ℓ. 19, 20.

"Corollary 1. — *Let ν be a positive measure on* X; *the set of measures μ such that $|\mu| \leqslant \nu$ is vaguely compact.*"

Let $\mathrm{H} = \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ |\mu| \leqslant \nu\}$; it suffices to show that H is (i) vaguely relatively compact, and (ii) vaguely closed.

Proof of (i). We verify criterion *d*) of Prop. 15. Let K be a compact subset of X. Since ν is a measure, there exists a finite constant $\mathrm{M_K} \geqslant 0$ such that

$$(*) \qquad |\nu(f)| \leqslant \mathrm{M_K}\|f\| \ \text{ for all } f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}).$$

Then, for every $\mu \in \mathrm{H}$ and $f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ we have

$$|\mu(f)| \leqslant |\mu|(|f|) \leqslant \nu(|f|) \leqslant \mathrm{M_K} \cdot \| \, |f| \, \| = \mathrm{M_K}\|f\|$$

by (13), the definition of H, and the relation (*), thus the criterion *d*) is satisfied.

Proof of (ii). Let (μ_j) be a directed family in H with $\mu_j \to \mu$ in $\mathscr{M}(\mathrm{X};\mathbf{C})$ vaguely; we are to show that $\mu \in \mathrm{H}$. Thus, assuming $f \in \mathscr{K}_+(\mathrm{X})$, it is to be shown that $|\mu|(f) \leqslant \nu(f)$. Given any $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ such that $|g| \leqslant f$, it will suffice by (12) to show that $|\mu(g)| \leqslant \nu(f)$. For each index j, we have

$$|\mu_j(g)| \leqslant |\mu_j|(|g|) \leqslant \nu(|g|) \leqslant \nu(f)$$

by the relations (13), $\mu_j \in \mathrm{H}$ and $|g| \leqslant f$; since $\mu_j(g) \to \mu(g)$ by the definition of the vague topology, it follows that $|\mu(g)| \leqslant \nu(f)$.

III.20, ℓ. 21, 22.

"Corollary 2. — *The set of measures* μ *such that* $\|\mu\| \leqslant a$ (a any finite number > 0) *is vaguely compact.*"

Proof #1. We are looking at a closed ball B in the dual $\mathscr{M}^1(\mathrm{X};\mathbf{C})$ of a normed space (the space $\mathscr{K}(\mathrm{X};\mathbf{C})$ equipped with the topology of uniform convergence in X) equipped with the topology $\sigma\big(\mathscr{M}^1(\mathrm{X};\mathbf{C}),\mathscr{K}(\mathrm{X};\mathbf{C})\big)$, for which B is known to be compact (TVS, III, §3, No. 4, Cor. 3 of Prop. 4). Since $\sigma\big(\mathscr{M}^1(\mathrm{X};\mathbf{C}),\mathscr{K}(\mathrm{X};\mathbf{C})\big)$ is the topology on $\mathscr{M}^1(\mathrm{X};\mathbf{C})$ induced by the vague topology on $\mathscr{M}(\mathrm{X};\mathbf{C})$, B is also a vaguely compact subset of $\mathscr{M}(\mathrm{X};\mathbf{C})$.

Proof #2. Write $\mathrm{H} = \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \|\mu\| \leqslant a\}$ (thus $\mathrm{H} \subset \mathscr{M}^1(\mathrm{X};\mathbf{C})$). We verify compactness by showing that (i) H is vaguely relatively compact, and (ii) H is vaguely closed.

(i) For all $\mu \in \mathrm{H}$ and $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ we have (since $\mu \in \mathscr{M}^1(\mathrm{X};\mathbf{C})$)

$$|\mu(f)| \leqslant \|\mu\|\,\|f\| \leqslant a\|f\|;$$

thus $\mathrm{H}(f) = \{\mu(f) : \mu \in \mathrm{H}\}$ is a bounded set in $\mathbf{C}$ for every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, therefore H is vaguely bounded (see the notes above for ℓ. 13), hence is vaguely relatively compact by Prop. 15.

(ii) Suppose (μ_j) is a directed family in H with $\mu_j \to \mu \in \mathscr{M}(\mathrm{X};\mathbf{C})$ vaguely. For every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and every index j, we have

$$|\mu_j(f)| \leqslant \|\mu_j\|\,\|f\| \leqslant a\|f\|;$$

since $\mu_j(f) \to \mu(f)$, we infer that $|\mu(f)| \leqslant a\|f\|$, and since $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ is arbitrary, we conclude that μ is bounded and $\|\mu\| \leqslant a$, that is, $\mu \in \mathrm{H}$.

III.20, ℓ. −14, −13.

"Corollary 3. — *If* X *is compact, the set of positive measures* μ *on* X *such that* $\|\mu\| = 1$ *is vaguely compact.*"

Write $\mathrm{H} = \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \mu \geqslant 0 \text{ and } \|\mu\| = 1\}$ for the set of measures in question. By No. 8, Cor. 2 of Prop. 10,

$$\begin{aligned}\mathrm{H} &= \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \mu \geqslant 0 \text{ and } \mu(1) = 1\} \\ &= \mathscr{M}_+(\mathrm{X}) \cap \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \mu(1) = 1\}.\end{aligned}$$

The first factor of the intersection is vaguely closed as noted in Prop. 14, and the second factor, being the inverse image of $\{1\}$ under the vaguely continuous mapping $\mu \mapsto \mu(1)$, is also vaguely closed, thus H is vaguely closed; being a subset of the vaguely compact set $\{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \|\mu\| \leqslant 1\}$ (Cor. 2), H is also vaguely compact.

III.20, ℓ. $-9, -8$.

"COROLLARY 4. — *In the space* $\mathscr{M}(\mathrm{X};\mathbf{C})$, *the mapping* $\mu \mapsto \|\mu\|$ *is lower semi-continuous for the vague topology.*"

For $a \in \mathbf{R}$, the set $\{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \|\mu\| \leqslant a\}$ is vaguely compact if $a \geqslant 0$ (Cor. 2) and empty if $a < 0$, and in either case is vaguely closed, hence the mapping $\mathscr{M}(\mathrm{X};\mathbf{C}) \rightarrow \mathbf{R}$ defined by $\mu \mapsto \|\mu\|$ is vaguely lower semi-continuous (GT, IV, §6, No. 2, Prop. 1).

III.21, ℓ. -15.

"*Each of these topologies is coarser than the next.*"

Let

$$\mathfrak{S}_1 = \text{all singletons } \{f\},\ f \in \mathrm{T};$$
$$\mathfrak{S}_2 = \text{all singletons } \{f\},\ f \in \mathscr{K}(\mathrm{X};\mathbf{C});$$
$$\mathfrak{S}_3 = \text{all strictly compact subsets of } \mathscr{K}(\mathrm{X};\mathbf{C});$$
$$\mathfrak{S}_4 = \text{all compact subsets of } \mathscr{K}(\mathrm{X};\mathbf{C}).$$

The inclusions $\mathfrak{S}_1 \subset \mathfrak{S}_2$ and $\mathfrak{S}_3 \subset \mathfrak{S}_4$ are obvious, and $\mathfrak{S}_2 \subset \mathfrak{S}_3$ because each $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ belongs to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ for some compact set K, whence the strict compactness of $\{f\}$. Thus $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \mathfrak{S}_4$, whence clearly $\mathscr{T}_1 \subset \mathscr{T}_2 \subset \mathscr{T}_3 \subset \mathscr{T}_4$ (convergence with respect to $\mathscr{T}_{i+1}$ implies convergence with respect to $\mathscr{T}_i$ $(i = 1, 2, 3)$).

Note that every set in $\mathfrak{S}_4$ is bounded in $\mathscr{K}(\mathrm{X};\mathbf{C})$, therefore $\mathscr{T}_i$ is the $\mathfrak{S}_i$-topology on $\mathscr{M}(\mathrm{X};\mathbf{C})$ for every i (TVS, III, §3, No. 7).

III.21, ℓ. -11 to -9.

"A vaguely bounded subset H of $\mathscr{M}(\mathrm{X};\mathbf{C})$ is equicontinuous (No. 9, Prop. 15), thus the first assertion follows from TVS, III, §3, No. 7, Prop. 9, and the second from GT, X, §2, No. 4, Th. 1."

Proof of (i). Since $i < j \Rightarrow \mathscr{T}_i \subset \mathscr{T}_j$ $(1 \leqslant i, j \leqslant 4)$, it is clear that every set bounded for $\mathscr{T}_j$ is bounded for $\mathscr{T}_i$; thus, to prove that the bounded sets for $\mathscr{T}_2, \mathscr{T}_3, \mathscr{T}_4$ are the same, it suffices to show that every set H that is bounded for $\mathscr{T}_2$ is bounded for $\mathscr{T}_4$. Let H be a vaguely bounded set. Since H is equicontinuous (Prop. 15) it is bounded for every $\mathfrak{S}$-topology (TVS, III, §3, No. 7, Prop. 9) and in particular for $\mathfrak{S}_4$.

Proof of (ii). Let H be a vaguely bounded subset of $\mathscr{M}(\mathrm{X};\mathbf{C})$; we are to show that the topologies $\mathscr{T}_i \cap \mathrm{H}$ induced on H by $\mathscr{T}_i$ $(i = 1, 2, 3, 4)$ are identical. For notational purposes let us write $\mathscr{X} = \mathscr{K}(\mathrm{X};\mathbf{C})$ (equipped with the direct limit topology) and $\mathscr{Y} = \mathbf{C}$ (equipped with the usual uniformity). We know that

$$\mathrm{H} \subset \mathscr{M}(\mathrm{X};\mathbf{C}) \subset \mathscr{C}(\mathscr{X};\mathscr{Y});$$

since H is an equicontinuous subset of $\mathscr{C}(\mathscr{X};\mathscr{Y})$ by Prop. 15, it follows that on H, the topology of pointwise convergence in the total subset T of $\mathscr{X} = \mathscr{K}(\mathrm{X};\mathbf{C})$ (equivalently, in its linear span, which is dense) coincides with the topology of compact convergence (GT, X, §2, No. 4, Th. 1), that is, $\mathscr{T}_1 \cap \mathrm{H} = \mathscr{T}_4 \cap \mathrm{H}$, whence the coincidence of all four topologies.

III.21, ℓ. −8 to −6.

"Recall that when X is *paracompact*, the topology of strictly compact convergence coincides with the topology of compact convergence (No. 1, Prop. 2)."

Because, for such a space X, every compact subset of $\mathscr{K}(\mathrm{X};\mathbf{C})$, being bounded, is strictly compact, i.e., is a compact subset of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ for some compact set K in X (No. 1, Prop. 2, (ii)).

III.22, ℓ. 3–6.

"Since every filter is the intersection of the ultrafilters finer than it (GT, I, §6, No. 4, Prop. 7), it suffices to show that if $\mathfrak{U}$ is an ultrafilter on $\mathscr{M}_+(\mathrm{X})$ that converges to a measure μ_0 for the topology $\mathscr{T}_1$, then it also converges to μ_0 for $\mathscr{T}_3$."

Clearly $\mathscr{T}_1 \subset \mathscr{T}_2 \subset \mathscr{T}_3$, so we need only show that the topology on $\mathscr{M}_+(\mathrm{X})$ induced by $\mathscr{T}_1$ is finer than that induced by $\mathscr{T}_3$, i.e., that $\mathscr{T}_1 \cap \mathscr{M}_+(\mathrm{X}) \supset \mathscr{T}_3 \cap \mathscr{M}_+(\mathrm{X})$. Thus, given a filter $\mathfrak{G}$ on $\mathscr{M}_+(\mathrm{X})$ that is convergent to $\mu_0 \in \mathscr{M}_+(\mathrm{X})$ for $\mathscr{T}_1 \cap \mathscr{M}_+(\mathrm{X})$ (in other words, $\mathfrak{G}$ contains the filter of neighborhoods of μ_0 for $\mathscr{T}_1 \cap \mathscr{M}_+(\mathrm{X})$), we are to show that $\mathfrak{G} \to \mu_0$ for the topology $\mathscr{T}_3 \cap \mathscr{M}_+(\mathrm{X})$. If $\mathfrak{U}$ is any ultrafilter on $\mathscr{M}_+(\mathrm{X})$ such that $\mathfrak{U} \supset \mathfrak{G}$, obviously $\mathfrak{U} \to \mu_0$ for $\mathscr{T}_1 \cap \mathscr{M}_+(\mathrm{X})$; since $\mathfrak{G}$ is the intersection of all such $\mathfrak{U}$, it will suffice to show that $\mathfrak{U} \to \mu_0$ for $\mathscr{T}_3 \cap \mathscr{M}_+(\mathrm{X})$ (GT, I, §7, No. 1, Prop. 2).

III.22, ℓ. 6.

"Let K be a compact subset of X..."

We are to show that $\mathfrak{U} \to \mu_0$ uniformly on each compact subset of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$.

III.22, ℓ. 6, 7.

"...by hypothesis, there exists a function $h \in \mathrm{V}$ that is $\geqslant 0$ on X and takes values > 0 on K"

Namely, the hypothesis that V satisfies the condition (P) of No. 7, Prop. 9.

III.22, ℓ. 8, 9.

"...it follows that every function $f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ may be written $f = gh$ with $g \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$"

Let $\mathrm{U} = \{x \in \mathrm{X} : \ h(x) > 0\}$; in particular, $\mathrm{K} \subset \mathrm{U}$. Define $g : \mathrm{X} \to \mathbf{C}$ by the formulas

$$g(x) = \begin{cases} \dfrac{f(x)}{h(x)} & \text{for } x \in \mathrm{U} \\ 0 & \text{for } x \in \mathrm{X} - \mathrm{K}\,; \end{cases}$$

g is defined unambiguously, since on $\mathrm{U} \cap (\mathrm{X} - \mathrm{K}) \subset \mathrm{X} - \mathrm{K}$ one has also $f(x) = 0$—which also establishes the formula $f = gh$. Since U and $\mathrm{X} - \mathrm{K}$ are open sets with union X, on each of which g has a continuous restriction, it follows that g is continuous on X (GT, I, §3, No. 2, Prop. 4), whence clearly $g \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$.

III.22, ℓ. 9.

"… $c = \inf_{x \in \mathrm{K}} h(x) > 0$ …"

Since K is compact and $h > 0$ on K, $c > 0$ by the theorem of Weierstrass (GT, IV, §6, No. 1, Th. 1).

III.22, ℓ. 9–11.

"By hypothesis, there exists a set $\mathrm{H}_0 \in \mathfrak{U}$ such that, for every measure $\mu \in \mathrm{H}_0$,

$$0 \leqslant \mu(h) \leqslant \mu_0(h) + 1 = b\,."$$

Recall that $h \in \mathrm{V}$ and that $\mathfrak{U} \to \mu_0$ pointwise in V.

III.22, ℓ. 15, 16.

"If $\mathfrak{U}_0$ is the ultrafilter induced by $\mathfrak{U}$ on H_0"

Namely $\mathfrak{U}_0 = \mathfrak{U}_{\mathrm{H}_0} = \{\mathrm{H} \cap \mathrm{H}_0 : \ \mathrm{H} \in \mathfrak{U}\}$ (GT, I, §6, No. 5, Prop. 9).

III.22, ℓ. 16, 17.

"…the image of $\mathfrak{U}_0$ under the mapping $\mu \mapsto h \cdot \mu$ is the base of an ultrafilter $\mathfrak{F}$ on H"

See GT, I, §6, No. 6, Prop. 10.

III.22, ℓ. 17, 18.

"…and since H is relatively compact for the topology of strictly compact convergence (Prop. 17 and No. 9, Prop. 15)"

Being vaguely bounded, H is vaguely relatively compact by Prop. 15. Thus, if $\overline{\mathrm{H}}$ is the vague closure of H in $\mathscr{M}(\mathrm{X}; \mathbf{C})$ (since $\mathscr{M}_+(\mathrm{X})$ is vaguely closed in $\mathscr{M}(\mathrm{X}; \mathbf{C})$ by Prop. 14, $\overline{\mathrm{H}}$ is also the closure of H in $\mathscr{M}_+(\mathrm{X})$ for the topology induced by the vague topology), we know that $\overline{\mathrm{H}}$ is vaguely compact, hence also vaguely bounded. Therefore by Prop. 17, (ii), the topology on $\overline{\mathrm{H}}$ induced by the topology of strictly compact convergence coincides with the topology induced by the vague topology; since the latter is compact, so is the former, i.e., $\overline{\mathrm{H}}$ is compact for the topology induced by $\mathscr{T}_3$, so its subset H is relatively compact for that topology.

III.22, ℓ. 18, 19.

"… $\mathfrak{F}$ is convergent to a measure ν_0 for this topology."

As in the preceding note, let $\overline{\mathrm{H}}$ be the vague closure of H in $\mathscr{M}(\mathrm{X};\mathbf{C})$, which we know to be compact for the topology induced by $\mathscr{T}_3$; in what follows, it is understood that $\overline{\mathrm{H}}$ bears this topology.

The filter $\mathfrak{F}$ on H is a filter base on $\overline{\mathrm{H}}$; by the compactness of $\overline{\mathrm{H}}$, $\mathfrak{F}$ admits a cluster point $\nu_0 \in \overline{\mathrm{H}}$ (GT, I, §9, No. 3, Prop. 7), hence there exists a filter $\mathfrak{G}$ on $\overline{\mathrm{H}}$ such that $\mathfrak{G} \supset \mathfrak{F}$ and $\mathfrak{G} \to \nu_0$ (GT, I, §7, No. 2, Prop. 4).

On the way to proving that $\mathfrak{F} \to \nu_0$ in $\overline{\mathrm{H}}$, we observe that if $\mathrm{G} \in \mathfrak{G}$ and $\mathrm{F} \in \mathfrak{F}$ then $\mathrm{G} \cap \mathrm{F} \in \mathfrak{F}$; for, for every $\mathrm{F}' \in \mathfrak{F}$ the intersection $(\mathrm{G}\cap\mathrm{F})\cap\mathrm{F}'$ is nonempty (it is an element of $\mathfrak{G}$), thus $\mathrm{G}\cap\mathrm{F}$ is a subset of H that intersects every set in the ultrafilter $\mathfrak{F}$, whence $\mathrm{G}\cap\mathrm{F} \in \mathfrak{F}$ (because $\{\mathrm{G}\cap\mathrm{F}\} \cup \mathfrak{F}$ is a base of a filter on H that, by the maximality of $\mathfrak{F}$, must coincide with $\mathfrak{F}$).

Now let U be any neighborhood of ν_0 in $\overline{\mathrm{H}}$; we are to show that U contains some set in $\mathfrak{F}$. Since $\mathfrak{G} \to \nu_0$, there exists a set $\mathrm{G} \in \mathfrak{G}$ such that $\mathrm{G} \subset \mathrm{U}$; choose any set $\mathrm{F} \in \mathfrak{F}$; then $\mathrm{G}\cap\mathrm{F} \in \mathfrak{F}$ by the foregoing, and $\mathrm{G}\cap\mathrm{F} \subset \mathrm{G} \subset \mathrm{U}$.

III.22, ℓ. 19–22.

"In other words, for any $\varepsilon > 0$ and any compact subset L of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$, there exists a subset N of H_0 belonging to $\mathfrak{U}$ such that, for every function $g \in \mathrm{L}$ and every pair of measures μ, μ' belonging to N, one has $|\langle g, h\cdot\mu\rangle - \langle g, h\cdot\mu'\rangle| \leqslant \varepsilon$"

With notations and assumptions as in the preceding Note, the sets $\mathrm{N} \in \mathfrak{U}$ such that $\mathrm{N} \subset \mathrm{H}_0$ are precisely the sets of $\mathfrak{U}_0$, thus the sets $h\cdot\mathrm{N} = \{h\cdot\mu :\ \mu \in \mathrm{N}\}$ form the base of a filter $\mathfrak{F}$ on H, and $\mathfrak{F}$, regarded as the base of a filter on $\overline{\mathrm{H}}$, is convergent (to ν_0) for the topology on $\overline{\mathrm{H}}$ induced by $\mathscr{T}_3$. The set $\mathscr{V} = \{(\nu,\nu') \in \overline{\mathrm{H}} \times \overline{\mathrm{H}} :\ |\nu(g) - \nu'(g)| \leqslant \varepsilon$ for all $g \in \mathrm{L}\}$ is an entourage for the uniformity associated with that topology; since $\mathfrak{F}$ is Cauchy for this uniformity, there exists a set $h\cdot\mathrm{N} \in \mathfrak{F}$ such that $h\cdot\mathrm{N} \times h\cdot\mathrm{N} \subset \mathscr{V}$, whence the assertion.

III.22, ℓ. 24, 25.

"Now, we saw above that the mapping $g \mapsto gh$ is an *automorphism* of the Banach space $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$."

From $\|gh\| \leqslant \|g\|\,\|h\|$ we see that the (linear) mapping is continuous; the $f = gh$ factorization property proved above shows that it is surjective, and the relation $\|g\| \leqslant c^{-1}\|f\|$ shows that it is injective, with continuous inverse mapping.

III.22, ℓ. 25, 26.

"We have thus shown that $\mathfrak{U}$ is a *Cauchy filter* on $\mathscr{M}_+(\mathrm{X})$ for the topology of strictly compact convergence."

Given any $\varepsilon > 0$ and any compact subset L' of $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$. By the preceding remark, there exists a compact subset L of $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$ such that $g \cdot \mathrm{L} = \mathrm{L}'$. Then choose N as in ℓ. 19–23; the inequality there shows that $|\langle f, \mu\rangle - \langle f, \mu'\rangle| \leqslant \varepsilon$ for all $f \in \mathrm{L}' = g \cdot \mathrm{L}$ and all $\mu, \mu' \in \mathrm{N}$, which is the desired Cauchy condition.

III.22, ℓ. 26, 27.

"*A fortiori*, it is a Cauchy filter for vague convergence"

Because $\mathscr{T}_3 \supset \mathscr{T}_2$.

III.22, ℓ. $-5, -4$.

"... moreover, since V is dense in $\mathscr{K}(\mathrm{X}; \mathbf{C})$, the hypothesis implies that $\mu_1 = \mu_0$"

For each $f \in \mathrm{V}$, we have $\mathfrak{U}(f) \to \mu_0(f)$ in $\mathbf{C}$ because $\mathfrak{U} \to \mu_0$ for $\mathscr{T}_1$; but $\mathfrak{U}(f) \to \mu_1(f)$ in $\mathbf{C}$ because $\mathfrak{U} \to \mu_1$ vaguely, whence $\mu_0(f) = \mu_1(f)$. Thus $\mu_0 = \mu_1$ on V; since V is dense in $\mathscr{K}(\mathrm{X}; \mathbf{C})$, $\mu_0 = \mu_1$ as claimed.

III.23, ℓ. 1–3.

"Corollary. — *If* X *is paracompact then the topologies induced on* $\mathscr{M}_+(\mathrm{X})$ *by the vague topology and the topology of compact convergence coincide.*"

Since X is paracompact, $\mathscr{T}_3 = \mathscr{T}_4$ (in the notation of Prop. 17), thus $\mathscr{T}_2 \subset \mathscr{T}_3 = \mathscr{T}_4$. But $\mathscr{T}_2 \cap \mathscr{M}_+(\mathrm{X}) = \mathscr{T}_3 \cap \mathscr{M}_+(\mathrm{X})$ by Prop. 18. Thus $\mathscr{T}_2 \cap \mathscr{M}_+(\mathrm{X}) = \mathscr{T}_4 \cap \mathscr{M}_+(\mathrm{X})$ as claimed.

§2. SUPPORT OF A MEASURE

III.23, ℓ. 11–13.

"...every continuous function with values in a topological vector space E, defined on Y and with compact support, may be extended by continuity to all of X, by giving it the value 0 on $\complement\mathrm{Y}$"

Suppose $g \in \mathscr{K}(\mathrm{Y}; \mathrm{E})$ and let f be the extension by 0 of g to X, that is,

$$f(x) = \begin{cases} g(x) & \text{for } x \in \mathrm{Y} \\ 0 & \text{for } x \in \mathrm{X} - \mathrm{Y}. \end{cases}$$

Say $g \in \mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E})$ where K is a compact subset of Y (hence also of X). Since $g = 0$ on $\mathrm{Y} - \mathrm{K}$, we have also

$$f(x) = \begin{cases} g(x) & \text{for } x \in \mathrm{Y} \\ 0 & \text{for } x \in \mathrm{X} - \mathrm{K}, \end{cases}$$

where Y and $\mathrm{X} - \mathrm{K}$ form an open covering of X; since the restrictions of f to Y and to $\mathrm{X} - \mathrm{K}$ are continuous, f is continuous on X (GT, I, §3, No. 2, Prop. 4) as claimed, and $f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) \subset \mathscr{K}(\mathrm{X};\mathrm{E})$.

III.23, ℓ. 14–16.

"...one can therefore identify in this way the space $\mathscr{K}(\mathrm{Y};\mathrm{E})$ with the linear subspace of $\mathscr{K}(\mathrm{X};\mathrm{E})$ formed by the continuous functions with compact support *contained in* Y.

CAUTION: If $f \in \mathscr{K}(\mathrm{X};\mathrm{E})$ and $f = 0$ on $\mathrm{X} - \mathrm{Y}$, it does not follow that f belongs to the subspace $\mathscr{K}(\mathrm{Y};\mathrm{E})$ just constructed, i.e., it does not follow that $\mathrm{Supp}\, f \subset \mathrm{Y}$. For example, if $f \in \mathscr{K}(\mathrm{X};\mathrm{E})$ and if the (open) set $\mathrm{Y} = \{x \in \mathrm{X} : f(x) \neq 0\}$ is not closed, then $f \notin \mathscr{K}(\mathrm{Y};\mathrm{E})$; for, the support of f is $\overline{\mathrm{Y}}$, and $\overline{\mathrm{Y}} \not\subset \mathrm{Y}$.

III.23, ℓ. 16–19.

"If μ is a measure on X, it is clear that the restriction of μ to $\mathscr{K}(\mathrm{Y};\mathbf{C})$ is a measure on Y, which is called the *restriction* of μ to the open subspace Y, or the measure *induced* on Y by μ, and is denoted $\mu|\mathrm{Y}$."

Proof #1: If K is a compact subset of Y, then K is also compact in X and the restriction of μ to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is known to be continuous for the topology of uniform convergence in X—in other words in K—or in Y, it comes to the same thing since $\mathrm{K} \subset \mathrm{Y} \subset \mathrm{X}$ and the functions in question are 0 outside of K.

Proof #2: Applying Th. 2 of §1, No. 5 to the real and imaginary parts of μ, we see that μ is a linear combination of four positive measures on X, therefore $\mu\big|\mathscr{K}(\mathrm{Y};\mathbf{C})$ is a linear combination of four linear forms whose restrictions to $\mathscr{K}(\mathrm{Y};\mathbf{R})$ are positive linear forms and so define real measures on Y (*loc. cit.*, Th. 1).

The question naturally arises, when E is locally convex does the topology τ on $\mathscr{K}(\mathrm{Y};\mathrm{E})$ induced by the direct limit topology τ_{X} on $\mathscr{K}(\mathrm{X};\mathrm{E})$ coincide with the direct limit topology τ_{Y} on $\mathscr{K}(\mathrm{Y};\mathrm{E})$? (Probably the answer is known, but it is not known to me.)

At any rate, it is easy to see that $\tau_{\mathrm{Y}} \supset \tau$. For, let

$$i : \mathscr{K}(\mathrm{Y};\mathrm{E}) \to \big(\mathscr{K}(\mathrm{X};\mathrm{E}),\, \tau_{\mathrm{X}}\big)$$

be the mapping $i(g) = g'$, where g' is the extension by 0 of g to X; i is the insertion mapping for identifying $\mathscr{K}(\mathrm{Y};\mathrm{E})$ as a linear subspace of $\mathscr{K}(\mathrm{X};\mathrm{E})$, and τ is the initial topology for i. For any compact subset K of Y, consider the insertion mapping

$$i_{\mathrm{K}} : \mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E}) \to \bigl(\mathscr{K}(\mathrm{Y};\mathrm{E}),\, \tau_{\mathrm{Y}}\bigr),$$

where $\mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E})$ bears the topology of uniform convergence in Y (equivalently in K)—which is also the topology on $\mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E})$ induced by τ_{Y} (§1, No. 1, Prop. 1, (i)). The composite mapping $i \circ i_{\mathrm{K}} : \mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E}) \to \bigl(\mathscr{K}(\mathrm{X};\mathrm{E}),\, \tau_{\mathrm{X}}\bigr)$,

$$\mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E}) \to \bigl(\mathscr{K}(\mathrm{Y};\mathrm{E}),\, \tau_{\mathrm{Y}}\bigr) \to \bigl(\mathscr{K}(\mathrm{X};\mathrm{E}),\, \tau_{\mathrm{X}}\bigr),$$

is continuous for the indicated topologies on $\mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E})$ and $\mathscr{K}(\mathrm{X};\mathrm{E})$: if $g_j, g \in \mathscr{K}(\mathrm{Y},\mathrm{K};\mathrm{E})$ with $g_j \to g$ uniformly in K (equivalently in Y) then $g'_j \to g'$ uniformly in X (equivalently in K), hence $g'_j \to g'$ in $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ and therefore in $\bigl(\mathscr{K}(\mathrm{X};\mathrm{E}),\, \tau_{\mathrm{X}}\bigr)$, whence the asserted continuity. Since this is true for every compact set $\mathrm{K} \subset \mathrm{Y}$, i is continuous because τ_{Y} is a direct limit topology, i.e., it is the locally convex final topology for the family of mappings i_{K} (TVS, II, §4, No. 4, Example II). But τ is the coarsest topology on $\mathscr{K}(\mathrm{Y};\mathrm{E})$ rendering i continuous, therefore $\tau_{\mathrm{Y}} \supset \tau$.

This inclusion is sufficient for proving that if $\mu : \mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathrm{F}$ is a continuous mapping of $\mathscr{K}(\mathrm{X};\mathrm{E})$ into a topological space F, then the restriction $\mu\big|\mathscr{K}(\mathrm{Y};\mathrm{E}) = \mu \circ i$,

$$\mathscr{K}(\mathrm{Y};\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathrm{F},$$

is continuous; for, it is certainly continuous when $\mathscr{K}(\mathrm{Y};\mathrm{E})$ bears the initial topology τ for i, hence *a fortiori* when $\mathscr{K}(\mathrm{Y};\mathrm{E})$ bears the (finer) direct limit topology τ_{Y}.

In particular, if μ is a continuous linear mapping of $\mathscr{K}(\mathrm{X};\mathrm{E})$ into a topological vector space F, then its restriction to $\mathscr{K}(\mathrm{Y};\mathrm{E})$ is also continuous. The special case $\mathrm{E} = \mathrm{F} = \mathbf{C}$ yields a third proof of the initial assertion about measures.

III.23, ℓ. 19, 20.

"The restrictions to Y of $|\mu|$, $\mathscr{R}\mu$ and $\mathscr{I}\mu$ are, respectively, $\big|\mu|\mathrm{Y}\big|$, $\mathscr{R}(\mu|\mathrm{Y})$ and $\mathscr{I}(\mu|\mathrm{Y})$, by virtue of §1, Nos. 5 and 6."

Write $\nu = \mu\big|\mathrm{Y}$. As in §1, No. 5, let $\overline{\mu}$ be the conjugate of μ, that is, $\overline{\mu}(f) = \overline{\mu(\overline{f})}$ for all $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$. In particular, $\overline{\mu}(g) = \overline{\mu(\overline{g})}$ for all $g \in \mathscr{K}(\mathrm{Y};\mathbf{C})$, whence it is clear that $\overline{\mu}\big|\mathrm{Y} = \overline{\nu}$. From

$$\mathscr{R}(\mu) = \frac{1}{2}(\mu + \overline{\mu}), \quad \mathscr{I}(\mu) = \frac{1}{2i}(\mu - \overline{\mu}),$$

restriction to $\mathscr{K}(\mathrm{Y};\mathbf{C})$ yields

$$\mathscr{R}(\mu)\big|\mathrm{Y} = \frac{1}{2}(\nu + \overline{\nu}) = \mathscr{R}(\nu)\,, \quad \mathscr{I}(\mu)\big|\mathrm{Y} = \frac{1}{2i}(\nu - \overline{\nu}) = \mathscr{I}(\nu)\,.$$

If $f \in \mathscr{K}_+(\mathrm{X})$ then by definition (§1, No. 6, formula (12))

$$(*) \qquad |\mu|(f) = \sup_{|g| \leqslant f,\, g \in \mathscr{K}(\mathrm{X};\mathbf{C})} |\mu(g)|\,.$$

If in particular $f \in \mathscr{K}_+(\mathrm{Y})$ then $|g| \leqslant f$ implies that $\mathrm{Supp}\, g \subset \mathrm{Supp}\, f \subset \mathrm{Y}$, thus the right side of (*) may be written

$$\sup_{|g| \leqslant f,\, g \in \mathscr{K}(\mathrm{Y};\mathbf{C})} |\nu(g)|\,,$$

which is the formula for $|\nu|(f)$; thus it follows from (*) that $|\mu|\big|\mathrm{Y} = |\nu|$ on $\mathscr{K}_+(\mathrm{Y})$, hence on $\mathscr{K}(\mathrm{Y};\mathbf{C})$.

III.23, ℓ. 20–22.

"If μ is real then the restrictions of μ^+ and μ^- to Y are, respectively, $(\mu|\mathrm{Y})^+$ and $(\mu|\mathrm{Y})^-$, by virtue of formula (8) of § 1, No. 5."

From the formulas (INT II.2)

$$\mu^+ = \frac{1}{2}(|\mu| + \mu)\,, \quad \mu^- = \frac{1}{2}(|\mu| - \mu)\,,$$

restriction to $\mathscr{K}(\mathrm{Y};\mathbf{C})$ yields (with $\nu = \mu\big|\mathrm{Y}$ as in the preceding Note)

$$\mu^+\big|\mathrm{Y} = \frac{1}{2}(|\nu| + \nu) = \nu^+\,, \quad \mu^-\big|\mathrm{Y} = \frac{1}{2}(|\nu| - \nu) = \nu^-\,.$$

More generally, if μ_1, μ_2 are real measures on X, and if $\nu_1 = \mu_1\big|\mathrm{Y}$, $\nu_2 = \mu_2\big|\mathrm{Y}$, then the formulas (INT II.2)

$$\sup(\mu_1, \mu_2) = \frac{1}{2}(\mu_1 + \mu_2 + |\mu_1 - \mu_2|)$$
$$\inf(\mu_1, \mu_2) = \frac{1}{2}(\mu_1 + \mu_2 - |\mu_1 - \mu_2|)$$

yield, on restriction to $\mathscr{K}(\mathrm{Y};\mathbf{C})$, the formulas

$$\sup(\mu_1, \mu_2)\big|\mathrm{Y} = \frac{1}{2}(\nu_1 + \nu_2 + |\nu_1 - \nu_2|) = \sup(\nu_1, \nu_2)$$
$$\inf(\mu_1, \mu_2)\big|\mathrm{Y} = \frac{1}{2}(\nu_1 + \nu_2 - |\nu_1 - \nu_2|) = \inf(\nu_1, \nu_2)\,.$$

III.23, ℓ. 23–26.

"One sees immediately that if Y and Z are two open sets in X such that $\mathrm{Y} \supset \mathrm{Z}$, and if $\mu|\mathrm{Y}$ and $\mu|\mathrm{Z}$ are the restrictions of μ to Y and Z, then $\mu|\mathrm{Z}$ is also the restriction of $\mu|\mathrm{Y}$ to the open subspace Z of the locally compact space Y."

If $h \in \mathscr{K}(\mathrm{Z};\mathbf{C})$ and $h^* \in \mathscr{K}(\mathrm{Y},\mathrm{Z};\mathbf{C})$ is the extension by 0 of h to Y, and if $h^{*\prime} \in \mathscr{K}(\mathrm{X},\mathrm{Y};\mathbf{C})$ is the extension by 0 of h^* to X, then in fact $h^{*\prime} \in \mathscr{K}(\mathrm{X},\mathrm{Z};\mathbf{C})$ is the extension by 0 of h to X, and so

$$(\mu|\mathrm{Z})(h) = \mu(h^{*\prime}) = (\mu|\mathrm{Y})(h^*) = \big((\mu|\mathrm{Y})|\mathrm{Z}\big)(h),$$

whence the assertion.

Writing $\mu_{\mathrm{Y}} = \mu|\mathrm{Y}$ (as in Ch. IV, §5, No. 7, Def. 4), the above assertion is that $(\mu_{\mathrm{Y}})_{\mathrm{Z}} = \mu_{\mathrm{Z}}$ when Y,Z are open sets in X with $\mathrm{Y} \supset \mathrm{Z}$. It follows that if Y,Z are arbitrary open sets in X, then

$$(\mu_{\mathrm{Y}})_{\mathrm{Y}\cap\mathrm{Z}} = \mu_{\mathrm{Y}\cap\mathrm{Z}} = (\mu_{\mathrm{Z}})_{\mathrm{Y}\cap\mathrm{Z}}.$$

III.23, ℓ. −2, −1, **III.24,** ℓ. 1.

"...the mapping

$$f \mapsto \int_0^1 \frac{f(x)}{x}\,dx$$

is a measure on Y"

If K is a nonempty compact subset of the topological subspace $\mathrm{Y} =]0,1[$ of $\mathbf{R}$ then K has a smallest element a and a largest b (apply the theorem of Weierstrass to the canonical injection of K into $\mathbf{R}$), thus $0 < a \leqslant x \leqslant b < 1$ and $1 < 1/b \leqslant 1/x \leqslant 1/a$ for all $x \in \mathrm{K}$. For every $f \in \mathscr{K}(\mathrm{Y},\mathrm{K};\mathbf{C})$, the function $x \mapsto f(x)/x$ $(x \in \mathrm{Y})$ is equal to 0 outside $[a,b]$, continuous and bounded, hence integrable on $]0,1[$ in the sense of FRV; writing

$$\nu(f) = \int_0^1 \frac{f(x)}{x}\,dx = \int_a^b \frac{f(x)}{x}\,dx$$

and $\mathrm{M} = \dfrac{b-a}{a}$, we have

$$|\nu(f)| \leqslant \int_a^b |f(x)|\cdot\frac{1}{x}\,dx \leqslant \|f\|\cdot\frac{1}{a}\cdot(b-a) = \mathrm{M}\|f\|$$

for all $f \in \mathscr{K}(\mathrm{Y},\mathrm{K};\mathbf{C})$. Thus $f \mapsto \nu(f)$, obviously a linear form on $\mathscr{K}(\mathrm{Y};\mathbf{C})$, is a measure on Y (§1, No. 3, (4)).

Note that since Y is the union of a sequence of compact intervals, it is trivially ν-moderated, that is, ν is a moderated measure (Ch. V, §1, No. 2, Def. 2), hence the concepts "ν-integrable" and "essentially ν-integrable" coincide (*loc. cit.*, No. 3, Cor. of Prop. 9).

III.24, ℓ. 2–4.

"However, this measure cannot be extended to a measure on $\mathbf{R}$ because, in the contrary case, its restriction to the set of functions $f \in \mathscr{K}(\mathrm{Y};\mathbf{C})$ such that $\|f\| \leqslant 1$ would be bounded"

To put the matter positively, if a measure ρ on the topological subspace $\mathrm{Y} =]0,1[$ of $\mathbf{R}$ can be extended to a measure μ on $\mathbf{R}$, then ρ is bounded. For, as noted above, Y is ρ-moderated and the concepts "ρ-integrable" and "essentially ρ-integrable" coincide. Since $[0,1]$ is compact, $[0,1] \cap \mathrm{Y} = \mathrm{Y}$ is ρ-integrable by Ch. V, §4, No. 2, Prop. 3) (specifically, the implication $d) \Rightarrow b)$), therefore ρ is bounded (Ch. IV, §4, No. 7, Prop. 12). See the next remark for the rest of the argument.

III.24, ℓ. 4.

"…but this is false."

In view of the preceding remark, to show that ν cannot be extended to a measure on $\mathbf{R}$ we need only observe that ν is not a bounded measure. To this end, for $n = 2,3,4,\ldots$ define $f_n \in \mathscr{K}(\mathrm{Y})$ to be the *piecewise linear* continuous function for which

$$f_n(x) = \begin{cases} 0 & \text{on }]0, \frac{1}{2n}] \\ 1 & \text{on } [\frac{1}{n}, 1 - \frac{1}{n}] \\ 0 & \text{on } [1 - \frac{1}{2n}, 1[\,. \end{cases}$$

Then the sequence

$$\nu(f_n) = \int_0^1 \frac{f_n(x)}{x}\,dx \geqslant \int_{1/n}^{1-1/n} \frac{1}{x}\,dx$$

is unbounded, whereas $\|f_n\| = 1$ for all n.

III.25, ℓ. 1.

"whence…"

In the preceding display (III.24, ℓ. -1) all terms are in $\mathscr{K}(\mathrm{Y}_{\alpha_i};\mathbf{C})$, hence one can apply μ_{α_i} then sum over i.

III.25, ℓ. 8–10.

"…the conclusion therefore follows at once from the definition of μ and from Prop. 6 of §1, No. 3."

Let $x \in \mathrm{X}$. By assumption there exists an index $\alpha_x \in \mathrm{A}$ such that $x \in \mathrm{Y}_{\alpha_x}$. Let K_x be a compact neighborhood of x such that $\mathrm{K}_x \subset \mathrm{Y}_{\alpha_x}$. Varying x, we obtain a family $(\mathrm{K}_x)_{x\in\mathrm{X}}$ whose interiors are a covering of X, and such that $\mathrm{K}_x \subset \mathrm{Y}_{\alpha_x}$ for all $x \in \mathrm{X}$. Since $\mathrm{K}_x \subset \mathrm{Y}_{\alpha_x}$ and since the restriction $\mu\big|\mathscr{K}(\mathrm{Y}_{\alpha_x};\mathbf{C}) = \mu_{\alpha_x}$ is by hypothesis a measure on Y_{α_x}, there exists a constant M_x such that

$$(*) \qquad |\mu(f)| = |\mu_{\alpha_x}(f)| \leqslant \mathrm{M}_x\|f\| \quad \text{for all } f \in \mathscr{K}(\mathrm{Y}_{\alpha_x},\mathrm{K}_x;\mathbf{C})$$

(§1, No. 3, (4)); note that we are concerned here only with the norm topology on $\mathscr{K}(\mathrm{Y}_{\alpha_x},\mathrm{K}_x;\mathbf{C})$—the direct limit topology on $\mathscr{K}(\mathrm{Y}_{\alpha_x};\mathbf{C})$ that induces it is not at issue here. If $f \in \mathscr{K}(\mathrm{X},\mathrm{K}_x;\mathbf{C})$, then $\operatorname{Supp} f \subset \mathrm{K}_x \subset \mathrm{Y}_{\alpha_x}$, so that f can be regarded as an element of $\mathscr{K}(\mathrm{Y}_{\alpha_x},\mathrm{K}_x;\mathbf{C})$, and hence satisfies (*). Thus (*) may be written

$$(**) \qquad |\mu(f)| \leqslant \mathrm{M}_x\|f\| \quad \text{for all } f \in \mathscr{K}(\mathrm{X},\mathrm{K}_x;\mathbf{C})\,.$$

Summarizing, we have a family $(\mathrm{K}_x)_{x\in\mathrm{X}}$ of compact subsets of X whose interiors form a covering of X, μ is a linear form on $\mathscr{K}(\mathrm{X};\mathbf{C})$, and for each $x \in \mathrm{X}$ there exists a constant M_x satisfying (**), consequently μ is a measure on X by Prop. 6 of §1, No. 3.

III.25, ℓ. 11–14.

"Corollary (Principle of localization). — *Let* μ *and* ν *be two measures on* X, *and let* (Y_α) *be a family of open sets of* X *such that, for every* α, *the restrictions to* Y_α *of* μ *and* ν *are equal; then the restrictions of* μ *and* ν *to* $\mathrm{Y} = \bigcup_\alpha \mathrm{Y}_\alpha$ *are equal.*"

As in the note for III.23, ℓ. 23–26, let us write μ_{Y} for $\mu\big|\mathrm{Y}$. Then (*loc. cit.*) for every α,

$$(\mu_{\mathrm{Y}})_{\mathrm{Y}_\alpha} = \mu_{\mathrm{Y}_\alpha} = \nu_{\mathrm{Y}_\alpha} = (\nu_{\mathrm{Y}})_{\mathrm{Y}_\alpha}\,;$$

thus, dropping down from X to Y, we can suppose that $\mathrm{Y} = \mathrm{X}$.

For every index α write $\mu_\alpha = \mu_{\mathrm{Y}_\alpha}$ and $\nu_\alpha = \nu_{\mathrm{Y}_\alpha}$. Then (*loc. cit.*) for every pair (α,β) of indices, we have

$$(\mu_\alpha)_{\mathrm{Y}_\alpha\cap\mathrm{Y}_\beta} = (\mu_{\mathrm{Y}_\alpha})_{\mathrm{Y}_\alpha\cap\mathrm{Y}_\beta} = \mu_{\mathrm{Y}_\alpha\cap\mathrm{Y}_\beta} = (\mu_{\mathrm{Y}_\beta})_{\mathrm{Y}_\alpha\cap\mathrm{Y}_\beta} = (\mu_\beta)_{\mathrm{Y}_\alpha\cap\mathrm{Y}_\beta}\,;$$

it follows from Prop. 1 that μ is the unique measure on X such that $\mu\big|\mathrm{Y}_\alpha = \mu_\alpha$ for all α. Similarly, ν is the unique measure on X such that $\nu\big|\mathrm{Y}_\alpha = \nu_\alpha$ for all α. But

$$\mu_\alpha = \mu\big|\mathrm{Y}_\alpha = \nu\big|\mathrm{Y}_\alpha = \nu_\alpha$$

for all α, whence $\mu = \nu$ by uniqueness.

III.25, ℓ. -9 to -7.

"To say that a point $x \in \mathrm{X}$ does not belong to the support of μ means that there exists an open neighborhood V of x such that the restriction of μ to V is zero"

With the notations preceding Def. 1, if $x \notin \mathrm{Supp}(\mu)$, that is, if $x \in \mathrm{U}_0 = \complement\,\mathrm{Supp}(\mu)$, then $\mathrm{V} = \mathrm{U}_0$ meets the requirement. Conversely, if there exists an open neighborhood V of x such that $\mathrm{V} \in \mathfrak{G}$ then $x \in \mathrm{V} \subset \mathrm{U}_0 = \complement\,\mathrm{Supp}(\mu)$, so $x \notin \mathrm{Supp}(\mu)$.

III.25, ℓ. -7 to -5.

"to say that x belongs to the support of μ therefore means that for *every* neighborhood V of x, there exists a function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, whose support is contained in V, such that $\mu(f) \neq 0$."

This is essentially the contrapositive form of the preceding assertion; marching through the details requires a review of some delicate definitions.

Suppose $x \in \mathrm{Supp}(\mu)$ and let V be any neighborhood of x; let U be an open set such that $x \in \mathrm{U} \subset \mathrm{V}$. Since $x \notin \complement\,\mathrm{Supp}(\mu) = \mathrm{U}_0$ and $x \in \mathrm{U}$, necessarily $\mathrm{U} \not\subset \mathrm{U}_0$; thus $\mathrm{U} \notin \mathfrak{G}$, that is, $\mu_{\mathrm{U}} \neq 0$, hence there exists a function $g \in \mathscr{K}(\mathrm{U};\mathbf{C})$ such that $\mu_{\mathrm{U}}(g) \neq 0$; if $f = g'$ is the extension by 0 of g to X, then $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, $\mathrm{Supp}\, f = \mathrm{Supp}\, g \subset \mathrm{U} \subset \mathrm{V}$, and $\mu(f) = \mu_{\mathrm{U}}(g) \neq 0$.

Conversely, suppose that for each neighborhood V of x, there exists a function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ such that $\mathrm{Supp}\, f \subset \mathrm{V}$ and $\mu(f) \neq 0$; we assert that $x \in \mathrm{Supp}(\mu)$. Assuming to the contrary that $x \in \complement\,\mathrm{Supp}(\mu) = \mathrm{U}_0$, then U_0 is an open neighborhood of x with $\mu_{\mathrm{U}_0} = 0$, therefore $\mu_{\mathrm{U}_0}(g) = 0$ for all $g \in \mathscr{K}(\mathrm{U}_0;\mathbf{C})$. For every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ such that $\mathrm{Supp}\, f \subset \mathrm{U}_0$, the function $g = f\big|\mathrm{U}_0$ belongs to $\mathscr{K}(\mathrm{U}_0;\mathbf{C})$ and f is the extension by 0 of g to X, therefore $\mu(f) = \mu_{\mathrm{U}_0}(g) = 0$; thus the original assumption on the point x is contradicted by its neighborhood $\mathrm{V} = \mathrm{U}_0$.

III.26, ℓ. $11, 12$.

"For, if the restriction of μ to an open set U is zero, then so is that of $|\mu|$ (resp. of μ^+ and μ^- when μ is real), and conversely."

Let U be an open set in X. Since $|\mu_{\mathrm{U}}| = |\mu|_{\mathrm{U}}$ (No. 1) one has

$$\mu_{\mathrm{U}} = 0 \quad\Leftrightarrow\quad |\mu_{\mathrm{U}}| = 0 \quad\Leftrightarrow\quad |\mu|_{\mathrm{U}} = 0\,,$$

whence $\mathrm{Supp}(\mu) = \mathrm{Supp}(|\mu|)$. If, moreover, μ is real, then

$$|\mu_{\mathrm{U}}| = |\mu|_{\mathrm{U}} = (\mu^+ + \mu^-)_{\mathrm{U}} = (\mu^+)_{\mathrm{U}} + (\mu^-)_{\mathrm{U}}\,,$$

therefore, by posititivy,

$$\mu_{\mathrm{U}} = 0 \quad \Leftrightarrow \quad (\mu^+)_{\mathrm{U}} = 0 \;\&\; (\mu^-)_{\mathrm{U}} = 0\,,$$

thus

$$\begin{aligned} \mathrm{U} \subset \complement\,\mathrm{Supp}(\mu) \quad &\Leftrightarrow \quad \mathrm{U} \subset \complement\,\mathrm{Supp}(\mu^+) \cap \complement\,\mathrm{Supp}(\mu^-) \\ &\Leftrightarrow \quad \mathrm{U} \subset \complement\big(\mathrm{Supp}(\mu^+) \cup \mathrm{Supp}(\mu^-)\big); \end{aligned}$$

letting $\complement\,\mathrm{Supp}(\mu)$ play the role of U yields the inclusion

$$\complement\,\mathrm{Supp}(\mu) \subset \complement\big(\mathrm{Supp}(\mu^+) \cup \mathrm{Supp}(\mu^-)\big),$$

and letting $\complement\big(\mathrm{Supp}(\mu^+) \cup \mathrm{Supp}(\mu^-)\big)$ play the role of U yields the reverse inclusion, whence equality.

III.26, ℓ. −11.

"The proposition is obvious from the definitions."

The set $\mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}})$ is an open set in U (hence in X) such that

$$0 = (\mu_{\mathrm{U}})_{\mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}})} = \mu_{\mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}})}\,,$$

therefore $\mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}}) \subset \mathrm{X} - \mathrm{Supp}(\mu)$, whence

$$\mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}}) \subset \mathrm{U} - \mathrm{U} \cap \mathrm{Supp}(\mu)\,,$$

and so $\mathrm{Supp}(\mu_{\mathrm{U}}) \supset \mathrm{U} \cap \mathrm{Supp}(\mu)$.

On the other hand, let $\mathrm{V} = \mathrm{U} - \mathrm{Supp}(\mu) = \mathrm{U} \cap \big(\mathrm{X} - \mathrm{Supp}(\mu)\big) = \mathrm{U} - \mathrm{U} \cap \mathrm{Supp}(\mu)$, which is open in both X and U. Since $\mathrm{V} \subset \mathrm{X} - \mathrm{Supp}(\mu)$, we have

$$0 = \mu_{\mathrm{V}} = \mu_{\mathrm{U} \cap \mathrm{V}} = (\mu_{\mathrm{U}})_{\mathrm{V}}\,,$$

therefore $\mathrm{V} \subset \mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}})$, that is, $\mathrm{U} - \mathrm{U} \cap \mathrm{Supp}(\mu) \subset \mathrm{U} - \mathrm{Supp}(\mu_{\mathrm{U}})$, and so

$$\mathrm{U} \cap \mathrm{Supp}(\mu) \supset \mathrm{Supp}(\mu_{\mathrm{U}})\,,$$

whence equality.

III.26, ℓ. −7 to −5.

"For, it is the intersection of the vaguely closed hyperplanes with equation $\mu(f) = 0$, where f runs over the set of functions in $\mathscr{K}(\mathrm{X}; \mathbf{C})$ whose support does not intersect F."

For a measure $\mu \in \mathscr{M}(\mathrm{X};\mathbf{C})$,

$$\begin{aligned}\mathrm{Supp}(\mu) \subset \mathrm{F} \quad &\Leftrightarrow \quad \complement\mathrm{F} \subset \complement\,\mathrm{Supp}(\mu)\\ &\Leftrightarrow \quad \mu_{\complement \mathrm{F}} = 0\\ &\Leftrightarrow \quad \mu_{\complement \mathrm{F}}(g) = 0 \quad \text{for all } g \in \mathscr{K}(\complement\mathrm{F};\mathbf{C})\\ &\Leftrightarrow \quad \mu(f) = 0 \quad \text{for all } f \in \mathscr{K}(\mathrm{X},\complement\mathrm{F};\mathbf{C})\\ &\Leftrightarrow \quad \mu \in \bigcap_{f \in \mathscr{K}(\mathrm{X},\complement\mathrm{F};\mathbf{C})} \{\nu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \nu(f) = 0\};\end{aligned}$$

for each f, the set $\{\nu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \nu(f) = 0\}$ is the kernel of the vaguely continuous linear form $\nu \mapsto \nu(f)$.

III.26, ℓ. -4 to -1.

"Suppose X is *not compact*: given a filter Φ on the space $\mathscr{M}(\mathrm{X};\mathbf{C})$ of measures on X, we shall say that the support of a measure μ *recedes indefinitely along* Φ if, for every compact subset K of X, there exists a set $\mathrm{M} \in \Phi$ such that $\mathrm{Supp}(\mu) \cap \mathrm{K} = \varnothing$ for every measure $\mu \in \mathrm{M}$."

Some thoughts on the definition: Let $\mathrm{X}' = \mathrm{X} \cup \{\omega\}$ be the one-point compactification of X (GT, I, §9, No. 8), with X viewed as an open subspace of X'. The open sets in X' are the open sets U in X together with the sets $\{\omega\} \cup (\mathrm{X} - \mathrm{K})$ with K compact in X (the latter are the only open sets in X' that contain ω). Thus the open neighborhoods of ω in X' are the sets of the form $\mathrm{U}' = \{\omega\} \cup (\mathrm{X} - \mathrm{K})$ for some compact subset K of X.

Thus, the concept defined above says that given any open neighborhood V' of ω in X' (equivalently, given any compact subset K of X), there exists a set $\mathrm{M} \in \Phi$ such that $\mathrm{Supp}(\mu) \subset \mathrm{V}'$ (resp. $\mathrm{Supp}(\mu) \cap \mathrm{K} = \varnothing$) for all $\mu \in \mathrm{M}$. It is the same to say that for *every* neighborhood V' of ω in X', such an M exists. Thus, writing $\mathrm{Supp}(\mathrm{M}) = \bigcup_{\mu \in \mathrm{M}} \mathrm{Supp}(\mu)$, the concept says that for every neighborhood V' of ω in X', there exists a set $\mathrm{M} \in \Phi$ such that $\mathrm{Supp}(\mathrm{M}) \subset \mathrm{V}'$; this might be expressed suggestively as

$$\lim_{\mathrm{M} \in \Phi} \mathrm{Supp}(\mathrm{M}) = \omega$$

or that $\mathrm{Supp}(\mu) \to \omega$ with respect to the filter Φ, symbolically

$$\lim{}_{\mu,\Phi}\, \mathrm{Supp}(\mu) = \omega\,.$$

At any rate, it is a property of Φ (not of Supp).

III.27, ℓ. 6, 7.

"...which proves the proposition."

To say that μ converges vaguely to a measure μ_0 with respect to a filter Φ on $\mathscr{M}(\mathrm{X};\mathbf{C})$ means that each vague neighborhood of μ_0 in $\mathscr{M}(\mathrm{X};\mathbf{C})$ contains some set $\mathrm{M} \in \Phi$, that is, $\Phi \supset \mathscr{V}_0$, where $\mathscr{V}_0$ is the filter of neighborhoods of μ_0 in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for the vague topology. Concisely, $\Phi \to \mu_0$ vaguely.

An equivalent condition (see the note for III.19, ℓ. 3–6) is that for each $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, one has

$$\mu_0(f) = \lim{}_{\mu,\Phi}\, \mu(f)\,;$$

the meaning of this is that for each $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ one considers the linear form $\mathrm{L}_f : \mathscr{M}(\mathrm{X};\mathbf{C}) \to \mathbf{C}$ defined by $\mathrm{L}_f(\mu) = \mu(f)$, and that

$$\lim{}_{\mu,\Phi}\, \mathrm{L}_f(\mu) = \mathrm{L}_f(\mu_0)\,;$$

that is, the filter on $\mathbf{C}$ with base $\mathrm{L}_f(\Phi)$ converges to $\mathrm{L}_f(\mu_0)$ in $\mathbf{C}$; here, $\mathrm{L}_f(\Phi) = \{\mathrm{L}_f(\mathrm{M}) : \ \mathrm{M} \in \Phi\}$, where

$$\mathrm{L}_f(\mathrm{M}) = \{\mathrm{L}_f(\mu) : \ \mu \in \mathrm{L}\} = \{\mu(f) : \ \mu \in \mathrm{M}\},$$

concisely and suggestively denoted $\mathrm{M}(f)$.

Equivalently, for every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and every $\varepsilon > 0$ there exists an $\mathrm{M} \in \Phi$ such that $|\mu(f) - \mu_0(f)| < \varepsilon$ for all $\mu \in \mathrm{M}$.

In particular, Φ converges to the measure 0 if and only if, for every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and every $\varepsilon > 0$, there exists a set $\mathrm{M} \in \Phi$ such that $|\mu(f)| < \varepsilon$ for all $\mu \in \mathrm{M}$; concisely, $\Phi \to 0$ vaguely.

The assertion of Prop. 7 is that if Φ is a filter on $\mathscr{M}(\mathrm{X};\mathbf{C})$ such that "the support of μ recedes indefinitely along Φ" then $\Phi \to 0$ vaguely. To prove this, suppose $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and $\varepsilon > 0$ (ε will play no role in the argument). Let K be the support of f. By hypothesis, there exists an $\mathrm{M} \in \Phi$ such that $\mathrm{Supp}(\mu) \cap \mathrm{K} = \varnothing$ for every $\mu \in \mathrm{M}$. In view of the preceding paragraph, it will obviously suffice to show that $\mu(f) = 0$ for every $\mu \in \mathrm{M}$. Fix $\mu \in \mathrm{M}$. We know that $\mathrm{Supp}(\mu) \cap \mathrm{K} = \varnothing$, that is, $\mathrm{K} \subset \complement\,\mathrm{Supp}(\mu)$. Writing $\mathrm{U} = \complement\,\mathrm{Supp}(\mu)$, we have $\mu_{\mathrm{U}} = 0$, that is, $\mu = 0$ on $\mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$; since $\mathrm{Supp}\, f = \mathrm{K} \subset \mathrm{U}$, we have $f \in \mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$, whence $\mu(f) = 0$, which completes the proof.

III.27, ℓ. 9, 10.

"By definition, if the support of a function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ *does not intersect* the support of a measure μ, then $\mu(f) = 0$"

To say that $\mathrm{Supp}(f) \cap \mathrm{Supp}(\mu) = \varnothing$ is to say that $\mathrm{Supp}(f) \subset \complement\,\mathrm{Supp}(\mu)$. Writing $\mathrm{U} = \complement\,\mathrm{Supp}(\mu)$, we have $\mu_{\mathrm{U}} = 0$, that is, $\mu = 0$ on $\mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$ and in particular $\mu(f) = 0$.

The condition $\mathrm{Supp}(f) \cap \mathrm{Supp}(\mu) = \varnothing$ can also be written

$$\begin{aligned} \mathrm{Supp}(\mu) \subset \complement\, \mathrm{Supp}(f) &= \complement\, \overline{\{x :\ f(x) \neq 0\}} \\ &= \mathrm{int}\big(\complement\, \{x :\ f(x) \neq 0\}\big) \\ &= \mathrm{int}\,\{x :\ f(x) = 0\} \end{aligned}$$

(where "int" means "interior") thus the observation is that

$$(*) \qquad \mathrm{Supp}(\mu) \subset \mathrm{int}\,\{x :\ f(x) = 0\} \quad \Rightarrow \quad \mu(f) = 0\,.$$

III.27, ℓ. 10, 11.

"...but the following more precise result is true:"

Prop. 8 is the proposition that, for a function $f \in \mathscr{K}(\mathrm{X}; \mathbf{C})$,

$$(**) \qquad \mathrm{Supp}(\mu) \subset \{x :\ f(x) = 0\} \quad \Rightarrow \quad \mu(f) = 0\,;$$

it is stronger than (*), in that it yields the same conclusion with a weaker hypothesis. It is more precise in the sense that it enlarges the set of functions for which the conclusion is true.

III.27, ℓ. 15, 16.

"... V is an open set containing S by hypothesis"

For, $\mathrm{S} = \mathrm{Supp}(\mu) \subset \{x :\ f(x) = 0\} \subset \mathrm{V}$.

III.27, ℓ. 19.

"Since the support of fh does not intersect S..."

$\mathrm{Supp}(fh) \subset \mathrm{Supp}(h) \subset \complement\, \mathrm{S}$.

III.27, ℓ. 20, 21.

"...therefore $|f - fh| \leqslant 2\varepsilon$ on X"

On V, $|f - fh| \leqslant |f| + |fh| \leqslant |f| + |f| < 2\varepsilon$, whereas on $\complement\,\mathrm{V}$, $|f - fh| = |f - f \cdot 1| = 0$.

III.27, ℓ. −3.

"... $|f(x)| \leqslant ah(x)$ on $\mathrm{Supp}(\mu)$"

On $\mathrm{Supp}(f) \cap \mathrm{Supp}(\mu)$, one has $|f(x)| \leqslant a = ah(x)$, whereas on $\complement\,\mathrm{Supp}(f) \cap \mathrm{Supp}(\mu)$ one has $|f(x)| = 0 \leqslant ah(x)$.

III.27, ℓ. −3, −2.

"...therefore

$$|\mu|(|f|) \leqslant a|\mu|(h) \leqslant a\|\mu\| \text{"}$$

The measure $|\mu|$ is positive, and on $\mathrm{Supp}(|\mu|) = \mathrm{Supp}(\mu)$ one has $ah - |f| \geqslant 0$, therefore $|\mu|(ah - |f|) \geqslant 0$ by Cor. 2; thus

$$|\mu|(|f|) \leqslant |\mu|(ah) = a|\mu|(h) \leqslant a\,\|\,|\mu|\,\| \cdot \|h\| = a\,\|\mu\| \cdot \|h\| \leqslant a\,\|\mu\|\,.$$

III.28, ℓ. 7.

"...then $g \leqslant bf/a$, whence $\mu(g) \leqslant b\mu(f)/a = 0$."

If $y \in \mathrm{V}$ then $f(y) \geqslant a$, whence $g(y) \leqslant b = b \cdot 1 \leqslant b \cdot \dfrac{f(y)}{a}$; whereas if $y \in \complement\mathrm{V}$ then $g(y) = 0$. Therefore $0 \leqslant \mu(g) \leqslant b\mu(f)/a = 0$, and so $\mu(g) = 0$.

The argument shows that for every $g \in \mathscr{K}_+(\mathrm{X})$ with $\operatorname{Supp} g \subset \mathrm{V}$, one has $\mu(g) = 0$. If $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with $\operatorname{Supp} g \subset \mathrm{V}$, then $|g| \in \mathscr{K}_+(\mathrm{X})$ and $\operatorname{Supp}|g| = \operatorname{Supp} g \subset \mathrm{V}$, therefore $\mu(|g|) = 0$ by the foregoing, and so $|\mu(g)| \leqslant \mu(|g|) = 0$. Thus $\mu(g) = 0$ for all $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with $\operatorname{Supp} g \subset \mathrm{V}$. *A fortiori*, $\mu(g) = 0$ for all $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with $\operatorname{Supp} g \subset \overset{\circ}{\mathrm{V}}$, in other words $\mu = 0$ on $\mathscr{K}(\mathrm{X}, \overset{\circ}{\mathrm{V}};\mathbf{C})$, that is, $\overset{\circ}{\mathrm{V}} \subset \complement\operatorname{Supp}(\mu)$, and in particular $x \in \complement\operatorname{Supp}(\mu)$. Thus $x \notin \operatorname{Supp}(\mu)$, as we wished to show.

III.28, ℓ. 11-13.

"...there exists an open neighborhood V of x_0 such that at every point of $\mathrm{V} \cap \mathrm{S}$, g is zero"

Write $\mathrm{N}(g) = \{x : \ g(x) \neq 0\}$, $\mathrm{Z}(g) = \complement\mathrm{N}(g) = \{x : \ g(x) = 0\}$; in particular, $\mathrm{T} = \overline{\mathrm{N}(g) \cap \mathrm{S}}$, and the assertion of the proposition is that $\mathrm{T} = \operatorname{Supp}(g \cdot \mu)$. By assumption, x_0 is not in the closure of $\mathrm{N}(g) \cap \mathrm{S}$, hence there exists an open neighborhood V of x_0 such that $\mathrm{V} \cap \mathrm{N}(g) \cap \mathrm{S} = \varnothing$, whence $\mathrm{V} \cap \mathrm{S} \subset \complement\mathrm{N}(g) = \mathrm{Z}(g)$—in other words, $g = 0$ on $\mathrm{V} \cap \mathrm{S}$.

III.28, ℓ. 13.

"...then fg is zero on S"

We know that $g = 0$ on $\mathrm{V} \cap \mathrm{S}$, and that $f = 0$ on $\complement\mathrm{V}$ hence on $\complement\mathrm{V} \cap \mathrm{S}$, therefore $fg = 0$ on $(\mathrm{V} \cap \mathrm{S}) \cup (\complement\mathrm{V} \cap \mathrm{S}) = \mathrm{S}$.

III.28, ℓ. 14, 15.

"...in other words, the restriction of $g \cdot \mu$ to V is zero."

The argument shows that $\mu(fg) = 0$ for all $f \in \mathscr{K}(\mathrm{X}, \mathrm{V};\mathbf{C})$, in other words $g \cdot \mu = 0$ on $\mathscr{K}(\mathrm{X}, \mathrm{V};\mathbf{C})$, whence $\mathrm{V} \subset \complement\operatorname{Supp}(g \cdot \mu)$ and in particular $x_0 \in \complement\operatorname{Supp}(g \cdot \mu)$. We have shown that $\complement\mathrm{T} \subset \complement\operatorname{Supp}(g \cdot \mu)$, that is, $\mathrm{T} \supset \operatorname{Supp}(g \cdot \mu)$. That's half the battle.

III.28, ℓ. 16–18.

"Conversely, assuming that the restriction of $g \cdot \mu$ to an open neighborhood W of a point $x_0 \in \mathrm{X}$ is zero, let us show that there does not exist a point of $\mathrm{W} \cap \mathrm{S}$ at which g is $\neq 0$."

One is assuming that $x_0 \in \mathrm{W}$, where W is open and $\mathrm{W} \subset \complement\operatorname{Supp}(g \cdot \mu)$; to put it another way, we are assuming that $x_0 \notin \operatorname{Supp}(g \cdot \mu)$ (a closed set) and that W is any open neighborhood of x_0 disjoint from $\operatorname{Supp}(g \cdot \mu)$. If we show that $g = 0$ on $\mathrm{W} \cap \mathrm{S}$, that is, $\mathrm{W} \cap \mathrm{S} \subset \mathrm{Z}(g) = \complement\, \mathrm{N}(g)$, that is, $\mathrm{W} \cap \mathrm{S} \cap \mathrm{N}(g) = \varnothing$, we will have shown that x_0 is not in the closure of $\mathrm{S} \cap \mathrm{N}(g)$, in other words $x_0 \notin \mathrm{T}$. Varying x_0, we will then have shown that $\complement\, \mathrm{Supp}(g \cdot \mu) \subset \complement\, \mathrm{T}$, that is, $\mathrm{Supp}(g \cdot \mu) \supset \mathrm{T}$, whence equality in view of the preceding note.

III.28, ℓ. 20–22.

"…but then every function $f \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ with support contained in U could be written $f = gh$, where $h \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ has support contained in $\mathrm{U} \subset \mathrm{W}$"

For such an f, write $\mathrm{Z}(f) = \{x : f(x) = 0\}$ and define

$$h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & \text{for } x \in \mathrm{U} \\ 0 & \text{for } x \in \mathrm{Z}(f)\,. \end{cases}$$

Since both formulas yield 0 on $\mathrm{U} \cap \mathrm{Z}(f)$, h is well-defined; and since $\complement\, \mathrm{Z}(f) = \mathrm{N}(f) \subset \mathrm{Supp}\, f \subset \mathrm{U}$, h is defined everywhere on X. As the restrictions of h to the closed sets U and $\mathrm{Z}(f)$ are continuous, h is continuous on X (GT, I, §3, No. 2, Prop. 4), and $\mathrm{Supp}\, h \subset \mathrm{U} \subset \mathrm{W}$.

III.28, ℓ. 22.

"…it would then follow that $\mu(f) = \mu(gh) = 0$"

We know that $\mathrm{W} \subset \complement\, \mathrm{Supp}(g \cdot \mu)$, that is, $g \cdot \mu = 0$ on $\mathscr{K}(\mathrm{X}, \mathrm{W}; \mathbf{C})$ and in particular $(g \cdot \mu)(h) = 0$, thus $0 = \mu(gh) = \mu(f)$.

III.28, ℓ. 23.

"…contrary to the hypothesis $y \in \mathrm{S}$."

It has just been argued that $\mu(f) = 0$ for every $f \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ with $\mathrm{Supp}\, f \subset \mathrm{U}$; *a fortiori* $\mu = 0$ on $\mathscr{K}(\mathrm{X}, \mathring{\mathrm{U}}; \mathbf{C})$, that is, $\mathring{\mathrm{U}} \subset \complement\, \mathrm{Supp}(\mu) = \complement\, \mathrm{S}$, and in particular $y \in \mathring{\mathrm{U}} \subset \complement\, \mathrm{S}$, contrary to $y \in \mathrm{W} \cap \mathrm{S}$. This completes the proof of Prop. 8, as observed at the end of the note for ℓ. 16–18.

III.28, ℓ. 24, 25.

"Note that T is contained in the intersection of the support S of μ and the support of g"

Writing $\mathrm{N}(g) = \{x : g(x) \neq 0\}$, one has

$$\mathrm{T} = \overline{\mathrm{S} \cap \mathrm{N}(g)} \subset \overline{\mathrm{S}} \cap \overline{\mathrm{N}(g)} = \mathrm{S} \cap \mathrm{Supp}\, g\,.$$

In the example that follows, $\mathrm{Supp}(\mu) = \{0\}$, $\mathrm{Supp}\, g = \mathbf{R}$ and $\mathrm{Supp}(g \cdot \mu) = \varnothing$.

III.28, ℓ. −9, −8.

"COROLLARY. — *In order that the measure* $g \cdot \mu$ *be zero, it is necessary and sufficient that* g *be zero on the support of* μ."

Writing $\mathrm{A} = \mathrm{Supp}(\mu) \cap \{x : g(x) \neq 0\}$, we know from Prop. 10 that $\mathrm{Supp}(g \cdot \mu) = \overline{\mathrm{A}}$. Since

$$g \cdot \mu = 0 \quad \Leftrightarrow \quad \mathrm{Supp}(g \cdot \mu) = \varnothing$$

and since

$$g = 0 \text{ on } \mathrm{Supp}(\mu) \quad \Leftrightarrow \quad \mathrm{A} = \varnothing\,,$$

the assertion of the corollary reduces to the triviality $\overline{\mathrm{A}} = \varnothing \Leftrightarrow \mathrm{A} = \varnothing$.

III.29, ℓ. 2–4.

"PROPOSITION 12. — *Let* a_i $(1 \leqslant i \leqslant n)$ *be distinct points in a locally compact space* X. *Every measure on* X *whose support is contained in the set of the* a_i *is a linear combination of the measures* ε_{a_i} $(1 \leqslant i \leqslant n)$."

Conversely, if $\mu = \sum_{i=1}^{n} c_i \varepsilon_{a_i}$ then (No. 2, Prop. 4)

$$\mathrm{Supp}(\mu) \subset \bigcup_{i=1}^{n} \mathrm{Supp}(c_i \varepsilon_{a_i}) \subset \bigcup_{i=1}^{n} \mathrm{Supp}(\varepsilon_{a_i}) = \{a_1, \ldots, a_n\}\,,$$

where the fact that $\mathrm{Supp}(\varepsilon_a) = \{a\}$ is shown by the following argument: if $\mathrm{U} = \mathrm{X} - \{a\}$ then for all $f \in \mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})$ one has $\varepsilon_a(f) = f(a) = 0$, thus $\mathrm{U} \subset \complement\, \mathrm{Supp}(\varepsilon_a)$, that is, $\mathrm{Supp}(\varepsilon_a) \subset \complement\, \mathrm{U} = \{a\}$; since $\mathrm{Supp}(\varepsilon_a) \neq \varnothing$ (because $\varepsilon_a \neq 0$) necessarily $\mathrm{Supp}(\varepsilon_a) = \{a\}$ (alternatively—cf. the note for III.25, ℓ. −7 to −5—for every neighborhood V of a, there exists a function $f \in \mathscr{K}(\mathrm{X}, \mathrm{V}; \mathbf{C})$ with $\varepsilon_a(f) = f(a) \neq 0$, whence $a \in \mathrm{Supp}(\varepsilon_a)$).

III.29, ℓ. 16–18.

"It suffices to prove that μ is orthogonal to the subspace V° of $\mathscr{K}(\mathrm{X}; \mathbf{C})$ orthogonal to V (TVS, II, §6, No. 3, Cor. 2 of Th. 1), that is, that the relations $\langle f, \varepsilon_a \rangle = 0$, where a runs over the support of μ, imply $\langle f, \mu \rangle = 0$ "

In view of Prop. 12, V is the linear subspace of $\mathscr{M}(\mathrm{X}; \mathbf{C})$ generated by the measures ε_a with $a \in \mathrm{Supp}(\mu)$; regarding $\mathscr{K}(\mathrm{X}; \mathbf{C})$ and $\mathscr{M}(\mathrm{X}; \mathbf{C})$ as being in a (separating) duality via the bilinear form

$$(f, \nu) \mapsto \langle f, \nu \rangle = \nu(f) \quad \big(f \in \mathscr{K}(\mathrm{X}; \mathbf{C}),\ \nu \in \mathscr{M}(\mathrm{X}; \mathbf{C})\big),$$

it follows that V° is a linear subspace of $\mathscr{K}(\mathrm{X}; \mathbf{C})$ and

$$\begin{aligned} \mathrm{V}^\circ &= \{f \in \mathscr{K}(\mathrm{X}; \mathbf{C}) : \langle f, \nu \rangle = 0 \text{ for all } \nu \in \mathrm{V}\} \\ &= \{f \in \mathscr{K}(\mathrm{X}; \mathbf{C}) : \langle f, \varepsilon_a \rangle = 0 \text{ for all } a \in \mathrm{Supp}(\mu)\} \end{aligned}$$

(TVS, II, §6, No. 3, Prop. 4, (iii) for the case of real scalars, *loc. cit.*, §8, No. 4 for the case of complex scalars). By the theorem on bipolars (TVS, II, §6, No. 3, Th. 1 and §8, No. 4), the polar $\mathrm{V}^{\circ\circ}$ of V° in $\mathscr{M}(\mathrm{X};\mathbf{C})$ is equal to the closure of V in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for the topology $\sigma\big(\mathscr{M}(\mathrm{X};\mathbf{C}),\mathscr{K}(\mathrm{X};\mathbf{C})\big)$, that is,

$$\overline{\mathrm{V}} = \mathrm{V}^{\circ\circ}\,,$$

where $\overline{\mathrm{V}}$ is the vague closure of V in $\mathscr{M}(\mathrm{X};\mathbf{C})$. Thus the assertion that μ is vaguely adherent to V is equivalent to the assertion that $\mu \in \mathrm{V}^{\circ\circ}$.

III.29, ℓ. 19.

"...but this is just Prop. 8 of No. 3."

The cited proposition states that

$$f = 0 \text{ on } \mathrm{Supp}(\mu) \quad\Rightarrow\quad \mu(f) = 0\,,$$

that is,

$$\langle f, \varepsilon_a\rangle = 0 \text{ for all } a \in \mathrm{Supp}(\mu) \quad\Rightarrow\quad \langle f, \mu\rangle = 0\,,$$

equivalently

$$f \in \mathrm{V}^{\circ} \quad\Rightarrow\quad \langle f, \mu\rangle = 0\,,$$

in other words $\mu \in \mathrm{V}^{\circ\circ}$.

Preparation for Corollary 1 *of Theorem* 1:

(I). *Let* $\mathrm{A} = \{a_1, \ldots, a_n\}$ *be a finite set of pairwise distinct points of a locally compact space* X. *In order that a measure* μ *on* X *have support* A, *it is necessary and sufficient that* $\mu = \sum_{k=1}^{n} c_k \varepsilon_{a_k}$ *with the* c_k *nonzero scalars, in which case*

$$\text{(i)} \qquad \|\mu\| = \sum_{k=1}^{n} |c_k|\,.$$

Proof. If μ has support A, we know from Prop. 12 that $\mu = \sum_{k=1}^{n} c_k \varepsilon_k$ for suitable scalars c_k. If a coefficient, say c_1, were equal to 0, then one would have $\mathrm{Supp}(\mu) \subset \{a_2, \ldots, a_n\}$ by the note for III.29, ℓ. 2–4, contrary to hypothesis.

Conversely, if $\mu = \sum_{k=1}^{n} c_k \varepsilon_k$ for suitable nonzero scalars c_k, then

$$\mathrm{Supp}(\mu) \subset \{a_1, \ldots, a_n\} = \mathrm{A}$$

by the note for III.29, ℓ. 2–4; we assert that $\mathrm{Supp}(\mu) = \mathrm{A}$. Assume to the contrary that some a_k is absent from $\mathrm{Supp}(\mu)$. Then there exists an open neighborhood U of a_k such that $\mathrm{U} \cap \mathrm{Supp}(\mu) = \varnothing$, whence $\mathrm{U} \subset \complement\,\mathrm{Supp}(\mu)$, and so $\mu(f) = 0$ for all $f \in \mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})$. Choose $f \in \mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})$ such that $f(a_k) \neq 0$. For all indices $j \neq k$ we have $a_j \in \complement\,\mathrm{U}$ and so $f(a_j) = 0$, thus

$$0 = \mu(f) = \sum_{j=1}^{n} c_j \varepsilon_{a_j}(f) = \sum_{j=1}^{n} c_j f(a_j) = c_k f(a_k) + \sum_{j \neq k} c_j \cdot 0 = c_k f(a_k),$$

contrary to $c_k \neq 0$ and $f(a_k) \neq 0$.

Suppose now that μ has the indicated representation. For every $f \in \mathscr{K}(\mathrm{X}; \mathbf{C})$,

$$|\mu(f)| = \Big| \sum_{k=1}^{n} c_k f(a_k) \Big| \leqslant \|f\| \sum_{k=1}^{n} |c_k|\,,$$

hence $\|\mu\| \leqslant \sum_{k=1}^{n} |c_k|$.

On the other hand, let $\mathrm{V}_1, \ldots, \mathrm{V}_n$ be pairwise disjoint neighborhoods of $a_1, \ldots, a_n$, respectively, and choose functions $f_k \in \mathscr{K}(\mathrm{X}; \mathbf{R})$ such that $0 \leqslant f_k \leqslant 1$, $f_k(a_k) = 1$ and $\mathrm{Supp}\, f_k \subset \mathrm{V}_k$; in particular,

$$f_k = 0 \text{ on } \bigcup_{j \neq k} \mathrm{V}_j \text{ for } k = 1, \ldots, n\,.$$

For each k write $|c_k| = \theta_k c_k$ with $|\theta_k| = 1$, and set

$$g = \sum_{k=1}^{n} \theta_k f_k\,;$$

then $g(a_k) = \theta_k f_k(a_k) = \theta_k$ for all k. If $x \in \mathrm{X}$ then, since the f_k have disjoint supports, at most one term in the sum for $g(x)$ can be nonzero, therefore $|g(x)| \leqslant 1$; but $|g(a_1)| = 1$, and so $\|g\| = 1$. Moreover,

$$\mu(g) = \sum_{k=1}^{n} c_k \varepsilon_{a_k}(g) = \sum_{k=1}^{n} c_k g(a_k) = \sum_{k=1}^{n} c_k \theta_k = \sum_{k=1}^{n} |c_k|\,,$$

thus $\|\mu\| \geqslant |\mu(g)| = \sum_{k=1}^{n} |c_k|$, which completes the proof of (i).

(II). *Let* $\mathrm{A} = \{a_1, \ldots, a_n\}$ *be a finite set of pairwise distinct points of a locally compact space* X, *and let* μ *be a measure on* X *with* $\mathrm{Supp}(\mu) \subset \mathrm{A}$. *Let* r *be a real number* > 0. *Then*

$$\text{(ii)} \qquad \|\mu\| \leqslant r \quad \Leftrightarrow \quad \mu \in \text{bal conv}\,\{r\varepsilon_{a_1}, \ldots, r\varepsilon_{a_n}\},$$

where bal conv $\{r\varepsilon_{a_1}, \ldots, r\varepsilon_{a_n}\}$ *denotes the balanced convex envelope of the set* $\{r\varepsilon_{a_1}, \ldots, r\varepsilon_{a_n}\}$ *in* $\mathscr{M}(\mathrm{X}; \mathbf{C})$.

Proof. Recall that the balanced convex envelope of a subset S of a vector space (real or complex) is the set of all finite linear combinations $\sum c_j u_j$, where $u_j \in \mathrm{S}$ and the c_j are scalars such that $\sum |c_j| \leqslant 1$ (TVS, II, §8, No. 2).

By (I), $\mu = \sum_{k=1}^{n} c_k \varepsilon_{a_k}$, where c_k is 0 when the point a_k does not belong to $\mathrm{Supp}(\mu)$, and

$$\|\mu\| \leqslant r \quad \Leftrightarrow \quad \sum_{k=1}^{n} \frac{|c_k|}{r} \leqslant 1\,.$$

Thus the formula

$$\mu = \sum_{k=1}^{n} \Bigl(\frac{c_k}{r}\Bigr) r\, \varepsilon_{a_k}$$

shows that if $\|\mu\| \leqslant r$ then μ belongs to the balanced convex envelope of $\{r\varepsilon_{a_1}, \ldots, r\varepsilon_{a_n}\}$.

Conversely, if $\mu = \sum_{k=1}^{n} b_k (r\varepsilon_{a_k})$ with $\sum_{k=1}^{n} |b_k| \leqslant 1$ then, by (i), $\|\mu\| = \sum_{k=1}^{n} |b_k r| = r \sum_{k=1}^{n} |b_k| \leqslant r$.

The following general lemma prepares the way for item (III) below.

Lemma. — *Let* F, G *be complex vector spaces in duality with respect to a bilinear form* $(x, y) \mapsto \langle x, y\rangle$ ($x \in \mathrm{F}$, $y \in \mathrm{G}$), *and let* M *be a "circled"* ($x \in \mathrm{M}$ *and* $|c| = 1$ *imply* $cx \in \mathrm{M}$) *subset of* F. *Then*:

(1) *The polar* M° *of* M *in* G *is given by the formula*

$$\mathrm{M}^\circ = \{y \in \mathrm{G} : \ |\langle x, y\rangle| \leqslant 1 \ \textit{for all} \ x \in \mathrm{M}\}\,.$$

(2) M° *is convex and balanced.*

(3) *If* N *is the balanced convex envelope of* M, *then* $\mathrm{N}^\circ = \mathrm{M}^\circ$.

(4) $\mathrm{M}^{\circ\circ} = \overline{\mathrm{N}}$ (*the closure of* N *for* $\sigma(\mathrm{F}, \mathrm{G})$), *which is the closed balanced convex envelope of* M.

Proof. Recall that a subset A of a complex vector space is said to be *balanced* if $cx \in A$ for all $x \in A$ and all scalars c with $|c| \leqslant 1$. Balanced sets are obviously circled. The polar of M is defined (for any subset M of F) by the formula

$$M^\circ = \{y \in G : \mathscr{R}\langle x, y\rangle \geqslant -1 \text{ for all } x \in M\}$$

(TVS, II, §8, No. 4, Def. 1).

(1) If $y \in G$ and $|\langle x, y\rangle| \leqslant 1$ for all $x \in M$, then for all $x \in M$ one has

$$|\mathscr{R}\langle x, y\rangle| \leqslant |\langle x, y\rangle| \leqslant 1,$$

whence $-1 \leqslant \mathscr{R}\langle x, y\rangle \leqslant 1$ and in particular $y \in M^\circ$.

Conversely, suppose $y \in M^\circ$ and $x \in M$. By assumption,

$$\mathscr{R}\langle cx, y\rangle \geqslant -1$$

for all scalars c with $|c| = 1$; if c is chosen so that $c\langle x, y\rangle = -|\langle x, y\rangle|$, then

$$-1 \leqslant \mathscr{R}\langle cx, y\rangle = \mathscr{R}\, c\langle x, y\rangle = -|\langle x, y\rangle|,$$

whence $|\langle x, y\rangle| \leqslant 1$.

(2) If $y = c_1 y_1 + c_2 y_2$, where $y_1, y_2 \in M^\circ$, $c_1 \geqslant 0$, $c_2 \geqslant 0$ and $c_1 + c_2 = 1$ then, for all $x \in M$,

$$\begin{aligned} |\langle x, y\rangle| &= |c_1\langle x, y_1\rangle + c_2\langle x, y_2\rangle| \\ &\leqslant c_1|\langle x, y_1\rangle| + c_2|\langle x, y_2\rangle| \leqslant c_1 \cdot 1 + c_2 \cdot 1 = 1, \end{aligned}$$

whence $y \in M^\circ$; thus M° is convex. If $y \in M^\circ$ and c is a scalar with $|c| \leqslant 1$ then, for all $x \in M$,

$$|\langle x, cy\rangle| = |c| \cdot |\langle x, y\rangle| \leqslant 1 \cdot 1 = 1,$$

whence $cy \in M^\circ$; thus M° is balanced.

(3) As noted at the beginning of the proof of (II), N is equal to the set of all finite linear combinations $x = \sum c_k x_k$ with $x_k \in M$ and $\sum |c_k| \leqslant 1$. For all such x, and for $y \in M^\circ$, one has

$$|\langle x, y\rangle| = \left|\sum c_k\langle x_k, y\rangle\right| \leqslant \sum |c_k| \cdot |\langle x_k, y\rangle| \leqslant \sum |c_k| \cdot 1 \leqslant 1,$$

whence $y \in N^\circ$ (by (1) applied to N in place of M); thus $M^\circ \subset N^\circ$. But $M \subset N$ implies $M^\circ \supset N^\circ$, therefore $M^\circ = N^\circ$.

(4) Since N is balanced and convex, its closure $\overline{N}$ for $\sigma(F, G)$ is also balanced and convex (TVS, II, §2, No. 6 and §8, No. 2), thus $\overline{N}$ is the

closed balanced convex envelope of M. Since N is convex, its closed convex envelope is also equal to $\overline{\mathrm{N}}$, and since $0 \in \mathrm{N}$ it follows from the theorem on bipolars that $\mathrm{N}^{\circ\circ} = \overline{\mathrm{N}}$. In view of (3), we have also $\mathrm{M}^{\circ\circ} = \overline{\mathrm{N}}$.

(III). *Let* S *be a subset of the locally compact space* X, *let* $r > 0$, *let*

$$\mathscr{B} = \{\nu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ \|\nu\| \leqslant r \ \textit{and} \ \mathrm{Supp}(\nu) \ \textit{is a finite subset of} \ \mathrm{S}\},$$

and let $\mathscr{B}^\circ$ *be the polar set of* $\mathscr{B}$ *in* $\mathscr{K}(\mathrm{X};\mathbf{C})$ *for the canonical duality* $\langle f, \nu\rangle = \nu(f)$. *Then*

$$\text{(iii)} \qquad \mathscr{B}^\circ = \{f \in \mathscr{K}(\mathrm{X};\mathbf{C}) : \ |\langle f, \varepsilon_a\rangle| \leqslant \frac{1}{r} \ \textit{for all} \ a \in \mathrm{S}\}.$$

Proof. Let $\mathscr{P} = \{r\varepsilon_a : \ a \in \mathrm{S}\}$, and let $\mathscr{P}_{\mathrm{bc}}$ be the balanced convex envelope of $\mathscr{P}$, that is, the set of all finite linear combinations $\sum c_k \nu_k$, where $\nu_k \in \mathscr{P}$ and $\sum |c_k| \leqslant 1$.

We assert that $\mathscr{B} = \mathscr{P}_{\mathrm{bc}}$. For, if $\nu \in \mathscr{B}$, so that $\|\nu\| \leqslant r$ and $\mathrm{Supp}(\nu) = \{a_1, \ldots, a_n\}$ with the a_k pairwise distinct elements of S, then by (II),

$$\nu \in \text{ bal conv}\{r\varepsilon_{a_1}, \ldots, r\varepsilon_{a_n}\} \subset \mathscr{P}_{\mathrm{bc}}.$$

Conversely, suppose $\nu \in \mathscr{P}_{\mathrm{bc}}$, so that $\nu = \sum_{k=1}^{n} c_k(r\varepsilon_{a_k})$ for suitable points $a_k \in \mathrm{S}$ and scalars c_k such that $\sum_{k=1}^{n} |c_k| \leqslant 1$; we can suppose (after gathering together coefficients of a same ε_{a_k}) that the points a_k are pairwise distinct (at work here is the triangle inequality for the partial sums of the c_k), so that the ε_{a_k} are linearly independent; then, omitting terms with zero coefficients, we can further suppose that the c_k are nonzero. Then, $\mathrm{Supp}(\nu) = \{a_1, \ldots, a_n\} \subset \mathrm{S}$ by (I), and $\|\nu\| \leqslant r$ by (II), thus $\nu \in \mathscr{B}$.

Now let $\{\mathrm{M} = c\nu : \ |c| = 1 \text{ and } \nu \in \mathscr{P}\}$ (the "circled envelope" of $\mathscr{P}$). Clearly M and $\mathscr{P}$ have the same balanced convex envelope, thus

$$\text{bal conv}\,\mathrm{M} = \mathscr{P}_{\mathrm{bc}} = \mathscr{B};$$

the vague closure $\overline{\mathscr{B}}$ of $\mathscr{B}$ is therefore equal to the vaguely closed balanced convex envelope of $\mathscr{B}$ as well as to the closed balanced convex envelope of M, and since both $\mathscr{B}$ and M are circled, it follows from the preceding Lemma that

$$\mathrm{M}^{\circ\circ} = \mathscr{B}^{\circ\circ} = \overline{\mathscr{B}}.$$

Then (TVS, II, §6, No. 3)

$$\mathrm{M}^\circ = (\mathrm{M}^{\circ\circ})^\circ = (\mathscr{B}^{\circ\circ})^\circ = \mathscr{B}^\circ,$$

thus

$$\begin{aligned}\mathscr{B}^\circ = \mathrm{M}^\circ &= \{ f \in \mathscr{K}(\mathrm{X};\mathbf{C}) : \ |\langle f, \nu\rangle| \leqslant 1 \ \text{ for all } \ \nu \in \mathrm{M}\,\} \\ &= \{ f \in \mathscr{K}(\mathrm{X};\mathbf{C}) : \ |\langle f, c(r\varepsilon_a)\rangle| \leqslant 1 \ \text{ for all } \ c \in \mathbf{T} \ \text{ and } \ a \in \mathrm{S}\,\} \\ &= \{ f \in \mathscr{K}(\mathrm{X};\mathbf{C}) : \ |\langle f, \varepsilon_a\rangle| \leqslant \frac{1}{r} \ \text{ for all } \ a \in \mathrm{S}\,\}\,,\end{aligned}$$

which establishes the formula (iii).

The argument shows incidentally that a measure μ on X is vaguely adherent to $\mathscr{B}$ if and only if $\mu \in \mathscr{B}^{\circ\circ}$; in view of the formula (iii), μ is vaguely adherent to $\mathscr{B}$ if and only if

$$f \in \mathscr{K}(\mathrm{X};\mathbf{C}) \ \& \ |\langle f, \varepsilon_a\rangle| \leqslant \frac{1}{r} \ \text{ for all } \ a \in \mathrm{S} \quad \Rightarrow \quad |\langle f, \mu\rangle| \leqslant 1\,,$$

that is,

$$f \in \mathscr{K}(\mathrm{X};\mathbf{C}) \ \& \ \|f\big|\mathrm{S}\| \leqslant \frac{1}{r} \quad \Rightarrow \quad |\mu(f)| \leqslant 1\,.$$

III.29, ℓ. 24–26.

"To prove the first assertion..."

This is covered by the preceding remarks, with $\mathrm{Supp}(\mu)$ in the role of S, $r = \|\mu\|$, and A in the role of $\mathscr{B}$.

III.29, ℓ. −8 to −5.

"...we note that

$$\liminf_{\nu\to\mu,\,\nu\in\mathrm{A}} \|\nu\| \geqslant \|\mu\|$$

since the function $\nu \mapsto \|\nu\|$ is lower semi-continuous for the vague topology (§1, No. 9, Cor. 4 of Prop. 15)"

One must adapt the cited Prop. 15 to the situation of a limit with respect to the relative topology induced on A by the vague topology. Let $\mathscr{V}$ be the set of neighborhoods of μ for the vague topology on $\mathscr{M}(\mathrm{X};\mathbf{C})$. By definition,

$$\liminf_{\nu\to\mu,\,\nu\in\mathrm{A}} \|\nu\| = \sup_{\mathrm{V}\in\mathscr{V}} \Big(\inf_{\nu\in\mathrm{V}\cap\mathrm{A}} \|\nu\| \Big).$$

Given $0 < h < \|\mu\|$, by the cited lower semi-continuity there exists a $\mathrm{V} \in \mathscr{V}$ such that $\nu \in \mathrm{V} \Rightarrow \|\nu\| > h$; in particular, $\nu \in \mathrm{V} \cap \mathrm{A} \Rightarrow \|\nu\| > h$, hence

$$h \leqslant \inf_{\nu\in\mathrm{V}\cap\mathrm{A}} \|\nu\| \leqslant \liminf_{\nu\to\mu,\,\nu\in\mathrm{A}} \|\nu\|\,,$$

and the desired inequality results from letting $h \to \|\mu\|$.

III.29, ℓ. −5, −4.

"...and the conclusion follows from the fact that $\|\nu\| \leqslant \|\mu\|$ for $\nu \in \mathrm{A}$ by definition."

Since $\|\nu\| \leqslant \|\mu\|$ for all $\nu \in \mathrm{A}$ by the definition of A, we have (notations as in the preceding note)

$$\limsup_{\nu\to\mu,\, \nu\in\mathrm{A}} \|\nu\| = \inf_{\mathrm{V}\in\mathscr{V}} \Bigl(\sup_{\nu\in\mathrm{V}\cap\mathrm{A}} \|\nu\|\Bigr) \leqslant \|\mu\| \leqslant \liminf_{\nu\to\mu,\, \nu\in\mathrm{A}} \|\nu\| ,$$

whence equality throughout (GT, IV, §5, No. 6, formula (11)); thus

$$\lim_{\nu\to\mu,\, \nu\in\mathrm{A}} \|\nu\| = \|\mu\|$$

by GT, IV, §5, No. 6, Cor. 1 of Th. 3 (the filter $\mathfrak{G}$ in question being the trace $\mathscr{V}\cap\mathrm{A}$ of $\mathscr{V}$ on A).

Note: It follows that the mapping $\mathrm{A}\cup\{\mu\} \to [0,+\infty[$ defined by $\nu \mapsto \|\nu\|$ is *continuous at* μ for the topology on $\mathrm{A}\cup\{\mu\}$ induced by the vague topology.

III.30, ℓ. 2–4.

"...for every ε such that $0<\varepsilon<1$, there exists, by virtue of Cor. 1, a measure ν_0 whose support is finite and contained in $\mathrm{Supp}(\mu)$ and for which $\nu_0-\mu\in\mathrm{V}$ and $\|\mu\| \geqslant \|\nu_0\| \geqslant (1-\varepsilon)\|\mu\|$."

With A as in Corollary 1, we know that

$$\lim_{\nu\to\mu,\, \nu\in\mathrm{A}} \|\nu\| = \|\mu\| ;$$

therefore, since $\|\mu\| > (1-\varepsilon)\|\mu\|$, there exists a vague neighborhood V' of 0 such that

$$\nu\in\mathrm{A}\,,\ \nu\in\mu+\mathrm{V}' \quad\Rightarrow\quad \|\nu\| > (1-\varepsilon)\|\mu\| .$$

Since μ is vaguely adherent to A, the vague neighborhood $\mathrm{V}\cap\mathrm{V}'$ of 0 contains $\nu_0-\mu$ for some $\nu_0\in\mathrm{A}$, whence both $\nu_0-\mu\in\mathrm{V}$ and $\|\mu\| \geqslant \|\nu_0\| > (1-\varepsilon)\|\mu\|$.

III.30, ℓ. 5.

"...and $\|\nu-\nu_0\| \leqslant \|\mu\|$ "

Since $\|\nu_0\| \geqslant (1-\varepsilon)\|\mu\| = \|\mu\| - \varepsilon\|\mu\|$, one has $\|\mu\| - \|\nu_0\| \leqslant \varepsilon\|\mu\|$, therefore

$$\nu-\nu_0 = \frac{\|\mu\|}{\|\nu_0\|}\cdot\nu_0 - \nu_0 = \Bigl(\frac{\|\mu\|}{\|\nu_0\|} - 1\Bigr)\nu_0 = \frac{\|\mu\|-\|\nu_0\|}{\|\nu_0\|}\cdot\nu_0 ,$$

whence $\|\nu - \nu_0\| = \|\mu\| - \|\nu_0\| \leqslant \varepsilon\|\mu\| < \|\mu\|$.

III.30, ℓ. 5, 6.

"...for ε sufficiently small we therefore have $\nu - \mu \in \mathrm{V} + \mathrm{V}$, whence the conclusion."

If $f \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ and $c > 0$, the set

$$\mathrm{W} = \{\rho \in \mathscr{M}(\mathrm{X}; \mathbf{C}) : \ |\rho(f)| < c\}$$

is an open neighborhood of 0 (for the vague topology); replacing f by $c^{-1}f$, one has

$$\mathrm{W} = \{\rho \in \mathscr{M}(\mathrm{X}; \mathbf{C}) : \ |\rho(f)| < 1\}.$$

The set of finite intersections of such sets is a fundamental system of neighborhoods of 0. Thus we can suppose that

$$\mathrm{V} = \{\rho \in \mathscr{M}(\mathrm{X}; \mathbf{C}) : \ |\rho(f_k)| < 1 \ \text{ for } \ k = 1, \ldots, n\},$$

where $f_k \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ for $k = 1, \ldots, n$.

Let us now impose a further condition on ε, namely, that it be sufficiently small that

$$\varepsilon\|\mu\| \cdot \|f_k\| < 1 \quad \text{for } \ k = 1, \ldots, n.$$

Then, the measure ν_0 constructed above will satisfy

$$|(\nu - \nu_0)(f_k)| \leqslant \|\nu - \nu_0\| \cdot \|f_k\| \leqslant \varepsilon\|\mu\| \cdot \|f_k\| < 1$$

for all k, whence $\nu - \nu_0 \in \mathrm{V}$. Thus

$$\nu - \mu = (\nu - \nu_0) + (\nu_0 - \mu) \in \mathrm{V} + \mathrm{V},$$

and so $\nu \in \mu + \mathrm{V} + \mathrm{V}$; since the $\mathrm{V} + \mathrm{V}$ form a fundamental system of neighborhoods of 0, we have shown that every neighborhood of μ contains a measure $\nu \in \mathrm{A}$ such that $\|\nu\| = \|\mu\|$.

III.30, ℓ. 10–13.

"...The same reasoning as in Cor. 2 shows that we can limit ourselves to proving that μ is in the vague closure of the convex set B formed by the positive measures with finite support contained in $\mathrm{Supp}(\mu)$ and with norm $\leqslant \|\mu\|$."

Recall that the real vector space $\mathscr{M}(\mathrm{X}; \mathbf{R})$ may be identified with the $\mathbf{R}$-linear subspace of $\mathscr{M}(\mathrm{X}; \mathbf{C})$ consisting of the complex measures ρ on X such that $\overline{\rho} = \rho$ (§1, No. 5). The identification is a homeomorphism for the vague topologies (induced by $\mathscr{K}(\mathrm{X};\mathbf{R})$ and $\mathscr{K}(\mathrm{X};\mathbf{C})$, respectively); for, if $\rho_j, \rho \in \mathscr{M}(\mathrm{X};\mathbf{R})$, then $\rho_j(g) \to \rho(g)$ for all $g \in \mathscr{K}(\mathrm{X};\mathbf{R})$ if and only if, for all $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$,

$$\rho_j(f) = \rho_j(\mathscr{R} f) + i\rho_j(\mathscr{I} f) \to \rho(\mathscr{R} f) + i\rho(\mathscr{I} f) = \rho(f)\,.$$

Thus if S is a subset of $\mathscr{M}(\mathrm{X};\mathbf{C})$ consisting of real measures, the vague closure of S in $\mathscr{M}(\mathrm{X};\mathbf{C})$ coincides with the vague closure of S in $\mathscr{M}(\mathrm{X};\mathbf{R})$. Moreover, it follows from §1, No. 8, Cor. 3 of Prop. 10 that $\|\rho\|$ for a real measure is the same whether calculated in $\mathscr{M}(\mathrm{X};\mathbf{C})$ or in $\mathscr{M}(\mathrm{X};\mathbf{R})$.

Let us write D for the set of all $\nu \in \mathscr{M}_+(\mathrm{X})$ such that $\mathrm{Supp}(\nu)$ is a finite subset of $\mathrm{Supp}(\mu)$ and $\|\nu\| = \|\mu\|$, and let $\overline{\mathrm{D}}$ be the vague closure of D. The convexity of D is assured by the computation $\|c\nu + (1-c)\nu'\| = c\|\nu\| + (1-c)\|\nu'\|$ for positive measures ν, ν' and for $0 \leqslant c \leqslant 1$ (§1, No. 8, Prop. 11). The assertion of Cor. 3 is that $\mu \in \overline{\mathrm{D}}$.

Suppose we can show that $\mu \in \overline{\mathrm{B}}$. Then (arguing as in the preceding note) given an open neighborhood V of 0 in $\mathscr{M}(\mathrm{X};\mathbf{C})$, say

$$\mathrm{V} = \{\rho \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ |\rho(f_k)| < 1 \quad \text{for } k = 1, \dots, n\}\,,$$

and given an $\varepsilon > 0$ such that $\varepsilon\|\mu\| \cdot \|f_k\| < 1$ for all k, choose $\nu_0 \in \mathrm{B}$ so that $\nu_0 - \mu \in \mathrm{V}$ and $\|\mu\| \geqslant \|\nu_0\| \geqslant (1-\varepsilon)\|\mu\|$. Setting $\nu = \dfrac{\|\mu\|}{\|\nu_0\|}\cdot\nu_0$, we have $\nu \geqslant 0$ (because $\nu_0 \geqslant 0$), $\|\nu\| = \|\mu\|$ (hence $\nu \in \mathrm{D}$) and $\|\nu - \nu_0\| \leqslant \|\mu\|$, whence $\nu - \nu_0 \in \mathrm{V}$ and $\nu - \mu \in \mathrm{V} + \mathrm{V}$. This shows that $\mu \in \overline{\mathrm{D}}$.

III.30, ℓ. 13, 14.

"Again, it suffices to establish that μ belongs to the polar set of B°, the polar set of B *in the space* $\mathscr{K}(\mathrm{X};\mathbf{R})$"

The point is that to exploit the argument of the preceding note, we must show that $\mu \in \overline{\mathrm{B}}$. It is clear that B is convex and that $0 \in \mathrm{B}$, thus $\overline{\mathrm{B}} = \mathrm{B}^{\circ\circ}$ by the theorem on bipolars for the duality between $\mathscr{M}(\mathrm{X};\mathbf{R})$ and $\mathscr{K}(\mathrm{X};\mathbf{R})$ (TVS, II, §6, No. 3, Th. 1).

III.30, ℓ. 15–17.

"...but this means that for $f \in \mathscr{K}(\mathrm{X};\mathbf{R})$ the relations $\langle f, \varepsilon_a\rangle \geqslant -1/\|\mu\|$ for all $a \in \mathrm{Supp}(\mu)$ imply $\langle f, \mu\rangle \geqslant -1$, which is a consequence of No. 3, Cor. 2 of Prop. 8."

We know, from item (I) of the Note for III.29, ℓ. 19, that B consists of 0 together with the set of all measures ν on X of the form

$$(*) \qquad \nu = \sum_{k=1}^{n} c_k \varepsilon_{a_k}\,,$$

where $a_1, \ldots, a_n$ are distinct points of $\mathrm{Supp}(\mu)$, $c_k > 0$ for $k = 1, \ldots, n$, and $\sum_{k=1}^{n} c_k = \|\nu\| \leqslant \|\mu\|$.

Given $f \in \mathscr{K}(\mathrm{X}; \mathbf{R})$, we assert that

$$f \in \mathrm{B}^\circ \quad \Leftrightarrow \quad \langle f, \varepsilon_a \rangle \geqslant -\frac{1}{\|\mu\|} \quad \text{for all } a \in \mathrm{Supp}(\mu).$$

For, if $f \in \mathrm{B}^\circ$, that is, if $\langle f, \nu \rangle \geqslant -1$ for all $\nu \in \mathrm{B}$, then in particular $\langle f, \|\mu\|\varepsilon_a \rangle \geqslant -1$ for all $a \in \mathrm{Supp}(\mu)$. Conversely, if f has the latter property and if $\nu \in \mathrm{B}$ is represented as in (*), then

$$\langle f, \nu \rangle = \sum c_k \langle f, \varepsilon_{a_k} \rangle \geqslant -\frac{1}{\|\mu\|} \sum c_k \geqslant -1,$$

whence $f \in \mathrm{B}^\circ$.

Thus, in order that μ belong to $\overline{\mathrm{B}} = \mathrm{B}^{\circ\circ}$, it is necessary and sufficient that

$$f \in \mathrm{B}^\circ \quad \Rightarrow \quad \langle f, \mu \rangle \geqslant -1,$$

that is,

$$\langle f, \varepsilon_a \rangle \geqslant -\frac{1}{\|\mu\|} \text{ for all } a \in \mathrm{Supp}(\mu) \quad \Rightarrow \quad \langle f, \mu \rangle \geqslant -1,$$

that is,

$$(**) \qquad f \geqslant -\frac{1}{\|\mu\|} \quad \text{on} \quad \mathrm{Supp}(\mu) \quad \Rightarrow \quad \mu(f) \geqslant -1.$$

To prove (**), choose $g \in \mathscr{K}(\mathrm{X}; \mathbf{R})$ such that $0 \leqslant g \leqslant 1$ and $g = 1$ on $\mathrm{Supp}(\mu)$, and in particular, $\|g\| = 1$. Then if $f \geqslant -\frac{1}{\|\mu\|}$ on $\mathrm{Supp}(\mu)$, one has $f + \frac{1}{\|\mu\|} g \geqslant 0$ on $\mathrm{Supp}(\mu)$, and since $\mu \geqslant 0$ it follows from Cor. 2 of No. 3, Prop. 8 that $\mu\big(f + \frac{1}{\|\mu\|} g\big) \geqslant 0$, that is, $\mu(f) \geqslant -\frac{1}{\|\mu\|}\mu(g)$; but $\mu(g) \leqslant \|\mu\| \cdot \|g\| = \|\mu\|$, whence $-\mu(g) \geqslant -\|\mu\|$, and finally $\mu(f) \geqslant -\frac{1}{\|\mu\|}\mu(g) \geqslant -1$.

III.30, ℓ. 23.

"…the conclusion therefore follows from Th. 1."

Let $\mathscr{V}$ be the linear subspace of $\mathscr{M}(\mathrm{X};\mathbf{C})$ generated by the point measures, that is, the set of finite linear combinations of the Dirac measures ε_a $(a \in \mathrm{X})$; it follows from Prop. 12 that $\mathscr{V}$ is equal to the set of measures on X with finite support (see item (I) in the note for III.29, ℓ. 19).

It is clear from Th. 1 that $\mathscr{V}$ is vaguely dense in $\mathscr{M}(\mathrm{X};\mathbf{C})$; we are to show that it is also dense for the (stronger) topology of strictly compact convergence; writing $\mathscr{W}$ for the closure of $\mathscr{V}$ for the topology of strictly compact convergence, we are to show that $\mathscr{W} = \mathscr{M}(\mathrm{X};\mathbf{C})$.

The strategy of the proof is to show that (1) $\mathscr{W}$ contains every bounded measure, and (2) the set of bounded measures is dense in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for the topology of strictly compact convergence.

(1) Let μ be a bounded measure. Then $\mathscr{R}\mu$ and $\mathscr{I}\mu$ are also bounded, hence μ is a linear combination of four bounded positive measures (§1, No. 8, Cor. 2 of Prop. 11); thus, in showing that $\mu \in \mathscr{W}$, we can suppose that $\mu \geqslant 0$. Let U be an open neighborhood of μ in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for the topology of strictly compact convergence; then $\mathrm{U} \cap \mathscr{M}_+(\mathrm{X})$ is an open neighborhood of μ in $\mathscr{M}_+(\mathrm{X})$ for that topology. By Prop. 18 of §1, No. 10, the topologies on $\mathscr{M}_+(\mathrm{X})$ induced by topology of strictly compact convergence and the vague topology are identical, therefore $\mathrm{U} \cap \mathscr{M}_+(\mathrm{X})$ is also an open neighborhood of μ for the vague topology, hence there exists a vaguely open set V in $\mathscr{M}(\mathrm{X};\mathbf{C})$ such that $\mathrm{U} \cap \mathscr{M}_+(\mathrm{X}) = \mathrm{V} \cap \mathscr{M}_+(\mathrm{X})$; by Cor. 3, $\mathrm{V} \cap \mathscr{M}_+(\mathrm{X})$ contains a measure ν with finite support (contained in $\mathrm{Supp}(\mu)$), thus $\nu \in \mathrm{U} \cap \mathscr{V}$ and we have shown that μ is in the closure of $\mathscr{V}$ for the topology of strictly compact convergence, that is, $\mu \in \mathscr{W}$.

(2) It remains to show that every measure μ is the limit, for the topology of strictly compact convergence, of bounded measures. Let $\mathscr{K}$ be the set of compact subsets K of X, which is directed upward by inclusion. For each $\mathrm{K} \in \mathscr{K}$, choose a function $g_{\mathrm{K}} \in \mathscr{K}(\mathrm{X};\mathbf{C})$ such that $g_{\mathrm{K}} = 1$ on K (no other properties are needed). Then, for each $\mathrm{K} \in \mathscr{K}$, the measure $g_{\mathrm{K}} \cdot \mu$ has (compact) support contained in $\mathrm{Supp}(g_{\mathrm{K}}) \cap \mathrm{Supp}(\mu)$ (No. 3, Prop. 10), hence is bounded (No. 3, Prop. 11), thus it will suffice to show that the directed family $(g_{\mathrm{K}} \cdot \mu)_{\mathrm{K} \in \mathscr{K}}$ converges to μ for the topology of strictly compact convergence, that is,

(∗) $g_{\mathrm{K}} \cdot \mu \to \mu$ uniformly on each strictly compact subset of $\mathscr{K}(\mathrm{X};\mathbf{C})$.

Let $\mathscr{A}$ be a strictly compact subset of $\mathscr{K}(\mathrm{X};\mathbf{C})$, say $\mathrm{K}_0 \in \mathscr{K}$ with $\mathscr{A} \subset \mathscr{K}(\mathrm{X}, \mathrm{K}_0;\mathbf{C})$; we are to show that $g_{\mathrm{K}} \cdot \mu \to \mu$ uniformly on $\mathscr{A}$. For every $\mathrm{K} \in \mathscr{K}$ with $\mathrm{K} \supset \mathrm{K}_0$, we have $g_{\mathrm{K}} \cdot \mu - \mu = 0$ on $\mathscr{K}(\mathrm{X}, \mathrm{K}_0;\mathbf{C})$; for, if $f \in \mathscr{K}(\mathrm{X}, \mathrm{K}_0;\mathbf{C})$ then $g_{\mathrm{K}} f = f$ (because $g_{\mathrm{K}} = 1$ on K, hence on K_0, whereas $f = 0$ outside K_0) and so

$$(g_{\mathrm{K}} \cdot \mu - \mu)(f) = \mu(g_{\mathrm{K}} f) - \mu(f) = \mu(f) - \mu(f) = 0\,;$$

in particular, $g_{\mathrm{K}} \cdot \mu - \mu = 0$ on $\mathscr{A}$ for all $\mathrm{K} \supset \mathrm{K}_0$ (because $\mathscr{A} \subset \mathscr{K}(\mathrm{X}, \mathrm{K}_0; \mathbf{C})$), thus $(*)$ is established with a vengeance.

One will notice that item (2) proved above is a consequence of the Cor. 4 for which it has served as a lemma.

Note also that given a measure $\mu \in \mathscr{M}(\mathrm{X}; \mathbf{C})$, the approximations in both (1) and (2) are accomplished with measures whose support is contained in $\mathrm{Supp}(\mu)$; thus μ is in the closure, for the topology of strictly compact convergence (call it $\mathscr{T}_3$, as in Prop. 17 of §1, No. 10) of the set of measures with finite support contained in $\mathrm{Supp}(\mu)$. In detail: Let U be an open neighborhood of μ for $\mathscr{T}_3$, and let $\nu \in \mathrm{U}$ be a bounded measure with support contained in $\mathrm{Supp}(\mu)$; since U is also a neighborhood of ν for $\mathscr{T}_3$, it contains a measure ρ with finite support contained in $\mathrm{Supp}(\nu)$, and hence in $\mathrm{Supp}(\mu)$.

III.30, ℓ. −12.

"The condition is obviously sufficient by virtue of Prop. 6 of No. 2."

With $\mathrm{F} = \mathrm{Supp}(\mu)$ in the cited Prop. 6, one concludes that the set

$$\mathscr{L} = \{\nu \in \mathscr{M}(\mathrm{X}; \mathbf{C}) : \ \mathrm{Supp}(\nu) \subset \mathrm{Supp}(\mu)\}$$

is a vaguely closed linear subspace of $\mathscr{M}(\mathrm{X}; \mathbf{C})$. For every $g \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ (indeed, for every $g \in \mathscr{C}(\mathrm{X}; \mathbf{C})$) it follows from No. 3, Prop. 10 that $\mathrm{Supp}(g \cdot \mu) \subset \mathrm{Supp}(\mu)$, thus $g \cdot \mu \in \mathscr{L}$. Since $\mathscr{L}$ is vaguely closed, it contains the vague closure of the set of all such measures $g \cdot \mu$; by hypothesis, ε_{x_0} belongs to that vague closure hence $\varepsilon_{x_0} \in \mathscr{L}$, that is,

$$\{x_0\} = \mathrm{Supp}(\varepsilon_{x_0}) \subset \mathrm{Supp}(\mu),$$

thus $x_0 \in \mathrm{Supp}(\mu)$.

III.30, ℓ. −10 to −6.

"...we are to prove that there exists a function $g \in \mathscr{K}(\mathrm{X}; \mathbf{C})$ such that $\|g \cdot \mu\| \leqslant 1$ and such that

$$|f_k(x_0) - \mu(g f_k)| \leqslant \delta$$

for $1 \leqslant k \leqslant n$."

One has $f_k(x_0) - \mu(g f_k) = \varepsilon_{x_0}(f_k) - (g \cdot \mu)(f_k) = (\varepsilon_{x_0} - g \cdot \mu)(f_k)$, thus the indicated inequalities state that the vague neighborhood of ε_{x_0} defined by

$$\{\nu \in \mathscr{M}(\mathrm{X}; \mathbf{C}) : \ |(\varepsilon_{x_0} - \nu)(f_k)| \leqslant \delta \quad (1 \leqslant k \leqslant n)\}$$

contains $g \cdot \mu$.

III.30, ℓ. −6, −5.

"Let U be a relatively compact open neighborhood of x_0 such that the oscillation of each of the f_k $(1 \leqslant k \leqslant n)$ on U is $\leqslant \delta/2$."

One could take U to be a relatively compact open neighborhood of x_0 such that $|f_k(x) - f_k(x_0)| \leqslant \delta/4$ for all $x \in \mathrm{U}$ and $k = 1, \ldots, n$. But in fact the proof requires only that U be a relatively compact open neighborhood of x_0 such that $|f_k(x) - f_k(x_0)| \leqslant \delta/2$ for all $x \in \mathrm{U}$ and $k = 1, \ldots, n$.

III.31, ℓ. 5.

"Since ν has its support in U, we may identify it with its restriction to U"

The matter is delicate.

Let us write simply ν_{U} for the restriction $\nu\big|\mathrm{U}$ of ν to ν (§2, No. 1); thus, for $g \in \mathscr{K}(\mathrm{U};\mathbf{C})$, $\nu_{\mathrm{U}}(g) = \nu(g')$, where g' is the extension by 0 of g to X. One knows (*loc. cit.*) that $g \mapsto g'$ is a vector space isomorphism (indeed, an algebra isomorphism) of $\mathscr{K}(\mathrm{U};\mathbf{C})$ onto the vector space $\mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$ of continuous functions on X with compact support contained in U, and is clearly an isometry: $\|g'\| = \|g\|$. But this is true for every measure ν and every open set U, and does not exploit the assumption that $\mathrm{Supp}(\nu) \subset \mathrm{U}$.

The text proposes to identify $\|\nu\|$ and $\|\nu_{\mathrm{U}}\|$. These are defined by the formulas

$$\begin{aligned}\|\nu\| &= \sup\{|\nu(f)| : f \in \mathscr{K}(\mathrm{X};\mathbf{C}),\ \|f\| \leqslant 1\}\\ \|\nu_{\mathrm{U}}\| &= \sup\{|\nu_{\mathrm{U}}(g)| : g \in \mathscr{K}(\mathrm{U};\mathbf{C}),\ \|g\| \leqslant 1\}\\ &= \sup\{|\nu(f)| : f \in \mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C}),\ \|f\| \leqslant 1\}.\end{aligned}$$

Since $\|\nu\|$ is defined by a larger class of functions f than is $\|\nu_{\mathrm{U}}\|$, we obviously have $\|\nu\| \geqslant \|\nu_{\mathrm{U}}\|$; the problem is to exploit the hypothesis $\mathrm{Supp}(\nu) \subset \mathrm{U}$ to prove that the two norms are equal. To this end, we employ an auxiliary function $f_0 \in \mathscr{K}(\mathrm{X};\mathbf{C})$.

Let $f_0 \in \mathscr{K}(\mathrm{X};\mathbf{C})$ be such that $0 \leqslant f_0 \leqslant 1$, $f_0 = 1$ on $\mathrm{Supp}(\nu)$ and $\mathrm{Supp}(f_0) \subset \mathrm{U}$ (hence $f_0 = 0$ on $\mathrm{X} - \mathrm{U}$). (The lemma at the end of this note reviews the construction of such a function.)

Proposition A. *For every* $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, *one has* $ff_0 \in \mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$ *and* $\nu(f) = \nu(ff_0)$; *thus if* $g = ff_0\big|\mathrm{U}$, *then* $g' = ff_0$ *and* $\nu_{\mathrm{U}}(g) = \nu(g') = \nu(ff_0) = \nu(f)$.

Proof. Let $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$. Then $\mathrm{Supp}(ff_0) \subset \mathrm{Supp}(f) \cap \mathrm{Supp}(f_0)$, thus $\mathrm{Supp}(ff_0) \subset \mathrm{Supp}(f_0) \subset \mathrm{U}$, whence $ff_0 \in \mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$. Since $ff_0 = f$ on $\mathrm{Supp}(\nu)$, one has $\nu(ff_0) = \nu(f)$ (§2, No. 3, Cor. 1 of Prop. 8).

Proposition B. $\|\nu_{\mathrm{U}}\| = \|\nu\|$.

Proof. For every $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, we have, with notations as in Prop. A,

$$\|g\| = \|g'\| = \|ff_0\| \leqslant \|f\| \cdot \|f_0\| \leqslant \|f\|,$$

therefore

$$|\nu(f)| = |\nu(ff_0)| = |\nu_{\mathrm{U}}(g)| \leqslant \|\nu_{\mathrm{U}}\| \cdot \|g\| \leqslant \|\nu_{\mathrm{U}}\| \cdot \|f\|,$$

whence $\|\nu\| \leqslant \|\nu_{\mathrm{U}}\|$. The reverse inequality was noted earlier.

The foregoing may be summarized as follows:

Proposition C. *If* U *is an open set in* X *and if* ν *is a measure on* X *with compact support contained in* U, *then*

$$\|\nu\| = \sup\{|\nu(f)| :\ f \in \mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C}),\ \|f\| \leqslant 1\}$$

(where $\|f\| = \sup_{x\in\mathrm{X}} |f(x)|$ and $\|\nu\| = \sup\{|\nu(f)| :\ f \in \mathscr{K}(\mathrm{X};\mathbf{C}),\ \|f\| \leqslant 1\}$).

This is the justification for "identifying ν with ν_{U}". The lemma used in the proof is as follows:

Lemma. *If, in a locally compact space* X, A *is a compact set and* U *is an open set such that* $\mathrm{A} \subset \mathrm{U}$, *then there exist* (i) *an open set* V *and a compact set* B *such that*

$$\mathrm{A} \subset \mathrm{V} \subset \mathrm{B} \subset \mathrm{U}$$

(briefly, every open neighborhood of the compact set A contains a compact neighborhood of A), *and* (ii) *a continuous function* $f : \mathrm{X} \to [0,1]$ *such that* $f = 1$ *on* A *and* $f = 0$ *on* $\complement\mathrm{B}$ (thus $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and $\mathrm{Supp}(f) \subset \mathrm{B}$, so that $f \in \mathscr{K}(\mathrm{X},\mathrm{B};\mathbf{C}) \subset \mathscr{K}(\mathrm{X},\mathrm{U};\mathbf{C})$).

Proof. (i) By local compactness, U contains a compact neighborhood of each of its points; a covering of A by finitely many such compact neighborhoods yields both V (the union of their interiors) and B (their union).

(ii) Since the compact space B is normal (GT, IX, §4, No. 1, Prop. 1), by Urysohn's theorem there exists a continuous function $g : \mathrm{B} \to [0,1]$ such that $g = 1$ on A and $g = 0$ on $\mathrm{B} - \mathrm{V}$ (*loc. cit.*, Th. 1). Defining $f : \mathrm{X} \to [0,1]$ by the formulas

$$f(x) = \begin{cases} g(x) & \text{for } x \in \mathrm{B} \\ 0 & \text{for } x \in \mathrm{X} - \mathrm{V} \end{cases}$$

(both formulas yield 0 on $\mathrm{B} \cap (\mathrm{X} - \mathrm{V}) = \mathrm{B} - \mathrm{V}$), f is continuous on X (GT, I, §3, No. 2, Prop. 4).

III.31, ℓ. 6–8.

"...the hypothesis $\|\nu\| = 1$ then implies that there exists a function $h \in \mathscr{K}(\mathrm{X}; \mathbf{C})$, with support contained in U, such that $\|h\| \leqslant 1$ and such that $|\alpha_k(1 - \nu(h))| \leqslant \delta/2$ for $1 \leqslant k \leqslant n$."

It is here that we need Prop. C of the preceding note: Since $\|\nu\| = 1$, there exists a function $h \in \mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})$ such that $\|h\| \leqslant 1$ and $|\nu(h)|$ is as close to $\|\nu\| = 1$ as we like, in particular, close enough so that

$$|\alpha_k(1 - |\nu(h)|)| \leqslant \delta/2 \quad \text{for } 1 \leqslant k \leqslant n\,;$$

replacing h by ch for suitable $|c| = 1$, we can suppose that $|\nu(h)| = \nu(h)$, whence the desired inequalities.

III.31, ℓ. 8, 9.

"The definition of U moreover shows that $|(\alpha_k - f_k(x))h(x)| \leqslant \delta/2$ for all $x \in \mathrm{U}$"

Recall that $\|h\| \leqslant 1$. Since the inequalities hold trivially for $x \in \complement\mathrm{U}$ (because $\mathrm{Supp}(h) \subset \mathrm{U}$), we have in fact $\|\alpha_k h - f_k h\| \leqslant \delta/2$.

III.31, ℓ. 9, 10.

"...since $\|\nu\| = 1$ and $\mathrm{Supp}(\nu) \subset \mathrm{U}$ we therefore have $|\nu((\alpha_k - f_k)h)| \leqslant \delta/2$"

For,

$$\big|\nu\big((\alpha_k - f_k)h\big)\big| \leqslant \|\nu\| \cdot \|(\alpha_k - f_k)h)\| = \|\alpha_k h - f_k h\| \leqslant \delta/2$$

by the preceding note.

III.31, ℓ. 10–12.

"...setting $g = g_0 h$,

$$|f_k(x_0) - \mu(gf_k)| \leqslant \delta \qquad \text{for } 1 \leqslant k \leqslant n\,."$$

For,

$$\begin{aligned} |f_k(x_0) - \mu(gf_k)| &= |f_k(x_0) - \mu(g_0 h f_k)| \\ &= |f_k(x_0) - (g_0 \cdot \mu)(f_k h)| \\ &= |\alpha_k - \nu(f_k h)| \\ &\leqslant \big|\alpha_k\big(1 - \nu(h)\big)\big| + \big|\nu(\alpha_k h - f_k h)\big| \leqslant \delta/2 + \delta/2\,. \end{aligned}$$

III.31, ℓ. 13.

"This proves the proposition, since $\|g \cdot \mu\| = \|(g_0 h) \cdot \mu\| \leqslant \|g_0 \cdot \mu\| = 1$."

For,

$$\|g \cdot \mu\| = \|(g_0 h) \cdot \mu\| = \|h \cdot (g_0 \cdot \mu)\| \leqslant \|h\| \cdot \|g_0 \cdot \mu\| \leqslant \|g_0 \cdot \mu\| = \|\nu\| = 1$$

(§1, No. 4 and No. 8, Prop. 12).

III.31, ℓ. 19–21.

"Conversely, if $\operatorname{Supp}(\nu) \subset \operatorname{Supp}(\mu)$ then ν is the vague limit of measures with *finite* support contained in $\operatorname{Supp}(\mu)$ (Th. 1), hence is in the vague closure of the set of measures $g \cdot \mu$ by Prop. 13."

Suppose first that ν has finite support contained in $\operatorname{Supp}(\mu)$; say $\nu = \sum_{k=1}^{n} c_k \varepsilon_{a_k}$ for suitable points $a_k \in \operatorname{Supp}(\mu)$ (Prop. 12). Note that the set

$$\mathscr{K}(\mathrm{X}; \mathbf{C}) \cdot \mu = \{ g \cdot \mu : \ g \in \mathscr{K}(\mathrm{X}; \mathbf{C}) \}$$

is a linear subspace of $\mathscr{M}(\mathrm{X}; \mathbf{C})$, hence its vague closure

$$\mathscr{V} = \overline{\{ g \cdot \mu : \ g \in \mathscr{K}(\mathrm{X}; \mathbf{C}) \}}$$

is also linear subspace of $\mathscr{M}(\mathrm{X}; \mathbf{C})$. It follows from Prop. 13 that $\varepsilon_a \in \mathscr{V}$ for every $a \in \operatorname{Supp}(\mu)$, in particular for every a_k $(k = 1, \ldots, n)$, hence $\nu \in \mathscr{V}$.

Now let ν be any measure with $\operatorname{Supp}(\nu) \subset \operatorname{Supp}(\mu)$. We know from Th. 1 that there exists a directed family (ν_j) of measures, with finite support contained in $\operatorname{Supp}(\mu)$, such that $\nu_j \to \nu$ vaguely; since $\nu_j \in \mathscr{V}$ by the foregoing, and $\mathscr{V}$ is vaguely closed, we conclude that $\nu \in \mathscr{V}$.

Remarks. $\mathscr{M}(\mathrm{X}; \mathbf{C})$ is of course a module over the ring $\mathscr{K}(\mathrm{X}; \mathbf{C})$, via the mapping $(f, \rho) \mapsto f \cdot \rho$; thus $\mathscr{V}$ is the vague closure of the cyclic submodule $\mathscr{K}(\mathrm{X}; \mathbf{C}) \cdot \mu$ of $\mathscr{M}(\mathrm{X}; \mathbf{C})$. In fact, $\mathscr{V}$ is a submodule of $\mathscr{M}(\mathrm{X}; \mathbf{C})$, that is,

$$h \in \mathscr{K}(\mathrm{X}; \mathbf{C}), \ \nu \in \mathscr{V} \quad \Rightarrow \quad h \cdot \nu \in \mathscr{V}.$$

For, if (g_j) is a directed family in $\mathscr{K}(\mathrm{X}; \mathbf{C})$ such that $g_j \cdot \mu \to \nu$ vaguely, then $h \cdot (g_j \cdot \mu) \to h \cdot \nu$ vaguely, since, for every $f \in \mathscr{K}(\mathrm{X}; \mathbf{C})$,

$$\big(h \cdot (g_j \cdot \mu)\big)(f) = \mu(h g_j f) = (g_j \cdot \mu)(hf) \to \nu(hf) = (h \cdot \nu)(f);$$

thus $(h \cdot g_j) \cdot \mu \to h \cdot \nu$ vaguely, which shows that $h \cdot \nu \in \mathscr{V}$. Thus $\mathscr{V}$ is the vaguely closed submodule of $\mathscr{M}(\mathrm{X}; \mathbf{C})$ generated by μ; the Corollary provides a remarkable characterization of it.

In particular, if $\operatorname{Supp}(\mu) = \mathrm{X}$, then $\mathscr{V} = \mathscr{M}(\mathrm{X}; \mathbf{C})$, so to speak $\mathscr{M}(\mathrm{X}; \mathbf{C})$ is a "topologically cyclic" module with generator μ. (Example: $\mathrm{X} = \mathbf{R}$ (or

$X = [0,1]$), with μ the Lebesgue measure.) Thus the corollary yields a striking characterization of "topologically cyclic" measures:

$$\overline{\mathscr{K}(X;\mathbf{C})\cdot\mu} = \mathscr{M}(X;\mathbf{C}) \quad \Leftrightarrow \quad \mathrm{Supp}(\mu) = X$$

(closure with respect to the vague topology on $\mathscr{M}(X;\mathbf{C})$).

Is $\mathscr{M}(X;\mathbf{C})$ a topological module over $\mathscr{K}(X;\mathbf{C})$ (for the direct limit topology on $\mathscr{K}(X;\mathbf{C})$ and the vague topology on $\mathscr{M}(X;\mathbf{C})$), i.e., is the bilinear mapping

$$\mathscr{K}(X;\mathbf{C}) \times \mathscr{M}(X;\mathbf{C}) \to \mathscr{M}(X;\mathbf{C})$$

defined by $(g,\rho) \mapsto g\cdot\rho$ continuous (jointly in the variables) for the indicated topologies? I suspect that the answer is usually "no" and that the case of $X = \mathbf{N}$ with the discrete topology may provide a counter-example (cf. TVS, III, §5, Exer. 3, TVS, IV, §1, Exer. 7, and the note for III.16, ℓ. 13–15). A counterexample with X compact is proposed in §1, Exer. 12 *d*) (possible candidate: the 1-point compactification of the discrete space $\mathbf{N}$?). The following is a more elementary observation:

Proposition. *The above mapping* $(g,\rho) \mapsto g\cdot\rho$ *is always separately continuous in the variables.*

Proof. If $\rho_j \to \rho$ vaguely then $g\cdot\rho_j \to g\cdot\rho$ vaguely; for, if $f \in \mathscr{K}(X;\mathbf{C})$ then

$$(g\cdot\rho_j)(f) = \rho_j(gf) \to \rho(gf) = (g\cdot\rho)(f)\,.$$

On the other hand, if $g_j \to g$ for the direct limit topology, the claim is that $g_j\cdot\rho \to g\cdot\rho$ vaguely for every measure ρ. This entails showing that for every $f \in \mathscr{K}(X;\mathbf{C})$ and every measure ρ,

$$\rho(g_j f) \to \rho(gf)\,;$$

for this, we need only show that the mapping $u : g \mapsto gf$ is a continuous endomorphism of $\mathscr{K}(X;\mathbf{C})$. To this end, if K is any compact subset of X, and $u_K : \mathscr{K}(X,K;\mathbf{C}) \to \mathscr{K}(X;\mathbf{C})$ is the canonical injection, it suffices to show that the composite mapping $u \circ u_K$,

$$\begin{gathered}\mathscr{K}(X,K;\mathbf{C}) \to \mathscr{K}(X;\mathbf{C}) \to \mathscr{K}(X;\mathbf{C})\\ g \mapsto g \mapsto fg\end{gathered}$$

is continuous. It is convenient to factor $u\circ u_K$ another way: let K_0 be the support of f, let $v : \mathscr{K}(X,K;\mathbf{C}) \to \mathscr{K}(X,K_0;\mathbf{C})$ be the mapping $v(g) = fg$, and consider the factorization $u \circ u_K = u_{K_0} \circ v$:

$$\begin{gathered}\mathscr{K}(X,K;\mathbf{C}) \to \mathscr{K}(X,K_0;\mathbf{C}) \to \mathscr{K}(X;\mathbf{C})\\ g \mapsto fg \mapsto fg\end{gathered}$$

The continuity of v ($\|fg\| \leqslant \|f\|\,\|g\|$) and that of u_{K_0} assures that of $u \circ u_{\mathrm{K}}$ and completes the proof of the Proposition.

III.31, ℓ. −9 to −7.

"For every $a \in \mathrm{N}$ and every neighborhood V of a, there exists a function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with support contained in V, equal to 1 at the point a and to 0 at the other points of N, whence $\mu(f) = h(a) \neq 0$."

Since N is discrete, $\{a\}$ is an open subset of N, hence there exists a neighborhood W of a in X such that $\mathrm{W} \cap \mathrm{N} = \{a\}$; replacing W by $\mathrm{V} \cap \mathrm{W}$, we can suppose that $\mathrm{W} \subset \mathrm{V}$. Choose $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ so that $f(a) = 1$ and $\mathrm{Supp}(f) \subset \mathrm{W}$; then $f = 0$ on $\complement\mathrm{W}$, hence on $\mathrm{N} \cap \complement\mathrm{W} = \mathrm{N} - \mathrm{W} = \mathrm{N} - \mathrm{W} \cap \mathrm{N} = \mathrm{N} - \{a\}$, that is, $f = 0$ at the points of N other than a; moreover, $\mathrm{Supp}(f) \subset \mathrm{W} \subset \mathrm{V}$ as desired. Finally,

$$\begin{aligned}
\mu(f) &= \sum_{x \in \mathrm{X}} h(x) f(x) && \text{(definition of } \mu \text{)} \\
&= \sum_{x \in \mathrm{N}} h(x) f(x) && (\mathrm{N} = \{x \in \mathrm{X} :\ h(x) \neq 0\,) \\
&= h(a) f(a) && (\,f = 0 \text{ on } \mathrm{N} - \{a\}\,) \\
&= h(a) \cdot 1 \neq 0 && (\,f(a) = 1 \text{ and } a \in \mathrm{N}\,).
\end{aligned}$$

To summarize: Assuming $a \in \mathrm{N}$, for every neighborhood V of a there exists a function $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ such that $\mathrm{Supp}(f) \subset \mathrm{V}$ and $\mu(f) \neq 0$, whence $a \in \mathrm{Supp}(\mu)$ (No. 2); thus $\mathrm{N} \subset \mathrm{Supp}(\mu)$.

III.31, ℓ. −5, −4.

"...for every function $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with support contained in W, we therefore have $\mu(g) = 0$, which proves that $b \notin \mathrm{Supp}(\mu)$."

Calculating as in the preceding note, we have

$$\begin{aligned}
\mu(g) &= \sum_{x \in \mathrm{N}} h(x) g(x) \\
&= 0 && \text{(because } g = 0 \text{ on } \complement\mathrm{W} \supset \mathrm{N}\text{)}.
\end{aligned}$$

To summarize: Assuming $b \notin \mathrm{N}$, the existence of such a neighborhood W of b assures that $b \notin \mathrm{Supp}(\mu)$ (No. 2). Thus $\complement\mathrm{N} \subset \complement\mathrm{Supp}(\mu)$, i.e., $\mathrm{Supp}(\mu) \subset \mathrm{N}$.

III.31, ℓ. −1 to **III.32,** ℓ. 1.

"...the restriction of μ to V_a is therefore a point measure with support $\{a\}$ (No. 2, Prop. 5)"

$\mathrm{Supp}(\mu|\mathrm{V}_a) = \mathrm{V}_a \cap \mathrm{Supp}(\mu) = \mathrm{V}_a \cap \mathrm{N} = \{a\}$; quote Prop. 12.

III.32, ℓ. 2.

"…is of the form $h(a)\varepsilon_a\ldots$"

The scalar coefficient yielded by the cited Prop. 12 is here written $h(a)$ in anticipation of the function h being defined.

III.32, ℓ. 2–4.

"Setting $h(x)=0$ at the points of $\complement\mathrm{N}$, and denoting by ν the measure defined by the masses $h(x)$, the principle of localization shows that $\nu=\mu$."

Let us first establish that the function $h:\mathrm{X}\to\mathbf{C}$ so defined is eligible to define a discrete measure. Indeed, for every compact subset K of X, $\mathrm{N}\cap\mathrm{K}$ is finite; for, since N is closed, $\mathrm{N}\cap\mathrm{K}$ is a compact subset of the discrete space N, hence is covered by finitely many of the open subsets $\{x\}$ $(x\in\mathrm{N})$ of N. It follows from the discussion in §1, No. 3, Example I that a discrete measure ν on X can be defined, for every $f\in\mathscr{K}(\mathrm{X};\mathbf{C})$, by the formula

$$\begin{aligned}\nu(f)&=\sum_{x\in\mathrm{X}}h(x)f(x)\\&=\sum_{a\in\mathrm{N}}h(a)f(a)\qquad\qquad(h=0\ \text{on}\ \complement\mathrm{N})\\&=\sum_{a\in\mathrm{N}\cap\mathrm{Supp}(f)}h(a)\varepsilon_a(f)\qquad(f=0\ \text{on}\ \complement\,\mathrm{Supp}(f))\end{aligned}$$

(a finite sum, since $\mathrm{Supp}(f)$ is compact).

Note that if $a\in\mathrm{N}$ then $\nu\big|\mathrm{V}_a=h(a)\varepsilon_a$; for, if $f\in\mathscr{K}(\mathrm{X},\mathrm{V}_a;\mathbf{C})$ then $f=0$ on $\complement\mathrm{V}_a$, hence on $\mathrm{N}-\mathrm{V}_a$, thus the above sum reduces to $h(a)\varepsilon_a(f)$, that is, $\nu(f)=h(a)\varepsilon_a(f)$. Thus,

$$\text{(i)}\qquad \nu\big|\mathrm{V}_a=h(a)\varepsilon_a=\mu\big|\mathrm{V}_a\quad\text{for all } a\in\mathrm{N}.$$

Now, N is *defined* to be equal to $\mathrm{Supp}(\mu)$, and the proof of "necessity" shows that $\mathrm{Supp}(\nu)=\mathrm{N}$. Thus

$$\text{(ii)}\qquad \mathrm{Supp}(\mu)=\mathrm{N}=\mathrm{Supp}(\nu).$$

For each $x\in\complement\mathrm{N}$ define $\mathrm{V}_x=\complement\mathrm{N}=\complement\,\mathrm{Supp}(\mu)=\complement\,\mathrm{Supp}(\nu)$; then

$$\text{(iii)}\qquad \nu\big|\mathrm{V}_x=0=\mu\big|\mathrm{V}_x\quad\text{for all } x\in\complement\mathrm{N}.$$

From (i) and (iii), we see that $(\mathrm{V}_x)_{x\in\mathrm{X}}$ is an open covering of X such that

$$\nu\big|\mathrm{V}_x=\mu\big|\mathrm{V}_x\quad\text{for all } x\in\mathrm{X},$$

therefore $\mu = \nu$ by the principle of localization (No. 1, Cor. of Prop. 1), thus μ is indeed discrete.

§3. INTEGRALS OF CONTINUOUS VECTOR-VALUED FUNCTIONS

III.32, ℓ. 14–16.

"… E'^*, *equipped with the weak topology* $\sigma(\mathrm{E}'^*, \mathrm{E}')$, *may be canonically identified with the completion of* E *equipped with the weakened topology* $\sigma(\mathrm{E}, \mathrm{E}')$."

In TVS, II, §6, No. 7, Prop. 9, set $\mathrm{F} = \mathrm{E}$ and $\mathrm{G} = \mathrm{E}'$, placed in duality via $\langle \mathbf{z}, \mathbf{z}' \rangle = \mathbf{z}'(\mathbf{z})$.

{In particular, regarding $\mathrm{E} \subset \widehat{\mathrm{E}}$ (the completion of E for the weakened topology $\sigma(\mathrm{E}, \mathrm{E}')$), if $\mathbf{z} \in \mathrm{E}$ then the corresponding element of E'^* is the linear form $\mathbf{z}' \mapsto \mathbf{z}'(\mathbf{z})$. The extension of the correspondence to general elements $\widehat{\mathbf{z}}$ of $\widehat{\mathrm{E}}$ entails identifying E' with $\widehat{\mathrm{E}}'$ (a continuous linear form $\mathbf{z}'$ on E being extended to $\widehat{\mathrm{E}}$ by continuity).}

The proof of the cited Prop. 9 is based on the fact that if G is any vector space over $\mathbf{C}$, and G^* is its algebraic dual equipped with the topology $\sigma(\mathrm{G}^*, \mathrm{G})$, then G^* is a complete topological vector space, a consequence of the observation that G^* *is isomorphic, as a topological vector space, to the product space* $\mathbf{C}^{\mathrm{I}}$, *where* I *is a set whose cardinality is equal to the dimension of* G (TVS, §6, No. 6, Cor. 2 of Prop. 8). An alternative proof of the latter observation is as follows.

Let $(e_\iota)_{\iota \in \mathrm{I}}$ be a basis of G indexed faithfully ($\iota \neq \kappa \Rightarrow e_\iota \neq e_\kappa$), and define a mapping $v : \mathrm{G}^* \to \mathbf{C}^{\mathrm{I}}$ by

$$v(y^*) = \big(y^*(e_\iota)\big)_{\iota \in \mathrm{I}} \quad \text{for } y^* \in \mathrm{G}^* .$$

The linearity of v follows directly from the definitions: for y^*, z^* in G^*,

$$\begin{aligned} v(y^* + z^*) &= \big((y^* + z^*)(e_\iota)\big)_{\iota \in \mathrm{I}} = \big(y^*(e_\iota) + z^*(e_\iota)\big)_{\iota \in \mathrm{I}} \\ &= \big(y^*(e_\iota)\big)_{\iota \in \mathrm{I}} + \big(z^*(e_\iota)\big)_{\iota \in \mathrm{I}} = v(y^*) + v(z^*) , \end{aligned}$$

and similarly $v(cy^*) = c\,v(y^*)$ for every scalar c. If $v(y^*) = 0$ then $y^*(e_\iota) = 0$ for all $\iota \in \mathrm{I}$, whence $y^* = 0$ by linearity; thus v is injective. Given any $(c_\iota)_{\iota \in \mathrm{I}} \in \mathbf{C}^{\mathrm{I}}$, the linear form $y^* \in \mathrm{G}^*$ for which $y^*(e_\iota) = c_\iota$ for all $\iota \in \mathrm{I}$ satisfies $v(y^*) = (c_\iota)_{\iota \in \mathrm{I}}$; thus v is surjective, hence is a vector space isomorphism.

The bicontinuity of v is perhaps most easily seen when convergence is cast in terms of directed families ("nets"). For each $y \in \mathrm{G}$, let $f_y : \mathrm{G}^* \to \mathbf{C}$ be the linear form defined by $f_y(y^*) = y^*(y)$ for all $y^* \in \mathrm{G}^*$; thus the topology $\sigma(\mathrm{G}^*, \mathrm{G})$ on G^* is by definition the initial topology for the family $(f_y)_{y \in \mathrm{G}}$ of linear forms. It follows that if $y^* \in \mathrm{G}^*$ and $\mathfrak{F}$ is a filter on G^*, then

$$\mathfrak{F} \to y^* \text{ in } \mathrm{G}^* \quad \Leftrightarrow \quad f_y(\mathfrak{F}) \to f_y(y^*) = y^*(y) \text{ for all } y \in \mathrm{G}$$

(GT, I, §7, No. 6, Prop. 10). Translated in terms of nets, this says that for a directed family (y^*_α) in G^*,

$$y^*_\alpha \to y^* \text{ in } \mathrm{G}^* \quad \Leftrightarrow \quad y^*_\alpha(y) \to y^*(y) \text{ in } \mathbf{C} \text{ for every } y \in \mathrm{G}\,.$$

Since every $y \in \mathrm{G}$ is a finite linear combination of the basis vectors e_ι, and since the linear operations in $\mathbf{C}$ are continuous, an equivalent condition is that

$$y^*_\alpha(e_\iota) \to y^*(e_\iota) \quad \text{for all } \iota \in \mathrm{I}\,,$$

that is, for every $\iota \in \mathrm{I}$, the ι'th coordinate of $v(y_\alpha{}^*)$ converges to the ι'th coordinate of $v(y^*)$; since the product topology on $\mathbf{C}^{\mathrm{I}}$ is the initial topology for the family of coordinate projections, this is in turn equivalent to the condition $v(y^*_\alpha) \to v(y^*)$ in $\mathbf{C}^{\mathrm{I}}$ (cf. J.L. Kelley, *General topology*, p. 91, Th. 4). Thus

$$y^*_\alpha \to y^* \text{ in } \mathrm{G}^* \quad \Leftrightarrow \quad v(y^*_\alpha) \to v(y^*) \text{ in } \mathbf{C}^{\mathrm{I}}\,,$$

whence the asserted bicontinuity of v. Since continuous linear mappings are automatically uniformly continuous (GT, III, §3, No. 1, Prop. 3), and since $\mathbf{C}^{\mathrm{I}}$ is complete (GT, II, §3, No. 5, Prop. 10), one concludes that G^* is complete.

III.32, ℓ. −16.

"...into $\mathbf{C}$ ..."

More precisely, into $\mathbf{R}$ or $\mathbf{C}$ as the case may be (cf. the preamble to the section).

III.32, ℓ. −16, −15.

"... (in other words the mapping $\mathbf{z}' \circ \mathbf{f}$, also denoted $\langle \mathbf{f}, \mathbf{z}' \rangle$) is continuous."

Since $\sigma(\mathrm{E}, \mathrm{E}')$ is the initial topology for the family of mappings $\mathbf{z}' : \mathrm{E} \to \mathbf{C}$ ($\mathbf{z}' \in \mathrm{E}$), the weak continuity of $\mathbf{f}$ can also be expressed by saying that $\mathbf{f} : \mathrm{X} \to \mathrm{E}$ is continuous when E is equipped with its weakened topology $\sigma(\mathrm{E}, \mathrm{E}')$.

III.32, ℓ. $-10, -9$.

"... they are, however, equal when E is finite-dimensional."

Assume E is a finite-dimensional Hausdorff topological vector space; then the weakened topology $\sigma(\mathrm{E}, \mathrm{E}')$ on E coincides with the original topology of E (TVS, I, §2, No. 3, Th. 2). Therefore every weakly continuous function $\mathbf{f} : \mathrm{X} \to \mathrm{E}$ is continuous (the converse is trivially true for any topological vector space E). We are to show that $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E}) \subset \mathscr{K}(\mathrm{X};\mathrm{E})$ (whence equality).

Let $\mathbf{f} \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$. By the preceding remark, $\mathbf{f}$ is continuous. Let $\mathbf{z}'_1, \ldots, \mathbf{z}'_n$ be a finite separating set in E' (e.g., a basis of E', which we know to be finite-dimensional with same dimension as E, by A, II, §7, No. 5, Th. 4 and TVS, I, §2, No. 3, Cor. 2 of Th. 2). Then, for $x \in \mathrm{X}$,

$$\mathbf{f}(x) \neq 0 \quad \Leftrightarrow \quad \mathbf{z}'_k\big(\mathbf{f}(x)\big) \neq 0 \text{ for some } k\,,$$

thus

$$\{x :\ \mathbf{f}(x) \neq 0\,\} = \bigcup_{k=1}^{n} \{x :\ (\mathbf{z}'_k \circ \mathbf{f})(x) \neq 0\,\}\,,$$

whence

$$\overline{\{x :\ \mathbf{f}(x) \neq 0\,\}} = \bigcup_{k=1}^{n} \overline{\{x :\ (\mathbf{z}'_k \circ \mathbf{f})(x) \neq 0\,\}}\,,$$

that is,

$$\operatorname{Supp}(\mathbf{f}) = \bigcup_{k=1}^{n} \operatorname{Supp}(\mathbf{z}'_k \circ \mathbf{f})\,;$$

by hypothesis, each term of the union is compact, whence $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$.

III.33, ℓ. 6–11.

"DEFINITION 1. — *For every function* $\mathbf{f} \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$ *we call integral of* $\mathbf{f}$ *with respect to* μ, *and denote by* $\int \mathbf{f}\,d\mu$ *or* $\int \mathbf{f}(x)\,d\mu(x)$, *or* $\int \mathbf{f}\mu$, *or* $\int \mathbf{f}(x)\mu(x)$, *the element of* E'^* *defined by*

$$\left\langle \int \mathbf{f}\,d\mu, \mathbf{z}' \right\rangle = \int \langle \mathbf{f}, \mathbf{z}' \rangle\,d\mu \quad \textit{for all } \mathbf{z}' \in \mathrm{E}'\,. \tag{1}$$

We note that even if E is *Hausdorff* and $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$, *one does not necessarily have* $\int \mathbf{f}\,d\mu \in \mathrm{E}$ (Exer. 1; cf. No. 3)."

It is noteworthy that the notation $\mu(\mathbf{f})$ is not proposed for $\int \mathbf{f}\,d\mu$ (a function, more precisely, a linear form on E'), nor for the generalization of $\int \mathbf{f}\,d\mu$ given in Ch. VI, §1, No. 1, Def. 1; yet it is authorized for the case of a vectorial measure $\mathbf{m} : \widetilde{\mathscr{K}}(\mathrm{X}) \to \mathrm{E}$ (E a Banach space) (Ch. VI, §2, No. 2, Def. 2), where $\int f\, d\mathbf{m} \in \mathrm{E}'^{*}$ is denoted $\mathbf{m}(f)$ for (in particular) $f \in \mathscr{K}(\mathrm{X})$.

For $\int \mathbf{f}\, d\mu$ to belong to E (after identifying each element $\mathbf{z}$ of E with the linear form on E' defined by $\mathbf{z}' \mapsto \mathbf{z}'(\mathbf{z})$ for all $\mathbf{z}' \in \mathrm{E}'$) means that there exists a vector $\mathbf{z_f} \in \mathrm{E}$ such that

$$\Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle = \big\langle \mathbf{z_f}, \mathbf{z}' \big\rangle \quad \text{for all } \mathbf{z}' \in \mathrm{E}' .$$

Conditions under which this occurs are discussed in No. 3, Cor. 2 of Prop. 7 (assuming E to be a quasi-complete Hausdorff locally convex space), but only for functions $\mathbf{f}$ in $\mathscr{K}(\mathrm{X};\mathrm{E})$. Criteria assuring that $\int \mathbf{f}\, d\mu$ belongs to E for functions $\mathbf{f} \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$ are given in Ch. VI, §1, No. 2, Cor. of Prop. 7 (with extra conditions on E and $\mathbf{f}$) and Prop. 8 (with extra conditions on μ and $\mathbf{f}$).

When E is a Banach space, so that the quasi-completeness condition cited above is trivially satisfied and $\int \mathbf{f}\, d\mu \in \mathrm{E}$ for every $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$, the notation $\mu(\mathbf{f})$ is authorized by Ch. IV, §4, No. 1, Def. 1, as well as by Ch. V, §1, No. 3, Def. 3; one would suppose that it would also be tolerated in the case that E is a quasi-complete Hausdorff locally convex space.

The issue does not come up in the case of a vector-valued measure $\mathbf{m}$ (Ch. VI, §2, No. 1, Def. 1), where $\mathbf{m}$ is given at the outset to be a continuous linear mapping of $\mathscr{K}(\mathrm{X})$ into a Banach space E and $\int f\, d\mathbf{m}$ is *defined* to be $\mathbf{m}(f)$ for $f \in \mathscr{K}(\mathrm{X})$ (*loc. cit.*, No. 2, Def. 2); similarly for the case of integration of a vector-valued function $\mathbf{f}$ with respect to a vector-valued measure $\mathbf{m}$ (*loc. cit.*, No. 7, Prop. 11 and the remark following its proof).

A final instance of the use of $\mu(\mathbf{f})$ occurs in the extension of the theory to Hausdorff topological spaces X, with E a Banach space (remarks following Prop. 16 in Ch. IX, §1, No. 10).

SUMMARY. I see no clear pattern in the foregoing, but sense a tendency: One writes $\int \mathbf{f}\, d\mu = \mu(\mathbf{f})$ either when this is the *definition* of $\int \mathbf{f}\ d\mu$ or when $\int \mathbf{f}\, d\mu$ turns out to be an element of the ambient locally convex space E; in the two cases cited above where the notation $\mu(\mathbf{f})$ is *not* proposed, $\int \mathbf{f}\, d\mu$ is an element of an algebraic dual E'^{*}.

At any rate, the issue seems unimportant; but the above tour of definitions may serve as a useful preview of forthcoming concepts.

III.33, ℓ. −8 to −6.

"... we then have $\mathbf{f}(x) = \sum\limits_{i=1}^{n} f_i(x)\mathbf{e}_i$ for all $x \in \mathrm{X}$, and

$$\int \mathbf{f}\, d\mu = \sum_{i=1}^{n} \mu(f_i)\mathbf{e}_i \,."$$

The first equality follows from the computation (for $k = 1, \dots, n$ and $x \in \mathrm{X}$)

$$\Big\langle \sum_{i=1}^{n} f_i(x)\mathbf{e}_i, \mathbf{e}'_k \Big\rangle = f_k(x) = (\mathbf{e}'_k \circ \mathbf{f})(x) = \mathbf{e}'_k\big(\mathbf{f}(x)\big) = \langle \mathbf{f}(x), \mathbf{e}'_k \rangle \,.$$

Thus $\mathbf{f} = \sum\limits_{i=1}^{n} f_i\mathbf{e}_i$. To prove the second equality, it suffices to show that for each i, the function $\mathbf{g}_i = f_i\mathbf{e}_i$ has integral $\mu(f_i)\mathbf{e}_i$. Indeed,

$$\begin{aligned}(\mathbf{z}' \circ \mathbf{g}_i)(x) &= \mathbf{z}'\big(\mathbf{g}_i(x)\big) = \mathbf{z}'\big(f_i(x)\mathbf{e}_i\big) \\ &= f_i(x)\mathbf{z}'(\mathbf{e}_i) = \langle \mathbf{e}_i, \mathbf{z}' \rangle f_i(x) \,,\end{aligned}$$

thus $\mathbf{z}' \circ \mathbf{g}_i = \langle \mathbf{e}_i, \mathbf{z}' \rangle f_i$, whence

$$\begin{aligned}\Big\langle \int \mathbf{g}_i \, d\mu, \mathbf{z}' \Big\rangle &= \int \langle \mathbf{g}_i, \mathbf{z}' \rangle \, d\mu = \int (\mathbf{z}' \circ \mathbf{g}_i) \, d\mu \\ &= \langle \mathbf{e}_i, \mathbf{z}' \rangle \int f_i \, d\mu = \mu(f_i)\langle \mathbf{e}_i, \mathbf{z}' \rangle \\ &= \langle \mu(f_i)\mathbf{e}_i, \mathbf{z}' \rangle \,,\end{aligned}$$

therefore $\int \mathbf{g}_i \, d\mu = \mu(f_i)\mathbf{e}_i$.

III.34, ℓ. 8–10.

"... we have

$$\int \mathbf{f} \, d\varepsilon_y = \mathbf{f}(y)$$

because $\int \langle \mathbf{f}, \mathbf{z}' \rangle \, d\varepsilon_y = \langle \mathbf{f}(y), \mathbf{z}' \rangle$ by definition."

For all $\mathbf{z}' \in \mathrm{E}'$,

$$\begin{aligned}\Big\langle \int \mathbf{f} \, d\varepsilon_y, \mathbf{z}' \Big\rangle &= \int \langle \mathbf{f}, \mathbf{z}' \rangle \, d\varepsilon_y = \int (\mathbf{z}' \circ \mathbf{f}) \, d\varepsilon_y \\ &= (\mathbf{z}' \circ \mathbf{f})(y) = \mathbf{z}'\big(\mathbf{f}(y)\big) = \langle \mathbf{f}(y), \mathbf{z}' \rangle \,.\end{aligned}$$

III.34, ℓ. 12, 13.

"... the duals E' and E'_1 are identical ..."

If $\pi : \mathrm{E} \to \mathrm{E}_1 = \mathrm{E}/\mathrm{N}$ is the canonical mapping and $\mathbf{z}'_1 \in \mathrm{E}'_1$ then $\mathbf{z}'_1 \circ \pi \in \mathrm{E}'$, whence a linear mapping $u : \mathrm{E}'_1 \to \mathrm{E}'$, $u(\mathbf{z}'_1) = \mathbf{z}'_1 \circ \pi$, which is injective since π is surjective. If $\mathbf{z}' \in \mathrm{E}'$ then its kernel $\overset{-1}{\mathbf{z}'}(0)$ is a closed subset of E containing 0, therefore $\overset{-1}{\mathbf{z}'}(0) \supset \mathrm{N}$; it follows that there is a unique linear form $\mathbf{z}'_1$ on E_1 such that $\mathbf{z}' = \mathbf{z}'_1 \circ \pi$. Since E_1 carries the final topology for π (TVS, II, §4, No. 4, Example I), the continuity of $\mathbf{z}'_1 \circ \pi = \mathbf{z}'$ implies that of $\mathbf{z}'_1$. Thus $\mathbf{z}'_1 \in \mathrm{E}'_1$ and $u(\mathbf{z}'_1) = \mathbf{z}'$, so u is surjective (hence is a vector space isomorphism).

III.34, ℓ. 13–15.

"... for a function f to belong to $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$, it is necessary and sufficient that $f_1 = \pi \circ f$ (where $\pi : \mathrm{E} \to \mathrm{E}_1$ is the canonical homomorphism) belong to $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E}_1)$, in which case $\int f\,d\mu = \int f_1\,d\mu$."

In the notations of the preceding note, if $\mathbf{z}'_1 \in \mathrm{E}'_1$ and $\mathbf{z}' = \mathbf{z}'_1 \circ \pi$, and if $f : \mathrm{X} \to \mathrm{E}$ and $f_1 = \pi \circ f : \mathrm{X} \to \mathrm{E}_1$, one has

$$\mathbf{z}' \circ f = (\mathbf{z}'_1 \circ \pi) \circ f = \mathbf{z}'_1 \circ (\pi \circ f) = \mathbf{z}'_1 \circ f_1 ;$$

thus $\mathbf{z}' \circ f = \mathbf{z}'_1 \circ f_1$, whence trivially $\mathbf{z}' \circ f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ if and only if $\mathbf{z}'_1 \circ f_1 \in \mathscr{K}(\mathrm{X};\mathbf{C})$. It follows that

$$f \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E}) \quad \Leftrightarrow \quad f_1 \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E}_1),$$

and in this case, for all such pairings $\mathbf{z}' \leftrightarrow \mathbf{z}'_1$, $f \leftrightarrow f_1$, one has

$$\mu(\mathbf{z}' \circ f) = \mu(\mathbf{z}'_1 \circ f_1),$$

that is,

$$\Big\langle \int f\,d\mu, \mathbf{z}' \Big\rangle = \Big\langle \int f_1\,d\mu, \mathbf{z}'_1 \Big\rangle ;$$

thus, when E' and E'_1 are identified via the pairing $\mathbf{z}' \leftrightarrow \mathbf{z}'_1$, $\int f\,d\mu$ and $\int f_1\,d\mu$ define the same element of the algebraic dual $(\mathrm{E}')^* = (\mathrm{E}'_1)^*$.

III.34, ℓ. -14 to -11.

"...the mapping $\mathbf{z}' \mapsto \mathscr{R}\mathbf{z}'$ which, to every continuous (complex) linear form $\mathbf{z}'$ on E, makes correspond the continuous (real) linear form $\mathbf{z} \mapsto \mathscr{R}\langle \mathbf{z}, \mathbf{z}' \rangle$ on E_0, is an $\mathbf{R}$-isomorphism of the dual E' onto the dual E'_0 of E_0 (TVS, II, §8, No. 1)."

To establish the notation for the next note, we review the construction of the isomorphism (TVS II.61). Most of what is going on is pure vector space theory; to keep the notation simple, it is best to leave the topology to the end.

To set the stage: by E'_0 the author means $(\mathrm{E}_0)'$, the dual of the topological vector space E_0, namely, the real vector space E_0 underlying E, equipped with the given topology on E. The assertion is that $(\mathrm{E}_0)'$ is isomorphic, as a real vector space, to $(\mathrm{E}')_0$, that is, the real vector space underlying the complex vector space E'; identifying $(\mathrm{E}_0)'$ and $(\mathrm{E}')_0$ via this isomorphism then renders the notation E'_0 unambiguous (alternatively, it allows E'_0 to denote either of the two spaces according to the context).

Suppose first that E is any complex vector space (no topology). If f is a (complex) linear form on E (that is, $f \in \mathrm{E}^*$) then the formula $g(\mathbf{z}) = \mathscr{R}\big(f(\mathbf{z})\big)$ defines an $\mathbf{R}$-linear form on E (concisely denoted $\mathscr{R}f$ —or $\mathscr{R} \circ f$, where $\mathscr{R} : \mathbf{C} \to \mathbf{R}$ is the $\mathbf{R}$-linear mapping $\mathscr{R}(a+ib) = a$), that is, a linear form on E_0, and so $g \in (\mathrm{E}_0)^*$ (the algebraic dual of E_0). Since

$$g(i\mathbf{z}) = \mathscr{R}\big(f(i\mathbf{z})\big) = \mathscr{R}\big(if(\mathbf{z})\big) = -\mathscr{I}\big(f(\mathbf{z})\big),$$

one has

$$(*) \qquad f(\mathbf{z}) = \mathscr{R}\big(f(\mathbf{z})\big) + i\mathscr{I}\big(f(\mathbf{z})\big) = g(\mathbf{z}) - ig(i\mathbf{z})\,.$$

The correspondence $f \mapsto g = \mathscr{R}f$ defines a mapping $\mathrm{E}^* \to (\mathrm{E}_0)^*$ that is clearly $\mathbf{R}$-*linear*; writing $u(f) = g = \mathscr{R}f$, the $\mathbf{R}$-linear mapping $u : \mathrm{E}^* \to (\mathrm{E}_0)^*$ can also be regarded as a *linear* mapping $(\mathrm{E}^*)_0 \to (\mathrm{E}_0)^*$.

From (*) it is clear that $g = 0 \Rightarrow f = 0$, therefore u is injective. To see that u is surjective, suppose g is any element of $(\mathrm{E}_0)^*$; define $f : \mathrm{E} \to \mathbf{C}$ by the formula

$$(**) \qquad f(\mathbf{z}) = g(\mathbf{z}) - i\,g(i\mathbf{z}) \qquad (\mathbf{z} \in \mathrm{E})\,.$$

It is routine to check that $f : \mathrm{E} \to \mathbf{C}$ is $\mathbf{R}$-linear; to see that it is $\mathbf{C}$-linear (i.e., that $f \in \mathrm{E}^*$) it suffices to check that $f(i\mathbf{z}) = i\,f(\mathbf{z})$, and this is shown by the computation

$$\begin{aligned} f(i\mathbf{z}) &= g(i\mathbf{z}) - i\,g(i \cdot i\mathbf{z}) = g(i\mathbf{z}) + i\,g(\mathbf{z}) \\ &= i\big(g(\mathbf{z}) - i\,g(i\mathbf{z})\big) = i\,f(\mathbf{z})\,. \end{aligned}$$

Thus $u : (\mathrm{E}^*)_0 \to (\mathrm{E}_0)^*$ is an isomorphism of real vector spaces, but we will write $u : \mathrm{E}^* \to (\mathrm{E}_0)^*$ when we want to regard u as an $\mathbf{R}$-linear bijection defined on a complex vector space.

Now suppose E is a topological vector space over $\mathbf{C}$. If f is continuous (i.e., if $f \in \mathrm{E}'$) it is clear from the formula $g = \mathscr{R}f$ that g is continuous (i.e., $g \in (\mathrm{E}_0)'$), thus u defines a mapping $(\mathrm{E}')_0 \to (\mathrm{E}_0)'$, linear and injective. Whereas if $g \in (\mathrm{E}_0)'$ is given, and if f is defined by (**), it follows from the continuity of $\mathbf{z} \mapsto i\mathbf{z}$ that f is continuous, that is, $f \in \mathrm{E}'$. Thus the mapping $(\mathrm{E}')_0 \to (\mathrm{E}_0)'$ is surjective, hence is an isomorphism of real vector spaces. In the notation of the cited assertion, $\mathbf{z}' \mapsto \mathscr{R}\mathbf{z}'$ ($\mathbf{z}' \in \mathrm{E}'$) defines an isomorphism of real vector spaces $(\mathrm{E}')_0 \to \mathrm{E}'_0$. The inverse mapping $v = u^{-1} : (\mathrm{E}_0)' \to (\mathrm{E}')_0$ is given, for $g \in (\mathrm{E}_0)'$, by $v(g) = \mathbf{z}' \in \mathrm{E}'$, where

$$\mathbf{z}'(\mathbf{z}) = g(\mathbf{z}) - i\, g(i\mathbf{z}) \quad \text{for all } \mathbf{z} \in \mathrm{E}\,;$$

in a somewhat more cluttered notation, for $\mathbf{z}'_0 \in (\mathrm{E}_0)'$, $v(\mathbf{z}'_0) \in (\mathrm{E}')_0$ is given by

$$\big(v(\mathbf{z}'_0)\big)(\mathbf{z}) = \mathbf{z}'_0(\mathbf{z}) - i\, \mathbf{z}'_0(i\mathbf{z}) \qquad \text{for all } \mathbf{z} \in \mathrm{E}\,,$$

that is,

$$\langle \mathbf{z}, v(\mathbf{z}'_0)\rangle = \langle \mathbf{z}, \mathbf{z}'_0\rangle - i\langle i\mathbf{z}, \mathbf{z}'_0\rangle$$

for all $\mathbf{z}'_0 \in (\mathrm{E}_0)'$ and $\mathbf{z} \in \mathrm{E}$.

III.34, $\ell.\ -11, -10$.

"... Similarly, the algebraic dual $\mathrm{E}_0'^*$ of the real vector space E'_0 may be canonically identified with the real space underlying the algebraic dual E'^* of E'."

As observed in the preceding note, E'_0 is to be interpreted as $(\mathrm{E}_0)'$; the assertion is that there is a canonical isomorphism $(\mathrm{E}_0)'^* \to (\mathrm{E}'^*)_0$ of real vector spaces.

The crux of the matter is purely algebraic: as shown in the preceding note, if F is any complex vector space, then the real vector spaces $(\mathrm{F}^*)_0$ and $(\mathrm{F}_0)^*$ are canonically isomorphic (where * denotes algebraic dual and 0 indicates the underlying real vector space structure), via the mapping $f \mapsto \mathscr{R}f$ ($f \in \mathrm{F}^*$).

Now let E be the given locally convex space over $\mathbf{C}$. By the preceding note, there is a vector space isomorphism $v : (\mathrm{E}_0)' \to (\mathrm{E}')_0$ that assigns to $\mathbf{z}'_0 \in (\mathrm{E}_0)'$ the element $v(\mathbf{z}'_0) \in \mathrm{E}'$ defined by

$$\big(v(\mathbf{z}'_0)\big)(\mathbf{z}) = \mathbf{z}'_0(\mathbf{z}) - i\, \mathbf{z}'_0(i\mathbf{z}) \qquad \text{for all } \mathbf{z} \in \mathrm{E}\,.$$

It follows that the transpose ${}^t v$ of v is an isomorphism

$${}^t v : \big((\mathrm{E}')_0\big)^* \to \big((\mathrm{E}_0)'\big)^* = (\mathrm{E}_0)'^*$$

of the algebraic duals (A, II, §2, No. 5), defined by ${}^t v(h) = h \circ v$ for all linear forms $h : (\mathrm{E}')_0 \to \mathbf{R}$.

On the other hand, by the second paragraph above (with F replaced by the complex vector space E') there exists an isomorphism w of real vector spaces,

$$w : (\mathrm{E}'^*)_0 \to \big((\mathrm{E}')_0\big)^*$$

defined by $w(f) = \mathscr{R}f$ for every linear form f on E'. Thus the composite mapping $\varphi = {}^t v \circ w$

$$(\mathrm{E}'^*)_0 \xrightarrow{\ w\ } \big((\mathrm{E}')_0\big)^* \xrightarrow{\ {}^t v\ } (\mathrm{E}_0)'^*$$

is the sought-for isomorphism of real vector spaces

$$\varphi : (\mathrm{E}'^*)_0 \to (\mathrm{E}_0)'^* .$$

The foregoing proof is efficient, but one yearns for a more direct construction that, given an element of E'^*, exhibits the corresponding element of $(\mathrm{E}_0)'^*$. Such a construction will now be given; the following (commutative) diagram will be helpful in keeping track of the argument.

$$\begin{array}{ccc} v(\mathbf{z}'_0) = \mathbf{z}' \in \mathrm{E}' & \xrightarrow{\quad f \quad} & \mathbf{C} \\ {\scriptstyle v = u^{-1}} \uparrow & & \downarrow {\scriptstyle \mathscr{R}} \\ \mathbf{z}'_0 \in (\mathrm{E}_0)' & \xrightarrow[\quad g \quad]{} & \mathbf{R} \end{array}$$

Given $f \in \mathrm{E}'^*$, the formula

$$g(\mathbf{z}'_0) = \big(\mathscr{R}f\big)\big(v(\mathbf{z}'_0)\big) \qquad \big(\mathbf{z}'_0 \in (\mathrm{E}_0)'\big)$$

defines a linear form on the real vector space $(\mathrm{E}_0)'$, that is, $g \in (\mathrm{E}_0)'^*$. We define $\psi : \mathrm{E}'^* \to (\mathrm{E}_0)'^*$ by $\psi(f) = g$. It is routine to verify that ψ is $\mathbf{R}$-linear.

ψ is injective: If $g = 0$ then $\big(\mathscr{R}f\big)\big(v(\mathbf{z}'_0)\big) = 0$ for all $\mathbf{z}'_0 \in (\mathrm{E}_0)'$, that is, $(\mathscr{R}f)(\mathbf{z}') = 0$ for all $\mathbf{z}' \in \mathrm{E}'$ (v is surjective); thus the real part of the complex linear form f is 0, whence $f = 0$ by a now-familiar argument.

ψ is surjective: Given $g \in (\mathrm{E}_0)'^*$, that is, a linear form $g : (\mathrm{E}_0)' \to \mathbf{R}$, we seek a linear form $f : \mathrm{E}' \to \mathbf{C}$ such that $\psi(f) = g$. For all $\mathbf{z}' \in \mathrm{E}'$ define

$$f(\mathbf{z}') = g(\mathscr{R}\mathbf{z}') - i\, g\big(\mathscr{R}(i\mathbf{z}')\big)$$

(where $\mathscr{R}(i\mathbf{z}')$ denotes the $\mathbf{R}$-linear form $\mathbf{z} \to \mathscr{R}\big(i\mathbf{z}'(\mathbf{z})\big) = -\mathscr{I}\big(\mathbf{z}'(\mathbf{z})\big)$ on E) It is routine to show that f is $\mathbf{R}$-linear and that $f(i\mathbf{z}') = i\, f(\mathbf{z}')$, hence that f is $\mathbf{C}$-linear, that is, $f \in \mathrm{E}'^*$. Now, $\mathscr{R}\big(f(\mathbf{z}')\big) = g(\mathscr{R}\mathbf{z}')$ for all $\mathbf{z}' \in \mathrm{E}'$; in other words, for all $\mathbf{z}'_0 \in (\mathrm{E}_0)'$,

$$\mathscr{R}\big(f(v(\mathbf{z}'_0))\big) = g\big(\mathscr{R}(v(\mathbf{z}'_0)\big) = g\big(u(v(\mathbf{z}'_0))\big) = g(\mathbf{z}'_0)\,,$$

that is, $\big(\mathscr{R}f\big)(v(\mathbf{z}_0')) = g(\mathbf{z}_0')$, which says that $\psi(f) = g$.

Inasmuch as $f \circ v = {}^t v(f)$ figures in both constructions, one suspects that $\varphi = \psi$. Indeed, for $f \in \mathrm{E}'^*$ and $\mathbf{z}_0' \in (\mathrm{E}_0)'$, one has

$$\begin{aligned}\big(\varphi(f)\big)(\mathbf{z}_0') &= \big(({}^t v \circ w)(f)\big)(\mathbf{z}_0') \\ &= \Big({}^t v\big(w(f)\big)\Big)(\mathbf{z}_0') = \big({}^t v(\mathscr{R}f)\big)(\mathbf{z}_0') \\ &= \big((\mathscr{R}f) \circ v\big)(\mathbf{z}_0') = \big(\mathscr{R}f\big)\big(v(\mathbf{z}_0')\big) = \big(\psi(f)\big)(\mathbf{z}_0'),\end{aligned}$$

whence $\varphi(f) = \psi(f)$ for all $f \in \mathrm{E}'^*$.

III.34, ℓ. -9 to -6.

"It follows that if μ is a *real measure* and $\mathbf{f}$ a mapping in $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$, the formula (1) is again valid when $\mathbf{f}$ is regarded as taking its values in E_0 and the canonical bilinear forms figuring in the two members as being, respectively, relative to the duality between E_0' and $\mathrm{E}_0'^*$ for the first member and the duality between E_0 and E_0' for the second."

In the swarm of identifications, it is easy to lose the ball in the sun. Here is what is going on. We are given a locally convex space E over $\mathbf{C}$, and a measure $\mu \in \mathscr{M}(\mathrm{X};\mathbf{C})$ such that the restriction of μ to $\mathscr{K}(\mathrm{X};\mathbf{R})$ is real-valued, hence defines a measure $\mu_0 \in \mathscr{M}(\mathrm{X};\mathbf{R})$. If $\mathbf{f} \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$, then $\int \mathbf{f}\,d\mu$ is defined to be a suitable element $\mathbf{z}'^*$ of E'^* (Def. 1). On the other hand, one can regard $\mathbf{f}$ as belonging to $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E}_0)$, where E_0 is the locally convex space over $\mathbf{R}$ underlying E; denoting it $\mathbf{f}_0$ when so regarded, one can define (analogously to Def. 1) its integral $\int \mathbf{f}_0\,d\mu_0$ to be a suitable element $\mathbf{z}_0'^*$ of $(\mathrm{E}_0')^*$. The assertion, stated simply, is that

$$\psi(\mathbf{z}'^*) = \mathbf{z}_0'^*,$$

where $\psi : \mathrm{E}'^* \to (\mathrm{E}_0)'^*$ is the $\mathbf{R}$-linear bijection described in the preceding note; so to speak, the identification of $(\mathrm{E}'^*)_0$ and $(\mathrm{E}_0)'^*$ via ψ transforms $\int \mathbf{f}\,d\mu$ into $\int \mathbf{f}_0\,d\mu_0$.

The first task is to paraphrase Definition 1 for the case of real measures. Let $\nu : \mathscr{K}(\mathrm{X};\mathbf{R}) \to \mathbf{R}$ be a real measure, let F be a locally convex space over $\mathbf{R}$ (not necessarily equal to E_0 for some locally convex space E over $\mathbf{C}$), and let $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{F})$ be the real vector space of all mappings $\mathbf{g} : \mathrm{X} \to \mathrm{F}$ that are weakly continuous and scalarly of compact support, that is, such for every $\mathbf{w}' \in \mathrm{F}'$ the function

$$x \mapsto \langle \mathbf{g}(x), \mathbf{w}' \rangle = \mathbf{w}'\big(\mathbf{g}(x)\big) = (\mathbf{w}' \circ \mathbf{g})(x) \qquad (x \in \mathrm{X})$$

belongs to $\mathscr{K}(\mathrm{X};\mathbf{R})$. For each $\mathbf{g} \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{F})$, the function

$$\mathbf{w}' \mapsto \int (\mathbf{w}' \circ \mathbf{g})(x)\,d\nu(x) = \int \langle \mathbf{g}(x), \mathbf{w}' \rangle\,d\nu(x) \qquad (\mathbf{w}' \in \mathrm{F}')$$

is evidently a linear form on F', that is, an element of F'^*; it is called the *integral* of $\mathbf{g}$ with respect to ν and is denoted $\int \mathbf{g}\,d\nu$, thus

$$(1') \qquad \Big\langle \int \mathbf{g}\,d\nu, \mathbf{w}' \Big\rangle = \int \langle \mathbf{g}, \mathbf{w}' \rangle\,d\nu \qquad \text{for all } \mathbf{w}' \in \mathrm{F}'.$$

We now apply the preceding paragraph to the case that $\mathrm{F} = \mathrm{E}_0$ and $\nu = \mu_0$ (as described in the first paragraph above). With $\mathbf{f} \in \widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E})$ regarded as an element $\mathbf{f}_0$ of $\widetilde{\mathscr{K}}(\mathrm{X};\mathrm{E}_0)$, we write $\int \mathbf{f}\,d\mu = \mathbf{z}'^* \in \mathrm{E}'^*$, and $\int \mathbf{f}_0\,d\mu_0$ for the element of $(\mathrm{E}_0)'^*$ such that

$$\Big\langle \int \mathbf{f}_0\,d\mu_0, \mathbf{z}'_0 \Big\rangle = \int \langle \mathbf{f}_0, \mathbf{z}'_0 \rangle\,d\mu_0 \qquad \text{for all } \mathbf{z}'_0 \in (\mathrm{E}_0)'.$$

In a sense, the preceding formula is all that the assertion in the text says: when $\mu \in \mathscr{M}(\mathrm{X};\mathbf{C})$ is a real measure, the formula (1) remains true (by definition!) with subscripts 0 installed. The proof that $\psi(\mathbf{z}'^*) = \int \mathbf{f}_0\,d\mu_0$ will show the relevance of the formula to the identifications of the preceding note.

Before proceeding, we note that

$$(*) \qquad \mathscr{R} \int f\,d\mu = \int (\mathscr{R} f)\,d\mu \qquad \text{for all } f \in \mathscr{K}(\mathrm{X};\mathbf{C})$$

(because $\int (\mathscr{R} f)\,d\mu$ and $\int (\mathscr{I} f)\,d\mu$ are real).

Let $\mathbf{z}'_0{}^* = \psi(\mathbf{z}'^*)$, where $\mathbf{z}'^* = \int \mathbf{f}\,d\mu$ and $\psi : \mathrm{E}'^* \to (\mathrm{E}_0)'^*$ is the $\mathbf{R}$-linear bijection of the preceding note. Our problem is to show that $\mathbf{z}'_0{}^* = \int \mathbf{f}_0\,d\mu_0$. By the definition of ψ, $\mathbf{z}'_0{}^*$ is the linear form on $(\mathrm{E}_0)'$ such that

$$\mathbf{z}'_0{}^*(\mathbf{z}'_0) = \big(\mathscr{R}\mathbf{z}'^*\big)\big(v(\mathbf{z}'_0)\big) \qquad \text{for all } \mathbf{z}'_0 \in (\mathrm{E}_0)',$$

where $v : (\mathrm{E}_0)' \to (\mathrm{E}')_0$ is the vector space isomorphism described in the preceding note. Thus, for all $\mathbf{z}_0' \in (\mathrm{E}_0)'$,

$$\begin{aligned}\langle \mathbf{z}_0'^*, \mathbf{z}_0' \rangle &= (\mathbf{z}_0'^*)(\mathbf{z}_0') \\ &= \big(\mathscr{R}\mathbf{z}'^*\big)\big(v(\mathbf{z}_0')\big) \\ &= \mathscr{R}\Big(\big(\mathbf{z}'^*\big)\big(v(\mathbf{z}_0')\big)\Big) \\ &= \mathscr{R}\big\langle \mathbf{z}'^*, v(\mathbf{z}_0')\big\rangle \\ &= \mathscr{R}\Big\langle \int \mathbf{f}\, d\mu, v(\mathbf{z}_0')\Big\rangle \\ &= \mathscr{R}\Big(\int \big\langle \mathbf{f}, v(\mathbf{z}_0')\big\rangle\, d\mu\Big) \\ &= \int \mathscr{R}\big\langle \mathbf{f}, v(\mathbf{z}_0')\big\rangle\, d\mu \qquad \text{(by (*))} \\ &= \int \mathscr{R}\big(v(\mathbf{z}_0') \circ \mathbf{f}\big) d\mu \\ &= \big\langle \mathscr{R}\big(v(\mathbf{z}_0') \circ \mathbf{f}\big),\ \mu\big\rangle,\end{aligned}$$

briefly,

$$(**) \qquad \langle \mathbf{z}_0'^*, \mathbf{z}_0' \rangle = \big\langle \mathscr{R}\big(v(\mathbf{z}_0') \circ \mathbf{f}\big),\ \mu\big\rangle.$$

But $\big(v(\mathbf{z}_0') \circ \mathbf{f}\big)(x) = \big(v(\mathbf{z}_0')\big)\big(\mathbf{f}(x)\big)$ for all $x \in \mathrm{X}$, and, by the definition of v,

$$\mathscr{R}\Big(\big(v(\mathbf{z}_0')\big)\big(\mathbf{f}(x)\big)\Big) = \mathbf{z}_0'\big(\mathbf{f}(x)\big),$$

thus $\mathscr{R}\big(v(\mathbf{z}_0') \circ \mathbf{f}\big) = \mathbf{z}_0' \circ \mathbf{f}$; substituting this in (**), we have, for all $\mathbf{z}_0' \in (\mathrm{E}_0)'$,

$$\begin{aligned}\langle \mathbf{z}_0'^*, \mathbf{z}_0' \rangle &= \langle \mathbf{z}_0' \circ \mathbf{f}, \mu \rangle \\ &= \langle \mathbf{z}_0' \circ \mathbf{f}_0, \mu_0 \rangle \qquad (\mathbf{z}_0' \circ \mathbf{f} \text{ is real-valued}) \\ &= \Big\langle \int \mathbf{f}_0\, d\mu_0, \mathbf{z}_0' \Big\rangle,\end{aligned}$$

thus $\int \mathbf{f}_0\, d\mu_0 = \mathbf{z}_0'^* = \psi(\mathbf{z}'^*) = \psi\big(\int \mathbf{f}\, d\mu\big)$.

III.35, ℓ. 6.

"... ${}^{tt}\mathbf{u}$ *extends* the mapping $\mathbf{u}$."

To each $\mathbf{y} \in \mathrm{E}$ corresponds the element $\widehat{\mathbf{y}} \in \mathrm{E}'^*$ defined by

$$\widehat{\mathbf{y}}(\mathbf{y}') = \mathbf{y}'(\mathbf{y}) = \langle \mathbf{y}, \mathbf{y}' \rangle \quad \text{for all } \mathbf{y}' \in \mathrm{E}'.$$

Similarly, each $\mathbf{z} \in \mathrm{F}$ defines $\widehat{\mathbf{z}} \in \mathrm{F}'^{*}$ by

$$\widehat{\mathbf{z}}(\mathbf{z}') = \mathbf{z}'(\mathbf{z}) = \langle \mathbf{z}, \mathbf{z}' \rangle \quad \text{for all } \mathbf{z}' \in \mathrm{F}' .$$

The assertion is that ${}^{tt}\mathbf{u}(\widehat{\mathbf{y}}) = \widehat{\mathbf{u}(\mathbf{y})}$ for all $\mathbf{y} \in \mathrm{E}$; indeed, ${}^{tt}\mathbf{u}(\widehat{\mathbf{y}}) = \widehat{\mathbf{y}} \circ {}^{t}u$ and, for all $\mathbf{z}' \in \mathrm{F}'$,

$$\begin{aligned}(\widehat{\mathbf{y}} \circ {}^{t}u)(\mathbf{z}') &= \widehat{\mathbf{y}}\big({}^{t}u(\mathbf{z}')\big) = \widehat{\mathbf{y}}(\mathbf{z}' \circ u) \\ &= (\mathbf{z}' \circ u)(\mathbf{y}) = \mathbf{z}'\big(\mathbf{u}(\mathbf{y})\big) = \widehat{\mathbf{u}(\mathbf{y})}(\mathbf{z}') .\end{aligned}$$

This observation is not needed for the Proposition 2 that follows it.

III.36, ℓ. 5, 6.

"If E is complex, we equip E with its underlying *real* vector space structure, which, as we have seen, does not modify the formula (1)."

Working through the identifications entailed in deriving the case for complex E from the case for the real space E_0 underlying E promises to be daunting. Instead, a more direct argument, patterned after the *proof* of the real case, is given in the Note for III.36, ℓ. 9–11.

III.36, ℓ. 7.

"(i) We know . . ."

We know it from TVS, II, §5, No. 3, Cor. 5 of Prop. 4, p. TVS II.39. Strictly speaking, by $\mathbf{f}(\mathrm{S})$ is here meant the canonical image

$$\mathrm{T} = \{\, \widehat{\mathbf{f}(x)} : \ x \in \mathrm{S} \,\}$$

of $\mathbf{f}(\mathrm{S})$ in E'^{*}.

III.36, ℓ. 9–11.

". . . it therefore suffices to prove that, for $\mathbf{z}' \in \mathrm{E}'$, the relation $\langle \mathbf{f}(x), \mathbf{z}' \rangle \geqslant 0$ for all $x \in \mathrm{S}$ implies

$$\left\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \right\rangle \geqslant 0 \text{ "}$$

The argument assumes that E is a locally convex space over $\mathbf{R}$ (after which, an argument for the complex case is given). Here the subset $\mathbf{f}(\mathrm{S})$ of E is being regarded as a subset of E'^{*}, by identifying $\mathbf{z} \in \mathrm{E}$ with the linear form $\widehat{\mathbf{z}}$ on E' defined by $\widehat{\mathbf{z}}(\mathbf{z}') = \mathbf{z}'(\mathbf{z}) = \langle \mathbf{z}, \mathbf{z}' \rangle$ $(\mathbf{z}' \in \mathrm{E}')$. Rather than make the identification, let us instead work directly with the canonical image of $\mathbf{f}(\mathrm{S})$ in E'^{*}, namely the set

$$\mathrm{T} = \{\, \widehat{\mathbf{f}(x)} : \ x \in \mathrm{S} \,\} \subset \mathrm{E}'^{*} .$$

Since E'^* is equipped with the weak topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ for the duality defined by the bilinear form $(f, \mathbf{z}') \mapsto \langle f, \mathbf{z}' \rangle = f(\mathbf{z}')$ ($f \in \mathrm{E}'^*$, $\mathbf{z}' \in \mathrm{E}'$), its dual for this topology is E', i.e., consists of the linear forms

$$f \mapsto f(\mathbf{z}') = \langle f, \mathbf{z}' \rangle \qquad (f \in \mathrm{E}'^*),$$

where $\mathbf{z}'$ varies over E'. Thus the half-spaces whose intersection is C are the half-spaces

$$\mathrm{H}_{\mathbf{z}'} = \{ f \in \mathrm{E}'^* : \ \langle f, \mathbf{z}' \rangle = f(\mathbf{z}') \geqslant 0 \}$$

defined by those $\mathbf{z}' \in \mathrm{E}'$ for which $\mathrm{T} \subset \mathrm{H}_{\mathbf{z}'}$, that is, for which

$$\big\langle \widehat{\mathbf{f}(x)}, \mathbf{z}' \big\rangle = \mathbf{z}'\big(\mathbf{f}(x)\big) \geqslant 0 \quad \text{for all } x \in \mathrm{S},$$

in other words $\mathbf{z}' \circ \mathbf{f} \geqslant 0$ on S. Writing A for the set of all such $\mathbf{z}'$, we have

$$\mathrm{C} = \bigcap_{\mathbf{z}' \in \mathrm{A}} \mathrm{H}_{\mathbf{z}'}.$$

To prove that $\int \mathbf{f}\, d\mu \in \mathrm{C}$, we need only show that $\int \mathbf{f}\, d\mu \in \mathrm{H}_{\mathbf{z}'}$ for every $\mathbf{z}' \in \mathrm{A}$, that is,

$$\mathbf{z}' \in \mathrm{A} \quad \Rightarrow \quad \int \mathbf{f}\, d\mu \in \mathrm{H}_{\mathbf{z}'},$$

in other words,

$$\mathrm{T} \subset \mathrm{H}_{\mathbf{z}'} \quad \Rightarrow \quad \int \mathbf{f}\, d\mu \in \mathrm{H}_{\mathbf{z}'},$$

i.e., that

$$\mathbf{z}' \circ \mathbf{f} \geqslant 0 \text{ on } \mathrm{S} \quad \Rightarrow \quad \Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle \geqslant 0.$$

By definition (see the formula (1)),

$$\Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle = \int \langle \mathbf{f}, \mathbf{z}' \rangle \, d\mu = \int (\mathbf{z}' \circ \mathbf{f})\, d\mu,$$

thus the implication to be proved may be written

$$\mathbf{z}' \circ \mathbf{f} \geqslant 0 \text{ on } \mathrm{S} \quad \Rightarrow \quad \int (\mathbf{z}' \circ \mathbf{f})\, d\mu \geqslant 0;$$

since μ is positive and $\mathbf{z}' \circ \mathbf{f} \in \mathscr{K}(\mathrm{X}; \mathbf{R})$, the implication follows at once from §2, No. 3, Cor. 2 of Prop. 8.

The complex case. The structure of the proof is the same as that of the real case. Assuming E is a locally convex space over $\mathbf{C}$, equip E'^* with the topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ relative to the duality defined by the bilinear form

$$(f, \mathbf{z}') \mapsto \langle f, \mathbf{z}' \rangle = f(\mathbf{z}') \qquad (f \in \mathrm{E}'^*,\ \mathbf{z}' \in \mathrm{E}') .$$

We again write

$$\mathrm{T} = \{\, \widehat{\mathbf{f}(x)} :\ x \in \mathrm{S} \,\} \subset \mathrm{E}'^* ,$$

where $\widehat{\mathbf{f}(x)}(\mathbf{z}') = \mathbf{z}'\big(\mathbf{f}(x)\big)$ for $\mathbf{z}' \in \mathrm{E}'$, and define C to be the closure in E'^* of the convex cone generated by T. The definition of C takes place in the context of the real vector space $(\mathrm{E}'^*)_0$ equipped with the topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$.

In the duality between E' and E'^*, the continuous linear forms on E'^* are the forms

$$f \mapsto f(\mathbf{z}') \qquad (f \in \mathrm{E}'^*) ,$$

where $\mathbf{z}'$ varies over E', thus the forms

$$(*) \qquad f \mapsto \mathscr{R}\big(f(\mathbf{z}')\big) \qquad (f \in \mathrm{E}'^*)$$

are continuous $\mathbf{R}$-linear forms on E'^*, hence are continuous linear forms on real space $(\mathrm{E}'^*)_0$ underlying E'^*. Conversely, if θ is a continuous linear form on $(\mathrm{E}'^*)_0$, then the formula

$$\eta(f) = \theta(f) - i\,\theta(if) \quad (f \in \mathrm{E}'^*)$$

defines a continuous linear form on E'^*, hence there exists $\mathbf{z}' \in \mathrm{E}'$ such that $\eta(f) = f(\mathbf{z}')$ for all $f \in \mathrm{E}'^*$, and so $\theta(f) = \mathscr{R}\big(f(\mathbf{z}')\big)$. Thus (*) describes all of the continuous linear forms on $(\mathrm{E}'^*)_0$. The closed half-spaces whose intersection is C (TVS, II, §5, No. 3, Cor. 5 of Prop. 4) are therefore the half-spaces

$$\mathrm{H}_{\mathbf{z}'} = \{ f \in \mathrm{E}'^* :\ \mathscr{R}\big(f(\mathbf{z}')\big) \geqslant 0 \,\}$$

defined by those $\mathbf{z}' \in \mathrm{E}'$ for which $\mathrm{T} \subset \mathrm{H}_{\mathbf{z}'}$, that is, for which

$$\mathscr{R}\big(\widehat{\mathbf{f}(x)}(\mathbf{z}')\big) \geqslant 0 \quad \text{for all } x \in \mathrm{S} ,$$

in other words $\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \geqslant 0$ on S. Writing A for the set of all such $\mathbf{z}'$, we have

$$\mathrm{C} = \bigcap_{\mathbf{z}' \in \mathrm{A}} \mathrm{H}_{\mathbf{z}'}$$

and, as argued in the real case, to prove that $\int \mathbf{f}\, d\mu \in \mathrm{C}$ we need only prove the implication

$$\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \geqslant 0 \text{ on } \mathrm{S} \quad \Rightarrow \quad \int \mathbf{f}\, d\mu \in \mathrm{H}_{\mathbf{z}'} ,$$

that is,

$$\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \geqslant 0 \text{ on } \mathrm{S} \quad \Rightarrow \quad \mathscr{R}\Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle \geqslant 0 ;$$

since

$$\mathscr{R}\Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle = \mathscr{R} \int \langle \mathbf{f}, \mathbf{z}' \rangle\, d\mu = \mathscr{R} \int (\mathbf{z}' \circ \mathbf{f})\, d\mu = \int \mathscr{R}(\mathbf{z}' \circ \mathbf{f})\, d\mu ,$$

and since μ is positive and $\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \in \mathscr{K}(\mathrm{X}; \mathbf{C})$, the last implication is a consequence of §2, No. 3, Cor. 2 of Prop. 8.

Postlude. How to derive the complex case from the real case more efficiently? Given E complex, the real case applied to E_0 yields $\int \mathbf{f}_0\, d\mu_0 \in \mathrm{C}_0$, where C_0 is the closed convex cone in $(\mathrm{E}_0)'^*$ generated by the canonical image of $\mathbf{f}_0(\mathrm{S})$ (μ and μ_0 have the same support), where $(\mathrm{E}_0)'^*$ is equipped with the topology $\sigma\big((\mathrm{E}_0)'^*, (\mathrm{E}_0)'\big)$. How to transform the relation $\int \mathbf{f}_0\, d\mu_0 \in \mathrm{C}_0$ into $\int \mathbf{f}\, d\mu \in \mathrm{C}$? In the note for III.34, ℓ. -9 to -6, an $\mathbf{R}$-linear bijection $\psi : \mathrm{E}'^* \to (\mathrm{E}_0)'^*$ was exhibited such that $\psi\big(\int \mathbf{f}\, d\mu\big) = \int \mathbf{f}_0\, d\mu_0$. Presumably $\psi(\mathrm{C}) = \mathrm{C}_0$; one needs to check that ψ transforms the canonical image of $\mathbf{f}(\mathrm{S})$ into the canonical image of $\mathbf{f}_0(\mathrm{S})$, and $\sigma(\mathrm{E}'^*, \mathrm{E}')$ into $\sigma\big((\mathrm{E}_0)'^*, (\mathrm{E}_0)'\big)$ . . . but that's where I lose the ball in the sun.

III.36, ℓ. 15.

"(ii) We know . . . "

We know it from TVS, II, §5, No. 3, Cor. 1 of Prop. 4, p. TVS II.38.

III.36, ℓ. 19.

". . . but this follows from §2, No. 3, Cor. 3 of Prop. 8."

I don't see how to apply the cited Cor. 3, where the hypothesis $|f(x)| \leqslant a$ (on $\mathrm{Supp}(\mu)$) imposes a constraint on $|f|$ with $a \geqslant 0$, whereas here the constraint is on f and a need not be $\geqslant 0$. However, the *proof* of Cor. 3 provides the necessary hint.

The real case. It is assumed that E is a locally convex space over $\mathbf{R}$ and that μ is a bounded positive measure on X. As in part (i), we argue directly with the canonical image

$$\mathrm{T} = \{\, \widehat{\mathbf{f}(x)} : \ x \in \mathrm{S} \,\} \subset \mathrm{E}'^*$$

of $\mathbf{f}(S)$ in E'^* rather than identifying E with its canonical image. The closed half-spaces in E'^* (for the topology $\sigma(E'^*, E')$) are the sets

$$H_{\mathbf{z}',a} = \{ f \in E'^* : \ f(\mathbf{z}') \leqslant a \} ,$$

where $\mathbf{z}' \in E'$ and $a \in \mathbf{R}$. Thus D is the intersection of the set of $H_{\mathbf{z}',a}$ defined by those pairs $(\mathbf{z}', a)$ for which $T \subset H_{\mathbf{z}',a}$ (TVS, II, §5, No. 3, Cor. 1 of Prop. 4, p. TVS II.38); writing A for the set of all such pairs, we have

$$D = \bigcap_{(\mathbf{z}',a)\in A} H_{\mathbf{z}',a} .$$

To prove that $\int \mathbf{f}\,d\mu \in \|\mu\| \cdot D$, we need only show that $\int \mathbf{f}\,d\mu \in \|\mu\| \cdot H_{\mathbf{z}',a}$ for every $(\mathbf{z}', a) \in A$, i.e., that

$$(\mathbf{z}', a) \in A \quad \Rightarrow \quad \int \mathbf{f}\,d\mu \in \|\mu\| \cdot H_{\mathbf{z}',a} ,$$

in other words that

$$(*) \qquad T \subset H_{\mathbf{z}',a} \quad \Rightarrow \quad \|\mu\|^{-1} \int \mathbf{f}\,d\mu \in H_{\mathbf{z}',a} .$$

The condition on the left side of (*) says that

$$\left\langle \widehat{\mathbf{f}(x)}, \mathbf{z}' \right\rangle \leqslant a \quad \text{for all } x \in S ,$$

that is,

$$\text{(a)} \qquad \mathbf{z}' \circ \mathbf{f} \leqslant a \quad \text{on } S .$$

and the condition on the right side says that

$$\text{(b)} \qquad \int (\mathbf{z}' \circ \mathbf{f})\,d\mu \leqslant a\|\mu\| ,$$

thus our problem is to show that (a) ⇒ (b).

Suppose $\mathbf{z}' \circ \mathbf{f} \leqslant a$ on S. Let $g \in \mathscr{K}(X; \mathbf{R})$ be such that $0 \leqslant g \leqslant 1$ on X and $g(x) = 1$ for $x \in \operatorname{Supp}(\mathbf{z}' \circ \mathbf{f})$ (§1, No. 2, Lemma 1). Then

$$\mathbf{z}' \circ \mathbf{f} = (\mathbf{z}' \circ \mathbf{f})g \leqslant ag \quad \text{on } S ,$$

that is, $ag - \mathbf{z}' \circ \mathbf{f} \geqslant 0$ on S, therefore $\int (ag - \mathbf{z}' \circ \mathbf{f})\,d\mu \geqslant 0$ by §2, No. 3, Cor. 2 of Prop. 8, thus

$$(**) \qquad \int (\mathbf{z}' \circ \mathbf{f})\,d\mu \leqslant a\mu(g) .$$

If we show that $a\mu(g) \leqslant a\|\mu\|$, the implication (a) $\Rightarrow$ (b) will be proved. At any rate, $0 \leqslant \mu(g) = |\mu(g)| \leqslant \|\mu\|\,\|g\| = \|\mu\|$.

case 1: If $a \geqslant 0$ then $a\mu(g) \leqslant a\|\mu\|$ by the preceding inequality, therefore (b) follows from (**).

case 2: If $a < 0$, then $\mathbf{z}' \circ \mathbf{f} \leqslant a < 0$ on S, therefore $\mathrm{S} \subset \mathrm{Supp}(\mathbf{z}' \circ \mathbf{f})$ (in particular, μ has compact support), and since $g = 1$ on $\mathrm{Supp}(\mathbf{z}' \circ \mathbf{f})$, we infer that $g = 1$ on S. If we show that $\mu(g) = \|\mu\|$ then (**) will again yield (b). Since $g \geqslant 0$, for every $h \in \mathscr{K}(\mathrm{X}; \mathbf{R})$ we have

$$-\|h\|g \leqslant hg \leqslant \|h\|g\,,$$

therefore

$$-\|h\|g \leqslant h \leqslant \|h\|g \quad \text{on } \mathrm{S}\,,$$

whence (since $\mu \geqslant 0$)

$$-\|h\|\mu(g) \leqslant \mu(h) \leqslant \|h\|\mu(g)\,;$$

thus $|\mu(h)| \leqslant \|h\|\mu(g)$ for all $h \in \mathscr{K}(\mathrm{X}; \mathbf{R})$, consequently $\|\mu\| \leqslant \mu(g)$. But

$$\mu(g) = |\mu(g)| \leqslant \|\mu\|\,\|g\| = \|\mu\|\,,$$

therefore $\mu(g) = \|\mu\|$, which completes the proof.

The complex case. Assume E is a locally convex space over $\mathbf{C}$. As observed in the complex case for (i), the continuous (for $\sigma(\mathrm{E}'^*, \mathrm{E}')$) $\mathbf{R}$-linear forms on E'^* are the forms

$$f \mapsto \mathscr{R}\big(f(\mathbf{z}')\big) \quad (f \in \mathrm{E}'^*)\,,$$

where $\mathbf{z}'$ varies over E', thus the closed half-spaces of $(\mathrm{E}'^*)_0$ are the sets

$$\mathrm{H}_{\mathbf{z}',a} = \{f \in \mathrm{E}'^* : \ \mathscr{R}\big(f(\mathbf{z}')\big) \leqslant a\}\,,$$

where $\mathbf{z}' \in \mathrm{E}'$ and $a \in \mathbf{R}$, and D is the intersection of the set of all $\mathrm{H}_{\mathbf{z}',a}$ defined by those pairs $(\mathbf{z}', a)$ for which $\mathrm{T} \subset \mathrm{H}_{\mathbf{z}',a}$ (TVS, §5, No. 3, Cor. 1 of Prop. 4, p. TVS II.38); writing A for the set of all such pairs, we have

$$\mathrm{D} = \bigcap_{(\mathbf{z}',a) \in \mathrm{A}} \mathrm{H}_{\mathbf{z}',a}\,.$$

To prove that $\int \mathbf{f}\,d\mu \in \|\mu\| \cdot \mathrm{D}$, we need only show that $\|\mu\|^{-1} \int \mathbf{f}\,d\mu \in \mathrm{H}_{\mathbf{z}',a}$ for every $(\mathbf{z}', a) \in \mathrm{A}$, i.e., that

$$(\mathbf{z}', a) \in \mathrm{A} \quad \Rightarrow \quad \|\mu\|^{-1} \int \mathbf{f}\,d\mu \in \mathrm{H}_{\mathbf{z}',a}\,,$$

in other words that

$$\mathrm{T} \subset \mathrm{H}_{\mathbf{z}',a} \quad \Rightarrow \quad \|\mu\|^{-1} \int \mathbf{f}\, d\mu \in \mathrm{H}_{\mathbf{z}',a}\,.$$

The condition on the left says that $\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \leqslant a$ on S, and the condition on the right says that $\int \mathscr{R}(\mathbf{z}' \circ \mathbf{f})\, d\mu \leqslant a\|\mu\|$, thus the problem is to show that

$$(\dagger) \qquad \mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \leqslant a \ \text{ on } \mathrm{S} \quad \Rightarrow \quad \int \mathscr{R}(\mathbf{z}' \circ \mathbf{f})\, d\mu \leqslant a\|\mu\|\,.$$

Suppose $\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) \leqslant a$ on S. Let $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with $0 \leqslant g \leqslant 1$ on X and $g = 1$ on $\operatorname{Supp}\mathscr{R}(\mathbf{z}' \circ \mathbf{f})$. Then

$$\big(a - \mathscr{R}(\mathbf{z}' \circ \mathbf{f})\big)g \geqslant 0 \quad \text{on } \mathrm{S}\,,$$

whence

$$\mathscr{R}(\mathbf{z}' \circ \mathbf{f}) = \mathscr{R}(\mathbf{z}' \circ \mathbf{f})g \leqslant ag \quad \text{on } \mathrm{S}\,,$$

therefore (since μ is positive)

$$(\dagger\dagger) \qquad \int \mathscr{R}(\mathbf{z}' \circ \mathbf{f})\, d\mu \leqslant a\mu(g)\,.$$

case 1: If $a \geqslant 0$ then $a\mu(g) \leqslant a\|\mu\|$ as in the real case, and (†) then follows from (††).

case 2: If $a < 0$ then $\mathrm{S} \subset \operatorname{Supp}\mathscr{R}(\mathbf{z}' \circ \mathbf{f})$ as in the real case, therefore $g = 1$ on S. Then, for every $h \in \mathscr{K}(\mathrm{X};\mathbf{C})$, we have $|h| \leqslant \|h\|g$ on S, hence $\mu(|h|) \leqslant \|h\|\mu(g)$; therefore

$$|\mu(h)| \leqslant |\mu|(|h|) = \mu(|h|) \leqslant \|h\|\mu(g)\,,$$

which shows that $\|\mu\| \leqslant \mu(g)$. But $\mu(g) \leqslant \|\mu\|\,\|g\| = \|\mu\|$, thus $\mu(g) = \|\mu\|$ and (†) again follows from (††).

III.36, ℓ. −9.

"…then ν is bounded"

Let U be an open set in X such that $\mathrm{K} \subset \mathrm{U}$ and $\overline{\mathrm{U}}$ is compact. By Prop. 5 of §2, No. 2,

$$\operatorname{Supp}(\nu) = \mathrm{U} \cap \operatorname{Supp}(\mu) \subset \mathrm{U} \subset \overline{\mathrm{U}}\,,$$

thus ν has compact support, hence is bounded (§2, No. 3, Prop. 11).

III.36, ℓ. −9.

"... and $\int \mathbf{f}\,d\mu = \int \mathbf{f}\,d\nu \in \|\nu\| \cdot \mathrm{D}$ by Prop. 4, (ii)."

Since $\mathrm{Supp}(\mathbf{f}) = \mathrm{K} \subset \mathrm{U}$, we have $\mathbf{f} \in \mathscr{K}(\mathrm{X}, \mathrm{U}; \mathrm{E})$, hence $\mathbf{f}\big|\mathrm{U} \in \mathscr{K}(\mathrm{U}; \mathrm{E})$ (see III.23, ℓ. 14–16). Thus by $\int \mathbf{f}\,d\nu$ (abuse of notation) is meant $\int (\mathbf{f}\big|\mathrm{U})\,d\nu = \int (\mathbf{f}\big|\mathrm{U})\,d(\mu\big|\mathrm{U})$ which, by the definition of $\mu\big|\mathrm{U}$, is equal to $\int \mathbf{f}\,d\mu$ since, for all $\mathbf{z}' \in \mathrm{E}'$,

$$\Big\langle \int (\mathbf{f}\big|\mathrm{U})\,d\nu, \mathbf{z}' \Big\rangle = \nu\big(\mathbf{z}' \circ (\mathbf{f}\big|\mathrm{U})\big) = \nu\big((\mathbf{z}' \circ \mathbf{f})\big|\mathrm{U}\big) = \mu(\mathbf{z}' \circ \mathbf{f}) = \Big\langle \int \mathbf{f}\,d\mu, \mathbf{z}' \Big\rangle .$$

By the cited Prop. 4, (ii), $\int (\mathbf{f}\big|\mathrm{U})\,d\nu \in \|\nu\| \cdot \mathrm{D}_1$, where D_1 is the closed convex envelope of (the canonical image of) $(\mathbf{f}\big|\mathrm{U})\big(\mathrm{Supp}(\nu)\big)$ in E'^*. Now,

$$(\mathbf{f}\big|\mathrm{U})\big(\mathrm{Supp}(\nu)\big) = (\mathbf{f}\big|\mathrm{U})\big(\mathrm{U} \cap \mathrm{Supp}(\mu)\big) = \mathbf{f}\big(\mathrm{U} \cap \mathrm{Supp}(\mu)\big) = \mathbf{f}(\mathrm{U} \cap \mathrm{S}) \subset \mathbf{f}(\mathrm{S}) ,$$

thus $\mathrm{D}_1 \subset \mathrm{D}$, and so

$$\int \mathbf{f}\,d\mu = \int (\mathbf{f}\big|\mathrm{U})\,d\nu \in \|\nu\| \cdot \mathrm{D}_1 \subset a \cdot \mathrm{D}$$

with $a = \|\nu\|$.

III.36, ℓ. −8, −7.

"The second result follows from this, since any complex measure may be written as $\mu_1 - \mu_2 + i\mu_3 - i\mu_4$, where the μ_j are positive."

Attention must be paid to the supports of the positive measures μ_j. As noted in §2, No. 2, Prop. 2, if ν is a real measure (so that $\nu = \nu^+ - \nu^-$), then

$$\mathrm{Supp}(\nu) = \mathrm{Supp}(|\nu|) = \mathrm{Supp}(\nu^+) \cup \mathrm{Supp}(\nu^-) ;$$

and if ν is a complex measure then $|\mathscr{R}\nu| \leqslant |\nu|$ and $|\mathscr{I}\nu| \leqslant |\nu|$ (§1, No. 6, (17)) show that the supports of $\mathscr{R}\nu$ and $\mathscr{I}\nu$—hence of their positive and negative parts—are contained in $\mathrm{Supp}(|\nu|) = \mathrm{Supp}(\nu)$. Thus ν has a representation of the indicated sort with $\mathrm{Supp}(\mu_j) \subset \mathrm{Supp}(\nu)$ for $1 \leqslant j \leqslant 4$. Then, by the previous result, for each j there exists a number $a_j > 0$ such that

$$\int \mathbf{f}\,d\mu_j \in a_j \cdot \mathrm{D}_j ,$$

where D_j is the closed convex envelope of $\mathbf{f}\big(\mathrm{Supp}(\mu_j)\big)$; but $\mathrm{Supp}(\mu_j) \subset \mathrm{Supp}(\nu)$ implies that $\mathrm{D}_j \subset \mathrm{D}$, thus

$$\int \mathbf{f}\,d\mu_j \in a_j \cdot \mathrm{D} \quad \text{for } 1 \leqslant j \leqslant 4 ,$$

and the second result follows from the relation

$$\int \mathbf{f}\,d\nu = \int \mathbf{f}\,d\mu_1 - \int \mathbf{f}\,d\mu_2 + i\int \mathbf{f}\,d\mu_3 - i\int \mathbf{f}\,d\mu_4$$

(Prop. 1).

III.36, ℓ. -1.

"... E'^* is *complete*"

For the uniform structure on E'^* derived from the weak topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ (TVS, II, §6, No. 7, Prop. 9).

III.37, ℓ. $1, 2$.

"... $\int \mathbf{f}\,d\mu \in \mathrm{C}$ for every measure μ belonging to the convex set H of positive measures on X of total mass equal to 1."

(As in previous notes, I regard C to be the closed convex envelope of the canonical image $\{\widehat{\mathbf{f}(x)} : x \in \mathrm{X}\}$ of $\mathbf{f}(\mathrm{X})$ in E'^*.) The point is that for $\mu \in \mathrm{H}$ one has $\|\mu\| = \mu(1) = 1$ (§1, No. 8, Cor. 2 of Prop. 10). By Prop. 4, (ii), $\int \mathbf{f}\,d\mu$ belongs to the closed convex envelope D of (the canonical image of) $\mathbf{f}(\mathrm{S})$ in E'^*, where $\mathrm{S} = \mathrm{Supp}(\mu)$, and since $\mathbf{f}(\mathrm{S}) \subset \mathbf{f}(\mathrm{X})$ implies $\mathrm{D} \subset \mathrm{C}$, one has

$$\int \mathbf{f}\,d\mu \in \|\mu\| \cdot \mathrm{D} = \mathrm{D} \subset \mathrm{C}$$

as claimed.

III.37, ℓ. 5–7.

"...the image of H_0 under the mapping $\mu \mapsto \int \mathbf{f}\,d\mu$ is the convex envelope C_0 of $\mathbf{f}(\mathrm{X})$ in E'^*."

If μ is a measure with finite support, say $\mu = \sum_{i=1}^{n} c_i \varepsilon_{x_i}$, where the x_i are distinct points of X (§2, No. 4, Prop. 12), then μ is bounded, $\|\mu\| = \sum_{i=1}^{n} |c_i|$ (see item (12) in the note for III.16, ℓ. 13–15) and $\mu \geqslant 0 \Leftrightarrow c_i \geqslant 0$ for all i, thus

$$\mu \in \mathrm{H}_0 \quad\Leftrightarrow\quad c_i \geqslant 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} c_i = 1\,;$$

that is, $\mu \in \mathrm{H}_0 \Leftrightarrow \mu$ is a convex combination of Dirac measures ε_x $(x \in \mathrm{X})$, and, for such a measure μ,

$$\int \mathbf{f}\,d\mu = \mu(\mathbf{f}) = \sum_{i=1}^{n} c_i \varepsilon_{x_i}(\mathbf{f}) = \sum_{i=1}^{n} c_i \mathbf{f}(x_i)\,,$$

whence $\left\{\int \mathbf{f}\,d\mu : \ \mu \in \mathrm{H}_0\right\} = \mathrm{C}_0$.

III.37, ℓ. 7, 8.

"... this mapping is continuous for the vague topology on $\mathscr{M}(\mathrm{X};\mathbf{C})$ and the topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ on E'^* ..."

If $\mu_j \to \mu$ vaguely, that is, if $\mu_j(g) \to \mu(g)$ for all $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$, then in particular

$$\mu_j(\mathbf{z}' \circ \mathbf{f}) \to \mu(\mathbf{z}' \circ \mathbf{f}) \quad \text{for all } \mathbf{z}' \in \mathrm{E}',$$

that is, $\left\langle \int \mathbf{f}\,d\mu_j, \mathbf{z}' \right\rangle \to \left\langle \int \mathbf{f}\,d\mu, \mathbf{z}' \right\rangle$ for all $\mathbf{z}' \in \mathrm{E}'$, in other words $\int \mathbf{f}\,d\mu_j \to \int \mathbf{f}\,d\mu$ in E'^* for $\sigma(\mathrm{E}'^*, \mathrm{E}')$.

III.37, ℓ. 9, 10.

"... the image of $\mathrm{H} = \overline{\mathrm{H}}_0$ is a *compact* convex set containing C_0 and contained in C "

Writing $\Phi : \mathscr{M}(\mathrm{X};\mathbf{C}) \to \mathrm{E}'^*$ for the mapping $\mu \mapsto \int \mathbf{f}\,d\mu$, which we know to be linear and continuous for the indicated topologies, we have

$$\mathrm{C}_0 = \Phi(\mathrm{H}_0) \subset \Phi(\mathrm{H}) = \Phi(\overline{\mathrm{H}}_0) \subset \overline{\Phi(\mathrm{H}_0)} = \overline{\mathrm{C}}_0 = \mathrm{C}.$$

III.37, ℓ. 17.

"... therefore $\mathrm{D} = \mathrm{D}^{\circ\circ}$ (TVS, II, §6, No. 3, Cor. 3 of Th. 1)."

Moreover, since D is balanced—hence symmetric or circled, according as E is real or complex—one has

$$\mathrm{D}^{\circ} = \{\mathbf{z}' \in \mathrm{E}' : \ |\langle \mathbf{z}, \mathbf{z}' \rangle| \leqslant 1 \ \text{ for all } \mathbf{z} \in \mathrm{D}\},$$
$$\mathrm{D}^{\circ\circ} = \{\mathbf{z} \in \mathrm{E} : \ |\langle \mathbf{z}, \mathbf{z}' \rangle| \leqslant 1 \ \text{ for all } \mathbf{z}' \in \mathrm{D}^{\circ}\}$$

(TVS, II, §6, No. 3 and §8, No. 4).

III.37, ℓ. 18, 19.

"It therefore suffices to prove that for every $\mathbf{z}' \in \mathrm{D}^{\circ}$,

$$\left|\left\langle \int \mathbf{f}\,d\mu, \mathbf{z}' \right\rangle\right| \leqslant \int (q \circ \mathbf{f})\ d|\mu| \text{ ''}$$

Let us first prove the inequality, then show why it is sufficient. By $\int \mathbf{f}\,d\mu \in \mathrm{E}$ is meant that there exists a vector $\mathbf{z}_0 \in \mathrm{E}$ such that $\int \mathbf{f}\,d\mu = \widehat{\mathbf{z}}_0$ (the canonical image of $\mathbf{z}_0$ in E'^*), that is, such that

$$\langle \mathbf{z}_0, \mathbf{z}' \rangle = \int (\mathbf{z}' \circ \mathbf{f})\,d\mu \quad \text{for all } \mathbf{z}' \in \mathrm{E}'.$$

Let $\mathbf{z}' \in \mathrm{D}^\circ$. This means that

$$\mathbf{z} \in \mathrm{D} \quad \Rightarrow \quad |\langle \mathbf{z}, \mathbf{z}' \rangle| \leqslant 1 \,,$$

that is,

$$(\alpha) \qquad q(\mathbf{z}) \leqslant 1 \quad \Rightarrow \quad |\langle \mathbf{z}, \mathbf{z}' \rangle| \leqslant 1 \,.$$

It follows that

$$(\beta) \qquad |\langle \mathbf{z}, \mathbf{z}' \rangle| \leqslant q(\mathbf{z}) \ \text{ for } \mathit{all} \ \mathbf{z} \in \mathrm{E} \,;$$

for, if $q(\mathbf{z}) = 0$ then $q(n\mathbf{z}) = nq(\mathbf{z}) = 0$ for $n = 1, 2, 3, \ldots$, whence $|\langle n\mathbf{z}, \mathbf{z}' \rangle| \leqslant 1$ for all n by (α), therefore $|\langle \mathbf{z}, \mathbf{z}' \rangle| = 0$ and (β) reduces to $0 \leqslant 0$; whereas if $q(\mathbf{z}) > 0$ then (β) results from applying (α) to the vector $\big(q(\mathbf{z})\big)^{-1}\mathbf{z}$. In particular,

$$|\langle \mathbf{f}(x), \mathbf{z}' \rangle| \leqslant q\big(\mathbf{f}(x)\big) \ \text{ for all } \ x \in \mathrm{X} \,.$$

that is, $|\mathbf{z}' \circ \mathbf{f}| \leqslant q \circ \mathbf{f}$; since both sides belong to $\mathscr{K}(\mathrm{X}; \mathbf{R})$ and since $|\mu| \geqslant 0$, we infer that

$$|\mu|(|\mathbf{z}' \circ \mathbf{f}|) \leqslant |\mu|(q \circ \mathbf{f}) \,;$$

but $|\mu(\mathbf{z}' \circ \mathbf{f})| \leqslant |\mu|(|\mathbf{z}' \circ \mathbf{f}|)$ by the inequality (13) of §1, No. 6, therefore

$$|\mu(\mathbf{z}' \circ \mathbf{f})| \leqslant |\mu|(q \circ \mathbf{f}) \,,$$

that is,

$$\Big| \Big\langle \int \mathbf{f} \, d\mu, \mathbf{z}' \Big\rangle \Big| \leqslant \int (q \circ \mathbf{f}) \, d|\mu| \,,$$

which is the asserted inequality; in other words,

$$(\gamma) \qquad |\langle \mathbf{z}_0, \mathbf{z}' \rangle| \leqslant \int (q \circ \mathbf{f}) \, d|\mu| \quad \text{for all } \ \mathbf{z}' \in \mathrm{D}^\circ \,.$$

Write $a = \int (q \circ \mathbf{f}) \, d|\mu|$; the assertion of the proposition is that

$$(\delta) \qquad q(\mathbf{z}_0) \leqslant a \,.$$

If $a = 0$ then, by (γ), $|\langle \mathbf{z}_0, \mathbf{z}' \rangle| = 0$ for all $\mathbf{z}' \in \mathrm{D}^\circ$, whence $|\langle n\mathbf{z}_0, \mathbf{z}' \rangle| = 0$ for all n and all $\mathbf{z}' \in \mathrm{D}^\circ$, therefore $n\mathbf{z}_0 \in \mathrm{D}^{\circ\circ} = \mathrm{D}$ for all n , that is, $q(n\mathbf{z}_0 \leqslant 1$ for all n and so $q(\mathbf{z}_0) = 0$; thus (δ) reduces to $0 \leqslant 0$. Whereas if $a > 0$ then it follows from (γ) that $|\langle a^{-1}\mathbf{z}_0, \mathbf{z}' \rangle| \leqslant 1$ for all $\mathbf{z}' \in \mathrm{D}^\circ$, thus

$a^{-1}\mathbf{z}_0 \in \mathrm{D}^{\circ\circ} = \mathrm{D}$, that is, $q(a^{-1}\mathbf{z}_0) \leqslant 1$; thus $q(\mathbf{z}_0) \leqslant a$ and (δ) is again verified.

We remark that if E is a Banach space, then the condition $\int \mathbf{f}\,d\mu \in \mathrm{E}$ is automatically verified (by Cor. 1 of Prop. 7 in No. 3 below), therefore

$$\left\| \int \mathbf{f}\,d\mu \right\| \leqslant \int \|\mathbf{f}(x)\|\,d|\mu|(x)$$

by the foregoing result; and if X is compact, $\left\| \int \mathbf{f}\,d\mu \right\| \leqslant \|\mathbf{f}\|_\infty \, \|\mu\|$.

III.38, *ℓ*. 3–5.

"... the closed convex envelope C of $\mathbf{f}(\mathrm{X})$ in E is then precompact (for the uniform structure induced by that of E) (TVS, II, §4, No. 1, Prop. 3)."

By the cited Prop. 3, the balanced convex envelope B_0 of $\mathbf{f}(\mathrm{X})$ in E is precompact, therefore so is its closure $\mathrm{B} = \overline{\mathrm{B}}_0$ (GT, II, §4, No. 2, Prop. 1); since B is closed and convex (it is the closed balanced convex envelope of $\mathbf{f}(\mathrm{X})$ by TVS, I, §1, No. 5, Prop. 2 and II, §2, No. 6, Prop. 14) it contains C, therefore C is also precompact (GT, II, §4, No. 2, Prop. 1).

In fact, C is compact, as noted in the Corollary of the cited Prop. 3.

III.38, *ℓ*. 8, 9.

"... C is the closed convex envelope of $\mathbf{f}(\mathrm{X})$ in E'^* for the topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ "

Let $\theta : \mathrm{E} \to \mathrm{E}'^*$ be the canonical mapping $\theta(\mathbf{z}) = \widehat{\mathbf{z}}$, where $\widehat{\mathbf{z}}(\mathbf{z}') = \langle \mathbf{z}, \mathbf{z}' \rangle = \mathbf{z}'(\mathbf{z})$ for all $\mathbf{z}' \in \mathrm{E}'$. Equip E with the weakened topology $\sigma(\mathrm{E}, \mathrm{E}')$, and E'^* with the topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$. Note that θ is continuous for these topologies; for, since $\sigma(\mathrm{E}'^*, \mathrm{E}')$ is the initial topology for the family of linear forms

$$f \mapsto f(\mathbf{z}') \quad (f \in \mathrm{E}'^*)$$

indexed by $\mathbf{z}' \in \mathrm{E}'$, it suffices to observe that the composite functions

$$\mathbf{z} \mapsto \theta(\mathbf{z}) = \widehat{\mathbf{z}} \mapsto \widehat{\mathbf{z}}(\mathbf{z}') = \mathbf{z}'(\mathbf{z})$$

are continuous for $\sigma(\mathrm{E}, \mathrm{E}')$. The assertion to be proved is that $\theta(\mathrm{C})$ is the closed convex envelope of $\theta\big(\mathbf{f}(\mathrm{X})\big)$ in E'^*.

Let C_0 be the convex envelope of $\mathbf{f}(\mathrm{X})$ in E. The closure of C_0 in E is the same for $\sigma(\mathrm{E}, \mathrm{E}')$ and the original topology of E (TVS, IV, §1, No. 2, Prop. 2), therefore C is also the closed convex envelope of $\mathbf{f}(\mathrm{X})$ in E for $\sigma(\mathrm{E}, \mathrm{E}')$, that is, $\mathrm{C} = \overline{\mathrm{C}}_0$ (closure for either topology).

Let D_0 be the convex envelope of $\theta\big(\mathbf{f}(\mathrm{X})\big)$ in E'^* and let $\mathrm{D} = \overline{\mathrm{D}}_0$ be its closure. Since θ is linear, $\theta(\mathrm{C}_0) = \mathrm{D}_0$. We know that C is compact in E for $\sigma(\mathrm{E},\mathrm{E}')$, therefore $\theta(\mathrm{C})$ is compact in E'^*, hence closed in E'^*. Thus

$$\theta(\mathrm{C}) = \theta(\overline{\mathrm{C}}_0) \subset \overline{\theta(\mathrm{C}_0)} = \overline{\mathrm{D}}_0 = \mathrm{D}\,;$$

but $\theta(\mathrm{C})$ is a closed convex set in E'^* that contains $\theta\big(\mathbf{f}(\mathrm{X})\big)$, therefore $\theta(\mathrm{C}) \supset \mathrm{D}$, and finally $\theta(\mathrm{C}) = \mathrm{D}$. In particular, $\mathrm{D} \subset \theta(\mathrm{E})$.

III.38, ℓ. 9, 10.

"...the proof is therefore concluded by the Corollary of Prop. 4 of No. 2."

By the cited corollary, there exist scalars a_1, a_2, a_3, a_4 such that

$$\int \mathbf{f}\, d\mu \in a_1\mathrm{D} - a_2\mathrm{D} + ia_3\mathrm{D} - ia_4\mathrm{D}\,,$$

where D is defined as in the preceding note, and that note shows that $a_k\mathrm{D} \subset \theta(\mathrm{E})$ for $1 \leqslant k \leqslant 4$, therefore $\int \mathbf{f}\, d\mu \in \theta(\mathrm{E})$; that is, $\int \mathbf{f}\, d\mu = \widehat{\mathbf{z}}_0$ for some $\mathbf{z}_0 \in \mathrm{E}$.

When $\int \mathbf{f}\, d\mu = \widehat{\mathbf{z}}_0$, $\mathbf{z}_0 \in \mathrm{E}$, it is useful to *redefine* $\int \mathbf{f}\, d\mu$ to be the unique element $\mathbf{z}_0 \in \mathrm{E}$ such that $\langle \mathbf{z}_0, \mathbf{z}' \rangle = \int \langle \mathbf{f}, \mathbf{z}' \rangle\, d\mu$ for all $\mathbf{z}' \in \mathrm{E}'$.

III.38, ℓ. 13–14.

"Since the duals of E and $\widehat{\mathrm{E}}$ are identical, it suffices to apply Prop. 7 while regarding $\mathbf{f}$ as taking its values in $\widehat{\mathrm{E}}$."

Strictly speaking, if E is regarded as a (dense) linear subspace of $\widehat{\mathrm{E}}$, and if $\mathbf{f}$ is regarded as taking its values in $\widehat{\mathrm{E}}$, that is, as an element of $\mathscr{K}(\mathrm{X};\widehat{\mathrm{E}})$, then $\int \mathbf{f}\, d\mu$ becomes (by Prop. 7) the unique element $\mathbf{w}_0 \in \widehat{\mathrm{E}}$ such that $\langle \mathbf{w}_0, \mathbf{w}' \rangle = \int \langle \mathbf{f}, \mathbf{w}' \rangle\, d\mu$ for all $\mathbf{w}' \in (\widehat{\mathrm{E}})'$. The fact that $\mathbf{w}'$ extends an element of E' is of no particular interest here.

However, to apply Prop. 7, it is necessary to note that $\widehat{\mathrm{E}}$ is locally convex. A roundabout proof using continuous semi-norms is indicated in TVS, II, §4, No. 1, remarks following the Corollary of Prop. 1; a brief direct proof, given in the book of J. Horváth (*Topological vector spaces and distributions*, vol. 1, p. 134, Addison-Wesley, Reading, Mass., 1966), is as follows.

We can regard E as a linear subspace of $\widehat{\mathrm{E}}$, equipped with the topology induced by that of $\widehat{\mathrm{E}}$. Let $\mathscr{N}$ be any fundamental system of neighborhoods of 0 in E, and let $\mathscr{N}_1$ be the set of closures in $\widehat{\mathrm{E}}$ of the sets $\mathrm{V} \in \mathscr{N}$. It will suffice to show that $\mathscr{N}_1$ is a fundamental system of neighborhoods of 0 in $\widehat{\mathrm{E}}$; for, the sets in $\mathscr{N}$ may be taken to be convex, and the closure of a convex set, in any topological vector space, is convex (TVS, II, §2, No. 6, Prop. 14).

At any rate, since E is dense in $\widehat{\mathrm{E}}$, the sets in $\mathscr{N}_1$ are neighborhoods of 0 in $\widehat{\mathrm{E}}$ (GT, I, §3, No. 1, Prop. 2). Let W be a closed neighborhood of 0 in $\widehat{\mathrm{E}}$; since such neighborhoods are fundamental (GT, II, §1, No. 2, Cor. 3 of Prop. 2 and III, §3, No. 1), it will suffice to find a set $\mathrm{V} \in \mathscr{N}$ such that $\overline{\mathrm{V}} \subset \mathrm{W}$ (closure in $\widehat{\mathrm{E}}$). Since E has the topology induced by $\widehat{\mathrm{E}}$, $\mathrm{W} \cap \mathrm{E}$ is a neighborhood of 0 in E; by assumption, there exists a set $\mathrm{V} \in \mathscr{N}$ such that $\mathrm{V} \subset \mathrm{W} \cap \mathrm{E}$, whence $\overline{\mathrm{V}} \subset \overline{\mathrm{W}} = \mathrm{W}$.

III.38, *ℓ*. 18.

"... hence bounded ..."

Every precompact subset of a locally convex space is bounded (TVS, III, §1, No. 2, Prop. 2).

An argument that does not appeal to the concept of total boundedness: since $\mathbf{f}(\mathrm{X})$ is compact, it is bounded (by a simplification of the proof of the Prop. 2 just cited, avoiding the concept of the completion of a uniform space), therefore its closed convex envelope is also bounded (TVS, III, §1, No. 2, Prop. 1).

III.38, *ℓ*. −11 to −9.

"2° E is the dual of a *barreled* Hausdorff locally convex space G, and E is equipped with an $\mathfrak{S}$-topology, where $\mathfrak{S}$ is a covering of G by bounded subsets (TVS, III, §4, No. 2, Cor. 4 of Th. 1)."

The adaptation of the cited Cor. 4 to the present notation is as follows:

Let G *and* F *be two locally convex spaces*, $\mathfrak{S}$ *a covering of* G *consisting of bounded subsets. If* G *is barreled and* F *is Hausdorff and quasi-complete, then the space* $\mathscr{L}_{\mathfrak{S}}(\mathrm{G};\mathrm{F})$ *is Hausdorff and quasi-complete.*

One takes $\mathrm{F} = \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ according as the spaces are real or complex, and one sets $\mathrm{E} = \mathscr{L}_{\mathfrak{S}}(\mathrm{G};\mathbf{K})$. Recall that $\mathscr{L}(\mathrm{G};\mathbf{K})$ is the vector space G' of continuous linear forms on G, and $\mathrm{E} = \mathscr{L}_{\mathfrak{S}}(\mathrm{G};\mathbf{K})$ is the vector space G' equipped with the topology of uniform convergence in the subsets $\mathrm{M} \in \mathfrak{S}$ of G.

III.38, *ℓ*. −8, −7.

"For example, Cor. 2 of Prop. 7 can be applied when E is the weak dual of a Banach space"

In the preceding note, one takes G to be a Banach space and $\mathfrak{S}$ to be the set of its 1-element subsets $\{\mathbf{y}\}$ $(\mathbf{y} \in \mathrm{G})$ (equivalently, the set of its finite subsets).

III.38, *ℓ*. −7, −6.

"... or a space of measures $\mathscr{M}(\mathrm{Y};\mathbf{C})$ equipped with the vague topology."

Here one takes (in the foregoing) $\mathrm{G} = \mathscr{K}(\mathrm{Y};\mathbf{C})$ equipped with the direct limit topology (a barreled space, by §1, No. 1, Prop. 2) and $\mathfrak{S}$ to be the set of its 1-element subsets $\{\mathbf{f}\}$ $(\mathbf{f} \in \mathscr{K}(\mathrm{Y};\mathbf{C}))$ (§1, No. 9).

III.39, ℓ. 2–5.

"Proposition 8. ..."

When $\int \mathbf{f}\,d\mu \in \mathrm{E}$ for all $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$ (for example, when E is quasi-complete, cf. No. 3, Cor. 2 of Prop. 7), it is a corollary of Prop. 8 that the mapping $\Phi_0 : \mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathrm{E}$ defined by $\Phi_0(\mathbf{f}) = \int \mathbf{f}\,d\mu$ is the unique continuous linear mapping $\mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathrm{E}$ such that $g\cdot\mathbf{a} \mapsto \mu(g)\cdot\mathbf{a}$ for every vector $a \in \mathrm{E}$ and every function $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$. For, since E is a topological subspace of $\widehat{\mathrm{E}}$, equivalently, the topology of E is the initial topology for the canonical injection $\iota : \mathrm{E} \to \widehat{\mathrm{E}}$, the continuity of Φ_0 is equivalent to that of $\iota \circ \Phi_0 = \Phi$ (GT, I, §2, No. 3, Prop. 4).

III.39, ℓ. 6–8.

"To prove the continuity of the mapping $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$, it suffices to show that its restriction to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is continuous for every compact subset K of X (TVS, II, §4, No. 4, Prop. 5)."

The issue has essentially been addressed in §1, No. 1 (see the note for III.2, ℓ. 7, 8); as some aspects are employed here for the first time, a review is in order.

In line with No. 3, Cor. 1 of Prop. 7, we regard $\int \mathbf{f}\,d\mu$ as the unique element $\mathbf{w}_0 \in \widehat{\mathrm{E}}$ such that $\langle \mathbf{w}_0, \mathbf{w}'\rangle = \int \langle \mathbf{f}, \mathbf{w}'\rangle\,d\mu$ for all $\mathbf{w}' \in (\widehat{\mathrm{E}})'$, so that $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$ is a linear mapping of $\mathscr{K}(\mathrm{X};\mathrm{E})$ into the locally convex space $\widehat{\mathrm{E}}$.

Let $\mathscr{T}_u$ be the topology on $\mathscr{K}(\mathrm{X};\mathrm{E})$ of uniform convergence in X. For every compact subset K of X, the topology on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is by definition the topology $\mathscr{T}_\mathrm{K}$ induced by $\mathscr{T}_u$, that is, the topology of uniform convergence (in X or in K, it comes to the same). On the other hand, the (direct limit) topology $\mathscr{T}$ on $\mathscr{K}(\mathrm{X};\mathrm{E})$ (§1, No. 1) is by definition the "final locally convex topology" for the family of mappings u_K, where K runs over the set of all compact subsets of X, and $u_\mathrm{K} : \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) \to \mathscr{K}(\mathrm{X};\mathrm{E})$ is the canonical injection; that is, $\mathscr{T}$ is the finest *locally convex* topology on $\mathscr{K}(\mathrm{X};\mathrm{E})$ for which all the u_K are continuous (TVS, II, §4, No. 4, Prop. 5 and *Example*).

One knows that if G is a locally convex space and $u : \mathscr{K}(\mathrm{X};\mathrm{E}) \to \mathrm{G}$ is a linear mapping, then u is continuous (for $\mathscr{T}$) if and only if $u \circ u_\mathrm{K}$ is continuous for every K (TVS, *loc. cit.*). The assertion at hand then follows from the observation that $u \circ u_\mathrm{K}$ is the restriction of u to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$.

{In particular, when $\mathrm{E} = \mathbf{C}$, a linear form on $\mathscr{K}(\mathrm{X};\mathrm{E})$ is a measure on X if and only if its restriction to every $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is continuous—as noted following the definition of a measure (§1, No. 3)}.

To conclude the review, since $\mathscr{T}_u$ obviously renders the u_K continuous, $\mathscr{T}$ is finer than $\mathscr{T}_u$, and the inclusion $\mathscr{T} \supset \mathscr{T}_u$ may be proper (for example, when $\mathrm{X} = \mathbf{N}$ equipped with the discrete topology; see item (9) in the note for III.16, ℓ. 13–15). Nevertheless, $\mathscr{T}$ and $\mathscr{T}_u$ induce the same topology $\mathscr{T}_{\mathrm{K}}$ on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$; for, since u_{K} is continuous when $\mathscr{K}(\mathrm{X};\mathrm{E})$ is equipped with $\mathscr{T}$, one has

$$\mathscr{T} \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) \subset \mathscr{T}_{\mathrm{K}}\,,$$

whereas, since $\mathscr{T}_u \subset \mathscr{T}$,

$$\mathscr{T}_{\mathrm{K}} = \mathscr{T}_u \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) \subset \mathscr{T} \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})\,,$$

whence equality throughout. Also, as noted in §1, No. 1, Prop. 1, $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is closed in $\mathscr{K}(\mathrm{X};\mathrm{E})$ for both $\mathscr{T}$ and $\mathscr{T}_u$.

III.39, ℓ. 8–11.

"We note that if the topology of E is defined by a family of semi-norms (q_α), that of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is defined by the family of semi-norms

$$p_\alpha(\mathbf{f}) = \sup_{x\in\mathrm{K}} q_\alpha\big(\mathbf{f}(x)\big)\,.\text{''}$$

The chain of reasoning is quite long; if the Corollary of Prop. 1 in TVS, II, §4, No. 1, its version for complex spaces, and the definition of the uniform structure of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ are well-digested, the following review can be omitted.

(1) If (N_α) is a fundamental system of neighborhoods of 0 in E, then the sets

$$\mathrm{V}_\alpha = \{(\mathbf{x}.\mathbf{y}) \in \mathrm{E}\times\mathrm{E} :\ \mathbf{x}-\mathbf{y} \in \mathrm{N}_\alpha\}$$

form a fundamental system of entourages for the uniform structure of E. For, the uniform structure of E is by definition that of its underlying topological group structure under addition (TVS, I, §1, No. 4 and GT, III, §3, No. 1, remark following Def. 1).

(2) Recall that in a complex topological vector space, the topology of the underlying real topological vector space is by definition the same as the original topology on the space (TVS, II, §8, No. 1; in particular, the open sets, closed sets, closure, interior, neighborhoods, continuity, uniform structure are the same–as well, of course, the convex sets).

The balanced sets for the complex structure are balanced for the real structure, but not *vice versa*; those for the complex structure are circled, whereas those for the real structure are symmetric. Sets absorbent for the complex structure are absorbent for the real structure—so to speak, **C**-absorbent ⇒ **R**-absorbent—but not *vice versa* (but neighborhoods of 0 are absorbent in both senses, since the **R**-neighborhoods and **C**-neighborhoods are the same).

(3) When E is complex, the official definition of locally convexity is that the underlying real topological vector space E_0 is locally convex, i.e., has a fundamental system of convex neighborhoods of 0 (TVS, II, §8, No. 2). In such a space, every neighborhood V of 0 in E contains a convex neighborhood of 0 in E; for, V is also a neighborhood of 0 in E_0, hence contains a convex neighborhood W of 0 in E_0, and W is also a neighborhood of 0 in E.

Thus one could have *defined* a complex topological vector space to be locally convex if it has a fundamental system of convex neighborhoods, and this would automatically have made E_0 locally convex. On the other hand, if one starts with a real locally convex space F admitting an automorphism u such that $u^2 = -1$ (1 the identity mapping of F), then F can be given the structure of a complex vector space E such that, with the same topology, E is a complex topological vector space and $E_0 = F$ (TVS, II, §8, No. 1, the paragraphs in fine print), and E is then automatically locally convex by the author's definition.

(4) A locally convex space (in particular, the given space E) has a fundamental system of closed, balanced, convex neighborhoods of 0 (TVS, II, §8, No. 2).

For, let N be a closed neighborhood of 0 in E (such neighborhoods are fundamental). Since N is also a neighborhood of 0 in E_0, it contains a convex neighborhood V of 0 in E_0. But V is also a neighborhood of 0 in E, hence it contains a balanced neighborhood W of 0 in E (TVS, I, §1, No. 5, Prop. 4). Then V contains the convex envelope U of W, and U is obviously balanced (cf. TVS, II, §2, No. 3, Cor. 1 of Prop. 8), therefore its closure $\overline{U}$ is balanced (TVS, I, §1, No. 5, Prop. 2) and convex (TVS, II, §2, No. 6, Prop. 14). Finally, the inclusions $W \subset U \subset V \subset N$ yield $W \subset \overline{U} \subset \overline{N} = N$, thus N contains a neighborhood $\overline{U}$ of 0 of the desired type.

(5) If p is a continuous semi-norm on a topological vector space, then the set $A = \{\mathbf{x} : p(\mathbf{x}) \leqslant 1\}$ is a closed, balanced, convex neighborhood of 0.

For, by continuity, A is closed, $\{\mathbf{x} : p(x) < 1\}$ is an open set containing 0, and A is obviously balanced and convex. Conversely:

(6) If G is any real or complex topological vector space and A is a closed, balanced, convex neighborhood of 0 in G, then:

(i) the interior $\overset{\circ}{A}$ of A is convex;

(ii) the closure of $\overset{\circ}{A}$ is equal to A;

(iii) there exists a (unique) semi-norm p on G such that

$$A = \{\mathbf{x} : p(\mathbf{x}) \leqslant 1\};$$

(iv) p is continuous (hence uniformly continuous) on G.

Items (i) and (ii) are consequences of TVS, II, §2, No. 6, Cor. 1 of Prop. 16.

Proof of (iii): As in TVS, II, §2, No. 11, Prop. 22, define a function $p = p_{\mathrm{A}} \geqslant 0$ (called the *gauge* of A) on the space G by

$$p(\mathbf{x}) = \inf\{\rho : \ \rho > 0\,,\ \mathbf{x} \in \rho\mathrm{A}\}\,;$$

since A is absorbent (it is a neighborhood of 0) one knows from the cited Prop. 22 that p is finite, $p(0) = 0$ and

$$p(\mathbf{x}+\mathbf{y}) \leqslant p(\mathbf{x}) + p(\mathbf{y})\,, \quad p(\lambda\mathbf{x}) = \lambda\, p(\mathbf{x}) \ \text{ for } \lambda > 0\,.$$

When G is real, A is symmetric, thus $\mathbf{x} \in \rho\mathrm{A} \Leftrightarrow -\mathbf{x} \in \rho\mathrm{A}$, whence $p(-\mathbf{x}) = p(\mathbf{x})$; and if $\lambda < 0$ then $p(\lambda\mathbf{x}) = p(-\lambda\mathbf{x}) = p(|\lambda|\mathbf{x}) = |\lambda| p(\mathbf{x})$, therefore

$$p(\lambda\mathbf{x}) = |\lambda|\, p(\mathbf{x}) \ \text{ for all scalars } \lambda\,.$$

The same formula holds when G is complex; for, writing $|\lambda| = \lambda\gamma$ with $|\gamma| = 1$, since A is circled we have $\gamma\mathrm{A} = \mathrm{A}$, therefore

$$|\lambda|\,\mathbf{z} = \lambda\gamma\,\mathbf{z} \in \rho\mathrm{A} \quad \Leftrightarrow \quad \lambda\mathbf{z} \in \rho\mathrm{A}\,,$$

whence $p(\lambda\mathbf{z}) = p(|\lambda|\mathbf{z}) = |\lambda|\, p(\mathbf{z})$. Thus p is a semi-norm in both cases, and, by Prop. 23 (*loc. cit.*), $\mathrm{A} = \{\mathbf{x} : \ p(\mathbf{x}) \leqslant 1\}$.

In general, a semi-norm p on a real or complex vector space is characterized by the set $\{\mathbf{x} : \ p(\mathbf{x}) \leqslant 1\}$. More generally, if p and q are semi-norms then

$$p \leqslant q \quad \Leftrightarrow \quad \{\mathbf{x} : \ p(\mathbf{x}) \leqslant 1\} \supset \{\mathbf{x} : \ q(\mathbf{x}) \leqslant 1\};$$

for, assuming that the inclusion on the right holds, given any vector $\mathbf{x}$ and, for any $\varepsilon > 0$, setting $\mathbf{y}_\varepsilon = \big(q(\mathbf{x}) + \varepsilon\big)^{-1}\mathbf{x}$, one has

$$q(\mathbf{y}_\varepsilon) = \big(q(\mathbf{x}) + \varepsilon\big)^{-1} q(\mathbf{x}) < 1\,,$$

therefore $p(\mathbf{y}_\varepsilon) \leqslant 1$ by hypothesis, and so $\big(q(\mathbf{x}) + \varepsilon\big)^{-1} p(\mathbf{x}) \leqslant 1$, that is, $p(\mathbf{x}) \leqslant q(\mathbf{x}) + \varepsilon$.

(iv) Given any $\varepsilon > 0$, one has

$$p(\mathbf{x}) \leqslant \varepsilon \quad \Leftrightarrow \quad p(\varepsilon^{-1}\mathbf{x}) \leqslant 1 \quad \Leftrightarrow \quad \varepsilon^{-1}\mathbf{x} \in \mathrm{A} \quad \Leftrightarrow \quad \mathbf{x} \in \varepsilon\mathrm{A}\,;$$

in particular, since $\varepsilon \mathrm{A}$ is a neighborhood of 0 in G, the implication $\mathbf{x} \in \varepsilon \mathrm{A} \Rightarrow p(\mathbf{x}) \leqslant \varepsilon$ shows that p is continuous at 0. The uniform continuity of p follows from the implication

$$\mathbf{x} - \mathbf{y} \in \varepsilon \mathrm{A} \quad \Rightarrow \quad |p(\mathbf{x}) - p(\mathbf{y})| \leqslant p(\mathbf{x} - \mathbf{y}) \leqslant \varepsilon\,,$$

which says that if $(\mathbf{x}, \mathbf{y})$ belongs to the entourage defined by the neighborhood $\varepsilon \mathrm{A}$ of 0 in G, then $\big(p(\mathbf{x}), p(\mathbf{y})\big)$ belongs to the entourage defined by the neighborhood $[-\varepsilon, \varepsilon]$ of 0 in $\mathbf{R}$.

Though it is not needed for the moment, this is a good place to record the fact that

$$\text{(v)} \qquad \{\mathbf{x} :\ p(\mathbf{x}) < 1\} = \overset{\circ}{\mathrm{A}}\,.$$

For, writing U for the set on the left, U is convex and it is open by the continuity of p, whence $\mathrm{U} \subset \overset{\circ}{\mathrm{A}}$; and $\overline{\mathrm{U}} \subset \overline{\mathrm{A}} = \mathrm{A}$. In fact, $\overline{\mathrm{U}} = \mathrm{A}$; for, if $\mathbf{x} \in \mathrm{A}$ and $0 < \alpha < 1$ then $p(\alpha \mathbf{x}) = \alpha\, p(\mathbf{x}) \leqslant \alpha < 1$, thus $\alpha \mathbf{x} \in \mathrm{U}$, and, since $\alpha \mathbf{x} \to \mathbf{x}$ as $\alpha \to 1$, every neighborhood of $\mathbf{x}$ contains such a vector $\alpha \mathbf{x}$, whence $\mathbf{x} \in \overline{\mathrm{U}}$. Since U is a convex open set, it is equal to the interior of its closure (TVS, II, §2, No. 7, Cor. 1 of Prop. 16), that is, $\mathrm{U} = \overset{\circ}{\mathrm{A}}$.

(7) Let (N_α) be a fundamental system of neighborhoods of 0 in E, so that, by (1), the sets

$$\mathrm{V}_\alpha = \{(\mathbf{x}, \mathbf{y}) \in \mathrm{E} \times \mathrm{E} :\ \mathbf{x} - \mathbf{y} \in \mathrm{N}_\alpha\}$$

are a fundamental system of entourages for the uniform structure of E.

If X is any nonempty set and $\mathscr{F} = \mathscr{F}(\mathrm{X}; \mathrm{E})$ is the vector space of all functions $\mathbf{f} : \mathrm{X} \to \mathrm{E}$ (with the pointwise operations), then (GT, X, §1, No. 1) the sets

$$\mathrm{W}_\alpha = \{(\mathbf{f}, \mathbf{g}) \in \mathscr{F} \times \mathscr{F} :\ \big(\mathbf{f}(x), \mathbf{g}(x)\big) \in \mathrm{V}_\alpha \ \text{ for all } \ x \in \mathrm{X}\}$$

are a fundamental system of entourages for a uniformity on $\mathscr{F}$ that induces on $\mathscr{F}$ the topology of uniform convergence in X, and the sets

$$\mathrm{U}_\alpha = \{\mathbf{f} \in \mathscr{F} :\ \mathbf{f}(x) \in \mathrm{N}_\alpha \ \text{ for all } \ x \in \mathrm{X}\}$$

form a fundamental system of neighborhoods of $0 \in \mathscr{F}$. This topology makes $\mathscr{F}$ a topological group under addition; for, given any index α, there exist indices β and γ such that

$$\mathrm{N}_\beta + \mathrm{N}_\beta \subset \mathrm{N}_\alpha \quad \text{and} \quad \mathrm{N}_\gamma \subset -\mathrm{N}_\alpha$$

(GT, III, §1, No. 2), whence $\mathrm{U}_\beta + \mathrm{U}_\beta \subset \mathrm{U}_\alpha$ and $\mathrm{U}_\gamma \subset -\mathrm{U}_\alpha$. However, $\mathscr{F}$ need not be a topological vector space: for example, if X is infinite and $\mathbf{f}_0 \in \mathscr{F}(\mathrm{X};\mathbf{R})$ is unbounded, then $\lambda\mathbf{f}_0$ does not converge to 0 as $\lambda \to 0$, indeed, $\sup_{x\in\mathrm{X}} |\lambda\mathbf{f}_0(x)| = +\infty$ for $\lambda \neq 0$ (TVS, I, §1, No. 1, *Example* 4).

For every subset $\mathscr{S}$ of $\mathscr{F}$, there is an induced uniformity: in the foregoing, one replaces "$\mathbf{f} \in \mathscr{F}$" by "$\mathbf{f} \in \mathscr{S}$"; thus if $\mathbf{g} \in \mathscr{S}$ then the sets

$$\{\mathbf{f} \in \mathscr{S} :\ \mathbf{f}(x) - \mathbf{g}(x) \in \mathrm{N}_\alpha \quad \text{for all } x \in \mathrm{X}\}$$

form a fundamental system of neighborhoods of $\mathbf{g}$ for the topology on $\mathscr{S}$ of uniform convergence in X. When X is a compact space, the linear subspace $\mathscr{S} = \mathscr{C}(\mathrm{X};\mathrm{E})$ is a topological vector space for this topology (TVS, *loc. cit.*). More generally, if X is a locally compact space and K is a compact subset of X, then $\mathscr{S} = \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is a topological vector space for the topology of uniform convergence in X, since $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ may be identified with a (closed) linear subspace of $\mathscr{C}(\mathrm{K};\mathrm{E})$ (§1, No. 1).

(8) Assume now that X is locally compact and K is a compact subset of X. We know from (7) that $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is a topological vector space for the topology of uniform convergence in X (equivalently, in K). By (4) and (6), we can suppose that the neighborhoods N_α of $0 \in \mathrm{E}$ are given by continuous semi-norms q_α,

$$\mathrm{N}_\alpha = \{\mathbf{x} \in \mathrm{E} :\ q_\alpha(\mathbf{x}) \leqslant 1\};$$

then (with U_α defined as in (7))

$$\mathrm{U}_\alpha = \{\mathbf{f} \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) :\ q_\alpha\big(\mathbf{f}(x)\big) \leqslant 1 \ \text{ for all } x \in \mathrm{X}\}.$$

For every $\mathbf{f} \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$, $q_\alpha\big(\mathbf{f}(\mathrm{X})\big) \cup \{0\} = q_\alpha\big(\mathbf{f}(\mathrm{K})\big) \cup \{0\}$ is a compact subset of $\mathbf{R}$, therefore the formula

$$p_\alpha(\mathbf{f}) = \sup_{x\in\mathrm{X}} q_\alpha\big(\mathbf{f}(x)\big) = \sup_{x\in\mathrm{K}} q_\alpha\big(\mathbf{f}(x)\big)$$

defines a (finite) semi-norm p_α on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$, such that

$$(*) \qquad \mathrm{U}_\alpha = \{\mathbf{f} \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E}) :\ p_\alpha(\mathbf{f}) \leqslant 1\}.$$

The neighborhoods U_α of 0 in $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ are thus convex, so $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is a locally convex space; moreover, it is clear from (*) that p_α is continuous at 0, hence uniformly continuous, by the argument in (6), (iv), therefore U_α is closed.

The formula (*) already establishes what we set out to prove: the topology of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$ is defined by the family of semi-norms (p_α).

III.39, ℓ. 13–16.

"…by No. 2, Prop. 6 we have, for every function $\mathbf{f} \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$,

$$q_\alpha\left(\int \mathbf{f}\, d\mu\right) = q_\alpha\left(\int h\mathbf{f}\, d\mu\right) \leqslant \int h(x) q_\alpha\big(\mathbf{f}(x)\big)\, d|\mu|(x) \leqslant |\mu|(h) \cdot p_\alpha(\mathbf{f})$$

(the q_α being extended by continuity to $\widehat{\mathrm{E}}$), which proves the continuity of $\mathbf{f} \mapsto \int \mathbf{f}\, d\mu$."

Superficially straightforward, the proof entails a subtle interplay between semi-norms on E, $\widehat{\mathrm{E}}$ and the corresponding function spaces; the details are as follows.

(1) It is more convenient to start from neighborhoods of 0 in $\widehat{\mathrm{E}}$. We know that $\widehat{\mathrm{E}}$ is locally convex (see the note for III.38, ℓ. 13,14). By the preceding note, $\widehat{\mathrm{E}}$ has a fundamental system (N_α^*) of closed, balanced, convex neighborhoods of 0, expressed by a family (q_α^*) of continuous semi-norms,

$$\mathrm{N}_\alpha^* = \{\mathbf{x}^* \in \widehat{\mathrm{E}} :\ q_\alpha^*(\mathbf{x}^*) \leqslant 1\}.$$

We can regard E as a dense linear subspace of $\widehat{\mathrm{E}}$, equipped with the relative topology; then the sets $\mathrm{N}_\alpha = \mathrm{N}_\alpha^* \cap \mathrm{E}$ form a fundamental system of closed, balanced, convex neighborhoods of 0 in E, and, setting $q_\alpha = q_\alpha^* \big| \mathrm{E}$, the q_α are continuous semi-norms on E such that

$$\mathrm{N}_\alpha = \{\mathbf{x} \in \mathrm{E} :\ q_\alpha(\mathbf{x}) \leqslant 1\}.$$

For $\mathbf{f} \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$ we write, as earlier in the proof,

$$p_\alpha(\mathbf{f}) = \sup_{\mathbf{x} \in \mathrm{K}} q_\alpha\big(\mathbf{f}(\mathbf{x})\big).$$

(2) With the above notations, the displayed inequalities in the text become, for all $\mathbf{f}^* \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \widehat{\mathrm{E}})$,

$$q_\alpha^*\left(\int \mathbf{f}^*\, d\mu\right) = q_\alpha^*\left(\int h\mathbf{f}^*\, d\mu\right) \leqslant \int h(x) q_\alpha^*\big(\mathbf{f}^*(x)\big)\, d|\mu|(x) \leqslant |\mu|(h) \cdot p_\alpha^*(\mathbf{f}^*),$$

where

(i) $$p_\alpha^*(\mathbf{f}^*) = \sup_{x \in \mathrm{K}} q_\alpha^*\big(\mathbf{f}^*(x)\big);$$

the first inequality follows from the cited Prop. 6 and the fact that $q_\alpha^* \circ (h \cdot \mathbf{f}^*) = h \cdot (q_\alpha^* \circ \mathbf{f}^*)$ as functions on X, whereas the second inequality follows from (i). From this, we need only retain

(ii) $$q_\alpha^*\left(\int \mathbf{f}^*\, d\mu\right) \leqslant |\mu|(h) \cdot p_\alpha^*(\mathbf{f}^*).$$

(3) Given $\mathbf{f} \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$, we propose to apply the inequality (ii) to the function $\mathbf{f}^* = \iota \circ \mathbf{f}$, where $\iota : \mathrm{E} \to \widehat{\mathrm{E}}$ is the canonical injection. Since $(\iota \circ \mathbf{f})(x) = \mathbf{f}(x) \in \mathrm{E}$, we have

$$p_\alpha^*(\iota \circ \mathbf{f}) = \sup_{x \in \mathrm{K}} q_\alpha^*\big(\mathbf{f}(x)\big) = \sup_{x \in \mathrm{K}} q_\alpha\big(\mathbf{f}(x)\big) = p_\alpha(\mathbf{f}),$$

thus the right side of (ii) becomes $|\mu|(h) \cdot p_\alpha(\mathbf{f})$. We wish to prove that

$$\text{(iii)} \qquad q_\alpha^*\left(\int \mathbf{f}\, d\mu\right) \leqslant |\mu|(h) \cdot p_\alpha(\mathbf{f}),$$

which will establish the continuity of (the restriction to $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$ of) the mapping $\mathbf{f} \mapsto \int \mathbf{f}\, d\mu$, since the p_α define the topology of $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$, and the q_α^* that of $\widehat{\mathrm{E}}$; thus we need only show that the elements $\int (\iota \circ \mathbf{f})\, d\mu$ and $\int \mathbf{f}\, d\mu$ of $\widehat{\mathrm{E}}$ are equal, and this follows from the fact that, for every $\mathbf{u} \in \widehat{\mathrm{E}}'$,

$$\begin{aligned}\left\langle \int (\iota \circ \mathbf{f})\, d\mu, \mathbf{u} \right\rangle &= \int \langle (\iota \circ \mathbf{f})(x), \mathbf{u} \rangle \, d\mu(x) \\ &= \int \langle \mathbf{f}(x), \mathbf{u} \rangle \, d\mu(x) = \left\langle \int \mathbf{f}\, d\mu, \mathbf{u} \right\rangle.\end{aligned}$$

III.39, ℓ. $-13, -12$.

"... with the notations of the statement,

$$\int \big(g(x) \cdot \mathbf{a}\big)\, d\mu(x) = \mu(g) \cdot \mathbf{a}\text{"}$$

Since $g \cdot \mathbf{a} \in \mathscr{K}(\mathrm{X}; \mathrm{E})$ (by the continuity of scalar multiplication in E), the symbol $\int (g \cdot \mathbf{a})\, d\mu$ is authorized by No. 1, Def. 1 for the element of E'^* such that, for all $\mathbf{z}' \in \mathrm{E}'$,

$$\begin{aligned}\left\langle \int (g \cdot \mathbf{a})\, d\mu, \mathbf{z}' \right\rangle &= \int \langle (g \cdot \mathbf{a})(x), \mathbf{z}' \rangle \, d\mu(x) \\ &= \int g(x) \langle \mathbf{a}, \mathbf{z}' \rangle \, d\mu(x) \\ &= \langle \mathbf{a}, \mathbf{z}' \rangle \int g(x)\, d\mu(x) \\ &= \langle \mu(g) \cdot \mathbf{a}, \mathbf{z}' \rangle;\end{aligned}$$

thus $\int (g \cdot \mathbf{a})\, d\mu$ is equal to the canonical image of $\mu(g) \cdot \mathbf{a}$ in $\widehat{\mathrm{E}}$, hence may be identified with the element $\mu(g) \cdot \mathbf{a}$.

III.39, ℓ. −5, −4.

"... *the mapping* $\mu \mapsto \int \mathbf{f}\, d\mu$ *of* $\mathscr{M}(\mathrm{X};\mathbf{C})$ *into* $\widehat{\mathrm{E}}$ ..."

Cor. 1 of Prop. 7 of No. 3 permits the definition of a mapping

$$\mathrm{B} : \mathscr{K}(\mathrm{X};\mathrm{E}) \times \mathscr{M}(\mathrm{X};\mathbf{C}) \to \widehat{\mathrm{E}}\,, \quad \mathrm{B}(\mathbf{f},\mu) = \int \mathbf{f}\, d\mu\,,$$

that is clearly bilinear.

The message of Props. 8 and 9: B is separately continuous when $\mathscr{K}(\mathrm{X};\mathrm{E})$ is equipped with the direct limit topology and $\mathscr{M}(\mathrm{X};\mathbf{C})$ with the topology of strictly compact convergence, and, as such a mapping, is characterized by its values $\mathrm{B}(g\cdot\mathbf{a},\varepsilon_x) = g(x)\mathbf{a}$ for $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$, $\mathbf{a}\in\mathrm{E}$ and $x \in \mathrm{X}$.

III.40, ℓ. 4—7.

"...consider the linear mapping $v : \mathbf{z}' \mapsto \langle \mathbf{f},\mathbf{z}'\rangle$ of E' into $\mathscr{K}(\mathrm{X};\mathbf{C})$, and let us show that the image under v of an *equicontinuous* subset H of E' is contained in a *strictly compact* subset of $\mathscr{K}(\mathrm{X};\mathbf{C})$."

From this point on, elements $\mathbf{z}' \in \mathrm{E}'$ may be regarded as acting also on $\widehat{\mathrm{E}}$ via their continuous extensions. It is simpler to replace E by $\widehat{\mathrm{E}}$: as observed at the end of the note for III.39, ℓ. 13–16, if $\iota : \mathrm{E} \to \widehat{\mathrm{E}}$ is the canonical injection, then $\int (\iota\circ\mathbf{f})\, d\mu = \int \mathbf{f}\, d\mu$, thus the two functions $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$ and $\iota\circ\mathbf{f} \in \mathscr{K}(\mathrm{X};\widehat{\mathrm{E}})$ produce the identical mapping $\mathscr{M}(\mathrm{X};\mathbf{C}) \to \widehat{\mathrm{E}}$.

To simplify the notation, we can therefore *assume henceforth that* E *is complete*, so that $\int \mathbf{f}\, d\mu \in \mathrm{E}$ for all $\mu \in \mathscr{M}(\mathrm{X};\mathbf{C})$. We also make the abbreviations $\mathscr{K} = \mathscr{K}(\mathrm{X};\mathbf{C})$ and $\mathscr{M} = \mathscr{M}(\mathrm{X};\mathbf{C})$, while retaining the full notations $\mathscr{K}(\mathrm{X};\mathrm{E})$ and $\mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$, $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ for K compact.

Since $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$, one has $v(\mathbf{z}') = \langle \mathbf{f},\mathbf{z}'\rangle = \mathbf{z}'\circ\mathbf{f} \in \mathscr{K}$ for all $\mathbf{z}' \in \mathrm{E}$; and if K is a compact set such that $\mathbf{f} \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathrm{E})$, for instance $\mathrm{K} = \mathrm{Supp}\,\mathbf{f}$, then $v(\mathbf{z}') \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ for all $\mathbf{z}' \in \mathrm{E}'$. The mapping $v : \mathrm{E}' \to \mathscr{K}$ is linear by a straightforward computation. We are to show that the set of functions

$$v(\mathrm{H}) = \{\mathbf{z}'\circ\mathbf{f} :\ \mathbf{z}' \in \mathrm{H}\} \subset \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) \subset \mathscr{K}$$

is contained in a strictly compact subset of $\mathscr{K}$, in other words, that the closure

$$\overline{v(\mathrm{H})}$$

of $v(\mathrm{H})$ in $\mathscr{K}$ (for the direct limit topology on $\mathscr{K}$) is strictly compact. Now, $v(\mathrm{H}) \subset \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ and we know that $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is a closed subset of $\mathscr{K}$ (§1, No. 1, Prop. 1), therefore $\overline{v(\mathrm{H})} \subset \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$; it follows (GT, I, §3, No. 1, Prop. 1) that

$$\overline{v(\mathrm{H})} = \overline{v(\mathrm{H})} \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$$

is the closure of $v(\mathrm{H})$ in $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ for the topology induced by that of $\mathscr{K}$, and since that topology is the norm topology (§1, No. 1, Prop. 1), $\overline{v(\mathrm{H})}$ is equal to the closure of $v(\mathrm{H})$ for the norm topology. Thus it will suffice to show that $\overline{v(\mathrm{H})}$ is compact for the norm topology.

Now, the norm topology on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ coincides with the topology of uniform convergence in K, with the topology of uniform convergence in X, and with the topology of compact convergence (the topology of uniform convergence in the compact subsets of X). One argues similarly that $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is a closed subset of $\mathscr{C}_c(\mathrm{X};\mathbf{C})$ (the space $\mathscr{C}(\mathrm{X};\mathbf{C})$ equipped with the topology τ_{cc} of compact convergence), that τ_{cc} induces on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ the norm topology, and that $\overline{v(\mathrm{H})}$ is also the closure of $v(\mathrm{H})$ in $\mathscr{C}_c(\mathrm{X};\mathbf{C})$ for τ_{cc}, that is, $\overline{v(\mathrm{H})} = \overline{v(\mathrm{H})}^{\tau_{\mathrm{cc}}}$.

Thus, the problem is to show that $v(\mathrm{H})$ is a relatively compact subset of $\mathscr{C}_c(\mathrm{X};\mathbf{C})$; by Ascoli's theorem (GT, X, §2, No. 5, Cor. 3 of Th. 2), it suffices to show that $v(\mathrm{H})$ is (i) equicontinuous, and (ii) pointwise bounded:

(i) Fix $x_0 \in \mathrm{X}$. Given any $\varepsilon > 0$, by equicontinuity of H there exists a neighborhood V of 0 in E such that

$$\mathbf{z} \in \mathrm{V} \quad \Rightarrow \quad |\mathbf{z}'(\mathbf{z})| \leqslant \varepsilon \quad \text{for all } \mathbf{z}' \in \mathrm{H},$$

and since $\mathbf{f}$ is continuous at x_0 there exists a neighborhood U of x_0 in X such that

$$x \in \mathrm{U} \quad \Rightarrow \quad \mathbf{f}(x) \in \mathbf{f}(x_0) + \mathrm{V},$$

whence $x \in \mathrm{U} \Rightarrow \big|\mathbf{z}'\big(\mathbf{f}(x) - \mathbf{f}(x_0)\big)\big| \leqslant \varepsilon$ for all $\mathbf{z}' \in \mathrm{H}$, that is,

$$\mathbf{z}' \in \mathrm{H} \quad \Rightarrow \quad \big|(\mathbf{z}' \circ \mathbf{f})(x) - (\mathbf{z}' \circ \mathbf{f})(x_0\big| \leqslant \varepsilon \quad \text{for all } x \in \mathrm{U},$$

whence the equicontinuity of the functions $\mathbf{z}' \circ \mathbf{f} \in v(\mathrm{H})$ at x_0.

(ii) Fix $x_0 \in \mathrm{X}$. We are to show that the set

$$\big(v(\mathrm{H})\big)(x_0) = \big\{\mathbf{z}'\big(\mathbf{f}(x_0)\big) : \mathbf{z}' \in \mathrm{H}\,\big\}$$

is bounded in $\mathbf{C}$. Indeed, since $\mathrm{H} \subset \mathrm{E}' = \mathscr{L}(\mathrm{E};\mathbf{C})$ is equicontinuous on E, H is relatively compact for the weak topology $\sigma(\mathrm{E}',\mathrm{E})$ on E' (TVS, III, §3, No. 4, Cor. 2 of Prop. 4). Since the mapping $\varphi : \mathrm{E}' \to \mathbf{C}$ defined by

$$\varphi(\mathbf{z}') = \mathbf{z}'\big(\mathbf{f}(x_0)\big) = \langle \mathbf{f}(x_0), \mathbf{z}' \rangle \qquad (\mathbf{z}' \in \mathrm{E}')$$

is continuous for $\sigma(\mathrm{E}', \mathrm{E})$, the relative compactness (i.e., boundedness) of $\big((v(\mathrm{H})\big)(x_0) = \varphi(\mathrm{H})$ follows from $\overline{\varphi(\mathrm{H})} \subset \varphi(\overline{\mathrm{H}})$ and the compactness of $\overline{\mathrm{H}}$.

To summarize, we have shown that the closure of $v(\mathrm{H})$ in $\mathscr{K}$ is contained in $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$ and is compact, hence is a strictly compact subset of $\mathscr{K}$.

III.40, ℓ. 11, 12.

"... u is none other than the restriction to $\mathscr{M}(\mathrm{X}; \mathbf{C})$ of the *transpose* ${}^t v$ (in the algebraic sense)

The conventions of the preceding note are in force; in particular, E is complete, and we abbreviate $\mathscr{K} = \mathscr{K}(\mathrm{X}; \mathbf{C})$, $\mathscr{M} = \mathscr{M}(\mathrm{X}; \mathbf{C})$, writing $\mathscr{K}^*$ and $\mathscr{M}^*$ for their algebraic duals. The linear mappings

$$u : \mathscr{M} \to \mathrm{E}, \ v : \mathrm{E}' \to \mathscr{K}$$

are defined by $u(\mu) = \int \mathbf{f}\, d\mu$ and $v(\mathbf{z}') = \mathbf{z}' \circ \mathbf{f}$ $\ (\mu \in \mathscr{M},\ \mathbf{z}' \in \mathrm{E}')$.

Equip E' with the topology $\sigma(\mathrm{E}', \mathrm{E}'^*)$, and $\mathscr{K}$ with the topology $\sigma(\mathscr{K}, \mathscr{K}^*)$; then v is continuous. For, if $f \in \mathscr{K}^*$ then $f \circ v \in \mathrm{E}'^*$ is continuous for $\sigma(\mathrm{E}', \mathrm{E}'^*)$

$$\mathrm{E}' \xrightarrow{\ v\ } \mathscr{K} \xrightarrow{\ f\ } \mathbf{C}$$

and since $\sigma(\mathscr{K}, \mathscr{K}^*)$ is the initial topology for the linear forms $f \in \mathscr{K}^*$ (TVS, II, §6, No. 2, Def. 2), it follows that v is continuous (GT, I, §2, No. 3, Prop. 4). The transposed linear mapping

$${}^t v : \mathscr{K}^* \to \mathrm{E}'^*$$

is thus given by ${}^t v : f \mapsto f \circ v$ (TVS, II, §6, No. 4, Prop. 5). {Moreover, ${}^t v$ is continuous for the topologies $\sigma(\mathscr{K}^*, \mathscr{K})$ and $\sigma(\mathrm{E}'^*, \mathrm{E}')$, and ${}^t({}^t v) = v$ (*loc. cit.*, Cor. of Prop. 5), but this is not needed here.}

Finally, $\mathscr{M} = \mathscr{K}' \subset \mathscr{K}^*$, and if $\mu \in \mathscr{M}$ then, for all $\mathbf{z}' \in \mathrm{E}'$,

$$\begin{aligned} \langle {}^t v(\mu), \mathbf{z}' \rangle &= \langle \mu, v(\mathbf{z}') \rangle = \langle \mu, \mathbf{z}' \circ \mathbf{f} \rangle \\ &= \Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle = \langle u(\mu), \mathbf{z}' \rangle, \end{aligned}$$

whence ${}^t v(\mu) = u(\mu)$ for all $\mu \in \mathscr{M}$. That is, ${}^t v \big| \mathscr{M} = u$.

III.40, ℓ. 12, 13.

"... its continuity therefore follows from the foregoing (TVS, IV, §1, No. 3, Prop. 6)."

The reference in the 1965 French original is inexplicit (*Esp. vect. top.*, chap. IV, 2e éd.), to a work not yet published—the bound edition of EVT did not appear until 1981. Unable to find a suitable substitute reference in the first edition at the time (1970's) that I first studied *Intégration*, I improvised a "direct proof" based on ingredients available in the first edition of Ch. III of *Esp. vect. top.* When I was preparing (1998) the published translation, Professor Jacques Dixmier guided me to the correct reference in TVS and sketched the indicated proof of the continuity of u ; an exposition of his proof (*Proof* #2) is given immediately after the following "direct proof".

Proof #1. The conventions of the preceding two notes are in force. Assuming V is a neighborhood of 0 in E , let us show that $\overset{-1}{u}(\mathrm{V})$ is a neighborhood of 0 in $\mathscr{M}$ for the topology τ_{scc} of strictly compact convergence on $\mathscr{M}$. We can suppose that V is convex, balanced and closed in E . For the canonical duality $\langle \mathbf{z}, \mathbf{z}' \rangle = \mathbf{z}'(\mathbf{z})$ between E and E′ , the polar of V in E′ is the set

$$\mathrm{V}^{\circ} = \{\mathbf{z}' \in \mathrm{E}' : \ |\mathbf{z}'(\mathbf{z})| \leqslant 1 \quad \text{for all } \mathbf{z} \in \mathrm{V}\}$$

(TVS, II, §8, No. 4); moreover, V° is convex, balanced, and closed for the weak topology $\sigma(\mathrm{E}', \mathrm{E})$, and, by the theorem on bipolars (*loc. cit.*, §6, No. 3, Cor. 3 of Th. 1), $\mathrm{V} = \mathrm{V}^{\circ\circ}$ (the polar of V° in E).

Let $\mathrm{H} = \mathrm{V}^{\circ} \subset \mathrm{E}'$; since $\mathrm{H}^{\circ} = \mathrm{V}$ is a neighborhood of 0 in E , H is equicontinuous on E (TVS, III, §3, No. 5, Prop. 7), therefore the closure $\overline{v(\mathrm{H})}$ of $v(\mathrm{H})$ in $\mathscr{K}$ is strictly compact (see (ii) of the Note for III.40, ℓ. 4–7).

Thus $\overline{v(\mathrm{H})} \in \mathfrak{S}$, the set of all strictly compact subsets of $\mathscr{K}$. The topology τ_{scc} on $\mathscr{M}$ is the topology of uniform convergence in the sets $\mathscr{S} \in \mathfrak{S}$. It follows (GT, X, §1, No. 2) that if $\mathscr{S} \in \mathfrak{S}$ and $\mathrm{D} = \{c \in \mathbf{C} : |c| \leqslant 1\}$ is the closed unit disc in $\mathbf{C}$, so that the set

$$\mathscr{D} = \{(c, d) \in \mathbf{C} \times \mathbf{C} : \ c - d \in \mathrm{D}\}$$

is an entourage for the uniformity of $\mathbf{C}$, then the set

$$\begin{aligned} \{(\mu, \nu) \in \mathscr{M} \times \mathscr{M} : \ &\big(\mu(g), \nu(g)\big) \in \mathscr{D} \ \text{ for all } g \in \mathscr{S}\} \\ &= \{(\mu, \nu) : \ |\mu(g) - \nu(g)| \leqslant 1 \ \text{ for all } g \in \mathscr{S}\} \end{aligned}$$

is an entourage for the uniformity on $\mathscr{M}$ of strictly compact convergence, and the set

$$(*) \qquad \{\mu \in \mathscr{M} : \ |\mu(g)| \leqslant 1 \ \text{ for all } g \in \mathscr{S}\}$$

is a neighborhood of 0 in $\mathscr{M}$ for τ_{scc} . In the canonical duality between $\mathscr{K}$ and $\mathscr{K}' = \mathscr{M}$, the set (*) is the polar $\mathscr{S}^{\circ}$ of $\mathscr{S}$ in $\mathscr{M}$. In particular, the set

$$\overline{v(\mathrm{H})}^{\,\circ} = \{\mu \in \mathscr{M} : \ |\mu(g)| \leqslant 1 \quad \text{for all } g \in \overline{v(\mathrm{H})}\}$$

is a neighborhood of 0 in $\mathscr{M}$ for τ_{scc}, and, since the $\mu \in \mathscr{M}$ are continuous on $\mathscr{K}$,

$$v(\mathrm{H})^\circ = \{\mu \in \mathscr{M} : \ |\mu(g)| \leqslant 1 \quad \text{for all } \ g \in v(\mathrm{H})\,\} = \overline{v(\mathrm{H})}^{\,\circ}\,.$$

Thus $v(\mathrm{H})^\circ$ is a neighborhood of 0 in $\mathscr{M}$ for τ_{scc}.

To conclude the proof, we need only show that

$$(**) \qquad \overset{-1}{u}(\mathrm{V}) = v(\mathrm{H})^\circ\,.$$

To that end, consider the linear mapping

$$v : \mathrm{E}' \to \mathscr{K}$$

in the context of the algebraic duals of E' and $\mathscr{K}$: as observed in the preceding note, v is continuous for the topologies $\sigma(\mathrm{E}', \mathrm{E}'^*)$ and $\sigma(\mathscr{K}, \mathscr{K}^*)$. Moreover, if ${}^t v : \mathscr{K}^* \to \mathrm{E}'^*$ is its transpose, then (TVS, II, §6, No. 4, Prop. 6),

$$(\dagger) \qquad v(\mathrm{H})^\circ = ({}^t v)^{-1}(\mathrm{H}^\circ)\,,$$

where the polars are taken in $\mathscr{K}^*$ and E'^*, respectively; note that $\mathscr{M} = \mathscr{K}' \subset \mathscr{K}^*$ and the $v(\mathrm{H})^\circ$ in (**) is the intersection with $\mathscr{M}$ of the $v(\mathrm{H})^\circ$ in (†). Finally,

$$\begin{aligned}
\overset{-1}{u}(v) &= \{\mu \in \mathscr{M} : \ u(\mu) \in \mathrm{V} = \mathrm{H}^\circ\,\} && (\text{polar in } \mathrm{E}\,) \\
&= \{\mu \in \mathscr{M} : \ ({}^t v)(\mu) \in \mathrm{H}^\circ\,\} && \big({}^t v\big|\mathscr{M} = u\big) \\
&= \{\mu \in \mathscr{M} : \ |\langle ({}^t v)(\mu), \mathbf{z}'\rangle| \leqslant 1 \quad \text{for all } \ \mathbf{z}' \in \mathrm{H}\,\} \\
&= \mathscr{M} \cap \{f \in \mathscr{K}^* : \ |\langle ({}^t v)(f), \mathbf{z}'\rangle| \leqslant 1 \quad \text{for all } \ \mathbf{z}' \in \mathrm{H}\,\} \\
&= \mathscr{M} \cap \{f \in \mathscr{K}^* : \ ({}^t v)(f) \in \mathrm{H}^\circ\,\} && (\text{polar in } \mathrm{E}'^*\,) \\
&= \mathscr{M} \cap ({}^t v)^{-1}(\mathrm{H}^\circ) \\
&= \mathscr{M} \cap v(\mathrm{H})^\circ && (\text{polar in } \mathscr{K}^*, \text{ by } (\dagger)\,) \\
&= \{\mu \in \mathscr{M} : \ |\langle \mu, g\rangle| \leqslant 1 \quad \text{for all } \ g \in v(\mathrm{H})\,\} \\
&= v(\mathrm{H})^\circ && (\text{polar in } \mathscr{M}\,)\,,
\end{aligned}$$

that is, (**) holds.

Proof #2. {For consistency of this intricate proof with the notation of the cited reference in TVS, IV, it is convenient to *interchange the original notations u and v in the proof*; otherwise, the conventions of the preceding two notes are in force (in particular, E is complete).}

Regard the pair of vector spaces $(\mathscr{K}, \mathscr{M})$ as put in duality by the bilinear form $(g, \mu) \mapsto \langle g, \mu \rangle = \int g\, d\mu$, and similarly for the pair $(\mathrm{E}, \mathrm{E}')$, put in duality by the bilinear form $(\mathbf{z}, \mathbf{z}') \mapsto \langle \mathbf{z}, \mathbf{z}' \rangle = \mathbf{z}'(\mathbf{z})$. We have linear mappings $u : \mathrm{E}' \to \mathscr{K}$ and $v : \mathscr{M} \to \mathrm{E}$, defined by the formulas (note the reversed roles of u and v)

$$u(\mathbf{z}') = \langle \mathbf{f}, \mathbf{z}' \rangle \quad \text{and} \quad v(\mu) = \int \mathbf{f}\, d\mu\,;$$

let us place them in the context of the foregoing dualities.

Let E_1 be the vector space E' equipped with the weak topology $\sigma(\mathrm{E}', \mathrm{E})$. The dual of E_1 may be identified with the vector space E, that is, $\mathrm{E}_1' = \mathrm{E}$ as vector spaces, and if E in turn is equipped with the weakened topology $\sigma(\mathrm{E}, \mathrm{E}')$, then the dual of E may be identified with the vector space E'; thus, with the first two primes interpreted in the sense of the dual pairing, $(\mathrm{E}_1')' = \mathrm{E}' = \mathrm{E}_1$ as vector spaces.

Similarly, let E_2 be the vector space $\mathscr{K}$ equipped with the weakened topology $\sigma(\mathscr{K}, \mathscr{M})$. Then the dual E_2' of E_2 may be identified with the vector space $\mathscr{M}$, and if $\mathscr{M}$ is in turn equipped with the vague topology $\sigma(\mathscr{M}, \mathscr{K})$, then the dual of $\mathscr{M}$ may be identified with the vector space $\mathscr{K}$; interpreting the primes appropriately, $(\mathrm{E}_2')' = \mathscr{K} = \mathrm{E}_2$ as vector spaces.

With these notations, we have

$$(*) \qquad u : \mathrm{E}_1 \to \mathrm{E}_2\,, \quad v : \mathrm{E}_2' \to \mathrm{E}_1'\,.$$

For every $\mathbf{z}' \in \mathrm{E}' = \mathrm{E}_1$ and $\mu \in \mathscr{M} = \mathrm{E}_2'$ we have (citing formula (1) of No. 1 for the second equality)

$$\langle u(\mathbf{z}'), \mu \rangle = \int \langle \mathbf{f}, \mathbf{z}' \rangle\, d\mu = \Big\langle \int \mathbf{f}\, d\mu, \mathbf{z}' \Big\rangle = \langle v(\mu), \mathbf{z}' \rangle\,;$$

it follows (TVS, II, §6, No. 4, Prop. 5) that, in the notations of (*), u and v are continuous for the topologies σ in question, and each is the *transpose* of the other (in the sense of *ibid.*, *Remark*), concisely ${}^t u = v$ (and ${}^t v = u$).

Our problem is to show that the linear mapping $v : \mathscr{M} \to \mathrm{E}$ remains continuous when $\mathscr{M}$ is equipped with the topology of strictly compact convergence (in general stronger than $\sigma(\mathrm{E}_2', \mathrm{E}_2) = \sigma(\mathscr{M}, \mathscr{K})$, the vague topology on $\mathscr{M}$), and E with its original topology (in general stronger than $\sigma(\mathrm{E}_1', \mathrm{E}_1) = \sigma(\mathrm{E}, \mathrm{E}')$, the weakened topology on E); this is fertile terrain for the $\mathfrak{S}$-topologies associated with dualities (TVS, III, §3, No. 5). Indeed, if $\mathfrak{S}_2$ is the set of all strictly compact subsets of $\mathrm{E}_2 = \mathscr{K}$, then the topology of $(\mathrm{E}_2')_{\mathfrak{S}_2}$ is the topology of strictly compact convergence in $\mathrm{E}_2' = \mathscr{M}$; and if $\mathfrak{S}_1$ is the set of all subsets of $\mathrm{E}_1 = \mathrm{E}'$ that are equicontinuous with respect to the original locally convex topology τ on E, then the topology of $(\mathrm{E}'_1)_{\mathfrak{S}_1}$ coincides with the topology τ on $\mathrm{E}'_1 = \mathrm{E}$ (*loc. cit.*, Cor. 1 of Prop. 7). Thus our problem is to show that the linear mapping

$$v : (\mathrm{E}'_2)_{\mathfrak{S}_2} \to (\mathrm{E}'_1)_{\mathfrak{S}_1}$$

is continuous for the indicated $\mathfrak{S}$-topologies. Now, $v = {}^t u$; in view of TVS, IV, §1, No. 3, Prop. 6, we need only show that:

1° $u : \mathrm{E}_1 \to \mathrm{E}_2$ is continuous for the weakened topologies of the E_i $(i = 1, 2)$;

2° the sets in $\mathfrak{S}_i$ are bounded in E_i $(i = 1, 2)$; and

3° if $\mathrm{A} \in \mathfrak{S}_1$ then there exist sets $\mathrm{A}_1, \ldots, \mathrm{A}_n$ in $\mathfrak{S}_2$ and a real number $\lambda > 0$ such that $\lambda \cdot u(\mathrm{A})$ is contained in the closed balanced convex envelope of $\mathrm{A}_1 \cup \ldots \cup \mathrm{A}_n$.

As to 1°: Since E_i bears the topology $\sigma(\mathrm{E}_i, \mathrm{E}'_i)$, that topology coincides with its weakened topology, so the assertion is immediate from the continuity property of u noted in connection with (*).

As to 2°: If $\mathrm{A} \in \mathfrak{S}_1$, that is, if $\mathrm{A} \subset \mathrm{E}'$ is equicontinuous for the original topology τ of E, then A is relatively compact for $\sigma(\mathrm{E}', \mathrm{E})$ (TVS, III, §3, No. 4, Cor. 2 of Prop. 4), therefore precompact and hence bounded (TVS, III, §1, No. 2, Prop. 2). If $\mathrm{A} \in \mathfrak{S}_2$, so that $\mathrm{A} \subset \mathscr{K}$ is (strictly) compact for the inductive limit topology, then it is also compact (hence bounded) for the weaker topology $\sigma(\mathscr{K}, \mathscr{M})$.

As to 3°: If $\mathrm{A} \in \mathfrak{S}_1$ then $u(\mathrm{A})$ is contained in a strictly compact subset A_1 of $\mathscr{K}$ (see the Note for III.40, ℓ. 4—7), thus condition 3° is verified with $n = 1$ and $\lambda = 1$ (and no need to take envelopes).

§4. PRODUCTS OF MEASURES

III.41, ℓ. 6, 7.

"It is immediate that the image under ω of $\mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$ is contained in $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}))$."

If $f : \mathrm{X} \times \mathrm{Y} \to \mathbf{C}$ then $\omega(f) : \mathrm{X} \to \mathscr{F}(\mathrm{Y}; \mathbf{C})$ is defined as follows: for each $x \in \mathrm{X}$, $\big(\omega(f)\big)(x) : \mathrm{Y} \to \mathbf{C}$ is defined by

$$\Big(\big(\omega(f)\big)(x)\Big)(y) = f(x, y) \quad \text{for all } y \in \mathrm{Y}.$$

If f is continuous then, for each $x \in \mathrm{X}$, the mapping $y \mapsto f(x, y)$ is continuous, hence $\big(\omega(f)\big)(x)$ is continuous on Y, and so $\omega(f) : \mathrm{X} \to \mathscr{C}(\mathrm{Y}; \mathbf{C})$.

Suppose now that $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$. For each $x \in \mathrm{X}$,

$$\Big(\big(\omega(f)\big)(x)\Big)(y) = f(x,y) = 0 \quad \text{for all } y \in \mathrm{Y} - \mathrm{L}\,,$$

therefore $\big(\omega(f)\big)(x) \in \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ for all $x \in \mathrm{X}$. Moreover, if $x \in \mathrm{X} - \mathrm{K}$, then

$$\Big(\big(\omega(f)\big)(x)\Big)(y) = f(x,y) = 0 \quad \text{for all } y \in \mathrm{Y}\,,$$

therefore $\big(\omega(f)\big)(x) = 0$, the zero element of $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$; thus

$$\omega(f) \in \mathfrak{F}\big(\mathrm{X}, \mathrm{K}; \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})\big).$$

To show that $\omega(f) \in \mathscr{K}\big(\mathrm{X}, \mathrm{K}; \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})\big)$, we need only show that $\omega(f)$ is continuous for the given topology on X and the norm topology on $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$. Since the norm topology on $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ coincides with the topology of uniform convergence in the compact subsets of Y, $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ is a topological subspace of $\mathscr{C}_c(\mathrm{Y}; \mathbf{C})$; the desired continuity of $\omega(f)$ is therefore a consequence of GT, X, §3, No. 4, Th. 3 (with $\mathrm{Z} = \mathbf{C}$ and $\widetilde{f} = \omega(f)$).

III.41, ℓ. 7–10.

"... if $\mathbf{u}$ is a continuous mapping of X into $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$, with support contained in K, then the mapping $(x,y) \mapsto \mathbf{u}(x)(y)$ of $\mathrm{X} \times \mathrm{Y}$ into $\mathbf{C}$ is continuous and has support contained in $\mathrm{K} \times \mathrm{L}$ "

Define $f : \mathrm{X} \times \mathrm{Y} \to \mathbf{C}$ by $f(x,y) = \big(\mathbf{u}(x)\big)(y)$. For every $x \in \mathrm{X}$,

$$\Big(\big(\omega(f)\big)(x)\Big)(y) = f(x,y) = \big(\mathbf{u}(x)\big)(y) \quad \text{for all } y \in \mathrm{Y}\,,$$

hence $\big(\omega(f)\big)(x) = \mathbf{u}(x)$ for all $x \in \mathrm{X}$; thus $\omega(f) = \mathbf{u}$. The continuity of $\omega(f) : \mathrm{X} \to \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}) \subset \mathscr{C}_c(\mathrm{Y}; \mathbf{C})$ then implies that of f, by GT, X, §3, No. 4, Th. 3 (the "conversely" part, with $\mathrm{Z} = \mathbf{C}$ and $\widetilde{f} = \omega(f)$), and it is clear that $f(x,y) \neq 0$ implies $(x,y) \in \mathrm{K} \times \mathrm{L}$.

III.41, ℓ. 14–18.

"... the image under ω, of

$$\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$$

identified with a subspace of $\mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$, is again the space $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ but this time identified canonically with a space of mappings of X into $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ (A, II, §7, No. 7, Cor. of Prop. 15)"

An element $f = \sum_{i=1}^{n} g_i \otimes h_i$ of the displayed tensor product space, viewed as a function on $\mathrm{X} \times \mathrm{Y}$ (the first identification referred to), is the function

$$f(x,y) = \sum_{i=1}^{n} g_i(x) h_i(y)\,.$$

Let us calculate its image under ω: for all $x \in \mathrm{X}$, $y \in \mathrm{Y}$,

$$\Big(\big(\omega(f)\big)(x)\Big)(y) = f(x,y) = \sum_{i=1}^{n} g_i(x) h_i(y)\,,$$

thus

$$\big(\omega(f)\big)(x) = \sum_{i=1}^{n} g_i(x) \cdot h_i \in \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}) \quad \text{for all } x \in \mathrm{X}\,.$$

Writing $\mathrm{E} = \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$, the preceding formula shows that the image of

$$\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}) = \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathrm{E}$$

under ω coincides with its interpretation in § 1, No. 2, Prop. 5 as a dense linear subspace of $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$ (the second identification referred to).

III.41, $\ell.\ -13, -12$.

"... the conclusion of (ii) follows from the fact that the restriction of ω is a topological isomorphism."

Let ω_0 be the indicated restriction; according to (i), the mapping

$$\omega_0 : \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C}) \to \mathscr{K}\big(\mathrm{X}, \mathrm{K}; \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})\big)$$

is an (isometric) isomorphism of topological vector spaces. Since ω_0 maps the subset $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ of its domain onto a dense subset of $\mathscr{K}\big(\mathrm{X}, \mathrm{K}; \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})\big)$, the isomorphism implies that

$$\overline{\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})} = \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})\,.$$

III.41, $\ell.\ -9, -8$.

"... the subspace $\mathscr{K}(\mathrm{X}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}; \mathbf{C})$ is *dense* in $\mathscr{K}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$."

Let $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$. Since $\mathrm{Supp}(f)$ is compact in $\mathrm{X} \times \mathrm{Y}$, one has $\mathrm{Supp}(f) \subset \mathrm{K} \times \mathrm{L}$ for suitable compact sets $\mathrm{K} \subset \mathrm{X}$, $\mathrm{L} \subset \mathrm{Y}$. Then $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$, so by Lemma 1, (ii), f belongs to the closure of $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ in $\mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$; but $\mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$ is a topological subspace of $\mathscr{K}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$ (§1, No. 1, Prop. 1, (i)), therefore

f belongs to the closure of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) \otimes_{\mathbf{C}} \mathscr{K}(\mathrm{Y},\mathrm{L};\mathbf{C})$ in $\mathscr{K}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$ (GT, I, §3, No. 1, Prop. 1).

The rest of this note is for use in No. 4.

Another way of putting the matter is that the set of functions (let us drop the "; **C**" for brevity)

$$f \otimes g \qquad \big(f \in \mathscr{K}(\mathrm{X}),\ g \in \mathscr{K}(\mathrm{Y})\big)$$

is *total* in $\mathscr{K}(\mathrm{X}\times\mathrm{Y})$ (TVS, I, §2, No. 1, Def. 1); by the Hahn-Banach theorem, a test for totality is that if ν is a continuous linear form on $\mathscr{K}(\mathrm{X}\times\mathrm{Y})$ (that is, a measure on $\mathrm{X}\times\mathrm{Y}$) such that

$$(*) \qquad \nu(f\otimes g) = 0 \ \text{ for all } f \in \mathscr{K}(\mathrm{X}),\ g \in \mathscr{K}(\mathrm{Y}),$$

then $\nu = 0$ (TVS, II, §8, No. 3, Cor. of Prop. 1).

We wish to sharpen this result to the following: If ν is a measure on $\mathrm{X}\times\mathrm{Y}$ such that $\nu(f\otimes g)=0$ as f runs over a total subset of $\mathscr{K}(\mathrm{X})$ and g runs over a total subset of $\mathscr{K}(\mathrm{Y})$, then $\nu = 0$.

Lemma. — *The bilinear function* $\mathscr{K}(\mathrm{X})\times\mathscr{K}(\mathrm{Y}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y})$ *defined by* $(f,g)\mapsto f\otimes g$ *is separately continuous* (for the direct limit topologies).

Proof. Fix $g \in \mathscr{K}(\mathrm{Y})$, say $g \in \mathscr{K}(\mathrm{Y},\mathrm{L})$, where L is a compact subset of Y. Let us write $u : \mathscr{K}(\mathrm{X}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y})$ for the linear mapping $u(f) = f\otimes g$; we are to show that u is continuous. Given any compact subset K of X, it suffices to show that the composite mapping

$$\mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y})$$

defined by $f \mapsto f \mapsto f\otimes g$ $(f \in \mathscr{K}(\mathrm{X},\mathrm{K}))$ is continuous (TVS, II, §4, No. 4, Prop. 5, (ii), *Example* II, and *Remark*). Writing $u_{\mathrm{K}} = u\big|\mathscr{K}(\mathrm{X},\mathrm{K})$, the problem is to show that the mapping

$$u_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y})$$

is continuous. For all $f \in \mathscr{K}(\mathrm{X},\mathrm{K})$ one has $f\otimes g \in \mathscr{K}(\mathrm{X}\times\mathrm{Y},\mathrm{K}\times\mathrm{L})$, whence a factorization

$$\mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y},\mathrm{K}\times\mathrm{L}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y})$$

of u_{K}. The second arrow being continuous by the definition of the direct limit topology, we are reduced to proving the continuity of the mapping

$$\mathscr{K}(\mathrm{X},\mathrm{K}) \to \mathscr{K}(\mathrm{X}\times\mathrm{Y},\mathrm{K}\times\mathrm{L})$$

defined by $f \mapsto f \otimes g$, and this is immediate from $\|f \otimes g\| = \|f\|\,\|g\|$. Similarly for the mappings $g \mapsto f \otimes g$.

Proposition. — *If* A (resp. B) *is a total subset of* $\mathscr{K}(\mathrm{X})$ (resp. $\mathscr{K}(\mathrm{Y})$), *then the set of functions*

$$f \otimes g \qquad (f \in \mathrm{A},\ g \in \mathrm{B})$$

is total in $\mathscr{K}(\mathrm{X} \times \mathrm{Y})$.

Proof. Let ν be a measure on $\mathrm{X} \times \mathrm{Y}$ such that $\nu(f \otimes g) = 0$ for all $f \in \mathrm{A}$ and $g \in \mathrm{B}$; we are to show that $\nu = 0$. Fix $g \in \mathrm{B}$. It follows from the Lemma that the function $f \mapsto \nu(f \otimes g)$ is a continuous linear form on $\mathscr{K}(\mathrm{X})$; since it vanishes on the total set A, it is identically zero. Thus $\nu(f \otimes g) = 0$ for all $f \in \mathscr{K}(\mathrm{X})$ and $g \in \mathrm{B}$. Similarly, fixing $f \in \mathscr{K}(\mathrm{X})$, the validity of $\nu(f \otimes g) = 0$ for all $g \in \mathrm{B}$ implies that $\nu(f \otimes g) = 0$ for all $g \in \mathscr{K}(\mathrm{Y})$. Thus $\nu = 0$.

Corollary. — *With notations as in the Proposition, if* ν *is a measure on* $\mathrm{X} \times \mathrm{Y}$ *such that* $\nu(f \otimes g) = 0$ *for all* $f \in \mathrm{A}$ *and* $g \in \mathrm{B}$, *then* $\nu = 0$.

III.42, ℓ. 3–5.

"…for $\mathbf{u} = \omega(f)$ and for every $y \in \mathrm{Y}$,

$$\left\langle \int \mathbf{u}(x)\,d\lambda(x), \varepsilon_y \right\rangle = \int \mathbf{u}(x)(y)\,d\lambda(x) = \int f(x,y)\,d\lambda(x)\,,$$

whence the lemma."

Write $\mathrm{E} = \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$; thus $\mathbf{u} \in \mathscr{K}(\mathrm{X}; \mathrm{E})$, where $\big(\mathbf{u}(x)\big)(y) = f(x,y)$ for all $x \in \mathrm{X}$, $y \in \mathrm{Y}$. For $y \in \mathrm{Y}$, the Dirac measure ε_y on Y is a continuous linear form on $\mathscr{K}(\mathrm{Y}; \mathbf{C})$; for clarity, let us write η_y for the restriction of ε_y to $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$. Since $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$ is a topological subspace of $\mathscr{K}(\mathrm{Y}; \mathbf{C})$, η_y is continuous on $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}) = \mathrm{E}$, thus $\eta_y \in \mathrm{E}'$. By the definition of the vectorial integral,

$$\left\langle \int \mathbf{u}\,d\lambda, \eta_y \right\rangle = \int (\eta_y \circ \mathbf{u})\,d\lambda \quad \text{for all } y \in \mathrm{Y}\,;$$

but $(\eta_y \circ \mathbf{u})(x) = \eta_y\big(\mathbf{u}(x)\big) = \varepsilon_y\big(\mathbf{u}(x)\big) = \big(\mathbf{u}(x)\big)(y) = f(x,y)$, thus $\eta_y \circ \mathbf{u} = f(\cdot, y)$ and so

$$\begin{aligned}\left\langle \int \mathbf{u}\,d\lambda, \eta_y \right\rangle &= \int f(\cdot, y)\,d\lambda \\ &= \int f(x,y)\,d\lambda(x) = h(y)\,,\end{aligned}$$

briefly

$$(*) \qquad h(y) = \Big\langle \int \mathbf{u}\, d\lambda, \eta_y \Big\rangle \quad \text{for all } y \in \mathrm{Y}\,.$$

If $y \notin \mathrm{L}$ then $f(\cdot, y)$ is the zero element of $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$ and so $h(y) = 0$, thus $h \in \mathscr{F}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$; it remains only to show that h is continuous on Y.

Now, $\int \mathbf{u}\, d\lambda \in \mathrm{E} = \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}) \subset \mathscr{K}(\mathrm{Y}; \mathbf{C})$, and $y \mapsto \varepsilon_y$ is a continuous mapping $\mathrm{Y} \to \mathscr{M}(\mathrm{Y}; \mathbf{C})$ for the vague topology on $\mathscr{M}(\mathrm{Y}; \mathbf{C})$ (§1, No. 9, Prop. 13), thus if $y \to y_0$ in Y then $\varepsilon_y \to \varepsilon_{y_0}$ vaguely, in particular $\varepsilon_y\big(\int \mathbf{u}\, d\lambda\big) = \varepsilon_{y_0}\big(\int \mathbf{u}\, d\lambda\big)$. But, for all $y \in \mathrm{Y}$,

$$\varepsilon_y\Big(\int \mathbf{u}\, d\lambda\Big) = \eta_y\Big(\int \mathbf{u}\, d\lambda\Big) = \Big\langle \int \mathbf{u}\, d\lambda, \eta_y \Big\rangle = h(y)$$

by (*), thus $h(y) \to h(y_0)$.

Alternate proof. Clearly (2) defines a function $h : \mathrm{Y} \to \mathbf{C}$ such that $h = 0$ on $\complement \mathrm{L}$, so we need only show that h is continuous. Indeed, as noted in the proof of Lemma 1 (with the roles of X and Y reversed), the mapping $\mathrm{Y} \to \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$ defined by $y \mapsto f(\cdot, y)$ is continuous (GT, X, §3, No. 4, Th. 3), $\mathscr{K}(\mathrm{X}, \mathrm{K}; \mathbf{C})$ is a topological subspace of $\mathscr{K}(\mathrm{X}; \mathbf{C})$, and λ is a continuous linear form on $\mathscr{K}(\mathrm{X}; \mathbf{C})$, so the composite mapping

$$y \mapsto f(\cdot, y) \mapsto \lambda\big(f(\cdot, y)\big) = h(y)$$

is continuous.

III.42, ℓ. 13.

"... whence $|\nu(f)| \leqslant a_{\mathrm{K}} b_{\mathrm{L}} \|f\|$."

In the notation of Lemma 2, $|h(y)| \leqslant a_{\mathrm{K}} \|f\|$ for all $y \in \mathrm{Y}$, therefore $|\mu(h)| \leqslant b_{\mathrm{L}} \|h\| \leqslant b_{\mathrm{L}} a_{\mathrm{K}} \|f\|$.

III.42, ℓ. 13, 14.

"The linear form ν on $\mathscr{K}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$ is thus a *measure* on $\mathrm{X} \times \mathrm{Y}$"

If A is any compact subset of $\mathrm{X} \times \mathrm{Y}$, and $\mathrm{K} \subset \mathrm{X}$, $\mathrm{L} \subset \mathrm{Y}$ are compact sets such that $\mathrm{A} \subset \mathrm{K} \times \mathrm{L}$, there exists a constant $\mathrm{M}_{\mathrm{A}} = a_{\mathrm{K}} b_{\mathrm{L}}$ such that $|\nu(f)| \leqslant \mathrm{M}_{\mathrm{A}} \|f\|$ for every $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$, hence for every $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{A}; \mathbf{C})$, whence the assertion (§1, No. 3, criterion following Def. 2).

III.42, ℓ. −2, −1.

$$\text{“(3)} \qquad \begin{aligned} \int f(x, y)\, d\nu(x, y) &= \int d\lambda(x) \int f(x, y)\, d\mu(y) \\ &= \int d\mu(y) \int f(x, y)\, d\lambda(x)\,.\text{”} \end{aligned}$$

By construction, ν is the unique measure on $\mathrm{X} \times \mathrm{Y}$ such that

$$\text{(i)} \quad \int g(x)h(y)\,d\nu(x,y) = \lambda(g)\mu(h) \quad \text{for all } g \in \mathscr{K}(\mathrm{X};\mathbf{C}),\ h \in \mathscr{K}(\mathrm{Y};\mathbf{C}),$$

and it satisfies

$$\text{(ii)} \qquad \nu(f) = \int \Big(\int f(x,y)\,d\lambda(x) \Big) d\mu(y) \quad \text{for all } f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y};\mathbf{C}).$$

Interchanging the roles of X and Y, there exists a unique measure ρ on $\mathrm{Y} \times \mathrm{X}$ such that

$$\text{(iii)} \quad \int h(y)g(x)\,d\rho(y,x) = \mu(h)\lambda(g) \quad \text{for all } h \in \mathscr{K}(\mathrm{Y};\mathbf{C}),\ g \in \mathscr{K}(\mathrm{X};\mathbf{C}),$$

and it satisfies

$$\text{(iv)} \qquad \rho(k) = \int \Big(\int k(y,x)\,d\mu(y) \Big) d\lambda(x) \quad \text{for all } k \in \mathscr{K}(\mathrm{Y} \times \mathrm{X};\mathbf{C}).$$

For each $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y};\mathbf{C})$ let $f' : \mathrm{Y} \times \mathrm{X} \to \mathbf{C}$ be the function defined by $f'(y,x) = f(x,y)$; since $(x,y) \mapsto (y,x)$ is a homeomorphism of $\mathrm{X} \times \mathrm{Y}$ onto $\mathrm{Y} \times \mathrm{X}$, $f \mapsto f'$ defines a vector space isomorphism

$$\mathscr{K}(\mathrm{X} \times \mathrm{Y};\mathbf{C}) \to \mathscr{K}(\mathrm{Y} \times \mathrm{X};\mathbf{C})$$

that is a homeomorphism for the respective direct limit topologies. Define

$$\nu' : \mathscr{K}(\mathrm{X} \times \mathrm{Y};\mathbf{C}) \to \mathbf{C}$$

by the formula $\nu'(f) = \rho(f')$. Then ν' is a continuous linear form, that is, a measure on $\mathrm{X} \times \mathrm{Y}$. In fact, $\nu' = \nu$ by the uniqueness part of Th. 1, since, for all $g \in \mathscr{K}(\mathrm{X};\mathbf{C})$ and $h \in \mathscr{K}(\mathrm{Y};\mathbf{C})$, writing $f(x,y) = g(x)h(y)$ one has

$$\begin{aligned}
\int g(x)h(y)\,d\nu'(x,y) &= \nu'(f) \\
&= \rho(f') \\
&= \int f'(y,x)d\rho(y,x) \\
&= \int f(x,y)d\rho(y,x) \\
&= \int h(y)g(x)d\rho(y,x) \\
&= \mu(h)\lambda(g) && \text{by (iii)} \\
&= \int g(x)h(y)\,d\nu(x,y) && \text{by (i)}.
\end{aligned}$$

Then, for every $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$, one has $\nu(f) = \nu'(f) = \rho(f')$, that is,

$$\begin{aligned}\int \Big(f(x,y)\,d\lambda(x)\Big)d\mu(y) &= \nu(f) && \text{by (ii)}\\ &= \rho(f')\\ &= \int \Big(\int f'(y,x)\,d\mu(y)\Big)d\lambda(x) && \text{by (iv)}\\ &= \int \Big(\int f(x,y)\,d\mu(y)\Big)d\lambda(x)\end{aligned}$$

as claimed.

III.44, ℓ. 1–4.

"Indeed, for every $\mathbf{z}' \in \mathrm{E}'$ we have

$$\begin{aligned}\Big\langle \iint \mathbf{f}\,d\lambda\,d\mu, \mathbf{z}'\Big\rangle &= \iint \langle \mathbf{f}, \mathbf{z}'\rangle\,d\lambda\,d\mu = \int d\mu \int \langle \mathbf{f}, \mathbf{z}'\rangle\,d\lambda\\ &= \int \Big\langle \int \mathbf{f}\,d\lambda, \mathbf{z}'\Big\rangle\,d\mu = \Big\langle \int d\mu \int \mathbf{f}\,d\lambda, \mathbf{z}'\Big\rangle\end{aligned}$$

by (4), whence (5)."

In slow motion: for all $\mathbf{z}' \in \mathrm{E}'$, one has

$$\begin{aligned}\Big\langle \iint \mathbf{f}\,d\lambda\,d\mu, \mathbf{z}'\Big\rangle &= \Big\langle \int \mathbf{f}\,d\nu, \mathbf{z}'\Big\rangle && \text{(definition of } \textstyle\iint \text{)}\\ &= \int \langle \mathbf{f}, \mathbf{z}'\rangle\,d\nu && \text{(§3, No. 1, Def. 1)}\\ &= \int \Big(\int \langle \mathbf{f}(x,y), \mathbf{z}'\rangle\,d\lambda(x)\Big)d\mu(y) && \text{(by (4))}\\ &= \int \Big(\int \langle \mathbf{f}(\cdot,y), \mathbf{z}'\rangle\,d\lambda\Big)d\mu(y) && \text{(cosmetics)}\\ &= \int \Big\langle \int \mathbf{f}(\cdot,y)\,d\lambda, \mathbf{z}'\Big\rangle d\mu(y) && \text{(§3, No. 1, Def. 1)}\\ &= \Big\langle \int \Big(\int \mathbf{f}(\cdot,y)\,d\lambda\Big)d\mu(y), \mathbf{z}'\Big\rangle && \text{(§3, No. 1, Def. 1)}\end{aligned}$$

whence

$$\begin{aligned}\int \mathbf{f}\,d\nu &= \int \Big(\int \mathbf{f}(\cdot,y)\,d\lambda\Big)d\mu(y)\\ &= \int \Big(\int \mathbf{f}(x,y)\,d\lambda(x)\Big)d\mu(y)\\ &= \int d\mu(y) \int \mathbf{f}(x,y)\,d\lambda(x)\,;\end{aligned}$$

the argument shows that $\int \mathbf{h}\,d\mu$, *a priori* an element of E'^*, is equal to $\int \mathbf{f}\,d\nu$ hence belongs to E. Similarly,

$$\Big\langle \int \mathbf{f}\,d\nu, \mathbf{z}' \Big\rangle = \int \Big(\int \langle \mathbf{f}(x,y), \mathbf{z}' \rangle \,d\mu(y) \Big) d\lambda(x) = \Big\langle \int \Big(\int \mathbf{f}(x,\cdot)\,d\mu \Big) d\lambda(x), \mathbf{z}' \Big\rangle,$$

whence

$$\int \mathbf{f}\,d\nu = \int \Big(\int \mathbf{f}(x,\cdot)\,d\mu \Big) d\lambda(x) = \int d\lambda(x) \int \mathbf{f}(x,y)\,d\mu(y)\,.$$

III.44, ℓ. −5.

"... which proves formula (6)."

For all $f \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$,

$$\begin{aligned}
\langle f, (g \otimes h) \cdot (\lambda \otimes \mu) \rangle &= \int \big(f \cdot (g \otimes h)\big) d(\lambda \otimes \mu) \\
&= \int \big(f(x,y) g(x) h(y)\big) d(\lambda \otimes \mu)(x,y) \\
&= \int \Big(\int f(x,y) g(x) h(y)\,d\mu(y) \Big) d\lambda(x) \qquad \text{by (3)} \\
&= \int \Big(\int f(x,y) h(y)\,d\mu(y) \Big) g(x)\,d\lambda(x) \\
&= \int \Big(\int f(x,\cdot)\,d(h \cdot \mu) \Big) d(g \cdot \lambda)(x) \\
&= \int f\,d\big((g \cdot \lambda) \otimes (h \cdot \mu)\big) \qquad \text{by (3)} \\
&= \langle f, (g \cdot \lambda) \otimes (h \cdot \mu) \rangle\,,
\end{aligned}$$

whence $(g \otimes h) \cdot (\lambda \otimes \mu) = (g \cdot \lambda) \otimes (h \cdot \mu)$.

III.45, ℓ. 1–3.

"... if U (resp. V) is an open set in X (resp. Y), then the restriction of $\lambda \otimes \mu$ to the product $\mathrm{U} \times \mathrm{V}$ is the product of the restrictions of λ to U and of μ to V"

We know from the first paragraph of §2, No. 1, that $\mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})$ may be identified with $\mathscr{K}(\mathrm{U}; \mathbf{C})$ via the mapping $f \mapsto f\big|\mathrm{U}$, and the measure $\lambda\big|\mathrm{U}$ on U is the result of transporting the restriction $\lambda\big|\mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})$ to a linear form on $\mathscr{K}(\mathrm{U}; \mathbf{C})$ via this identification; that is,

$$(\lambda\big|\mathrm{U})(f\big|\mathrm{U}) = \lambda(f) \quad \text{for all } f \in \mathscr{K}(\mathrm{X}, \mathrm{U}; \mathbf{C})\,.$$

From another perspective, if, for $g \in \mathscr{K}(\mathrm{U};\mathbf{C})$, g' denotes the extension by 0 of g to X, then

$$(\lambda\big|\mathrm{U})(g) = \lambda(g') \quad \text{for all } g \in \mathscr{K}(\mathrm{U};\mathbf{C})$$

(see the note for III.23, ℓ. 16–17). Similarly for the relation between μ and $\mu\big|\mathrm{V}$, and between $\lambda \otimes \mu$ and $(\lambda \otimes \mu)\big|\mathrm{U} \times \mathrm{V}$.

Both $(\lambda\big|\mathrm{U}) \otimes (\mu\big|\mathrm{V})$ and $(\lambda \otimes \mu)\big|\mathrm{U} \times \mathrm{V}$ are measures on $\mathrm{U} \times \mathrm{V}$; if $g \in \mathscr{K}(\mathrm{U};\mathbf{C})$ and $h \in \mathscr{K}(\mathrm{V};\mathbf{C})$, then $g \otimes h \in \mathscr{K}(\mathrm{U} \times \mathrm{V};\mathbf{C})$, $g' \otimes h' = (g \otimes h)'$ on $\mathrm{X} \times \mathrm{Y}$, and

$$\begin{aligned}
[(\lambda\big|\mathrm{U}) \otimes (\mu\big|\mathrm{V})](g \otimes h) &= (\lambda\big|\mathrm{U})(g) \cdot (\mu\big|\mathrm{V})(h) \\
&= \lambda(g')\mu(h') \\
&= (\lambda \otimes \mu)(g' \otimes h') \\
&= \big(\lambda \otimes \mu\big)\big((g \otimes h)'\big) \\
&= [(\lambda \otimes \mu)\big|\mathrm{U} \times \mathrm{V})](g \otimes h)\,,
\end{aligned}$$

therefore $(\lambda\big|\mathrm{U}) \otimes (\mu\big|\mathrm{V}) = (\lambda \otimes \mu)\big|\mathrm{U} \times \mathrm{V}$ by the uniqueness part of No. 1, Th. 1.

III.45, ℓ. 6–8.

"... which proves the proposition, on taking into account the definition of the support of a measure (§2, No. 2)."

Let $\mathrm{U} = \mathrm{X} - \operatorname{Supp} \lambda$. Then $\lambda\big|\mathrm{U} = 0$ and, writing $\nu = \lambda \otimes \mu$, we have

$$\nu\big|\mathrm{U} \times \mathrm{Y} = (\lambda\big|\mathrm{U}) \otimes (\mu\big|\mathrm{Y}) = 0 \otimes \mu = 0\,,$$

therefore $\mathrm{U} \times \mathrm{Y} \subset \complement \operatorname{Supp} \nu$, thus

$$\operatorname{Supp} \nu \subset \complement(\mathrm{U} \times \mathrm{Y}) = (\complement\,\mathrm{U}) \times \mathrm{Y} = (\operatorname{Supp} \lambda) \times \mathrm{Y}\,.$$

Similarly $\operatorname{Supp} \nu \subset \mathrm{X} \times \operatorname{Supp} \mu$, thus

$$\operatorname{Supp} \nu \subset [(\operatorname{Supp} \lambda) \times \mathrm{Y}] \cap [\mathrm{X} \times (\operatorname{Supp} \mu)] = (\operatorname{Supp} \lambda) \times (\operatorname{Supp} \mu)\,.$$

To prove the reverse inclusion, suppose $(x, y) \in \complement \operatorname{Supp} \nu$; it will suffice to show that either $x \in \complement \operatorname{Supp} \lambda$ or $y \in \complement \operatorname{Supp} \mu$. Let $\mathrm{U} \subset \mathrm{X}$ and $\mathrm{V} \subset \mathrm{Y}$ be open sets such that

$$(x, y) \in \mathrm{U} \times \mathrm{V} \subset \complement \operatorname{Supp} \nu\,.$$

Then $\nu\big|\mathrm{U} \times \mathrm{V} = 0$, thus $(\lambda\big|\mathrm{U}) \otimes (\mu\big|\mathrm{V}) = (\lambda \otimes \mu)\big|\mathrm{U} \times \mathrm{V} = 0$, therefore $\lambda\big|\mathrm{U} = 0$ or $\mu\big|\mathrm{V} = 0$, whence $\mathrm{U} \subset \complement \operatorname{Supp} \lambda$ or $\mathrm{V} \subset \complement \operatorname{Supp} \mu$, and so $x \in \complement \operatorname{Supp} \lambda$ or $y \in \complement \operatorname{Supp} \mu$.

III.47, ℓ. 3.

"There exist ..."

By §1, No. 8, Prop. 10.

III.47, ℓ. 13–17.

"PROPOSITION 6. — *When* $\mathscr{M}(\mathrm{X};\mathbf{C})$, $\mathscr{M}(\mathrm{Y};\mathbf{C})$ *and* $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$ *are equipped with the topology of strictly compact convergence* (§1, No. 10), *the bilinear mapping* $(\lambda,\mu)\mapsto\lambda\otimes\mu$ *of* $\mathscr{M}(\mathrm{X};\mathbf{C})\times\mathscr{M}(\mathrm{Y};\mathbf{C})$ *into* $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$ *is hypocontinuous for the set of vaguely bounded subsets of* $\mathscr{M}(\mathrm{X};\mathbf{C})$ *and* $\mathscr{M}(\mathrm{Y};\mathbf{C})$ (TVS, III, §5, No. 3)."

The assertion is that if $\mathfrak{S}$ and $\mathfrak{T}$ are the sets of vaguely bounded subsets of $\mathscr{M}(\mathrm{X};\mathbf{C})$ and $\mathscr{M}(\mathrm{Y};\mathbf{C})$, respectively, then the bilinear mapping $u:(\lambda,\mu)\mapsto\lambda\otimes\mu$ is $(\mathfrak{S},\mathfrak{T})$-hypocontinuous (in the sense of TVS, III, §5, No. 3), that is, (i) it is separately continuous (for the topologies of strictly compact convergence), (ii) for every $\mathrm{M}\in\mathfrak{S}$ and every neighborhood W of 0 in $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$, there exists a neighborhood V of 0 in $\mathscr{M}(\mathrm{Y};\mathbf{C})$ such that $u(\mathrm{M}\times\mathrm{V})\subset\mathrm{W}$, and (ii$'$) for every $\mathrm{N}\in\mathfrak{T}$ and every neighborhood W of 0 in $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$, there exists a neighborhood U of 0 in $\mathscr{M}(\mathrm{X};\mathbf{C})$ such that $u(\mathrm{U}\times\mathrm{N})\subset\mathrm{W}$.

(The conditions (i) and (ii) express that φ is $\mathfrak{S}$-hypocontinuous, whereas (i) and (ii$'$) express that it is $\mathfrak{T}$-hypocontinuous.)

Remarks. — Suppose E, F, G are locally convex spaces, $\varphi:\mathrm{E}\times\mathrm{F}\to\mathrm{G}$ is bilinear, and $\mathfrak{S}$ (resp. $\mathfrak{T}$) is a set of bounded subsets of E (resp. F). If φ is continuous ("jointly") then it is $(\mathfrak{S},\mathfrak{T})$-hypocontinuous for all such $\mathfrak{S}$ and $\mathfrak{T}$ (TVS, III, §5, No. 3, discussion following Def. 2). Thus $(\mathfrak{S},\mathfrak{T})$-hypocontinuity is in general stronger than separate continuity and weaker than continuity. According to Exercise 3 for §4, the bilinear mapping $u:(\lambda,\mu)\mapsto\lambda\otimes\mu$ is *continuous* (not just separately) for the topologies of strictly compact convergence. It then follows from the foregoing that u is $(\mathfrak{S},\mathfrak{T})$-hypocontinuous when $\mathfrak{S}$ (resp. $\mathscr{T}$) is any set of sets that are bounded for the topology τ_{scc} of strictly compact convergence; but the sets bounded for τ_{scc} are the same as the vaguely bounded sets (§1, No. 10, Prop. 17), thus one recovers Prop. 6, enhanced with joint continuity (for τ_{scc}).

III.47, ℓ. −14 to −11.

"the mapping φ of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathscr{K}(\mathrm{Y},\mathrm{L};\mathbf{C}))\times\mathscr{M}(\mathrm{X};\mathbf{C})$ into $\mathscr{K}(\mathrm{Y},\mathrm{L};\mathbf{C})$, such that $\varphi(g,\lambda)$ is the function h defined by $h(y)=\int g(x,y)\,d\lambda(x)$, is *separately continuous* by virtue of §3, No. 4, Props. 8 and 9."

There is here an "abuse of notation": a function

$$g\in\mathscr{K}(\mathrm{X}\times\mathrm{Y},\mathrm{K}\times\mathrm{L};\mathbf{C})$$

is being identified with the function

$$\omega(g) \in \mathscr{K}\big(\mathrm{X}, \mathrm{K}; \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})\big)$$

of No. 1, Lemma 1; explicitly, $\big(\omega(g)\big)(x)$ is the function $y \mapsto g(x, y)$, that is, $\big(\omega(g)\big)(x) = g(x, \cdot)$. For clarity, let us maintain the distinction.

Write $\mathrm{E} = \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$, a Banach space; thus $\omega(g) \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$ for $g \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$. Define a (bilinear) mapping

$$\varphi : \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C}) \times \mathscr{M}(\mathrm{X}; \mathbf{C}) \to \mathrm{E}$$

by $\varphi(g, \lambda) = \int \omega(g)\, d\lambda = \int g(x, \cdot)\, d\lambda(x) \in \widehat{\mathrm{E}} = \mathrm{E}$ (§3, No. 3, Cor. 2 of Prop. 7).

For fixed $g \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})$, hence fixed $\omega(g) \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$, the mapping

$$\lambda \mapsto \varphi(g, \lambda) = \int \omega(g)\, d\lambda \qquad \big(\lambda \in \mathscr{M}(\mathrm{X}; \mathbf{C})\big)$$

is continuous for the topology of strictly compact convergence (§3, No. 4, Prop. 9).

For fixed $\lambda \in \mathscr{M}(\mathrm{X}; \mathbf{C})$, the mapping

$$\omega(g) \mapsto \int \omega(g)\, d\lambda \qquad \big(\omega(g) \in \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})\big)$$

is continuous for the norm topology (§3, No. 4, Prop. 8), and since $g \mapsto \omega(g)$ is an isometry $\mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C}) \to \mathscr{K}(\mathrm{X}, \mathrm{K}; \mathrm{E})$ (No. 1, Lemma 1), the mapping

$$g \mapsto \int \omega(g)\, d\lambda = \varphi(g, \lambda) \qquad \big(g \in \mathscr{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{K} \times \mathrm{L}; \mathbf{C})\big)$$

is also continuous for the norm topology. Thus φ is separately continuous.

III.47, ℓ. $-4, -3$.

"...the conditions $\lambda \in \mathrm{B}$ and $\mu \in \mathrm{C}^\circ$ therefore imply $\lambda \otimes \mu \in \mathrm{A}^\circ$."

Polars are being taken; what are the dualities involved?

As to C°: We have $\mathrm{C} \subset \mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C}) \subset \mathscr{K}(\mathrm{Y}; \mathbf{C})$; as C is a compact subset of $\mathscr{K}(\mathrm{Y}, \mathrm{L}; \mathbf{C})$, it is a strictly compact subset of $\mathscr{K}(\mathrm{Y}; \mathbf{C})$ (for the direct limit topology). The duality in question is the canonical duality

$$(h, \mu) \mapsto \langle h, \mu \rangle = \mu(h) = \int h\, d\mu$$

between $\mathscr{K}(\mathrm{Y};\mathbf{C})$ and $\mathscr{K}(\mathrm{Y};\mathbf{C})' = \mathscr{M}(\mathrm{Y};\mathbf{C})$, thus

$$\text{(i)} \qquad \mathrm{C}^\circ = \{\mu \in \mathscr{M}(\mathrm{Y};\mathbf{C}) : \ \mathscr{R}\langle h, \mu\rangle \geqslant -1 \ \text{ for all } h \in \mathrm{C}\,\}$$

(TVS, II, §8, No. 4, Def. 1).

As to A°: Similarly $\mathrm{A} \subset \mathscr{K}(\mathrm{X}\times\mathrm{Y}, \mathrm{K}\times\mathrm{L};\mathbf{C})$ is a strictly compact subset of $\mathscr{K}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$, and

$$\text{(ii)} \qquad \mathrm{A}^\circ = \{\nu \in \mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C}) : \ \mathscr{R}\langle f, \nu\rangle \geqslant -1 \ \text{ for all } f \in \mathrm{A}\,\}\,.$$

Let $f \in \mathscr{K}(\mathrm{X}\times\mathrm{Y}, \mathrm{K}\times\mathrm{L};\mathbf{C})$, $\lambda \in \mathscr{M}(\mathrm{X};\mathbf{C})$, $\mu \in \mathscr{M}(\mathrm{Y};\mathbf{C})$. Then $\varphi(f,\lambda) = \int f(x,\cdot)\,d\lambda(x) \in \mathscr{K}(\mathrm{Y},\mathrm{L};\mathbf{C})$, and

$$\begin{aligned}\langle f, \lambda\otimes\mu\rangle &= \int \Big(\int f(x,y)\,d\lambda(x)\Big)\,d\mu(y) \qquad \text{(by (3) of No. 1)}\\ &= \int \big(f(x,\cdot)\,d\lambda(x)\big)\,d\mu \\ &= \langle \varphi(f,\lambda), \mu\rangle\,.\end{aligned}$$

If $f \in \mathrm{A}$, $\lambda \in \mathrm{B}$ and $\mu \in \mathrm{C}^\circ$, then $\varphi(f,\lambda) \in \mathrm{C}$ and, by (i),

$$(*) \qquad \mathscr{R}\langle f, \lambda\otimes\mu\rangle = \mathscr{R}\langle \varphi(f,\lambda), \mu\rangle \geqslant -1\,;$$

the validity of (*) for all $f \in \mathrm{A}$ says, by (ii), that $\lambda\otimes\mu \in \mathrm{A}^\circ$. This proves the assertion

$$(\lambda,\mu) \in \mathrm{B}\times\mathrm{C}^\circ \quad\Rightarrow\quad \lambda\otimes\mu \in \mathrm{A}^\circ\,.$$

So to speak, $\mathrm{B}\otimes\mathrm{C}^\circ \subset \mathrm{A}^\circ$.

One notes that if $\mu \in \mathscr{M}(\mathrm{Y};\mathbf{C})$ is such that $\lambda\otimes\mu \in \mathrm{A}^\circ$ for all $\lambda \in \mathrm{B}$, then the validity of (*) for all $f \in \mathrm{A}$ and $\lambda \in \mathrm{B}$ shows, since $\varphi(f,\lambda)$ then runs over $\varphi(\mathrm{A}\times\mathrm{B}) = \mathrm{C}$, that $\mu \in \mathrm{C}^\circ$. Thus if $\mu \in \mathscr{M}(\mathrm{Y};\mathbf{C})$, then

$$\lambda\otimes\mu \in \mathrm{A}^\circ \text{ for all } \lambda \in \mathrm{B} \quad\Rightarrow\quad \mu \in \mathrm{C}^\circ\,,$$

so to speak, $\mathrm{B}\otimes\mu \subset \mathrm{A}^\circ \Rightarrow \mu \in \mathrm{C}^\circ$. Combining this with the preceding paragraph, we see that

$$\mathrm{C}^\circ = \{\mu \in \mathscr{M}(\mathrm{Y};\mathbf{C}) : \ \mathrm{B}\otimes\mu \subset \mathrm{A}^\circ\,\}\,.$$

Sets such as C and A figure in the topologies of strictly compact convergence on $\mathscr{M}(\mathrm{Y};\mathbf{C})$ and $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$, respectively. Crucial for the rest of the proof is that the sets C° (resp. A°) form a fundamental system of neighborhoods of 0 in $\mathscr{M}(\mathrm{Y};\mathbf{C})$ (resp. $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$). The details are given in the next note; in preparation, we give here the following:

PROPOSITION . — *If* Z *is a locally compact space and* A *is a strictly compact subset of* $\mathscr{K}(\mathrm{Z};\mathbf{C})$ *then the closed balanced convex envelope of* A *is strictly compact* (*therefore the closed balanced envelope and the closed convex envelope of* A *are strictly compact*).

Say $\mathrm{A}\subset\mathscr{K}(\mathrm{Z},\mathrm{K};\mathbf{C})$, K a compact subset of Z. Since $\mathscr{K}(\mathrm{Z},\mathrm{K};\mathbf{C})$ is a closed topological subspace of $\mathscr{K}(\mathrm{Z};\mathbf{C})$ and is a Banach space, the assertion is immediate from the following lemma:

Lemma. — *If* A *is a compact subset of a complete Hausdorff locally convex space* G, *then the closed balanced convex envelope of* A *is compact.*

Proof. Let A_1 be the balanced envelope of A, A_2 the convex envelope of A_1. Then A_1 is compact (TVS, I, §1, No. 5, Prop. 3), and A_2, being the set of all convex combinations of elements of A_1 (TVS, II, §2, No. 3, Cor. 1 of Prop. 8), is obviously balanced, thus A_2 is the balanced convex envelope of A; therefore A_2 is precompact (TVS, II, §4, No. 1, Prop. 3), that is, the completion $\widehat{\mathrm{A}}_2$ of A_2 is compact. Since G is complete, $\widehat{\mathrm{A}}_2$ may be identified with the closure $\overline{\mathrm{A}}_2$ of A_2 in G (GT, II, §3, No. 9, Cor. 1 of Prop. 18), thus $\overline{\mathrm{A}}_2$ is compact; since, moreover, $\overline{\mathrm{A}}_2$ is balanced and convex (TVS, I, §1, No. 5, Prop. 2 and II, §2, No. 6, Prop. 14), it is clear that $\overline{\mathrm{A}}_2$ is the closed balanced convex envelope of A. (The lemma remains true in any Hausdorff locally convex space G provided that the compact subset A is contained in a complete balanced convex subset of G, cf. TVS, II, §4, No. 1, Cor. of Prop. 3.)

III.47, ℓ. −3 to −1.

"In view of the definition of the topology of strictly compact convergence, this proves the proposition (TVS, III, §5, No. 3, Def. 2)."

The formula $u(\lambda,\mu)=\lambda\otimes\mu$ defines a bilinear mapping

$$u:\mathscr{M}(\mathrm{X};\mathbf{C})\times\mathscr{M}(\mathrm{Y};\mathbf{C})\to\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})\,.$$

Let us take on faith (details later) that the sets of the form A° (resp. C°) are a fundamental system of neighborhoods of 0 in $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$ (resp. $\mathscr{M}(\mathrm{Y};\mathbf{C})$) for the topology of strictly compact convergence. As in the note for III.47, ℓ. 13–17, let us write $\mathfrak{S}$ (resp. $\mathfrak{T}$) for the set of vaguely bounded subsets of $\mathscr{M}(\mathrm{X};\mathbf{C})$ (resp. $\mathscr{M}(\mathrm{Y};\mathbf{C})$).

Given a neighborhood $\mathrm{W}=\mathrm{A}^\circ$ of 0 in $\mathscr{M}(\mathrm{X}\times\mathrm{Y};\mathbf{C})$ and a set $\mathrm{B}\in\mathfrak{S}$ that is vaguely closed (not a restriction, since, in any real or complex topological vector space, the closure of a bounded set is bounded—because the closed neighborhoods of 0 are fundamental), we have produced a neighborhood C° of 0 in $\mathscr{M}(\mathrm{Y};\mathbf{C})$ such that $u(\mathrm{B}\times\mathrm{C}^\circ)\subset\mathrm{A}^\circ$, informally

$B \otimes C^\circ \subset A^\circ$. In particular, if $\lambda \in \mathscr{M}(X;\mathbf{C})$ and $B = \{\lambda\}$, one sees that $\mu \mapsto \lambda \otimes \mu$ is continuous. A symmetric argument shows that $\lambda \mapsto \lambda \otimes \mu$ is continuous for each μ (one begins with mappings

$$\omega' : \mathscr{K}(X \times Y, K \times L;\mathbf{C}) \to \mathscr{K}(Y, L;\mathscr{K}(X, K;\mathbf{C}))$$

and $\varphi' : \mathscr{K}(X \times Y, K \times L;\mathbf{C}) \times \mathscr{M}(Y;\mathbf{C}) \to \mathscr{K}(X, K;\mathbf{C})$), whence separate continuity, and so the $\mathfrak{S}$-hypocontinuity of u is established. A symmetric argument proves $\mathfrak{T}$-hypocontinuity.

The details. The following argument for X will apply as well to Y and $X \times Y$. To simplify the notations, we make the abbreviations $\mathscr{K} = \mathscr{K}(X;\mathbf{C})$, $\mathscr{M} = \mathscr{M}(X;\mathbf{C})$; $\mathscr{K}$ is equipped with the direct limit topology, so that $\mathscr{M} = \mathscr{K}'$. A priori, the only obvious topology on $\mathscr{M}$ is the vague topology, that is, the topology $\sigma(\mathscr{M};\mathscr{K})$ derived from the canonical duality

$$(f,\lambda) \mapsto \langle f,\lambda\rangle = \int f\,d\lambda \qquad (f \in \mathscr{K},\ \lambda \in \mathscr{M}).$$

But $\mathscr{M}$ is a set of functions on $\mathscr{K}$, and "$\lambda_j \to \lambda$ vaguely" means simply that $\lambda_j(f) \to \lambda(f)$ for each $f \in \mathscr{K}$, whence $\lambda_j \to \lambda$ uniformly on every finite subset of $\mathscr{K}$; thus vague convergence in $\mathscr{M}$ means uniform convergence in the finite subsets of $\mathscr{K}$. By modifying the sets of subsets of $\mathscr{K}$ considered, one opens the door for other topologies on $\mathscr{M}$. To wit:

Let $\mathscr{A}$ be the set of all strongly compact subsets A of $\mathscr{K}$, and let $\mathscr{S}$ be the set of all subsets of $\mathscr{K}$ that are contained in some $A \in \mathscr{A}$; since $\mathscr{A}$ is closed under finite unions, the same is true of $\mathscr{S}$, thus $\mathscr{S}$ is a bornology on $\mathscr{K}$ (TVS, III, §1, No. 1, Def. 1). One knows (see the Proposition at the end of the preceding note) that if $A \in \mathscr{A}$, say $A \subset \mathscr{K}(X, K;\mathbf{C})$, then the closed balanced convex envelope of A in $\mathscr{K}$ is also compact, is contained in $\mathscr{K}(X, K;\mathbf{C})$, hence belongs to $\mathscr{A}$; the same is therefore true of every subset of A, hence (varying A) for every set $S \in \mathscr{S}$. In particular, if $S \in \mathscr{S}$ then the closure of S belongs to $\mathscr{S}$ (indeed, to $\mathscr{A}$). Moreover, if $S \in \mathscr{S}$ and $c \in \mathbf{C}$, then $cS \in \mathscr{S}$. These properties may be summarized by saying that the bornology $\mathscr{S}$ on $\mathscr{K}$ is *adapted* (*loc. cit.*, No. 2, Def. 4), and, in particular, it is *convex* (*loc. cit.*, No. 1, Def. 2). {Another example: When $\mathscr{M}$ is equipped with the vague topology, the set $\mathfrak{S}$ of (vaguely) bounded subsets of $\mathscr{M}$ is an adapted bornology on $\mathscr{M}$ (*loc. cit.*, No. 2, Prop. 1).}

Now, $\mathscr{M} \subset \mathfrak{F}(\mathscr{K};\mathbf{C})$ and $\mathscr{S}$ is a set of subsets of $\mathscr{K}$; the *topology of strictly compact convergence* on $\mathscr{M}$ is by definition the $\mathscr{S}$-topology, that is, the topology of uniform convergence in the sets of $\mathscr{S}$ (GT, X, §1, No. 2). Equipped with this topology, $\mathscr{M} = \mathscr{K}'$ is also denoted $\mathscr{M}_{\mathrm{scc}} = \mathscr{L}_{\mathscr{S}}(\mathscr{K};\mathbf{C}) = \mathscr{M}_{\mathscr{S}}(X;\mathbf{C})$, and is itself a locally convex space (TVS, III, §3, No. 1, Cor. of Prop. 1).

We now describe a fundamental system of entourages for the uniformity of $\mathscr{S}$-convergence. A fundamental system of entourages for the uniformity of $\mathbf{C}$ is given by the sets

$$\mathrm{V}_\varepsilon = \{(c,d) \in \mathbf{C} \times \mathbf{C} : \quad |c-d| \leqslant \varepsilon\} \qquad (\varepsilon > 0)\,;$$

if $\mathrm{S} \in \mathscr{S}$, a fundamental system of entourages for the topology of uniform convergence in S is given by the sets

$$\mathrm{W}_{\varepsilon,\mathrm{S}} = \{(\lambda,\lambda') \in \mathscr{M} \times \mathscr{M} : \quad |\lambda(f) - \lambda'(f)| \leqslant \varepsilon \ \text{ for all } f \in \mathrm{S}\} \quad (\varepsilon > 0)\,;$$

a fundamental system of entourages for the uniformity of $\mathscr{S}$-convergence is therefore given by the finite intersections

$$\mathrm{W}_{\varepsilon_1,\,\mathrm{S}_1} \cap \cdots \cap \mathrm{W}_{\varepsilon_n,\,\mathrm{S}_n}$$

(GT, X, §1, No. 2, *Remark* 2). If $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\mathrm{S} = \mathrm{S}_1 \cup \cdots \cup \mathrm{S}_n$, then

$$\mathrm{W}_{\varepsilon,\,\mathrm{S}} \subset \mathrm{W}_{\varepsilon_1,\,\mathrm{S}_1} \cap \cdots \cap \mathrm{W}_{\varepsilon_n,\,\mathrm{S}_n}\,;$$

it follows that the sets $\mathrm{W}_{\varepsilon,\,\mathrm{S}}$ form a fundamental system of entourages. But if $\mathrm{S} \in \mathscr{S}$ and A is the closed balanced convex envelope of S, then $\mathrm{A} \in \mathscr{S}$ and $\mathrm{W}_{\varepsilon,\,\mathrm{A}} \subset \mathrm{W}_{\varepsilon,\,\mathrm{S}}$; thus the sets

$$\mathrm{W}_{\varepsilon,\,\mathrm{A}} \qquad (\varepsilon > 0\,,\ \mathrm{A} \in \mathscr{A} \ \text{ balanced and convex})$$

form a fundamental system of entourages. Now,

$$|\lambda(f) - \lambda'(f)| \leqslant \varepsilon \quad \Leftrightarrow \quad |\lambda(\varepsilon^{-1}f) - \lambda'(\varepsilon^{-1}f)| \leqslant 1\,,$$

whence $\mathrm{W}_{\varepsilon,\,\mathrm{A}} = \mathrm{W}_{1,\,\varepsilon^{-1}\mathrm{A}}$; since the $\varepsilon^{-1}\mathrm{A}$ run over the closed, balanced convex elements of $\mathscr{A}$ as A does, the sets

$$\mathrm{W}_{1,\,\mathrm{A}} \qquad (\mathrm{A} \in \mathscr{A} \ \text{ balanced and convex})$$

form a fundamental system of entourages. Therefore the sets

$$\mathrm{U}_\mathrm{A} = \{\lambda \in \mathscr{M} : \ |\lambda(f)| \leqslant 1 \ \text{ for all } f \in \mathrm{A}\}\,,$$

as A runs over the set of balanced, convex, strictly compact subsets of K, form a fundamental system of neighborhoods of 0 in $\mathscr{M}$ for the topology of $\mathscr{S}$-convergence, that is, the topology of strictly compact convergence. In the canonical duality between the vector spaces $\mathscr{K}$ and $\mathscr{M}$,

$$(*) \qquad \mathrm{U}_\mathrm{A} = \mathrm{A}^\circ$$

(TVS, II, §8, No. 4); thus the sets (*), as A runs over the set of balanced, convex, strictly compact subsets of $\mathscr{K}$, form a *fundamental system of neighborhoods of* $0 \in \mathscr{M}$ *for the topology of strictly compact convergence.*

Incidentally, by the theorem on bipolars (TVS, II, §6, No. 3, Th. 1), $\mathrm{A}^{\circ\circ}$ is the closure of A in $\mathscr{K}$ for the weakened topology $\sigma(\mathscr{K}, \mathscr{M})$. But the strongly closed convex set A is also closed for $\sigma(\mathscr{K}, \mathscr{M})$ (*loc. cit.*, Cor. 3 of Th. 1); thus $\mathrm{A}^{\circ\circ} = \mathrm{A}$.

III.48, ℓ. 1, 2.

"The conclusion of Prop. 6 is no longer valid when the topology of strictly compact convergence is replaced by the vague topology (Exer. 2 *c*))."

A surprising result, since u is separately continuous for the vague topology (Prop. 5).

In the cited exercise, $\mathrm{X} = \mathrm{Y} = [0,1]$, and $\mathscr{M}(\mathrm{X}; \mathbf{C})$, $\mathscr{M}(\mathrm{X} \times \mathrm{X}; \mathbf{C})$ are equipped with the vague topology. Write $u(\lambda, \mu) = \lambda \otimes \mu$ as in the preceding note. Let $\mathfrak{S}$ be the set of vaguely bounded subsets of $\mathscr{M}(\mathrm{X}; \mathbf{C})$. The set $\mathrm{B} = \{\mu \in \mathscr{M}(\mathrm{X}; \mathbf{C}) : \ \|\mu\| \leqslant 1\}$ is vaguely compact (§1, No. 9, Cor. 2 of Prop. 15), hence vaguely bounded. If u were $\mathfrak{S}$-hypocontinuous for the vague topology, then the restriction of u to $\mathrm{B} \times \mathscr{M}(\mathrm{X}; \mathbf{C})$ would be continuous (TVS, III, §5, No. 3, Prop. 4), contrary to part *c*) of the exercise.

III.48, ℓ. 2–6.

"... if B (resp. C) is a vaguely bounded subset of $\mathscr{M}(\mathrm{X}; \mathbf{C})$ (resp. $\mathscr{M}(\mathrm{Y}; \mathbf{C})$), then the image of $\mathrm{B} \times \mathrm{C}$ under the mapping $(\lambda, \mu) \mapsto \lambda \otimes \mu$ is vaguely bounded in $\mathscr{M}(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$ and therefore the restriction of this mapping to $\mathrm{B} \times \mathrm{C}$ is vaguely continuous ..."

Let $u : \mathscr{M}(\mathrm{X}) \times \mathscr{M}(\mathrm{Y}) \to \mathscr{M}(\mathrm{X} \times \mathrm{Y})$ be the bilinear mapping $u(\lambda, \mu) = \lambda \otimes \mu$, and let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be the set of vaguely bounded subsets of $\mathscr{M}(\mathrm{X})$ (resp. $\mathscr{M}(\mathrm{Y})$); we know from Prop. 6 that u is $(\mathfrak{S}, \mathfrak{T})$-hypcontinuous when each of $\mathscr{M}(\mathrm{X})$, $\mathscr{M}(\mathrm{Y})$ and $\mathscr{M}(\mathrm{X} \times \mathrm{Y})$ is equipped with the topology τ_{scc} of strictly compact convergence.

From §1, No. 10, Prop. 17, we know that $\mathfrak{S}$ is also the set of subsets of X that are bounded for τ_{scc}, and similarly for $\mathfrak{T}$ and Y. By TVS, III, §5, No. 3, Props. 4, 5, $u(\mathrm{B} \times \mathrm{C})$ is bounded for τ_{scc}—hence for the vague topology—and the restriction of u to $\mathrm{B} \times \mathrm{C}$ is continuous when each of B, C is equipped with the topology induced by τ_{scc}, and $\mathrm{B} \times \mathrm{C}$ with the product topology. Since B, C and $u(\mathrm{B} \times \mathrm{C})$ are vaguely bounded, the topologies induced on them by τ_{scc} and the vague topology are identical by the Prop. 17 cited above; and the product topology on $\mathrm{B} \times \mathrm{C}$ coincides with the topology induced by the product of the vague topologies on $\mathscr{M}(\mathrm{X})$ and $\mathscr{M}(\mathrm{Y})$ (GT, I, §4, No. 1, Cor. of Prop. 3); therefore the restriction $u|\mathrm{B} \times \mathrm{C}$ is continuous when all spaces in sight are equipped with the vague topology.

III.48, ℓ. 10–14.

"The set of linear combinations of complex functions of the form

$$(x_1, x_2, \dots, x_n) \mapsto f_1(x_1)f_2(x_2)\cdots f_n(x_n)\,,$$

where $f_i \in \mathscr{K}(\mathrm{X}_i;\mathbf{C})$ $(1 \leqslant i \leqslant n)$, may be identified canonically with the tensor product $\bigotimes_{i=1}^{n} \mathscr{K}(\mathrm{X}_i;\mathbf{C})$, and it follows from Lemma 1 of No. 1, by induction on n, that this tensor product is *dense* in $\mathscr{K}(\mathrm{X};\mathbf{C})$."

The case $n = 2$ is proved in the remarks following the cited Lemma 1.

Assume inductively that all is well for for $n-1$ and consider the case of n. Write $\mathrm{Y} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_{n-1}$. The canonical homeomorphism of $\mathrm{X}_1 \times\cdots\times \mathrm{X}_n$ with $\mathrm{Y}\times\mathrm{X}_n$ permits the identification of $\mathscr{K}(\mathrm{X}_1\times\cdots\times\mathrm{X}_n)$ with $\mathscr{K}(\mathrm{Y}\times\mathrm{X}_n)$. By the induction hypothesis, the set of functions $f_1\otimes\cdots\otimes f_{n-1}$ is total in $\mathscr{K}(\mathrm{Y})$, therefore the set of functions

$$f_1 \otimes \cdots \otimes f_n = (f_1 \otimes \cdots \otimes f_{n-1}) \otimes f_n \qquad (\,f_i \in \mathscr{K}(\mathrm{X}_i) \text{ for } i = 1, \dots, n)$$

is total in $\mathscr{K}(\mathrm{Y}\times\mathrm{X}_n) = \mathscr{K}(\mathrm{X}_1 \times \cdots \times \mathrm{X}_n)$ (see the Proposition in the note for III.41, ℓ. −9, −8).

III.49, ℓ. −13 to −11.

"The integral notation and formula (14) may be extended in an obvious way to functions $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{E})$ with values in a Hausdorff locally convex space E, such that $\mathbf{f}(\mathrm{X})$ is contained in a complete convex subset of E."

Preparatory to computing the case $n = 3$, let us review the *Remark* in No. 1 (which is the case $n = 2$). In the context of a measure $\nu = \lambda \otimes \mu$ on $\mathrm{X}\times\mathrm{Y}$, if $\mathbf{f} \in \mathscr{K}(\mathrm{X}\times\mathrm{Y};\mathrm{E})$ is such that $\mathbf{f}(\mathrm{X}\times\mathrm{Y}) \subset \mathrm{C} \subset \mathrm{E}$ with C complete and convex, so that $\int \mathbf{f}\,d\nu \in \mathrm{E}$ by §3, No. 3, Prop. 7, then

$$\int \mathbf{f}\,d\nu = \int d\mu(y) \int \mathbf{f}(x,y)\,d\lambda(x) = \int d\lambda(x) \int \mathbf{f}(x,y)\,d\mu(y)$$

in the sense that

(i) for each $y \in \mathrm{Y}$, the integral $\int \mathbf{f}(x,y)\,d\lambda(x)$ exists and is an element of E,

(ii) the function $\mathbf{h} : \mathrm{Y} \to \mathrm{E}$ defined by $\mathbf{h}(y) = \int \mathbf{f}(x,y)\,d\lambda(x)$ belongs to $\mathscr{K}(\mathrm{Y};\mathrm{E})$, and

(iii) $\int \mathbf{h}\,d\mu = \int \mathbf{f}\,d\nu$;

thus, for each $\mathbf{z}' \in \mathrm{E}'$,

$$\begin{aligned}\int \langle \mathbf{f}, \mathbf{z}'\rangle\,d\nu = \Big\langle \int \mathbf{f}\,d\nu, \mathbf{z}'\Big\rangle &= \Big\langle \int d\mu(y) \int \mathbf{f}(x,y)\,d\lambda(x), \mathbf{z}'\Big\rangle \\ &= \int \Big\langle \int \mathbf{f}(x,y)\,d\lambda(x), \mathbf{z}'\Big\rangle d\mu(y) \\ &= \int \Big(\int \langle \mathbf{f}(x,y), \mathbf{z}'\rangle\,d\lambda(x)\Big) d\mu(y)\,,\end{aligned}$$

and similarly

$$\begin{aligned}\Big\langle \int \mathbf{f}\,d\nu, \mathbf{z}'\Big\rangle &= \Big\langle \int d\lambda(x) \int \mathbf{f}(x,y)\,d\mu(y), \mathbf{z}'\Big\rangle \\ &= \int \Big\langle \int \mathbf{f}(x,y)\,d\mu(y), \mathbf{z}'\Big\rangle d\lambda(x) \\ &= \int \Big(\int \langle \mathbf{f}(x,y), \mathbf{z}'\rangle\,d\mu(y)\Big) d\lambda(x)\,.\end{aligned}$$

Now consider locally compact spaces $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$, and let $\mathrm{Z} = \mathrm{X}_1 \times \mathrm{X}_2 \times \mathrm{X}_3$, canonically identified with $(\mathrm{X}_1 \times \mathrm{X}_2) \times \mathrm{X}_3$ and $\mathrm{X}_1 \times (\mathrm{X}_2 \times \mathrm{X}_3)$. Let μ_i be a measure on X_i $(i = 1, 2, 3)$ and let $\mathbf{f} \in \mathscr{K}(\mathrm{Z}; \mathrm{E})$ be such that $\mathbf{f}(\mathrm{Z}) \subset \mathrm{C}$ for some complete convex subset C of E. Write $\nu = \mu_1 \otimes \mu_2 \otimes \mu_3 = (\mu_1 \otimes \mu_2) \otimes \mu_3 = \mu_1 \otimes (\mu_2 \otimes \mu_3)$. We know that $\int \mathbf{f}\,d\nu \in \mathrm{E}$.

From the preliminary remarks, with $\lambda = \mu_1 \otimes \mu_2$ and $\mu = \mu_3$, we have

$$\begin{aligned}\int \mathbf{f}\,d\nu = \int \mathbf{f}\,d\big((\mu_1 \otimes \mu_2) \times \mu_3\big) &= \int d\mu_3(x_3) \int \mathbf{f}(x_1, x_2, x_3)\,d(\mu_1 \otimes \mu_2)(x_1, x_2) \\ &= \int d\mu_3(x_3) \Big(\int d\mu_2(x_2) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_1(x_1)\Big) \\ &= \int d\mu_3(x_3) \int d\mu_2(x_2) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_1(x_1)\,,\end{aligned}$$

as well as

$$\begin{aligned}\int \mathbf{f}\,d\nu &= \int d\mu_3(x_3) \Big(\int d\mu_1(x_1) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_2(x_2)\Big) \\ &= \int d\mu_3(x_3) \int d\mu_1(x_1) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_2(x_2)\,.\end{aligned}$$

This accounts for the permutations (3,2,1) and (3,1,2). On the other hand,

$$\begin{aligned}\int \mathbf{f}\,d\nu = \int \mathbf{f}\,d\big((\mu_1 \otimes \mu_2) \otimes \mu_3\big) &= \int d(\mu_1 \otimes \mu_2)(x_1, x_2) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_3(x_3) \\ &= \int d\mu_1(x_1) \Big(\int d\mu_2(x_2) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_3(x_3)\Big) \\ &= \int d\mu_1(x_1) \int d\mu_2(x_2) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_3(x_3)\,,\end{aligned}$$

as well as

$$\begin{aligned}\int \mathbf{f}\,d\nu &= \int d\mu_2(x_2) \Big(\int d\mu_1(x_1) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_3(x_3)\Big) \\ &= \int d\mu_2(x_2) \int d\mu_1(x_1) \int \mathbf{f}(x_1, x_2, x_3)\,d\mu_3(x_3)\,,\end{aligned}$$

accounting for the permutations (1,2,3) and (2,1,3). Finally,

$$\int \mathbf{f}\, d\nu = \int \mathbf{f} d\big(\mu_1 \otimes (\mu_2 \otimes \mu_3)\big)$$

leads to formulas for the permutations (2,3,1) and (3,2,1).

III.50, ℓ. −14, −13.

"It is clear that $(\mathscr{M}(\mathrm{X}_\alpha;\mathbf{C}),(p_{\alpha\beta})_*)$ is an *inverse system* of vector spaces, and that $((p_\alpha)_*)$ is an *inverse system* of linear mappings ..."

If $\alpha \leqslant \beta$, so that $p_{\alpha\beta}:\mathrm{X}_\beta \to \mathrm{X}_\alpha$, then, by the foregoing,

$$(p_{\alpha\beta})_* : \mathscr{M}(\mathrm{X}_\beta;\mathbf{C}) \to \mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})$$

is a linear mapping, continuous for the vague topology.

If $\alpha \leqslant \beta \leqslant \gamma$ then, from $p_{\alpha\gamma} = p_{\alpha\beta}\circ p_{\beta\gamma}$ (GT, I, §4, No. 4) we infer that, for every $f \in \mathscr{C}(\mathrm{X}_\gamma;\mathbf{C})$,

$$\begin{aligned} {p_{\alpha\gamma}}'(f) &= f\circ p_{\alpha\gamma} = (f\circ p_{\alpha\beta})\circ p_{\beta\gamma} \\ &= {p_{\beta\gamma}}'(f\circ p_{\alpha\beta}) = {p_{\beta\gamma}}'\big({p_{\alpha\beta}}'(f)\big)\,, \end{aligned}$$

thus ${p_{\alpha\gamma}}' = {p_{\beta\gamma}}'\circ {p_{\alpha\beta}}'$; taking transpose,

(I) $$(p_{\alpha\gamma})_* = (p_{\alpha\beta})_*\circ(p_{\beta\gamma})_* \quad \text{when } \alpha\leqslant\beta\leqslant\gamma\,.$$

Since $p_{\alpha\alpha}$ is the identity mapping of X_α,

(II) $$(p_{\alpha\alpha})_* \text{ is the identity mapping of } \mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})\,.$$

Thus, assuming $\mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})$ equipped with the vague topology, the family $\big(\mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})\big)$ of topological vector spaces and the family $\big((p_{\alpha\beta})_*\big)$ of continuous linear mappings form an inverse system

(†) $$\big(\mathscr{M}(\mathrm{X}_\alpha;\mathbf{C}),\ (p_{\alpha\beta})_*\big)$$

(S, III, §7, No. 1 and GT, *loc. cit.*).

Recall that the inverse limit

$$\mathrm{X} = \varprojlim \mathrm{X}_\alpha$$

is a closed subspace of the compact space $\prod_\alpha \mathrm{X}_\alpha$ (GT, I, §8, No. 2, Cor. 2 of Prop. 7), the canonical mapping

$$p_\alpha : \mathrm{X}\to\mathrm{X}_\alpha$$

of X *into* X_α is defined to be the restriction to X of the projection mapping pr_α of the product space into X_α (GT, I, §4, No. 4). When X_α is nonempty for every α, the compact space X is also nonempty (GT, I, §9, No. 6, Prop. 8).

From the relation $p_\alpha = p_{\alpha\beta} \circ p_\beta$ when $\alpha \leqslant \beta$, one infers that

$$(*) \qquad (p_\alpha)_* = (p_{\alpha\beta})_* \circ (p_\beta)_* \quad \text{when } \alpha \leqslant \beta .$$

The sense in which the term "inverse system" is applied to the family of mappings

$$(p_\alpha)_* : \mathscr{M}(X;\mathbf{C}) \to \mathscr{M}(X_\alpha;\mathbf{C})$$

is explained in S, III, §7, No. 2, *Remark* 2 (reviewed at the end of this note), but what the property (*) is called is not important; what is important is the role that it plays in Definition 2 below, as follows.

Consider the inverse limit of the system (†), namely

$$\varprojlim \mathscr{M}(X_\alpha;\mathbf{C})\,;$$

its elements are the families $(\mu_\alpha) \in \prod_\alpha \mathscr{M}(X_\alpha;\mathbf{C})$ such that $\mu_\alpha = (p_{\alpha\beta})_*(\mu_\beta)$ when $\alpha \leqslant \beta$.

For example, suppose μ is a measure on the compact space $X = \varprojlim X_\alpha$. For each α, define a linear form μ_α on $\mathscr{C}(X_\alpha;\mathbf{C})$ by

$$\mu_\alpha(f) = \mu(f \circ p_\alpha) \qquad \bigl(f \in \mathscr{C}(X_\alpha;\mathbf{C})\bigr);$$

since, for $f \in \mathscr{C}(X_\alpha;\mathbf{C})$, $f \circ p_\alpha$ belongs to $\mathscr{C}(X;\mathbf{C})$, one has

$$|\mu_\alpha(f)| \leqslant \|\mu\| \cdot \|f \circ p_\alpha\| \leqslant \|\mu\| \cdot \|f\| ,$$

thus μ_α is a (bounded) measure on the compact space X_α such that $\|\mu_\alpha\| \leqslant \|\mu\|$. By definition, for all $f \in \mathscr{C}(X_\alpha;\mathbf{C})$ one has

$$\begin{aligned}\mu_\alpha(f) &= \mu(f \circ p_\alpha) = \mu\bigl({p_\alpha}'(f)\bigr)\\ &= (\mu \circ {p_\alpha}')(f) = [{}^t({p_\alpha}')(\mu)](f) = \bigl((p_\alpha)_*(\mu)\bigr)(f) ,\end{aligned}$$

whence $(p_\alpha)_*(\mu) = \mu_\alpha$ for all α. It then follows from (*) that, for $\alpha \leqslant \beta$,

$$\begin{aligned}(p_{\alpha\beta})_*(\mu_\beta) &= (p_{\alpha\beta})_*\bigl((p_\beta)_*(\mu)\bigr)\\ &\bigl((p_{\alpha\beta})_* \circ (p_\beta)_*\bigr)(\mu) = (p_\alpha)_*(\mu) = \mu_\alpha ,\end{aligned}$$

thus the family (μ_α) is (in the sense of Def. 2) an inverse system of measures admitting μ as inverse limit. One can therefore define a (linear) mapping

$$\Phi : \mathscr{M}(\mathrm{X};\mathbf{C}) \to \varprojlim \mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})$$

by the formula

$$\Phi(\mu) = \big((p_\alpha)_*(\mu)\big) \qquad \big(\mu \in \mathscr{M}(\mathrm{X};\mathbf{C})\big).$$

By Def. 2 the range of Φ consists of the set of all inverse systems of measures that admit an inverse limit. A preview of Prop. 8:

By (i) of Prop. 8, Φ is injective.

By (ii) of Prop. 8, if (μ_α) is an inverse system admitting an inverse limit μ, then the family $(\|\mu_\alpha\|)$ is bounded (indeed, $(\mu_\alpha) = \Phi(\mu)$ satisfies $\|\mu_\alpha\| \leqslant \|\mu\|$ for all α as noted above).

By (iii) of Prop. 8, if the $p_{\alpha\beta}$ are surjective, then every inverse system (μ_α) for which $(\|\mu_\alpha\|)$ is bounded has an inverse limit; in this case, the range of Φ is precisely the set of elements (μ_α) of $\varprojlim \mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})$ for which $(\|\mu_\alpha\|)$ is bounded, and Φ is a bijection of $\mathscr{M}(\mathrm{X};\mathbf{C})$ onto this set (a linear subspace of $\varprojlim \mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})$).

By (iv) of Prop. 8, if the $p_{\alpha\beta}$ are surjective, then Φ is a bijection of $\mathscr{M}_+(\mathrm{X};\mathbf{C})$ onto the set of elements (μ_α) of $\varprojlim \mathscr{M}(\mathrm{X}_\alpha;\mathbf{C})$ such that $\mu_\alpha \geqslant 0$ for all α, and, when $\mu \in \mathscr{M}_+(\mathrm{X};\mathbf{C})$, $\|\mu_\alpha\| = \|\mu\|$ for all α.

To summarize, the family of mappings $\big((p_\alpha)_*\big)$ and their property (*) are the link between inverse systems of measures on the X_α and possible measures μ on the space $\mathrm{X} = \varprojlim \mathrm{X}_\alpha$ that may correspond to them.

We now review the justification for calling the family $\big((p_\alpha)_*\big)$ an "inverse system" of mappings.

Let $(\mathrm{E}_\alpha, f_{\alpha\beta})$ *be an inverse system,* $\mathrm{E} = \varprojlim \mathrm{E}_\alpha$, *and* $f_\alpha : \mathrm{E} \to \mathrm{E}_\alpha$ *the canonical mapping.*

{In the application at hand, $\mathrm{E}_\alpha = \mathscr{M}(\mathrm{X}_\alpha)$ (we drop the " ;**C**" for brevity), $f_{\alpha\beta} = (p_{\alpha\beta})_*$, $\mathrm{E} = \varprojlim \mathscr{M}(\mathrm{X}_\alpha)$ and f_α the restriction of $\mathrm{pr}_\alpha : \prod_\beta \mathscr{M}(\mathrm{X}_\beta) \to \mathscr{M}(\mathrm{X}_\alpha)$ to $\varprojlim \mathscr{M}(\mathrm{X}_\beta)$. Thus $f_\alpha\big((\mu_\beta)\big) = \mu_\alpha$ for every inverse system of measures $\mu_\beta \in \mathscr{M}(\mathrm{X}_\beta)$.}

Let $(\mathrm{F}_\alpha, g_{\alpha\beta})$ *be a second inverse system,* $\mathrm{F} = \varprojlim \mathrm{F}_\alpha$, $g_\alpha : \mathrm{F} \to \mathrm{F}_\alpha$ *the canonical mapping.*

{In the application at hand, $\mathrm{F}_\alpha = \mathscr{M}(\mathrm{X})$ for all α, $g_{\alpha\beta}$ is the identity mapping on $\mathscr{M}(\mathrm{X})$; the elements (μ_α) of $\varprojlim \mathrm{F}_\alpha$ are then the "constant families" (μ_α) where, for some $\mu \in \mathscr{M}(\mathrm{X})$, $\mu_\alpha = \mu$ for all α; and the canonical mapping $g_\alpha : \varprojlim \mathrm{F}_\beta \to \mathrm{F}_\alpha$ assigns to a constant family (μ) the value μ. In effect, $\varprojlim \mathrm{F}_\alpha$ is the "diagonal" of the product space $\prod \mathrm{F}_\alpha$; one may identify $\varprojlim \mathrm{F}_\alpha$ with $\mathrm{F} = \mathscr{M}(\mathrm{X})$, and g_α with the identity mapping id on $\mathscr{M}(\mathrm{X})$.}

Suppose that for each α *we are given a mapping* $u_\alpha : \mathrm{F}_\alpha \to \mathrm{E}_\alpha$ *such that, when* $\alpha \leqslant \beta$, *the diagram*

$$\begin{array}{ccc} \mathrm{F}_\beta & \xrightarrow{u_\beta} & \mathrm{E}_\beta \\ {\scriptstyle g_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle f_{\alpha\beta}} \\ \mathrm{F}_\alpha & \xrightarrow[u_\alpha]{} & \mathrm{E}_\alpha \end{array}$$

is commutative, that is, $u_\alpha \circ g_{\alpha\beta} = f_{\alpha\beta} \circ u_\beta$ *when* $\alpha \leqslant \beta$. *One then says that the family* (u_α) *is an* **inverse system** *of mappings of* $(\mathrm{F}_\alpha, g_{\alpha\beta})$ *into* $(\mathrm{E}_\alpha, f_{\alpha\beta})$ (S, III, §7, No. 2).

{In the application at hand, $u_\alpha = (p_\alpha)_*$, the diagram takes the form

$$\begin{array}{ccc} \mathscr{M}(\mathrm{X}) & \xrightarrow{(p_\beta)_*} & \mathscr{M}(\mathrm{X}_\beta) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle (p_{\alpha\beta})_*} \\ \mathscr{M}(\mathrm{X}) & \xrightarrow[(p_\alpha)_*]{} & \mathscr{M}(\mathrm{X}_\alpha) \end{array}$$

whose commutativity reduces to the relation

$$(*) \qquad (p_\alpha)_* = (p_{\alpha\beta})_* \circ (p_\beta)_*$$

whenever $\alpha \leqslant \beta$.}

By S, III, §7, No. 2, Cor. 1 of Prop. 1, *there exists a unique mapping* $u : \mathrm{F} \to \mathrm{E}$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{F} & \xrightarrow{u} & \mathrm{E} \\ {\scriptstyle g_\alpha}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ \mathrm{F}_\alpha & \xrightarrow[u_\alpha]{} & \mathrm{E}_\alpha \end{array}$$

is commutative for all α. *The mapping* u *is then called the* **inverse limit** *of the family* (u_α), *written* $u = \varprojlim u_\alpha$ (S, III, §7, No. 2).

{In the application at hand, the diagram takes the form

$$\begin{array}{ccc} \mathscr{M}(\mathrm{X}) & \xrightarrow{\ u\ } & \varprojlim \mathscr{M}(\mathrm{X}_\beta) \\ \Big\downarrow \mathrm{id} & & \Big\downarrow f_\alpha \\ \mathscr{M}(\mathrm{X}) & \xrightarrow[(p_\alpha)_*]{} & \mathscr{M}(\mathrm{X}_\alpha) \end{array}$$

where f_α is the restriction of $\mathrm{pr}_\alpha : \prod \mathscr{M}(\mathrm{X}_\beta) \to \mathscr{M}(\mathrm{X}_\alpha)$ to $\varprojlim \mathscr{M}(\mathrm{X}_\beta)$. The property required of u is that $f_\alpha\big(u(\mu)\big) = (p_\alpha)_*(\mu)$ for all α; since $\Phi(\mu) = \big((p_\beta)_*(\mu)\big)$ has this property, $\Phi = u$ by the uniqueness of u. Thus $\big((p_\alpha)_*\big)$ is an **inverse system of mappings** and $\Phi = \varprojlim (p_\alpha)_*$.}

{Another application of inverse limits occurs in the theory of Souslin spaces (GT, IX, §6, No. 5).}

III.51, ℓ. 10–13.

"... which shows that

$$g + h = (g_\beta \circ p_{\beta\alpha} + h_\gamma \circ p_{\gamma\alpha}) \circ p_\alpha$$

belongs to F ..."

In slow motion,

$$(*) \qquad \begin{aligned} g + h &= g_\beta \circ p_\beta + h_\gamma \circ p_\gamma \\ &= g_\beta \circ (p_{\beta\alpha} \circ p_\alpha) + h_\gamma \circ (g_{\gamma\alpha} \circ p_\alpha) \\ &= (g_\beta \circ p_{\beta\alpha}) \circ p_\alpha + (h_\gamma \circ p_{\gamma\alpha}) \circ p_\alpha \,. \end{aligned}$$

Now, $A = g_\beta \circ p_{\beta\alpha}$ and $B = h_\gamma \circ p_{\gamma\alpha}$ are continuous functions $\mathrm{X}_\alpha \to \mathbf{C}$, that is, elements of $\mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$; therefore their sum is defined and belongs to $\mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$, say $A + B = f_\alpha \in \mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$, and, by the definition of sum,

$$(A + B)(x_\alpha) = A(x_\alpha) + B(x_\alpha) \quad \text{for all } x_\alpha \in \mathrm{X}_\alpha \,.$$

In particular,

$$\big(A + B\big)\big(p_\alpha(x)\big) = A\big(p_\alpha(x)\big) + B\big(p_\alpha(x)\big) \quad \text{for all } x \in \mathrm{X} \,,$$

that is, $(A + B) \circ p_\alpha = A \circ p_\alpha + B \circ p_\alpha$ as functions $\mathrm{X} \to \mathbf{C}$. In view of (*),

$$g + h = A \circ p_\alpha + B \circ p_\alpha = (A + B) \circ p_\alpha = f_\alpha \circ p_\alpha,$$

where $f_\alpha = A + B \in \mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$, consequently $g + h \in \mathrm{F}$.

III.52, ℓ. 10.

"... $(p_\alpha)_*(\mu) = \mu_\alpha$ for all $\alpha \in \mathrm{I}$ ..."

For every $f_\alpha \in \mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$,

$$\mu_\alpha(f_\alpha) = \lambda(f_\alpha \circ p_\alpha) = \mu(f_\alpha \circ p_\alpha) = \big((p_\alpha)_*(\mu)\big)(f_\alpha)\,,$$

whence $\mu_\alpha = (p_\alpha)_*(\mu)$.

III.52, ℓ. 13.

"... the relation $\mu_\alpha = (p_{\alpha\beta})_*(\mu_\beta)$ implies that $\mu_\alpha(1) = \mu_\beta(1)$ ..."

Write $1_\alpha \in \mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$ for the constant function equal to 1. The point is that $1_\alpha \circ p_{\alpha\beta} = 1_\beta$, thus

$$\mu_\beta(1_\beta) = \mu_\beta(1_\alpha \circ p_{\alpha\beta}) = \big((p_{\alpha\beta})_*(\mu_\beta)\big)(1_\alpha) = \mu_\alpha(1_\alpha)\,.$$

III.52, ℓ. 15, 16.

"... the subspace F obviously satisfies the property (P) of §1, No. 7, Prop. 9 ..."

The constant function 1 belongs to F because $1 = 1_\alpha \circ p_\alpha$, where $1_\alpha \in \mathscr{C}(\mathrm{X}_\alpha; \mathbf{C})$ is the constant function equal to 1, and $f = 1$ trivially satisfies the requirements of (P) in the cited Prop. 9.

III.52, ℓ. -13 to -10.

"We know that $(\mathrm{X}_\mathrm{J}, \mathrm{pr}_{\mathrm{JK}})$ is an inverse system of compact spaces, and that the inverse limit of the system of continuous mappings (pr_J) is a *homeomorphism* of X onto the inverse limit space $\varprojlim \mathrm{X}_\mathrm{J}$, permitting one to identify these two spaces (GT, I, §4, No. 4 and S, III, §7, No. 2, *Remark* 3)."

I had great difficulty applying the cited references, making many mistakes along the way; be suspicious of every move in the following exposition.

To recapitulate the notations, we are given a family of nonempty compact spaces, and

$$\mathrm{X} = \prod_{\lambda \in \mathrm{L}} \mathrm{X}_\lambda$$

is the product topological space, the topology being defined as the initial topology for the family $(\mathrm{pr}_\lambda)_{\lambda \in \mathrm{L}}$ of projection mappings

$$\mathrm{pr}_\lambda : \mathrm{X} \to \mathrm{X}_\lambda\,.$$

No order relation on the index set L is assumed.

Let $\mathfrak{F}$ be the set of all nonempty finite subsets J of L; $\mathfrak{F}$ is an increasing directed set for the order relation $\mathrm{J} \subset \mathrm{K}$ on $\mathfrak{F}$. For each $\mathrm{J} \in \mathfrak{F}$ one writes

$$\mathrm{X}_\mathrm{J} = \prod_{\lambda \in \mathrm{J}} \mathrm{X}_\lambda\,,$$

and the mapping

$$\mathrm{pr}_{\mathrm{J}} : \mathrm{X} \to \mathrm{X}_{\mathrm{J}}$$

is defined, for $x = (x_\lambda)_{\lambda \in \mathrm{L}}$ in X by

$$\mathrm{pr}_{\mathrm{J}}\, x = (x_\lambda)_{\lambda \in \mathrm{J}} = (\mathrm{pr}_\lambda\, x)_{\lambda \in \mathrm{J}} ;$$

so to speak, pr_{J} "masks out" the coordinates $\lambda \in \mathrm{L} - \mathrm{J}$. If $\mathrm{J}, \mathrm{K} \in \mathfrak{F}$ with $\mathrm{J} \subset \mathrm{K}$, then

$$\mathrm{pr}_{\mathrm{JK}} : \mathrm{X}_{\mathrm{K}} \to \mathrm{X}_{\mathrm{J}}$$

is defined to be the mapping $(x_\lambda)_{\lambda \in \mathrm{K}} \mapsto (x_\lambda)_{\lambda \in \mathrm{J}}$ (thus masking out the coordinates $\lambda \in \mathrm{K} - \mathrm{J}$); also $\mathrm{pr}_{\mathrm{JJ}} = \mathrm{id}$ on X_{J} and, when $\mathrm{I} \subset \mathrm{J} \subset \mathrm{K} \in \mathfrak{F}$,

$$\mathrm{pr}_{\mathrm{IJ}} \circ \mathrm{pr}_{\mathrm{JK}} = \mathrm{pr}_{\mathrm{IK}} .$$

To summarize,

$$(\mathrm{X}_{\mathrm{J}}, \mathrm{pr}_{\mathrm{JK}})$$

is an inverse system of sets relative to the index set $\mathfrak{F}$ (S, III, §7, No. 1). Writing $\mathrm{Z} = \prod_{\mathrm{J} \in \mathfrak{F}} \mathrm{X}_{\mathrm{J}}$, the inverse limit

$$\varprojlim \mathrm{X}_{\mathrm{J}}$$

is the subset of the compact space Z consisting of the elements $(z_{\mathrm{J}})_{\mathrm{J} \in \mathfrak{F}} \in \mathrm{Z}$ such that

$$\mathrm{pr}_{\mathrm{JK}}\, z_{\mathrm{K}} = z_{\mathrm{J}} \quad \text{when } \mathrm{J} \subset \mathrm{K} ;$$

explicitly, if $z_{\mathrm{K}} = (a_\mu)_{\mu \in \mathrm{K}}$, where $a_\mu \in \mathrm{X}_\mu$ for all $\mu \in \mathrm{K}$, then $\mathrm{pr}_{\mathrm{JK}}\, z_{\mathrm{K}} = (a_\mu)_{\mu \in \mathrm{J}}$. In particular, if $\lambda \in \mathrm{K}$ and $\mathrm{J} = \{\lambda\}$, then

$$\mathrm{pr}_{\{\lambda\}\mathrm{K}}\, z_{\mathrm{K}} = (a_\mu)_{\mu \in \{\lambda\}} = a_\lambda ;$$

thus, when $\lambda \in \mathrm{K}$, the effect of $\mathrm{pr}_{\{\lambda\}\mathrm{K}}$ on an element z_{K} of X_{K} is to single out the λ'th coordinate of z_{K}.

For all $x = (x_\lambda)_{\lambda \in \mathrm{L}} \in \mathrm{X}$, $\mathrm{pr}_{\mathrm{JK}}(\mathrm{pr}_{\mathrm{K}}\, x) = \mathrm{pr}_{\mathrm{JK}}\big((x_\lambda)_{\lambda \in \mathrm{K}}\big) = (x_\lambda)_{\lambda \in \mathrm{J}} = \mathrm{pr}_{\mathrm{J}}\, x$, thus

(*) $$\mathrm{pr}_{\mathrm{JK}} \circ \mathrm{pr}_{\mathrm{K}} = \mathrm{pr}_{\mathrm{J}}$$

as functions $\mathrm{X} \to \mathrm{X}_{\mathrm{J}}$. We write

$$\mathrm{Y} = \varprojlim \mathrm{X}_{\mathrm{J}}$$

and, for $\mathrm{J} \in \mathfrak{F}$,

$$p_{\mathrm{J}} : \mathrm{Y} \to \mathrm{X}_{\mathrm{J}}$$

for the restriction to Y of the J'th coordinate projection mapping $\mathrm{Z} \to \mathrm{X}_{\mathrm{J}}$.

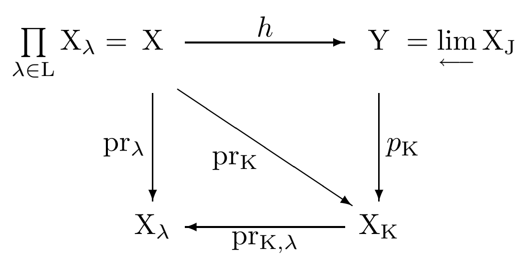
By GT, I, §9, No. 6, Prop. 8, $\mathrm{Y} = \varprojlim \mathrm{X}_{\mathrm{J}}$ is compact and nonempty; our problem is to show that X is homeomorphic to Y. At any rate, there is a natural mapping $h : \mathrm{X} \to \mathrm{Y}$, defined by

$$h(x) = (\mathrm{pr}_{\mathrm{J}}\, x)_{\mathrm{J} \in \mathfrak{F}}\,,$$

where $h(x) \in \mathrm{Y}$ by (*); explicitly, if $\mathrm{J} \subset \mathrm{K}$ then

$$\mathrm{pr}_{\mathrm{JK}}(\mathrm{pr}_{\mathrm{K}}\, x) = (\mathrm{pr}_{\mathrm{JK}} \circ \mathrm{pr}_{\mathrm{K}})x = \mathrm{pr}_{\mathrm{J}}\, x\,.$$

For every $\mathrm{K} \in \mathfrak{F}$ and every $\lambda \in \mathrm{K}$, we have the commutative diagram



where $\mathrm{pr}_{\mathrm{K},\lambda}$ is the λ'th coordinate projection on the product space X_{K}. The equality $p_{\mathrm{K}} \circ h = \mathrm{pr}_{\mathrm{K}}$ expresses the definition of h, and $\mathrm{pr}_{\mathrm{K},\lambda} \circ \mathrm{pr}_{\mathrm{K}} = \mathrm{pr}_{\lambda}$ is clear from the definition of $\mathrm{pr}_{\mathrm{K},\lambda}$.

We prove that h is a homeomorphism in three steps:

(i) h is injective;

(ii) h is continuous;

(iii) h is surjective.

Proof of (i): If $h(x) = h(x')$ then, for every $\mathrm{J} \in \mathfrak{F}$, $\mathrm{pr}_{\mathrm{J}}\, x = \mathrm{pr}_{\mathrm{J}}\, x'$; thus if $\lambda \in \mathrm{L}$ and if J is an element of $\mathfrak{F}$ containing λ, then

$$\mathrm{pr}_{\mathrm{J},\lambda}(\mathrm{pr}_{\mathrm{J}}\, x) = \mathrm{pr}_{\mathrm{J},\lambda}(\mathrm{pr}_{\mathrm{J}}\, x')\,,$$

that is, $\mathrm{pr}_{\lambda}\, x = \mathrm{pr}_{\lambda}\, x'$.

Proof of (ii): Y bears the initial topology for the family of mappings $(p_{\mathrm{K}})_{\mathrm{K} \in \mathfrak{F}}$, and $p_{\mathrm{K}} \circ h = \mathrm{pr}_{\mathrm{K}}$ is continuous for every K, therefore h is continuous.

Proof of (iii): Given $y = (y_{\mathrm{J}})_{\mathrm{J} \in \mathfrak{F}} \in \mathrm{Y}$, we seek an element $x \in \mathrm{X}$ such that $h(x) = y$, that is, $\mathrm{pr}_{\mathrm{J}}\, x = y_{\mathrm{J}}$ for all $\mathrm{J} \in \mathfrak{F}$.

Given any $\lambda \in \mathrm{L}$, we have $\{\lambda\} \in \mathfrak{F}$ and $y_{\{\lambda\}} \in \mathrm{X}_{\{\lambda\}} = \mathrm{X}_{\lambda}$, therefore $y_{\{\lambda\}} = x_{\lambda}$ for a unique $x_{\lambda} \in \mathrm{X}_{\lambda}$. Define $x = (x_{\lambda})_{\lambda \in \mathrm{L}}$.

As noted earlier, if K is any element of $\mathfrak{F}$ that contains λ, then the effect of $\mathrm{pr}_{\{\lambda\}\mathrm{K}}$ on an element z_{K} of X_{K} is to single out its λ'th coordinate; in particular, $\mathrm{pr}_{\{\lambda\}\mathrm{K}}\, y_{\mathrm{K}} = y_{\{\lambda\}} = x_\lambda$ shows that the λ'th coordinate of y_{K} is x_λ.

Now let J be any element of $\mathfrak{F}$ and let us show that $\mathrm{pr}_{\mathrm{J}}\, x = y_{\mathrm{J}}$. Say $y_{\mathrm{J}} = (a_\lambda)_{\lambda\in\mathrm{J}}$, where $a_\lambda \in \mathrm{X}_\lambda$ for all $\lambda \in \mathrm{J}$. For every $\lambda \in \mathrm{J}$, we know from the preceding paragraph that the λ'th coordinate of y_{J} is x_λ, that is, $a_\lambda = x_\lambda$, whence $y_{\mathrm{J}} = (x_\lambda)_{\lambda\in\mathrm{J}} = \mathrm{pr}_{\mathrm{J}}\, x$.

III.52, ℓ. −9 to −6.

"Since the projections $\mathrm{pr}_{\mathrm{J,K}}$ are surjective, it follows from Prop. 8 that the set $\mathscr{M}(\mathrm{X};\mathbf{C})$ (resp. $\mathscr{M}_+(\mathrm{X})$) may be identified with the set of inverse systems (μ_{J}) such that the family of norms $(\|\mu_{\mathrm{J}}\|)$ is bounded (resp. such that the μ_{J} are all positive, and necessarily of the same total mass)."

Both notations $\mathrm{pr}_{\mathrm{JK}}$ and $\mathrm{pr}_{\mathrm{J,K}}$ are employed in the *Example* to denote the canonical projection $\mathrm{X}_{\mathrm{K}} \to \mathrm{X}_{\mathrm{J}}$. The former, without a separating comma, is consistent with the general notation for an inverse system, but the latter is used here three times.

By the discussion at the beginning of No. 5, to a continuous mapping $p : \mathrm{A} \to \mathrm{B}$ between compact spaces there is associated a mapping $p_* : \mathscr{M}(\mathrm{A};\mathbf{C}) \to \mathscr{M}(\mathrm{B};\mathbf{C})$, $p_* = {}^t(p')$. The correspondence evidently satisfies $(p\circ q)_* = p_* \circ q_*$ and $(1_{\mathrm{A}})_* = 1_{\mathscr{M}(\mathrm{A};\mathbf{C})}$. (In the language of categories, $p \mapsto p_*$ defines a covariant functor between the appropriate categories.) When p is surjective, p' preserves norm, hence so does p_*.

As shown in the preceding note, if $\mathrm{X} = \prod_{\lambda\in\mathrm{L}} \mathrm{X}_\lambda$ and $\mathrm{Y} = \varprojlim \mathrm{X}_{\mathrm{J}}$ $(\mathrm{J}\in\mathfrak{F})$, then the mapping $h : \mathrm{X} \to \mathrm{Y}$ defined by $h(x) = (\mathrm{pr}_{\mathrm{J}}\, x)_{\mathrm{J}\in\mathfrak{F}}$ is a homeomorphism. It follows that the mapping

$$\text{(i)} \qquad h_* : \mathscr{M}(\mathrm{X}) \to \mathscr{M}(\mathrm{Y})$$

defined, for $\mu \in \mathscr{M}(\mathrm{X}))$ and $g \in \mathscr{C}(\mathrm{Y})$ by

$$\langle h_*(\mu), g\rangle = \langle {}^t(h')(\mu), g\rangle = \langle \mu, h'(g)\rangle = \langle \mu, g\circ h\rangle$$

is an isomorphism of vector spaces such that $\mu \geqslant 0 \Leftrightarrow h_*(\mu) \geqslant 0$, and $\|h_*(\mu)\| = \|\mu\|$ (because h' preserves norm).

On the other hand, by the note for III.50, ℓ. −14, −13, there is a linear mapping

$$\text{(ii)} \qquad \Phi : \mathscr{M}(\mathrm{Y}) \to \varprojlim \mathscr{M}(\mathrm{X}_{\mathrm{J}}),$$

defined by $\Phi(\nu) = \big((p_{\mathrm{J}})_*(\nu)\big)_{\mathrm{J}\in\mathfrak{F}}$, that is injective (by (i) of Prop. 8), the range $\Phi\big(\mathscr{M}(\mathrm{Y})\big)$ of Φ is the set of inverse systems $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ of measures on the $\mathrm{X_J}$ for which the norms $\|\mu_{\mathrm{J}}\|$ are bounded (by (iii) of Prop. 8; see also the note for III. 50, $\ell.\ -14, -13$); and, by (iv) of Prop. 8, $\Phi\big(\mathscr{M}_+(\mathrm{X})\big)$ is the set of all inverse systems $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ for which $\mu_{\mathrm{J}} \geqslant 0$ for all J, in which case $\|\mu_{\mathrm{J}}\|$ is constant, specifically, if $\Phi(\mu) = (\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ then $\mu \geqslant 0$ and $\|\mu_{\mathrm{J}}\| = \|\mu\|$ for all $\mathrm{J} \in \mathfrak{F}$.

Combining the mappings (i) and (ii), we have a mapping

$$\text{(iii)} \qquad \theta = \Phi \circ h_* : \mathscr{M}(\mathrm{X}) \to \varprojlim \mathscr{M}(\mathrm{X_J})$$

such that θ is linear, injective and preserves positivity, its range $\theta\big(\mathscr{M}(\mathrm{X})\big)$ is the set of all inverse systems $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ of measures on the $\mathrm{X_J}$ for which the norms $\|\mu_{\mathrm{J}}\|$ are bounded; and $\theta\big(\mathscr{M}_+(\mathrm{X})\big)$ is the set of all inverse systems $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ for which $\mu_{\mathrm{J}} \geqslant 0$ for all J, in which case, if $\theta(\mu) = (\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$, one has $\|\mu_{\mathrm{J}}\| = \|\mu\|$ for all J.

Let us calculate $\theta(\mu)$ explicitly for $\mu \in \mathscr{M}(\mathrm{X})$. Writing $\nu = h_*(\mu) \in \mathscr{M}(\mathrm{Y})$, we have

$$\theta(\mu) = \big(\Phi \circ h_*\big)(\mu) = \Phi\big(h_*(\mu)\big) = \Phi(\nu) = \big((p_{\mathrm{J}})_*(\nu)\big)_{\mathrm{J}\in\mathfrak{F}},$$

where $(p_{\mathrm{J}})_*(\nu) \in \mathscr{M}(\mathrm{X_J})$ is the measure on $\mathrm{X_J}$ whose effect on a function $f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X_J})$ is given by

$$\begin{aligned}\langle (p_{\mathrm{J}})_*(\nu), f_{\mathrm{J}}\rangle &= \langle \nu, f_{\mathrm{J}} \circ p_{\mathrm{J}}\rangle \\ &= \langle h_*(\mu), f_{\mathrm{J}} \circ p_{\mathrm{J}}\rangle \\ &= \langle \mu, (f_{\mathrm{J}} \circ p_{\mathrm{J}}) \circ h\rangle \\ &= \langle \mu, f_{\mathrm{J}} \circ (p_{\mathrm{J}} \circ h)\rangle \\ &= \langle (p_{\mathrm{J}} \circ h)_*(\mu), f_{\mathrm{J}}\rangle,\end{aligned}$$

whence $(p_{\mathrm{J}})_*(\nu) = (p_{\mathrm{J}} \circ h)_*(\mu)$, and so

$$\theta(\mu) = \big((p_{\mathrm{J}} \circ h)_*(\mu)\big)_{\mathrm{J}\in\mathfrak{F}};$$

thus the inverse family $\theta(\mu) = (\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ is given by

$$\text{(iv)} \qquad \mu_{\mathrm{J}} = (p_{\mathrm{J}} \circ h)_*(\mu) \quad \text{for all } \mathrm{J} \in \mathfrak{F}.$$

Now, $p_{\mathrm{J}} \circ h = \mathrm{pr_J}$ by the definitions of h and p_{J}, thus the meaning of (iv) is that, for all $\mathrm{J} \in \mathfrak{F}$ and all $f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X_J})$, one has

$$\begin{aligned}\langle \mu_{\mathrm{J}}, f_{\mathrm{J}}\rangle &= \langle (p_{\mathrm{J}} \circ h)_*(\mu), f_{\mathrm{J}}\rangle \\ &= \langle \mu, f_{\mathrm{J}} \circ (p_{\mathrm{J}} \circ h)\rangle = \langle \mu, f_{\mathrm{J}} \circ \mathrm{pr_J}\rangle,\end{aligned}$$

that is,

(v) $\mu_{\mathrm{J}}(f_{\mathrm{J}}) = \mu(f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J}})$ for all $\mathrm{J} \in \mathfrak{F}$ and $f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X}_{\mathrm{J}})$.

We may regard (v) in two lights:

(a) given any $\mu \in \mathscr{M}(\mathrm{X})$, the formulas (v) define an inverse system $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ of measures on the X_{J}, so to speak providing a *representation of* μ as the inverse limit of a family of measures on the X_{J}, concisely $\mu = (\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$;

(b) given any inverse system $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ of measures on the X_{J}, linked by the formulas

$$(\mathrm{pr}_{\mathrm{JK}})_*(\mu_{\mathrm{K}}) = \mu_{\mathrm{J}} \quad \text{when } \mathrm{J} \subset \mathrm{K} \in \mathfrak{F},$$

that is,

$$\langle \mu_{\mathrm{J}}, f_{\mathrm{J}} \rangle = \langle \mu_{\mathrm{K}}, f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{JK}} \rangle \quad \text{when } \mathrm{J} \subset \mathrm{K} \in \mathfrak{F} \text{ and } f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X}_{\mathrm{J}}),$$

then there exists a unique measure $\mu \in \mathscr{M}(\mathrm{X})$ satisfying (v), providing, so to speak, a *representation of the inverse system* $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ of measures on the X_{J} as a measure μ on X, concisely $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}} = \mu$. It is in the sense of this correspondence that one "identifies" measures on X with inverse systems of measures on the X_{J}. In view of Prop. 8, the inverse systems $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ on the X_{J} that correspond to measures on X are precisely those for which the norms $\|\mu_{\mathrm{J}}\|$ are bounded; and an inverse system $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ corresponds to a positive measure $\mu \in \mathscr{M}_+(\mathrm{X})$ if and only if $\mu_{\mathrm{J}} \geqslant 0$ for all $\mathrm{J} \in \mathfrak{F}$, a condition that implies that the norms $\|\mu_{\mathrm{J}}\| = \mu_{\mathrm{J}}(1_{\mathrm{X}_{\mathrm{J}}})$ are all equal to each other (if $\mathrm{J}, \mathrm{K} \in \mathfrak{F}$ then $\mathrm{J}\cup\mathrm{K} \in \mathfrak{F}$ provides a link between $\|\mu_{\mathrm{J}}\|$ and $\|\mu_{\mathrm{K}}\|$) hence are bounded, so that the system $(\mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ corresponds to a measure $\mu \geqslant 0$ on X and in fact $\|\mu_{\mathrm{J}}\| = \mu_{\mathrm{J}}(1_{\mathrm{X}_{\mathrm{J}}}) = \mu(1_{\mathrm{X}}) = \|\mu\|$ for all $\mathrm{J} \in \mathfrak{F}$.

III.52, ℓ. −3.

For "subsets of I" read "subsets of L".

III.52, ℓ. −3 to −1.

"... we have, by virtue of formula (14) of No. 4,

$$\mu_{\mathrm{K}}(f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J,K}}) = \mu_{\mathrm{J}}(f_{\mathrm{J}}) \cdot \prod_{\lambda \in \mathrm{K} - \mathrm{J}} \mu_\lambda(1)\,. \text{"} \tag{17}$$

One is assuming that $\mathrm{J} \subset \mathrm{K}$, and f_{J} is a given function in $\mathscr{C}(\mathrm{X}_{\mathrm{J}}; \mathbf{C})$. Define $f_{\mathrm{K}} = f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J,K}}$. If $\mathrm{J} = \mathrm{K}$, the formula reduces to $\mu_{\mathrm{J}}(f_{\mathrm{J}}) = \mu_{\mathrm{J}}(f_{\mathrm{J}})$. Let us assume that J has j elements, K has k elements, and $j < k$. To simplify the notation, let us suppose that

$$\mathrm{J} = \{1, \ldots, j\}, \quad \mathrm{K} = \{1, \ldots, j, j+1, \ldots, k\}.$$

Then for each fixed $(x_1, \ldots, x_j) \in \mathrm{X_J}$ and every $(x_{j+1}, \ldots, x_k) \in \mathrm{X_{K-J}}$, one has

$$f_{\mathrm{K}}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k) = f_{\mathrm{J}}(x_1, \ldots, x_j) \cdot 1\,;$$

by the cited formula (14), successive integration with respect to the variables $x_{j+1}, \ldots, x_k$ produces the factors $\mu_{j+1}(1), \ldots, \mu_k(1)$, thus

$$(*) \quad \int f_{\mathrm{K}}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k)\, d\mu_{\mathrm{K-J}} = f_{\mathrm{J}}(x_1, \ldots, x_j) \cdot \prod_{i \in \mathrm{K-J}} \mu_i(1)\,.$$

Denote groups of variables by $\mathbf{x} = (x_1, \ldots, x_j, x_{j+1}, \ldots, x_k) \in \mathrm{X_K}$, $\mathbf{y} = (x_1, \ldots, x_j) \in \mathrm{X_J}$, $\mathbf{z} = (x_{j+1}, \ldots, x_k) \in \mathrm{X_{K-J}}$, making the identification $\mathbf{x} = (\mathbf{y}, \mathbf{z})$. Then

$$f_{\mathrm{K}}(\mathbf{x}) = f_{\mathrm{K}}(\mathbf{y}, \mathbf{z}) = f_{\mathrm{J}}(\mathbf{y}) \cdot 1\,,$$

and the formula (*) may be written

$$\int f_{\mathrm{K}}(\mathbf{y}, \mathbf{z})\, d\mu_{\mathrm{K-J}}(\mathbf{z}) = f_{\mathrm{J}}(\mathbf{y}) \cdot \prod_{i \in \mathrm{K-J}} \mu_i(1)\,.$$

Then, citing (14) for the case of two variables,

$$\begin{aligned}
\mu_{\mathrm{K}}(f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J,K}}) = \mu_{\mathrm{K}}(f_{\mathrm{K}}) &= \int f_{\mathrm{K}}\, d(\mu_{\mathrm{J}} \otimes \mu_{\mathrm{K-J}}) \\
&= \int d\mu_{\mathrm{J}}(\mathbf{y}) \Big(\int f_{\mathrm{K}}(\mathbf{y}, \mathbf{z})\, d\mu_{\mathrm{K-J}}(\mathbf{z}) \Big) \\
&= \int d\mu_{\mathrm{J}}(\mathbf{y}) \Big(f_{\mathrm{J}}(\mathbf{y}) \cdot \prod_{i \in \mathrm{K-J}} \mu_i(1) \Big) \\
&= \mu_{\mathrm{J}}(f_{\mathrm{J}}) \cdot \prod_{i \in \mathrm{K-J}} \mu_i(1)\,.
\end{aligned}$$

III.53, ℓ. 1, 2.

"For (μ_{J}) to be an inverse system of measures, it is therefore necessary and sufficient that either $\mu_\lambda = 0$ for all $\lambda \in \mathrm{L}$ or $\mu_\lambda(1) = 1$ for all $\lambda \in \mathrm{L}$."

As in the preceding notes on the *Example*, $\mathfrak{F}$ denotes the set of all nonempty finite subsets J of L. The condition (17) may be written

$$\big((\mathrm{pr}_{\mathrm{J,K}})_*(\mu_{\mathrm{K}})\big)(f_{\mathrm{J}}) = \mu_{\mathrm{J}}(f_{\mathrm{J}}) \cdot \prod_{\lambda \in \mathrm{K-J}} \mu_\lambda(1)$$

whenever $\mathrm{J} \subset \mathrm{K} \in \mathfrak{F}$ and $f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X}_{\mathrm{J}}; \mathbf{C})$, that is,

$$(\mathrm{pr}_{\mathrm{J,K}})_*(\mu_{\mathrm{K}}) = \Big(\prod_{\lambda \in \mathrm{K} - \mathrm{J}} \mu_\lambda(1)\Big) \cdot \mu_{\mathrm{J}} \quad \text{when } \mathrm{J} \subset \mathrm{K} \in \mathfrak{F};$$

in order that $(\mu_{\mathrm{J}})_{\mathrm{J} \in \mathfrak{F}}$ be an inverse system of measures on the X_{J}, it is therefore necessary and sufficient that

$$(*) \qquad \mu_{\mathrm{J}} = \Big(\prod_{\lambda \in \mathrm{K} - \mathrm{J}} \mu_\lambda(1)\Big) \cdot \mu_{\mathrm{J}} \quad \text{when } \mathrm{J} \subset \mathrm{K} \in \mathfrak{F}.$$

If $\mu_\lambda(1) = 1$ for all $\lambda \in \mathrm{L}$, it is obvious that (*) holds, therefore $(\mu_{\mathrm{J}})_{\mathrm{J} \in \mathfrak{F}}$ is an inverse system. Whereas if $\mu_\lambda = 0$ for all $\lambda \in \mathrm{L}$, then $\mu_{\mathrm{J}} = 0$ for all $\mathrm{J} \in \mathfrak{F}$ and $(\mu_{\mathrm{J}})_{\mathrm{J} \in \mathfrak{F}}$ is trivially an inverse system.

Conversely, suppose (μ_{J}) is an inverse system, that is, satisfies the condition (*). We are to show that either $\mu_\lambda = 0$ for all $\lambda \in \mathrm{L}$ or $\mu_\lambda(1) = 1$ for all $\lambda \in \mathrm{L}$; assuming there exists a $\lambda_0 \in \mathrm{L}$ such that $\mu_{\lambda_0} \neq 0$, it will suffice to show that $\mu_\lambda(1) = 1$ for all $\lambda \in \mathrm{L}$. Let λ be any element of L such that $\lambda \neq \lambda_0$. Set $\mathrm{J} = \{\lambda_0\}$, $\mathrm{K} = \{\lambda_0, \lambda\}$. Then $\mu_{\mathrm{J}} = \mu_{\lambda_0}$ and the condition (*) reads $\mu_{\lambda_0} = \mu_\lambda(1) \cdot \mu_{\lambda_0}$, whence $\mu_\lambda(1) = 1$. In particular $\mu_\lambda \neq 0$, so the roles of λ and λ_0 can be interchanged, leading to $\mu_{\lambda_0} = 1$, consequently $\mu_\lambda(1) = 1$ for all $\lambda \in \mathrm{L}$.

III.53, ℓ. 5, 6.

"We retain the notations of the *Example* of No. 5, so that in particular $\mu_{\mathrm{J}} = \bigotimes_{\lambda \in \mathrm{J}} \mu_\lambda$ for every finite subset J of L."

As in the preceding notes, we write $\mathfrak{F}$ for the set of all nonempty finite subsets $\mathrm{J}, \mathrm{K}, \ldots$ of L. The following remarks, or something like them, should have been made at the outset of the *Example* of No. 5.

(i) What does $\mathrm{X}_{\mathrm{J}} = \prod_{\lambda \in \mathrm{J}} \mathrm{X}_\lambda$ mean? It is the set of all (finite) families $(x_\lambda)_{\lambda \in \mathrm{J}}$ such that $x_\lambda \in \mathrm{X}_\lambda$ for all $\lambda \in \mathrm{J}$, equipped with the initial topology for a family of canonical projections. Its elements are described independently of any ordering that might exist in L or in J. If $\mathrm{J} \subset \mathrm{K} \in \mathfrak{F}$ then the formulas

$$\mathrm{X}_{\mathrm{K}} = \mathrm{X}_{\mathrm{J}} \times \mathrm{X}_{\mathrm{K} - \mathrm{J}} = \mathrm{X}_{\mathrm{K} - \mathrm{J}} \times \mathrm{X}_{\mathrm{J}}$$

are a consequence of the associative law for products (GT, I, §4, No. 1, Prop. 2), the equalities expressing canonical homeomorphisms. Thus if $\mathbf{y} \in \mathrm{X}_{\mathrm{J}}$ and $\mathbf{z} \in \mathrm{X}_{\mathrm{K} - \mathrm{J}}$ then $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{z}, \mathbf{y})$ are representations of a same element $\mathbf{x}$ of X_{K}.

(ii) If $f_\lambda \in \mathscr{C}(\mathrm{X}_\lambda;\mathbf{C})$ for all $\lambda \in \mathrm{J}$, what does $\bigotimes\limits_{\lambda\in\mathrm{J}} f_\lambda$ mean? It is the function f on X_J defined by

$$f\big((x_\lambda)_{\lambda\in\mathrm{J}}\big) = \prod_{\lambda\in\mathrm{J}} f_\lambda(x_\lambda)\,,$$

the product being independent of any hypothetical ordering of the λ's.

(iii) If $\mu_\lambda \in \mathscr{M}(\mathrm{X}_\lambda;\mathbf{C})$ for all $\lambda \in \mathrm{J}$, what does $\bigotimes\limits_{\lambda\in\mathrm{J}} \mu_\lambda$ mean? It is the unique measure μ_J on X_J characterized by the relations

$$\mu_\mathrm{J}\Big(\bigotimes_{\lambda\in\mathrm{J}} f_\lambda\Big) = \prod_{\lambda\in\mathrm{J}} \mu_\lambda(f_\lambda) \qquad (\, f_\lambda \in \mathscr{C}(\mathrm{X}_\lambda;\mathbf{C}) \text{ for all } \lambda\in\mathrm{J}\,)\,,$$

the product being independent of any consideration of ordering. If $\mathrm{J} \subset \mathrm{K} \in \mathfrak{F}$ then the formulas

$$\mu_\mathrm{K} = \mu_\mathrm{J} \otimes \mu_{\mathrm{K}-\mathrm{J}} = \mu_{\mathrm{K}-\mathrm{J}} \otimes \mu_\mathrm{J}$$

exhibit three representations of the same measure on X_K, an instance of the associative law for finite products of measures (No. 4, Prop. 7).

III.53, ℓ. -5 to -2.

"It then follows from Prop. 8 of No. 5 that there exists a positive measure μ' on X of total mass 1, such that $\mu'(f_\mathrm{J} \circ \mathrm{pr}_\mathrm{J}) = \mu'_\mathrm{J}(f_\mathrm{J})$ for every finite subset J of L and every function $f_\mathrm{J} \in \mathscr{C}(\mathrm{X}_\mathrm{J};\mathbf{C})$."

As in previous notes, we write $\mathfrak{F}$ for the set of all nonempty finite subsets of L. Since $\mu'_\lambda(1) = 1$ for all λ, the measures

$$\mu'_\mathrm{J} = \bigotimes_{\lambda\in\mathrm{J}} \mu'_\lambda$$

form an inverse system of measures on the X_J by the last sentence of No. 5. It is clear that for every $\mathrm{J} \in \mathfrak{F}$, one has $\mu'_\mathrm{J} \geqslant 0$ and $\mu'_\mathrm{J}(1) = \prod\limits_{\lambda\in\mathrm{J}} \mu'_\lambda(1) = 1$. By (iv) of the cited Prop. 8 there exists an inverse limit measure $\mu' = \varprojlim \mu'_\mathrm{J}$ on X that is $\geqslant 0$ and satisfies $\|\mu'\| = \mu'(1) = \mu'_\mathrm{J}(1) = 1$ for all $\mathrm{J} \in \mathfrak{F}$. In particular,

$$\mu'(f_\mathrm{J} \circ \mathrm{pr}_\mathrm{J}) = \mu'_\mathrm{J}(f_\mathrm{J}) \quad \text{ for all } \mathrm{J} \in \mathfrak{F} \text{ and } f_\mathrm{J} \in \mathscr{C}(\mathrm{X}_\mathrm{J};\mathbf{C})$$

(see the formula (v) in the note for III.52, ℓ. -13 to -10).

III.53, ℓ. −2 to **III.54,** ℓ. 3.

"The positive measure

$$\mu = \Bigl(\prod_{\lambda\in\mathrm{L}} \mu_\lambda(1)\Bigr)\,\mu'$$

then meets the requirements, since

$$\mu_{\mathrm{J}}(f_{\mathrm{J}}) = \mu'_{\mathrm{J}}(f_{\mathrm{J}}) \cdot \prod_{\lambda\in\mathrm{J}} \mu_\lambda(1)\,,$$

$$\prod_{\lambda\in\mathrm{L}} \mu_\lambda(1) = \prod_{\lambda\in\mathrm{J}} \mu_\lambda(1) \cdot \prod_{\lambda\in\mathrm{L}-\mathrm{J}} \mu_\lambda(1)\,."$$

The factorization of $\prod_{\lambda\in\mathrm{L}} \mu_\lambda(1)$ in the preceding line is a special case of an associativity theorem for multipliable families (GT, IV, §7, No. 5, *Remark*).

For all $\mathrm{J} \in \mathfrak{F}$ we have

$$\begin{aligned}\mu'_{\mathrm{J}} &= \bigotimes_{\lambda\in\mathrm{J}} \mu'_\lambda = \bigotimes_{\lambda\in\mathrm{J}} \bigl(\mu_\lambda(1)\bigr)^{-1}\mu_\lambda \\ &= \Bigl(\prod_{\lambda\in\mathrm{J}} \mu_\lambda(1)\Bigr)^{-1} \cdot \bigotimes_{\lambda\in\mathrm{J}} \mu_\lambda = \Bigl(\prod_{\lambda\in\mathrm{J}} \mu_\lambda(1)\Bigr)^{-1} \cdot \mu_{\mathrm{J}}\,,\end{aligned}$$

whence

$$(*) \qquad \mu_{\mathrm{J}} = \Bigl(\prod_{\lambda\in\mathrm{J}} \mu_\lambda(1)\Bigr) \cdot \mu'_{\mathrm{J}}\,;$$

therefore, for all $f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X}_{\mathrm{J}};\mathbf{C})$,

$$\begin{aligned}\mu(f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J}}) &= \Bigl(\prod_{\lambda\in\mathrm{L}} \mu_\lambda(1)\Bigr)\mu'(f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J}}) && \text{(definition of } \mu\text{)} \\ &= \Bigl(\prod_{\lambda\in\mathrm{L}} \mu_\lambda(1)\Bigr)\mu'_{\mathrm{J}}(f_{\mathrm{J}}) && \text{(preceding note)} \\ &= \Bigl(\prod_{\lambda\in\mathrm{J}} \mu_\lambda(1) \cdot \mu'_{\mathrm{J}}(f_{\mathrm{J}})\Bigr) \cdot \prod_{\lambda\in\mathrm{L}-\mathrm{J}} \mu_\lambda(1) \\ &= \mu_{\mathrm{J}}(f_{\mathrm{J}}) \cdot \prod_{\lambda\in\mathrm{L}-\mathrm{J}} \mu_\lambda(1) && \text{(by (*))}\end{aligned}$$

whence the formula (18). And, since $\mu'(1) = 1$, the defining formula

$$\mu = \Big(\prod_{\lambda\in L} \mu_\lambda(1)\Big)\mu'$$

yields the formula (19).

III.54, ℓ. 11–13.

"This follows at once from the formulas (18) and (19) and the associativity of the product for multipliable families in $\mathbf{R}_+$ (GT, IV, §7, No. 5, *Remark*)."

(The term "at once" comes in various sizes. Am I overlooking a dramatic simplification? The unpleasant length of the following argument carries with it an increased risk of error.)

By the cited *Remark*, for every $\rho \in R$ the subfamily $(\mu_\lambda(1))_{\lambda\in L_\rho}$ is multipliable, the family of products $\Big(\prod_{\lambda\in L_\rho} \mu_\lambda(1)\Big)_{\rho\in R}$ is multipliable, and

$$\prod_{\rho\in R}\Big(\prod_{\lambda\in L_\rho} \mu_\lambda(1)\Big) = \prod_{\lambda\in L} \mu_\lambda(1)\,. \tag{1}$$

Therefore, by Prop. 9, for each $\rho \in R$ there exists the positive measure

$$\nu_\rho = \bigotimes_{\lambda\in L_\rho} \mu_\lambda \quad \text{on} \quad Y_\rho = \prod_{\lambda\in L_\rho} X_\lambda \tag{2}$$

satisfying

$$\nu_\rho(f_J \circ \mathrm{pr}_J) = \mu_J(f_J) \prod_{\lambda\in L_\rho - J} \mu_\lambda(1) \tag{3}$$

for every finite $J \subset L_\rho$ and every $f_J \in \mathscr{C}(X_J;\mathbf{C})$, and one has

$$\nu_\rho(1) = \prod_{\lambda\in L_\rho} \mu_\lambda(1)\,; \tag{4}$$

moreover, since the family $\big(\nu_\rho(1)\big)_{\rho\in R}$ is multipliable, Prop. 9 yields a positive measure

$$\nu = \bigotimes_{\rho\in R} \nu_\rho \quad \text{on} \quad \prod_{\rho\in R} Y_\rho = X \tag{5}$$

such that, writing

$$\nu_{\mathrm{S}} = \bigotimes_{\rho \in \mathrm{S}} \nu_\rho \quad \text{for finite } \mathrm{S} \subset \mathrm{R}\,, \tag{6}$$

one has

$$\nu(g_{\mathrm{S}} \circ \mathrm{pr}_{\mathrm{S}}) = \nu_{\mathrm{S}}(g_{\mathrm{S}}) \prod_{\rho \in \mathrm{R} - \mathrm{S}} \nu_\rho(1) \tag{7}$$

for finite $\mathrm{S} \subset \mathrm{R}$ and all $g_{\mathrm{S}} \in \mathscr{C}(\mathrm{Y}_{\mathrm{S}}; \mathbf{C})$, where

$$\mathrm{Y}_{\mathrm{S}} = \prod_{\rho \in \mathrm{S}} \mathrm{Y}_\rho = \prod_{\rho \in \mathrm{S}} \Big(\prod_{\lambda \in \mathrm{L}_\rho} \mathrm{X}_\lambda \Big) = \prod_{\lambda \in \bigcup_{\rho \in \mathrm{S}} \mathrm{L}_\rho} \mathrm{X}_\lambda \tag{8}$$

and, by (4) and (1),

$$\nu(1) = \prod_{\rho \in \mathrm{R}} \nu_\rho(1) = \prod_{\lambda \in \mathrm{L}} \mu_\lambda(1)\,. \tag{9}$$

On the other hand, by Prop. 9 there exists the product measure

$$\mu = \bigotimes_{\lambda \in \mathrm{L}} \mu_\lambda \quad \text{on} \quad \mathrm{X} = \prod_{\lambda \in \mathrm{L}} \mathrm{X}_\lambda \tag{10}$$

satisfying

$$\mu(f_{\mathrm{J}} \circ \mathrm{pr}_{\mathrm{J}}) = \mu_{\mathrm{J}}(f_{\mathrm{J}}) \prod_{\lambda \in \mathrm{L} - \mathrm{J}} \mu_\lambda(1) \tag{11}$$

for all finite $\mathrm{J} \subset \mathrm{L}$ and all $f_{\mathrm{J}} \in \mathscr{C}(\mathrm{X}_{\mathrm{J}}; \mathbf{C})$, and one has

$$\mu(1) = \prod_{\lambda \in \mathrm{L}} \mu_\lambda(1) \qquad (= \nu(1) \text{ by } (9))\,. \tag{12}$$

By (5) and (2),

$$\nu = \bigotimes_{\rho \in \mathrm{R}} \nu_\rho = \bigotimes_{\rho \in \mathrm{R}} \Big(\bigotimes_{\lambda \in \mathrm{L}_\rho} \mu_\lambda \Big) \quad \text{on} \quad \prod_{\rho \in \mathrm{R}} \mathrm{Y}_\rho = \mathrm{X}\,; \tag{13}$$

since $\mu = \bigotimes_{\lambda \in \mathrm{L}} \mu_\lambda$ on X, the problem is to show that $\nu = \mu$. So to speak, $\prod_{\lambda \in \mathrm{L}} \mathrm{X}_\lambda$ and $\prod_{\rho \in \mathrm{R}} \mathrm{Y}_\rho$ are two representations of the same space X; we are to show that μ and ν are two representations of the same measure on X.

To this end, let J be a finite subset of $\mathrm{L} = \bigcup_{\rho\in\mathrm{R}} \mathrm{L}_\rho$ and let $f_\mathrm{J} \in \mathscr{C}(\mathrm{X}_\mathrm{J};\mathbf{C})$. The set J can intersect only finitely many of the mutually disjoint sets L_ρ; let S be a nonempty finite subset of R such that $\mathrm{J} \subset \bigcup_{\rho\in\mathrm{S}} \mathrm{L}_\rho$, which we can suppose to be minimal, so that $\mathrm{J}\cap\mathrm{L}_\rho \neq \varnothing$ for every $\rho\in\mathrm{S}$. Then

$$\mathrm{J} = \bigcup_{\rho\in\mathrm{S}} \mathrm{J}\cap\mathrm{L}_\rho = \bigcup_{\rho\in\mathrm{S}} \mathrm{J}_\rho\,, \tag{14}$$

where the sets $\mathrm{J}_\rho = \mathrm{J}\cap\mathrm{L}_\rho$ $(\rho\in\mathrm{S})$ are pairwise disjoint and nonempty.

In view of (8) and the fact $\mathrm{J}\subset \bigcup_{\rho\in\mathrm{S}} \mathrm{L}_\rho$, setting $\mathrm{T} = \bigcup_{\rho\in\mathrm{S}} \mathrm{L}_\rho$ one can write

$$\mathrm{Y}_\mathrm{S} = \prod_{\lambda\in\mathrm{T}} \mathrm{X}_\lambda = \mathrm{X}_\mathrm{J} \times \prod_{\lambda\in\mathrm{T}-\mathrm{J}} \mathrm{X}_\lambda\,,$$

hence one can define a function $g_\mathrm{S}\in\mathscr{C}(\mathrm{Y}_\mathrm{S};\mathbf{C})$ by the formula

$$g_\mathrm{S} = f_\mathrm{J}\circ\overline{\mathrm{pr}}_\mathrm{J}\,, \tag{15}$$

where $\overline{\mathrm{pr}}_\mathrm{J}$ is the J'th projection in the context of the space Y_S. Writing pr_S for the S'th projection in the context of $\prod_{\rho\in\mathrm{R}} \mathrm{Y}_\rho = \mathrm{X} = \mathrm{Y}_\mathrm{S} \times \prod_{\rho\in\mathrm{R}-\mathrm{S}} \mathrm{Y}_\rho$, one then has

$$g_\mathrm{S}\circ\mathrm{pr}_\mathrm{S} = (f_\mathrm{J}\circ\overline{\mathrm{pr}}_\mathrm{J})\circ\mathrm{pr}_\mathrm{S} = f_\mathrm{J}\circ(\overline{\mathrm{pr}}_\mathrm{J}\circ\mathrm{pr}_\mathrm{S}) = f_\mathrm{J}\circ\mathrm{pr}_\mathrm{J}\,; \tag{16}$$

here pr_J is the J'th projection in the context of

$$\mathrm{X} = \prod_{\lambda\in\mathrm{L}} \mathrm{X}_\lambda = \mathrm{X}_\mathrm{J} \times \prod_{\lambda\in\mathrm{L}-\mathrm{J}} \mathrm{X}_\lambda\,,$$

thus

$$g_\mathrm{S}\circ\mathrm{pr}_\mathrm{S} = f_\mathrm{J}\circ\mathrm{pr}_\mathrm{J} \tag{17}$$

on the space $\prod_{\rho\in\mathrm{R}} \mathrm{Y}_\rho = \mathrm{X} = \prod_{\lambda\in\mathrm{L}} \mathrm{X}_\lambda$ (see (5) and (10)); this is the link between the presentations ν and μ of the measure on X. Then

$$\begin{aligned} \nu(f_\mathrm{J}\circ\mathrm{pr}_\mathrm{J}) &= \nu(g_\mathrm{S}\circ\mathrm{pr}_\mathrm{S}) && \text{(by (17))} \\ &= \nu_\mathrm{S}(g_\mathrm{S}) \prod_{\rho\in\mathrm{R}-\mathrm{S}} \nu_\rho(1) && \text{(by (7))}\,. \end{aligned} \tag{18}$$

Consider again the decomposition $\mathrm{J} = \bigcup_{\rho\in\mathrm{S}} \mathrm{J}_\rho$ of (14). Let $s = \operatorname{card}\mathrm{S}$, $j = \operatorname{card}\mathrm{J}$. Since each J_ρ contains at least one element of J, one has $s \leqslant j$. Let us index S as follows:

$$\mathrm{S} = \{\rho_1, \dots, \rho_s\} \qquad (s = \operatorname{card}\mathrm{S}).$$

Then J is the disjoint union

$$\mathrm{J} = \mathrm{J}_{\rho_1} \cup \cdots \cup \mathrm{J}_{\rho_s},$$

which yields a factorization

$$\begin{aligned} \mathrm{X}_\mathrm{J} &= \mathrm{X}_{\mathrm{J}_{\rho_1}} \times \cdots \times \mathrm{X}_{\mathrm{J}_{\rho_s}} = \prod_{i=1}^{s} \mathrm{X}_{\mathrm{J}_{\rho_i}} \\ &= \prod_{\rho\in\mathrm{S}} \Big(\prod_{\lambda\in\mathrm{J}_\rho} \mathrm{X}_\lambda \Big) = \prod_{\rho\in\mathrm{S}} \mathrm{X}_{\mathrm{J}_\rho}. \end{aligned} \tag{19}$$

This is the crucial step; functions on X_J ostensibly of j variables $x_\lambda \in \mathrm{X}_\lambda$ $(\lambda \in \mathrm{J})$ in the context of $\mathrm{X} = \prod_{\lambda\in\mathrm{L}} \mathrm{X}_\lambda$, may be regarded as functions of s variables $y_\rho \in \mathrm{X}_{\mathrm{J}_\rho} = \prod_{\lambda\in\mathrm{J}_\rho} \mathrm{X}_\lambda = \Big(\prod_{\lambda\in\mathrm{L}_\rho} \mathrm{X}_\lambda \Big)_{\mathrm{J}_p} = (\mathrm{Y}_\rho)_{\mathrm{J}_\rho}$ $(\rho \in \mathrm{S})$, where the last equality is in the context of $\mathrm{Y}_\rho = \prod_{\lambda\in\mathrm{L}_\rho} \mathrm{X}_\lambda$ and $\mathrm{J}_\rho \subset \mathrm{L}_\rho$.

Among the functions in $\mathscr{C}(\mathrm{X}_\mathrm{J}; \mathbf{C}) = \mathscr{C}\Big(\prod_{\rho\in\mathrm{S}} \mathrm{X}_{\mathrm{J}_\rho}; \mathbf{C} \Big)$ are the "elementary functions"

$$f_\mathrm{J} = \bigotimes_{\rho\in\mathrm{S}} f_{\mathrm{J}_\rho} \qquad \big(\rho \in \mathrm{S},\ f_{\mathrm{J}_\rho} \in \mathscr{C}(\mathrm{X}_{\mathrm{J}_\rho}; \mathbf{C}) \big), \tag{20}$$

which form a total set of continuous functions on X_J (GT, X, §4, No. 3, Th. 4); such a function acts on a point

$$x = (x_\lambda)_{\lambda\in\mathrm{J}} = \big((x_\lambda)_{\lambda\in\mathrm{J}_\rho} \big)_{\rho\in\mathrm{S}} \in \mathrm{X}_\mathrm{J} = \prod_{\rho\in\mathrm{S}} \mathrm{X}_{\mathrm{J}_\rho}$$

as follows:

$$f_\mathrm{J}(x) = \prod_{\rho\in\mathrm{S}} f_{\mathrm{J}_\rho}\big((x_\lambda)_{\lambda\in\mathrm{J}_\rho} \big).$$

We now construct a function $g_\mathrm{S} \in \mathscr{C}(\mathrm{Y}_\mathrm{S}; \mathbf{C})$ such that (15) holds, that is, $f_\mathrm{J} \circ \overline{\mathrm{pr}}_\mathrm{J} = g_\mathrm{S}$, where $\overline{\mathrm{pr}}_\mathrm{J}$ is the projection $\mathrm{Y}_\mathrm{S} \to \mathrm{X}_\mathrm{J}$ described there, and f_J is the function defined in (20).

For each $\rho \in \mathrm{S}$ define a function $g_\rho \in \mathscr{C}(\mathrm{Y}_\rho; \mathbf{C})$ by the formula

$$(21) \qquad g_\rho = f_{\mathrm{J}_\rho} \circ \overline{\mathrm{pr}}_{\mathrm{J}_\rho} ,$$

where $\overline{\mathrm{pr}}_{\mathrm{J}_\rho}$ is the projection $\mathrm{Y}_\rho \to \mathrm{X}_{\mathrm{J}_\rho}$ in the context of

$$\mathrm{Y}_\rho = \prod_{\lambda \in \mathrm{L}_\rho} \mathrm{X}_\lambda = \prod_{\lambda \in \mathrm{J}_\rho} \mathrm{X}_\lambda \times \prod_{\lambda \in \mathrm{L}_\rho - \mathrm{J}_\rho} \mathrm{X}_\lambda = \mathrm{X}_{\mathrm{J}_\rho} \times \prod_{\lambda \in \mathrm{L}_\rho - \mathrm{J}_\rho} \mathrm{X}_\lambda .$$

Define $g_\mathrm{S} \in \mathscr{C}(\mathrm{Y}_\mathrm{S}; \mathbf{C}) = \mathscr{C}(\prod_{\rho \in \mathrm{S}} \mathrm{Y}_\rho; \mathbf{C})$ by the formula

$$(22) \qquad g_\mathrm{S} = \bigotimes_{\rho \in \mathrm{S}} g_\rho .$$

We now show that $f_\mathrm{J} \circ \overline{\mathrm{pr}}_\mathrm{J} = g_\mathrm{S}$, where the projection $\overline{\mathrm{pr}}_\mathrm{J} : \mathrm{Y}_\mathrm{S} \to \mathrm{X}_\mathrm{J}$ has been described in connection with (15). Let $y \in \mathrm{Y}_\mathrm{S} = \prod_{\rho \in \mathrm{S}} \mathrm{Y}_\rho$, say $y = (y_\rho)_{\rho \in \mathrm{S}}$, where, for $\rho \in \mathrm{S}$, $y_\rho \in \mathrm{Y}_\rho$ and $y_\rho = (x_\lambda)_{\lambda \in \mathrm{L}_\rho}$ with $x_\lambda \in \mathrm{X}_\lambda$ for $\lambda \in \mathrm{L}_\rho$. Then (recall that $\mathrm{J} = \bigcup_{\rho \in \mathrm{S}} \mathrm{J}_\rho$ with $\mathrm{J}_\rho \subset \mathrm{L}_\rho$)

$$(23) \qquad \overline{\mathrm{pr}}_\mathrm{J}\, y = (x_\lambda)_{\lambda \in \mathrm{J}} = \big((x_\lambda)_{\lambda \in \mathrm{J}_\rho}\big)_{\rho \in \mathrm{S}} \in \prod_{\rho \in \mathrm{S}} \mathrm{X}_{\mathrm{J}_\rho} = \mathrm{X}_\mathrm{J} ,$$

therefore

$$(24) \qquad f_\mathrm{J}(\overline{\mathrm{pr}}_\mathrm{J}\, y) = \Big(\bigotimes_{\rho \in \mathrm{S}} f_{\mathrm{J}_\rho}\Big)\Big(\big((x_\lambda)_{\lambda \in \mathrm{J}_\rho}\big)_{\rho \in \mathrm{S}}\Big) = \prod_{\rho \in \mathrm{S}} f_{\mathrm{J}_\rho}\big((x_\lambda)_{\lambda \in \mathrm{J}_\rho}\big);$$

but, for $\rho \in \mathrm{S}$, by (21) one has

$$g_\rho(y_\rho) = f_{\mathrm{J}_\rho}(\overline{\mathrm{pr}}_{\mathrm{J}_\rho}\, y_\rho) = f_{\mathrm{J}_\rho}\big((x_\lambda)_{\lambda \in \mathrm{J}_\rho}\big) ,$$

whence, by (24),

$$f_\mathrm{J}(\overline{\mathrm{pr}}_\mathrm{J}\, y) = \prod_{\rho \in \mathrm{S}} g_\rho(y_\rho) = \Big(\bigotimes_{\rho \in \mathrm{S}} g_\rho\Big)\big((y_\rho)_{\rho \in \mathrm{S}}\big) = g_\mathrm{S}(y) ,$$

and so

$$(25) \qquad g_\mathrm{S} = f_\mathrm{J} \circ \overline{\mathrm{pr}}_\mathrm{J} \in \mathscr{C}(\mathrm{Y}_\mathrm{S}; \mathbf{C})$$

(that is, (15) holds), in other words, taking into account (20) and (22),

$$\bigotimes_{\rho\in\mathrm{S}} g_\rho = \Big(\bigotimes_{\rho\in\mathrm{S}} f_{\mathrm{J}_\rho}\Big) \circ \overline{\mathrm{pr}}_{\mathrm{J}}\,. \tag{26}$$

We now consider $\nu_{\mathrm{S}} = \bigotimes_{\rho\in\mathrm{S}} \nu_\rho \in \mathscr{M}(\prod_{\rho\in\mathrm{S}} \mathrm{Y}_\rho;\mathbf{C}) = \mathscr{M}(\mathrm{Y}_{\mathrm{S}};\mathbf{C})$ (see (6)) and apply it to the function $g_{\mathrm{S}} \in \mathscr{C}(\mathrm{Y}_{\mathrm{S}};\mathbf{C}) = \mathscr{C}(\prod_{\rho\in\mathrm{S}} \mathrm{Y}_\rho;\mathbf{C})$:

$$\begin{aligned}
\nu_{\mathrm{S}}(g_{\mathrm{S}}) &= \Big(\bigotimes_{\rho\in\mathrm{S}} \nu_\rho\Big)\Big(\bigotimes_{\rho\in\mathrm{S}} g_\rho\Big) && \big(\text{by (6) and (22)}\big)\\
&= \prod_{\rho\in\mathrm{S}} \nu_\rho(g_\rho) && \big(\text{by (12) of No. 4}\big)\\
&= \prod_{\rho\in\mathrm{S}} \nu_\rho\big(f_{\mathrm{J}_\rho}\circ \overline{\mathrm{pr}}_{\mathrm{J}_\rho}\big) && \big(\text{by (21)}\big)\\
&= \prod_{\rho\in\mathrm{S}} \Big[\mu_{\mathrm{J}_\rho}\big(f_{\mathrm{J}_\rho}\big)\cdot \prod_{\lambda\in\mathrm{L}_\rho - \mathrm{J}_\rho} \mu_\lambda(1)\Big] && \big(\text{by (3)}\big)\\
&= \Big[\prod_{\rho\in\mathrm{S}} \mu_{\mathrm{J}_\rho}\big(f_{\mathrm{J}_\rho}\big)\Big]\cdot\Big[\prod_{\rho\in\mathrm{S}}\Big(\prod_{\lambda\in\mathrm{L}_\rho - \mathrm{J}_\rho} \mu_\lambda(1)\Big)\Big]\\
&= \Big[\Big(\bigotimes_{\rho\in\mathrm{S}} \mu_{\mathrm{J}_\rho}\Big)\Big(\bigotimes_{\rho\in\mathrm{S}} f_{\mathrm{J}_\rho}\Big)\Big]\cdot\Big[\prod_{\rho\in\mathrm{S}}\Big(\prod_{\lambda\in\mathrm{L}_\rho - \mathrm{J}_\rho} \mu_\lambda(1)\Big)\Big]\\
&= \mu_{\mathrm{J}}(f_{\mathrm{J}}) \prod_{\lambda\in\mathrm{M}-\mathrm{J}} \mu_\lambda(1)\,,
\end{aligned}$$

where $\bigotimes_{\rho\in\mathrm{S}} \mu_{\mathrm{J}_\rho} = \bigotimes_{\rho\in\mathrm{S}}\big(\bigotimes_{\lambda\in\mathrm{J}_\rho}\mu_\lambda\big) = \mu_{\mathrm{J}}$ by No. 4, Prop. 7, and where $\mathrm{M} = \bigcup_{\rho\in\mathrm{S}} \mathrm{L}_\rho$, so that by disjointness one has

$$\mathrm{M} - \mathrm{J} = \bigcup_{\rho\in\mathrm{S}} \mathrm{L}_\rho - \bigcup_{\rho\in\mathrm{S}} \mathrm{J}_\rho = \bigcup_{\rho\in\mathrm{S}} (\mathrm{L}_\rho - \mathrm{J}_\rho)\,, \tag{27}$$

and the previous display may be abbreviated

$$\nu_{\mathrm{S}}(g_{\mathrm{S}}) = \mu_{\mathrm{J}}(f_{\mathrm{J}}) \prod_{\lambda\in\mathrm{M}-\mathrm{J}} \mu_\lambda(1)\,. \tag{28}$$

Noting that (15) has been verified, and substituting (28) into (18), we then have (for the "elementary functions" $f_{\rm J}$)

$$
\begin{aligned}
\nu(f_{\rm J}\circ {\rm pr_J}) &= \nu_{\rm S}(g_{\rm S}) \prod_{\rho\in {\rm R-S}} \nu_\rho(1) \\
&= \Big[\mu_{\rm J}(f_{\rm J}) \prod_{\lambda\in {\rm M-J}} \mu_\lambda(1)\Big] \cdot \prod_{\rho\in{\rm R-S}} \nu_\rho(1) \qquad (29)\\
&= \Big[\mu_{\rm J}(f_{\rm J}) \prod_{\lambda\in {\rm M-J}} \mu_\lambda(1)\Big] \cdot \Big[\prod_{\rho\in{\rm R-S}} \Big(\prod_{\lambda\in {\rm L}_\rho} \mu_\lambda(1)\Big)\Big],
\end{aligned}
$$

the second and third equalities by (28) and (4), respectively; the indices λ for which $\mu_\lambda(1)$ occurs as a factor in (29) are those in

$$
\begin{aligned}
({\rm M-J}) \cup \Big(\bigcup_{\rho\in{\rm R-S}} {\rm L}_\rho\Big) &= \Big(\bigcup_{\rho\in{\rm S}} {\rm L}_\rho - \bigcup_{\rho\in{\rm S}} {\rm J}_\rho\Big) \cup \Big(\bigcup_{\rho\in{\rm R-S}} {\rm L}_\rho\Big) \quad \text{(by (27))}\\
&= \Big(\bigcup_{\rho\in{\rm S}} {\rm L}_\rho \cup \bigcup_{\rho\in{\rm R-S}} {\rm L}_\rho\Big) - \bigcup_{\rho\in{\rm R}} {\rm J}_\rho \\
&= \Big(\bigcup_{\rho\in{\rm R}} {\rm L}_\rho\Big) - {\rm J} \\
&= {\rm L - J},
\end{aligned}
$$

whence (29) becomes

$$
(30) \qquad \nu(f_{\rm J}\circ {\rm pr_J}) = \mu_{\rm J}(f_{\rm J}) \prod_{\lambda\in{\rm L-J}} \mu_\lambda(1) = \mu(f_{\rm J}\circ{\rm pr_J}) \qquad \text{(by (11))}.
$$

We have thus shown that for every finite ${\rm J} \subset {\rm L}$, and, behind the scenes, the finite ${\rm S}\subset{\rm R}$ chosen preparatory to (14), one has

$$
\nu(f_{\rm J}\circ {\rm pr_J}) = \mu(f_{\rm J}\circ{\rm pr_J})
$$

for all $f_{\rm J}$ running over a suitable set of "elementary functions" total in $\mathscr{C}({\rm X_J};\mathbf{C})$, hence

$$
(*) \qquad \nu(f\circ{\rm pr_J}) = \mu(f\circ{\rm pr_J})
$$

for all finite linear combinations of such $f_{\rm J}$'s. If now $f \in \mathscr{C}({\rm X_J};\mathbf{C})$ is arbitrary, it is the uniform limit of a sequence f_n of such linear combinations. whence

$$
\|f_n\circ{\rm pr_J} - f\circ{\rm pr_J}\| = \|(f_n - f)\circ{\rm pr_J}\| \leqslant \|f_n - f\| \to 0,
$$

consequently (*) holds for all $f\in\mathscr{C}({\rm X_J};\mathbf{C})$ by the continuity of ν and μ. Such functions

$$
f\circ{\rm pr_J} \qquad \big({\rm J}\subset{\rm L} \text{ finite, } f\in\mathscr{C}({\rm X_J};\mathbf{C})\big)
$$

form a dense linear subspace of $\mathscr{C}({\rm X};\mathbf{C})$ by Lemma 3 of No. 5, therefore $\nu = \mu$ on $\mathscr{C}({\rm X};\mathbf{C})$ by continuity.

Ouf!

CHAPTER IV

Extension of a measure. L^p spaces

§1. UPPER INTEGRAL OF A POSITIVE FUNCTION

IV.1, ℓ. −6, −5.

"*Lemma. — Every function* $f \in \mathscr{I}_+$ *is the upper envelope of the set (directed for the relation* $\leqslant$ *) of all functions* $g \in \mathscr{K}_+$ *such that* $g \leqslant f$ *.*"

Let $\mathscr{G}$ be the set of all $g \in \mathscr{K}_+$ such that $g \leqslant f$ (for example, $0 \in \mathscr{G}$) and let $g_0 = \sup \mathscr{G}$ be its upper envelope, $g_0(x) = \sup_{g \in \mathscr{G}} g(x)$ for all $x \in \mathrm{X}$. Since $g, g' \in \mathscr{G} \Rightarrow \sup(g, g') \in \mathscr{G}$, $\mathscr{G}$ is directed upward for $\leqslant$.

It is obvious that $g_0 \leqslant f$; we are to show that $g_0 = f$. Fix $x \in \mathrm{X}$ and let us show that $g_0(x) \geqslant f(x)$. Given any $a \in \mathbf{R}$ with $f(x) > a$ it will suffice to find $g \in \mathscr{G}$ such that $g(x) \geqslant a$, for then $g_0(x) \geqslant g(x) \geqslant a$ and letting $a \to f(x)$ yields $g_0(x) \geqslant f(x)$. If $a \leqslant 0$ then $g = 0$ satisfies $g(x) = 0 \geqslant a$ as desired; so we can suppose that $a > 0$.

Since $0 < a < f(x)$ it follows from lower semi-continuity that there exists a neighborhood V of x such that $f > a$ on V (GT, IV, §6, No. 2, Def. 1) and we can suppose that V is compact. It follows from GT, IX, §1, No. 5, Th. 2 that there exists a continuous function $u : \mathrm{X} \to [0, 1]$ such that $u(x) = 0$ and $u = 1$ on $\complement \mathrm{V}$; then $g = a \cdot (1 - u)$ satisfies $0 \leqslant g(y) \leqslant a$ for all $y \in \mathrm{X}$, $g(x) = a$, and $g = 0$ on $\complement \mathrm{V}$. Since V is closed, $\operatorname{Supp} g \subset \mathrm{V}$, and since V is compact, $g \in \mathscr{K}_+$. Finally $g \in \mathscr{G}$; for, if $y \in \mathrm{V}$ then $g(y) \leqslant a < f(y)$ (recall that $f > a$ on V) and if $y \in \complement \mathrm{V}$ then $g(y) = 0 \leqslant f(y)$. Thus the requirements $g \in \mathscr{G}$ and $g(x) \geqslant a$ are fulfilled.

Special case: Suppose $f = \varphi_{\mathrm{U}}$ is the characteristic function of an open set U of X (lower semi-continuous by GT, IV, §6, No. 2, Cor. of Prop. 1). In the foregoing proof, $0 < a < f(x)$ implies $x \in \mathrm{U}$, so one can suppose in addition that $\mathrm{V} \subset \mathrm{U}$; then the constructed function g belongs to $\mathscr{K}_+(\mathrm{X}, \mathrm{U}; \mathbf{C})$, thus

$$\varphi_{\mathrm{U}} = \sup\{g \in \mathscr{K}_+ : \ g \leqslant \varphi_{\mathrm{U}} \, , \ \operatorname{Supp} g \subset \mathrm{U}\} \, .$$

IV.2, ℓ. −13.

"... which again proves the relation $\mu^*(f) = \sum_{x\in \mathrm{X}} \alpha(x)f(x)$."

Recall that $\mu \in \mathscr{M}_+(\mathrm{X})$, X a discrete space, $\mathscr{K}_+$ is equal to the linear span of the characteristic functions $\varphi_{\{x\}}$ $(x \in \mathrm{X})$ with finite real coefficients $\geqslant 0$, and $\alpha(x) = \mu(\varphi_{\{x\}})$ $(x \in \mathrm{X})$; thus every $g \in \mathscr{K}_+$ has the form

$$g = \sum_{x\in \mathrm{M}} g(x)\varphi_{\{x\}}$$

with M a finite subset of X, and one has

$$\mu(g) = \sum_{x\in \mathrm{M}} \alpha(x)g(x)\,.$$

For a fixed function $f : \mathrm{X} \to \overline{\mathbf{R}}$ with $f \geqslant 0$, we wish to show that

$$(*) \qquad \mu^*(f) = \sum_{x\in \mathrm{X}} \alpha(x)f(x)\,,$$

where, by convention, $\alpha(x)f(x) = 0$ when $\alpha(x) = 0$ and $f(x) = +\infty$. Let

$$\mathrm{D} = \{x \in \mathrm{X} :\ \alpha(x) = 0 \ \text{and}\ f(x) = +\infty\,\}\,;$$

thus $x \in \mathrm{D}$ implies $\alpha(x)f(x) = 0$, and the right side S of (*) may be written

$$\mathrm{S} = \sum_{x\in \mathrm{X}-\mathrm{D}} \alpha(x)f(x)\,.$$

We are to show that $\mu^*(f) = \mathrm{S}$. By definition,

$$\mu^*(f) = \sup_{g\in\mathscr{K}_+,\, g\leqslant f} \mu(g)\,.$$

Let

$$\mathrm{A} = \{x \in \mathrm{X} :\ f(x) = +\infty\,\}\,.$$

case 1: There exists an $x_0 \in \mathrm{A}$ such that $\alpha(x_0) > 0$.

Then for every positive integer n one has $0 < n < +\infty = f(x_0)$, so that the function $g_n = n \cdot \varphi_{\{x_0\}}$ satisfies $g_n \in \mathscr{K}_+$, $0 \leqslant g_n \leqslant f$, and the sequence $\mu(g_n) = n \cdot \alpha(x_n)$ is unbounded, thus $\mu(g_n) \leqslant \mu^*(f)$ shows that $\mu^*(f) = +\infty$. On the other hand, $\alpha(x_0)f(x_0) = +\infty$ is a term on the right side of (*), therefore $\mathrm{S} = +\infty$ and the equation (*) holds.

case 2: $\alpha(x) = 0$ for all $x \in \mathrm{A}$.

Then $\mathrm{D} = \mathrm{A}$, so that

$$\mathrm{S} = \sum_{x \in \mathrm{X} - \mathrm{A}} \alpha(x) f(x),$$

where, for every $x \in \mathrm{X} - \mathrm{A}$, both $\alpha(x)$ and $f(x)$ are finite.

Let us first show that $\mathrm{S} \leqslant \mu^*(f)$. Let M be any finite subset of $\mathrm{X} - \mathrm{A}$; then $g = \sum_{x \in \mathrm{M}} f(x)\varphi_{\{x\}}$ belongs to $\mathscr{K}_+$ and $g \leqslant f$, therefore $\mu(g) \leqslant \mu^*(f)$, that is,

$$\sum_{x \in \mathrm{M}} \alpha(x) f(x) \leqslant \mu^*(f);$$

the validity of this inequality for every finite $\mathrm{M} \subset \mathrm{X} - \mathrm{A}$ shows that $\mathrm{S} \leqslant \mu^*(f)$.

To prove the reverse inequality, let $g \in \mathscr{K}_+$ be such that $g \leqslant f$ and let us show that $\mu(g) \leqslant \mathrm{S}$. Let M be the support of g, so that

$$g = \sum_{x \in \mathrm{M}} g(x)\varphi_{\{x\}} \quad \text{and} \quad \mu(g) = \sum_{x \in \mathrm{M}} \alpha(x) g(x).$$

If $x \in \mathrm{M} \cap \mathrm{A}$ then $f(x) = +\infty$ and $\alpha(x) = 0$ by assumption, so that $\alpha(x) f(x) = 0$ by convention, consequently (recalling that $g \leqslant f$)

$$\mu(g) = \sum_{x \in \mathrm{M}} \alpha(x) g(x) \leqslant \sum_{x \in \mathrm{M}} \alpha(x) f(x) = \sum_{x \in \mathrm{M} - \mathrm{A}} \alpha(x) f(x) \leqslant \mathrm{S}$$

(the last inequality because $\mathrm{M} - \mathrm{A} \subset \mathrm{X} - \mathrm{A} \subset \mathrm{X} - \mathrm{D}$).

IV.3, ℓ. 11, 12.

"... since H is directed, so is Φ, and $f = \sup_{\varphi \in \Phi} \varphi$."

If $g, g' \in \mathrm{H}$ with $g \leqslant g'$, clearly $\Phi_g \subset \Phi_{g'}$. Suppose $\varphi_1, \varphi_2 \in \Phi$, say $\varphi_1 \in \Phi_{g_1}$, $\varphi_2 \in \Phi_{g_2}$. Choose $g \in \mathrm{H}$ so that $g_1 \leqslant g$ and $g_2 \leqslant g$. Then $\varphi_k \in \Phi_{g_k} \subset \Phi_g$ for $k = 1, 2$, therefore $\sup(\varphi_1, \varphi_2) \in \Phi_g \subset \Phi$, thus Φ is directed upward for $\leqslant$. The assertion that

$$\sup_{g \in \mathrm{H}} g = \sup_{\varphi \in \Phi} \varphi,$$

that is (in view of the *Lemma*)

$$\sup_{g \in \mathrm{H}} \Big(\sup_{\varphi \in \Phi_g} \varphi \Big) = \sup_{\varphi \in \bigcup_{g \in \mathrm{H}} \Phi_g} \varphi,$$

is an application of the following result (a straightforward consequence of the definition of supremum):

Let E *be an ordered* (i.e., 'partially ordered') *space in which every nonempty subset has a supremum* (for example $E = \overline{\mathbf{R}}$, $[0, +\infty]$ or $\overline{\mathbf{R}}^{T}$, T any set). *If* $(x_i)_{i \in I}$ *is a family of elements of* E *and if* $(I_j)_{j \in J}$ *is a family of nonempty subsets of* I *such that* $I = \bigcup_{j \in J} I_j$, *then*

$$\sup_{i \in I} x_i = \sup_{j \in J} \Big(\sup_{i \in J_i} x_i \Big)$$

('associativity of sups'; S, III, §1, No. 9, Prop. 7).

IV.3, ℓ. 12, 13.

"Since $\psi \leqslant f$, ψ is the upper envelope of the set of functions $\inf(\psi, \varphi)$ as φ runs over Φ ..."

Since $\inf(\psi, \varphi) \leqslant \psi$ for all $\varphi \in \Phi$, obviously

(*) $$\sup_{\varphi \in \Phi} \big(\inf(\psi, \varphi) \big) \leqslant \psi .$$

Given any $x \in X$, the problem is to show that

(†) $$\sup_{\varphi \in \Phi} \Big(\min \big(\psi(x), \varphi(x) \big) \Big) = \psi(x) .$$

case 1: $\psi(x) = f(x)$.

Then $\min\big(\psi(x), \varphi(x)\big) = \min\big(f(x), \varphi(x)\big) = \varphi(x)$ for all $\varphi \in \Phi$, and one knows that $\sup_{\varphi \in \Phi} \varphi(x) = f(x)$, thus (†) reduces to $f(x) = \psi(x)$.

case 2: $\psi(x) < f(x)$.

Since $\sup_{\varphi \in \Phi} \varphi = f$, there exists $\varphi_0 \in \Phi$ such that $\psi(x) < \varphi_0(x) \leqslant f(x)$. Then $\min\big(\psi(x), \varphi_0(x)\big) = \psi(x)$, consequently

$$\sup_{\varphi \in \Phi} \Big(\min \big(\psi(x), \varphi(x) \big) \Big) \geqslant \min \big(\psi(x), \varphi_0(x) \big) = \psi(x) ,$$

whence equality by (*).

IV.4, ℓ. −7, −6.

"Moreover, $\mu^*(X) = \|\mu\|$, as is shown by formula (22) of Ch. III, §1, No. 8."

The cited formula reads $\|\mu\| = \sup_{0 \leqslant f \leqslant 1,\, f \in \mathscr{K}(X)} |\mu|(f)$ for *any* measure μ; when $\mu \geqslant 0$ this simplifies to

$$\|\mu\| = \sup_{0 \leqslant f \leqslant 1,\, f \in \mathscr{K}(X)} \mu(f) = \mu^*(1) = \mu^*(\varphi_X) = \mu^*(X)$$

by the definitions.

IV.11, ℓ. 5, 6.

"These propositions are the translations of Props. 10 and 13 and of Th. 3 of No. 3 for characteristic functions of sets."

Let $\mathrm{A} = \bigcup_n \mathrm{A}_n$. Then $\varphi_{\mathrm{A}} \leqslant \sum_n \varphi_{\mathrm{A}_n}$, hence

$$\mu^*(\varphi_{\mathrm{A}}) \leqslant \mu^*\Big(\sum_n \varphi_{\mathrm{A}_n}\Big) \leqslant \sum_n \mu^*(\varphi_{\mathrm{A}_n})$$

by Props. 10 and 13, whence Prop. 18. If, moreover, (A_n) is increasing, then (φ_{A_n}) is increasing and $\varphi_{\mathrm{A}} = \sup_n \varphi_{\mathrm{A}_n}$, therefore $\mu^*(\varphi_{\mathrm{A}}) = \sup_n \mu^*(\varphi_{\mathrm{A}_n})$ by Th. 3.

§2. NEGLIGIBLE FUNCTIONS AND SETS

IV.12, ℓ. −7, −6.

"For, by Prop. 3, in order that an open set G be negligible, it is necessary and sufficient that $\mathrm{G} \cap \mathrm{S} = \varnothing$, that is, $\mathrm{G} \subset \complement \mathrm{S}$."

For G open, write $f = \varphi_{\mathrm{G}}$, which is lower semi-continuous and $\geqslant 0$. Then

$$\begin{aligned} |\mu|^*(f) = 0 \quad &\Leftrightarrow \quad f = 0 \text{ on } \mathrm{S} \qquad \text{(Prop. 3)} \\ &\Leftrightarrow \quad \mathrm{S} \subset \complement \mathrm{G} \\ &\Leftrightarrow \quad \mathrm{G} \subset \complement \mathrm{S} ; \end{aligned}$$

and, since $\complement \mathrm{S}$ is open, it is eligible to play the role of G .

IV.13, ℓ. 5, 6.

"... therefore φ_{N} is negligible (No. 1, Props. 1 and 2)."

Each nf is negligible (Prop. 1), hence $\sup_n nf$ is negligible (Prop. 2), hence φ_{N} is negligible (Prop. 1), thus N is negligible. If $\boldsymbol{P}\,\{x\}$ is the property « $f(x) = 0$ », then $\mathrm{N} = \{x \in \mathrm{X} : \text{ not } \boldsymbol{P}\,\{x\}\}$; thus the negligibility of N means that $\boldsymbol{P}\,\{x\}$ almost everywhere, that is, $f(x) = 0$ almost everywhere.

IV.13, ℓ. 8, 9.

"... therefore f is negligible (No. 1, Props. 1 and 2)."

Each $n\varphi_{\mathrm{N}}$ is negligible (Prop. 1), hence $\sup_n n\varphi_{\mathrm{N}}$ is negligible (Prop. 2), hence f is negligible (Prop. 1).

IV.14, ℓ. 10, 11.

"Corollary. — *If* φ *is a mapping of* $\prod_n F_n$ *into a set* G, *then the mappings* $\varphi\big((f_n)\big)$ *and* $\varphi\big((g_n)\big)$ *of* X *into* G *are equivalent.*"

If $x \in X$ then $\big(f_n(x)\big) \in \prod_n F_n$. The mapping $\varphi\big((f_n)\big) : X \to G$ of the Corollary is the composite function $X \to \prod_n F_n \to G$ defined by

$$x \mapsto \big(f_n(x)\big) \mapsto \varphi\Big(\big(f_n(x)\big)\Big);$$

let us abbreviate it $u = \varphi\big((f_n)\big)$, thus

$$u(x) = \varphi\Big(\big(f_n(x)\big)\Big) \qquad (x \in X).$$

Similarly, let $v = \varphi\big((g_n)\big)$ be the function

$$v(x) = \varphi\Big(\big(g_n(x)\big)\Big) \qquad (x \in X).$$

Let H be a negligible subset of X as described in Prop. 8. If $x \notin H$ then $f_n(x) = g_n(x)$ for all n, therefore $\big(f_n(x)\big) = \big(g_n(x)\big)$ as elements of $\prod_n F_n$, consequently $\varphi\Big(\big(f_n(x)\big)\Big) = \varphi\Big(\big(g_n(x)\big)\Big)$, that is, $u(x) = v(x)$; thus $u = v$ almost everywhere, that is, u and v are equivalent, i.e., $\varphi\big((f_n)\big)$ and $\varphi\big((g_n)\big)$ are equivalent.

The assumption on the f_n, g_n is that $\widetilde{f}_n = \widetilde{g}_n$ for all n; the conclusion of the corollary is that the classes of $\varphi\big((f_n)\big)$ and $\varphi\big((g_n)\big)$ are equal. The class of $\varphi\big((f_n)\big)$ is denoted $\varphi\big((\widetilde{f}_n)\big)$; that is, $\varphi\big((\widetilde{f}_n)\big)$ is defined to be the class $\widetilde{\varphi\big((f_n)\big)}$. Thus, the message of the corollary is that if $\widetilde{f}_n = \widetilde{g}_n$ for all n, then $\varphi\big((\widetilde{f}_n)\big) = \varphi\big((\widetilde{g}_n)\big)$.

Another way of viewing the conclusion is that the *class* of $\varphi\big((f_n)\big)$ is unchanged if each f_n is replaced by a function equivalent to it.

IV.14, ℓ. −7 to −5.

"...for, this is equivalent to saying that there exists a negligible set H in X such that, for every $x \notin H$, $\rho_n(f(x), g(x)) = 0$ for all n, that is, $f(x) = g(x)$."

To put it another way, from

$$\{x : f(x) = g(x)\} = \bigcap_n \{x : \rho_n\big(f(x), g(x)\big) = 0\},$$

we have $\{x:\ f(x) \neq g(x)\} = \bigcup_n \{x:\ \rho_n\big(f(x), g(x)\big) \neq 0\}$, and the set on the left is negligible if and only if each term of the union is negligible.

IV.15, ℓ. 5, 6.

"... for it to be negligible, it is necessary and sufficient that it not intersect the support of μ (No. 2, Prop. 5)."

Write $\mathrm{G} = \{x:\ f(x) \neq g(x)\}$, an open subset of X since since F is Hausdorff (GT, I, §8, No. 1, Prop. 2). Then

$$\begin{aligned} f(x) = g(x) \ \text{ a.e.} \quad &\Leftrightarrow \quad \mathrm{G} \text{ negligible} \\ &\Leftrightarrow \quad \mathrm{G} \subset \complement\,\mathrm{Supp}(\mu) \quad \text{(No. 2, Prop. 5)} \\ &\Leftrightarrow \quad \mathrm{Supp}(\mu) \subset \complement\,\mathrm{G} = \{x:\ f(x) = g(x)\} \\ &\Leftrightarrow \quad f(x) = g(x) \ \text{ for all } x \in \mathrm{Supp}(\mu)\,. \end{aligned}$$

IV.17, ℓ. −3 to −1.

"For f to be negligible, it is necessary and sufficient that $|f|$ be negligible (or that both f^+ and f^- be negligible)."

If $f(x)$ is defined, one defines $|f|(x) = |f(x)|$, $f^+(x) = \sup\big(f(x), 0\big)$, and $f^-(x) = \sup\big(-f(x), 0\big) = -\inf\big(f(x), 0\big)$. Thus f, $|f|$, f^+, f^- have the same domain of definition; $f^+(x)$ and $f^-(x)$ can't both be equal to $+\infty$, hence $f = f^+ - f^-$; and $|f| = f^+ + f^-$. The assertions about negligibility are then clear.

§3. L^p SPACES

IV.19, ℓ. 20–24.

"We extend Def. 1 to the case of numerical functions, *finite or not*, defined on X, by again setting

$$\mathrm{N}_p(f) = \left(\int^* |f|^p \, d|\mu|\right)^{1/p}$$

for such a function f. One sees immediately that the relations (3) and (4) also hold for these functions when $f + g$ is defined on X and $\alpha \neq 0$."

It is convenient to simultaneously treat the proof of Prop. 2.

If $f, g : \mathrm{X} \to \overline{\mathbf{R}}$ and $|f| \leqslant |g|$ almost everywhere in X, then $|f|^p \leqslant |g|^p$ almost everywhere, therefore $|\mu|^*(|f|^p) \leqslant |\mu|^*(|g|^p)$ by §2, No. 3, Prop. 6 and §1, No. 3, Prop. 10, hence $\big(|\mu|^*(|f|^p)\big)^{1/p} \leqslant \big(|\mu|^*(|g|^p)\big)^{1/p}$; thus

(i) $$|f| \leqslant |g| \ \text{ a.e.} \quad \Rightarrow \quad \mathrm{N}_p(f) \leqslant \mathrm{N}_p(g)\,.$$

Incidentally, $N_p(\alpha f) = |\alpha| \cdot N_p(f)$ even if $\alpha = 0$, with the convention $0 \cdot (+\infty) = 0$.

Note that if $\mathbf{f} : X \to F$ (F a Banach space, real or complex) then, by definition,

(ii) $$N_p(\mathbf{f}) = N_p(|\mathbf{f}|)\,.$$

It follows from (i) and (ii) that if $\mathbf{f}, \mathbf{g} : X \to F$ and $|\mathbf{f}| \leqslant |\mathbf{g}|$ a.e., then

(iii) $$N_p(\mathbf{f}) \leqslant N_p(\mathbf{g})\,.$$

If $f, g : X \to \overline{\mathbf{R}}$ and $f + g$ is defined, then $|f + g| \leqslant |f| + |g|$, hence (by (i) and No. 1, Prop. 1)

$$N_p(|f+g|) \leqslant N_p(|f| + |g|) \leqslant N_p(|f|) + N_p(|g|)\,,$$

thus

(iv) $$f, g : X \to \overline{\mathbf{R}},\ f + g \text{ defined} \quad \Rightarrow \quad N_p(f+g) \leqslant N_p(f) + N_p(g)\,.$$

If $\mathbf{f}, \mathbf{g} : X \to F$ then $|\mathbf{f} + \mathbf{g}| \leqslant |\mathbf{f}| + |\mathbf{g}|$ by the triangle inequality in F, therefore (by (i) and (iv))

$$N_p(|\mathbf{f}+\mathbf{g}|) \leqslant N_p(|\mathbf{f}| + |\mathbf{g}|) \leqslant N_p(|\mathbf{f}|) + N_p(|\mathbf{g}|)\,,$$

thus

(v) $$\mathbf{f}, \mathbf{g} : X \to F \quad \Rightarrow \quad N_p(\mathbf{f}+\mathbf{g}) \leqslant N_p(\mathbf{f}) + N_p(\mathbf{g})\,.$$

IV.19, ℓ. −1 to **IV.20,** ℓ. 1.

"... the definition of $N_p(f)$, and Th. 3 of §1, No. 3, show that $N_p(f) = \sup_n N_p(g_n)$."

By the definition of f, $g_n \uparrow f$, hence $(g_n)^p \uparrow f^p$, hence $|\mu|^*\big((g_n)^p\big) \uparrow |\mu|^*(f^p)$ by Th.3 of §1, No. 3, hence $\Big(|\mu|^*\big((g_n)^p\big)\Big)^{1/p} \uparrow \big(|\mu|^*(f^p)\big)^{1/p}$ as claimed.

IV.20, ℓ. 6.

"The proposition follows at once from Th. 1 of §2, No. 3."

$$\begin{aligned}
\mathbf{f}, \mathbf{g} \text{ equivalent} \quad &\Leftrightarrow \quad \mathbf{f} - \mathbf{g} = 0 \ \text{ a.e.} && (\S 2, \text{ No. } 4)\\
&\Leftrightarrow \quad |\mathbf{f} - \mathbf{g}|^p = 0 \text{ a.e. for some } p \geqslant 1\\
&\Leftrightarrow \quad |\mathbf{f} - \mathbf{g}|^p = 0 \text{ a.e. for all } p \geqslant 1\\
&\Leftrightarrow \quad |\mu|^*(|\mathbf{f} - \mathbf{g}|^p) = 0 \text{ for some (all) } p \geqslant 1 && (\S 2, \text{ Th. } 1)\\
&\Leftrightarrow \quad \big(|\mu|^*(|\mathbf{f} - \mathbf{g}|^p)\big)^{1/p} = 0 \ \text{ for some (all) } p \geqslant 1\,,
\end{aligned}$$

that is, $\mathbf{f}$ and $\mathbf{g}$ are equivalent if and only if $\mathrm{N}_p(\mathbf{f}-\mathbf{g})=0$ for some (hence for all) $p \geqslant 1$.

IV.20, ℓ. 16–19.

"One can then define $\mathrm{N}_p(\mathbf{f})$ for a function with values in F (resp. in $\overline{\mathbf{R}}$) defined almost everywhere in X, by setting $\mathrm{N}_p(\mathbf{f}) = \mathrm{N}_p(\widetilde{\mathbf{f}})$; it is then clear that the relations (3) and (4) again hold (assuming $\alpha \neq 0$ and $f+g$ defined almost everywhere, in the case of numerical functions, finite or not)."

Recall (§2, No. 5) that if f is any function defined almost everywhere in X, with values in a set G (not necessarily vectorial or numerical), the class $\widetilde{f}$ is defined as follows: let f' be any *extension* of f to all of X, with values in G, and let $\widetilde{f}$ be the set of all functions $g : \mathrm{X} \to \mathrm{G}$ such that $g = f'$ almost everywhere in X, that is, $\widetilde{f}$ is the equivalence class of f' in $\mathscr{F}(\mathrm{X};\mathrm{G})$, for the relation of equality almost everywhere; if f'' is another extension of f to X, then f'' is equivalent to f', whence it is clear that $\widetilde{f}$ is independent of the particular extension f' of f to X.

When G is equal to the Banach space F or to $\overline{\mathbf{R}}$, and f' is any extension of f to all of X, then $\mathrm{N}_p(f')$ is defined (Def. 1 and the remarks following Prop. 2) and is independent of the particular extension f' (§2, No. 3, Prop. 6), thus the definition $\mathrm{N}_p(\widetilde{f}) = \mathrm{N}_p(\widetilde{f'})$ $(= \mathrm{N}_p(f'))$ depends only on f. The author *defines* $\mathrm{N}_p(f)$ to be equal to $\mathrm{N}_p(\widetilde{f})$; but what it all comes down to is that $\mathrm{N}_p(f)$ may unambiguously be defined to be equal to $\mathrm{N}_p(g)$, where g is any function equivalent to any extension of f to X.

If $\mathbf{f},\mathbf{g}$ are functions defined almost everywhere in X with values in F, then $\mathbf{f}+\mathbf{g}$ is defined almost everywhere and if $\mathbf{f}'$ and $\mathbf{g}'$ are extension of $\mathbf{f}$ and $\mathbf{g}$ to X, then $\mathbf{f}'+\mathbf{g}'$ extends $\mathbf{f}+\mathbf{g}$ and the relation (4) for $\mathbf{f}'$ and $\mathbf{g}'$ says precisely $\mathrm{N}_p(\mathbf{f}+\mathbf{g}) \leqslant \mathrm{N}_p(\mathbf{f}) + \mathrm{N}_p(\mathbf{g})$. Similarly for (3).

If f,g are defined almost everywhere in X and if $f+g$ is defined almost everywhere, then there exists a negligible set N such that f, g and $f+g$ are defined on $\complement \mathrm{N}$; if f' (resp. g') is a function on X that extends $f|\complement \mathrm{N}$ (resp. $g|\complement \mathrm{N}$), then f', g' and $f'+g'$ clearly represent the equivalence classes $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{f+g}$, respectively, consequently the relation (4) for f' and g' says precisely $\mathrm{N}_p(f+g) \leqslant \mathrm{N}_p(f) + \mathrm{N}_p(g)$. Similarly for (3).

IV.21, ℓ. 4–6.

"This terminology extends at once to the case that the functions $\mathbf{f}_n$ and the function $\mathbf{f}$ are only defined almost everywhere (or have values in $\overline{\mathbf{R}}$, and are defined and finite almost everywhere)."

When the functions $\mathbf{f}_n, \mathbf{f}$ have values in the Banach space F, there exists a negligible set N on whose complement the $\mathbf{f}_n$ and $\mathbf{f}$ are defined; in particular (for a pair of such functions) linear operations on the classes of such functions are readily defined, and the notation $N_p(\mathbf{f}_n - \mathbf{f})$ is available (see the preceding Note).

When the functions take their values in $\overline{\mathbf{R}}$ and are, moreover, finite almost everywhere, then there exists a negligible set N on whose complement the functions are defined and finite-valued; one then proceeds as in the preceding paragraph. Note in this connection that if $\mathbf{f}$ is a function defined almost everywhere in X and taking values in $\overline{\mathbf{R}}$, and if $N_p(\mathbf{f}) < +\infty$ (where $N_p(\mathbf{f})$ is defined as in the preceding Note), then $\mathbf{f}$ is finite-valued almost everywhere (§2, No. 3, Prop. 7).

IV.21, ℓ. 7–9.

"... the closure of 0 in this space is the linear subspace $\mathscr{N}_F$ of *negligible* mappings of X into F (No. 2, Prop. 3)."

The closure in question is the set $\{\mathbf{f} \in \mathscr{F}_F^p : N_p(\mathbf{f}) = 0\}$ (TVS, II, §1, No. 2, Prop. 2), and $N_p(\mathbf{f}) = 0$ if and only if $\mathbf{f}$ is negligible (§2, No. 1, Def. 1). It follows that $\mathscr{N}_F$ is contained in each of the spaces $\mathscr{L}_F^p$ to be defined in No. 4 (Def. 2).

IV.21, ℓ. −13, −12.

"From this, it follows that $N_p(\mathbf{f} - \mathbf{f}_0) \leqslant \varepsilon N_p(h)$, whence the proposition."

Since, for every $\varepsilon > 0$, there exists an $M \in \mathfrak{B}$ such that

$$(*) \qquad |\mathbf{f}(x) - \mathbf{f}_0(x)| \leqslant \varepsilon \quad \text{for all } \mathbf{f} \in M \text{ and } x \in X,$$

and since every $\mathbf{f} \in M$ is 0 on $\complement K$, it is clear that $\mathbf{f}_0 = 0$ on $\complement K$, therefore (*) yields

$$|\mathbf{f}(x) - \mathbf{f}_0(x)| \leqslant \varepsilon h(x) \quad \text{for all } \mathbf{f} \in M \text{ and } x \in X,$$

whence

$$(**) \qquad N_p(\mathbf{f} - \mathbf{f}_0) \leqslant \varepsilon N_p(h) \quad \text{for all } f \in M;$$

since $N_p(h) < +\infty$ one has $N_p(\mathbf{f} - \mathbf{f}_0) < +\infty$, therefore $\mathbf{f} - \mathbf{f}_0 \in \mathscr{F}_F^p$ and so $\mathbf{f}_0 = \mathbf{f} - (\mathbf{f} - f_0) \in \mathscr{F}_F^p$. Since, given any $\varepsilon > 0$ there exists an $M \in \mathfrak{B}$ satisfying (**), the convergence in mean of order p of $\mathfrak{B}$ to $\mathbf{f}_0$ is proved. Incidentally, the role of h can be played equally well by φ_K (§1, No. 4, Cor. of Prop. 16).

IV.24, ℓ. 5–7.

"Proposition 7. — *For a function* $\mathbf{f}$ *to belong to* $\mathscr{L}_F^p$, *it is necessary and sufficient that, for every* $\varepsilon > 0$, *there exist a continuous function* $\mathbf{g}$ *with compact support such that* $N_p(\mathbf{f} - \mathbf{g}) \leqslant \varepsilon$."

If $\mathbf{f}$ has the indicated property, then for any such $\mathbf{g}$ one clearly has $\mathbf{f}-\mathbf{g} \in \mathscr{F}^p_{\mathrm{F}}$, hence $\mathbf{f} = (\mathbf{f}-\mathbf{g}) + \mathbf{g} \in \mathscr{F}^p_{\mathrm{F}}$; thus $\mathbf{f}$ belongs to the closure of $\mathscr{K}(\mathrm{X};\mathrm{F})$ in $\mathscr{F}^p_{\mathrm{F}}$, that is, $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$. The converse is immediate from Def. 2.

IV.24, ℓ. 15, 16.

"For, f is finite almost everywhere and $\mathrm{N}_p(f-g) \leqslant \mathrm{N}_p(h-g) \leqslant \varepsilon$; Prop. 7 therefore shows that f is p-th power integrable."

Since the functions g, h of the statement are finite almost everywhere, so is f; redefining f on a negligible set, we can suppose that it is everywhere defined and finite. Then, given $\varepsilon > 0$, choose g and h as in the statement of the proposition; replacing them by equivalent functions, we can suppose that $g, h \in \mathscr{L}^p$. Then all three functions are everywhere defined, finite, and satisfy $g \leqslant f \leqslant h$ almost everywhere in X; and since $0 \leqslant f - g \leqslant h - g$ almost everywhere, $\mathrm{N}_p(f-g) \leqslant \mathrm{N}_p(h-g) \leqslant \varepsilon$. Choose $g_0 \in \mathscr{K}(\mathrm{X};\mathbf{R})$ with $\mathrm{N}_p(g-g_0) \leqslant \varepsilon$ (Def. 2); then

$$\mathrm{N}_p(f-g_0) = \mathrm{N}_p\big((f-g)+(g-g_0)\big) \leqslant \mathrm{N}_p(f-g) + \mathrm{N}_p(g-g_0) \leqslant \varepsilon + \varepsilon\,,$$

therefore $f \in \mathscr{L}^p$ by Prop. 7, hence the original function f (before redefinition) is p-th power integrable.

IV.24, ℓ. −6, −5.

"... parts 2° and 3° then follow from Prop. 6 of No. 3 and the fact that $\mathscr{L}^p_{\mathrm{F}}$ is closed in $\mathscr{F}^p_{\mathrm{F}}$."

By 1° and the cited Prop. 6, the series with general term $\mathbf{f}_{n_{k+1}}(x) - \mathbf{f}_{n_k}(x)$ is absolutely convergent for almost every x in X, and the function $\mathbf{g} : \mathrm{X} \to \mathrm{F}$ defined by

$$\mathbf{g}(x) = \sum_k [\mathbf{f}_{n_{k+1}}(x) - \mathbf{f}_{n_k}(x)] = \lim_k \mathbf{f}_{n_k}(x) - \mathbf{f}_{n_1}(x)$$

at the points x where the series converges, and by $\mathbf{g}(x) = 0$ at the remaining points x, belongs to $\mathscr{F}^p_{\mathrm{F}}$, and, as $k \to \infty$,

$$\sum_{j=1}^{k-1} [\mathbf{f}_{n_{j+1}} - \mathbf{f}_{n_j}] = \mathbf{f}_{n_k} - \mathbf{f}_{n_1} \to \mathbf{g} \quad \text{in } \mathscr{F}^p_{\mathrm{F}}\,,$$

that is, in mean of order p; $\mathbf{g} \in \mathscr{L}^p_{\mathrm{F}}$ because the $\mathbf{f}_{n_k} - \mathbf{f}_{n_1}$ belong to $\mathscr{L}^p_{\mathrm{F}}$ and $\mathscr{L}^p_{\mathrm{F}}$ is closed in $\mathscr{F}^p_{\mathrm{F}}$. Thus, setting $\mathbf{f} = \mathbf{g} + \mathbf{f}_{n_1}$, one has $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$, $\mathbf{f}_{n_k}(x) \to \mathbf{f}(x)$ for almost every x in X, and $\mathbf{f}_{n_k} \to \mathbf{f}_{n_1} + \mathbf{g} = \mathbf{f}$ in $\mathscr{L}^p_{\mathrm{F}}$, that is, in mean of order p. Since $(\mathbf{f}_n)$ is Cauchy in $\mathscr{L}^p_{\mathrm{F}}$ and $\mathbf{f}_{n_k} \to \mathbf{f}$ in $\mathscr{L}^p_{\mathrm{F}}$, it follows that $\mathbf{f}_n \to \mathbf{f}$ in $\mathscr{L}^p_{\mathrm{F}}$ (an easy application of the triangle inequality; for the abstract principle, see GT, II, §3, No. 2, Cor. 3 of Prop. 5). Whence 3°.

IV.24, ℓ. −2, −1.

"... there exists a lower semi-continuous function $g \geqslant h + |\mathbf{f}_{n_1}|$ such that $\mathrm{N}_p(g) < +\infty$, which completes the proof."

By definition, $h(x) = \sum_k |\mathbf{f}_{n_{k+1}(x)} - \mathbf{f}_{n_k}(x)|$ for all $x \in \mathrm{X}$, thus $h = \sum_k |\mathbf{f}_{n_{k+1}} - \mathbf{f}_{n_k}|$ is the sum of the positive numerical functions $|\mathbf{f}_{n_{k+1}} - \mathbf{f}_{n_k}|$. By Th. 1 of No. 2,

$$\mathrm{N}_p(h) \leqslant \sum_k \mathrm{N}_p(|\mathbf{f}_{n_{k+1}} - \mathbf{f}_{n_k}|) = \sum_k \mathrm{N}_p(\mathbf{f}_{n_{k+1}} - \mathbf{f}_{n_k}) < +\infty\,,$$

thus $h \in \mathscr{F}^p$. For every positive integer k and every $x \in \mathrm{X}$,

$$\sum_{j=1}^{k-1} \big(\mathbf{f}_{n_{j+1}}(x) - \mathbf{f}_{n_j}(x)\big) = \mathbf{f}_{n_k}(x) - \mathbf{f}_{n_1}(x)\,,$$

thus $\mathbf{f}_{n_k}(x) = \mathbf{f}_{n_1}(x) + \sum_{j=1}^{k-1} \big(\mathbf{f}_{n_{j+1}}(x) - \mathbf{f}_{n_j}(x)\big)$, therefore

$$|\mathbf{f}_{n_k}(x)| \leqslant |\mathbf{f}_{n_1}(x)| + \sum_{j=1}^{k-1} \big|\mathbf{f}_{n_{j+1}}(x) - \mathbf{f}_{n_j}(x)\big| \leqslant |\mathbf{f}_{n_1}(x)| + h(x)\,,$$

briefly $|\mathbf{f}_{n_k}| \leqslant |\mathbf{f}_{n_1}| + h$.

From $|\mathbf{f}_{n_1}| \in \mathscr{L}^p$ and $h \in \mathscr{F}^p$ we have $|\mathbf{f}_{n_1}| + h \in \mathscr{F}^p$, thus

$$\mathrm{N}_p(|\mathbf{f}_{n_1}| + h) < +\infty\,,$$

whence $|\mu|^*\big((|\mathbf{f}_{n_1}| + h)^p\big) < +\infty$. By the definition of $|\mu|^*$ (§1, No. 3, Def. 3) there exists a lower semi-continuous function $u \geqslant 0$ such that

$$(|\mathbf{f}_{n_1}| + h)^p \leqslant u \quad \text{and } |\mu|^*(u) < +\infty\,.$$

Since $t \mapsto t^{1/p}$ is continuous and increasing (GT, IV, §3, No. 3, with the convention $(+\infty)^{1/p} = +\infty$) it is easy to see that $u^{1/p}$ is also lower semi-continuous (more generally, see GT, IV, §6, Exer. 4). Writing $g = u^{1/p}$, we have $|\mathbf{f}_{n_1}| + h \leqslant g$ and

$$\mathrm{N}_p(g) = \big(|\mu|^*(g^p)\big)^{1/p} = \big(|\mu|^*(u)\big)^{1/p} < +\infty\,,$$

thus $g \in \mathscr{F}^p$. Finally, $|\mathbf{f}_{n_k}| \leqslant |\mathbf{f}_{n_1}| + h \leqslant g$, thus $|\mathbf{f}_{n_k}(x)| \leqslant g(x)$ for all $x \in \mathrm{X}$. Incidentally, since $g \in \mathscr{F}^p$ is lower semi-continuous, in fact $g \in \mathscr{L}^p$ (see below, §4, No. 4, Prop. 5).

IV.25, ℓ. −3.

"... the theorem is proved."

Recall that the functions in $\mathscr{L}^p_{\mathrm{F}}$ (in particular $\mathbf{f}$) are defined *everywhere* in X; and they are characterized as the functions in $\mathscr{F}^p_{\mathrm{F}}$ that are in the closure of $\mathscr{K}_{\mathrm{F}}$ for the N_p-topology. Since $|(\mathbf{u} \circ \mathbf{f})(x)| \leqslant \|\mathbf{u}\| \cdot |\mathbf{f}(x)|$ for all $x \in \mathrm{X}$, that is, $|\mathbf{u} \circ \mathbf{f}| \leqslant \|\mathbf{u}\| \cdot |\mathbf{f}|$, one has $\mathrm{N}_p(\mathbf{u} \circ \mathbf{f}) \leqslant \|\mathbf{u}\| \cdot \mathrm{N}_p(\mathbf{f}) < +\infty$, thus $\mathbf{u} \circ \mathbf{f} \in \mathscr{F}^p_{\mathrm{G}}$; and the argument in the text shows that $\mathbf{u} \circ \mathbf{f}$ is in the closure of $\mathscr{K}_{\mathrm{G}}$, whence $\mathbf{u} \circ \mathbf{f} \in \mathscr{L}^p_{\mathrm{G}}$.

Theorem 4 extends to functions defined almost everywhere, as follows. If $\mathbf{f}$ is p-th power integrable but is defined only almost everywhere in X, there exists an $\mathbf{f}' \in \mathscr{L}^p_{\mathrm{F}}$ such that $\mathbf{f} = \mathbf{f}'$ almost everywhere (remarks following No. 4, Def. 2). Then $\mathbf{u} \circ \mathbf{f} = \mathbf{u} \circ \mathbf{f}'$ almost everywhere, where $\mathbf{u} \circ \mathbf{f}' \in \mathscr{L}^p_{\mathrm{G}}$ by the theorem, hence $\mathbf{u} \circ \mathbf{f}$ is p-th power integrable.

IV.25, ℓ. −2, −1.

"COROLLARY 1."

In Theorem 4, take G to be the field of scalars of F, and $\mathbf{u} = \mathbf{a}' \in \mathrm{F}'$.

IV.26, ℓ. 4, 5.

"This follows from the fact that the mapping $t \mapsto \mathbf{a}t$ of $\mathbf{R}$ into F is continuous."

The field of scalars $\mathbf{K}$ of F can be either $\mathbf{R}$ or $\mathbf{C}$. We can suppose that $k = 1$. Thus, assuming $\mathbf{a} \in \mathrm{F}$ and $f \in \mathscr{L}^p$, one defines $\mathbf{f} = \mathbf{a}f$ by the formula $\mathbf{f}(x) = f(x)\mathbf{a}$. Define $\mathbf{u} : \mathbf{K} \to \mathrm{F}$ by $\mathbf{u}(t) = t\mathbf{a}$; then $\mathbf{u}$ is a continuous linear mapping and

$$(\mathbf{u} \circ f)(x) = \mathbf{u}\big(f(x)\big) = f(x)\mathbf{a} = \mathbf{f}(x),$$

whence $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$ by Theorem 4 (with the substitutions of $\mathbf{K}$ for F, and F for G).

IV.26, ℓ. 10.

"This follows at once from Cors. 1 and 2 of Th. 4."

Equip F with its unique compatible Hausdorff topology (TVS, I, §2, No. 3, Th. 2) and let F′ be the dual space; since every linear form on F is continuous (*loc. cit.*, Cor. 2 of Th. 2), the topological dual coincides with the algebraic dual. Let $(\mathbf{e}'_k)_{1 \leqslant k \leqslant n}$ be the basis of F′ dual to $(\mathbf{e}_k)_{1 \leqslant k \leqslant n}$; then

$$\langle \mathbf{f}(x), \mathbf{e}'_k \rangle = f_k(x) \qquad (x \in \mathrm{X}),$$

thus $f_k = \langle \mathbf{f}, \mathbf{e}'_k \rangle$. If $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$ then $f_k \in \mathscr{L}^p$ for all k by Cor. 1 of Th. 4; the converse is immediate from Cor. 2.

IV.27, ℓ. −8 to −6.

"... the image of the section filter $\mathfrak{F}$ of H under this mapping is therefore a base of a Cauchy filter on $\mathbf{R}$."

The set of all $\|\widetilde{\mathbf{f}}\|_p$ ($\widetilde{f} \in \mathrm{H}$) has a finite supremum λ and is directed upward by the relation $\leqslant$ on H; that is, the mapping $\Phi : \mathrm{H} \to \mathbf{R}$ defined by $\Phi(\widetilde{\mathbf{f}}) = \|\widetilde{\mathbf{f}}\|_p$ is increasing and has a finite supremum λ. The sets

$$\mathrm{H}(\widetilde{\mathbf{g}}) = \{\widetilde{\mathbf{f}} \in \mathrm{H} : \ \widetilde{\mathbf{f}} \geqslant \widetilde{\mathbf{g}}\} \qquad (\widetilde{\mathbf{g}} \in \mathrm{H})$$

are a base for the section filter $\mathfrak{F}$ of the directed set H. The sets

$$\Phi\big(\mathrm{H}(\widetilde{\mathbf{g}})\big) = \{\|\widetilde{\mathbf{f}}\|_p : \ \widetilde{\mathbf{f}} \in \mathrm{H}, \ \widetilde{\mathbf{f}} \geqslant \widetilde{\mathbf{g}}\} \qquad (\widetilde{\mathbf{g}} \in \mathrm{H})$$

form a base of a filter $\mathfrak{G}$ on $\mathbf{R}$ (GT, I, §6, No. 6).

Given any $\varepsilon > 0$, choose $\widetilde{g} \in \mathrm{H}$ so that $\|\widetilde{g}\| \geqslant \lambda - \varepsilon$; then

$$\widetilde{\mathbf{f}} \in \mathrm{H}, \ \widetilde{\mathbf{f}} \geqslant \widetilde{\mathbf{g}} \quad \Rightarrow \quad \lambda - \varepsilon \leqslant \|\widetilde{\mathbf{g}}\| \leqslant \|\widetilde{\mathbf{f}}\| \leqslant \lambda ,$$

thus $\Phi\big(\mathrm{H}(\widetilde{\mathbf{g}})\big) \subset [\lambda - \varepsilon, \lambda] \subset [\lambda - \varepsilon, \lambda + \varepsilon]$. This shows that the filter $\mathfrak{G}$ is convergent (to λ), hence is Cauchy.

IV.28, ℓ. 10–12.

"... every function $f \geqslant 0$ in $\mathscr{L}^p$ is the limit (for convergence in mean of order p) of a sequence of continuous functions $\geqslant 0$ with compact support, by the continuity of the mapping $g \mapsto |g|$ on $\mathscr{L}^p$ (Prop. 11)."

By the definition of $\mathscr{L}^p$, there exists a sequence (f_n) in $\mathscr{K}$ such that $f_n \to f$ (in mean of order p). Since $|f_n| \to |f| = f$ by the cited Prop. 11, one can suppose without loss of generality that $f_n \geqslant 0$ for all n. Similarly, since $g - f \in \mathscr{L}^p$ and $g - f \geqslant 0$, there exists a sequence (h_n) in $\mathscr{K}$ with $h_n \geqslant 0$ and $h_n \to g - f$. Then

$$f_n + h_n \to f + (g - f) = g ;$$

setting $g_n = f_n + h_n$, we have $0 \leqslant f_n \leqslant g_n$, where $f_n, g_n \in \mathscr{K}_+$ and $f_n \to f$, $g_n \to g$ (in mean of order p). By the special case considered earlier,

$$\big(\mathrm{N}_p(g_n - f_n)\big)^p \leqslant \big(\mathrm{N}_p(g_n)\big)^p - \big(\mathrm{N}_p(f_n)\big)^p \quad \text{for all } n ,$$

and passage to the limit yields (9).

IV.31, ℓ. 11, 12.

"... $\lim_{\mathfrak{F}} \mathrm{N}_p(\mathbf{f} - \mathbf{f}_\alpha)$ exists and is equal to the common limit 0 of all of the sequences $(\mathrm{N}_p(\mathbf{f} - \mathbf{f}_{\alpha_n}))$."

Explicitly: Given $\varepsilon > 0$, we seek an index n such that $\mathrm{N}_p(\mathbf{f} - \mathbf{f}_\alpha) < \varepsilon$ for all $\alpha \in \mathrm{A}_n$. Assume to the contrary that for every n there exists an $\alpha_n \in \mathrm{A}_n$ such that $\mathrm{N}_p(\mathbf{f} - \mathbf{f}_{\alpha_n}) \geqslant \varepsilon$. This contradicts the fact, already shown, that $\lim\limits_{n \to \infty} \mathrm{N}_p(\mathbf{f} - \mathbf{f}_{\alpha_n}) = 0$.

§4. INTEGRABLE FUNCTIONS AND SETS

IV.33, ℓ. 13.

"... we have $|\mu|^*(|\mathbf{f}|) = \sum_{x\in X} |\alpha(x)|\cdot|\mathbf{f}(x)| < +\infty$ (§1, No. 3, *Example*)"

In the case that F is replaced by $\overline{\mathbf{R}}$ and $\mathbf{f}$ is a numerical function, this implies that $\alpha(x) = 0$ when $\mathbf{f}(x) = \pm\infty$. In any case, $\alpha(x) = \mu(\varphi_{\{x\}})$ is finite since $\varphi_{\{x\}}$ is a continuous function with compact support (X being discrete).

IV.33, ℓ. 16.

"... at the points $x \in M$ where $|\mathbf{f}|$ is finite"

In the vector-valued case, $|\mathbf{f}(x)|$ is finite for every $x \in X$. Apparently the comment is intended for the case of numerical functions $\mathbf{f}$; similarly for the subsequent remark "by the conventions that have been made".

IV.33, ℓ. -10.

"... whence the second assertion."

The argument shows that as $\mathbf{g} \to \mathbf{f}$ in mean, $\mu(\mathbf{g}) \to \sum_{x\in X} \alpha(x)\mathbf{f}(x)$; but $\mu(\mathbf{g}) \to \int \mathbf{f}\, d\mu$ by the definition of $\int \mathbf{f}\, d\mu$.

IV.34, ℓ. 14.

"... whence the proposition."

Since $f \in \mathscr{L}^1$ there exists a sequence $f_n \in \mathscr{K}$ such that $f_n \to f$ in mean, and since $|f_n| \to |f| = f$ in mean (§3, No. 5, Prop. 11) we can suppose that $f_n \geqslant 0$. From $N_1(f_n - f) \to 0$ one infers $N_1(f_n) \to N_1(f)$ (N_1 is a semi-norm on $\mathscr{L}^1$); but

$$N_1(f_n) = |\mu|^*(|f_n|) = |\mu|(f_n) \to |\mu|(f)$$

by the definition of $|\mu|(f)$, thus $|\mu|(f) = \lim_n N_1(f_n) = N_1(f) = \int^* f\, d\,|\mu|$.

IV.34, ℓ. 15–17.

"Corollary 1."

The positive function $f = |\mathbf{f}|$ belongs to $\mathscr{L}^1$ by §3, No. 5, Prop. 11, and $N_1(\mathbf{f}) = N_1(f)$.

IV.35, ℓ. 3, 4.

"This follows at once from the inequality (1) by passage to the limit, on taking into account (3) and the continuity of $\mathrm{N}_1(\mathbf{f})$ on $\mathscr{L}^1_{\mathrm{F}}$."

By the definition of "$\mathbf{f}$ integrable", there exists a sequence $\mathbf{f}_n \in \mathscr{K}_{\mathrm{F}}$ such that

$$(*) \qquad \mathrm{N}_1(\mathbf{f}_n - \mathbf{f}) \to 0\,,$$

and one defines

$$\text{(i)} \qquad \int \mathbf{f}\,d\mu = \lim_n \int \mathbf{f}_n\,d\mu\,.$$

But also $\big||\mathbf{f}_n| - |\mathbf{f}|\big| \leqslant |\mathbf{f}_n - \mathbf{f}|$, therefore $\mathrm{N}_1(|\mathbf{f}_n| - |\mathbf{f}|) \leqslant \mathrm{N}_1(\mathbf{f}_n - \mathbf{f}) \to 0$, whence $|\mathbf{f}| \in \mathscr{L}^1$ and

$$\text{(ii)} \qquad \int |\mathbf{f}|\,d|\mu| = \lim_n \int |\mathbf{f}_n|\,d|\mu|\,.$$

By (1),

$$\text{(iii)} \qquad \Big|\int \mathbf{f}_n\,d\mu\Big| \leqslant \int |\mathbf{f}_n|\,d|\mu| = \mathrm{N}_1(\mathbf{f}_n)\,;$$

since (*) implies $\mathrm{N}_1(\mathbf{f}_n) \to \mathrm{N}_1(\mathbf{f})$, passage to the limit in (iii) yields

$$\Big|\int \mathbf{f}\,d\mu\Big| \leqslant \int |\mathbf{f}|\,d|\mu| = \mathrm{N}_1(\mathbf{f})$$

by (i) and (ii).

IV.35, ℓ. 9–11.

"...the relation (6), being valid for every $\mathbf{f} \in \mathscr{K}_{\mathrm{F}}$, extends to every integrable function $\mathbf{f}$ by the principle of extension of identities"

From $|(\mathbf{u} \circ \mathbf{f})(x)| = \big|\mathbf{u}\big(\mathbf{f}(x)\big)\big| \leqslant \|\mathbf{u}\| \cdot |\mathbf{f}(x)|$ for all $x \in \mathrm{X}$, we have $|\mathbf{u} \circ \mathbf{f}| \leqslant \|\mathbf{u}\| \cdot |\mathbf{f}|$, whence $\mathrm{N}_1(\mathbf{u} \circ \mathbf{f}) \leqslant \|\mathbf{u}\|\,\mathrm{N}_1(\mathbf{f})$; thus the linear mapping $\mathbf{f} \mapsto \mathbf{u} \circ \mathbf{f}$ is continuous in mean, therefore $\mathbf{f} \mapsto \int (\mathbf{u} \circ \mathbf{f})\,d\mu$ is a continuous linear mapping $\mathscr{L}^1_{\mathrm{F}} \to \mathrm{G}$.

On the other hand, $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$ is a continuous linear mapping $\mathscr{L}^1_{\mathrm{F}} \to \mathrm{F}$, hence the composite $\mathbf{f} \mapsto \mathbf{u}\big(\int \mathbf{f}\,d\mu\big)$ is also a continuous linear mapping $\mathscr{L}^1_{\mathrm{F}} \to \mathrm{G}$.

Summarizing, the mappings

$$\mathbf{f} \mapsto \int (\mathbf{u} \circ \mathbf{f})\,d\mu \quad \text{and} \quad \mathbf{f} \mapsto u\Big(\int \mathbf{f}\,d\mu\Big)$$

are continuous mappings of $\mathscr{L}^1_{\mathrm{F}}$ into the Hausdorff space G; since they agree on the dense subspace $\mathscr{K}_{\mathrm{F}}$ (cf. Ch. III, §3, No. 2, Prop. 2 and No. 3, Cor. 2 of Prop. 7), they are identical by the "Principle of extension of identities" (GT, I, §8, No. 1, Cor. 1 of Prop. 2).

IV.35, ℓ. -1.

$$\int \left(\sum_{k=1}^{n} \mathbf{a}_k f_k \right) d\mu = \sum_{k=1}^{n} \mathbf{a}_k \int f_k \, d\mu \,. \tag{8}$$

Let $\mathbf{K}$ ($= \mathbf{R}$ or $\mathbf{C}$) be the field of scalars for the Banach space F. Assume first that the $f_k \in \mathscr{L}^1$ are defined and finite-valued on all of X.

By linearity, we can suppose that $n = 1$. Thus, given $\mathbf{a} \in \mathrm{F}$ and a function $f \in \mathscr{L}^1$ defined and finite-valued on all of X, the formula $(\mathbf{a}f)(x) = f(x)\mathbf{a}$ $(x \in \mathrm{X})$ defines a function $\mathbf{a}f : \mathrm{X} \to \mathrm{F}$. Let $\mathbf{u} : \mathbf{K} \to \mathrm{F}$ be the continuous linear mapping defined by $\mathbf{u}(t) = t\mathbf{a}$ $(t \in \mathbf{K})$. Then

$$(\mathbf{u} \circ f)(x) = \mathbf{u}\big(f(x)\big) = f(x)\mathbf{a} = (\mathbf{a}f)(x) \quad (x \in \mathrm{X})\,,$$

hence by Th. 1, $\mathbf{a}f \in \mathscr{L}^1_{\mathrm{F}}$ and

$$\int \mathbf{a}f \, d\mu = \Big(\int f \, d\mu \Big) \mathbf{a} \,.$$

(The notation $\mathbf{a}f$ presumably suggests that $\mathbf{a}$ is a 'constant coefficient' for the function f.)

When the f_k are not necessarily finite-valued, let $g_k \in \mathscr{L}^1$ be finite-valued with $g_k = f_k$ almost everywhere, apply the foregoing to the g_k, and interpret (8) in the light of the comments at the end of No. 1.

IV.36, ℓ. 7.

"For, $\mathfrak{B}$ converges in mean to $\mathbf{f}_0$ (§3, No. 3, Prop. 4)."

By the cited Prop. 4, $\mathbf{f}_0 \in \mathscr{F}^1_{\mathrm{F}}$ and $\mathfrak{B} \to \mathbf{f}_0$ in mean (of order 1). But the functions $\mathbf{f} \in \mathrm{M} \in \mathfrak{B}$ belong to $\mathscr{L}^1_{\mathrm{F}}$, and $\mathscr{L}^1_{\mathrm{F}}$ is closed in $\mathscr{F}^1_{\mathrm{F}}$ (it is by definition the closure of $\mathscr{K}_{\mathrm{F}}$ in $\mathscr{F}^1_{\mathrm{F}}$ for convergence in mean), therefore $\mathbf{f}_0 \in \mathscr{L}^1_{\mathrm{F}}$; and since $\lim_{\mathfrak{B}} \mathbf{f} = \mathbf{f}_0$ in mean, $\int \mathbf{f}_0 \, d\mu = \lim_{\mathfrak{B}} \int \mathbf{f} \, d\mu$ by the continuity (for convergence in mean) of $\mathbf{g} \mapsto \int \mathbf{g} \, d\mu$ ($\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}$).

IV.36, ℓ. 14.

" $g_n = f_n + f_1^-$ "

Caution . The functions f_n are assumed to be integrable, but they need not belong to $\mathscr{L}^1$ (§3, No. 4) hence may have infinite values. For example, if $f_1(x) = -\infty$ then $f_1^-(x) = \sup(-f_1(x), 0) = +\infty$ and the sum $f_1(x)+f_1^-(x)$ is not defined. (This problem did not arise in §3, No. 6, Cor. 2 of Th. 5, because the functions f_n were assumed to belong to $\mathscr{L}^p$.)

The cure: If the f_n are replaced by equivalent functions f_n' in $\mathscr{L}^1$ then the upper envelope f' of the f_n' is equivalent to f; f' may admit $+\infty$ as a value, but there is no problem with $g' = f' + f_1'^-$ since f_1' is finite-valued. To assure that the sequence (f_n') is increasing, let A_n be a negligible set on whose complement $f_n' = f_n$, and redefine the f_n' to be 0 on $\bigcup_n \mathrm{A}_n$. Finally, $g_n' = f_n' + f_1'^- \geqslant f_1' + f_1'^- = f_1'^+$, thus the functions g_n' belong to $\mathscr{L}^1$ and are $\geqslant 0$. The proof of Prop. 4 carries through for the f_n' as indicated (see the next note), and the validity of Prop. 4 for the f_n follows at once from the function equivalences.

IV.36, ℓ. 15, 16.

"... the proposition follows from Th. 5 of §3, No. 6."

As observed in the preceding note, we can suppose that the f_n are finite-valued, that is, belong to $\mathscr{L}^1$. Since (recall that $f_n + f_1^- \geqslant 0$) the norms

$$\mathrm{N}_1(f_n + f_1^-) = \int (f_n + f_1^-)\, d|\mu| = \int f_n\, d|\mu| + \int f_1^-\, d|\mu|$$

have a finite upper bound, it follows from the cited Th. 5 that the function $g = f + f_1^-$ is integrable, therefore $f = g - f_1^-$ is integrable. Th. 5 also yields

$$\mathrm{N}_1(f + f_1^-) = \sup_n \mathrm{N}_1(f_n + f_1^-)\,,$$

that is,

$$\int (f + f_1^-)\, d|\mu| = \sup_n \int (f_n + f_1^-)\, d|\mu|\,,$$

whence

$$\int f\, d|\mu| = \sup_n \int f_n\, d|\mu|\,,$$

thus $\int f_n\, d|\mu| \to \int f\, d|\mu|$, therefore $\int (f - f_n)\, d|\mu| \to 0$; but $f - f_n \geqslant 0$ and, citing Prop. 2 of No. 2, one has

$$\Big| \int (f - f_n)\, d\mu \Big| \leqslant \int |f - f_n|\, d|\mu| = \int (f - f_n)\, d|\mu| \to 0\,,$$

whence $\int f_n\, d\mu \to \int f\, d\mu$.

IV.36, ℓ. −8.

"The theorem follows from Lebesgue's theorem (§3, No. 7, Cor. of Th. 6)..."

Here the functions $\mathbf{f}_\alpha$ belong to $\mathscr{L}^1_{\mathrm{F}}$ (F a Banach space), and $\mathbf{f} : \mathrm{X} \to \mathrm{F}$; one applies the cited Cor. with $p = 1$, to conclude that $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}$ and that $\mathbf{f}_\alpha \to \mathbf{f}$ in mean with respect to $\mathfrak{F}$, whence $\int \mathbf{f}_\alpha \, d\mu \to \int \mathbf{f} \, d\mu$ in F with respect to $\mathfrak{F}$ by the continuity of the integral.

IV.36, ℓ. −5 to **IV.37,** ℓ. 5.

"Corollary 1."

Here Ω will play the role of A in Th. 2, and the role of $\mathfrak{F}$ will be played by the neighborhood filter of t_0 in Ω (which, by hypothesis, has a countable base). For each $t \in \Omega$ define $\mathbf{f}_t : \mathrm{X} \to \mathrm{F}$ by the formula

$$\mathbf{f}_t(x) = \mathbf{f}(x,t) \qquad (x \in \mathrm{X}) .$$

By the hypothesis *a*), we have $\mathbf{f}_t \in \mathscr{L}^1_{\mathrm{F}}$ for every $t \in \Omega$. The strategy is to apply Th. 2 to the family $(\mathbf{f}_t)_{t \in \Omega}$, where Ω is filtered by $\mathfrak{F}$.

By the hypothesis *b*), for each $x \in \mathrm{X}$ we have

$$\mathbf{f}_t(x) = \mathbf{f}(x,t) \to \mathbf{f}(x,t_0) = \mathbf{f}_{t_0}(x) \quad \text{as } t \to t_0 ;$$

in other words, the family $(\mathbf{f}_t)_{t \in \Omega}$ converges pointwise in X to $\mathbf{f}_{t_0}$ with respect to the filter $\mathfrak{F}$.

By the hypothesis *c*), for each $t \in \mathrm{U}$ we have

$$|\mathbf{f}_t(x)| = |\mathbf{f}(x,t)| \leqslant g(x) \quad \text{for all } x \in \mathrm{X} .$$

Since the neighborhoods V of t_0 with $\mathrm{V} \subset \mathrm{U}$ form a base for $\mathfrak{F}$, it follows from Th. 2 that $\mathbf{f}_{t_0} \in \mathscr{L}^1_{\mathrm{F}}$ (which we already know from *a*)) and that

$$\int \mathbf{f}_{t_0} \, d\mu = \lim_{t,\,\mathfrak{F}} \int \mathbf{f}_t \, d\mu ,$$

in other words

$$\int \mathbf{f}(x,t) \, d\mu(x) \to \int \mathbf{f}(x,t_0) \, d\mu(x) \quad \text{in F as } t \to t_0 ,$$

which is the desired continuity property at t_0.

Incidentally, the continuity in *b*) and the inequality in *c*) need only hold for almost every x in X.

IV.37, ℓ. 6–13.

"Corollary 2."

The points $x \in \mathrm{X}$ for which the series with general term $\mathbf{f}_n(x)$ does not converge form a negligible set; redefining all of the $\mathbf{f}_n$ to be 0 at such x, we can suppose that $\mathbf{f}$ is defined everywhere in X. For each positive integer n, define

$$\mathbf{g}_n = \sum_{k=1}^{n} \mathbf{f}_k\,;$$

then $\mathbf{g}_n \in \mathscr{L}^1_{\mathrm{F}}$, $\mathbf{g}_n \to \mathbf{f}$ pointwise in X, and

$$|\mathbf{g}_n(x)| \leqslant g(x) \quad \text{for almost every } x \text{ in } \mathrm{X} \quad (n = 1, 2, 3, \ldots).$$

Taking $\mathfrak{F}$ to be the Fréchet filter on the set $\mathrm{A} = \{1, 2, 3, \ldots\}$ of indices, it follows from Th. 2 that $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}$ and

$$\int \mathbf{f}\,d\mu = \lim_{n,\,\mathfrak{F}} \int \mathbf{g}_n\,d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int \mathbf{f}_k\,d\mu = \sum_{n=1}^{\infty} \int \mathbf{f}_n\,d\mu$$

as claimed.

IV.38, ℓ. 8.

"We may limit ourselves to the case of lower semi-continuous functions"

For, if H is a set of upper semi-continuous functions directed for $\geqslant$, then $-\mathrm{H} = \{-f : f \in \mathrm{H}\}$ is a set of lower semi-continuous directed for $\leqslant$.

However, the case of lower semi-continuous functions of arbitrary sign, directed for $\leqslant$, will be reduced to the case of Cor. 2 for sets of functions $\geqslant 0$, thus will entail the consideration of sets H of both types: lower semi-continuous and directed for $\leqslant$ as well as upper semi-continuous and directed for $\geqslant$.

IV.38, ℓ. 10–12.

"…the upper (resp. lower) envelope of the f^+ (resp. f^-), for $f \in \mathrm{H}$, is equal to g^+ (resp. g^-)."

Let $h = \sup_{f\in\mathrm{H}} f^+$. Since $f \leqslant g$ implies $f^+ = \sup(f, 0) \leqslant \sup(g, 0) = g^+$, it is obvious that $h \leqslant g^+$. To prove that $g^+ \leqslant h$, let $x \in \mathrm{X}$. Since $g(x) = \sup_{f\in\mathrm{H}} f(x)$, there exists a sequence $f_n \in \mathrm{H}$ such that $g(x) = \sup_n f_n(x)$, and since H is directed for $\leqslant$ we can suppose that the sequence (f_n) is increasing. If $g(x) = +\infty$ then $f_n(x) > 0$ from some index onward, therefore

$$g^+(x) = +\infty = \sup_n f_n^+(x) \leqslant \sup_{f\in\mathrm{H}} f^+(x) = h(x)\,;$$

if $g(x) < +\infty$ then $f_n(x) \to g(x)$ in $\mathbf{R}$, whence $f_n^+(x) \to g^+(x)$, so that $g^+(x) = \sup_n f_n^+(x) \leqslant h(x)$.

IV.38, ℓ. 14.

"... then $\int f^+ \, d|\mu| \leqslant \int f \, d|\mu| + \int f_0^- \, d|\mu|$ "

If $f \geqslant f_0$ then $f^- = \sup(-f, 0) \leqslant \sup(-f_0, 0) = f_0^-$; since $f = f^+ - f^-$, and f and f_0 are finite almost everywhere, one has, for almost every x,

$$f^+(x) = f(x) + f^-(x) \leqslant f(x) + f_0^-(x),$$

whence the asserted inequality (which shows that $\sup_{f \in \mathrm{H}} \int f^+ \, d|\mu| < +\infty$).

IV.38, ℓ. 15, 16.

"... we are reduced to proving the two assertions of the corollary when H consists of *positive* functions."

Suppose H consists of lower semi-continuous functions and is directed for $\leqslant$, satisfying $\sup_{f \in \mathrm{H}} \int f \, d|\mu| < +\infty$. Since $f_1 \leqslant f_2$ implies $f_1^+ \leqslant f_2^+$ and $f_1^- \geqslant f_2^-$, it is clear that the set

$$\mathrm{H}^+ = \{f^+ : f \in \mathrm{H}\} \quad (\text{resp. } \mathrm{H}^- = \{f^- : f \in \mathrm{H}\})$$

consists of lower (resp. upper) semi-continuous functions (GT, IV, §6, No. 2, Prop. 2 and its analog for upper semi-continuous functions) and is directed for $\leqslant$ (resp. $\geqslant$). Moreover, from $f \geqslant f_0$ one infers that, for almost every x,

$$f^-(x) = f^+(x) - f(x) \geqslant f_0^+(x) - f(x),$$

whence

$$\int f^- \, d|\mu| \geqslant \int f_0^+ \, d|\mu| - \int f \, d|\mu|,$$

which shows that $\inf_{f \in \mathrm{H}} \int f^- \, d|\mu| > -\infty$.

Thus, the validity of both parts of Cor. 2 for functions $\geqslant 0$ will imply that

$$\int g^+ \, d\mu = \lim_{f \in \mathrm{H}} \int f^+ \, d\mu \quad \text{and} \quad \int g^- \, d\mu = \lim_{f \in \mathrm{H}} \int f^- \, d\mu,$$

as well as

$$\int g^+ \, d|\mu| = \sup_{f \in \mathrm{H}} \int f^+ \, d|\mu| \quad \text{and} \quad \int g^- \, d|\mu| = \inf_{f \in \mathrm{H}} \int f^- \, d|\mu|,$$

whence

$$\int g \, d\mu = \int g^+ \, d\mu - \int g^- \, d\mu = \lim_{f \in \mathrm{H}} \int (f^+ - f^-) \, d\mu = \lim_{f \in \mathrm{H}} \int f \, d\mu,$$

as well as

$$\begin{aligned}\int g\,d|\mu| &= \int (g^+ - g^-)\,d|\mu| = \int g^+\,d|\mu| - \int g^-\,d|\mu| \\ &= \sup_{f\in \mathrm{H}} \int f^+\,d|\mu| - \inf_{f\in \mathrm{H}} \int f^-\,d|\mu| \\ &= \sup_{f\in \mathrm{H}} \int^+ f^+\,d|\mu| + \sup_{f\in \mathrm{H}} \int (-f^-)\,d|\mu| \\ &= \lim_{f\in \mathrm{H}} \int^+ f^+\,d|\mu| + \lim_{f\in \mathrm{H}} \int (-f^-)\,d|\mu| \\ &= \lim_{f\in \mathrm{H}} \Big(\int f^+\,d|\mu| + \int (-f^-)\,d|\mu| \Big) \\ &= \lim_{f\in \mathrm{H}} \int f\,d|\mu| = \sup_{f\in \mathrm{H}} \int f\,d|\mu|\,.\end{aligned}$$

To summarize: The validity of Cor. 2 for functions $\geqslant 0$ implies the validity of the 'lower semi-continuous half' of Cor. 2 for functions of arbitrary sign (hence the validity of the corollary as stated).

IV.38, ℓ. -9 to -7.

"If H is directed for $\geqslant$ and consists of upper semi-continuous integrable functions f such that $0 \leqslant f \leqslant f_1$ with $f_1 \in \mathrm{H}$, then there exists a lower semi-continuous integrable function h such that $f_1 \leqslant h$ "

Given a set H of upper semi-continuous functions $\geqslant 0$, directed for $\geqslant$ and such that $\inf_{f\in\mathrm{H}} \int f\,d|\mu| > -\infty$, one chooses any $f_1 \in \mathrm{H}$; then the lower envelope g of H is equal to the lower envelope of the cofinal set

$$\mathrm{H}_1 = \{ f \in \mathrm{H} :\ 0 \leqslant f \leqslant f_1 \}\,;$$

also,

$$\inf_{f\in \mathrm{H}_1} \int f\,d|\mu| = \inf_{f\in \mathrm{H}} \int f\,d|\mu| > -\infty\,,$$

and if we can prove that g is integrable and that

$$\int g\,d\mu = \lim_{f\in \mathrm{H}_1} \int f\,d\mu \quad \text{and} \quad \int g\,d|\mu| = \inf_{f\in \mathrm{H}_1} \int f\,d|\mu|\,,$$

it will follow that

$$\int g\,d\mu = \lim_{f\in \mathrm{H}} \int f\,d\mu \quad \text{and} \quad \int g\,d|\mu| = \inf_{f\in \mathrm{H}} \int f\,d|\mu|\,.$$

Replacing H by H_1, we can suppose that $0 \leqslant f \leqslant f_1$ for all $f \in \mathrm{H}$.

Since f_1 is positive and integrable, $|\mu|^*(f) < +\infty$ (No. 2, Prop. 1), hence there exists a lower semi-continuous function h such that $f_1 \leqslant h$ and $|\mu|^*(h) < +\infty$ (§1, No. 3, Def. 3), and h is integrable by Prop. 5.

IV.38, ℓ. −6.

"... we may write $f = h - f'$ ..."

Not quite; if $f(x) < +\infty$ but $h(x) = +\infty$, then by definition $f'(x) = h(x) - f(x) = +\infty$, so that $h - f'$ is not defined at x. However, since h is integrable it is finite almost everywhere, so that $f = h - f'$ almost everywhere (which is good enough for calculating integrals).

IV.38, ℓ. −5, −4.

"... the f' form a directed set, for $\leqslant$, of lower semi-continuous integrable functions $\geqslant 0$ "

It is clear that the f' are $\geqslant 0$. Since H is directed for $\geqslant$, it suffices to show that if $u, v \in \mathrm{H}$ and $u \leqslant v \leqslant f_1$, then $u' \geqslant v'$. At any rate, $0 \leqslant u \leqslant v \leqslant f_1 \leqslant h$; if $v(x) < +\infty$ then also $u(x) < +\infty$, therefore

$$u'(x) = h(x) - u(x) \geqslant h(x) - v(x) = v'(x)\,,$$

whereas if $v(x) = +\infty$ then $v'(x) = 0 \leqslant u'(x)$.

Finally, to show that f' is lower semi-continuous ($f \in \mathrm{H}$), we revisit the proof of a result on sums of lower semi-continuous functions (GT, IV, §6, No. 2, Prop. 2). Recall that $0 \leqslant f \leqslant f_1 \leqslant h$ (where h depends only on f_1). The basic idea: h and $-f$ are lower semi-continuous on X, $h - f = h + (-f)$ is defined almost everywhere and is lower semi-continuous at every point x where it is defined, and $f' = h - f$ at almost every point of X; but we have to show that f' is lower semi-continuous at *every* point of X.

Fix $x \in \mathrm{X}$ and suppose that $f'(x) > k \in \overline{\mathbf{R}}$; we seek a neighborhood V of x in X such that $f' > k$ on V. From $f'(x) > k$ we know that $k < +\infty$. If $k < 0$ (in particular, if $k = -\infty$) then, since $f' \geqslant 0$ on X, one has $f' > k$ on the neighborhood $\mathrm{V} = \mathrm{X}$ of x. Suppose $k \geqslant 0$, so that $0 \leqslant k < +\infty$ and $f'(x) > k \geqslant 0$. Since $f'(x) > 0$, we know that $f(x)$ must be finite and $f'(x) = h(x) - f(x)$ (possibly equal to $+\infty$). Thus $h(x) - f(x) > k \geqslant 0$. It follows that

$$k - h(x) < -f(x)$$

(trivial if $h(x) = +\infty$, elementary algebra if $h(x)$ is finite). As in the cited Prop. 2, let r and s be (finite) real numbers such that

$$k = r + s\,, \quad r < h(x)\,, \quad s < -f(x)\,;$$

explicitly, choose s so that

$$k - h(x) < s < -f(x)\,,$$

note that $k - s < h(x)$ (trivial if $h(x) = +\infty$, elementary algebra if $h(x)$ is finite) and set $r = k - s$. If V_1 and V_2 are neighborhoods of x such that

$$h > r \text{ on } V_1 \quad \text{and} \quad -f > s \text{ on } V_2\,,$$

then both inequalities hold on $V = V_1 \cap V_2$; and since $f < -s < +\infty$ on V, one has $f' = h - f$ on V and

$$f'(y) = h(y) + \big(-f(y)\big) > r + s = k \quad \text{for all } y \in V\,,$$

briefly $f' > k$ on V.

IV.38, ℓ. −2.

"... we can apply to them what has been proved above ..."

"them" is the set $H' = \{f' : f \in H\}$, consisting of lower semi-continuous integrable functions $\geqslant 0$, and H' is directed for the relation $\leqslant$. Recall that the lower envelope of H is denoted g in the statement of Cor. 2. Let us write $\widehat{g}$ for the upper envelope of H'. {The text uses the notation g', but this is confusing since g' is not derived from g in the same way that f' is derived from f; more about this later.}

As just shown, $\sup_{f \in H} \int f'\,d|\mu| < -\infty$, thus, by the case of Cor. 2 for positive lower semi-continuous functions, we know that $\widehat{g}$ is integrable and that

$$\int \widehat{g}\,d\mu = \lim_{f\in H} \int f'\,d\mu \quad \text{and} \quad \int \widehat{g}\,d|\mu| = \sup_{f\in H} \int f'\,d|\mu|\,.$$

Recall that $0 \leqslant f \leqslant f_1 \leqslant h$ for all $f \in H$. Note that

$$0 \leqslant f' \leqslant h \quad \text{for all } f \in H\,;$$

for, if $f(x) = +\infty$ then $f'(x) = 0 \leqslant h(x)$, whereas if $f(x) < +\infty$ then $f'(x) = h(x) - f(x)$ is $\leqslant h$ because $f \geqslant 0$, and $\geqslant 0$ because $f \leqslant h$. It follows that $0 \leqslant \widehat{g} \leqslant h$ everywhere in X. Since h is integrable, the set

$$A = \{x \in X : \ h(x) = +\infty\}$$

is negligible. If $x \in \complement A$ then $0 \leqslant f(x) \leqslant h(x) < +\infty$ $(f \in H)$, and $f'(x) = h(x) - f(x)$, $g(x)$ and $\widehat{g}(x)$ are finite, therefore

$$\begin{aligned}\widehat{g}(x) = \sup_{f\in H} f'(x) = \sup_{f\in H}[h(x) - f(x)] = h(x) + \sup_{f\in H}\big(-f(x)\big) \\ = h(x) - \inf_{f\in H} f(x) = h(x) - g(x)\,;\end{aligned}$$

thus $g = h - \widehat{g}$ almost everywhere, and since h and $\widehat{g}$ are integrable, so is g. {Incidentally, the argument shows that if g' were to be defined to be $h(x) - g(x)$ when $g(x) < +\infty$ and to be 0 otherwise, then $g' = \widehat{g}$ almost everywhere.} Moreover, since, for every $f \in \mathrm{H}$, $f' = h - f$ almost everywhere, we have

$$\begin{aligned}\int g\,d\mu &= \int h\,d\mu - \int \widehat{g}\,d\mu = \int h\,d\mu - \lim_{f\in\mathrm{H}} \int f'\,d\mu \\ &= \int h\,d\mu - \lim_{f\in\mathrm{H}} \Big(\int h\,d\mu - \int f\,d\mu \Big),\end{aligned}$$

therefore $\lim\limits_{f\in\mathrm{H}} \int f\,d\mu$ exists and is equal to $\int g\,d\mu$; and

$$\begin{aligned}\int g\,d|\mu| &= \int h\,d|\mu| - \int \widehat{g}\,d|\mu| = \int h\,d|\mu| - \sup_{f\in\mathrm{H}} \int f'\,d|\mu| \\ &= \int h\,d|\mu| - \sup_{f\in\mathrm{H}} \Big(\int h\,d|\mu| - \int f\,d|\mu| \Big) \\ &= \int h\,d|\mu| - \int h\,d|\mu| - \sup_{f\in\mathrm{H}} \Big(- \int f\,d|\mu| \Big) = \inf_{f\in\mathrm{H}} \int f\,d|\mu|\,,\end{aligned}$$

which completes the proof of Cor. 2.

IV.39, $\ell.\ 18, 19$.

"The condition is *sufficient* by a general criterion for integrability (§3, No. 4, Prop. 8), Prop. 5 and its Corollary 1."

Note that if $0 \leqslant g \leqslant f \leqslant h$ with h lower semi-continuous and integrable, and g is upper semi-continuous, then

$$\int^* g\,d|\mu| = \mathrm{N}_1(g) \leqslant \mathrm{N}_1(h) = \int^* h\,d|\mu|\,;$$

but $\int^* h\,d|\mu| = \int h\,d|\mu| < +\infty$ (No. 2, Prop. 1), thus $\int^* g\,d|\mu| < +\infty$, hence g is integrable by Cor. 1 of Prop. 5. By hypothesis, given any $\varepsilon > 0$ there exist such g and h with $\int (h-g)\,d|\mu| \leqslant \varepsilon$, that is,

$$\mathrm{N}_1(h-g) = \int^* |h-g|\,d|\mu| = \int (h-g)\,d|\mu| \leqslant \varepsilon\,,$$

whence f is integrable (§3, No. 4, Prop. 8).

IV.39, $\ell.\ -16, -15$.

"...since $u(x)$ is everywhere finite, it follows that $(u(x) - v(x))^+ \leqslant f(x) \leqslant u(x) + v(x)$ for all $x \in \mathrm{X}$."

From $-v(x) \leqslant f(x) - u(x) \leqslant v(x)$, we have

$$u(x) - v(x) \leqslant f(x) \leqslant u(x) + v(x);$$

also $0 \leqslant f(x)$, therefore $\sup(u(x) - v(x), 0) \leqslant f(x)$.

IV.39, ℓ. −14.

"The functions $g = (u - v)^+$ and $h = u + v$ meet the requirements."

At any rate, $0 \leqslant g \leqslant f \leqslant h$. Since u is continuous—hence lower semi-continuous—and since v is lower semi-continuous, it follows that $h = u+v$ is also lower semi-continuous (GT, IV, §6, No. 2, Prop. 2); since v is integrable by Prop. 5, $h = u + v$ is integrable.

Since $-v$ is upper semi-continuous and u is continuous, $u - v$ and $(u - v)^+ = \sup(u - v, 0)$ are upper semi-continuous (by the upper semi-continuous analog of the cited Prop. 2 from GT). Moreover, the function $g = (u - v)^+$ has compact support: if $u(x) = 0$ then from $-v \leqslant 0$ we infer that $(u(x) - v(x))^+ = 0$; therefore

$$(u - v)^+(x) \neq 0 \quad \Rightarrow \quad u(x) \neq 0,$$

whence $\operatorname{Supp}(u - v)^+ \subset \operatorname{Supp} u$. Also, g is finite-valued everywhere in X; for, if $v(x) = +\infty$ then $g(x) = 0$.

From $h - g = (u + v) - (u - v)^+$ and $u - v \leqslant (u - v)^+$ we see that

$$0 \leqslant h - g = (u + v) - (u - v)^+ \leqslant (u + v) - (u - v) = 2v,$$

whence

$$\mathrm{N}_1(h - g) \leqslant 2\,\mathrm{N}_1(v) \leqslant 2 \cdot \frac{\varepsilon}{2} = \varepsilon,$$

that is, $\int (h - g)\, d|\mu| \leqslant \varepsilon$ as desired.

IV.40, ℓ. 2, 3.

"…since $g \leqslant f$, g is integrable by Prop. 4 of No. 3"

The g_n are integrable (§3, No. 5, Cor. of Prop. 12) and $g_n \uparrow g \leqslant f$; since $\int g_n\, d|\mu| \leqslant \int f\, d|\mu| < +\infty$ for all n, it follows from Prop. 4 of No. 3 that g is integrable and that $\int g\, d|\mu| = \lim_n \int g_n\, d|\mu|$.

IV.40, ℓ. −15.

"*step function*"

The French original is "*fonction en escalier*" (literally "function in steps"), signifying a (finite) linear combination of characteristic functions of subintervals of $\mathbf{R}$ (FVR, II, §1, No. 3). Later in the section (No. 9, Def. 4) a more general concept is introduced, signifying a function $\mathrm{X} \to \mathrm{F}$

(F a Banach space) that is a finite sum $\sum_i \mathbf{a}_i \varphi_{\mathrm{M}_i}$, where the $\mathbf{a}_i$ belong to F and the M_i belong to a given clan Φ of subsets of X — so to speak, a linear combination of characteristic functions of sets of Φ, with 'coefficients' $\mathbf{a}_i$ in F; this concept is translated as "step function with respect to Φ", or "Φ-step function" (in the original, "*fonction* Φ-*étagée*" or "*fonction étagée sur les ensembles de* Φ"). A fuller discussion of how this double usage of "step function" came about is given in the note for **IV.66,** ℓ. −15, −14; for the present, suffice it to note that "step function" in the sense of FRV is intended also at the following places in *Integration*: the footnote on p. xi of Vol. I, and, in the present chapter, the hints for Exers. 29 *d*) and 30 for §5, and Exer. 17 for §6.

IV.40, ℓ. −12 to −8.

"It follows that if $\mathbf{f}$ is a *regulated* function on $\mathbf{R}$ with compact support (FRV, II, §1, No. 3), then $\mathbf{f}$ is integrable, because it is the uniform limit of a sequence of step functions $\mathbf{g}_n$ with support contained in a fixed compact set (No. 3, Prop. 3); moreover, $\int \mathbf{f}\,d\mu = \lim_{n\to\infty} \int \mathbf{g}_n\,d\mu$."

From the boldface notation and the references, we can suppose that $\mathbf{f} : \mathbf{R} \to \mathrm{F}$, F a Banach space.

Let $[a,b]$ be a compact interval in $\mathbf{R}$ such that $\operatorname{Supp}\mathbf{f} \subset [a,b]$ and let $\mathbf{g}_n : \mathbf{R} \to \mathrm{F}$ be a sequence of step functions such that $\mathbf{g}_n \to \mathbf{f}$ uniformly in $[a,b]$ (FRV, II, §1, No. 3, Def. 3). Multiplying $\mathbf{g}_n$ by the characteristic function of $[a,b]$, one can suppose that $\operatorname{Supp}\mathbf{g}_n \subset [a,b]$ for all n; then by Prop. 3 of No. 3, $\mathbf{f}$ is integrable and $\int \mathbf{f}\,d\mu = \lim_{n\to\infty} \int \mathbf{g}_n\,d\mu$.

IV.40, ℓ. −6, −5.

"... the integral $\int \mathbf{f}\,d\mu$ is equal to the integral $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ defined in FRV, II, §2, No. 1."

The following remark will be useful in the proof: *If* $\mathbf{f} : \mathbf{R} \to \mathrm{F}$ *is a regulated function* (F *a Banach space*) *such that* $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ *exists, then, for every closed interval* $[a,b]$ *of* $\mathbf{R}$, $\int_{-\infty}^{+\infty} (\varphi_{[a,b]}\mathbf{f})(x)\,dx$ *exists and*

$$\int_{-\infty}^{+\infty} (\varphi_{[a,b]}\mathbf{f})(x)\,dx = \int_a^b \mathbf{f}(x)\,dx\,.$$

Proof. One knows that $\mathbf{f}$ has a primitive $\mathbf{F} : \mathbf{R} \to \mathrm{F}$ (FRV, II, §1, No. 3, Th. 2), thus $\int_a^b \mathbf{f}(x)\,dx$ exists and is equal to $\mathbf{F}(b) - \mathbf{F}(a)$ (FVR, II, §1, No. 4).

Now, $\varphi_{[a,b]}\mathbf{f}$ is also regulated (FRV, II, §1, No. 3, Th. 3). The core observation is that

$$\int_a^b (\varphi_{[a,b]}\mathbf{f})(x)\,dx = \int_a^b \mathbf{f}(x)\,dx\,.$$

Indeed, let $\mathbf{G} : \mathbf{R} \to \mathrm{F}$ be a primitive of $\varphi_{[a,b]}\mathbf{f}$, and consider the restrictions $\mathbf{G}_0$, $\mathbf{F}_0$, $(\varphi_{[a,b]}\mathbf{f})_0$, $\mathbf{f}_0$ of $\mathbf{G}$, $\mathbf{F}$, $\varphi_{[a,b]}\mathbf{f}$, $\mathbf{f}$ to $[a,b]$. Then $\mathbf{G}_0$, $\mathbf{F}_0$ are continuous and, for all but countably many x in $]a,b[$,

$$\mathbf{G}_0'(x) = (\varphi_{[a,b]}\mathbf{f})_0(x) = 1 \cdot \mathbf{f}_0(x) = \mathbf{f}(x)$$

and

$$\mathbf{F}_0'(x) = \mathbf{f}_0(x) = \mathbf{f}(x)\,,$$

therefore $\mathbf{G}_0$ and $\mathbf{F}_0$ differ by a constant vector (FRV, II, §1, Prop. 1) and

$$\begin{aligned}\int_a^b (\varphi_{[a,b]}\mathbf{f})(x)\,dx &= \mathbf{G}(b) - \mathbf{G}(a) = \mathbf{G}_0(b) - \mathbf{G}_0(a)\\ &= \mathbf{F}_0(b) - \mathbf{F}_0(a) = \mathbf{F}(b) - \mathbf{F}(a) = \int_a^b \mathbf{f}(x)\,dx\,.\end{aligned}$$

Next, we note that if $[r,s]$ is a closed interval of $\mathbf{R}$ that contains $[a,b]$, then

$$(\dagger) \qquad \int_r^s (\varphi_{[a,b]}\mathbf{f})(x)\,dx = \int_a^b \mathbf{f}(x)\,dx\,.$$

For, by formula (6) in FRV, II, §1, No. 5, one has

$$\begin{aligned}&\int_r^s (\varphi_{[a,b]}\mathbf{f})(x)\,dx = \\ &\qquad \int_r^a (\varphi_{[a,b]}\mathbf{f})(x)\,dx + \int_a^b (\varphi_{[a,b]}\mathbf{f})(x)\,dx + \int_b^s (\varphi_{[a,b]}\mathbf{f})(x)\,dx\,;\end{aligned}$$

the first and third terms of the right member are 0 because $\varphi_{[a,b]}\mathbf{f}$ is equal to 0 on $]r,a[$ and on $]b,s[$ (then consider its restrictions to $[r,a]$ and $[b,s]$ as in the foregoing discussion to argue that the zero function serves as primitives for these restrictions), whereas the middle term is equal to $\int_a^b \mathbf{f}(x)\,dx$, thus (†) is verified.

Now let $\mathfrak{K}$ be the set of all compact intervals $[r,s]$ of $\mathbf{R}$, which is directed for $\subset$. For every $[r,s] \in \mathfrak{K}$ satisfying $[r,s] \supset [a,b]$, the equality (†) holds; since such intervals $[r,s]$ are cofinal in $\mathfrak{K}$, it follows that the limit

$$\lim_{[r,s]\in\mathfrak{K}} \int_r^s (\varphi_{[a,b]}\mathbf{f})(x)\,dx$$

exists and is equal to $\int_a^b \mathbf{f}(x)\,dx$, consequently

$$\int_{-\infty}^{+\infty} (\varphi_{[a,b]}\mathbf{f})(x)\,dx = \int_a^b \mathbf{f}(x)\,dx$$

by the definition of the left member (FRV, II, §2, No. 1); if, moreover, $\operatorname{Supp}\mathbf{f} \subset [a,b]$, then $\varphi_{[a,b]}\mathbf{f} = \mathbf{f}$ and we have the following:

$$(\dagger\dagger) \qquad \int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx = \int_a^b \mathbf{f}(x)\,dx \quad \text{when } \operatorname{Supp}\mathbf{f} \subset [a,b]\,.$$

To return to the assertion of **IV.40**, $\ell.\ -6, -5$, suppose first that $\mathbf{f} = \mathbf{a}f$, where $\mathbf{a} \in \mathrm{F}$ and f is the characteristic function of an interval I in $\mathbf{R}$ with finite end-points $c \leqslant d$. Then, by No. 2, Cor. 2 of Th. 1,

$$\int \mathbf{f}\,d\mu = \mathbf{a}\int f\,d\mu = (d-c)\mathbf{a}\,.$$

But also $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx = (d-c)\mathbf{a}$. For, if $F : \mathbf{R} \to \mathbf{R}$ is the continuous piecewise linear function

$$F(x) = \begin{cases} 0 & \text{for } x \leqslant c \\ x-c & \text{for } c < x < d \\ d-c & \text{for } x \geqslant d \end{cases}$$

then $F'(x) = f(x)$ except at the points c, d, thus the function $\mathbf{F} : \mathbf{R} \to \mathrm{F}$ defined by $\mathbf{F} = \mathbf{a}F$ is a primitive for $\mathbf{f}$ in $\mathbf{R}$ (FRV, II, §1, No. 1, Def. 1), and, for every closed interval $[r,s]$ containing I,

$$\int_r^s \mathbf{f}(x)\,dx = \mathbf{F}(s) - \mathbf{F}(r) = (d-c)\mathbf{a} - 0$$

(FRV, II, §1, No. 4, paragraph preceding (2)), therefore

$$\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx = \lim_{r\to-\infty,\, s\to+\infty} \int_r^s \mathbf{f}(x)\,dx = (d-c)\mathbf{a},$$

and so

$$(*) \qquad \int \mathbf{f}\,d\mu = \int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$$

for all such $\mathbf{f}$ (FRV, II, §2, No. 1, formula (1)). It follows by linearity that (*) holds for every step function with compact support.

Now let $\mathbf{f} : \mathbf{R} \to \mathrm{F}$ be a regulated function with compact support and let $(\mathbf{g}_n)$ be a sequence of step functions with support contained in a fixed compact interval $[c, d]$, such that $\mathbf{g}_n \to \mathbf{f}$ uniformly in $\mathbf{R}$ (see the preceding note). By (*),

$$(**) \qquad \int \mathbf{g}_n \, d\mu = \int_{-\infty}^{+\infty} \mathbf{g}_n(x) \, dx \quad \text{for all } n\,;$$

we know that $\int \mathbf{f} \, d\mu = \lim_{n\to\infty} \int \mathbf{g}_n \, d\mu$, and it follows from FRV, II, §3, No. 1, Cor. 1 of Prop. 1 that

$$\int_{-\infty}^{+\infty} \mathbf{f}(x) \, dx = \int_c^d \mathbf{f}(x) \, dx = \lim_{n\to\infty} \int_c^d \mathbf{g}_n(x) \, dx = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \mathbf{g}_n(x) \, dx$$

(the first and last equalities by (††)), so passage to the limit in (**) shows that (*) holds for every regulated function $\mathbf{f}$ with compact support.

IV.40, ℓ. −1 to **IV.41,** ℓ. 1.

"... therefore $\int |\mathbf{f}| \, d\mu = \lim_{n\to\infty} \int_{-n}^{n} |\mathbf{f}(x)| \, dx$ by Th. 2 of No. 3"

By the cited Th. 2, $\int |\mathbf{f}| \, d\mu = \lim_{n\to\infty} \int \varphi_{\mathrm{I}_n} |\mathbf{f}| \, d\mu$. Since $\varphi_{\mathrm{I}_n} |\mathbf{f}|$ is regulated and has compact support,

$$(*) \qquad \int \varphi_{\mathrm{I}_n} |\mathbf{f}| \, d\mu = \int_{-\infty}^{+\infty} (\varphi_{\mathrm{I}_n} |\mathbf{f}|)(x) \, dx$$

by the preceding *Example.* With primitive-function arguments, one shows, as at the beginning of the preceding note, that

$$\int_{-n}^{n} (\varphi_{\mathrm{I}_n} |\mathbf{f}|)(x) \, dx = \int_{-n}^{n} |\mathbf{f}|(x) \, dx\,,$$

next that

$$\int_r^s (\varphi_{\mathrm{I}_n} |\mathbf{f}|)(x) \, dx = \int_{-n}^{n} |\mathbf{f}|(x) \, dx \quad \text{when } [r, s] \supset [\,-n, n]\,,$$

whence the limit

$$\lim_{r\to-\infty,\, s\to+\infty} \int (\varphi_{\mathrm{I}_n} |\mathbf{f}|)(x) \, dx$$

exists, and

$$\int_{-\infty}^{+\infty} (\varphi_{\mathrm{I}_n} |\mathbf{f}|(x) \, dx = \int_{-n}^{n} |\mathbf{f}|(x) \, dx\,;$$

therefore by (*),

$$\int \varphi_{\mathrm{I}_n} |\mathbf{f}| \, d\mu = \int_{-n}^{n} |\mathbf{f}|(x) \, dx \,,$$

and passage to the limit yields the desired formula.

IV.41, ℓ. 1, 2.

"...thus, the integral $\int_{-\infty}^{+\infty} \mathbf{f}(x) \, dx$ is *absolutely convergent* (FRV, II, §2, No. 3)."

Note that if $f : \mathbf{R} \to \mathbf{R}$ is a regulated function such that $f \geqslant 0$, and F is a primitive of f, then F is an increasing function; for, if $a \leqslant b$ then

$$F(b) - f(a) = \int_a^b f(x) \, dx \geqslant 0$$

by the theorem of the mean (FRV, I, §1, No. 5, Prop. 6), and it follows that for intervals $[a, b] \subset [r, s]$ one has $\int_a^b f(x) \, dx \leqslant \int_r^s f(x) \, dx$ (FRV, II, §1, No. 5, formula (6)).

In particular, if F is a primitive for the regulated function $|\mathbf{f}|$ then F is increasing. Thus if $\mathfrak{K}$ is the set of compact intervals $[r, s]$ of $\mathbf{R}$, directed by $\subset$, then the function

$$[r, s] \mapsto \int_r^s |\mathbf{f}|(x) \, dx$$

is increasing. As shown in the preceding note,

$$\int |\mathbf{f}| \, d\mu = \lim_{n \to \infty} \int_{-n}^{n} |\mathbf{f}|(x) \, dx \,;$$

since the intervals $[-n, n]$ are cofinal in $\mathfrak{K}$, it follows that

$$\sup_{[r,s] \in \mathfrak{K}} \int_r^s |\mathbf{f}|(x) \, dx = \sup_n \int_{-n}^{n} |\mathbf{f}|(x) \, dx = \lim_{n \to \infty} \int_{-n}^{n} |\mathbf{f}|(x) \, dx = \int |\mathbf{f}| \, d\mu \,,$$

in other words,

$$\lim_{r \to -\infty, \, s \to +\infty} \int_r^s |\mathbf{f}|(x) \, dx = \int |\mathbf{f}| \, d\mu < +\infty \,.$$

Thus, in the terminology following FRV, II, §2, No. 1, Def. 1, the integral of $|\mathbf{f}|$ over the interval $(-\infty, +\infty)$ is convergent and

$$\int_{-\infty}^{+\infty} |\mathbf{f}|(x) \, dx = \int |\mathbf{f}| \, d\mu \,;$$

that is, in the language of FRV, II, §2, No. 3, Def. 2, "the integral of $\mathbf{f}$ is absolutely convergent". It then follows from *loc. cit.*, Prop. 4, that $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ exists; more precisely, see the next note.

IV.41, ℓ. 2.

"Moreover, $\int \mathbf{f}\,d\mu = \int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ by Th. 2 of No. 3."

As remarked at the end of the preceding note, $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ exists (by FRV, II, §2, No. 3, Prop. 4). Thus, writing $\mathfrak{K}$ for the set of compact intervals $[a,b]$ of $\mathbf{R}$, directed by $\subset$,

$$\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx = \lim_{[a,b]\in\mathfrak{K}} \int_a^b \mathbf{f}(x)\,dx\,.$$

On the other hand, by the cited Th. 2 (with $|\mathbf{f}|$ playing the role of g; the intervals $[-n,n]$ form a countable base for $\mathfrak{K}$),

$$\int \mathbf{f}\,d\mu = \lim_{[a,b]\in\mathfrak{K}} \int \varphi_{[a,b]}\mathbf{f}\,d\mu\,.$$

Since $\varphi_{[a,b]}\mathbf{f}$ is regulated and has compact support,

$$\int \varphi_{[a,b]}\mathbf{f}\,d\mu = \int_{-\infty}^{+\infty} (\varphi_{[a,b]}\mathbf{f})(x)\,dx$$

by the preceding *Example*, and, as shown at the beginning of the note for IV.40, ℓ. $-6,-5$,

$$\int_{-\infty}^{+\infty} (\varphi_{[a,b]}\mathbf{f})(x)\,dx = \int_a^b \mathbf{f}(x)\,dx\,,$$

therefore

$$\int \mathbf{f}\,d\mu = \lim_{[a,b]\in\mathfrak{K}} \int \varphi_{[a,b]}\mathbf{f}\,d\mu = \lim_{[a,b]\in\mathfrak{K}} \int_a^b \mathbf{f}(x)\,dx = \int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx\,.$$

IV.41, ℓ. 2–4.

"Conversely, suppose that $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ is absolutely convergent; again, by Th. 2 of No. 3, $\int \mathbf{f}\,d\mu = \int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$."

Assuming $\mathbf{f}$ is a regulated function such that $\int_{-\infty}^{+\infty} |\mathbf{f}|(x)\,dx$ is convergent (i.e., exists; equivalently, the integrals $\int_{-n}^{n} |\mathbf{f}|(x)\,dx$ are bounded), the problem is to show that $\mathbf{f}$ is μ-integrable. (We know from FRV, II, §2, No. 3, Prop. 4 that $\int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ exists, but no use of this fact will be made.)

We will show that $\int^*|\mathbf{f}|\,d\mu < +\infty$, so that the function $|\mathbf{f}|$ can play the role of g in the cited Th. 2. The numerical function $g = |\mathbf{f}|$ is regulated and $\geqslant 0$. Let $\mathrm{I}_n = [-n, n]$ $(n = 1, 2, 3, \ldots)$. For every n, the function $\varphi_{\mathrm{I}_n} g$ is regulated, positive, and has compact support; by the above *Example*, $\varphi_{\mathrm{I}_n} g$ is μ-integrable and

$$\int \varphi_{\mathrm{I}_n} g\,d\mu = \int_{-\infty}^{+\infty} (\varphi_{\mathrm{I}_n} g)(x)\,dx\,.$$

Since $\int_{-\infty}^{+\infty} g(x)\,dx$ exists by hypothesis, we have

$$\int_{-\infty}^{+\infty} (\varphi_{\mathrm{I}_n} g)(x)\,dx = \int_{-n}^{n} g(x)\,dx$$

by the remark at the beginning of the note for **IV.40**, $\ell.\ -6, -5$; thus $\int \varphi_{\mathrm{I}_n} g\,d\mu = \int_{-n}^{n} g(x)\,dx$, and

$$\mu^*(\varphi_{\mathrm{I}_n} g) = \int \varphi_{\mathrm{I}_n} g\,d\mu = \int_{-n}^{n} g(x)\,dx \leqslant \int_{-\infty}^{+\infty} g(x)\,dx < +\infty \quad \text{for all } n$$

(the first equality by No. 2, Prop. 1); since $\varphi_{\mathrm{I}_n} g \uparrow g$ pointwise, by §1, No. 3, Th. 3 one has

$$\mu^*(g) = \sup_n \mu^*(\varphi_{\mathrm{I}_n} g) \leqslant \int_{-\infty}^{+\infty} g(x)\,dx < +\infty\,,$$

qualifying g for its role in the cited Th. 2: each $\varphi_{\mathrm{I}_n}\mathbf{f}$ is μ-integrable (it is regulated and has compact support), $|\varphi_{\mathrm{I}_n}\mathbf{f}| = \varphi_{\mathrm{I}_n} g \leqslant g$ for all n, and $\varphi_{\mathrm{I}_n}\mathbf{f} \to \mathbf{f}$ pointwise in $\mathbf{R}$, hence $\mathbf{f}$ is μ-integrable (and $\int \mathbf{f}\,d\mu = \lim_n \int \varphi_{\mathrm{I}_n}\mathbf{f}\,d\mu$). The formula $\int \mathbf{f}\,d\mu = \int_{-\infty}^{+\infty} \mathbf{f}(x)\,dx$ then follows from the preceding note.

IV.41, $\ell.\ -10$ to -8.

"... for a set to be *negligible*, it is necessary and sufficient that it be of *measure zero with respect to* $|\mu|$."

Every negligible function is integrable; for, the set $\mathscr{N}$ of negligible functions (the functions f such that $\mathrm{N}_1(f) = 0$, equivalently $\mathrm{N}_p(f) = 0$ for $1 \leqslant p < +\infty$) is the closure of $\{0\}$ in the topological vector space $\mathscr{F}^p$ (§3, No. 3), consequently $\mathscr{N} \subset \mathscr{L}^p$ for $1 \leqslant p < +\infty$ (*loc. cit.*, No. 4, Def. 2).

IV.42, $\ell.\ 4, 5$.

"1° *If* A *and* B *are two integrable sets such that* $\mathrm{B} \subset \mathrm{A}$, *then the set* $\mathrm{C} = \mathrm{A} - \mathrm{B}$ *is integrable* ..."

It follows that the condition $B \subset A$ can be omitted; for, $A \cup B$ is integrable by Prop. 6, and $A - B = (A \cup B) - B$.

IV.42, ℓ. 10.

"$\varphi_A = \inf_n \varphi_{A_n}$, therefore A is integrable (No. 3, Prop. 4)."

For the cited Prop. 4 to be applicable, the sequence (A_n) must be decreasing; thus the crux of the matter is to show that if A and B are integrable then $A \cap B$ is integrable, so that A_n may be replaced by $\bigcap_{k=1}^{n} A_k$. Since

$$\varphi_{A \cap B} = \varphi_A \varphi_B = \inf(\varphi_A, \varphi_B),$$

the integrability of $A \cap B$ is immediate from §3, No. 5, Cor. of Prop. 12.

Alternate proof: $A \cup B$ is integrable by Prop. 6, therefore $A \cup B - B$ is integrable by 1° of Prop. 7; but $A \cup B - B = A - B$, thus $A - B$ is integrable without the restriction $B \subset A$, and $A \cap B = A - (A - B)$ is integrable. This shows that the set Φ of integrable sets is a (Boolean) ring of sets, that is, a clan (No. 9, Prop. 17 below).

The property 2° of Φ (closure under countable intersections) makes it, so to speak, a "δ-ring" ("δ" as in "G_δ"). In general, Φ is not closed under countable unions, hence is not a σ-ring: if (A_n) is a sequence in Φ such that $|\mu|(A_n)$ is unbounded, then obviously the set $\bigcup_{n=1}^{\infty} A_n$ is not integrable. On the other hand, if the function $A \mapsto |\mu|(A)$ on Φ is bounded, then it follows from Prop. 8 below that Φ is closed under countable unions, hence is a σ-ring, and the formulas in this No. show that μ and $|\mu|$ are 'measures' in the set-function sense (see the 2nd Note for IV.85, ℓ. 19–27).

One notes that, in a δ-ring $\mathfrak{R}$, if (A_n) is a sequence that is 'bounded above' in the sense that there exists a set $B \in \mathfrak{R}$ such that $A_n \subset B$ for all n, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{R}$; for, $B - A_n \in \mathfrak{R}$ for all n, and $\bigcup_{n=1}^{\infty} A_n = B - \bigcap_{n=1}^{\infty} (B - A_n)$.

IV.43, ℓ. −15, −14.

"COROLLARY 1. — *Every compact set is integrable*; *every relatively compact open set is integrable.*"

(i) If A is compact (hence closed) then $|\mu|^*(A) < +\infty$ (§1, No. 4, Cor. of Prop. 16), hence A is integrable by Prop. 10.

(ii) If A is relatively compact, then $|\mu|^*(A) < +\infty$ by §1, same Cor.; if, moreover, A is open, then A is integrable by Prop. 10.

IV.43, ℓ. −13, −12.

"Corollary 2. — *For every positive measure* μ *on* X, $A \mapsto \mu^*(A)$ *is a capacity on* X (cf. GT, IX, §6, No. 9, *Example*)."

We have $\mu^* : \mathscr{P}(X) \to \overline{\mathbf{R}}$, where μ^* is monotonic (§1, No. 4, Prop. 16), $A_n \uparrow A$ implies $\mu^*(A_n) \uparrow \mu^*(A)$ (*loc. cit.*, Prop. 17), and if $A_n \downarrow A$ with the A_n (hence also A) compact, then $\mu(A_n) \downarrow \mu(A)$ (by Cor. 1 and No. 5, Cor. of Prop. 7). These are the defining properties of a capacity (GT, IX, §6, No. 9, Def. 8). {In TG (apparently published after GT), the axiom (CA_{III}) for a capacity f is changed to

$$f(K) = \inf\{f(U) : K \subset U, \ U \text{ open}\}$$

for every compact set K, such capacities being called 'continuous on the right'; this version of the axiom is also satisfied by μ^* (§1, No. 4, Prop. 19).}

IV.43, ℓ. −2, −1.

"... the proposition therefore follows from Prop. 10."

Alternate arrangement (with proof of the formulas). Let $H = \{\varphi_G : G \in \mathfrak{G}\}$; then H is directed for the relation $\leqslant$, and φ_A is the upper envelope of H.

The φ_G are lower semi-continuous and integrable, and, by hypothesis, $\sup_{G\in\mathfrak{G}} \int \varphi_G \, d|\mu| < +\infty$; therefore (No. 4, Cor. 2 of Prop. 5), φ_A is integrable, $\int \varphi_A \, d\mu = \lim_{G\in\mathfrak{G}} \int \varphi_G \, d\mu$ and $\int \varphi_A \, d|\mu| = \sup_{G\in\mathfrak{G}} \int \varphi_G \, d|\mu|$.

IV.44, ℓ. 7.

"... we are thus reduced to Prop. 11."

As the sets $H \in \mathfrak{F}$ with $H \subset H_0$ are cofinal, we can suppose that $H \subset U$ for all $H \in \mathfrak{F}$. By Prop. 11, the set

$$U \cap \complement B = U \cap \bigcup_{H\in\mathfrak{F}} \complement H = \bigcup_{H\in\mathfrak{F}} (U - H)$$

is integrable, therefore so is $B = U - (U - B)$ by No. 5, Prop. 7, and

$$\begin{aligned} \mu(U) - \mu(B) = \mu(U - B) &= \lim_{H,\mathfrak{F}} \mu(U - H) \\ &= \lim_{H,\mathfrak{F}} [\mu(U) - \mu(H)] = \mu(U) - \lim_{H,\mathfrak{F}} \mu(H), \end{aligned}$$

whence $\mu(B) = \lim_{H,\mathfrak{F}} \mu(H)$; similarly,

$$\begin{aligned} |\mu|(U) - |\mu|(B) = |\mu|(U - B) &= |\mu|\Big(\bigcup_{H\in\mathfrak{F}} (U - H)\Big) \\ &= \sup_{H\in\mathfrak{F}} |\mu|(U - H) = \sup_{H\in\mathfrak{F}} [|\mu|(U) - |\mu|(H)] \\ &= |\mu|(U) - \inf_{H\in\mathfrak{F}} |\mu|(H), \end{aligned}$$

whence $|\mu|(\mathrm{B}) = \inf_{\mathrm{H}\in\mathfrak{F}} |\mu|(\mathrm{H})$.

IV.44, ℓ. −11.

" $f \leqslant \varphi_{\mathrm{K}} + \delta\varphi_{\mathrm{B}}$ "

For $f(x) = 0$, the inequality is trivial. Suppose $f(x) > 0$. Then $\varphi_{\mathrm{A}} \geqslant f$ yields $\varphi_{\mathrm{A}}(x) > 0$, so $\varphi_{\mathrm{A}}(x) = 1$, $x \in \mathrm{A}$; if $x \notin \mathrm{K}$ (that is, $f(x) < \delta$) then $x \in \mathrm{A} - \mathrm{K} = \mathrm{B}$ and the desired inequality reduces to $f(x) \leqslant 0 + \delta \cdot 1$, whereas if $x \in \mathrm{K}$ then $\varphi_{\mathrm{K}}(x) = 1$ and the desired inequality results from $f(x) \leqslant \varphi_{\mathrm{A}}(x) = 1 = \varphi_{\mathrm{K}}(x)$.

IV.45, ℓ. 1–3.

"The condition is sufficient, because it says that, for the topology of convergence in mean, φ_{A} is in the closure of the set of integrable functions φ_{K} (K an arbitrary compact subset of A)."

Observe first that $\varphi_{\mathrm{A}} \in \mathscr{F}^1$: for $\varepsilon = 1$ choose a compact set $\mathrm{K}_1 \subset \mathrm{A}$ with $|\mu|^*(\mathrm{A} - \mathrm{K}_1) \leqslant 1$; then

$$\begin{aligned}|\mu|^*(\mathrm{A}) &= |\mu|^*\big((\mathrm{A} - \mathrm{K}_1) \cup \mathrm{K}_1\big) \\ &\leqslant |\mu|^*(\mathrm{A} - \mathrm{K}_1) + |\mu|^*(\mathrm{K}_1) \leqslant 1 + |\mu|^*(\mathrm{K}_1) < +\infty,\end{aligned}$$

thus $\varphi_{\mathrm{A}} \in \mathscr{F}^1$. The sufficiency of the condition then follows from the fact that $\mathscr{L}^1$ is closed in $\mathscr{F}^1$ (§3, No. 4, Def. 2).

The set $\mathfrak{F}$ of all compact sets $\mathrm{K} \subset \mathrm{A}$ is directed for $\subset$, and from $|\mu|(\mathrm{A}) - |\mu|(\mathrm{K}) = |\mu|(\mathrm{A} - \mathrm{K})$ and the monotonicity of $|\mu|$, one infers that

$$|\mu|(\mathrm{A}) = \lim_{\mathrm{K},\mathfrak{F}} |\mu|(\mathrm{K}) = \sup_{\mathrm{K}\in\mathfrak{F}} |\mu|(\mathrm{K});$$

and then (No. 5, Prop. 7) $|\mu(\mathrm{A}) - \mu(\mathrm{K})| = |\mu(\mathrm{A} - \mathrm{K})| \leqslant |\mu|(\mathrm{A} - \mathrm{K})$ yields $\mu(\mathrm{A}) = \lim_{\mathrm{K},\mathfrak{F}} \mu(\mathrm{K})$.

IV.45, ℓ. −11, −10.

" φ_{U} is the upper envelope of the set H of functions $f \in \mathscr{K}_+$ such that $f \leqslant \varphi_{\mathrm{U}}$ and $\mathrm{Supp}(f) \subset \mathrm{U}$ (cf. the proof of §1, No. 1, Lemma)"

Clearly $\mathrm{H} = \{f \in \mathscr{K}_+ : 0 \leqslant f \leqslant 1 \text{ and } \mathrm{Supp}\, f \subset \mathrm{U}\}$, and H is directed for the relation $\subset$. Writing

$$g(x) = \sup_{f\in\mathrm{H}} f(x) \quad (x \in \mathrm{X}),$$

we are to show that $g = \varphi_{\mathrm{U}}$. Since $f \leqslant \varphi_{\mathrm{U}}$ for all $f \in \mathrm{H}$, obviously $g \leqslant \varphi_{\mathrm{U}}$.

Conversely, assuming $x \in \mathrm{X}$, let us show that $\varphi_{\mathrm{U}}(x) \leqslant g(x)$. If $x \notin \mathrm{U}$ then $\varphi_{\mathrm{U}}(x) = 0 \leqslant g(x)$. Whereas if $x \in \mathrm{U}$, then there exists an $f \in \mathscr{K}_+$

such that $0 \leqslant f \leqslant 1$, $f(x) = 1$ and $f = 0$ on $\complement\mathrm{U}$: for, let V be a compact neighborhood of x such that $\mathrm{V} \subset \mathrm{U}$, and let f be a continuous function $\mathrm{X} \to [0,1]$ such that $f(x) = 1$ and $f = 0$ on $\complement\mathrm{V}$, whence $\mathrm{Supp}\, f \subset \mathrm{V} \subset \mathrm{U}$; thus $f \in \mathrm{H}$ and so $g(x) \geqslant f(x) = 1 = \varphi_{\mathrm{U}}(x)$.

IV.45, $\ell.\ -8, -7$.

"... the corollary is then immediate from the fact that if $f \in \mathrm{H}$ and $\mathrm{K} = \mathrm{Supp}\,(f)$, then $f \leqslant \varphi_{\mathrm{K}} \leqslant \varphi_{\mathrm{U}}$."

From $\varphi_{\mathrm{U}} = g = \sup_{f\in \mathrm{H}} f$ and Th. 1 of §1, No. 1, we have

$$(*) \qquad |\mu|^*(\mathrm{U}) = |\mu|^*(\varphi_{\mathrm{U}}) = \sup_{f\in\mathrm{H}} |\mu|(f)\,.$$

For $f \in \mathrm{H}$, writing $\mathrm{K}_f = \mathrm{Supp}\, f$, we know that $\mathrm{K}_f \subset \mathrm{U}$, so that

$$f = \varphi_{\mathrm{K}_f} f \leqslant \varphi_{\mathrm{K}_f} \cdot 1 \leqslant \varphi_{\mathrm{U}}\,;$$

in view of Cor. 1 of Prop. 10,

$$|\mu|^*(\mathrm{U}) \geqslant |\mu|^*(\mathrm{K}_f) = |\mu|(\mathrm{K}_f) \geqslant |\mu|(f)\,,$$

and it follows from (*) that

$$|\mu|^*(\mathrm{U}) \geqslant \sup_{f\in\mathrm{H}} |\mu|(\mathrm{K}_f) \geqslant \sup_{f\in\mathrm{H}} |\mu|(f) = |\mu|^*(\mathrm{U})\,,$$

whence

$$|\mu|^*(\mathrm{U}) = \sup_{f\in\mathrm{H}} |\mu|(\mathrm{K}_f)\,.$$

But $|\mu|(\mathrm{K}) \leqslant |\mu|^*(\mathrm{U})$ for *all* compact $\mathrm{K} \subset \mathrm{U}$, so *a fortiori*

$$|\mu|^*(\mathrm{U}) = \sup\{|\mu|(\mathrm{K}) :\ \mathrm{K} \subset \mathrm{U}\ \text{compact}\}.$$

IV.46, $\ell.\ 7, 8$.

"For, we have seen that $|\mu|^*(\mathrm{X}) = \|\mu\|$ (§1, No. 2); the proposition therefore follows from Prop. 10 of No. 6."

By the remarks following §1, No. 2, Def. 2, $|\mu|^*(\mathrm{X}) = \|\,|\mu|\,\|$, and $\|\,|\mu|\,\| = \|\mu\|$ (Ch. III, §1, No. 8, Cor. 1 of Prop. 10); μ is bounded if and only if $\|\mu\| < +\infty$ (*loc. cit.*, III.16, $\ell.\ -11, -10$), hence if and only if $|\mu|^*(\mathrm{X}) < +\infty$, that is (No. 6, Prop. 10), X is μ-integrable. This means, in turn, that $\varphi_{\mathrm{X}} = 1$ is μ-integrable (No. 5, Def. 2), in which case $\int 1\, d|\mu| = |\mu|^*(1)$ (No. 2, Prop. 1), that is, $\int d|\mu| = \|\mu\|$, and one writes

$$|\mu|(\mathrm{X}) = \int \varphi_{\mathrm{X}}\, d|\mu| = \int d|\mu|$$

(No. 5, Def. 2).

IV.46, ℓ. −8, −7.

"... if $\mathbf{f} \in \mathrm{M} \cap \mathrm{N}$ then $\mathrm{N}_p((\mathbf{f}-\mathbf{f}_0)\varphi_{\mathrm{K}}) \leqslant \varepsilon$ and $\mathrm{N}_p((\mathbf{f}-\mathbf{f}_0)\varphi_{\complement\mathrm{K}}) \leqslant 2a\varepsilon$ "

From $\mathbf{f} \in \mathrm{N}$ we see that $|\mathbf{f}-\mathbf{f}_0|\varphi_{\mathrm{K}} \leqslant \varepsilon\big(|\mu|(\mathrm{K})\big)^{-1/p}\varphi_{\mathrm{K}}$, whence

$$\mathrm{N}_p(|\mathbf{f}-\mathbf{f}_0|\varphi_{\mathrm{K}}) \leqslant \varepsilon\big(|\mu|(\mathrm{K})\big)^{-1/p}\mathrm{N}_p(\varphi_{\mathrm{K}}) = \varepsilon\big(|\mu|(\mathrm{K})\big)^{-1/p}\big(|\mu|(\mathrm{K})\big)^{1/p} = \varepsilon\,.$$

On the other hand, $|\mathbf{f}| \leqslant a$ (because $\mathbf{f} \in \mathrm{M}$), and $|\mathbf{f}_0| \leqslant a$, thus $|\mathbf{f}-\mathbf{f}_0| \leqslant |\mathbf{f}| + |\mathbf{f}_0| \leqslant 2a$; then $|\mathbf{f}-\mathbf{f}_0|\varphi_{\complement\mathrm{K}} \leqslant 2a\,\varphi_{\complement\mathrm{K}}$, whence

$$\mathrm{N}_p(|\mathbf{f}-\mathbf{f}_0|\varphi_{\complement\mathrm{K}}) \leqslant 2a\,\mathrm{N}_p(\varphi_{\complement\mathrm{K}}) = 2a\big((|\mu|(\complement\mathrm{K})\big)^{1/p} \leqslant 2a\varepsilon\,.$$

IV.46, ℓ. −1 to **IV.47,** ℓ. 3.

"It is clear that the sets M_{K} form a filter base $\mathfrak{B}$ on $\mathscr{L}^p_{\mathrm{F}}$, that the functions belonging to M_{K} are uniformly bounded, and that $\mathfrak{B}$ converges uniformly to $\mathbf{f}$ on every compact subset of X, whence the corollary."

The functions $h\mathbf{f}$ belong to $\mathscr{K}_{\mathrm{F}}$, hence to $\mathscr{L}^p_{\mathrm{F}}$ (§3, No. 4, Def. 2), thus $\mathrm{M}_{\mathrm{K}} \subset \mathscr{L}^p_{\mathrm{F}}$ for every compact set $\mathrm{K} \subset \mathrm{X}$. If $\mathrm{K}_1, \mathrm{K}_2$ are compact and K is a compact set such that $\mathrm{K} \supset \mathrm{K}_1 \cup \mathrm{K}_2$, then $\mathrm{M}_{\mathrm{K}} \subset \mathrm{M}_{\mathrm{K}_1} \cap \mathrm{M}_{\mathrm{K}_2}$ (because $h = 1$ on $\mathrm{K} \Rightarrow h = 1$ on K_1 and $h = 1$ on K_2), so that the M_{K} form a filter base on $\mathscr{L}^p_{\mathrm{F}}$. The uniform boundedness of the $h\mathbf{f}$ is clear from the boundedness of $\mathbf{f}$ and the fact that $\|h\| = 1$.

Given any compact $\mathrm{K}_0 \subset \mathrm{X}$, we are to show that $\mathfrak{B} \to \mathbf{f}$ uniformly on K_0 (it will then follow from Prop. 13 that $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$ and that $\mathfrak{B} \to \mathbf{f}$ in mean of order p). Given any $\varepsilon > 0$, suppose $\mathrm{K} \supset \mathrm{K}_0$, K compact, and let $h\mathbf{f} \in \mathrm{M}_{\mathrm{K}}$, where $h \in \mathscr{K}$, $0 \leqslant h \leqslant 1$ and $h = 1$ on K; then

$$x \in \mathrm{K}_0 \ \Rightarrow\ x \in \mathrm{K} \ \Rightarrow\ h(x) = 1 \ \Rightarrow\ (h\mathbf{f}-\mathbf{f})(x) = 0\,,$$

thus $\sup\limits_{x\in\mathrm{K}_0} |(h\mathbf{f}-\mathbf{f})(x)| = 0 < \varepsilon$ and the claimed uniform convergence on K_0 holds trivially.

IV.47, ℓ. 4, 5.

" $\int \mathbf{f}\,d\mu$ is the limit with respect to $\mathfrak{B}$ of the integrals $\int h\mathbf{f}\,d\mu$."

The integral is (by definition) a continuous linear mapping $\mathscr{L}^1_{\mathrm{F}} \to \mathrm{F}$ (No. 1, Def. 1).

IV.47, ℓ. 13–17.

"the mapping $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$ is *continuous* on the Banach space $\mathscr{C}^b(\mathrm{X};\mathrm{F})$; its restriction to the closure $\mathscr{C}^0(\mathrm{X};\mathrm{F})$ of $\mathscr{K}(\mathrm{X};\mathrm{F})$ in $\mathscr{C}^b(\mathrm{X};\mathrm{F})$, that is, to the space of continuous functions tending to 0 at the point at infinity

(Ch. III, §1, No. 2, Prop. 3), is therefore the *extension by continuity* of the integral to $\mathscr{C}^0(\mathrm{X};\mathrm{F})$."

The notations $\mathscr{C}^b(\mathrm{X};\mathrm{F})$ and $\mathscr{C}^0(\mathrm{X};\mathrm{F})$ are established in Ch. III, §1, No. 2. Measures are defined in the following subsection (*loc. cit.*, No. 3), the set $\mathscr{M}(\mathrm{X};\mathbf{C})$ of all (complex) measures μ being defined to be the vector space dual to the topological vector space $\mathscr{K}(\mathrm{X};\mathbf{C})$ (equipped with the direct limit topology).

Bounded measures μ are defined *loc. cit.*, No. 8, the definition of $\mu(f)$ being limited to $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$; it is observed that the linear subspace $\mathscr{M}^1(\mathrm{X};\mathbf{C})$ of $\mathscr{M}(\mathrm{X};\mathbf{C})$ consisting of the bounded measures μ may be regarded as the dual of the vector space $\mathscr{K}(\mathrm{X};\mathbf{C})$ equipped with the topology defined by the norm $\|f\| = \sup_{x\in\mathrm{X}} |f(x)|$, but the domain of μ remains $\mathscr{K}(\mathrm{X};\mathbf{C})$. The possibility of extending μ to $\mathscr{C}^b(\mathrm{X};\mathbf{C})$, and in particular to $\mathscr{C}^0(\mathrm{X};\mathbf{C})$, is not taken up, but it is is latent in the observation that if $g \in \mathscr{C}^b(\mathrm{X};\mathbf{C})$ then $g \cdot \mu$ is a bounded measure (*ibid.*, Prop. 12), whence $\varphi_{\mathrm{X}} = 1$ is $(g \cdot \mu)$-integrable (Prop. 12 of the present No. 7), inviting the definition $\mu(g) = \int 1\, d(g\cdot\mu)$ (No. 1, Def. 1). This is accomplished in the present No., more generally for $\mathbf{f} \in \mathscr{C}^b(\mathrm{X};\mathrm{F})$ (Cor. of Prop. 13): $\mathscr{C}^b(\mathrm{X};\mathrm{F}) \subset \mathscr{L}^p_{\mathrm{F}}(\mathrm{X},\mu)$ and, by the inequality (20), the mapping $\mathbf{f} \mapsto \mu(\mathbf{f})$ so defined is a linear mapping $\mathscr{C}^b(\mathrm{X};\mathrm{F}) \to \mathrm{F}$ continuous for the norm topology on $\mathscr{C}^b(\mathrm{X};\mathrm{F})$.

Write $L(f) = \int \mathbf{f}\, d\mu$ (μ a bounded measure, $\mathbf{f} \in \mathscr{C}^b(\mathrm{X};\mathrm{F})$), which is a linear mapping $\mathscr{C}^b(\mathrm{X};\mathrm{F}) \to \mathrm{F}$ continuous for the norm topology, and let $L_0 = L\big|\mathscr{C}^0(\mathrm{X};\mathrm{F})$ be the restriction of L to $\mathscr{C}^0(\mathrm{X};\mathrm{F})$. One has

$$\mathscr{K}(\mathrm{X};\mathrm{F}) \subset \mathscr{C}^0(\mathrm{X};\mathrm{F}) \subset \mathscr{C}^b(\mathrm{X};\mathrm{F}) \subset \mathscr{L}^1_{\mathrm{F}}(\mathrm{X},\mu)$$

(for μ bounded); since $\mathscr{K}(\mathrm{X};\mathrm{F})$ is by definition dense in $\mathscr{C}^0(\mathrm{X};\mathrm{F})$ for the norm topology, and since L_0 is continuous for that topology, it is clear that L_0 is the extension by continuity of the integral $\mathbf{f} \mapsto \int \mathbf{f}\, d\mu$ ($f \in \mathscr{K}(\mathrm{X};\mathrm{F})$) to $\mathscr{C}^0(\mathrm{X};\mathrm{F})$.

IV.47, ℓ. $-11, -10$.

"Let μ be a measure on X whose support $\mathrm{S} = \mathrm{Supp}(\mu)$ is *compact*; the open set $\mathrm{X} - \mathrm{S}$ is *negligible* (§2, No. 2, Prop. 5)."

In particular, φ_{X} and φ_{S} are equivalent, whence

$$|\mu|^*(\mathrm{X}) = |\mu|^*(\mathrm{S}) = |\mu|(\mathrm{S}) < +\infty\,,$$

thus X is integrable (No. 6, Prop. 10) and so μ is bounded (No. 7, Prop. 12), as already observed in Ch. III, §2, No. 3, Prop. 11.

IV.47, ℓ. −5, −4.

"If, moreover, $\mathbf{f}$ is *bounded on* S, it follows from (20) that

$$\left|\int \mathbf{f}\,d\mu\right| \leqslant \|\mu\| \cdot \sup_{x\in \mathrm{S}} |\mathbf{f}(x)|\,. \text{"} \tag{22}$$

We know that μ is bounded (see the preceding note). Let $\mathrm{M} = \sup_{x\in\mathrm{S}} |\mathbf{f}(x)|$; by assumption, $\mathrm{M} < +\infty$. It is also assumed that $\mathbf{f}$ is μ-integrable, equivalently, $\mathbf{f}\varphi_{\mathrm{S}}$ is μ-integrable. Then $|\mathbf{f}\varphi_{\mathrm{S}}| \leqslant \mathrm{M}\varphi_{\mathrm{S}} \leqslant \mathrm{M}\varphi_{\mathrm{X}}$ and

$$\left|\int \mathbf{f}\,d\mu\right| = \left|\int \mathbf{f}\varphi_{\mathrm{S}}\,d\mu\right| \leqslant \int |\mathbf{f}\varphi_{\mathrm{S}}|\ d|\mu| \leqslant \mathrm{M}\,|\mu|(1) = \mathrm{M}\,\|\mu\|\,,$$

where the first inequality holds by the inequality (5) of No. 2, Prop. 5 (cited also in the proof of (20), which, strictly speaking, is stated only for *continuous* bounded functions).

IV.47, ℓ. −3 to −1.

"In particular, if $\mathbf{f}$ is *continuous* on X then $\mathbf{f}$ is μ-integrable, since $\mathbf{f}h \in \mathscr{K}(\mathrm{X};\mathrm{F})$ for every function $h \in \mathscr{K}(\mathrm{X};\mathbf{R})$ equal to 1 on S (Ch. III, §1, No. 2, Lemma 1)."

Let h be any such function (which exists by the cited lemma). Then $\mathbf{f} - \mathbf{f}h = 0$ on S; since $\complement\mathrm{S}$ is μ-negligible (§2, No. 2, Prop. 5), it follows that $|\mathbf{f} - \mathbf{f}h| = 0$ almost everywhere, consequently $\mathrm{N}_1(\mathbf{f} - \mathbf{f}h) = 0$ (§2, No. 3, Prop. 6). Thus, $\mathbf{f} - \mathbf{f}h$ is negligible for μ (§2, No. 1, Def. 1) hence belongs to $\mathscr{L}^1_{\mathrm{F}}(\mathrm{X},\mu)$ (see the note for **IV.21,** ℓ. 7–9); but $\mathbf{f}h \in \mathscr{K}(\mathrm{X};\mathrm{F}) \subset \mathscr{L}^1_{\mathrm{F}}(\mathrm{X},\mu)$ (§3, No. 4, Def. 2), therefore $\mathbf{f} = (\mathbf{f} - \mathbf{f}h) + \mathbf{f}h \in \mathscr{L}^1_{\mathrm{F}}(\mathrm{X},\mu)$. (The same argument shows that $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}(\mathrm{X},\mu)$ for $1 \leqslant p < +\infty$.)

Thus, when μ has compact support, every $\mathbf{f} \in \mathscr{C}(\mathrm{X};\mathrm{F})$ is μ-integrable; since such a function is bounded on every compact subset of X, and in particular on $\mathrm{S} = \mathrm{Supp}(\mu)$, the inequality (22) holds for every $\mathbf{f} \in \mathscr{C}(\mathrm{X};\mathrm{F})$.

IV.48, ℓ. 9–11.

"…the mapping $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$ of $\mathscr{C}(\mathrm{X};\mathrm{F})$ into F is continuous for the topology of compact convergence."

The locally convex topology in question is defined by the family of semi-norms $p_{\mathrm{K}}(\mathbf{f}) = \sup_{x\in\mathrm{K}} |\mathbf{f}(x)|$, where $\mathrm{K} \subset \mathrm{X}$ is compact (GT, X, §1, No. 3, Example III and No. 6; TVS, II, §1, No. 2 and §4, No. 1, Cor. of Prop. 1; and the note for **III.39,** ℓ. 8–11). The inequality (22) shows that the linear mapping $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$ ($\mathbf{f} \in \mathscr{C}(\mathrm{X};\mathrm{F})$) is continuous for the topology defined by the semi-norm p_{S} (S the support of μ), hence for the (finer) topology defined by all the p_{K}.

IV.48, ℓ. 12–14.

"Then, there is a compact set $K \subset X$ and a number $a > 0$ such that $|\mu(\mathbf{f})| \leqslant a \cdot \sup_{x \in K} |\mathbf{f}(x)|$ for every function $\mathbf{f} \in \mathscr{K}(X; F)$ "

It follows from the hypothesis that the formula $q(\mathbf{f}) = \left| \int \mathbf{f}\, d\mu \right|$ defines a semi-norm on $\mathscr{K}(X; F)$ continuous for the topology of compact convergence; as that topology is defined by the family of semi-norms $p_K(\mathbf{f}) = \sup_{x \in K} |\mathbf{f}(x)|$ with $K \subset X$ compact (see the preceding note), at issue is the characterization of the continuous semi-norms on $\mathscr{K}(X; F)$ (and $\mathscr{C}(X; F)$) for the topology. As the topology of a topological vector space can be defined by a set of semi-norms if and only if it is locally convex (TVS, II, §4, No. 1, Cor. of Prop. 1), it is of interest (and notationally simpler) to consider the general case.

Let E be a vector space (over $\mathbf{R}$ or $\mathbf{C}$) and let p be a semi-norm on E. The sets

$$V(p, \alpha) = \{\mathbf{z} \in E : \ p(\mathbf{z}) \leqslant \alpha\} \quad (\alpha > 0)$$

form a fundamental system of neighborhoods of 0 for a locally convex topology τ_p on E, and τ_p is the coarsest topology on E that makes p continuous and is compatible with the additive group structure of E (TVS, II, §1, No. 2).

Lemma 1. *For a semi-norm* q *on* E, *the following conditions are equivalent*:

(a) q *is continuous for* τ_p ;

(b) $\tau_q \subset \tau_p$;

(c) *there exists a constant* $M > 0$ *such that* $q \leqslant Mp$ *on* E.

Proof. (b) ⇒ (a): q is continuous for τ_q, hence *a fortiori* for the finer topology τ_p.

(a) ⇒ (b): The topology τ_p is compatible with the additive structure of E and makes q continuous, hence τ_q is coarser than τ_p.

(c) ⇒ (a): For every $\alpha > 0$, clearly $p(\mathbf{z}) \leqslant \alpha/M \Rightarrow q(\mathbf{z}) \leqslant \alpha$, thus

$$\{\mathbf{z} : \ q(\mathbf{z}) \leqslant \alpha\} \supset \{\mathbf{z} : \ p(\mathbf{z}) \leqslant \alpha/M\};$$

since the right side is a neighborhood of 0 for τ_p (by the continuity of p for τ_p) so is the left side, consequently q is continuous at 0 for τ_p. It then follows from $|q(\mathbf{z}) - q(\mathbf{z}')| \leqslant q(\mathbf{z} - \mathbf{z}')$ that q is (uniformly) continuous for τ_p.

(a) ⇒ (c): Let $D = \{\lambda : \ |\lambda| \leqslant 1\}$, a neighborhood of 0 in the field of scalars. By hypothesis, q is continuous at 0 for τ_p, hence the set

$$V(q, 1) = \{\mathbf{z} : \ q(\mathbf{z}) \leqslant 1\} = \overset{-1}{q}(D)$$

is a neighborhood of 0 for τ_p; it follows that there exists a scalar $\alpha > 0$ such that

$$\{\mathbf{z} :\ q(\mathbf{z}) \leqslant 1\} \supset \{\mathbf{z} :\ p(\mathbf{z}) \leqslant \alpha\}.$$

Let $\mathrm{M} = 1/\alpha$. If $\mathbf{z} \in \mathrm{E}$ and $\varepsilon > 0$ is arbitrary, then the vector $\mathbf{z}' = \alpha\big(p(\mathbf{z}) + \varepsilon\big)^{-1}\mathbf{z}$ satisfies

$$p(\mathbf{z}') = \alpha\big(p(\mathbf{z}) + \varepsilon\big)^{-1} p(\mathbf{z}) < \alpha,$$

therefore $q(\mathbf{z}') \leqslant 1$, that is, $\alpha\big(p(\mathbf{z}) + \varepsilon\big)^{-1} q(\mathbf{z}) \leqslant 1$, in other words $q(\mathbf{z}) \leqslant \mathrm{M}\big(p(\mathbf{z}) + \varepsilon\big)$; since ε is arbitrary, $q(\mathbf{z}) \leqslant \mathrm{M}p(\mathbf{z})$.

Lemma 2. *Let Γ be a set of semi-norms on E and let τ be the (locally convex) topology on E defined by Γ, that is, the coarsest topology τ on E that is compatible with the additive structure of E and is such that every $p \in \Gamma$ is continuous* (TVS, II, §1, No. 2).

Assume, moreover, that if $p_1, \ldots, p_n$ belong to Γ then there exists a $p \in \Gamma$ such that $p \geqslant \sup_{1 \leqslant i \leqslant n} p_i$ (the upper envelope of $p_1, \ldots, p_n$). Then, for a semi-norm q on E, the following conditions are equivalent:

(i) *q is continuous for τ;*

(ii) *there exist a $p \in \Gamma$ and a constant $\mathrm{M} > 0$ such that $q \leqslant \mathrm{M}p$.*

Proof. Clearly τ is the coarsest topology, compatible with the additive structure of E, for which $\tau \supset \tau_p$ for all $p \in \Gamma$.

(ii) $\Rightarrow$ (i): By Lemma 1, q is continuous for τ_p, hence *a fortiori* for the finer topology τ.

(i) $\Rightarrow$ (ii): The sets $\mathrm{D}_\alpha = \{\lambda :\ |\lambda| \leqslant \alpha\}$ $(\alpha > 0)$ form a base for the neighborhoods of 0 in the field of scalars. By the definition of τ, a neighborhood base at $0 \in \mathrm{E}$ for τ is given by the finite intersections

$$\overset{-1}{p}_1(\mathrm{D}_{\alpha_1}) \cap \cdots \cap \overset{-1}{p}_n(\mathrm{D}_{\alpha_n}),$$

where $p_1, \ldots, p_n \in \Gamma$ and $\alpha_1, \ldots, \alpha_n$ are > 0 (TVS, *loc. cit.*). If $\alpha = \min(\alpha_1, \ldots, \alpha_n)$, then $\mathrm{D}_\alpha \subset \mathrm{D}_{\alpha_i}$ for all i, therefore the sets

$$\overset{-1}{p}_1(\mathrm{D}_\alpha) \cap \cdots \cap \overset{-1}{p}_n(\mathrm{D}_\alpha) \qquad (\alpha > 0,\ p_i \in \Gamma \text{ for } 1 \leqslant i \leqslant n)$$

also form a neighborhood base at 0 for τ. By assumption, there exists a $p \in \Gamma$ such that $p \geqslant \sup_{1 \leqslant i \leqslant n} p_i$. Then

$$p(\mathbf{z}) \leqslant \alpha \quad \Rightarrow \quad p_i(\mathbf{z}) \leqslant \alpha \text{ for } i = 1, \ldots, n,$$

thus $\overset{-1}{p}(\mathrm{D}_\alpha) \subset \overset{-1}{p}_i(\mathrm{D}_\alpha)$ for $i = 1, \ldots, n$, whence

$$\overset{-1}{p}(\mathrm{D}_\alpha) \subset \overset{-1}{p}_1(\mathrm{D}_\alpha) \cap \cdots \cap \overset{-1}{p}_n(\mathrm{D}_\alpha)\,,$$

therefore the sets

$$\overset{-1}{p}(\mathrm{D}_\alpha) \qquad (\,\alpha > 0\,,\ p \in \Gamma\,)$$

also form a neighborhood base at 0 for τ.

By assumption, q is continuous at 0 for τ, hence the set $\overset{-1}{q}(\mathrm{D}_1) = \{\mathbf{z} :\ q(\mathbf{z}) \leqslant 1\,\}$ is a neighborhood of 0 for τ; by the foregoing, there exist an $\alpha > 0$ and a $p \in \Gamma$ such that $\overset{-1}{q}(\mathrm{D}_1) \supset \overset{-1}{p}(\mathrm{D}_\alpha)$, that is,

$$\{\mathbf{z} :\ q(\mathbf{z}) \leqslant 1\,\} \supset \{\mathbf{z} :\ p(\mathbf{z}) \leqslant \alpha\,\}\,,$$

therefore, as argued in the proof of "(a) ⇒ (c)" of Lemma 1, $q \leqslant \mathrm{M}p$ with $\mathrm{M} = 1/\alpha$.

Consider now the space $\mathscr{K}(\mathrm{X};\mathrm{F})$ (or $\mathscr{C}(\mathrm{X};\mathrm{F})$) equipped with the topology of compact convergence, that is, the topology defined by the family of semi-norms

$$p_{\mathrm{K}}(\mathbf{f}) = \sup_{x \in \mathrm{K}} |\mathbf{f}(x)| \qquad (\,\mathrm{K} \subset \mathrm{X} \text{ compact}).$$

If $\mathrm{K}_1, \ldots, \mathrm{K}_n$ are compact and $\mathrm{K} = \bigcup_{i=1}^{n} \mathrm{K}_i$, then $p_{\mathrm{K}} \geqslant p_{\mathrm{K}_i}$ for $i = 1, \ldots, n$, therefore $p_{\mathrm{K}} \geqslant \sup_{1 \leqslant i \leqslant n} p_{\mathrm{K}_i}$; in view of Lemma 2, if q is a semi-norm on the space, then q is continuous for the topology of compact convergence if and only if $q \leqslant a \cdot p_{\mathrm{K}}$ for some $a > 0$ and compact set K, whence the original assertion of the text.

IV.48, ℓ. 16, 17.

" $\mu(h) = 0$ for every function $h \in \mathscr{K}(\mathrm{X};\mathbf{C})$ whose support does not intersect K "

Recall that $\mu(h\mathbf{a}) = \mu(h)\mathbf{a}$ (Ch. III, §3, No. 4, Prop. 8).

IV.48, ℓ. 17, 18.

"... which proves that $\mathrm{Supp}(\mu) \subset \mathrm{K}$."

By the foregoing argument, $\mu(h) = 0$ for every $h \in \mathscr{K}(\mathrm{X};\mathbf{C})$ such that $\mathrm{Supp}(h) \subset \complement\mathrm{K}$; this means that the measure induced by μ on the open set $\complement\mathrm{K}$ is 0 (Ch. III, §2, No. 1), therefore $\complement\mathrm{K} \subset \complement\mathrm{Supp}(\mu)$ by the definition of $\mathrm{Supp}(\mu)$ (*loc. cit.*, No. 2, Def. 1).

IV.48, ℓ. 22–24.

"The set of measures on X with compact support may therefore be identified with the *dual* $\mathscr{C}'(\mathrm{X};\mathbf{C})$ of the Hausdorff locally convex space $\mathscr{C}(\mathrm{X};\mathbf{C})$."

Write $\mathscr{M}_{\mathrm{c}}$ for the set of all measures on X with compact support. Since the support of the sum of two measures is the union of their supports (Ch. III, §2, No. 2, Prop. 4), it is clear that $\mathscr{M}_{\mathrm{c}}$ is a linear subspace of the space $\mathscr{M}(\mathrm{X};\mathbf{C})$ of all measures on X. According to Prop. 14, for each measure μ with compact support, every $\mathbf{f} \in \mathscr{C}(\mathrm{X};\mathbf{C})$ is μ-integrable, the linear form $\mathbf{f} \mapsto \int \mathbf{f}\,d\mu$ is continuous for the topology τ_{cc} of compact convergence on $\mathscr{C}(\mathrm{X};\mathbf{C})$ (hence is an element of the dual space $\mathscr{C}'(\mathrm{X};\mathbf{C})$), and it is the only element of $\mathscr{C}'(\mathrm{X};\mathbf{C})$ that extends the linear form $\mu : \mathscr{K}(\mathrm{X};\mathbf{C}) \to \mathbf{C}$.

For each $\mu \in \mathscr{M}_{\mathrm{c}}$ let us write $\mu' \in \mathscr{C}'(\mathrm{X};\mathbf{C})$ for its unique τ_{cc}-continuous extension to $\mathscr{C}(\mathrm{X};\mathbf{C})$. The mapping $\mathscr{M}_{\mathrm{c}} \to \mathscr{C}'(\mathrm{X};\mathbf{C})$ defined by $\mu \mapsto \mu'$ is linear; for instance, if $\mu_1, \mu_2 \in \mathscr{M}_{\mathrm{c}}$ then $\mu_1' + \mu_2'$ extends $\mu_1 + \mu_2$, therefore $(\mu_1 + \mu_2)' = \mu_1' + \mu_2'$. The mapping is injective; for, the restriction of μ' to $\mathscr{K}(\mathrm{X};\mathbf{C})$ is μ, so if $\mu' = 0$ then $\mu = 0$. Thus $\mu \mapsto \mu'$ is an injective linear mapping $\mathscr{M}_{\mathrm{c}} \to \mathscr{C}'(\mathrm{X};\mathbf{C})$; our problem is to show that it is also surjective.

Thus, given $\nu \in \mathscr{C}'(\mathrm{X};\mathbf{C})$, we seek a measure μ on X with compact support such that $\mu' = \nu$. Let $\mu = \nu \big| \mathscr{K}(\mathrm{X};\mathbf{C})$ and let us show that μ is a measure on X with compact support. It suffices to show that μ is a measure, for, by Prop. 14, compactness of its support is assured by the existence of the extension ν; and then $\mu' = \nu$, because ν extends μ.

Given any compact subset K of X, it suffices to show that there exists a constant $\mathrm{M} > 0$ such that

$$(*) \qquad |\mu(f)| \leqslant \mathrm{M} \cdot \|f\| \quad \text{for all } f \in \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}),$$

where $\|f\| = \sup_{x \in \mathrm{X}} |f(x)|$ (Ch. III, §1, No. 3). Now, the topology τ_{cc} on $\mathscr{C}(\mathrm{X};\mathbf{C})$ is defined by the set of semi-norms $p_{\mathrm{L}}(f) = \sup_{x \in \mathrm{L}} |f(x)|$ (L a compact subset of X), and the semi-norm $q(f) = |\nu(f)|$ ($f \in \mathscr{C}(\mathrm{X};\mathbf{C})$) is continuous for τ_{cc}; as shown in the note for **IV.48,** ℓ. 9–10, there exist a compact set L and a constant $\mathrm{M} > 0$ such that $q \leqslant \mathrm{M} \cdot p_{\mathrm{L}}$ on $\mathscr{C}(\mathrm{X};\mathbf{C})$, hence on $\mathscr{K}(\mathrm{X};\mathbf{C})$. Thus, for $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$, we have

$$|\mu(f)| = |\nu(f)| = q(f) \quad \text{and} \quad p_{\mathrm{L}}(f) \leqslant \|f\|,$$

therefore $|\mu(f)| \leqslant \mathrm{M} \cdot p_{\mathrm{L}}(f) \leqslant \mathrm{M} \cdot \|f\|$; in particular, (*) holds.

The argument shows incidentally that a measure μ with compact support is continuous on $\mathscr{K}(\mathrm{X};\mathbf{C})$ for the norm topology—not news, since μ is bounded (Ch. III, §2, No. 3, Prop. 11).

IV.48, ℓ. −13 to −11.

"... the topology of $\mathscr{C}(\mathrm{X};\mathbf{C})$ can be defined by the countable family of semi-norms $p_n(f) = \sup_{x\in \mathrm{K}_n} |f(x)|$, therefore $\mathscr{C}(\mathrm{X};\mathbf{C})$ is a *Fréchet space* in this case."

Following GT, X, §1, No. 6, we write $\mathscr{C}_c(\mathrm{X};\mathbf{C})$ for $\mathscr{C}(\mathrm{X};\mathbf{C})$ equipped with the topology of compact convergence (τ_{cc}). We know that τ_{cc} is defined by the semi-norms $p_{\mathrm{K}}(f) = \sup_{x\in\mathrm{K}} |f(x)|$ ($\mathrm{K} \subset \mathrm{X}$ compact), and the sets

$$\{f:\ p_{\mathrm{K}}(f) \leqslant \varepsilon\} \qquad (\mathrm{K}\subset \mathrm{X} \text{ compact},\ \varepsilon > 0\}$$

form a fundamental system of neighborhoods of 0 in $\mathscr{C}_c(\mathrm{X};\mathbf{C})$ (see the note for IV.48, ℓ. 12–14). To show that τ_{cc} is defined by the semi-norms p_n, it suffices to show that if K is any compact set in X then there exists an index n such that $p_{\mathrm{K}} \leqslant p_n$; for, this will imply that

$$\{f:\ p_n(f) \leqslant \varepsilon\} \subset \{f:\ p_{\mathrm{K}}(f) \leqslant \varepsilon\},$$

hence that the sets

$$\{f:\ p_n(f) \leqslant \varepsilon\} \qquad (\varepsilon > 0,\ n = 1,2,3,\dots)$$

form a fundamental system of neighborhoods of 0 for τ_{cc}.

If K is any compact set in X, then

$$\mathrm{K} \subset \mathrm{X} = \bigcup_{n=1}^{\infty} \mathrm{K}_n = \bigcup_{n=1}^{\infty} \overset{\circ}{\mathrm{K}}_{n+1},$$

thus the sets $\overset{\circ}{\mathrm{K}}_n$ form an open covering of K; since $\overset{\circ}{\mathrm{K}}_1 \subset \overset{\circ}{\mathrm{K}}_2 \subset \overset{\circ}{\mathrm{K}}_3 \subset \cdots$ and since K is covered by finitely many of the $\overset{\circ}{\mathrm{K}}_n$, one has $\mathrm{K} \subset \overset{\circ}{\mathrm{K}}_n$ for some n, whence $\mathrm{K} \subset \mathrm{K}_n$ and so $p_{\mathrm{K}} \leqslant p_{\mathrm{K}_n} = p_n$.

It follows that the sets

$$\{f:\ p_n(f) \leqslant 1/m\} \qquad (m, n = 1,2,3,\dots)$$

form a countable fundamental system of neighborhoods of 0 for τ_{cc}, and their intersection is visibly equal to $\{0\}$, therefore $\mathscr{C}_c(\mathrm{X};\mathbf{C})$ is metrizable (TVS, I, §3, No. 1); being locally convex and complete, it is a Fréchet space (TVS, II, §4, No. 1), hence is barreled (TVS, III, §4, No. 1, Cor. of Prop. 2).

IV.49, ℓ. 11.

"... and proves the proposition."

Write $\mathrm{V} = \{\mu \in \mathscr{C}'(\mathrm{X};\mathbf{C}) : |\mu(f) - \varepsilon_{x_0}(f)| \leqslant \delta \text{ for all } f \in \mathrm{H}\}$, where H is a compact subset of $\mathscr{C}_c(\mathrm{X};\mathbf{C})$, and $\delta > 0$; such sets form a fundamental system of neighborhoods of ε_0 in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ for the topology of uniform convergence in the compact subsets of $\mathscr{C}_c(\mathrm{X};\mathbf{C})$. The argument produces a neighborhood U of x_0 in X such that

$$x \in \mathrm{U} \;\Rightarrow\; |\varepsilon_x(f) - \varepsilon_{x_0}(f)| \leqslant \delta \quad \text{for all } f \in \mathrm{H},$$

that is, $x \in \mathrm{U} \Rightarrow \varepsilon_x \in \mathrm{V}$, whence the continuity of $x \mapsto \varepsilon_x$ at x_0 for the indicated topology on $\mathscr{C}'(\mathrm{X};\mathbf{C})$.

IV.49, ℓ. 17, 18.

"It is clear that on L, the topology induced by $\mathscr{T}$ is finer than the topology induced by $\mathscr{T}'$."

As in the note for **IV.48,** ℓ. 22-24, we write $\mathscr{M}_{\mathrm{c}}$ for the set of all measures on X with compact support, and $\mu \mapsto \mu'$ for the linear bijection $\mathscr{M}_{\mathrm{c}} \to \mathscr{C}'(\mathrm{X};\mathbf{C})$ described there.

Caution . There is defined on $\mathscr{M}(\mathrm{X};\mathbf{C})$ a topology of compact convergence (uniform convergence in the subsets of $\mathscr{K}(\mathrm{X};\mathbf{C})$ that are compact for the direct limit topology); the topology it induces on the subset $\mathscr{M}_{\mathrm{c}}$ of $\mathscr{M}(\mathrm{X};\mathbf{C})$ is not to be confused with the topology $\mathscr{T}$ on the dual space $\mathscr{C}'(\mathrm{X};\mathbf{C}) = \{\mu' : \mu \in \mathscr{M}_{\mathrm{c}}\}$ of $\mathscr{C}(\mathrm{X};\mathbf{C})$ ($\mathscr{T}$ is the topology of uniform convergence in the compact subsets of $\mathscr{C}(\mathrm{X};\mathbf{C})$, where $\mathscr{C}(\mathrm{X};\mathbf{C})$ is equipped with the topology of uniform convergence in the compact subsets of X). Thus, although it is possible to identify the vector space $\mathscr{M}_{\mathrm{c}}$ with the vector space $\mathscr{C}'(\mathrm{X};\mathbf{C})$, in the present discussion it is useful to maintain the distinction between them, that is, between $\mu \in \mathscr{M}_{\mathrm{c}}$ (a linear form on $\mathscr{K}(\mathrm{X};\mathbf{C})$) and μ' (a linear form on $\mathscr{C}(\mathrm{X};\mathbf{C})$).

{Incidentally, L is a vaguely closed linear subspace of $\mathscr{M}(\mathrm{X};\mathbf{C})$ (Ch. III, §2, No. 2, Prop. 6), but I see no way to invoke Prop. 17 of Ch. III, §1, No. 10).}

Since $\mathrm{L} \subset \mathscr{M}_{\mathrm{c}} \subset \mathscr{M}(\mathrm{X};\mathbf{C})$, we may write

$$\mathrm{L}' = \{\mu' : \mu \in \mathrm{L}\} \subset \mathscr{C}'(\mathrm{X};\mathbf{C}).$$

We have $\mathrm{L} \subset \mathscr{M}(\mathrm{X};\mathbf{C})$ and $\mathscr{T}'$ is a topology on $\mathscr{M}(\mathrm{X};\mathbf{C})$; let us write $\mathscr{T}' \cap \mathrm{L}$ for the topology on L induced by $\mathscr{T}'$.

On the other hand, $\mathrm{L}' \subset \mathscr{C}'(\mathrm{X};\mathbf{C})$ and $\mathscr{T}$ is a topology on $\mathscr{C}'(\mathrm{X};\mathbf{C})$; we write $\mathscr{T} \cap \mathrm{L}'$ for the topology on L' induced by $\mathscr{T}$.

Proposition 16 asserts, informally, that $\mathscr{T} \cap \mathrm{L} = \mathscr{T}' \cap \mathrm{L}$; we interpret this to mean that the mapping $\mu \mapsto \mu'$ $(\mu \in \mathrm{L})$ is a homeomorphism $\mathrm{L} \to \mathrm{L}'$ for the topologies $\mathscr{T}' \cap \mathrm{L}$ and $\mathscr{T} \cap \mathrm{L}'$.

In the present note, the task is to show that "$\mathscr{T} \cap \mathrm{L}$ is finer than $\mathscr{T}' \cap \mathrm{L}$"; we interpret this to mean that the inverse mapping $\mathrm{L}' \to \mathrm{L}$ defined by $\mu' \mapsto \mu$ $(\mu \in \mathrm{L})$ is continuous for the topologies $\mathscr{T} \cap \mathrm{L}'$ and $\mathscr{T}' \cap \mathrm{L}$. It will suffice to prove that the bijection $\mathscr{C}'(\mathrm{X};\mathbf{C}) \to \mathscr{M}_{\mathrm{c}}$ defined by $\mu' \mapsto \mu$ is continuous for $\mathscr{T}$ and $\mathscr{T}' \cap \mathscr{M}_{\mathrm{c}}$ (for, this will imply the asserted continuity for an arbitrary subset L of $\mathscr{M}_{\mathrm{c}}$, since $\mathscr{T}' \cap \mathrm{L} = (\mathscr{T}' \cap \mathscr{M}_{\mathrm{c}}) \cap \mathrm{L}$). Informally, we will show that if $\mu' \to \mu'_0$ in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ for the topology of uniform convergence in the compact subsets of $\mathscr{C}(\mathrm{X};\mathbf{C})$, then $\mu \to \mu_0$ in $\mathscr{M}_{\mathrm{c}}$ for the topology of strictly compact convergence. {The argument can be formalized using the concept of convergence of nets, as exposed in S. Willard's *General topology* [Addison–Wesley, Reading, 1970], p. 75, but it is simpler to argue with neighborhoods of zero in locally convex spaces.} We will need the following two lemmas.

Lemma 1. *If* J *is a compact subset of* X, *then the topology of* $\mathscr{C}(\mathrm{X};\mathbf{C})$ *induces on* $\mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$ *the norm topology.*

Proof. $\mathscr{C}(\mathrm{X};\mathbf{C})$ bears the topology τ_{cc} of uniform convergence in the compact subsets of X, thus the assertion is that $\tau_{\mathrm{cc}} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$ is the topology of uniform convergence in X (equivalently, in J), in other words the norm topology, which we shall denote by τ_{n}. Thus, the assertion is that $\tau_{\mathrm{cc}} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}) = \tau_{\mathrm{n}}$.

Let G be a compact subset of X, let $\varepsilon > 0$, and write

$$\mathrm{V}(\mathrm{G},\varepsilon) = \{f \in \mathscr{C}(\mathrm{X};\mathbf{C}) : \ |f(x)| \leqslant \varepsilon \text{ for all } x \in \mathrm{G}\};$$

such sets form a fundamental system of neighborhoods of 0 for τ_{cc}. Then

$$\begin{aligned}\mathrm{V}(\mathrm{G},\varepsilon) \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}) &= \{f \in \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}) : \ |f(x)| \leqslant \varepsilon \text{ for all } x \in \mathrm{G}\} \\ &\supset \{f \in \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}) : \ |f(x)| \leqslant \varepsilon \text{ for all } x \in \mathrm{X}\},\end{aligned}$$

which shows that $\mathrm{V}(\mathrm{G},\varepsilon) \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$ is a neighborhood of 0 for τ_{n}, whence $\tau_{\mathrm{cc}} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}) \subset \tau_{\mathrm{n}}$.

On the other hand, for $\varepsilon > 0$ write

$$\mathrm{W}(\varepsilon) = \{f \in \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}) : \ |f(x)| \leqslant \varepsilon \text{ for all } x \in \mathrm{X}\};$$

such sets form a fundamental system of neighborhoods of 0 for τ_{n}. Then

$$\mathrm{W}(\varepsilon) = \{f \in \mathscr{C}(\mathrm{X};\mathbf{C})) : \ |f(x)| \leqslant \varepsilon \text{ for all } x \in \mathrm{J}\} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C}),$$

which shows that $\mathrm{W}(\varepsilon)$ is a neighborhood of 0 for $\tau_{\mathrm{cc}} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$, whence $\tau_{\mathrm{n}} \subset \tau_{\mathrm{cc}} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$.

Lemma 2. *If* S *is a strictly compact subset of* $\mathscr{K}(\mathrm{X};\mathbf{C})$, *then* S *is compact in* $\mathscr{C}(\mathrm{X};\mathbf{C})$.

Proof. Let J be a compact subset of X such that S is a compact subset of $\mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$ for the norm topology. By Lemma 1, S is compact in $\mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$ for the topology $\tau_{\mathrm{cc}} \cap \mathscr{K}(\mathrm{X},\mathrm{J};\mathbf{C})$, whence it is elementary that S is compact in $\mathscr{C}(\mathrm{X};\mathbf{C})$ for τ_{cc}.

We are ready to prove that the (bijective) linear mapping $\Phi : \mathscr{C}'(\mathrm{X};\mathbf{C}) \to \mathscr{M}_{\mathrm{c}}$ defined by $\Phi(\mu') = \mu$ is continuous for the topologies $\mathscr{T}$ and $\mathscr{T}' \cap \mathscr{M}_{\mathrm{c}}$. Given a strictly compact subset S of $\mathscr{K}(\mathrm{X};\mathbf{C})$, and $\varepsilon > 0$, let

$$\mathrm{U}(\mathrm{S},\varepsilon) = \{\mu \in \mathscr{M}(\mathrm{X};\mathbf{C}) : \ |\mu(f)| \leqslant \varepsilon \ \text{ for all } \ f \in \mathrm{S}\,\};$$

such sets form a fundamental system of neighborhoods of 0 in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for $\mathscr{T}'$, hence the sets

$$\mathrm{U}(\mathrm{S},\varepsilon) \cap \mathscr{M}_{\mathrm{c}} = \{\mu \in \mathscr{M}_{\mathrm{c}} : \ |\mu(f)| \leqslant \varepsilon \ \text{ for all } \ f \in \mathrm{S}\,\}$$

form a fundamental system of neighborhoods of 0 in $\mathscr{M}_{\mathrm{c}}$ for $\mathscr{T}' \cap \mathscr{M}_{\mathrm{c}}$. Since S is a compact subset of $\mathscr{C}(\mathrm{X};\mathbf{C})$ (Lemma 2), the set

$$\mathrm{V} = \{\mu' \in \mathscr{C}'(\mathrm{X};\mathbf{C}) : \ |\mu'(f)| \leqslant \varepsilon \ \text{ for all } \ f \in \mathrm{S}\,\}$$

is a neighborhood of 0 in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ for $\mathscr{T}$; but $\mu' \big| \mathscr{K}(\mathrm{X};\mathbf{C}) = \mu$, so

$$\begin{aligned} \mathrm{V} &= \{\mu' \in \mathscr{C}'(\mathrm{X};\mathbf{C}) : \ |\mu(f)| \leqslant \varepsilon \ \text{ for all } \ f \in \mathrm{S}\,\} \\ &= \{\mu' \in \mathscr{C}'(\mathrm{X};\mathbf{C}) : \ \mu \in \mathrm{U}(\mathrm{S},\varepsilon) \cap \mathscr{M}_{\mathrm{c}}\,\} \\ &= \overset{-1}{\Phi}\big(\mathrm{U}(\mathrm{S},\varepsilon) \cap \mathscr{M}_{\mathrm{c}}\big), \end{aligned}$$

thus $\overset{-1}{\Phi}\big(\mathrm{U}(\mathrm{S},\varepsilon) \cap \mathscr{M}_{\mathrm{c}}\big)$ is a neighborhood of 0 in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ for $\mathscr{T}$, whence the asserted continuity of Φ.

IV.49, ℓ. 19, 20.

"It is clear that the set H′ of functions fh, where f runs over H, is strictly compact in $\mathscr{K}(\mathrm{X};\mathbf{C})$"

The formula $\Psi(f) = fh$ defines a linear mapping $\Psi : \mathscr{C}(\mathrm{X};\mathbf{C}) \to \mathscr{C}(\mathrm{X};\mathbf{C})$, and $\Psi(\mathrm{H}) = \mathrm{H}'$. Moreover, if G is the support of h, then Ψ takes all of its values in $\mathscr{K}(\mathrm{X},\mathrm{G};\mathbf{C})$ and in particular $\mathrm{H}' \subset \mathscr{K}(\mathrm{X},\mathrm{G};\mathbf{C})$. It will suffice to prove that Ψ is continuous (for the topology τ_{cc} of compact convergence); for, this will imply that $\mathrm{H}' = \Psi(\mathrm{H})$ is compact in $\mathscr{C}(\mathrm{X};\mathbf{C})$ for τ_{cc}, whence H′ is compact in $\mathscr{K}(\mathrm{X},\mathrm{G};\mathbf{C})$ for the induced topology

$\tau_{cc} \cap \mathscr{K}(X, G; \mathbf{C})$, in other words, for the norm topology on $\mathscr{K}(X, G; \mathbf{C})$ (Lemma 1 of the preceding note), whence the strict compactness of H'.

For each compact subset J of X, let p_J be the semi-norm on $\mathscr{C}(X; \mathbf{C})$ defined by $p_J(f) = \sup_{x \in J} |f(x)|$; the topology τ_{cc} of $\mathscr{C}(X; \mathbf{C})$ is the topology defined by the semi-norms p_J. For $J \subset X$ compact and $\varepsilon > 0$, write

$$V(J, \varepsilon) = \{ f \in \mathscr{C}(X; \mathbf{C}) : \ p_J(f) \leqslant \varepsilon \};$$

the sets $V(J, \varepsilon)$ form a fundamental system of neighborhoods of 0 for τ_{cc}. For every $f \in \mathscr{C}(X; \mathbf{C})$ one has $p_J(fh) \leqslant p_J(f) \cdot \|h\|$, therefore

$$\{ f \in \mathscr{C}(X; \mathbf{C}) : \ p_J(f) \leqslant \varepsilon / \|h\| \} \subset \{ f \in \mathscr{C}(X; \mathbf{C}) : \ p_J(fh) \leqslant \varepsilon \},$$

in other words

$$V(J, \varepsilon/\|h\|) \subset \{ f \in \mathscr{C}(X; \mathbf{C}) : \ \Psi(f) \in V(J, \varepsilon) \} = \overset{-1}{\Psi}\big(V(J, \varepsilon)\big);$$

this shows that, for every J and ε, $\overset{-1}{\Psi}\big(V(J, \varepsilon)\big)$ is a neighborhood of 0 in $\mathscr{C}(X; \mathbf{C})$ for τ_{cc}, whence the continuity of Ψ.

IV.49, ℓ. 21.

" for every measure $\mu \in L$, $\mu(f) = \mu(fh)$ for every function $f \in H$ "

As $f \in \mathscr{C}(X; \mathbf{C})$ and $fh \in \mathscr{K}(X; \mathbf{C})$, a review of the notations is in order. In the expression $\mu(fh)$, μ is the original linear form on $\mathscr{K}(X; \mathbf{C})$. As noted at the beginning of the No., since μ has compact support, f is μ-integrable in the sense of No. 1, Def. 1 ($f - fh = 0$ on $K \supset \operatorname{Supp}(\mu)$ and $\complement\big(\operatorname{Supp}(\mu)\big)$ is negligible (§2, No. 2, Prop. 5), therefore $f = fh$ μ-almost everywhere) and, by Prop. 14, $f \mapsto \int f \, d\mu$ ($f \in \mathscr{C}(X; \mathbf{C})$) is the unique linear form on $\mathscr{C}(X; \mathbf{C})$ that is continuous for the topology of compact convergence and extends the original linear form $f \mapsto \mu(f)$ ($f \in \mathscr{K}(X; \mathbf{C})$). One also writes $\mu(f)$ for $\int f \, d\mu$ ($f \in \mathscr{C}(X; \mathbf{C})$), so that $\mu(f) = \mu(fh)$ (because $f = fh$ μ-almost everywhere).

In the note for **IV.48,** ℓ. 22–24, we have introduced the ephemeral notation μ' for the linear form in $\mathscr{C}(X; \mathbf{C})$ so defined, as an aid to understanding the proofs; thus $\mu'(f) = \mu(f)$ ($f \in \mathscr{C}(X; \mathbf{C})$) by the definition of μ', and $\mu'(f) = \mu(f) = \mu(fh) = \mu'(fh)$.

IV.49, ℓ. 21, 22.

"... whence the conclusion."

The linear bijection $\Phi : \mathscr{C}'(X; \mathbf{C}) \to \mathscr{M}_c$, $\Phi(\mu') = \mu$ ($\mu \in \mathscr{M}_c$) was introduced in the note for **IV.49,** ℓ. 17–18, and shown to be continuous for the topologies $\mathscr{T}$ and $\mathscr{T}' \cap \mathscr{M}_c$, whence the continuity of the linear bijection $\Phi\big|L' : L' \to L$ for the topologies $\mathscr{T} \cap L'$ and $\mathscr{T}' \cap L$; the desired conclusion is that $\Phi\big|L'$ is a homeomorphism, thus we must show that the inverse mapping $(\Phi\big|L')^{-1} : L \to L'$ is continuous for $\mathscr{T}' \cap L$ and $\mathscr{T} \cap L'$. Let us abbreviate $\Theta = (\Phi\big|L')^{-1}$.

For every compact subset H of $\mathscr{C}(X; \mathbf{C})$ and every $\varepsilon > 0$, write

$$W(H, \varepsilon) = \{\mu' \in \mathscr{C}'(X; \mathbf{C}) : \ |\mu'(f)| \leqslant \varepsilon \ \text{ for all } \ f \in H\};$$

such sets form a fundamental system of neighborhoods of 0 in $\mathscr{C}'(X; \mathbf{C})$ for the topology $\mathscr{T}$, thus the sets

$$W(H, \varepsilon) \cap L' = \{\mu' \in L' : \ |\mu'(f)| \leqslant \varepsilon \ \text{ for all } \ f \in H\}$$

form a fundamental system of neighborhoods of 0 in L' for the topology $\mathscr{T} \cap L'$.

Given any compact set H in $\mathscr{C}(X; \mathbf{C})$, choose $h \in \mathscr{K}(X; \mathbf{C})$ as in the text, so that the set $Hh = \{fh : \ f \in H\}$ is a strictly compact subset of $\mathscr{K}(X; \mathbf{C})$, and, when $\mu \in L$, $\mu'(f) = \mu'(fh) = \mu(fh)$ for all $f \in H$. Then the set

$$V(Hh, \varepsilon) = \{\mu \in \mathscr{M}(X; \mathbf{C}) : \ |\mu(fh)| \leqslant \varepsilon \ \text{ for all } \ f \in H\}$$

is a neighborhood of 0 in $\mathscr{M}(X; \mathbf{C})$ for the topology $\mathscr{T}'$, so that the set

$$V(Hh, \varepsilon) \cap L = \{\mu \in L : \ |\mu(fh)| \leqslant \varepsilon \ \text{ for all } \ f \in H\}$$

is a neighborhood of 0 in L for $\mathscr{T}' \cap L$. But, when $\mu \in L$, $\mu(fh) = \mu'(f)$ for all $f \in H$, thus

$$\begin{aligned} V(Hh, \varepsilon) \cap L &= \{\mu \in L : \ |\mu'(f)| \leqslant \varepsilon \ \text{ for all } \ f \in H\} \\ &= \{\mu \in L : \ \mu' \in W(H, \varepsilon) \cap L'\} \\ &= \{\Phi(\mu') : \ \mu' \in W(H, \varepsilon) \cap L'\} \\ &= \{(\Phi\big|L')(\mu') : \ \mu' \in W(H, \varepsilon) \cap L'\} \\ &= (\Phi\big|L')\big(W(H, \varepsilon) \cap L'\big) = \Theta^{-1}\big(W(H, \varepsilon) \cap L'\big); \end{aligned}$$

this shows that $\Theta^{-1}\big(W(H, \varepsilon) \cap L'\big)$ is a neighborhood of 0 in L for $\mathscr{T}' \cap L$, whence the continuity of Θ.

IV.49, ℓ. -12 to -8.

"For, let H be a subset of $\mathscr{C}(X; \mathbf{C})$ consisting of functions that are uniformly bounded on K; there exists a number $c > 0$ such that $|\mu(f)| \leqslant c \cdot \|\mu\| \leqslant ac$ for every function $f \in H$ and every measure $\mu \in B$, by virtue of (22); therefore $\mathrm{B} \subset ac\mathrm{H}^\circ$ in the dual $\mathscr{C}'(\mathrm{X};\mathbf{C})$ of $\mathscr{C}(\mathrm{X};\mathbf{C})$, which proves the equicontinuity of B ”

We retain the notations introduced in the note for **IV.48,** ℓ. 22–24, thus maintaining the distinction between a measure μ on X with compact support and its unique extension to a linear form μ' on $\mathscr{C}(\mathrm{X};\mathbf{C})$ continuous for the topology τ_{cc} of compact convergence; as observed in the cited note, $\mu \mapsto \mu'$ is a linear bijection $\mathscr{M}_{\mathrm{c}} \to \mathscr{C}'(\mathrm{X};\mathbf{C})$, where $\mathscr{M}_{\mathrm{c}}$ is the vector space of all measures on X with compact support. In that light, since $\mathrm{B} \subset \mathscr{M}_{\mathrm{c}}$ we interpret the assertions of the corollary to mean that the subset $\mathrm{B}' = \{\mu' : \ \mu \in \mathrm{B}\}$ of $\mathscr{C}'(\mathrm{X};\mathbf{C})$ acts equicontinuously on $\mathscr{C}(\mathrm{X};\mathbf{C})$, and that B' is compact for the topology $\mathscr{T}$.

The set H of the statement is a parameter at our disposal; we need only one such set that “works”. For simplicity, let $c > 0$ (for even greater simplicity let $c = 1$) and let

$$\begin{aligned}\mathrm{H} &= \{f \in \mathscr{C}(\mathrm{X};\mathbf{C}) : \ |f(x)| \leqslant c \ \text{ for all } \ x \in \mathrm{K}\} \\ &= \{f \in \mathscr{C}(\mathrm{X};\mathbf{C}) : \ p_{\mathrm{K}}(f) \leqslant c\},\end{aligned}$$

where $p_{\mathrm{K}}(f) = \sup_{x\in\mathrm{K}} |f(x)|$ defines a semi-norm on $\mathscr{C}(\mathrm{X};\mathbf{C})$; recalling that the topology of $\mathscr{C}(\mathrm{X};\mathbf{C})$ can be defined by such semi-norms, one knows that p_{K} is a continuous semi-norm and so H is a neighborhood of 0 in $\mathscr{C}(\mathrm{X};\mathbf{C})$. Moreover, H is a balanced subset of $\mathscr{C}(\mathrm{X};\mathbf{C})$; therefore, in the canonical duality

$$(f,\mu') \mapsto \mu'(f) \qquad \big(f \in \mathscr{C}(\mathrm{X};\mathbf{C}), \ \mu' \in \mathscr{C}'(\mathrm{X};\mathbf{C})\big),$$

the polar of H in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ is given by

$$\mathrm{H}^\circ = \{\mu' \in \mathscr{C}'(\mathrm{X};\mathbf{C}) : \ |\mu'(f)| \leqslant 1 \ \text{ for all } \ f \in \mathrm{H}\}$$

(TVS, II, §8, No. 4), H° is also balanced, and $\mathrm{H} \subset (\mathrm{H}^\circ)^\circ$ (*loc. cit.*, § 6, No. 3).

Now, for every $\mu \in \mathscr{M}_{\mathrm{c}}$ and every $f \in \mathscr{C}(\mathrm{X};\mathbf{C})$, f is μ-integrable (that is, $f \in \mathscr{L}^1_{\mathbf{C}}(\mathrm{X},\mu)$), the inequality (22) holds, and $\mu'(f) = \mu(f) = \int f\,d\mu$ defines, via Prop. 14, the element μ' of $\mathscr{C}'(\mathrm{X};\mathbf{C})$; thus $|\mu'(f)| = |\mu(f)| \leqslant \|\mu\| \cdot p_{\mathrm{S}}(f)$, where S is the support of μ. When $\mu \in \mathrm{B}$ and $f \in \mathrm{H}$, so that $\mathrm{S} \subset \mathrm{K}$, $\|\mu\| \leqslant a$ and $p_{\mathrm{K}}(f) \leqslant c$, we have $p_{\mathrm{S}} \leqslant p_{\mathrm{K}}$ and

$$|\mu'(f)| \leqslant \|\mu\| \cdot p_{\mathrm{S}}(f) \leqslant \|\mu\| \cdot p_{\mathrm{K}}(f) \leqslant ac\,,$$

that is, $|(ac)^{-1}\mu'(f)| \leqslant 1$; thus

$$(ac)^{-1}\mu' \subset \mathrm{H}^\circ \ \text{ for all } \ \mu' \in \mathrm{B}'\,,$$

that is, $(ac)^{-1}B' \subset H^\circ$.

Since H is a neighborhood of 0 in $\mathscr{C}(X;\mathbf{C})$ and $(H^\circ)^\circ \supset H$, $(H^\circ)^\circ$ is also a neighborhood of 0 in $\mathscr{C}(X;\mathbf{C})$, therefore H° is equicontinuous on $\mathscr{C}(X;\mathbf{C})$ (TVS, III, §3, No. 5, Prop. 7, (iii) ⇒ (i), with $E = \mathscr{C}(X;\mathbf{C})$ and $M = H^\circ$), hence so is its subset $(ac)^{-1}B'$, whence also B'.

IV.49, ℓ. −8 to −4.

"... the fact that B is compact for $\mathscr{T}$ follows from the fact that, on B, $\mathscr{T}$ and the vague topology induce the same topology (Prop. 16 and Ch. III, §1, No. 10, Prop. 17) and the fact that B is vaguely compact (Ch. III, §1, No. 9, Cor. 2 of Prop. 15 and §2, No. 2, Prop. 6)."

As explained at the beginning of the preceding note, we interpret the assertion to mean that $B' = \{\mu' : \mu \in B\}$ is a compact subset of $\mathscr{C}'(X;\mathbf{C})$ for the topology $\mathscr{T}$. Let us write

$$L = \{\mu \in \mathscr{M}(X;\mathbf{C}) : \operatorname{Supp}(\mu) \subset K\}$$
$$M = \{\mu \in \mathscr{M}(X;\mathbf{C}) : \|\mu\| \leqslant a\};$$

thus $B = L \cap M$. Since L is vaguely closed in $\mathscr{M}(X;\mathbf{C})$ (Ch. III, §2, No. 2, Prop. 6) and M is vaguely compact (Ch. III, §1, No. 9, Cor. 2 of Prop. 15), $B = L \cap M$ is vaguely compact, hence vaguely bounded (TVS, III, §1, No. 2, Prop. 2). In the notations of Ch. III, §1, No. 10, Prop. 17, B is compact, hence bounded, for the vague topology $\mathscr{T}_2$, therefore the induced topologies $\mathscr{T}_2 \cap B$ and $\mathscr{T}_3 \cap B$ are identical, where $\mathscr{T}_3 = \mathscr{T}'$ is the topology of strictly compact convergence; thus the topology $\mathscr{T}' \cap B = \mathscr{T}_2 \cap B$ is compact, that is, B *is compact for* $\mathscr{T}'$.

We know from Prop. 16 that the mapping $\mu \mapsto \mu'$ defines a homeomorphism $L \to L'$ for $\mathscr{T}' \cap L$ and $\mathscr{T} \cap L'$ (see the note for **IV.49,** ℓ. 21, 22), whence, since $B \subset L$, an induced homeomorphism $B \to B'$ for $\mathscr{T}' \cap B$ and $\mathscr{T} \cap B'$. Since B is compact for $\mathscr{T}'$, that is, the topology $\mathscr{T}' \cap B$ is compact, it follows that B' is compact for $\mathscr{T}$.

IV.50, ℓ. 3–6.

"For, on the set B of measures ν such that $\operatorname{Supp}(\nu) \subset \operatorname{Supp}(\mu)$ and $\|\nu\| \leqslant \|\mu\|$, the topology induced by the vague topology is identical to the topology induced by $\mathscr{T}$, and the corollary therefore follows from Ch. III, §2, No. 4, Cors. 2 and 3 of Th. 1."

We retain the conventions of the preceding note. Setting $K = \operatorname{Supp}(\mu)$ and $a = \|\mu\|$, we are in the situation of Cor. 1; thus the set

$$B = \{\nu \in \mathscr{M}(X;\mathbf{C}) : \operatorname{Supp}(\nu) \subset \operatorname{Supp}(\mu) \text{ and } \|\nu\| \leqslant \|\mu\|\}$$

is compact for the topology $\mathscr{T}'$ on $\mathscr{M}(X;\mathbf{C})$, and the induced (compact) topology $\mathscr{T}' \cap B$ coincides with $\mathscr{T}_2 \cap B$, where $\mathscr{T}_2$ is the vague topology on $\mathscr{M}(\mathrm{X};\mathbf{C})$. Let

$$\mathrm{C} = \{\nu \in \mathrm{B} : \ \|\nu\| = \|\mu\|\,\}$$
$$\mathrm{C}^+ = \{\nu \in \mathrm{C} : \ \nu \geqslant 0\,\}\,.$$

By Cor. 2 of Th. 1 of Ch. III, §2, No. 4, μ belongs to the closure of C in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for the vague topology, hence for $\mathscr{T}'$; and when $\mu \geqslant 0$, μ belongs to the closure of C^+ in $\mathscr{M}(\mathrm{X};\mathbf{C})$ for the vague topology (*loc. cit.*, Cor. 3), hence for $\mathscr{T}'$.

As observed in the preceding note, the mapping $\nu \mapsto \nu'$ defines a homeomorphism $\mathrm{B} \to \mathrm{B}'$ for the topologies $\mathscr{T}' \cap \mathrm{B}$ and $\mathscr{T} \cap \mathrm{B}'$, therefore μ' belongs to the closure of C' in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ for $\mathscr{T}$; and when $\mu \geqslant 0$, μ' belongs to the closure of $(\mathrm{C}^+)'$ in $\mathscr{C}'(\mathrm{X};\mathbf{C})$ for $\mathscr{T}$.

IV.53, *ℓ*. −8, −7.

"... for every compact subset S of X, μ and ν take on the same values in $\mathscr{K}(\mathrm{X},\mathrm{S};\mathbf{C})$."

Let $f \in \mathscr{K}(\mathrm{X},\mathrm{S};\mathbf{C})$. By Cor. 2, if $\varepsilon > 0$ there exists a finite linear combination

$$g = \sum_i \lambda_i \varphi_{\mathrm{K}_i}$$

such that $\|f - g\| \leqslant \varepsilon$, where the K_i are compact subsets of S, and it is clear from the hypothesis that $\mu(g) = \nu(g)$. Then $f - g$ is μ-integrable and $f - g = (f - g)\varphi_{\mathrm{S}}$, so

$$|f - g| = |f - g|\varphi_{\mathrm{S}} \leqslant \varepsilon\,\varphi_{\mathrm{S}}\,,$$

therefore (No. 2, Prop. 2)

$$|\mu(f) - \mu(g)| = |\mu(f - g)| \leqslant |\mu|(|f - g|) \leqslant \varepsilon\,|\mu|(\mathrm{S})\,,$$

where $|\mu|(\mathrm{S}) < +\infty$. Letting $\varepsilon = 1/n$, construct a sequence (g_n) of such linear combinations satisfying $\|f - g_n\| \leqslant 1/n$; then

$$|\mu(f) - \mu(g_n)| \leqslant \frac{1}{n}\,|\mu|(\mathrm{S})\,,$$

thus $\mu(g_n) \to \mu(f)$. Similarly $\nu(g_n) \to \nu(f)$. But $\mu(g_n) = \nu(g_n)$ for all n, whence $\mu(f) = \nu(f)$.

IV.55, *ℓ*. 10.

"... the I(K, U) form a *base* for the topology $\mathscr{T}$ "

The formula displayed in ℓ. 7 shows that the set of all $I(K,U)$ is closed under finite intersections, therefore the set of unions of arbitrary families of sets $I(K,U)$ is the set of all open sets for $\mathscr{T}$. In particular, for each $M \subset X$, the set of all $I(K,U)$ for which $K \subset M \subset U$ forms a fundamental system of neighborhoods of M. Note that $\mathscr{T}$ exists independently of any function α.

IV.55, ℓ. 12, 13.

"... in $\mathfrak{P}(X)$, the set of compact subsets of X is *dense*."

Let $\mathfrak{K}$ be the set of all compact subsets of X. If $\mathscr{O}$ is any nonempty open subset of $\mathfrak{P}(X)$ for $\mathscr{T}$, then $\mathscr{O}$ contains some set $I(K,U)$, whence $K \in I(K,U) \subset \mathscr{O}$; thus, every nonempty open subset of $\mathfrak{P}(X)$ contains an element of $\mathfrak{K}$.

IV.55, ℓ. 13.

"The condition (PC_{II}) expresses that Φ is *dense* in $\mathfrak{P}(X)$ "

If $\mathscr{O}$ is any nonempty open subset of $\mathfrak{P}(X)$ then $\mathscr{O}$ contains some set $I(K,U)$, and $I(K,U) \cap \Phi \neq \varnothing$ by (PC_{II}), whence $\mathscr{O} \cap \Phi \neq \varnothing$.

IV.55, ℓ. 14, 15.

"... condition (PM_{IV}) expresses that the function α is *continuous* on Φ for the topology induced by $\mathscr{T}$."

Let $M \in \Phi$. Given $\varepsilon > 0$, choose K and U as in (PM_{IV}); then $I(K,U)$ is a neighborhood of M in $\mathfrak{P}(X)$ for $\mathscr{T}$, and

$$N \in I(K,U) \cap \Phi \quad \Rightarrow \quad |\alpha(N) - \alpha(M)| \leqslant \varepsilon\,;$$

thus $I(K,U) \cap \Phi$ is a neighborhood of M in Φ (for the induced topology $\mathscr{T} \cap \Phi$) that is mapped by α into the ε-neighborhood of $\alpha(M)$ in $\mathbf{R}$.

IV.55, ℓ. 15–17.

"Th. 4 of No. 6 expresses that the function $M \mapsto \mu(M)$ is *continuous* on the clan of μ-integrable sets, for the topology induced by $\mathscr{T}$."

Here, μ can be any measure on X (not necessarily positive). Write Ψ for the set of all μ-integrable subsets of X; it follows from Props. 6 and 7 of No. 5 that Ψ is a clan, hence satisfies (PC_I). We are to show that the function $\Psi \to \mathbf{C}$ defined by $A \mapsto \mu(A)$ $(A \in \Psi)$ is continuous for the topology $\mathscr{T} \cap \Psi$ induced on Ψ by the topology $\mathscr{T}$ on $\mathfrak{P}(X)$.

Given $M \in \Psi$, let us show continuity at M. Given any $\varepsilon > 0$, choose K and G as in the cited Th. 4, so that $M \in I(K,G)$ and $|\mu|(G - K) \leqslant \varepsilon$. Then

$$N \in I(K,G) \cap \Psi \quad \Rightarrow \quad |\mu|(G - N) \leqslant |\mu|(G - K) \leqslant \varepsilon\,,$$

and in particular $|\mu|(\mathrm{G} - \mathrm{M}) \leqslant \varepsilon$, whence

$$\begin{aligned}|\mu(\mathrm{N}) - \mu(\mathrm{M})| &\leqslant |\mu(\mathrm{N}) - \mu(\mathrm{G})| + |\mu(\mathrm{G}) - \mu(\mathrm{M})| \\ &= |\mu(\mathrm{G} - \mathrm{N})| + |\mu(\mathrm{G} - \mathrm{M})| \\ &\leqslant |\mu|(\mathrm{G} - \mathrm{N}) + |\mu|(\mathrm{G} - \mathrm{M}) \leqslant 2\varepsilon\,.\end{aligned}$$

(This proves the asserted continuity, but Th. 4 says more.)

IV.55, ℓ. 19, 20.

"We denote by $\overline{\Phi}$ the set of subsets $\mathrm{M} \subset \mathrm{X}$ such that $\alpha(\mathrm{N})$ tends to a finite limit as N tends to M (for the topology $\mathscr{T}$) while remaining in Φ"

$\overline{\Phi}$ is not to be confused with the closure of Φ in $\mathfrak{P}(\mathrm{X})$ (it excludes the elements of the closure where the limit is either infinite or fails to exist), the closure of Φ being equal to $\mathfrak{P}(\mathrm{X})$ by $(\mathrm{PC_{II}})$.

IV.55, ℓ. 20–22.

"...we may then extend α in only one way to a *continuous* mapping $\overline{\alpha}$ of $\overline{\Phi}$ into $\mathbf{R}$ (GT, I, §8, No. 5, Th. 1)."

To apply the cited Th. 1, we must check that $\Phi \subset \overline{\Phi}$. One is assuming here that Φ and α satisfy the conditions in the first sentence of Th. 5, and that there exists a measure μ on X such that the sets of Φ are μ-integrable and $\mu(\mathrm{M}) = \alpha(\mathrm{M})$ for all $\mathrm{M} \in \Phi$ (since α is positive, it will turn out that μ is necessarily positive). The problem is to show that μ is uniquely determined by its values on Φ.

(1) One first observes that every μ-integrable set $\mathrm{M} \subset \mathrm{X}$ belongs to $\overline{\Phi}$. For, since Φ is dense in $\mathfrak{P}(\mathrm{X})$ for $\mathscr{T}$ (by $(\mathrm{PC_{II}})$), in particular M belongs to the closure of Φ; therefore if $\mathfrak{V}$ is the filter of neighborhoods $\mathscr{N}$ of M in $\mathfrak{P}(\mathrm{X})$, then its trace $\mathfrak{V} \cap \Phi$ on Φ is a filter on Φ (GT, I, §6, No. 5, *Example*). The assertion is that the filter base $\alpha(\mathfrak{V} \cap \Phi)$ on $\mathbf{R}$ converges to an element of $\mathbf{R}$ (GT, I, §7, No. 3, Prop. 7).

Now, μ is continuous for $\mathscr{T}$ on the clan Ψ of μ-integrable sets (see the note for ℓ. 15–17); the continuity of μ at $\mathrm{M} \in \Psi$ means that the filter base $\mu(\mathfrak{V} \cap \Psi)$ converges to $\mu(\mathrm{M}) \in \mathbf{R}$ (GT, I, §7, No. 4, Prop. 9). Thus, given any $\varepsilon > 0$, there exists a neighborhood $\mathscr{N}$ of M in $\mathfrak{P}(\mathrm{X})$ such that

$$|\mu(\mathrm{N}) - \mu(\mathrm{M})| \leqslant \varepsilon \quad \text{for all } \mathrm{N} \in \mathscr{N} \cap \Psi\,;$$

we are assuming that $\Phi \subset \Psi$ and that $\mu = \alpha$ on Φ, therefore

$$|\alpha(\mathrm{N}) - \mu(\mathrm{M})| \leqslant \varepsilon \quad \text{for all } \mathrm{N} \in \mathscr{N} \cap \Phi \in \mathfrak{V} \cap \Phi\,,$$

and we have shown that the filter base $\alpha(\mathfrak{V} \cap \Phi)$ converges to $\mu(\mathrm{M}) \in \mathbf{R}$. {Indeed, $\mu(\mathrm{M}) \in \mathbf{R}^+$ since $\alpha \geqslant 0$. Thus $\mu(\mathrm{M}) \geqslant 0$ for every μ-integrable set M. In particular, $\mu(K) \geqslant 0$ for all compact $K \subset X$, therefore $\mu \geqslant 0$ on $\mathscr{E}(\Psi)$, where $\mathscr{E}(\Psi)$ is the **R**-linear span of the characteristic functions of the sets in Ψ (No. 9, Prop. 18). It follows that if $f \in \mathscr{K}_+(X)$, there exists a sequence (g_n) of functions in $\mathscr{E}(\Psi)$ such that $0 \leqslant g_n \leqslant f$ for all n and $g_n \to f$ uniformly (No. 10, Prop. 19); since $\operatorname{Supp}(g_n) \subset \operatorname{Supp}(f)$, it follows that $\mu(f) = \lim_{n\to\infty} \mu(g_n) \geqslant 0$ (Ch. III, §1, No. 3), thus $\mu \geqslant 0$.}

(2) By (1), we have $\Phi \subset \Psi \subset \overline{\Phi}$; in particular $\Phi \subset \overline{\Phi}$, so we may speak of extensions of α to $\overline{\Phi}$.

Since Φ is dense in $\mathfrak{P}(X)$ for $\mathscr{T}$, it is dense in $\overline{\Phi}$ for $\mathscr{T} \cap \overline{\Phi}$; and since $\alpha : \Phi \to \mathbf{R}$ is such that, for every $M \in \overline{\Phi}$, the limit

$$\lim_{N\in\Phi,\, N\to M} \alpha(N)$$

exists in **R** (by the definition of $\overline{\Phi}$), it follows that α is uniquely extendible to a continuous mapping $\overline{\alpha} : \overline{\Phi} \to \mathbf{R}$ (GT, I, §8, No. 5, Th. 1), explicitly,

$$\overline{\alpha}(M) = \lim_{N\in\Phi,\, N\to M} \alpha(N) \qquad \text{for all } M \in \overline{\Phi}.$$

(3) If K is any compact set in X then, by (1), $\overline{\alpha}(K) = \mu(K)$; since μ is characterized by its values for compact sets (No. 10, Cor. 3 of Prop. 19), the uniqueness of $\overline{\alpha}$ implies that of μ.

IV.55, ℓ. −16, −15.

"Without assuming the existence of μ, we are now going to study the set $\overline{\Phi}$ and the extension $\overline{\alpha}$ of α to $\overline{\Phi}$."

For the rest of the proof, $(\mathrm{PM_{IV}})$ is in force, consequently α is continuous on Φ for the topology $\mathscr{T} \cap \Phi$ induced by $\mathscr{T}$ on Φ (see the note for ℓ. 14, 15). Then, for every $M \in \Phi$, the limit

$$\lim_{N\in\Phi,\, N\to M} \alpha(N)$$

exists and is equal to $\alpha(M)$, consequently $\Phi \subset \overline{\Phi}$. The formula

$$\overline{\alpha}(M) = \lim_{N\in\Phi,\, N\to M} \alpha(N) \qquad (M \in \overline{\Phi})$$

then provides a unique continuous extension of α to $\overline{\Phi}$ for the topology $\mathscr{T} \cap \overline{\Phi}$ (GT, I, §8, No. 5, Th. 1).

IV.56, ℓ. 8–10.

"For, let U be an open set belonging to $\overline{\Phi}$; for every $\varepsilon > 0$ there exists a compact set $K \subset U$ such that, for every set $M \in \Phi$ satisfying $K \subset M \subset U$, one has $|\overline{\alpha}(U) - \alpha(M)| \leqslant \varepsilon$, whence $|\overline{\alpha}(U) - \overline{\alpha}(K)| \leqslant \varepsilon$ "

Since $\mathrm{U} \in \overline{\Phi}$, by the definition of $\overline{\Phi}$ and $\overline{\alpha}$ one has

$$\lim_{\mathrm{N}\in\Phi,\,\mathrm{N}\to\mathrm{U}} \alpha(\mathrm{N}) = \overline{\alpha}(\mathrm{U}) \in \mathbf{R}\,.$$

Thus, given any $\varepsilon > 0$, there exists a neighborhood $\mathrm{I}(\mathrm{K},\mathrm{V})$ of U in $\mathfrak{P}(\mathrm{X})$ such that

$$\mathrm{M} \in \Phi \cap \mathrm{I}(\mathrm{K},\mathrm{V}) \quad\Rightarrow\quad |\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{U})| \leqslant \varepsilon\,;$$

since $\mathrm{U} \in \mathrm{I}(\mathrm{K},\mathrm{V})$, one has $\mathrm{K} \subset \mathrm{U} \subset \mathrm{V}$, whence $\mathrm{I}(\mathrm{K},\mathrm{U}) \subset \mathrm{I}(\mathrm{K},\mathrm{V})$, consequently

$$\mathrm{M} \in \Phi \cap \mathrm{I}(\mathrm{K},\mathrm{U}) \quad\Rightarrow\quad |\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{U})| \leqslant \varepsilon\,,$$

that is,

$$(*) \qquad \mathrm{M} \in \Phi\,,\quad \mathrm{K} \subset \mathrm{M} \subset \mathrm{U} \quad\Rightarrow\quad |\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{U})| \leqslant \varepsilon\,.$$

By 3°, one has

$$\overline{\alpha}(\mathrm{K}) = \inf_{\mathrm{M}\in\Phi,\,\mathrm{M}\supset\mathrm{K}} \alpha(\mathrm{M})\,;$$

by the monotonicity of $\overline{\alpha}$ established in 3°, the infimum is unchanged if we restrict the M to be subsets of U—for, by $(\mathrm{PC_{II}})$ there exists an $\mathrm{N} \in \Phi$ with $\mathrm{K} \subset \mathrm{N} \subset \mathrm{U}$, and we may replace the M's by the $\mathrm{M} \cap \mathrm{N}$'s—therefore the inequality $|\overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K})| \leqslant \varepsilon$ is a consequence of (*).

IV.56, *ℓ*. 10–12.

"...if K′ is any compact set contained in U, then $\mathrm{K} \subset \mathrm{K}\cup\mathrm{K}' \subset \mathrm{U}$, whence $|\overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K}\cup\mathrm{K}')| \leqslant \varepsilon$ and so $\overline{\alpha}(\mathrm{U}) \geqslant \overline{\alpha}(\mathrm{K}\cup\mathrm{K}') - \varepsilon \geqslant \overline{\alpha}(\mathrm{K}') - \varepsilon$"

With U, ε and K as in the preceding note, we also know that $\mathrm{K}\cup\mathrm{K}' \in \overline{\Phi}$, hence

$$\overline{\alpha}(\mathrm{K}\cup\mathrm{K}') = \lim_{\mathrm{M}\in\Phi,\,\mathrm{M}\to\mathrm{K}\cup\mathrm{K}'} \alpha(\mathrm{M})\,.$$

Given $\delta > 0$, choose a neighborhood $\mathrm{I}(\mathrm{K}^*,\mathrm{U}^*)$ of $\mathrm{K}\cup\mathrm{K}'$ such that

$$(**) \qquad \mathrm{M} \in \Phi \cap \mathrm{I}(\mathrm{K}^*,\mathrm{U}^*) \quad\Rightarrow\quad |\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{K}\cup\mathrm{K}')| \leqslant \delta\,.$$

In particular, $\mathrm{K}\cup\mathrm{K}' \in \mathrm{I}(\mathrm{K}^*,\mathrm{U}^*)$, so

$$\mathrm{K}^* \subset \mathrm{K}\cup\mathrm{K}' \subset \mathrm{U}^*\,;$$

replacing U^* by $\mathrm{U}^*\cap\mathrm{U}$ (which still contains $\mathrm{K}\cup\mathrm{K}'$), we can suppose that $\mathrm{U}^* \subset \mathrm{U}$.

By $(\mathrm{PC_{II}})$ there exists a set $\mathrm{M} \in \Phi$ with

$$\mathrm{K}\cup\mathrm{K}' \subset \mathrm{M} \subset \mathrm{U}^*\,.$$

Since $\mathrm{K} \subset \mathrm{K} \cup \mathrm{K}' \subset \mathrm{M} \subset \mathrm{U}^* \subset \mathrm{U}$, by (*) of the preceding note one has

$$|\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{U})| \leqslant \varepsilon\,,$$

and since $\mathrm{K}^* \subset \mathrm{K} \cup \mathrm{K}' \subset \mathrm{M} \subset \mathrm{U}^*$, by (**) one has

$$|\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{K} \cup \mathrm{K}')| \leqslant \delta\,,$$

therefore

$$|\overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K} \cup \mathrm{K}')| \leqslant |\overline{\alpha}(\mathrm{U}) - \alpha(\mathrm{M})| + |\alpha(\mathrm{M}) - \overline{\alpha}(\mathrm{K} \cup \mathrm{K}')| \leqslant \varepsilon + \delta\,,$$

concisely $|\overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K} \cup \mathrm{K}')| \leqslant \varepsilon + \delta$; since δ was introduced after U, ε, K and K' were fixed, letting $\delta \to 0$ yields the first inequality of the statement.

It follows that $\overline{\alpha}(\mathrm{U}) \geqslant \overline{\alpha}(\mathrm{K} \cup \mathrm{K}') - \varepsilon \geqslant \overline{\alpha}(\mathrm{K}') - \varepsilon$ (the latter inequality by the monotonicity of $\overline{\alpha}$ for compact sets). In particular, $\overline{\alpha}(\mathrm{K}') \leqslant \overline{\alpha}(\mathrm{U}) + \varepsilon$, which is needed for the next note.

IV.56, ℓ. 12, 13.

"... $\overline{\alpha}(\mathrm{U})$ is therefore indeed equal to the supremum of the numbers $\overline{\alpha}(\mathrm{K})$ as K runs over the set of compact subsets of U."

With notations as in the preceding note, the argument shows that if an open set U belongs to $\overline{\Phi}$, then for every $\varepsilon > 0$ and every compact set $\mathrm{K}' \subset \mathrm{U}$, one has

$$\overline{\alpha}(\mathrm{K}') \leqslant \overline{\alpha}(\mathrm{U}) + \varepsilon$$

(the K and M in the argument play auxiliary roles, and can be forgotten here); fixing K' and varying ε, one has $\overline{\alpha}(\mathrm{K}') \leqslant \overline{\alpha}(\mathrm{U})$, and since K' is an arbitrary compact subset of U, we conclude that

$$\sup_{\mathrm{K}' \text{ compact, } \mathrm{K}' \subset \mathrm{U}} \overline{\alpha}(\mathrm{K}') \leqslant \overline{\alpha}(\mathrm{U})\,.$$

But, by the note for ℓ. 8–10, given any $\varepsilon > 0$ there exists a compact subset K' of U such that the difference between $\overline{\alpha}(\mathrm{K}')$ and $\overline{\alpha}(\mathrm{U})$ is as small as we like, therefore the preceding displayed inequality is in fact an equality.

IV.56, ℓ. 17–19.

"... by $(\mathrm{PM}''_{\mathrm{IV}})$, for every set $\mathrm{M} \in \Phi$ such that $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$, there exists a compact set $\mathrm{K}' \subset \mathrm{M}$ such that

$$\alpha(\mathrm{M}) \leqslant \overline{\alpha}(\mathrm{K}') + \varepsilon \leqslant b + \varepsilon \text{''}$$

Let M be any set in Φ such that $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$ (such M exist by $(\mathrm{PC}_{\mathrm{II}})$). Then $(\mathrm{PM}''_{\mathrm{IV}})$ provides a compact set $\mathrm{K}' \subset \mathrm{M}$ such that $\alpha(\mathrm{N}) \geqslant \alpha(\mathrm{M}) - \varepsilon$

for every $N \in \Phi$ with $N \supset K'$; but by 3°, $K' \in \overline{\Phi}$ and $\overline{\alpha}(K')$ is the infimum of $\alpha(N)$ over all such N, therefore $\overline{\alpha}(K') \geqslant \alpha(M) - \varepsilon$, whence the first of the asserted inequalities—whereas the second follows from the definition of b.

IV.56, ℓ. 20.

"... therefore $b - \varepsilon \leqslant \alpha(M) \leqslant b + \varepsilon$, which proves that $U \in \overline{\Phi}$."

We know that $b - \varepsilon \leqslant \overline{\alpha}(K)$ by the choice of K; but $\overline{\alpha}(K) \leqslant \alpha(M)$ because $K \subset M$ (see the definition of $\overline{\alpha}(K)$ in 3°), and $\alpha(M) \leqslant b + \varepsilon$ by the preceding note.

Thus, given any $\varepsilon > 0$ one has found a neighborhood $I(K, U)$ of U in $\mathfrak{P}(X)$ such that

$$M \in \Phi \cap I(K, U) \quad \Rightarrow \quad |\alpha(M) - b| \leqslant \varepsilon ;$$

this shows that

$$\lim_{M \in \Phi,\ M \to U} \alpha(M) = b \in \mathbf{R},$$

therefore $U \in \overline{\Phi}$, and $\overline{\alpha}(U) = b$ (as one also knows from the proof of "necessity" of the condition $b < +\infty$).

IV.56, ℓ. −8 to −3.

"The definition of $\overline{\Phi}$ and $\overline{\alpha}$ can now be transformed as follows (taking into account (PC_{II})): in order that $M \in \overline{\Phi}$, it is necessary and sufficient that, for every $\varepsilon > 0$, there exist a compact set K and an open set $U \in \overline{\Phi}$ such that $K \subset M \subset U$ and $\overline{\alpha}(U) - \overline{\alpha}(K) \leqslant \varepsilon$; $\overline{\alpha}(M)$ is, moreover, the *infimum* of the $\overline{\alpha}(U)$ for the open sets $U \in \overline{\Phi}$ containing M, and the *supremum* of the $\overline{\alpha}(K)$ for the compact sets $K \subset M$."

One is assuming that the hypotheses of Th. 5, including (PM_{IV}), are fulfilled, and that, as shown in 3°:

(i) Every compact set $K \subset X$ belongs to $\overline{\Phi}$, with $\overline{\alpha}(K) \in \mathbf{R}^+$ defined by the formula

$$\overline{\alpha}(K) = \inf_{P \in \Phi,\ P \supset K} \alpha(P) ;$$

$\overline{\alpha}$ is monotone, subadditive and additive on the set $\mathfrak{K}$ of all compact subsets of X, that is, $\overline{\alpha}(K_1) \leqslant \overline{\alpha}(K_2)$ when $K_1 \subset K_2$, and $\overline{\alpha}(K_1 \cup K_2) \leqslant \overline{\alpha}(K_1) + \overline{\alpha}(K_2)$, with equality when K_1, K_2 are disjoint. Moreover, as shown in 4°:

(ii) Denoting by $\mathfrak{U}$ the set of all open sets U in X for which

$$\sup_{K \in \mathfrak{K},\ K \subset U} \overline{\alpha}(K) < +\infty ,$$

an open set U in X belongs to $\overline{\Phi}$ if and only if $U \in \mathfrak{U}$; that is, writing $\mathscr{U}$ for the set of all open sets in X, one has $\mathscr{U} \cap \overline{\Phi} = \mathfrak{U}$. Moreover,

$$\overline{\alpha}(U) = \sup_{K \in \mathfrak{K},\ K \subset U} \overline{\alpha}(K) \quad \text{for all } U \in \mathfrak{U},$$

and $\overline{\alpha}$ is monotone and subadditive on $\mathfrak{U}$.

To summarize, $\mathfrak{K} \subset \overline{\Phi}$, $\mathfrak{U} = \mathscr{U} \cap \overline{\Phi}$, and $\overline{\alpha}$ has the indicated properties on $\mathfrak{K}$ and $\mathfrak{U}$.

We are to show:

(A) A set $M \subset X$ belongs to $\overline{\Phi}$ if and only if it satisfies the following condition: for every $\varepsilon > 0$, there exist sets $K \in \mathfrak{K}$ and $U \in \mathfrak{U}$ such that $K \subset M \subset U$ and $|\overline{\alpha}(U) - \overline{\alpha}(K)| \leqslant \varepsilon$. Moreover:

For every $M \in \overline{\Phi}$, one has

$$\text{(B)} \qquad \overline{\alpha}(M) = \sup_{K \in \mathfrak{K},\ K \subset M} \overline{\alpha}(K) = \inf_{U \in \mathfrak{U},\ U \supset M} \overline{\alpha}(U).$$

Let us write $\widetilde{\Phi}$ for the set of all $M \subset X$ that satisfy the condition in (A); we are to show that $\widetilde{\Phi} = \overline{\Phi}$ and that the formulas (B) hold for every $M \in \overline{\Phi}$. The proof is organized into three parts: $\overline{\Phi} \subset \widetilde{\Phi}$, $\widetilde{\Phi} \subset \overline{\Phi}$, and the formulas (B).

Proof of $\overline{\Phi} \subset \widetilde{\Phi}$. Let $M \in \overline{\Phi}$ and let $\varepsilon > 0$. Since $\alpha(N) \to \overline{\alpha}(M)$ as $N \to M$ while remaining in Φ, there exists a neighborhood $I(K, U)$ of M in $\mathfrak{P}(X)$ such that

$$(*) \qquad N \in \Phi \cap I(K, U) \quad \Rightarrow \quad |\alpha(N) - \overline{\alpha}(M)| \leqslant \varepsilon/2.$$

We assert that $U \in \overline{\Phi}$ (that is, $U \in \mathfrak{U}$). To this end, let us show that, as K' runs over all compact subsets of U, the $\overline{\alpha}(K')$ have a finite upper bound. Let $K' \subset U$ be compact. Then $K \cup K' \subset U$, and by (PC_{II}) there exists a set $N \in \Phi$ such that

$$K \cup K' \subset N \subset U.$$

Since $K' \subset N \in \Phi$, one has $\overline{\alpha}(K') \leqslant \alpha(N)$ by 3°; but $K \subset N \subset U$, therefore $\alpha(N) \leqslant \overline{\alpha}(M) + \varepsilon/2$ by $(*)$, whence $\overline{\alpha}(K') \leqslant \overline{\alpha}(M) + \varepsilon/2$, so that $\overline{\alpha}(M) + \varepsilon/2$ serves as an upper bound for the $\overline{\alpha}(K')$ ($K' \in \mathfrak{K}$, $K' \subset U$), which proves that $U \in \overline{\Phi}$. Then, by 4°, $\overline{\alpha}(U) = \sup \overline{\alpha}(K')$ over all such K', whence $\overline{\alpha}(U) \leqslant \overline{\alpha}(M) + \varepsilon/2$.

Since $K \subset U \in \overline{\Phi}$, it follows from 3° that $\overline{\alpha}(K) \leqslant \overline{\alpha}(U)$. On the other hand, it follows from $(*)$ that $|\overline{\alpha}(K) - \overline{\alpha}(M)| \leqslant \varepsilon/2$; for, one knows from 3° that $\overline{\alpha}(K)$ is the infimum of the $\alpha(P)$ over all $P \in \Phi$ such that $P \supset K$, and by the monotonicity of α one can restrict P to be a subset of any fixed N occurring in $(*)$ (an argument made in the note for ℓ. 8–10). Then

$$0 \leqslant \overline{\alpha}(U) - \overline{\alpha}(K) = [\overline{\alpha}(U) - \overline{\alpha}(M)] + [\overline{\alpha}(M) - \overline{\alpha}(K)] \leqslant \varepsilon/2 + \varepsilon/2,$$

concisely $\overline{\alpha}(U) - \overline{\alpha}(K) \leqslant \varepsilon$, which proves that $M \in \widetilde{\Phi}$, and so $\overline{\Phi} \subset \widetilde{\Phi}$.

Proof of $\widetilde{\Phi} \subset \overline{\Phi}$. First, a useful lemma:

Lemma 1. If $\mathrm{K} \subset \mathrm{N} \subset \mathrm{U}$, where $\mathrm{N} \in \Phi$, K is compact and U is an open set with $\mathrm{U} \in \overline{\Phi}$ (that is, $\mathrm{U} \in \mathfrak{U}$), then $\overline{\alpha}(\mathrm{K}) \leqslant \alpha(\mathrm{N}) \leqslant \overline{\alpha}(\mathrm{U})$.

Proof. That $\overline{\alpha}(\mathrm{K}) \leqslant \alpha(\mathrm{N})$ was shown in 3°.

To prove that $\alpha(\mathrm{N}) \leqslant \overline{\alpha}(\mathrm{U})$ we apply ($\mathrm{PM_{IV}}$) to N: given any $\varepsilon > 0$, there exists a neighborhood $\mathrm{I}(\mathrm{K}_0, \mathrm{U}_0)$ of N in $\mathfrak{P}(\mathrm{X})$ such that

$$\mathrm{N}' \in \Phi \cap \mathrm{I}(\mathrm{K}_0, \mathrm{U}_0) \quad \Rightarrow \quad |\alpha(\mathrm{N}') - \alpha(\mathrm{N})| \leqslant \varepsilon .$$

But $\mathrm{I}(\mathrm{K}, \mathrm{U})$ is also a neighborhood of N, therefore so is

$$\mathrm{I}(\mathrm{K} \cup \mathrm{K}_0, \mathrm{U} \cap \mathrm{U}_0) = \mathrm{I}(\mathrm{K}, \mathrm{U}) \cap \mathrm{I}(\mathrm{K}_0, \mathrm{U}_0) ;$$

replacing $\mathrm{I}(\mathrm{K}_0, \mathrm{U}_0)$ by $\mathrm{I}(\mathrm{K} \cup \mathrm{K}_0, \mathrm{U} \cap \mathrm{U}_0)$, we can suppose that

$$\mathrm{K} \subset \mathrm{K}_0 \subset \mathrm{N} \subset \mathrm{U}_0 \subset \mathrm{U} ,$$

and since $\mathrm{U} \in \overline{\Phi}$ it is clear from the criterion in 4° that also $\mathrm{U}_0 \in \overline{\Phi}$.

We assert that $|\overline{\alpha}(\mathrm{U}_0) - \alpha(\mathrm{N})| \leqslant \varepsilon$. We know from 4° that $\overline{\alpha}(\mathrm{U}_0) = \sup \overline{\alpha}(\mathrm{K}')$, where K' runs over all compact subsets of U_0; since also $\mathrm{K}_0 \subset \mathrm{U}_0$, whence $\mathrm{K}' \subset \mathrm{K}_0 \cup \mathrm{K}' \subset \mathrm{U}_0$, and since $\overline{\alpha}$ is monotone for compact sets, $\overline{\alpha}(\mathrm{U}_0)$ is also the supremum of the $\overline{\alpha}(\mathrm{K}')$ where K' is compact and $\mathrm{K}_0 \subset \mathrm{K}' \subset \mathrm{U}$, thus it will suffice to show that $|\overline{\alpha}(\mathrm{K}') - \alpha(\mathrm{N})| \leqslant \varepsilon$ for every such K'.

Now, given such a K', we know from 3° that $\overline{\alpha}(\mathrm{K}') = \inf \alpha(\mathrm{N}')$, where $\mathrm{N}' \in \Phi$ and $\mathrm{N}' \supset \mathrm{K}'$. But $\mathrm{K}' \subset \mathrm{U}_0$, so by ($\mathrm{PC_{II}}$) there exists an $\mathrm{N}'' \in \Phi$ with $\mathrm{K}' \subset \mathrm{N}'' \subset \mathrm{U}_0$. Then for every $\mathrm{N}' \in \Phi$ with $\mathrm{N}' \supset \mathrm{K}'$, one has $\mathrm{K}' \subset \mathrm{N}' \cap \mathrm{N}'' \subset \mathrm{U}_0$; thus, by the monotonicity of α, it suffices to consider only $\mathrm{N}' \in \Phi$ with $\mathrm{K}' \subset \mathrm{N}' \subset \mathrm{U}_0$. For such an N', we need only show that $|\alpha(\mathrm{N}') - \alpha(\mathrm{N})| \leqslant \varepsilon$. Indeed,

$$\mathrm{K} \subset \mathrm{K}_0 \subset \mathrm{K}' \subset \mathrm{N}' \subset \mathrm{U}_0 ,$$

thus $\mathrm{N}' \in \Phi \cap \mathrm{I}(\mathrm{K}_0, \mathrm{U}_0)$, therefore $|\alpha(\mathrm{N}') - \alpha(\mathrm{N})| \leqslant \varepsilon$ by the choice of K_0 and U_0, which completes the verification of $|\overline{\alpha}(\mathrm{U}_0) - \alpha(\mathrm{N})| \leqslant \varepsilon$.

To summarize: given any $\varepsilon > 0$, we have found an open set $\mathrm{U}_0 \in \overline{\Phi}$ such that

$$\mathrm{N} \subset \mathrm{U}_0 \subset \mathrm{U} \quad \text{and} \quad |\overline{\alpha}(\mathrm{U}_0) - \alpha(\mathrm{N})| \leqslant \varepsilon .$$

With $\varepsilon = 1/n$ $(n = 1, 2, 3, \ldots)$ choose an open set $\mathrm{U}_n \in \overline{\Phi}$ such that $\mathrm{N} \subset \mathrm{U}_n \subset \mathrm{U}$ and $|\overline{\alpha}(\mathrm{U}_n) - \alpha(\mathrm{N})| \leqslant 1/n$. Then $\overline{\alpha}(\mathrm{U}_n) \to \alpha(\mathrm{N})$; but $\overline{\alpha}(\mathrm{U}_n) \leqslant \overline{\alpha}(\mathrm{U})$ because $\overline{\alpha}$ is monotone on $\mathfrak{U}$, so passage to the limit yields $\alpha(\mathrm{N}) \leqslant \overline{\alpha}(\mathrm{U})$ and the lemma is proved. ◇

Assuming now that $\mathrm{M} \in \widetilde{\Phi}$ (that is, M satisfies the condition in (A)), let us show that $\mathrm{M} \in \overline{\Phi}$. By assumption, given any $\varepsilon > 0$, there exists a neighborhood $\mathrm{I(K,U)}$ of M with K compact and U an open set belonging to $\overline{\Phi}$. We assert that if N, N' is any pair of elements of $\Phi \cap \mathrm{I(K,U)}$, then

$$|\alpha(\mathrm{N}) - \alpha(\mathrm{N}')| \leqslant \varepsilon\,;$$

indeed, from $\mathrm{K} \subset \mathrm{N} \subset \mathrm{U}$ and $\mathrm{K} \subset \mathrm{N}' \subset \mathrm{U}$, the lemma yields

$$\alpha(\mathrm{N}) \in [\overline{\alpha}(\mathrm{K}), \overline{\alpha}(\mathrm{U})] \quad \text{and} \quad \alpha(\mathrm{N}') \in [\overline{\alpha}(\mathrm{K}), \overline{\alpha}(\mathrm{U})]\,,$$

whence $|\alpha(\mathrm{N}) - \alpha(\mathrm{N}')| \leqslant \overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K}) \leqslant \varepsilon$.

This shows that the image, under α, of the trace on Φ of the neighborhood filter of M, is Cauchy in $\mathbf{R}$, hence is convergent to an element of $\mathbf{R}$. In other words $\mathrm{M} \in \overline{\Phi}$, which completes the proof that $\widetilde{\Phi} \subset \overline{\Phi}$, and hence that $\widetilde{\Phi} = \overline{\Phi}$.

Proof of the formulas (B). The formulas are an easy consequence of the following generalization of Lemma 1:

Lemma 2. If $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$, where K is compact, $\mathrm{M} \in \overline{\Phi}$, and $\mathrm{U} \in \overline{\Phi}$ is open, then $\overline{\alpha}(\mathrm{K}) \leqslant \overline{\alpha}(\mathrm{M}) \leqslant \overline{\alpha}(\mathrm{U})$.

Proof. If $\mathrm{N} \in \Phi \cap \mathrm{I(K,U)}$, then $\overline{\alpha}(\mathrm{K}) \leqslant \alpha(\mathrm{N}) \leqslant \overline{\alpha}(\mathrm{U})$ by Lemma 1, thus

$$(**) \qquad \alpha\big(\Phi \cap \mathrm{I(K,U)}\big) \subset [\overline{\alpha}(\mathrm{K}), \overline{\alpha}(\mathrm{U})]\,.$$

Since Φ is dense in $\mathfrak{P}(\mathrm{X})$, every neighborhood of M in $\mathfrak{P}(\mathrm{X})$ intersects Φ, and since $\mathrm{I(K,U)}$ is a neighborhood of M, clearly every neighborhood of M intersects $\Phi \cap \mathrm{I(K,U)}$; it follows that if $\mathrm{I(K',U')}$ is any neighborhood of M, then $\overline{\Phi} \cap \mathrm{I(K',U')}$ (which is a basic neighborhood of M in $\overline{\Phi}$) intersects $\Phi \cap \mathrm{I(K,U)}$, thus M belongs to the closure of $\Phi \cap \mathrm{I(K,U)}$ in $\overline{\Phi}$. Since $\overline{\alpha} : \overline{\Phi} \to \mathbf{R}$ is continuous and extends α, it follows from $(**)$ that (the long overbars indicate closure in $\mathbf{R}$)

$$\overline{\alpha}(\mathrm{M}) \in \overline{\overline{\alpha}\big(\Phi \cap \mathrm{I(K,U)}\big)} = \overline{\alpha\big(\Phi \cap \mathrm{I(K,U)}\big)} \subset [\overline{\alpha}(\mathrm{K}), \overline{\alpha}(\mathrm{U})]\,,$$

that is, $\overline{\alpha}(\mathrm{K}) \leqslant \overline{\alpha}(\mathrm{M}) \leqslant \overline{\alpha}(\mathrm{U})$. ♢

Given any $\mathrm{M} \in \overline{\Phi}$, define

$$r = \inf\{\overline{\alpha}(\mathrm{U}) :\ \mathrm{M} \subset \mathrm{U} \in \overline{\Phi},\ \mathrm{U}\ \text{open}\}\,,$$
$$s = \sup\{\overline{\alpha}(\mathrm{K}) :\ \mathrm{K} \subset \mathrm{M},\ \mathrm{K}\ \text{compact}\}\,;$$

we are to show that $r = s = \overline{\alpha}(\mathrm{M})$.

Since $\mathrm{M} \in \overline{\Phi} = \widetilde{\Phi}$, we know that M satisfies the condition in (A): given any $\varepsilon > 0$, there exist $\mathrm{K} \in \mathfrak{K}$ and $\mathrm{U} \in \mathfrak{U}$ such that $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$ and $\overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K}) \leqslant \varepsilon$; by the above lemma,

$$\overline{\alpha}(\mathrm{K}) \leqslant \overline{\alpha}(\mathrm{M}) \leqslant \overline{\alpha}(\mathrm{U}) .$$

Obviously $\overline{\alpha}(\mathrm{K}) \leqslant s$ and $r \leqslant \overline{\alpha}(\mathrm{U})$.

For every compact subset K' of M, one has $\mathrm{K}' \subset \mathrm{M} \subset \mathrm{U}$, therefore $\overline{\alpha}(\mathrm{K}') \leqslant \overline{\alpha}(\mathrm{M})$ by Lemma 2; consequently $s \leqslant \overline{\alpha}(\mathrm{M})$. On the other hand, for every set $\mathrm{U}' \in \mathfrak{U}$ such that $\mathrm{M} \subset \mathrm{U}'$, one has $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}'$, therefore $\overline{\alpha}(\mathrm{M}) \leqslant \overline{\alpha}(\mathrm{U}')$ by Lemma 2; consequently $\overline{\alpha}(\mathrm{M}) \leqslant r$. Thus

$$\overline{\alpha}(\mathrm{K}) \leqslant s \leqslant \overline{\alpha}(\mathrm{M}) \leqslant r \leqslant \overline{\alpha}(\mathrm{U}) ,$$

whence $0 \leqslant r - s \leqslant \overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K}) \leqslant \varepsilon$. Since ε is arbitrary, $r - s = 0$; thus $s \leqslant \overline{\alpha}(\mathrm{M} \leqslant r$ with $s = r$, whence equality throughout.

An immediate consequence of either of these formulas for $\overline{\alpha}(\mathrm{M})$: $\overline{\alpha}$ is *monotone* on $\overline{\Phi}$.

IV.57, ℓ. 7, 8.

"By the foregoing, we have $\overline{\alpha}(\mathrm{U}) \leqslant \overline{\alpha}(\mathrm{K}) + \overline{\alpha}(\mathrm{U} - \mathrm{K})$."

Since $\mathrm{U} - \mathrm{K}$ is open and $\mathrm{U} - \mathrm{K} \subset \mathrm{U} \in \overline{\Phi}$, $\mathrm{U} - \mathrm{K} \in \overline{\Phi}$ by the criterion in 4°; apply the preceding assertion ('conditional subadditivity') to $\mathrm{U} = \mathrm{K} \cup (\mathrm{U} - \mathrm{K})$.

IV.57, ℓ. 11.

"... $\overline{\alpha}(\mathrm{K}) + \overline{\alpha}(\mathrm{U} - \mathrm{K}) \leqslant \overline{\alpha}(\mathrm{U})$."

By the previously displayed formula, $\overline{\alpha}(\mathrm{K}') \leqslant \overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K})$ for all compact sets $\mathrm{K}' \subset \mathrm{U} - \mathrm{K} \subset \overline{\Phi}$, whence $\overline{\alpha}(\mathrm{U} - \mathrm{K}) \leqslant \overline{\alpha}(\mathrm{U}) - \overline{\alpha}(\mathrm{K})$.

IV.57, ℓ. −16.

"... $\mathrm{K}'' \subset \mathrm{M} \cap \complement \mathrm{N} \subset \mathrm{U}''$ ... "

From $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$ and $\complement \mathrm{U}' \subset \complement \mathrm{N} \subset \complement \mathrm{K}'$, one has

$$\mathrm{K}'' = \mathrm{K} \cap \complement \mathrm{U}' \subset \mathrm{M} \cap \complement \mathrm{N} \subset \mathrm{U} \cap \complement \mathrm{K}' = \mathrm{U}'' .$$

IV.57, ℓ. −16, −15.

"... $\mathrm{U}'' - \mathrm{K}''$ is contained in the union of $\mathrm{U} \cap \complement \mathrm{K}$ and $\mathrm{U}' \cap \complement \mathrm{K}'$ ... "

For,

$$\begin{aligned} \mathrm{U}'' - \mathrm{K}'' &= \mathrm{U}'' \cap \complement (\mathrm{K} \cap \complement \mathrm{U}') = \mathrm{U}'' \cap (\complement \mathrm{K} \cup \mathrm{U}') \\ &= (\mathrm{U}'' \cap \complement \mathrm{K}) \cup (\mathrm{U}'' \cap \mathrm{U}') \\ &= (\mathrm{U} \cap \complement \mathrm{K}' \cap \complement \mathrm{K}) \cup (\mathrm{U} \cap \complement \mathrm{K}' \cap \mathrm{U}') \\ &\subset (\mathrm{U} \cap \complement \mathrm{K}) \cup (\mathrm{U}' \cap \complement \mathrm{K}') , \end{aligned}$$

whence $\overline{\alpha}(U'' - K'') \leqslant \overline{\alpha}(U - K) + \overline{\alpha}(U' - K')$ by the subadditivity of $\overline{\alpha}$ for open sets belonging to $\overline{\Phi}$.

IV.57, ℓ. $-12, -11$.

"... $M \cup N$ belongs to $\overline{\Phi}$."

This completes the proof that $\overline{\Phi}$ is a clan (No. 9, Prop. 17). Consequently $\overline{\alpha}$ is subadditive on $\overline{\Phi}$ by the 'conditional subadditivity' (p. IV.56, ℓ. $-2, -1$) proved earlier.

IV.57, ℓ. $-9, -8$.

"... since ε is arbitrary, we have $\overline{\alpha}(M \cup N) = \overline{\alpha}(M) + \overline{\alpha}(N)$."

Since ε is arbitrary, we have $\overline{\alpha}(M \cup N) \geqslant \overline{\alpha}(M) + \overline{\alpha}(N)$, and the reverse inequality holds by the subadditivity of $\overline{\alpha}$.

IV.57, ℓ. -3 to -1.

"Since β is positive, $|\beta(f)| \leqslant \overline{\alpha}(K) \cdot \|f\|$ for every function $f \in \mathscr{E}(\overline{\Phi})$ whose support is contained in K "

For such a function f, $-\|f\|\,\varphi_K \leqslant f \leqslant \|f\|\,\varphi_K$, whence

$$-\|f\|\,\beta(\varphi_K) \leqslant \beta(f) \leqslant \|f\|\,\beta(\varphi_K),$$

where $\beta(\varphi_K) = \overline{\alpha}(K)$.

IV.58, ℓ. $1, 2$.

"... it may therefore be extended to a positive continuous linear form $\overline{\beta}_K$ on $\mathscr{G}(K)$."

A detail to be checked: the positivity of $\overline{\beta}$.

Recall that $\mathscr{E}(\overline{\Phi})$ is a Riesz space for the pointwise ordering of functions (No. 9, second paragraph following Def. 4); so is its linear subspace

$$\mathscr{E}(\overline{\Phi}) \cap \mathfrak{F}(X, K; \mathbf{R})$$

(the functions in $\mathscr{E}(\overline{\Phi})$ whose support is contained in K), whose closure $\mathscr{G}(K)$ in the Banach space $\mathscr{B}(X; \mathbf{R})$ of bounded functions on X (TVS, I, §1, No. 4, *Examples*) is also a Riesz space. Note that $\mathscr{E}(\overline{\Phi}) \cap \mathfrak{F}(X, K; \mathbf{R}) = \mathscr{E}(\overline{\Phi}) \cap \mathscr{G}(K)$.

Let $g \in \mathscr{G}(K)$ and choose a sequence $f_n \in \mathscr{E}(\overline{\Phi}) \cap \mathscr{G}(K)$ such that $\|f_n - g\| \to 0$. Then $|g| \in \mathscr{G}(K)$ and $\|\,|f_n| - |g|\,\| \to 0$, therefore

$$\overline{\beta}_K(|g|) = \lim_n \overline{\beta}_K(|f_n|) = \lim_n \beta_K(|f_n|) \geqslant 0.$$

In particular, if $g \geqslant 0$, that is, $g = |g|$, then $\overline{\beta}_K(g) \geqslant 0$.

IV.58, ℓ. $3, 4$.

"... the restriction of $\overline{\beta}_{K_1}$ to $\mathscr{G}(K)$ is identical to $\overline{\beta}_K$ "

If $g \in \mathscr{G}(\mathrm{K})$ and $f_n \in \mathscr{E}(\overline{\Phi}) \cap \mathscr{G}(\mathrm{K})$ with $\|f_n - g\| \to 0$, then also $g \in \mathscr{G}(\mathrm{K}_1)$ and $f_n \in \mathscr{E}(\overline{\Phi}) \cap \mathfrak{F}(\mathrm{X}, \mathrm{K}_1; \mathbf{R}) = \mathscr{E}(\overline{\Phi}) \cap \mathscr{G}(\mathrm{K}_1)$, therefore $\overline{\beta}_{\mathrm{K}_1}(g) = \lim_n \beta(f_n) = \overline{\beta}_{\mathrm{K}}(g)$.

IV.58, ℓ. 8, 9.

"...the *restriction* to $\mathscr{K}$ of the positive linear form $\overline{\beta}$ is therefore a positive *measure* μ."

Every positive linear form on $\mathscr{K}(\mathrm{X}; \mathbf{R})$ is a positive measure (Ch. III, §1, No. 5, Th. 1).

IV.58, ℓ. 10.

"there exists an open set $\mathrm{U} \in \overline{\Phi}$ such that $\mathrm{K} \subset \mathrm{U}$, $\mu(\mathrm{U}) \leqslant \mu(\mathrm{K}) + \varepsilon$ "

There exists an open set V with $\mathrm{K} \subset \mathrm{V} \subset \overline{\mathrm{V}} \subset \mathrm{U}$ and $\overline{\mathrm{V}}$ compact; replacing U by V, we can suppose that U is relatively compact, thereby assuring that every function whose support is contained in U has compact support.

IV.58, ℓ. 13.

"Then $\mu(\mathrm{K}) \leqslant \mu(f) \leqslant \mu(\mathrm{U}) \leqslant \mu(\mathrm{K}) + \varepsilon$ "

To justify the notation $\mu(f)$ at this stage, we must assume that f has compact support; this is assured by arranging that U be relatively compact (or by constructing f to have compact support). Moreover, the equality $\mu(f) = \overline{\beta}(f)$ will shortly be needed.

{Absent this precaution, we do not know that f is μ-integrable; of course, *after* it has been shown that every open set belonging to $\overline{\Phi}$ is μ-integrable, a bounded continuous function with support contained in such a set will be integrable (No. 4, Prop. 5).}

IV.58, ℓ. 15.

"...we see that $|\mu(\mathrm{K}) - \overline{\alpha}(\mathrm{K})| \leqslant \varepsilon$ "

From the above inequalities, we extract (noting that $\mu(f) = \overline{\beta}(f)$ by the definition of μ)

$$\mu(\mathrm{K}) \leqslant \mu(f) \leqslant \mu(\mathrm{K}) + \varepsilon$$

and $\overline{\alpha}(\mathrm{K}) \leqslant \overline{\beta}(f) = \mu(f) \leqslant \overline{\alpha}(\mathrm{K}) + \varepsilon$, that is,

$$-\overline{\alpha}(\mathrm{K}) - \varepsilon \leqslant -\mu(f) \leqslant -\overline{\alpha}(\mathrm{K}).$$

Adding term-by-term the displayed inequalities, we get

$$[\mu(\mathrm{K}) - \overline{\alpha}(\mathrm{K})] - \varepsilon \leqslant 0 \leqslant [\mu(\mathrm{K}) - \overline{\alpha}(\mathrm{K})] + \varepsilon;$$

adding $\mu(\mathrm{K}) - \overline{\alpha}(\mathrm{K})$ throughout then yields $-\varepsilon \leqslant \mu(\mathrm{K}) - \overline{\alpha}(\mathrm{K}) \leqslant \varepsilon$.

IV.58, ℓ. 18, 19.

"... the open sets belonging to $\overline{\Phi}$ are none other than the μ-integrable open sets"

Write $\mathfrak{K}$ for the set of all compact subsets of X. One knows that $\mu(\mathrm{K}) = \overline{\alpha}(\mathrm{K})$ for all $\mathrm{K} \in \mathfrak{K}$. Let U be an open subset of X and let

$$b = \sup_{\mathrm{K}\in\mathfrak{K},\ \mathrm{K}\subset\mathrm{U}} \overline{\alpha}(\mathrm{K}) = \sup_{\mathrm{K}\in\mathfrak{K},\ \mathrm{K}\subset\mathrm{U}} \mu(\mathrm{K})\,.$$

By 4°,

$$\mathrm{U} \in \overline{\Phi} \quad\Leftrightarrow\quad b < +\infty\,,$$

in which case $\overline{\alpha}(\mathrm{U}) = b$. On the other hand, by Prop. 10 of No. 6,

$$\mathrm{U} \text{ is } \mu\text{-integrable} \quad\Leftrightarrow\quad \mu^*(\mathrm{U}) < +\infty\,,$$

in which case $\mu(\mathrm{U}) = \mu^*(\mathrm{U})$ by definition. But $\mu^*(\mathrm{U}) = b$ (No. 6, Cor. 4 of Th. 4), thus

$$\mathrm{U} \text{ is } \mu\text{-integrable} \quad\Leftrightarrow\quad b < +\infty \quad\Leftrightarrow\quad \mathrm{U} \in \overline{\Phi}\,,$$

in which case $\mu(\mathrm{U}) = b = \overline{\alpha}(\mathrm{U})$.

IV.58, ℓ. 21.

"... the μ-integrable sets are the sets of $\overline{\Phi}$"

Write $\mathfrak{K}$ for the set of all compact subsets of X, and let M be any subset of X.

As shown in 5°, $\mathrm{M} \in \overline{\Phi}$ if and only if it satisfies the following condition:

(A) For every $\varepsilon > 0$, there exist a compact set K and an open set $\mathrm{U} \in \overline{\Phi}$ such that $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$ and $\overline{\alpha}(\mathrm{U} - \mathrm{K}) \leqslant \varepsilon$.

By the foregoing discussion, we know that an open set U belongs to $\overline{\Phi}$ if and only if it is μ-integrable, and that $\mu(\mathrm{U}) = \overline{\alpha}(\mathrm{U})$ for all open sets $\mathrm{U} \in \overline{\Phi}$. Thus the condition (A) may be expressed in terms of μ as follows:

(A′) For every $\varepsilon > 0$, there exist a compact set K and an integrable open set U such that $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$ and $\mu(\mathrm{U} - \mathrm{K}) \leqslant \varepsilon$.

But (A′) holds if and only if M is μ-integrable (No. 6, Th. 4), thus

$$\mathrm{M} \in \overline{\Phi} \quad\Leftrightarrow\quad \mathrm{M} \text{ is } \mu\text{-integrable.}$$

Moreover, when $\mathrm{M} \in \overline{\Phi}$ one has, as shown in 5°,

$$\overline{\alpha}(\mathrm{M}) = \sup_{\mathrm{K}\in\mathfrak{K},\ \mathrm{K}\subset\mathrm{M}} \overline{\alpha}(\mathrm{K})\,;$$

whereas when M is μ-integrable,

$$\mu(\mathrm{M}) = \sup_{\mathrm{K}\in\mathfrak{K},\ \mathrm{K}\subset\mathrm{M}} \mu(\mathrm{K}),$$

as observed in No. 6, Cor. 1 of Th. 4 (see the note for **IV.45,** ℓ. 1–3). Since $\mu = \overline{\alpha}$ on $\mathfrak{K}$, one concludes that $\mu(\mathrm{M}) = \overline{\alpha}(\mathrm{M})$.

IV.58, ℓ. 22.

" $\mu^*(\mathrm{U}) = \sup_{\mathrm{M}\in\Phi,\,\mathrm{M}\subset\mathrm{U}} \alpha(\mathrm{M})$ for every open set U "

Every $\mathrm{M} \in \Phi$ is μ-integrable and $\alpha(\mathrm{M}) = \overline{\alpha}(\mathrm{M}) = \mu(\mathrm{M}) = \mu^*(\mathrm{M})$; if, moreover, $\mathrm{M} \subset \mathrm{U}$, then $\alpha(\mathrm{M}) = \mu^*(\mathrm{M}) \leqslant \mu^*(\mathrm{U})$, so the supremum in question is $\leqslant \mu^*(\mathrm{U})$.

Write $\mathfrak{K}$ for the set of all compact subsets of X. By Cor. 4 of Th. 4 of No. 6,

$$(*) \qquad \mu^*(\mathrm{U}) = \sup_{\mathrm{K}\in\mathfrak{K},\ \mathrm{K}\subset\mathrm{U}} \mu(\mathrm{K}).$$

But if $\mathrm{K} \in \mathfrak{K}$ and $\mathrm{K} \subset \mathrm{U}$, by ($\mathrm{PC_{II}}$) there exists a set $\mathrm{M} \in \Phi$ such that $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$, therefore $\mu^*(\mathrm{K}) \leqslant \mu^*(\mathrm{M}) \leqslant \mu^*(\mathrm{U})$; since K and M are μ-integrable, and $\mu(\mathrm{M}) = \alpha(\mathrm{M})$, one has

$$\mu(\mathrm{K}) \leqslant \alpha(\mathrm{M}) \leqslant \mu^*(\mathrm{U}),$$

whence it is clear from $(*)$ that $\mu^*(\mathrm{U}) = \sup_{\mathrm{M}\in\Phi,\ \mathrm{M}\subset\mathrm{U}} \alpha(\mathrm{M})$.

IV.58, ℓ. $-6, -5$.

"The conditions ($\mathrm{PC_I}$), ($\mathrm{PC_{II}}$), ($\mathrm{PM_I}$), ($\mathrm{PM_{II}}$), ($\mathrm{PM_{III}}$) and ($\mathrm{PM''_{IV}}$) are then satisfied."

Recall that Ψ is a tribe, namely, the tribe generated by the set Φ of all compact subsets of X (GT, IX, §6, No. 3, Def. 4). The property (i) of β is called *complete additivity* (last sentence of No. 5); one notes the following consequences of (i) and (ii):

a) $\beta(\varnothing) = 0$. For, $\beta(\varnothing) < +\infty$ by (ii), and $\beta(\varnothing) = \beta\big(\bigcup_{k=1}^{\infty} \varnothing\big) = \sum_{k=1}^{\infty} \beta(\varnothing)$ by (i).

b) β is (finitely) *additive* (in the sense of No. 9, Def. 5), by *a*) and (i).

c) β is *monotone.* For, if $\mathrm{B}_1 \subset \mathrm{B}_2$ then $\mathrm{B}_2 = \mathrm{B}_1 \cup (\mathrm{B}_2 - \mathrm{B}_1)$ is a disjoint union, whence, by *b*), $\beta(\mathrm{B}_2) = \beta(\mathrm{B}_1) + \beta(\mathrm{B}_2 - \mathrm{B}_1) \geqslant \beta(\mathrm{B}_1)$; if, moreover, $\beta(\mathrm{B}_1)$ is finite, then $\beta(\mathrm{B}_2 - \mathrm{B}_1) = \beta(\mathrm{B}_2) - \beta(\mathrm{B}_1)$ (β is 'conditionally subtractive').

d) β is *subadditive*. For,

$$\beta(\mathrm{B}_1 \cup \mathrm{B}_2) = \beta(\mathrm{B}_1) + \beta(\mathrm{B}_2 - \mathrm{B}_1) \leqslant \beta(\mathrm{B}_1) + \beta(\mathrm{B}_2)$$

by *b*) and *c*).

e) If $(\mathrm{B}_k)_{k\geqslant 1}$ is an increasing sequence in Ψ and $\mathrm{B} = \bigcup_{k=1}^{\infty} \mathrm{B}_k$, then

$$\beta(\mathrm{B}) = \sup_k \beta(\mathrm{B}_k) = \lim_k \beta(\mathrm{B}_k) \quad \text{in } \overline{\mathbf{R}}.$$

For, setting $\mathrm{C}_1 = \mathrm{B}_1$ and $\mathrm{C}_n = \mathrm{B}_n - \mathrm{B}_{n-1}$ for $n \geqslant 2$, the C_n are pairwise disjoint and $\bigcup_{n=1}^{\infty} \mathrm{C}_n = \mathrm{B}$, therefore

$$\begin{aligned}\beta(\mathrm{B}) = \sum_{n=1}^{\infty} \beta(\mathrm{C}_n) = \sup_n \sum_{k=1}^{n} \beta(\mathrm{C}_k) &= \sup_n \beta\Big(\bigcup_{k=1}^{n} \mathrm{C}_k\Big) \\ &= \sup_n \beta(\mathrm{B}_n) = \lim_n \beta(\mathrm{B}_n)\end{aligned}$$

by the theorem on monotone limits (GT, IV, §5, No. 2, Th. 2).

f) For any sequence $(\mathrm{B}_k)_{k\geqslant 1}$ in Ψ, $\beta\Big(\bigcup_{k=1}^{\infty} \mathrm{B}_k\Big) \leqslant \sum_{k=1}^{\infty} \beta(\mathrm{B}_k)$ ('complete subadditivity'). For, writing $\mathrm{C}_n = \bigcup_{k=1}^{n} \mathrm{B}_k$ for $n = 1, 2, 3, \ldots$ and $\mathrm{C} = \bigcup_{k=1}^{\infty} \mathrm{B}_k$, by *e*) and *d*) one has

$$\beta(\mathrm{C}) = \sup_n \beta(\mathrm{C}_n) \leqslant \sup_n \sum_{k=1}^{n} \beta(\mathrm{B}_k) = \sum_{k=1}^{\infty} \beta(\mathrm{B}_k)\,.$$

g) If $(\mathrm{C}_n)_{n\geqslant 1}$ is a decreasing sequence in Ψ such that $\beta(\mathrm{C}_1) < +\infty$, then $\beta\Big(\bigcap_{n=1}^{\infty} \mathrm{C}_n\Big) = \inf_n \beta(\mathrm{C}_n) = \lim_n \beta(\mathrm{C}_n)$. For, writing $\mathrm{C} = \bigcap_{n=1}^{\infty} \mathrm{C}_n$ and $\mathrm{B}_n = \mathrm{C}_1 - \mathrm{C}_n$ for all n, the sequence (B_n) in Ψ is increasing and $\mathrm{C}_1 - \mathrm{C} = \bigcup_{n=1}^{\infty} \mathrm{B}_n$, so by *c*) and *e*) one has

$$\begin{aligned}\beta(\mathrm{C}_1) - \beta(\mathrm{C}) = \beta(\mathrm{C}_1 - \mathrm{C}) &= \sup_n \beta(\mathrm{B}_n) \\ &= \sup_n [\beta(\mathrm{C}_1) - \beta(\mathrm{C}_n)] = \beta(\mathrm{C}_1) - \inf_n \beta(\mathrm{C}_n)\,.\end{aligned}$$

As for the assertions at hand concerning the restriction α of β to the set Φ of compact subsets of X:

(PC_I) If K_1, K_2 are compact, then so are $K_1 \cup K_2$ and $K_1 \cap K_2$.

(PC_{II}) Let $M = K$.

(PM_I) Follows from *c*) above.

(PM_{II}) Follows from *d*) above.

(PM_{III}) Follows from *b*) above.

(PM''_{IV}) Let $K = M$.

Thus, to apply Theorem 5, it remains only to verify the condition (PM'_{IV}); this is the condition whose verification requires that X have a countable base.

IV.58, ℓ. -5 to -3.

"Then K is the intersection of a decreasing sequence $(U_1, U_2, \dots)$ of relatively compact open sets of X (GT, IX, §2, No. 5, Prop. 7)."

Since X is locally compact and has a countable base, it is metrizable (GT, IX, §2, No. 9, Cor. of Prop. 16). Let $(W_n)_{n \geqslant 1}$ be a sequence of open sets such that $K = \bigcap_{n=1}^{\infty} W_n$ (*loc. cit.*, No. 5, Prop. 7). For each n, there exists a relatively compact open set V_n such that $K \subset V_n \subset W_n$, and the open sets $U_n = V_1 \cap \cdots \cap V_n$ meet the requirements.

Let us proceed directly to the objective: the verification of the condition (PM'_{IV}). Given any element of Φ—in other words any compact set K—and any $\varepsilon > 0$, we seek an open set U such that for every set $N \in \Phi$ contained in U—that is, for every compact subset N of U—one has $\beta(N) \leqslant \beta(K) + \varepsilon$.

With $K = \bigcap_{n=1}^{\infty} U_n$ as above, one knows that $\beta(U_n) < \infty$; for, all open sets and closed sets are Borel sets, and $\beta(U_n) \leqslant \beta(\overline{U}_n) < +\infty$ by the hypothesis (ii) of the Corollary. Therefore $\beta(U_n) \to \beta(K)$ by item *g*) in the note for **IV.58,** ℓ. $-6, -5$. Choose an index m such that $\beta(U_m) \leqslant \beta(K) + \varepsilon$. Then, for every compact $N \subset U_m$, one has

$$\beta(N) \leqslant \beta(U_m) \leqslant \beta(K) + \varepsilon,$$

thus $U = U_m$ meets the requirements of (PM'_{IV}).

IV.59, ℓ. 1.

"This proves that the condition (PM'_{IV}) is satisfied."

See the preceding note.

IV.59, ℓ. 3, 4.

"Since every open set U of X is the union of an increasing sequence of compact subsets, we have $\mu^*(U) = \beta(U)$."

One is assuming that X is locally compact and has a countable base, therefore X is countable at infinity (GT, IX, §2, No. 9, Cor. of Prop. 16). A subset of U is compact in X if and only if it is compact in U for the induced topology. {This is immediate from the definition of compact subset (GT, I, §9, No. 3, Def. 2) and the transitivity of induced topologies (GT, I, §3, No. 1).} Since U is a neighborhood of each of its points, it follows that the subspace U is itself locally compact and has a countable base, hence is countable at infinity; therefore $U = \bigcup_{n=1}^{\infty} K_n$ with the K_n compact in U, hence also in X. Replacing K_n by $K_1 \cup \cdots \cup K_n$, one can suppose that the sequence (K_n) is increasing. Then

$$\mu^*(U) = \sup_n \mu(K_n) = \sup_n \beta(K_n) = \beta(U)\,,$$

the first equality by §1, No. 2, Prop. 7, and the last by item *e*) of the note for **IV.58,** ℓ. −6, −5.

IV.59, ℓ. 6, 7.

"... if B is an element of Ψ contained in L, then B is μ-integrable"

Let $\mathscr{S}$ be the tribe of μ-integrable subsets of L. The set of all $A \subset X$ such that $A \cap L \in \mathscr{S}$ is easily seen to be a tribe that contains every compact subset of X, hence contains the tribe Ψ of all Borel sets in X; that is, $B \cap L$ is μ-integrable for every $B \in \Psi$. In particular, if $B \in \Psi$ is contained in L, then $B = B \cap L$ is μ-integrable.

IV.59, ℓ. 8, 9.

"Since $\beta(U) = \mu^*(U)$ and $\beta(K) = \mu(K)$, we see that $|\mu^*(B) - \beta(B)| \leqslant 2\varepsilon$."

For,

$$(*) \qquad \mu^*(B) - \beta(B) = [\mu^*(B) - \mu(K)] + [\mu(K) - \beta(B)]\,,$$

where

$$0 \leqslant \mu^*(B) - \mu^*(K) = \mu^*(B) - \mu(K) \leqslant \mu^*(U) - \mu(K) \leqslant \varepsilon$$

and, since $\beta(B) - \mu(K) = \beta(B) - \beta(K) = \beta(B - K) \geqslant 0$,

$$\begin{aligned} 0 \leqslant \beta(B) - \mu(K) = \beta(B - K) &\leqslant \beta(U - K) \\ &= \beta(U) - \beta(K) = \mu^*(U) - \mu(K) \leqslant \varepsilon\,, \end{aligned}$$

therefore, taking absolute values in $(*)$ yields $|\mu^*(B) - \beta(B)| \leqslant \varepsilon + \varepsilon$; since ε is arbitrary, $\mu^*(B) = \beta(B)$.

Since L is an arbitrary compact subset of X, the argument shows that $\mu^*(B) = \beta(B)$ for every relatively compact Borel set B.

IV.59, ℓ. 10, 11.

"...every Borel set C of X is the union of a sequence of pairwise disjoint, relatively compact Borel sets"

Write $X = \bigcup_{n=1}^{\infty} K_n$ with (K_n) an increasing sequence of compact sets. Then

$$X = K_1 \cup \bigcup_{n=2}^{\infty} (K_n - K_{n-1})$$

expresses the Borel set X as such a union, so the decomposition

$$C = (C \cap K_1) \cup \bigcup_{n=2}^{\infty} C \cap \Big(K_n - K_{n-1}\Big)$$

meets the requirements.

IV.59, ℓ. 11.

"...whence $\beta(C) = \mu^*(C)$."

Write $C = \bigcup_{n=1}^{\infty} B_n$, with (B_n) a sequence of pairwise disjoint, relatively compact Borel sets. As shown above, the B_n are μ-integrable and $\beta(B_n) = \mu^*(B_n) = \mu(B_n)$, so the assertion follows from the complete additivity of μ (No. 5, Prop. 6) and β.

IV.59, ℓ. 11, 12.

"The uniqueness of μ follows at once from Th. 5."

Suppose also ν is a measure on X such that $\nu^*(B) = \beta(B)$ for all $B \in \Psi$. In particular, for every compact set K one has

$$\nu(K) = \nu^*(K) = \beta(K) = \mu^*(K) = \mu(K),$$

thus $\nu\big|\Phi = \mu\big|\Phi = \alpha$, so $\mu = \nu$ by the uniqueness part of Th. 5.

§5. MEASURABLE FUNCTIONS AND SETS

IV.60, ℓ. 6, 7.

"... therefore the restriction of f to H is continuous."

GT, I, §3, No. 2, Prop. 4.

IV.60, ℓ. 17.

" $|\mu|(\mathrm{K} - \mathrm{K}_0) \leqslant \sum_{n=1}^{\infty} \varepsilon/2^n$ "

Write $\mathrm{A}_n = \mathrm{K} - \mathrm{K}_n$ and $\mathrm{A} = \mathrm{K} - \mathrm{K}_0$, and cite §4, No. 5, Cor. of Prop. 8.

IV.60, ℓ. −14 to −11.

"... it comes to the same to say that a measurable set A is a set such that, for every compact set K, there exist a negligible set $\mathrm{N} \subset \mathrm{K}$ and a partition (K_n) of $\mathrm{K} - \mathrm{N}$ formed by a sequence of compact sets each of which is contained either in $\mathrm{K} \cap \mathrm{A}$ or in $\mathrm{K} \cap \complement \mathrm{A}$."

Suppose A is measurable, that is, φ_{A} is measurable, and let K be any compact set in X. By Def. 1, there exist a negligible set $\mathrm{N} \subset \mathrm{K}$ and a partition (K_n) of $\mathrm{K} - \mathrm{N}$ into a sequence of compact sets K_n such that $\varphi_{\mathrm{A}}|\mathrm{K}_n$ is continuous for each n. But 0, 1 are the only possible values of φ_{A}; let

$$\mathrm{K}'_n = \mathrm{K}_n \cap \overset{-1}{\varphi}_{\mathrm{A}}(1), \quad \mathrm{K}''_n = \mathrm{K}_n \cap \overset{-1}{\varphi}_{\mathrm{A}}(0).$$

Then $\mathrm{K}'_n, \mathrm{K}''_n$ partition K_n into subsets that are closed (by the continuity of $\varphi_{\mathrm{A}}|\mathrm{K}_n$) hence compact, with $\mathrm{K}'_n \subset \mathrm{A}$ and $\mathrm{K}''_n \subset \complement \mathrm{A}$. Thus N, together with the compact sets (K'_n), (K''_n) satisfy the condition in the assertion.

The converse is immediate from Def. 1, since φ_{A} is constant (hence continuous) on each of the sets $\mathrm{K} \cap \mathrm{A}$ and $\mathrm{K} \cap \complement \mathrm{A}$.

IV.60, ℓ. −7.

"The condition is necessary ..."

Suppose A is measurable and let K be any compact set in X. As noted following Def. 2, there exists a partition

$$\mathrm{K} = \mathrm{N} \cup \bigcup_{n=1}^{\infty} \mathrm{K}'_n \cup \bigcup_{n=1}^{\infty} \mathrm{K}''_n,$$

where N is negligible, the K'_n and K''_n are compact, and $\mathrm{K}'_n \subset \mathrm{A}$, $\mathrm{K}''_n \subset \complement \mathrm{A}$ for all n. Then $\mathrm{A} \cap \mathrm{K}''_n = \varnothing$ for all n, so

$$(*) \qquad \mathrm{A} \cap \mathrm{K} = (\mathrm{A} \cap \mathrm{N}) \cup \bigcup_{n=1}^{\infty} \mathrm{K}'_n ;$$

$\mathrm{A} \cap \mathrm{N}$ is negligible, hence integrable with $|\mu|(\mathrm{A} \cap \mathrm{N}) = 0$; and, for all n,

$$\sum_{i=1}^{n} |\mu|(\mathrm{K}'_i) = |\mu|\Big(\bigcup_{i=1}^{n} \mathrm{K}'_i\Big) \leqslant |\mu|(\mathrm{K}) < +\infty ,$$

therefore $\bigcup_{n=1}^{\infty} \mathrm{K}'_n$ is integrable (§4, No. 5, Cor. of Prop. 8). Thus each of the two terms on the right side of $(*)$ is integrable, hence so is $\mathrm{A} \cap \mathrm{K}$.

IV.60, ℓ. −5.

"The condition is sufficient . . ."

Let K be any compact set in X. By hypothesis $\mathrm{A} \cap \mathrm{K}$ is integrable, hence there exists a partition

$$\mathrm{A} \cap \mathrm{K} = \mathrm{N}' \cup \bigcup_{n=1}^{\infty} \mathrm{K}'_n ,$$

where N' is negligible and the K'_n are compact (§4, No. 6, Cor. 2 of Th. 4). But $(\complement \mathrm{A}) \cap \mathrm{K} = \mathrm{K} - \mathrm{A} \cap \mathrm{K}$ is also integrable (§4, No. 5, Prop. 7), so there is also a partition

$$\complement \mathrm{A} \cap \mathrm{K} = \mathrm{N}'' \cup \bigcup_{n=1}^{\infty} \mathrm{K}''_n$$

with N'' negligible and the K''_n compact. Then $\mathrm{N} = \mathrm{N}' \cup \mathrm{N}''$ is negligible and

$$\mathrm{K} = (\mathrm{A} \cap \mathrm{K}) \cup (\complement \mathrm{A} \cap \mathrm{K}) = \mathrm{N} \cup \bigcup_{n=1}^{\infty} \mathrm{K}'_n \cup \bigcup_{n=1}^{\infty} \mathrm{K}''_n ,$$

where the K'_n, K''_n $(n = 1, 2, 3, \ldots)$ partition $\mathrm{K} - \mathrm{N}$ into compact sets such that $\mathrm{K}'_n \subset \mathrm{A}$ and $\mathrm{K}''_n \subset \complement \mathrm{A}$ for all n, therefore A is measurable by the remark following Def. 2.

It follows from Prop. 3 that every integrable set A is measurable; for, by Prop. 7, 2° of §4, No. 5, $\mathrm{A} \cap \mathrm{K}$ is integrable for every compact set K.

IV.60, ℓ. −2.

"*The open sets and the closed sets are measurable.*"

The measurability of closed sets is obvious from Prop. 3; as for open sets, it suffices to prove the following:

$$\mathrm{A} \text{ measurable} \quad \Rightarrow \quad \complement \mathrm{A} \text{ measurable.}$$

For, if A is measurable and K is any compact set in X, then $\complement \mathrm{A} \cap \mathrm{K} = \mathrm{K} - \mathrm{A} \cap \mathrm{K}$ is the difference of two integrable sets, hence is integrable (§4, No. 5, Prop. 7).

Prop. 3 yields a brief proof of the following:

"COROLLARY 3." —*If* μ *is a measure on the locally compact space* X, *then*:

(i) *The set of* μ*-measurable subsets of* X *is a tribe.*

(ii) *Every Borel set in* X *is* μ*-measurable.*

These assertions are proved in No. 4 below as corollaries of Egoroff's theorem (Cors. 2 and 3 of Th. 2). For the definition of a tribe, see the footnote on p. IV.50.

(i) If (A_n) is a sequence of measurable sets, then the set $\mathrm{A} = \bigcap_n \mathrm{A}_n$ is measurable. For, if K is any compact subset of X, the set $\mathrm{A} \cap \mathrm{K} = \bigcap_n (\mathrm{A}_n \cap \mathrm{K})$ is the intersection of a sequence of integrable sets, hence is integrable (§4, No. 5, Prop. 7). It follows by complementation that the set $\bigcup_n \mathrm{A}_n$ is also measurable.

(ii) The tribe of measurable sets contains every open set in X by Cor. 1, hence it contains the tribe they generate (GT, IX, §6, No. 3, Def. 4).

IV.61, ℓ. 3, 4.

"... it suffices to verify that every relatively compact Souslin set A is μ-integrable."

Let A be a Souslin subset of the metrizable locally compact space X, and let μ be any measure on X. We are to show that A is μ-measurable; given any compact subset K of X, it suffices by Prop. 3 to show that $\mathrm{A} \cap \mathrm{K}$ is μ-integrable. Since K is a metrizable compact subspace of X, it is Polish (GT, IX, §6, No. 1, Cor. of Prop. 2) hence Souslin (*loc. cit.*, No. 2), therefore $\mathrm{A} \cap \mathrm{K}$ is a Souslin subset of X (*loc. cit.*, Prop. 8) and is obviously relatively compact; thus, one is reduced to showing that a relatively compact Souslin subset of X is μ-integrable.

IV.61, ℓ. 4, 5.

"... such a set A is capacitable for $|\mu|^*$ (GT, IX, §6, No. 9, Th. 5)."

That $|\mu|^*$ is a capacity on X is noted in §4, No. 6, Cor. 2 of Prop. 10. (See the note for **IV.43,** ℓ. −13, −12, where it is noted that the definition of 'capacity' is different in GT and TG, but that with either definition, $|\mu|^*$ has the desired properties. In both cases, the property asserted for A is

$$|\mu|^*(\mathrm{A}) = \sup\{|\mu|^*(\mathrm{K}) : \ \mathrm{K} \subset \mathrm{A}\,, \ \mathrm{K} \text{ compact}\}\,,$$

established in GT, IX, §6, No. 9, Th. 5 and TG, IX, §6, No. 10, Th. 6, respectively.

For the purposes of the proof at hand, one need read no further; the rest of this note is devoted to the comparative analysis of the two definitions.

In GT, IX, §6, No. 9, Def. 8, a *capacity* on (a Hausdorff space) X is defined to be a function $f : \mathfrak{P}(\mathrm{X}) \to \overline{\mathbf{R}}$ satisfying the following axioms:

($\mathrm{CA_I}$) If $\mathrm{A} \subset \mathrm{B}$, then $f(\mathrm{A}) \leqslant f(\mathrm{B})$.

($\mathrm{CA_{II}}$) If (A_n) is any increasing sequence of subsets of X, then

$$f\Big(\bigcup_n \mathrm{A}_n\Big) = \sup_n f(\mathrm{A}_n)\,.$$

($\mathrm{CA_{III}}$) If (K_n) is any decreasing sequence of *compact* subsets of X, then

$$f\Big(\bigcap_n \mathrm{K}_n\Big) = \inf_n f(\mathrm{K}_n)\,.$$

Whereas in TG, IX, §6, No. 10, Def. 9, a function $f : \mathfrak{P}(\mathrm{X}) \to \overline{\mathbf{R}}$ is called a 'capacity' if it merely satisfies ($\mathrm{CA_I}$) and ($\mathrm{CA_{II}}$); to compensate for the absence of ($\mathrm{CA_{III}}$), a *capacity continuous on the right* (for short, a CR-*capacity* is required in addition to satisfy the axiom

($\mathrm{CA'_{III}}$) For every compact subset K of X and every number $a > f(\mathrm{K})$, there exists an open set U containing K such that $f(\mathrm{U}) < a$.

The prime $'$ in ($\mathrm{CA'_{III}}$) does not appear in TG but is added here to avoid confusion. It is shown (TG, *loc. cit.*, *Remark*) that every CR-capacity is a capacity in the sense of GT (for short, a GT-*capacity*); so to speak, axiom ($\mathrm{CA'_{III}}$) is (in the presence of ($\mathrm{CA_I}$)) stronger than ($\mathrm{CA_{III}}$).

Note that ($\mathrm{CA'_{III}}$) is, in the presence of ($\mathrm{CA_I}$), equivalent to the following condition:

($\mathrm{CA''_{III}}$) For every compact subset K of X,

$$f(\mathrm{K}) = \inf\{f(\mathrm{U}) : \ \mathrm{K} \subset \mathrm{U}\,, \ \mathrm{U} \text{ open}\}\,.$$

For, let α be the infimum on the right side; if U is any open set containing K (at least $\mathrm{U} = \mathrm{X}$ qualifies) then $f(\mathrm{K}) \leqslant f(\mathrm{U})$, whence $f(\mathrm{K}) \leqslant \alpha$.

($\mathrm{CA'_{III}}$) $\Rightarrow$ ($\mathrm{CA''_{III}}$): If $f(\mathrm{K}) = +\infty$ then also $\alpha = +\infty$. If $f(\mathrm{K}) < +\infty$, let a be any real number such that $f(\mathrm{K}) < a < +\infty$; since $a > f(\mathrm{K})$, by ($\mathrm{CA'_{III}}$) there exists an open set U such that $\mathrm{K} \subset \mathrm{U}$ and $f(\mathrm{U}) < a$, whence, by the definition of α, $\alpha \leqslant f(\mathrm{U}) < a$, and varying a yields $\alpha \leqslant f(\mathrm{K})$.

$(\mathrm{CA}''_{\mathrm{III}}) \Rightarrow (\mathrm{CA}'_{\mathrm{III}})$: Let K be a compact subset of X and let $a > f(\mathrm{K})$. By assumption, $f(\mathrm{K}) = \alpha$, that is, $f(\mathrm{K})$ is the *greatest* lower bound of the $f(\mathrm{U})$ for U open and $\mathrm{K} \subset \mathrm{U}$; since $a > f(\mathrm{K})$, a is not a lower bound for the $f(\mathrm{U})$, so there must exist an open set $\mathrm{U} \supset \mathrm{K}$ such that $f(\mathrm{U}) < a$.

Already noted above is that the function $|\mu|^* : \mathfrak{P}(\mathrm{X}) \to \overline{\mathbf{R}}$ (in other words, μ^* for any positive measure μ) is a GT-capacity; in view of the preceding equivalence, it is also a CR-capacity, indeed, the formula

$$|\mu|^*(\mathrm{A}) = \inf\{|\mu|^*(\mathrm{U}) :\ \mathrm{A} \subset \mathrm{U}\,,\ \mathrm{U}\ \text{open}\}$$

holds for every subset A of X (§2, No. 4, Prop. 19).

For f either a GT-capacity or a CR-capacity, a subset A of X is said to be *capacitable* if it satisfies the condition

$$f(\mathrm{A}) = \sup\{f(\mathrm{K}) :\ \mathrm{K} \subset \mathrm{A}\,,\ \mathrm{K}\ \text{compact}\}$$

(GT, IX, §6, No. 9, Def. 9) or (TG, IX, §6, No. 10, Def. 10). For $f = |\mu|^*$ the condition reads

$$|\mu|^*(\mathrm{A}) = \sup\{|\mu|(\mathrm{K}) :\ \mathrm{K} \subset \mathrm{A}\,,\ \mathrm{K}\ \text{compact}\}\,.$$

If f is a GT-capacity on a metrizable space X, then every relatively compact Souslin subset A of X is capacitable for f (GT, *loc. cit.*, Th. 5), whereas if f is a CR-capacity on a Hausdorff space X, then every Souslin subset A of X is capacitable for f (TG, *loc. cit.*, Th. 6).

IV.61, ℓ. 5, 6.

"Therefore, for every $\varepsilon > 0$ there exists a compact subset K of A such that $|\mu|^*(\mathrm{A}) \leqslant |\mu|^*(\mathrm{K}) + \varepsilon$"

Since $|\mu|^*(\mathrm{A}) < +\infty$ (A is relatively compact), this is immediate from the definition of capacitability (GT, IX, §6, No. 9, Def. 9; see the discussion in the preceding note).

IV.61, ℓ. 6–8.

"Let U be a relatively compact open set in X containing A such that

$$|\mu|(\mathrm{U}) = |\mu|^*(\mathrm{U}) \leqslant |\mu|^*(\mathrm{A}) + \varepsilon\,."$$

Since $\overline{\mathrm{A}}$ is compact, $|\mu|^*(\mathrm{A}) \leqslant |\mu|^*(\overline{\mathrm{A}}) < +\infty$, hence there exists an open set V with $\mathrm{A} \subset \mathrm{V}$ and $|\mu|^*(\mathrm{V}) \leqslant |\mu|^*(\mathrm{A}) + \varepsilon$ (§1, No. 4, Prop. 19). On the other hand, there exists a relatively compact open set W with $\overline{\mathrm{A}} \subset \mathrm{W}$

(every point of $\overline{\mathrm{A}}$ has a relatively compact open neighborhood; extract a finite subcovering of $\overline{\mathrm{A}}$). Let $\mathrm{U} = \mathrm{V} \cap \mathrm{W}$. Then $\mathrm{A} = \mathrm{A} \cap \overline{\mathrm{A}} \subset \mathrm{V} \cap \mathrm{W} = \mathrm{U}$, where U is relatively compact (because W is) and $|\mu|^*(\mathrm{U}) \leqslant |\mu|^*(\mathrm{V}) \leqslant |\mu|^*(\mathrm{A}) + \varepsilon$.

IV.61, ℓ. 17–21.

"It follows at once (§4, No. 9, Lemma) that there exist a negligible set $\mathrm{N} \subset \mathrm{K}$ and a finite partition of $\mathrm{K} - \mathrm{N}$ formed of integrable sets M_j such that each of the sets $\mathrm{K} \cap \mathrm{V}_{x_i}$ is the union of a subset of N and a certain number of the M_j, and such that on each of the M_j, f is equal to one of the functions g_{x_i}."

Say i runs from 1 to r. For brevity, write $\mathrm{V}_i = \mathrm{V}_{x_i}$ and $g_i = g_{x_i}$ for $i = 1, \ldots, r$. By hypothesis, for each i one has $f = g_i$ almost everywhere in V_i, so there exists a negligible set $\mathrm{N}_i \subset \mathrm{V}_i$ such that $f = g_i$ on $\mathrm{V}_i - \mathrm{N}_i$. Write $\mathrm{A}_i = \mathrm{V}_i - \mathrm{N}_i$; as the difference of integrable sets, A_i is integrable. Thus

$$(*) \qquad \mathrm{V}_i = \mathrm{N}_i \cup \mathrm{A}_i \quad (1 \leqslant i \leqslant r),$$

a disjoint union with N_i negligible and A_i integrable.

For every r-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ with $\varepsilon_i = \pm 1$, write

$$\mathrm{V}_\varepsilon = \mathrm{V}_1^{\varepsilon_1} \cap \cdots \cap \mathrm{V}_r^{\varepsilon_r},$$

with the conventions $\mathrm{V}_i^1 = \mathrm{V}_i$ and $\mathrm{V}_i^{-1} = \mathrm{X} - \mathrm{V}_i$. The 2^r sets V_ε partition X:

$$\mathrm{X} = (\mathrm{V}_1 \cup \complement \mathrm{V}_1) \cap (\mathrm{V}_2 \cup \complement \mathrm{V}_2) \cap \cdots \cap (\mathrm{V}_r \cup \complement \mathrm{V}_r) = \bigcup_\varepsilon \mathrm{V}_\varepsilon .$$

Let $\varepsilon_0 = (-1, \ldots, -1)$; since

$$\mathrm{V}_{\varepsilon_0} = \complement \mathrm{V}_1 \cap \cdots \cap \complement \mathrm{V}_r = \complement (\mathrm{V}_1 \cup \cdots \cup \mathrm{V}_r),$$

the $2^r - 1$ sets V_ε $(\varepsilon \neq \varepsilon_0)$ are integrable (each being contained in at least one V_i), pairwise disjoint, and

$$\bigcup_{i=1}^r \mathrm{V}_i = \bigcup_{\varepsilon \neq \varepsilon_0} \mathrm{V}_\varepsilon, \quad \mathrm{V}_i = \bigcup_{\varepsilon_i = 1} \mathrm{V}_\varepsilon .$$

Since $\mathrm{K} \subset \bigcup_{i=1}^r \mathrm{V}_i$, one has

$$(\dagger) \qquad \mathrm{K} = \mathrm{K} \cap \bigcup_{i=1}^r \mathrm{V}_i = \bigcup_{\varepsilon \neq \varepsilon_0} \mathrm{K} \cap \mathrm{V}_\varepsilon, \quad \mathrm{K} \cap \mathrm{V}_i = \bigcup_{\varepsilon_i = 1} \mathrm{K} \cap \mathrm{V}_\varepsilon .$$

The plan is to decompose each of the sets $K \cap V_\varepsilon$ $(\varepsilon \neq \varepsilon_0)$ in the form of (∗)—the disjoint union of a negligible part and an integrable part—then decompose each integrable part as the disjoint union of a negligible set and a sequence of compact sets on which f is continuous.

To begin with, consider V_{ε_1}, where $\varepsilon_1 = (1, \ldots, 1)$; citing (∗),

$$V_{\varepsilon_1} = V_1 \cap \cdots \cap V_r = \bigcap_{i=1}^{r} (N_i \cup A_i) = N_{\varepsilon_1} \cup \bigcap_{i=1}^{r} A_i ,$$

where N_{ε_1} is the union of $2^r - 1$ sets of the form $B_1 \cap \cdots \cap B_r$, each B_i being equal to either N_i or A_i, and at least one of the B_i being equal to N_i. Writing $A_{\varepsilon_1} = \bigcap_{i=1}^{r} A_i$, one has

$$V_{\varepsilon_1} = N_{\varepsilon_1} \cup A_{\varepsilon_1} ,$$

where N_{ε_1} is negligible, A_{ε_1} is integrable, and $A_{\varepsilon_1} \subset A_i$ for at least one value of i (in this instance, for every i).

Consider now any ε such that $\varepsilon \neq \varepsilon_0$ and $\varepsilon \neq \varepsilon_1$ (that is, ε_i is equal to 1 for at least one value of i and to -1 for at least one value). Then

$$V_\varepsilon = \Big(\bigcap_{\varepsilon_i = 1} V_i \Big) \cap \Big(\bigcap_{\varepsilon_i = -1} \complement V_i \Big) = \Big(\bigcap_{\varepsilon_i = 1} V_i \Big) \cap \complement \Big(\bigcup_{\varepsilon_i = -1} V_i \Big) = R \cap \complement S ,$$

where $R = \bigcap_{\varepsilon_i = 1} V_\varepsilon$ and $S = \bigcup_{\varepsilon_i = -1} V_\varepsilon$. By the argument used for V_{ε_1}, one can write R as a disjoint union $R = N_0 \cup A$ with N_0 negligible, A integrable and $A \subset A_i$ for some i (indeed, $A = \bigcap_{\varepsilon_i = 1} A_i$). Then

$$\begin{aligned} V_\varepsilon &= R \cap \complement S = (N_0 \cup A) \cap \complement S \\ &= (N_0 \cap \complement S) \cup (A \cap \complement S) = (N_0 \cap \complement S) \cup (A - A \cap S) , \end{aligned}$$

where $N_0 \cap \complement S$ is negligible, $A - A \cap S$ is integrable and is contained in some A_i (indeed, in $\bigcap_{\varepsilon_i = 1} A_i$). We write $N_\varepsilon = N_0 \cap \complement S$ and $A_\varepsilon = A - A \cap S$. To summarize: for every $\varepsilon \neq \varepsilon_0$ we may write

$$V_\varepsilon = N_\varepsilon \cup A_\varepsilon$$

with N_ε negligible, A_ε integrable, $N_\varepsilon \cap A_\varepsilon = \varnothing$ and A_ε contained in some A_i (indeed, in $\bigcap_{\varepsilon_i = 1} A_i$), so that, for such i, $f = g_i$ on A_i. Then

$$(\dagger\dagger) \qquad (\forall\ \varepsilon \neq \varepsilon_0) \quad K \cap V_\varepsilon = (K \cap N_\varepsilon) \cup (K \cap A_\varepsilon) ,$$

a disjoint union with $K \cap N_\varepsilon$ negligible, $K \cap A_\varepsilon$ integrable, and $K \cap A_\varepsilon$ contained in some A_i (indeed, in $\bigcap_{\varepsilon_i = 1} A_i$). Setting

$$N = \bigcup_{\varepsilon \neq \varepsilon_0} K \cap N_\varepsilon \,,$$

we have a disjoint union

$$K = \bigcup_{\varepsilon \neq \varepsilon_0} K \cap V_\varepsilon = N \cup \bigcup_{\varepsilon \neq \varepsilon_0} K \cap A_\varepsilon \,,$$

where N is negligible, every $K \cap A_\varepsilon$ $(\varepsilon \neq \varepsilon_0)$ is integrable and is contained in some A_i.

Let M_j $(1 \leqslant j \leqslant 2^r - 1)$ be any enumeration of the sets $K \cap A_\varepsilon$ $(\varepsilon \neq \varepsilon_0)$. We then have a disjoint union

$$K = N \cup \bigcup_j M_j \,, \tag{†††}$$

where N is negligible, the M_j are integrable and, for each j, there exists an index i such that $f = g_i$ on M_j.

{Though it is not needed for the rest of the proof of Prop. 4, we verify that for each i $(1 \leqslant i \leqslant r)$, $K \cap V_i = K \cap V_{x_i}$ has the form

$$K \cap V_i = N'_i \cup \bigcup_{j \in J_i} M_j \,,$$

where $N'_i \subset N$ and J_i is some set of the indices j $(1 \leqslant j \leqslant 2^r - 1)$. Indeed, as observed in (†),

$$K \cap V_i = \bigcup_{\varepsilon_i = 1} K \cap V_\varepsilon \,;$$

in view of (††) this may be written

$$K \cap V_i = \Big(\bigcup_{\varepsilon_i = 1} K \cap N_\varepsilon \Big) \cup \Big(\bigcup_{\varepsilon_i = 1} K \cap A_\varepsilon \Big),$$

where the $K \cap N_\varepsilon$ are all subsets of N, and each $K \cap A_\varepsilon$ is equal to one of the M_j. (The set J_i of indices in question depends on how the $K \cap A_\varepsilon$ are enumerated.)}

IV.61, ℓ. −9.

"… f is measurable."

The conclusion of the proof of Prop. 4 is a matter of refining (twice) the partition (†††) of K of the preceding note, so as to exhibit the measurability of f. Since each M_j is integrable, there exists a partition

$$M_j = N_j \cup \bigcup_{n\in\mathbf{N}} K_{nj}$$

with N_j negligible[1] and $(K_{nj})_{n\in\mathbf{N}}$ a sequence of pairwise disjoint compact sets (§4, No. 6, Cor. 2 of Th. 4). Also, for each j, there exists an index i such that $f = g_i$ on M_j, hence $f = g_i$ on K_{nj} for all $n \in \mathbf{N}$; since g_i is measurable and K_{nj} is compact, there exists a further partition

$$K_{nj} = P_{nj} \cup \bigcup_{m\in\mathbf{N}} K_{mnj}$$

with P_{nj} negligible and $(K_{mnj})_{m\in\mathbf{N}}$ a sequence of pairwise disjoint compact sets such that the restriction to K_{mnj} of g_i —hence also of f —is continuous (No. 1, Def. 1). The partition (†††) doubly refined now takes the form

$$\begin{aligned} K &= N \cup \bigcup_j \Big(N_j \cup \bigcup_{n\in\mathbf{N}} K_{nj} \Big) \\ &= \Big(N \cup \bigcup_j N_j \Big) \cup \bigcup_j \bigcup_{n\in\mathbf{N}} \Big(P_{nj} \cup \bigcup_{m\in\mathbf{N}} K_{mnj} \Big) \\ &= \Big(N \cup \bigcup_j N_j \cup \bigcup_{j,n} P_{nj} \Big) \cup \bigcup_{j,n,m} K_{mnj} \\ &= N' \cup \bigcup_{j,n,m} K_{mnj} \end{aligned}$$

with $N' = N \cup \bigcup_j N_j \cup \bigcup_{j,n} P_{nj}$ negligible and the K_{mnj} compact sets such that $f\big|K_{mnj}$ is continuous; since K is an arbitrary compact subset of X, f is measurable (No. 1, Def. 1).

IV.61, ℓ. −4.

"By the principle of localization, every locally negligible set is *measurable.*"

Let $A \subset X$ be locally measurable. Given any compact set K in X, it suffices to show that $A \cap K$ is integrable (No. 1, Prop. 3). Indeed, $A \cap K$ is negligible by Prop. 5 below (which is derived directly from Def. 3 without the

[1]The N_j's here are different from the N_i's occurring in the preceding note (an ephemeral notation on the way to the partition (†††)).

intervention of Prop. 4); but every negligible set N is integrable: $\varphi_{\mathrm{N}} = 0$ μ-almost everywhere, therefore $\varphi_{\mathrm{N}} \in \mathscr{L}^p$ for all $p \geqslant 1$ (§3, No. 4, remarks following Def. 2) and so φ_{N} is integrable (§4, No. 1, Def. 1) with integral 0 (*loc. cit.*).

The proof via Prop. 4 is overkill, but here it is: Let A be a locally negligible set and let $f = \varphi_{\mathrm{A}}$. For every $x \in \mathrm{X}$, there exists a neighborhood V_x of x such that $\mathrm{V}_x \cap \mathrm{A}$ is negligible; dropping down to a compact neighborhood of x contained in V_x, we can suppose that V_x is compact, hence integrable. Set $g_x = 0$ (the zero function on X); then $\varphi_{\mathrm{A}}\big|\mathrm{V}_x = 0$ except on the negligible set $\mathrm{V}_x \cap \mathrm{A}$, thus $f = g_x$ almost everywhere in V_x, and so f is measurable by Prop. 4.

IV.62, ℓ. 10, 11.

"Conversely, if it is satisfied then A is contained in an integrable open set G "

Since $|\mu|^*(\mathrm{A}) < +\infty$, there exists an open set G with $\mathrm{A} \subset \mathrm{G}$ and $|\mu|^*(\mathrm{G}) < +\infty$ (§1, No. 4, Prop. 19), and an open set with finite outer measure is integrable (§4, No. 6, Prop. 10).

IV.62, ℓ. 18–20.

"...if G is locally negligible then $|\mu|(\mathrm{K}) = 0$ for every compact set K contained in G "

For, $\mathrm{K} = \mathrm{K} \cap \mathrm{G}$ is negligible by Prop. 5.

Thus, $|\mu|^*(\mathrm{G}) = \sup\{|\mu|(\mathrm{K}) : \mathrm{K} \subset \mathrm{G},\ \mathrm{K} \text{ compact}\} = 0$ (§4, No. 6, Cor. 4 of Th. 4), whence $\mathrm{G} \subset \complement\,\mathrm{Supp}(\mu)$ (§2, No. 2, Prop. 5).

IV.62, ℓ. −12.

"*If the set* N *of points of discontinuity of* f *is locally negligible*..."

The proof shows, more generally, that if N is any locally negligible set such that $f\big|\mathrm{X} - \mathrm{N}$ is continuous, then f is measurable.

IV.62, ℓ. −9, −8.

"...for every $\varepsilon > 0$, there exists a compact set $\mathrm{K}_1 \subset \mathrm{K} - (\mathrm{K} \cap \mathrm{N})$ such that $|\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant \varepsilon$ "

Let $\varepsilon > 0$. Since $|\mu|^*(\mathrm{K} \cap \mathrm{N}) = 0$, there exists an open set U with $\mathrm{K} \cap \mathrm{N} \subset \mathrm{U}$ and $|\mu|(\mathrm{U}) \leqslant \varepsilon$ (§1, No. 4, Prop. 19). Let

$$\mathrm{K}_1 = \mathrm{K} \cap \complement\,\mathrm{U} = \mathrm{K} - \mathrm{U} = \mathrm{K} - \mathrm{K} \cap \mathrm{U};$$

then K_1 is compact, $\mathrm{K}_1 \subset \mathrm{K}$, and $\mathrm{K} - \mathrm{K}_1 = \mathrm{K} \cap \mathrm{U} \subset \mathrm{U}$, whence $|\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant |\mu|(\mathrm{U}) \leqslant \varepsilon$. Since $\mathrm{K} \cap \mathrm{N} \subset \mathrm{K} \cap \mathrm{U}$, so that $\mathrm{K}_1 = \mathrm{K} - \mathrm{K} \cap \mathrm{U} \subset \mathrm{K} - \mathrm{K} \cap \mathrm{N} = \mathrm{K} - \mathrm{N}$, and since $f\big|\mathrm{X} - \mathrm{N}$ is continuous, $f\big|\mathrm{K}_1$ is also continuous; thus f is measurable by No. 1, Prop. 1.

Note that it suffices to assume the weaker condition that N is a locally negligible set such that $f|X - N$ is continuous (i.e., f need not be continuous as a function on X at the points of $X - N$). Consider, for example, $X = \mathbf{R}$, μ the Lebesgue measure on $\mathbf{R}$, and $f = \varphi_{\mathbf{Q}}$ the characteristic function of the rationals. Since $\mathbf{Q}$ is negligible and $f|\mathbf{R} - \mathbf{Q} = 0$ is continuous, f is measurable by the foregoing argument, despite being nowhere continuous.

IV.62, ℓ. −6, −5.

"« $\boldsymbol{P}\{x\}$ *locally almost everywhere* (*with respect to* μ) »"

Let $N = \{x : x \in X \text{ and } \operatorname{non} \boldsymbol{P}\{x\}\}$; then Prop. 5 yields the criterion

$$\begin{aligned} \boldsymbol{P}\{x\} \text{ locally almost everywhere in } X & \\ &\Leftrightarrow \quad N \text{ is locally negligible} \\ &\Leftrightarrow \quad (\forall \text{ compact } K)\ K \cap N \text{ is negligible} \\ &\Leftrightarrow \quad (\forall \text{ compact } K)\ \{x : x \in K \text{ and } \operatorname{non} \boldsymbol{P}\{x\}\} \text{ is negligible} \\ &\Leftrightarrow \quad (\forall \text{ compact } K)\ \boldsymbol{P}\{x\} \text{ almost everywhere in } K. \end{aligned}$$

IV.63, ℓ. 11–13.

"PROPOSITION 6."

It is convenient to change the notation: assuming $g : X \to F$ is measurable and $f : X \to F$ is a function equal locally almost everywhere to g, let us show that f is also measurable. Let $N = \{x \in X : f(x) \neq g(x)\}$, a locally negligible set. The plan is to cite the principle of localization (Prop. 4).

For each $x \in X$ let V_x be any compact (hence integrable) neighborhood of x and let $g_x = g$. Then $V_x \cap N$ is negligible (Prop. 5) and, on the subset $V_x - V_x \cap N$ of $X - N$, $f = g = g_x$, thus $f = g_x$ almost everywhere in V_x. Quote Prop. 4.

IV.64, ℓ. 6–9.

"COROLLARY 4."

One will observe that the same proof works with $\mathbf{R}$ replaced by $\mathbf{C}$. It is tacit that F has its unique compatible Hausdorff topology (TVS, I, §2, No. 3, Th. 2).

Sufficiency. Assume each $f_k : X \to F$ is measurable. For each k, $\mathbf{e}_k f_k : X \to F$ is the composite $X \to \mathbf{R} \to F$, where

$$x \mapsto f_k(x) \mapsto f_k(x) \cdot \mathbf{e}_k \qquad (x \in X);$$

since $x \mapsto f_k(x)$ is measurable, and $c \mapsto c \cdot \mathbf{e}_k$ is continuous, the composite $\mathbf{e}_k f_k$ is measurable by Th. 1, so $\mathbf{f} = \sum_{k=1}^{n} \mathbf{e}_k f_k$ is measurable by Cor. 3.

(Here, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ can be any finite list of vectors, not necessarily linear independent, in any topological vector space.)

Necessity. Assume $\mathbf{f} = \sum_{k=1}^{n} \mathbf{e}_k f_k$ is measurable. Let $u : \mathrm{F} \to \mathbf{R}^n$ be the linear mapping such that $u(\mathbf{e}_1), \ldots, u(\mathbf{e}_n)$ is the canonical basis of $\mathbf{R}^n$. Since F is Hausdorff, u is continuous (even bicontinuous, by Th. 2 of TVS, §2, No. 3). For all $x \in \mathrm{X}$,

$$u\big(\mathbf{f}(x)\big) = u\Big(\sum_{k=1}^{n} \mathbf{e}_k f_k(x)\Big) = (f_1(x), \ldots, f_n(x)) .$$

If $\mathrm{pr}_k : \mathbf{R}^n \to \mathbf{R}$ is the k'th coordinate projection, then $\mathrm{pr}_k\big(u(\mathbf{f}(x)\big) = f_k(x)$ for all x, thus

$$f_k = (\mathrm{pr}_k \circ u) \circ \mathbf{f} ;$$

since $\mathbf{f}$ is measurable and $\mathrm{pr}_k \circ u$ is continuous, f_k is measurable by Th. 1.

IV.64, ℓ. 10–13.

"Corollary 5."

The mapping $[\mathbf{f} \cdot \mathbf{g}] : \mathrm{F} \times \mathrm{G} \to \mathrm{H}$ is the composite $\mathrm{X} \to \mathrm{F} \times \mathrm{G} \to \mathrm{H}$ defined by

$$x \mapsto \big(\mathbf{f}(x), \mathbf{g}(x)\big) \mapsto [\mathbf{f}(x) \cdot \mathbf{g}(x)] ,$$

where $\mathbf{f}$ and $\mathbf{g}$ are measurable and $(u, v) \to [u \cdot v]$ is continuous, so the composite is measurable by Th. 1.

The notation $[\mathbf{f} \cdot \mathbf{g}]$ is suggestive of a bilinear operation, but in fact bilinearity plays no role in the proof and can be omitted from the hypothesis.

IV.64, ℓ. -5.

"The first assertion obviously follows from the second"

With notations as in 2°, f is continuous on K_1; cite No. 1, Prop. 1.

IV.65, ℓ. 4, 5.

" $\mathrm{B}_{n,r}$ is a countable union of compact sets contained in K_0, hence is integrable"

For any pair $(\alpha, \beta) \in \mathrm{A} \times \mathrm{A}$, write $\mathrm{F}_{\alpha,\beta} : \mathrm{K}_0 \to \mathbf{R}$ for the function

$$\mathrm{F}_{\alpha,\beta}(x) = d\big(f_\alpha(x), f_\beta(x)\big) \qquad (x \in \mathrm{K}_0) .$$

Then $\mathrm{F}_{\alpha,\beta}$ is continuous, so the set

$$\{x \in \mathrm{K}_0 :\ d\big(f_\alpha(x), f_\beta(x)\big) \geqslant 1/r\} = \overset{-1}{\mathrm{F}}_{\alpha,\beta}([1/r, +\infty[)$$

is a closed, hence compact, subset of K_0; by definition,

$$(*) \qquad B_{n,r} = \bigcup_{(\alpha,\beta)\in A_n\times A_n} \overset{-1}{F}_{\alpha,\beta}([1/r,+\infty[)\,,$$

a countable union of compact subsets of K_0. If $(S_n)_{n\geqslant 1}$ is any enumeration of these compact subsets then, for each index m, the set $T_m = \bigcup_{k=1}^{m} S_k$ is compact, hence integrable (§4, No. 6, Cor. 1 of Prop. 10), with

$$|\mu|(T_m) \leqslant |\mu|(K_0) < +\infty \quad \text{for all } m\,,$$

therefore $B_{n,r} = \bigcup_{m=1}^{\infty} T_m$ is integrable (§4, No. 5, Prop. 8).

Note that since A_n is a decreasing function of n, it is clear from $(*)$ that, for fixed r, $B_{n,r}$ is also a decreasing function of n; and for fixed n, $B_{n,r}$ is an increasing function of r.

IV.65, ℓ. 6–8.

"If r is fixed, the intersection of the decreasing sequence of sets $B_{n,r}$ $(n = 1, 2, \ldots)$ has measure zero, since $f_\alpha(x)$ tends to $f(x)$ almost everywhere in K_0 with respect to the filter $\mathfrak{F}$"

Set $B_r = \bigcap_{n=1}^{\infty} B_{n,r}$. Since $(B_{n,r})_{n\geqslant 1}$ is a decreasing sequence of integrable sets, we know that B_r is integrable and $|\mu|(B_{n,r}) \to |\mu|(B_r)$ as $n\to\infty$ (§4, No. 5, Cor. of Prop. 7).

We are to show that $|\mu|(B_r) = 0$; since $K_0 \cap N$ is negligible (No. 2, Prop. 5) and $B_r \subset K_0$, it will suffice to show that $B_r \subset N$.

Let $x \in B_r$ and assume to the contrary that $x \in X - N$. Then, $f_\alpha(x) \to f(x)$ with respect to $\mathfrak{F}$, hence the family $\big(f_\alpha(x)\big)_{\alpha\in A}$ is Cauchy with respect to $\mathfrak{F}$; since the A_n form a base for $\mathfrak{F}$, given any $\varepsilon > 0$ there exists an index n such that

$$(\alpha,\beta) \in A_n \times A_n \quad\Rightarrow\quad d\big(f_\alpha(x), f_\beta(x)\big) < \varepsilon\,.$$

In particular, let $\varepsilon = 1/r$ and let n be such an index, so that

$$d\big(f_\alpha(x), f_\beta(x)\big) < 1/r \quad \text{for all } (\alpha,\beta) \in A_n \times A_n\,;$$

the existence of such an n means that x does not belong to $B_{n,r}$, contrary to $x \in B_r \subset B_{n,r}$.

Thus $\lim_{n\to\infty} |\mu|(B_{n,r}) = |\mu|(B_r) = 0$.

IV.65, ℓ. 8.

"... thus $\lim_{n\to\infty} |\mu|(B_{n,r}) = 0$"

See the preceding note.

IV.65, ℓ. 13–16.

"Let C be the complement of B in K_0; by construction, $f_\alpha(x)$ converges *uniformly* to $f(x)$ in C with respect to the filter $\mathfrak{F}$, and since the restrictions of the f_α to C are continuous, so is the restriction of f to C."

Of course $B \subset K_0$. Recall that $K_0 \cap N$ is negligible; for reasons that will be clear in the following argument, it is convenient to take C to be the complement of $B \cup (K_0 \cap N)$ in K_0. Writing $N_0 = K_0 \cap N$, an elementary calculation yields

$$C = K_0 - (B \cup N_0) = (K_0 - B) \cap (X - N)\,.$$

Note that C is integrable; for, B is integrable and N_0 is negligible (hence integrable), whence the integrability of $B \cup N_0$ and then of $K_0 - (B \cup N_0) = C$ (§4, No. 5, Props. 6 and 7). Moreover, $K_0 - C = B \cup N_0$, whence $|\mu|(K_0 - C) \leqslant |\mu|(B) + |\mu|(N_0) = |\mu|(B) \leqslant \varepsilon/4$.

Let $x \in C$. Since $x \in X - N$, we know that $f_\alpha(x) \to f(x)$ with respect to $\mathfrak{F}$. Also,

$$\begin{aligned} x \in K_0 - B = K_0 \cap \complement B &= K_0 \cap \complement\Big(\bigcup_{r=1}^{\infty} B_{n_r,r}\Big) \\ &= K_0 \cap \bigcap_{r=1}^{\infty} \complement B_{n_r,r} = \bigcap_{r=1}^{\infty} (K_0 - B_{n_r,r})\,. \end{aligned}$$

Thus, for all $x \in C$ and all r, we have $x \in K_0 - B_{n_r,r}$, therefore (by the definition of $B_{n_r,r}$)

$$d\big(f_\alpha(x), f_\beta(x)\big) < 1/r \quad \text{for all } \alpha, \beta \in A_{n_r,r}\,.$$

In other words: given any integer $r \geqslant 1$, for every $(\alpha, \beta) \in A_{n_r,r} \times A_{n_r,r}$ we have

$$d\big(f_\alpha(x), f_\beta(x)\big) < 1/r \quad \text{for all } x \in C\,,$$

whence $\sup_{x \in C} d\big(f_\alpha(x), f_\beta(x)\big) \leqslant 1/r$; this shows that the family of restrictions $f_\alpha|C$ is uniformly Cauchy with respect to $\mathfrak{F}$, and since $f_\alpha \to f$ pointwise in C it follows that $f_\alpha|C \to f|C$ uniformly with respect to $\mathfrak{F}$ (GT, X, §1, No. 5, Prop. 5). And since $C \subset K_0$ and the $f_\alpha|K_0$ are continuous, we conclude that $f|C$ is continuous.

Since C is integrable, there exists a compact set $K_1 \subset C$ such that $|\mu|(C - K_1) \leqslant \varepsilon/4$ (§4, No. 6, Cor. 1 of Th. 4), and since $f\big|C$ is continuous, so is $f\big|K_1$. Now, $K_1 \subset C \subset K_0 \subset K$, so it remains only to check that $|\mu|(K - K_1) \leqslant \varepsilon$. Indeed,

$$K - K_1 = (K - K_0) \cup (K_0 - C) \cup (C - K_1),$$

therefore

$$|\mu|(K - K_1) = |\mu|(K - K_0) + |\mu|(K_0 - C) + |\mu|(C - K_1) \leqslant \varepsilon/2 + \varepsilon/4 + \varepsilon/4.$$

IV.65, ℓ. −15, −14.

" f is the limit of the sequence (f_{α_n}) locally almost everywhere, hence is measurable"

We can suppose that the sequence $(A_n)_{n \in \mathbf{N}}$ is decreasing; then

$$n \geqslant m \quad \Rightarrow \quad \alpha_n \in A_n \subset A_m.$$

Let $x \in X - N$. By hypothesis, $f_\alpha(x) \to f(x)$ with respect to $\mathfrak{F}$; thus, given any $\varepsilon > 0$, there exists an index m such that

$$\alpha \in A_m \quad \Rightarrow \quad d\big(f_\alpha(x), f(x)\big) \leqslant \varepsilon$$

(GT, I, §7, No. 3, Prop. 7), consequently

$$n \geqslant m \quad \Rightarrow \quad \alpha_n \in A_m \quad \Rightarrow \quad d\big(f_{\alpha_n}(x), f(x)\big) \leqslant \varepsilon.$$

This shows that the sequence $\big(f_{\alpha_n}(x)\big)_{n \in \mathbf{N}}$ converges in F to $f(x)$ for every $x \in X - N$. Switching to $\mathbf{N}$ as index set and to the Fréchet filter on $\mathbf{N}$ (GT, I, §6, No. 1, *Example* 3), Th. 2 becomes applicable: $f_{\alpha_n}\big|X - N \to f\big|X - N$ pointwise with respect to the Fréchet filter (that is, for each $x \in X - N$ the elementary filter associated with the sequence $\big(f_{\alpha_n}(x)\big)_{n \in \mathbf{N}}$ converges in F to $f(x)$), therefore f extended in any manner to all of X is measurable by 1° of Th. 2. In this context, 2° of Th. 2 holds, but restricted to the sequence of functions f_{α_n} (the classical form of Egoroff's theorem).

As for condition 2° as stated in the text, recall that its proof entails the countability of the sets A_n (see (∗) in the note for ℓ. 4–5).

It is perhaps useful to state the classical sequential form explicitly:

Theorem 2′. — Let X be a locally compact space, μ a measure on X, and (f_n) a sequence of measurable mappings of X into a metrizable space F.

Assume that there exists a locally negligible subset N of X such that $\lim_{n\to\infty} f_n(x)$ exists in F for every $x \in \mathrm{X} - \mathrm{N}$. Then:

1° If $f : \mathrm{X} \to \mathrm{F}$ is any mapping such that

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in \mathrm{X} - \mathrm{N},$$

then f is measurable.

2° For every compact subset K of X and every $\varepsilon > 0$, there exists a compact set $\mathrm{K}_1 \subset \mathrm{K}$ such that $|\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant \varepsilon$ and such that the restrictions of the f_n to K_1 are continuous and converge uniformly to f on K_1.

In the language of Th. 2 and its proof, $\mathrm{A} = \mathbf{N}$ and $\mathfrak{F}$ is the filter on $\mathbf{N}$ generated by the sets $\mathrm{A}_n = \{k \in \mathbf{N} : k \geqslant n\}$. Another way of packaging the result:

Theorem 2″. — Let X be a locally compact space, μ a measure on X, and (f_n) a sequence of measurable mappings of X into a metrizable space F.

If $f : \mathrm{X} \to \mathrm{F}$ is a mapping such that $f_n(x) \to f(x)$ locally almost everywhere in X (i.e., for x in the complement of a locally negligible set), then:

1° f is measurable.

2° Same as in Th. 2′ above.

IV.66, ℓ. −15.

"… f is measurable."

Recall that $\mathrm{A}_i \cap \mathrm{K}$ is integrable (No. 1, Prop. 3), whence (§4, No. 6, Cor. 2 of Th. 4) the existence of the partitions

$$\mathrm{A}_i \cap \mathrm{K} = \mathrm{N}_i \cup \bigcup_{n=1}^{\infty} \mathrm{K}_{in} \quad (i = 1, \dots, m).$$

The $\mathrm{A}_i \cap \mathrm{K}$ are pairwise disjoint, with union K (because $\bigcup_{i=1}^{m} \mathrm{A}_i = \mathrm{X}$), therefore

$$\mathrm{K} = \bigcup_{i=1}^{m} \mathrm{A}_i \cap \mathrm{K} = \Big(\bigcup_{i=1}^{m} \mathrm{N}_i\Big) \cup \bigcup_{i=1}^{m} \bigcup_{n=1}^{\infty} \mathrm{K}_{in},$$

where the first union is negligible and the K_{in} are pairwise disjoint compact sets with $f = a_i$ on K_{in}, thus f is measurable by No. 1, Def. 1.

IV.66, ℓ. −15, −14.

"By an abuse of language ..."

Heretofore, the term "step function" (*fonction étagée*) was used only for functions with values in a Banach space (§4, No. 9, Def. 4), whose additive structure admits representing step functions by means of characteristic functions and confers algebraic structure on the set of step functions.

There is another sense in which the language is abused: if $f : \mathrm{X} \to \mathrm{F}$ is a step function (with respect to some clan) and f is a measurable function (with respect to the given measure on X) then f need not be a 'measurable step function' in the sense of the text. For example, if $\mathrm{F} = \{a, b\}$ is the 2-point space whose only open sets are F and ∅, then every function $\mathrm{X} \to \mathrm{F}$ is continuous, hence measurable; but if X has a non-measurable subset A, then the function $f : \mathrm{X} \to \mathrm{F}$ that assigns the value a to the points of A and the value b to the points of $\mathrm{X} - \mathrm{A}$ is a step function with respect to the clan $\{\varnothing, \mathrm{A}, \mathrm{X} - \mathrm{A}, \mathrm{X}\}$ but is not a measurable step function.

However, if F is a T_1-space (all 1-point subsets closed) then all is well: every measurable function $f : \mathrm{X} \to \mathrm{F}$ with finite range $f(\mathrm{X})$ is a measurable step function, because $\overset{-1}{f}(a)$ is measurable for all $a \in \mathrm{F}$ by Prop. 7 below.

The term used for a linear combination of characteristic functions of intervals is *fonction en escalier* (FVR, II, §1, No. 3, Def. 2); having translated it as "step function" for the footnote on p. ix of Vol. I and in the *Example* in §4, No. 4 (p. **IV.40**), when I arrived at *fonction étagée*, I did not notice the difference between the French terms and translated it as "step function" too. There's a *real* abuse of language, somewhat mitigated by the fact that "fonction en escalier" is a special case of "fonction étagée"; alas, I noticed *my* abuse of language only when exploring, for these notes, why the author thought he had abused it.

For the record, I can find just 5 places in *Integration* where "step function" is to be interpreted in the FRV sense:

page xi of Vol. 1: footnote.

IV.40: Ch. IV, §4, No. 4, *Example* (once in italics, once in roman).

IV.133: Ch. IV, §5, the hint for Exer. 29 *d*). (Regrettably, in part *c*) of the same exercise, "step function" is to be interpreted as *fonction étagée*; part of the exercise is to guess the clan, the tribe of measurable functions being a promising suspect.)

IV.134: Ch. IV, §5, the hint for Exer. 30 (twice).

IV.139: Ch. IV, §6, Exer. 17.

The following comments should help prevent misunderstandings:

a) a "step function" in the sense of FRV is defined on an interval I of **R** and is a step function with respect to the clan Φ generated by the set of subintervals of I;

b) for a step function whose domain is not an interval of $\mathbf{R}$, the clan Φ in question must be either signaled explicitly or be inferable from the context;

c) ambiguity can occur only when the domain of the function is an interval of $\mathbf{R}$; if such a function does not fall under case *a*), the clan Φ in question should be explicitly stated or be inferable from the context.

If not "step function", what were the alternatives for "fonction étagée"? Harrap's Unabridged gives "tiered" or "terraced" as possibilities for "étagée". The frequent translation of "ét..." as "st..." suggests "staged". In the terminology of P.R. Halmos' *Measure theory*, a "fonction Φ-étagée" (Φ a clan) is a "simple function" with respect to a ring Φ of subsets of a set (*op. cit.*, p. 84), whereas T.H. Hildebrandt's *Introduction to the theory of integration* employs "finite-step function", with "step function" reserved for functions with a countable number of values (*op. cit.*, p. 208).

After consulting twenty or so textbooks on Integration and Real Analysis, I can report the following usage:

(1) "step function" (or "step map"), for "fonction en escalier": Asplund & Bungart, Burrill & Knudsen, Dieudonné, Hewitt & Stromberg, K. Hoffman, Lang, McShane & Botts, K.A. Ross, Royden, Stromberg, Zaanen. The overwhelming choice. {Not all books treated the topic, and Hildebrandt employs the interesting term "staircase function", with "interval step function" as possible alternative (*op. cit.*, p. 218).}

(2) "step function" (or "step map") with respect to a class of sets, for "fonction étagée": Dinculeanu, L.M. Graves, Lang, McShane, Segal & Kunze, Zaanen.

(2′) "simple function" for "fonction étagée": Berberian, Burrill & Knudsen, Halmos, Hewitt & Stromberg, Munroe, McShane & Botts, Royden, Rudin, Saks, Stromberg. Somewhat in the lead, but it translates "étagée" poorly.

Two possible solutions seem to me to stand out for the usage in *Integration*:

(A) "step function" for "étagée", and "interval step function" for "en escalier";

(B) "simple function" for "étagée" and "step function" for "en escalier".

Were *Integration* to receive a second edition, (A) would be easy to implement—insert 'interval' at the 5 places noted above (twice on p. **IV.40** and on p. **IV.134**). This would highlight, not contradict, the usage in FRV, which clearly has first claim on "step function".

IV.66, ℓ. −7, −6.

"The condition is sufficient by Egoroff's theorem and the principle of localization."

Assuming the condition, we are to show that f is measurable. Let $x \in \mathrm{X}$ and let K be a compact (hence integrable) neighborhood of x. In order to invoke the principle of localization (No. 2, Prop. 4), we seek a measurable function g on X such that $f = g$ almost everywhere in K.

By assumption, there exists a sequence of measurable step functions $g_n : \mathrm{X} \to \mathrm{F}$ such that $g_n \to f$ almost everywhere in K. Let N be a negligible subset of K such that $g_n \to f$ on the set $\mathrm{A} = \mathrm{K} - \mathrm{N}$, which is measurable (indeed, integrable; see the comment at the end of the note for IV.60, ℓ. −5).

We will define a sequence of measurable step functions $h_n : \mathrm{X} \to \mathrm{F}$ that converges to the desired function g. Fix a point $a \in \mathrm{F}$. For each n, define h_n by

$$h_n(y) = \begin{cases} g_n(y) & \text{if } y \in \mathrm{A} \\ a & \text{if } y \in \mathrm{X} - \mathrm{A}\,. \end{cases}$$

Since $g_n(\mathrm{X})$ is finite, so is $h_n(\mathrm{X})$. To show that h_n is a measurable step function, we need only show that if b is any value of h_n, then $\overset{-1}{h}_n(b)$ is a measurable set; indeed, if $b \neq a$ then

$$\overset{-1}{h}_n(b) = \{y \in \mathrm{A} : \ g_n(y) = b\} = \mathrm{A} \cap \overset{-1}{g}_n(b)\,,$$

whereas if $b = a$ then

$$\overset{-1}{h}_n(b) = \overset{-1}{h}_n(a) = \{y \in \mathrm{A} : \ g_n(y) = b\} \cup (\mathrm{X} - \mathrm{A}) = \Big(\mathrm{A} \cap \overset{-1}{g}_n(a)\Big) \cup (\mathrm{X} - \mathrm{A})\,.$$

If $y \in \mathrm{A}$ then $h_n(y) = g_n(y) \to f(y)$, and if $y \in \mathrm{X} - \mathrm{A}$ then $h_n(y) = a$ for all n; defining

$$g(y) = \begin{cases} f(y) & \text{if } y \in \mathrm{A} = \mathrm{K} - \mathrm{N} \\ a & \text{if } y \in \mathrm{X} - \mathrm{A}\,, \end{cases}$$

we have $h_n \to g$ pointwise in X; since the h_n are measurable, g is measurable by Egoroff's theorem (No. 4, Th. 2) and the assertion is proved.

IV.66, ℓ. −2 to **IV.67,** ℓ. 2.

"... for each K_i with index $i \leqslant n$, there exists a finite partition of K_i into integrable sets A_{ij} $(1 \leqslant j \leqslant q_i)$ sufficiently small that the oscillation of f on each of the A_{ij} is $\leqslant 1/n$ (§4, No. 9, Lemma)"

To accommodate special cases, it is convenient to rearrange slightly the notations. We have a partition

$$\mathrm{K} = \mathrm{N} \cup \bigcup_i \mathrm{K}_i$$

with N negligible and (K_i) a countable family of pairwise disjoint compact sets such that the restrictions $f|\mathrm{K}_i$ are all continuous. The problem is to construct a sequence (g_n) of measurable step functions $g_n : \mathrm{X} \to \mathrm{F}$ such that, for every i, $g_n \to f$ almost everywhere in K_i. For technical purposes, fix a point $a \in \mathrm{F}$.

(i) If K is negligible then every sequence (g_n) of measurable step functions meets the requirements: K_i is a negligible subset of K_i, and $g_n \to f$ vacuously on $\mathrm{K}_i - \mathrm{K}_i = \varnothing$. (For example, let every g_n be the constant mapping $g_n(x) = a$ for all $x \in \mathrm{X}$.)

Assuming K is not negligible, we can assume henceforth that every negligible K_i has been assimilated in N, so that every remaining K_i is non-negligible, hence nonempty. We are going to construct a sequence (g_n) of measurable step functions such that $g_n \to f$ at every point of $\mathrm{K} - \mathrm{N}$ (hence almost everywhere in K).

(ii) Suppose there are only finitely many K_i, say $\mathrm{K}_1, \ldots, \mathrm{K}_r$. Then $\mathrm{K} = \mathrm{N} \cup \mathrm{K}'$, where $\mathrm{K}' = \bigcup_{i=1}^{r} \mathrm{K}_i$ is compact and $f|\mathrm{K}'$ is continuous (GT, I, §3, No. 2, Prop. 4), thus we are reduced to the case $r = 1$.

Suppose then that $\mathrm{K} = \mathrm{N} \cup \mathrm{K}'$ is a partition of K with N negligible and K' a nonempty compact set such that $f|\mathrm{K}'$ is continuous.

Given any integer $n \geqslant 1$ let us construct a measurable step function g_n as follows. If $x \in \mathrm{K}'$ then, by the continuity of f at x in K', there exists a compact neighborhood V_x of x in X such that

$$d\big(f(y), f(x)\big) \leqslant 1/2n \quad \text{for all } y \in \mathrm{V}_x \cap \mathrm{K}' ;$$

since K' is compact, $\mathrm{K}' \subset \mathrm{V}_{x_1} \cup \cdots \cup \mathrm{V}_{x_r}$ for suitable $x_1, \ldots, x_r$ in K'. Then

$$\mathrm{K}' = \bigcup_{k=1}^{r} \mathrm{V}_{x_k} \cap \mathrm{K}' ,$$

where each of the sets $\mathrm{V}_{x_k} \cap \mathrm{K}'$ $(1 \leqslant k \leqslant r)$ is nonempty (it contains at least x_k) and compact (hence integrable) and on which the oscillation of f is $\leqslant 1/n$; for, if $y, z \in \mathrm{V}_{x_k} \cap \mathrm{K}'$ then

$$d\big(f(y), f(z)\big) \leqslant d\big(f(y), f(x_k)\big) + d\big(f(x_k), f(z)\big) \leqslant 1/2n + 1/2n .$$

By the Lemma of §4, No. 9 there exists in the clan generated by the $\mathrm{V}_{x_k} \cap \mathrm{K}'$ (hence in the clan of integrable sets and in the tribe of measurable sets) a finite family $\mathscr{A}$ of pairwise disjoint sets such that each $\mathrm{V}_{x_k} \cap \mathrm{K}'$ may be expressed as the union of a subfamily of $\mathscr{A}$. Let $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_q$ be a listing (without repetitions) of those nonempty members of $\mathscr{A}$ that occur in the expressions of $\mathrm{V}_{x_1} \cap \mathrm{K}', \ldots, \mathrm{V}_{x_r} \cap \mathrm{K}'$; then

$$\bigcup_{j=1}^{q} \mathrm{A}_j = \bigcup_{k=1}^{r} \mathrm{V}_{x_k} \cap \mathrm{K}' = \mathrm{K}',$$

where the A_j are nonempty measurable (and integrable) sets on each of which f has oscillation $\leqslant 1/n$. For $j = 1, \ldots, q$ choose $a_j \in f(\mathrm{A}_j)$ and define a function $g_n : \mathrm{X} \to \mathrm{F}$ with finite range by

$$g_n(y) = \begin{cases} a_j & \text{if } y \in \mathrm{A}_j \text{ for some } j \\ a & \text{if } y \in \mathrm{X} - \mathrm{K}'; \end{cases}$$

g_n is a measurable step function since (keeping in mind that, although the A_j are pairwise disjoint, the a_j need not be distinct, and any or all of them may be equal to a)

$$\overset{-1}{g_n}(a) = (\mathrm{X} - \mathrm{K}') \cup \bigcup_{a_k = a} \mathrm{A}_k$$

(that is, the union of $\mathrm{X} - \mathrm{K}'$ and those A_k for which $a_k = a$), and $\overset{-1}{g_n}(a_j)$ is equal to

$$\bigcup_{a_k = a_j} \mathrm{A}_k \quad \text{or} \quad (\mathrm{X} - \mathrm{K}') \cup \bigcup_{a_k = a_j} \mathrm{A}_k$$

according as $a_j \neq a$ or $a_j = a$, respectively.

Given any $y \in \mathrm{K}'$, say $y \in \mathrm{A}_j$, then

$$g_n(y) = a_j \in f(\mathrm{A}_j) \quad \text{and} \quad f(y) \in f(\mathrm{A}_j);$$

since f has oscillation $\leqslant 1/n$ on A_j, we have $d\big(g_n(y), f(y)\big) \leqslant 1/n$.

To summarize: for each integer $n \geqslant 1$ we have constructed a measurable step function $g_n : \mathrm{X} \to \mathrm{F}$ such that

$$d\big(g_n(y), f(y)\big) \leqslant 1/n \quad \text{for all} \quad y \in \mathrm{K}';$$

fixing $y \in \mathrm{K}'$ and letting $n \to \infty$, we have $g_n(y) \to f(y)$, thus $g_n \to f$ pointwise on $\mathrm{K}' = \mathrm{K} - \mathrm{N}$, whence $g_n \to f$ almost everywhere in K.

(iii) Finally, suppose $\mathrm{K} = \mathrm{N} \cup \bigcup_{i=1}^{\infty} \mathrm{K}_i$ is a partition of K with N negligible and (K_i) an infinite sequence of non-negligible compact sets such that every $f|\mathrm{K}_i$ is continuous.

For every $n \geqslant 1$ let $\mathrm{J}_n = \bigcup_{i=1}^{n} \mathrm{K}_i$, so that (J_n) is an increasing sequence of compact sets such that $\bigcup_{n=1}^{\infty} \mathrm{J}_n = \mathrm{K} - \mathrm{N}$ and $f|\mathrm{J}_n$ is continuous for every n.

For each n, the technique of the preceding case shows that there exists a measurable step function g_n such that

$$(*) \qquad d\big(g_n(y), f(y)\big) \leqslant 1/n \quad \text{for all} \quad y \in \mathrm{J}_n\,,$$

and, say, $g_n(y) = a$ for all $y \in \bigcup_{i=n+1}^{\infty} \mathrm{K}_i = (\mathrm{K} - \mathrm{N}) - \mathrm{J}_n$. Given any $y \in \mathrm{K} - \mathrm{N} = \bigcup_{n=1}^{\infty} \mathrm{J}_n$, we assert that $g_n(y) \to f(y)$. Say $y \in \mathrm{J}_{n_0}$. Given any $\varepsilon > 0$, choose an integer n_1 such that $1/n_1 < \varepsilon$; increasing n_1 if necessary, we can suppose that $n_1 > n_0$. Then, for all $n \geqslant n_1$ we have

$$y \in \mathrm{J}_{n_0} \subset \mathrm{J}_{n_1} \subset \mathrm{J}_n\,,$$

therefore by $(*)$, $d\big(g_n(y), f(y)\big) \leqslant 1/n \leqslant 1/n_1 < \varepsilon$. Thus $g_n \to f$ pointwise on $\mathrm{K} - \mathrm{N}$, hence almost everywhere in K.

IV.67, ℓ. 5, 6.

"It is clear that the sequence $\big(g_n(x)\big)$ converges to $f(x)$ at every point of K not belonging to N."

See the preceding note.

IV.67, ℓ. 12.

"With notations as in the proof of Th. 3 ..."

The notations in the note before the last are readily adapted to the present corollary, following the analysis into cases (i)–(iii).

(i) If K is negligible, define $\mathbf{g}_n = 0$ for all n.

(ii) If $\mathrm{K} = \mathrm{N} \cup \mathrm{K}'$ is a partition of K with N negligible, K′ compact and $\mathbf{f}|\mathrm{K}'$ continuous, given any integer $n \geqslant 1$ construct a function $\mathbf{g}_n : \mathrm{X} \to \mathrm{F}$ as follows. As in the previous note, let $\mathrm{K}' = \mathrm{A}_1 \cup \cdots \cup \mathrm{A}_q$ be a partition of K′ such that the A_j are nonempty measurable sets on each of which $\mathbf{f}$ has oscillation $\leqslant 1/n$. For each $j = 1, \ldots, q$ choose a point $x_j \in \mathrm{A}_j$ and let $\mathbf{a}_j = \mathbf{f}(x_j)$. If j is an index for which $|\mathbf{a}_j| > 1/n$, that is, $n|\mathbf{a}_j| - 1 > 0$, define

$$\mathbf{a}'_j = \big(1 - \frac{1}{n|\mathbf{a}_j|}\big)\mathbf{a}_j \in \mathrm{F}\,,$$

and note that $|\mathbf{a}'_j| = |\mathbf{a}_j| - 1/n$, $\mathbf{a}'_j - \mathbf{a}_j = -\dfrac{1}{n|\mathbf{a}_j|}\cdot\mathbf{a}_j$ and $|\mathbf{a}'_j - \mathbf{a}_j| = 1/n$.
Define

$$\mathbf{g}_n(x) = \begin{cases} 0 & \text{if } x \in \mathrm{X} - \mathrm{K}' = (\mathrm{X} - \mathrm{K}) \cup \mathrm{N} \\ 0 & \text{if } x \in \mathrm{A}_j \text{ and } |\mathbf{a}_j| \leqslant 1/n \\ \mathbf{a}'_j & \text{if } x \in \mathrm{A}_j \text{ and } |\mathbf{a}_j| > 1/n\,. \end{cases}$$

Each of the finitely many values of $\mathbf{g}_n$ is assumed on a measurable subset of X:

$$\overset{-1}{\mathbf{g}}_n(0) = (\mathrm{X} - \mathrm{K}') \cup \bigcup_{|\mathbf{a}_j| \leqslant 1/n} \mathrm{A}_j\,,$$

and if $|\mathbf{a}_j| > 1/n$ then

$$\overset{-1}{\mathbf{g}}_n(\mathbf{a}'_j) = \bigcup_{\mathbf{a}_k = \mathbf{a}_j} \mathrm{A}_k\,,$$

thus $\mathbf{g}_n$ is a measurable step function. We assert that

$$(*) \qquad |\mathbf{g}_n(x) - \mathbf{f}(x)| \leqslant 2/n \quad \text{for all } x \in \mathrm{K}'\,;$$

for, if $x \in \mathrm{A}_j$ and $|\mathbf{a}_j| \leqslant 1/n$ then $\mathbf{g}_n(x) = 0$ and

$$|\mathbf{g}_n(x) - \mathbf{f}(x)| = |\mathbf{f}(x)| \leqslant |\mathbf{f}(x) - \mathbf{f}(x_j)| + |\mathbf{f}(x_j)| \leqslant 1/n + |\mathbf{a}_j| \leqslant 1/n + 1/n\,,$$

whereas if $|\mathbf{a}_j| > 1/n$ then

$$\begin{aligned} |\mathbf{g}_n(x) - \mathbf{f}(x)| = |\mathbf{a}'_j - \mathbf{f}(x)| &\leqslant |\mathbf{a}'_j - \mathbf{a}_j| + |\mathbf{a}_j - \mathbf{f}(x)| \\ &= 1/n + |\mathbf{f}(x_j) - \mathbf{f}(x)| \leqslant 1/n + 1/n\,. \end{aligned}$$

Having constructed such a $\mathbf{g}_n$ for every n, it is clear from $(*)$ that $\mathbf{g}_n(x) \to \mathbf{f}(x)$ for every $x \in \mathrm{K}'$.

Fix an index $n \geqslant 1$ and assume the foregoing notations for $\mathbf{g}_n$; let us show that

$$(**) \qquad |\mathbf{g}_n(x)| \leqslant |\mathbf{f}(x)| \quad \text{for all } x \in \mathrm{X}\,.$$

If $\mathbf{g}_n(x) = 0$ this is trivial, so we can suppose that $x \in \mathrm{A}_j$ for some index j with $|\mathbf{a}_j| > 1/n$. Then

$$|\mathbf{g}_n(x)| = |\mathbf{a}'_j| = |\mathbf{a}_j| - 1/n = |\mathbf{f}(x_j)| - 1/n\,;$$

but

$$|\mathbf{f}(x_j)| - |\mathbf{f}(x)| \leqslant |\mathbf{f}(x_j) - \mathbf{f}(x)| \leqslant 1/n\,,$$

thus $|\mathbf{f}(x_j)| \leqslant |\mathbf{f}(x)| + 1/n$, therefore

$$|\mathbf{g}_n(x)| = |\mathbf{f}(x_j)| - 1/n \leqslant (|\mathbf{f}(x)| + 1/n) - 1/n = |\mathbf{f}(x)|.$$

(iii) Finally, suppose $\mathrm{K} = \mathrm{N} \cup \bigcup_{i=1}^{\infty} \mathrm{K}_i$ is a partition of K, with N negligible and (K_i) an infinite sequence of non-negligible compact sets such that $\mathbf{f}\big|\mathrm{K}_i$ is continuous for all i. For every $n \geqslant 1$ let $\mathrm{J}_n = \bigcup_{i=1}^{n} \mathrm{K}_i$, so that (J_n) is an increasing sequence of compact sets such that $\bigcup_{n=1}^{\infty} \mathrm{J}_n = \mathrm{K} - \mathrm{N}$ and $\mathbf{f}\big|\mathrm{J}_n$ is continuous for every n.

For each n, the technique of the preceding case (with K' replaced by J_n) shows that there is a measurable step function $\mathbf{g}_n : \mathrm{X} \to \mathrm{F}$ such that

$$(*) \qquad |\mathbf{g}_n(x) - \mathbf{f}(x)| \leqslant 2/n \quad \text{for all } x \in \mathrm{J}_n,$$

$\mathbf{g}_n = 0$ on $(\mathrm{X} - \mathrm{K}) \cup \mathrm{N} \cup \bigcup_{i=n+1}^{\infty} \mathrm{K}_i = (\mathrm{X} - \mathrm{K}) \cup (\mathrm{K} - \mathrm{J}_n) = \mathrm{X} - \mathrm{J}_n$, and $|\mathbf{g}_n(x)| \leqslant |\mathbf{f}(x)|$ for all $x \in \mathrm{X}$. It follows from (∗), as in the previous note, that $\mathbf{g}_n(x) \to \mathbf{f}(x)$ for every x in $\bigcup_{n=1}^{\infty} \mathrm{J}_n = \mathrm{K} - \mathrm{N}$.

IV.67, ℓ. $-13, -12$.

"... the proof can then be concluded as in Th. 3 without modification."

Except that, writing $\mathrm{J}_n = \bigcup_{i=1}^{n} \mathrm{K}_i$ as in the note for **IV.66,** ℓ. -2ff, we now have $\bigcup_{n=1}^{\infty} \mathrm{J}_n = \mathrm{X} - \mathrm{N}$ (i.e., "K" is out of the picture). When F is a Banach space, one can arrange, as in Cor. 1, that $|g_n(x)| \leqslant |f(x)|$ for all $x \in \mathrm{X}$ (see the preceding note).

IV.67, ℓ. -1.

"... which proves that $\overset{-1}{f}(\mathrm{A})$ is measurable."

With $\mathrm{B} = \overset{-1}{f}(\mathrm{A})$ in mind, the proof depends on the following characterization of measurability:

Lemma. For a subset B of X, the following conditions are equivalent:
(a) B is measurable;
(b) for every compact subset K of X, there exists a partition

$$\mathrm{K} \cap \mathrm{B} = \mathrm{N}' \cup \bigcup_n \mathrm{K}'_n$$

of $K \cap B$ with N' negligible and (K'_n) a sequence of (pairwise disjoint) compact sets.

Proof. (a) ⇒ (b): Assuming B measurable and $K \subset X$ compact, by the remarks following No. 1, Def. 2 there exists a partition

$$K = N \cup \bigcup_n K_n$$

with N negligible and (K_n) a sequence of (pairwise disjoint) compact sets such that, for each n, either $K_n \subset B$ or $K_n \subset \complement B$. Writing $N' = N \cap B$ and $K'_n = K_n \cap B$ we have a partition

$$K \cap B = N' \cup \bigcup_n K'_n$$

with N' negligible and K'_n ($= K_n$ or $\varnothing$) compact.

(b) ⇒ (a): To show that B is measurable, it will suffice to show that, given any compact set $K \subset X$, the set $K \cap B$ is integrable (No. 1, Prop. 3). By assumption, we have a partition

$$K \cap B = N' \cup \bigcup_n K'_n$$

with N' negligible (hence integrable) and the K'_n compact (hence integrable). For every n we have (§4, No. 5, Prop. 6)

$$\sum_{i=1}^{n} |\mu|(K'_i) = |\mu|\Big(\bigcup_{i=1}^{n} K'_i\Big) \leqslant |\mu|(K) < +\infty\,,$$

therefore $\bigcup_n K'_n$ is integrable (*loc. cit.*, Cor. of Prop. 8), hence so is $K \cap B$.

IV.68, ℓ. 8–10.

"... in the notations of Th. 3, condition b) is satisfied by taking H to be the countable set formed by the values of all of the functions g_n."

A proof can be based directly on the definition of 'measurable function'. Given a compact set $K \subset X$, we seek a negligible set $N \subset K$ and a countable set $H \subset F$ such that $f(K - N) \subset \overline{H}$. Since f is measurable, there exist a negligible set $N \subset K$ and a sequence (K_n) of compact sets such that $K - N = \bigcup_n K_n$ and $f|K_n$ is continuous for every n (No. 1, Def. 1; that the K_n can be taken to be pairwise disjoint is not needed here). For each n, $f(K_n)$ is a compact metrizable subspace of F, hence has a countable base (GT, IX, §2, No. 9, Prop. 16) and therefore a countable subset H_n such that

$\overline{\mathrm{H}}_n = f(\mathrm{K}_n)$ (*loc. cit.*, No. 8, Prop. 12; since $f(\mathrm{K}_n)$ is a closed subset of F, closure in $f(\mathrm{K}_n)$ means the same as closure in F). Then $\mathrm{H} = \bigcup_n \mathrm{H}_n$ is a countable subset of F such that (with closure in the sense of F)

$$\overline{\mathrm{H}} \supset \bigcup_n \overline{\mathrm{H}}_n = \bigcup_n f(\mathrm{K}_n) = f\Big(\bigcup_n \mathrm{K}_n\Big) = f(\mathrm{K} - \mathrm{N})\,.$$

One can arrange that $\mathrm{H} \subset f(\mathrm{K} - \mathrm{N})$. For, $f(\mathrm{K} - \mathrm{N}) \subset \overline{\mathrm{H}}$, where $\overline{\mathrm{H}}$ is a metrizable subspace of F containing a countable dense subset, therefore $\overline{\mathrm{H}}$ has a countable base (GT, IX, §2, No. 8, Prop. 12); then the subspace $f(\mathrm{K} - \mathrm{N})$ of F also has a countable base (transitivity of induced topologies), hence has a countable dense subset H′ (*loc. cit.*, No. 8, Prop. 12 again). Thus $\mathrm{H}' \subset f(\mathrm{K} - \mathrm{N}) \subset \overline{\mathrm{H}'}$.

A tiny supplement to Th. 4: If $f : \mathrm{X} \to \mathrm{F}$ is a measurable function, where F is a Souslin space in the sense of TG, IX, §6, No. 2, Def. 2,(*) one can again prove that b) holds, and one can arrange that $\mathrm{H} \subset f(\mathrm{K} - \mathrm{N})$. For, $f(\mathrm{K}_n)$ is a compact subset of F, hence is closed in F, hence is a Souslin subspace of F (*loc. cit.*, Prop. 5), hence is metrizable (TG, IX, App. 1, Cor. 2 of Prop. 3); thus the earlier arguments are applicable.

The condition b) can be formulated in terms of neighborhoods:

Remark. — If X is a locally compact space equipped with a measure, and F is any topological space, the following conditions on a function $f : \mathrm{X} \to \mathrm{F}$ are equivalent:

b) for every compact set $\mathrm{K} \subset \mathrm{X}$, there exist a negligible set $\mathrm{N} \subset \mathrm{K}$ and a countable set $\mathrm{H} \subset \mathrm{F}$ such that $f(\mathrm{K} - \mathrm{N}) \subset \overline{\mathrm{H}}$;

b′) for every point $x \in \mathrm{X}$, there exist a neighborhood V_x of x, a negligible set $\mathrm{N}_x \subset \mathrm{V}_x$, and a countable set $\mathrm{H}_x \subset \mathrm{F}$ such that $f(\mathrm{V}_x - \mathrm{N}_x) \subset \overline{\mathrm{H}}_x$.

When F is metrizable, one can arrange that $\mathrm{H} \subset f(\mathrm{K} - \mathrm{N})$ (or that $\mathrm{H}_x \subset f(\mathrm{V}_x - \mathrm{N}_x)$).

Proof. b) ⇒ b′): Given any $x \in \mathrm{X}$, let V_x be a compact neighborhood of x and apply b) to $\mathrm{K} = \mathrm{V}_x$.

b′) ⇒ b): Let K be a compact set in X . For each $x \in \mathrm{K}$, choose V_x, N_x and H_x as in b′); say

$$\mathrm{K} \subset \mathrm{V}_{x_1} \cup \cdots \cup \mathrm{V}_{x_n}\,.$$

Writing $\mathrm{N} = \mathrm{K} \cap (\mathrm{N}_{x_1} \cup \cdots \cup \mathrm{N}_{x_n})$ and $\mathrm{H} = \mathrm{H}_{x_1} \cup \cdots \cup \mathrm{H}_{x_n}$, N is a negligible subset of K such that

$$\mathrm{K} - \mathrm{N} \subset (\mathrm{V}_{x_1} - \mathrm{N}_{x_1}) \cup \cdots \cup (\mathrm{V}_{x_n} - \mathrm{N}_{x_n})\,,$$

(*) In TG (resp. GT), a Souslin space is defined to be a Hausdorff space (resp. metrizable space) that is a continuous image of a Polish space.

H is a countable subset of F, and

$$f(\mathrm{K}-\mathrm{N}) \subset f(\mathrm{V}_{x_1}-\mathrm{N}_{x_1}) \cup \cdots \cup f(\mathrm{V}_{x_n}-\mathrm{N}_{x_n}) \subset \overline{\mathrm{H}}_1 \cup \cdots \cup \overline{\mathrm{H}}_n = \overline{\mathrm{H}}.$$

When F is metrizable, by an earlier argument we can arrange that $\mathrm{H} \subset f(\mathrm{K}-\mathrm{N})$.

IV.68, ℓ. 13, 14.

"... which we arrange in a sequence (a_n)."

Before having chosen a compact set K, fix a point $b \in \mathrm{F}$ for use in all choices of K.

Note that for every $a \in \mathrm{X}$, $\overset{-1}{f}(a)$ is a measurable set; for, writing $\mathrm{B}_r(a)$ for the closed ball of radius r and center a, $\overset{-1}{f}\big(\mathrm{B}_r(a)\big)$ is measurable by the assumption a), therefore the set

$$\overset{-1}{f}(a) = \overset{-1}{f}\Big(\bigcap_{p=1}^{\infty} \mathrm{B}_{1/p}(a)\Big) = \bigcap_{p=1}^{\infty} \overset{-1}{f}\big(\mathrm{B}_{1/p}(a)\big)$$

is measurable (No. 4, Cor. 2 of Th. 2).

Given any compact set $\mathrm{K} \subset \mathrm{X}$, it will suffice to show that there exists a measurable function $g_{\mathrm{K}} : \mathrm{X} \to \mathrm{F}$ such that $g_{\mathrm{K}} = f$ almost everywhere in K; it will then follow from the local compactness of X that f is measurable (see the note for ℓ. 23–25). If K is negligible, the constant function $x \mapsto b$ will serve for g_{K}. Assume K is non-negligible.

By b), there exist a negligible set $\mathrm{N} \subset \mathrm{K}$ and a countable set $\mathrm{H} \subset \mathrm{X}$ such that $f(\mathrm{K}-\mathrm{N}) \subset \overline{\mathrm{H}}$. By assumption, $\mathrm{K}-\mathrm{N} \neq \varnothing$. As observed in the preceding note, we can suppose that $\mathrm{H} \subset f(\mathrm{K}-\mathrm{N}) \subset \overline{\mathrm{H}}$.

If H is finite then $\mathrm{H} = \overline{\mathrm{H}} = f(\mathrm{K}-\mathrm{N})$. Write $f(\mathrm{K}-\mathrm{N}) = \{a_1, \ldots, a_n\}$ with the a_i distinct, and let $\mathrm{A}_i = (\mathrm{K}-\mathrm{N}) \cap \overset{-1}{f}(a_i)$, which is a measurable set. Then the A_i form a partition of $\mathrm{K}-\mathrm{N}$ and the function $g_{\mathrm{K}} : \mathrm{X} \to \mathrm{F}$ defined by

$$g_{\mathrm{K}}(x) = \begin{cases} a_i & \text{if } x \in \mathrm{A}_i \text{ for some (unique) } i \in \{1,, \ldots, n\} \\ b & \text{if } x \in (\mathrm{X}-\mathrm{K}) \cup \mathrm{N} \end{cases}$$

is a measurable step function such that $g_{\mathrm{K}} = f$ on $\mathrm{K}-\mathrm{N}$.

Suppose henceforth that H is infinite, and write $\mathrm{H} = \{a_1, a_2, a_3, \ldots\}$ with $a_n \in f(\mathrm{K}-\mathrm{N})$ for all n (distinct, if we like, but it is not necessary). For later use, for each n choose $x_n \in \mathrm{K}-\mathrm{N}$ with $f(x_n) = a_n$.

IV.68, ℓ. 15.

"It follows from condition a) that $A_{n,p}$ is measurable."

We continue with the notations established in the preceding note. Fix an integer $p \geqslant 1$. For every integer $n \geqslant 1$,

$$\begin{aligned} A_{n,p} &= \{x \in K - N : \ d\big(f(x), a_n\big) \leqslant 1/p\} \\ &= \{x \in K - N : \ f(x) \in B_{1/p}(a_n)\} = (K - N) \cap \overset{-1}{f}\big(B_{1/p}(a_n)\big), \end{aligned}$$

which is a measurable set. For later use, we observe that

$$K - N = \bigcup_{n=1}^{\infty} A_{n,p} \, .$$

For, if $x \in K - N$, so that $f(x) \in f(K - N) \subset \overline{H}$, there exists an element of H, say a_n, such that $d\big(f(x), a_n\big) \leqslant 1/p$, whence $x \in A_{n,p}$. Thus $K - N \subset \bigcup_{n=1}^{\infty} A_{n,p}$, and the reverse inclusion is immediate from the definition of the $A_{n,p}$.

Incidentally, with $x_n \in K - N$ chosen in the preceding note so that $f(x_n) = a_n$, one obviously has $x_n \in A_{n,p}$, thus every $A_{n,p}$ is nonempty.

IV.68, ℓ. 18, 19.

"... the sets $B_{n,p}$ are measurable, and those that are nonempty form a partition of $K - N$."

For fixed p, we have $K - N = \bigcup_{k=1}^{\infty} A_{n,p}$ by the preceding note; the $B_{n,p}$ are the result of 'disjointifying' the $A_{n,p}$ in the usual way: $B_{1,p} = A_{1,p}$ and

$$B_{n+1,p} = \bigcup_{k=1}^{n+1} A_{k,p} - \bigcup_{k=1}^{n} A_{k,p} = A_{n+1,p} - \bigcup_{k=1}^{n} A_{k,p} \, .$$

At least $B_{1,p} = A_{1,p}$ is guaranteed to be nonempty: $x_1 \in A_{1,p}$ (see the preceding note).

IV.68, ℓ. 19–21.

"Let $g_{m,p}$ be the function equal to a_i on the set $B_{i,p}$ for $1 \leqslant i \leqslant m$ and equal to a constant $b \in F$ on the complement of the union of these sets"

For every integer $m \geqslant 1$, write $C_{m,p} = \bigcup_{i=1}^{m} B_{i,p}$; some of the $B_{i,p}$ may be empty, but at least $B_{1,p} = A_{1,p} \neq \varnothing$ (see the preceding notes), therefore

$C_{m,p}$ is nonempty hence eligible to be part of the domain of a function. One has

$$K - N = \bigcup_{i=1}^{\infty} B_{i,p} = C_{m,p} \cup \bigcup_{i=m+1}^{\infty} B_{i,p} .$$

Define $g_{m,p} : X \to F$ by

$$g_{m,p}(x) = \begin{cases} b & \text{if } x \in X - C_{m,p} = (X - K) \cup N \cup \bigcup_{i=m+1}^{\infty} B_{i,p} \\ a_i & \text{if } x \in B_{i,p} \text{ for some (unique) } i \in \{1, , \ldots , m\} . \end{cases}$$

IV.68, ℓ. 21–23.

"... as m tends to infinity, $g_{m,p}$ converges pointwise to the function f_p equal to a_n on $B_{n,p}$ $(n \geqslant 1)$ and to b on $N \cup \complement K$ "

Continuing the notations of the preceding note, the sets $C_{m,p}$ $(m = 1, 2, 3, \ldots)$ form an increasing sequence with union $K - N$.

As $B_{n,p}$ may be empty for some n, the definition of f_p must be adjusted. Let's just calculate $\lim\limits_{m\to\infty} g_{m,p}(x)$. If $x \in (X - K) \cup N$ then $g_{m,p}(x) = b$ for every m, so the limit is b. On the other hand, if $x \in K - N$, then $x \in B_{n,p}$ for a unique n; clearly $g_{m,p}(x) = g_{n,p}(x) = a_n$ for all $m \geqslant n$ (by the uniqueness mentioned in the definition of $g_{m,p}$), thus the limit is a_n. So the pointwise limit of the $g_{m,p}$ as $m \to \infty$ is the function $f_p : X \to F$ defined by

$$f_p(x) = \begin{cases} b & \text{if } x \in (X - K) \cup N \\ a_n & \text{if } x \in B_{n,p} \text{ for some (unique) } n . \end{cases}$$

Thus only the nonempty $B_{n,p}$ figure in the definition of f_p (but their union is still $K - N$); if $B_{n,p} = \varnothing$ then a_n does not get used in the definition (but, being an element of $f(K - N)$, it will necessarily show up as a limit of values of the f_p).

At any rate, f_p is measurable, being the pointwise limit of a sequence of measurable step functions (No. 4, Th. 2).

IV.68, ℓ. 23–25.

"As p tends to infinity, $f_p(x)$ tends to $f(x)$ for every $x \in K - N$, and to b for $x \in N \cup \complement K$ "

Let $x \in K - N$. Given any integer $p \geqslant 1$, x belongs to $B_{n,p}$ for some n. We know that for all $m \geqslant n$, $g_{m,p}(x) = g_{n,p}(x) = a_n$ and so

$$d\big(f(x), g_{m,p}(x)\big) = d\big(f(x), a_n\big) \leqslant 1/p ;$$

letting $m \to \infty$, we have $d\big(f(x), f_p(x)\big) \leqslant 1/p$, whence $f_p(x) \to f(x)$ as $p \to \infty$. On the other hand, if $x \in (\mathrm{X} - \mathrm{K}) \cup \mathrm{N}$ then $f_p(x) = b$ for all p, therefore $f_p(x) \to b$ trivially. Thus the function $g_{\mathrm{K}} : \mathrm{X} \to \mathrm{F}$ defined by

$$g_{\mathrm{K}}(x) = \begin{cases} b & \text{if } x \in (\mathrm{X} - \mathrm{K}) \cup \mathrm{N} \\ f(x) & \text{if } x \in \mathrm{K} - \mathrm{N} \end{cases}$$

is the pointwise limit of the sequence of measurable functions f_p, hence is measurable.

To summarize, given any compact set $\mathrm{K} \subset \mathrm{X}$, we have found a measurable function $g_{\mathrm{K}} : \mathrm{X} \to \mathrm{F}$ such that $f = g_{\mathrm{K}}$ almost everywhere in K. Applying this to $\mathrm{K} = \mathrm{V}_x$, where V_x is a compact (hence integrable) neighborhood of any given point $x \in \mathrm{X}$, f is measurable by the Principle of localization (No. 2, Prop. 4).

IV.68, ℓ. 25, 26.

"... the limit of the f_p is therefore measurable, and the principle of localization proves that f itself is measurable."

See the preceding note.

IV.69, ℓ. 10.

"... and Th. 4 may be applied."

Some reflections on the topology of $\overline{\mathbf{R}}$ are in order. By definition, it is the topology generated by the open intervals of $\overline{\mathbf{R}}$ (GT, IV, §4, No. 2). Then $\overline{\mathbf{R}}$ is homeomorphic to every nondegenerate closed interval $[a, b]$ of $\mathbf{R}$ (*loc. cit.*, Prop. 2), hence is a metrizable compact space. Explicitly, let $u : \mathbf{R} \to]-1, 1[$ be the order-preserving homeomorphism defined (for example) by

$$u(t) = \frac{t}{1 + |t|} \qquad (t \in \mathbf{R})\,;$$

defining $u(-\infty) = -1$ and $u(+\infty) = +1$, one obtains an order-preserving homeomorphism $u : \overline{\mathbf{R}} \to [-1, 1]$. It follows that the metric on $\overline{\mathbf{R}}$ defined by

$$d(a, b) = |u(a) - u(b)| \qquad (a, b \in \overline{\mathbf{R}})$$

is compatible with its topology. The 'closed balls' in $[-1, 1]$ (for the absolute value metric) are its closed subintervals I, therefore the closed balls in $\overline{\mathbf{R}}$ for the metric d are the sets $\overset{-1}{u}(\mathrm{I})$, that is—since u is an order isomorphism—*the closed subintervals of* $\overline{\mathbf{R}}$.

Thus, to verify condition a) of Th. 4, it suffices to show that $\overset{-1}{f}([a, b])$ is measurable for every closed interval $[a, b]$ of $\overline{\mathbf{R}}$.

Condition b) is trivially verified: since D is dense in $\overline{\mathbf{R}}$ (because $\mathbf{R}$ is dense in $\overline{\mathbf{R}}$), for every compact set $\mathrm{K} \subset \mathrm{X}$ one may take $\mathrm{N} = \varnothing$ and $\mathrm{H} = \mathrm{D}$.

IV.69, ℓ. 13.

"COROLLARY."

If f is lower semi-continuous then, for every $\mathbf{a} \in \mathbf{R}$ the set

$$\{x : \ -f(x) \geqslant a\} = \{x : \ f(x) \leqslant -a\}$$

is closed (GT, IV, §6, No. 2, Prop. 1), therefore $-f$ is measurable (cite Prop. 8 with $\mathrm{D} = \mathbf{Q}$) hence so is f (No. 3, Th. 1 and the continuity of $c \mapsto -c$ in $\overline{\mathbf{R}}$).

IV.69, ℓ. $-9, -8$.

"It is clear that $d\big(f(x), g_n(x)\big) \leqslant 2/n$ for all $x \in \mathrm{X}$."

Say $x \in \mathrm{A}_k$, so that $f(x) \in \mathrm{B}_k$. Say $x \in \mathrm{C}_i \subset \mathrm{A}_k$, so that

$$g_n(x) = f(c_i) \in f(\mathrm{A}_k) \subset \mathrm{B}_k .$$

Thus $f(x), g_n(x) \in \mathrm{B}_k$, where $\operatorname{diam} \mathrm{B}_k \leqslant 2/n$.

IV.69, ℓ. -2.

"The condition being obviously necessary (No. 3, Th. 1) ..."

The scalar function in question is $\mathbf{a}'_n \circ \mathbf{f}$, so its measurability is a special case of No. 3, Th. 1; the following argument is perhaps more transparent:

Proposition. If $f : \mathrm{X} \to \mathrm{F}$ is measurable and $u : \mathrm{F} \to \mathrm{G}$ is continuous, then $u \circ f : \mathrm{X} \to \mathrm{G}$ is measurable (where X is a locally compact space equipped with a measure, and F, G are any topological spaces).

Proof. Immediate from the definition of measurability (No. 1, Def. 1).

But beware: if $u : \mathrm{X} \to \mathrm{X}$ is continuous and $f : \mathrm{X} \to \mathrm{F}$ is measurable, then the composite function $f \circ u : \mathrm{X} \to \mathrm{F}$ need not be measurable (No. 3, *Remark* 2).

IV.70, ℓ. 1–3.

"... it will suffice by the principle of localization to prove that, for every compact subset K of X and every closed ball $\mathrm{B} \subset \mathrm{F}$, with center $\mathbf{a}$ and radius r, the set $\mathrm{A} = \mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})$ is measurable"

The objective is to show that $\overset{-1}{\mathbf{f}}(\mathrm{B})$ is measurable. More generally,

Proposition. For a subset $\mathrm{S} \subset \mathrm{X}$, the following conditions are equivalent:

a) S is measurable;
b) $K \cap S$ is measurable for every compact set $K \subset X$;
c) $K \cap S$ is integrable for every compact set $K \subset X$.

Proof. *a*) ⇔ *c*): No. 1, Prop. 3.

c) ⇒ *b*): Suppose S satisfies *c*). Given $K \subset X$ compact. To prove that $K \cap S$ is measurable, it suffices by *c*) ⇒ *a*) to show that for every compact $K' \subset X$, the set $K' \cap (K \cap S)$ is integrable; indeed, $K' \cap (K \cap S) = (K' \cap K) \cap S$ is integrable by *c*).

b) ⇒ *c*): Suppose S satisfies *b*). Given $K \subset X$ compact. We know that $K \cap S$ is measurable, hence $K \cap S = K \cap (K \cap S)$ is integrable by *a*) ⇒ *c*).

The following corollary is sharpened in No. 6, Th. 5 below.

Corollary. For a subset $S \subset X$,

$$S \text{ integrable} \quad \Rightarrow \quad S \text{ measurable.}$$

Proof. Suppose S is integrable. Given any compact set $K \subset X$, one knows that K is integrable (§4, No. 6, Cor. 1 of Prop. 10), hence so is $K \cap S$ (§4, No. 5, Prop. 7, 2°), thus S is measurable by the above proposition.

IV.70, ℓ. 3, 4.

"... for every $\mathbf{z} \in F$,

$$|\mathbf{z}| = \sup_n |\langle \mathbf{z}, \mathbf{a}'_n \rangle| / |\mathbf{a}'_n| \text{ "}$$

Since F is separable, the unit ball $B' = \{\mathbf{z}' \in F' : |\mathbf{a}'| \leqslant 1\}$ of F' is a metrizable compact space for the weak topology $\sigma(F', F)$, hence has a countable subset D' that is dense in B' for the weak topology (TVS, III, §3, No. 4, Cor. 2 of Prop. 6).

We assume given such a countable subset, indexed as a sequence, $D' = \{\mathbf{a}'_n : n = 1, 2, 3, \dots\}$, having the further property that for every n, the function

$$(*) \qquad x \mapsto \langle \mathbf{f}(x), \mathbf{a}'_n \rangle \quad (x \in X)$$

is measurable. The asserted formula suggests replacing D' by $D = D' - \{0\}$; to be sure that the weak closure of D includes 0 (hence is equal to B'), one can add to the original subset D' the vectors $(1/n)\mathbf{a}'_m$ $(n = 2, 3, \dots)$, where $\mathbf{a}'_m$ is a nonzero element of D', noting that the functions $(*)$ for the added vectors are also measurable.

In other words, we can suppose that the vectors $\mathbf{a}'_n$ are all nonzero. The asserted formula is then the formula (5) of TVS, IV, §1, No. 3, Cor. of Prop. 8; as the D of that corollary is part of the conclusion rather than part of the hypothesis, it may be useful to review the proof. For every $\mathbf{z} \in \mathrm{F}$ define

$$\rho(\mathbf{z}) = \sup_n |\langle \mathbf{z}, \mathbf{a}'_n \rangle| / |\mathbf{a}'_n| = \sup_{\mathbf{a}' \in \mathrm{D}} |\langle \mathbf{z}, \mathbf{a}' \rangle| / |\mathbf{a}'|\,;$$

we are to show that $\rho(\mathbf{z}) = |\mathbf{z}|$. Fix $\mathbf{z} \in \mathrm{F}$. For every n,

$$|\langle \mathbf{z}, \mathbf{a}'_n \rangle| / |\mathbf{a}'_n| = |\mathbf{a}'_n(\mathbf{z})| / |\mathbf{a}'_n| \leqslant (|\mathbf{a}'_n| \cdot |\mathbf{z}|) / |\mathbf{a}'_n| = |\mathbf{z}|\,,$$

therefore $\rho(\mathbf{z}) \leqslant |\mathbf{z}|$. On the other hand,

$$|\mathbf{z}| = \sup_{\mathbf{a}' \in \mathrm{B}'} |\langle \mathbf{z}, \mathbf{a}' \rangle|$$

(*loc. cit.*, Prop. 8, (i)); to show that $|\mathbf{z}| \leqslant \rho(\mathbf{z})$, it suffices to show that $|\langle \mathbf{z}, \mathbf{a}' \rangle| \leqslant \rho(\mathbf{z})$ for every $\mathbf{a}' \in \mathrm{B}'$. Indeed, if $\mathbf{a}' \in \mathrm{B}'$ then $\mathbf{a}'$ belongs to the weak closure of D, hence there exists a sequence in D, say $\mathbf{a}'_{n_k}$ $(k = 1, 2, 3, \dots)$ such that $\mathbf{a}'_{n_k} \to \mathbf{a}'$ weakly as $k \to \infty$. In particular $\langle \mathbf{z}, \mathbf{a}'_{n_k} \rangle \to \langle \mathbf{z}, \mathbf{a}' \rangle$, hence $|\langle \mathbf{z}, \mathbf{a}'_{n_k} \rangle| \to |\langle \mathbf{z}, \mathbf{a}' \rangle|$; but

$$|\langle \mathbf{z}, \mathbf{a}'_{n_k} \rangle| = (|\langle \mathbf{z}, \mathbf{a}'_{n_k} \rangle| / |\mathbf{a}'_{n_k}|) \cdot |\mathbf{a}'_{n_k}| \leqslant |\langle \mathbf{z}, \mathbf{a}'_{n_k} \rangle| / |\mathbf{a}'_{n_k}| \leqslant \rho(\mathbf{z})\,,$$

and passage to the limit yields $|\langle \mathbf{z}, \mathbf{a}' \rangle| \leqslant \rho(\mathbf{z})$.

Thus, writing $\mathbf{b}'_n = \mathbf{a}'_n / |\mathbf{a}'_n|$ (a unit vector in F'), one has

$$|\mathbf{z}| = \sup_n |\langle \mathbf{z}, \mathbf{b}'_n \rangle|$$

for all $\mathbf{z} \in \mathrm{F}$.

IV.70, ℓ. 5, 6.

" A is thus the intersection of K and the sets defined by

$$|\langle \mathbf{f}(x), \mathbf{a}'_n \rangle - \langle \mathbf{a}, \mathbf{a}'_n \rangle| \leqslant r\, |\mathbf{a}'_n| \text{ "}$$

To simplify the notation slightly, write $\mathbf{b}'_n = \mathbf{a}'_n / |\mathbf{a}'_n|$ as in the preceding note. Recalling that B is the closed ball $\mathrm{B}_r(\mathbf{a})$,

$$\begin{aligned}
\mathrm{A} = \mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B}) &= \{x \in \mathrm{K} : \ \mathbf{f}(x) \in \mathrm{B}\} \\
&= \{x \in \mathrm{K} : \ |\mathbf{f}(x) - \mathbf{a}| \leqslant r\} \\
&= \{x \in \mathrm{K} : \ \sup_n |\langle \mathbf{f}(x) - \mathbf{a}, \mathbf{b}'_n \rangle| \leqslant r\} \\
&= \{x \in \mathrm{K} : \ |\langle \mathbf{f}(x), \mathbf{b}'_n \rangle - \langle \mathbf{a}, \mathbf{b}'_n \rangle| \leqslant r \quad \text{for all } n\} \\
&= \mathrm{K} \cap \bigcap_{n=1}^{\infty} \{x : \ \langle \mathbf{f}(x), \mathbf{b}'_n \rangle \leqslant r + \langle \mathbf{a}, \mathbf{b}'_n \rangle\} \\
&\qquad \cap \bigcap_{n=1}^{\infty} \{x : \ \langle \mathbf{f}(x), \mathbf{b}'_n \rangle \geqslant -r + \langle \mathbf{a}, \mathbf{b}'_n \rangle\};
\end{aligned}$$

since each function $x \mapsto \langle \mathbf{f}(x), \mathbf{b}'_n \rangle = \langle \mathbf{f}(x), \mathbf{a}'_n \rangle / |\mathbf{a}'_n|$ is by assumption measurable, A is the intersection of (countably many) measurable sets by Prop. 7, hence is measurable (No. 4, Cor. 2 of Th. 2).

IV.70, ℓ. 16–18.

"... we may (on account of b)) suppose, after modifying $\mathbf{f}$ if necessary on a negligible set, that $\mathbf{f}(\mathrm{K}) \subset \overline{\mathrm{H}}$, where H is a countable subset of F."

The objective is to show that $\mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})$ is measurable (where $\mathrm{B} = \mathrm{B}_r(\mathbf{a})$ is the ball with center $\mathbf{a}$ and radius r). By b), there exists a negligible set $\mathrm{N} \subset \mathrm{K}$ and a countable set $\mathrm{H} \subset \mathrm{F}$ such that $\mathbf{f}(\mathrm{K} - \mathrm{N}) \subset \overline{\mathrm{H}}$. Fix an element $\mathbf{b}$ of $\overline{\mathrm{H}}$ and define $\mathbf{g} : \mathrm{X} \to \mathrm{F}$ by

$$\mathbf{g}(x) = \begin{cases} \mathbf{b} & \text{for } x \in \mathrm{N} \\ \mathbf{f}(x) & \text{for } x \in \mathrm{X} - \mathrm{N}. \end{cases}$$

Then $\mathbf{g} = \mathbf{f}$ almost everywhere, therefore $\mathbf{g}$ also satisfies the condition a) (No. 2, Prop. 6), and it satisfies condition b) for every compact set $\mathrm{K}' \subset \mathrm{X}$ (not just for K) because the union of two negligible sets is negligible. Moreover, for the particular compact set K, we have

$$\mathbf{g}(\mathrm{K}) = \mathbf{g}(\mathrm{N}) \cup \mathbf{g}(\mathrm{K} - \mathrm{N}) = \{\mathbf{b}\} \cup \mathbf{f}(\mathrm{K} - \mathrm{N}) \subset \overline{\mathrm{H}}.$$

If we succeed in showing that $\mathrm{K} \cap \overset{-1}{\mathbf{g}}(\mathrm{B})$ is measurable, it will follow from

$$\begin{aligned} \mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B}) &= [(\mathrm{K} - \mathrm{N}) \cap \overset{-1}{\mathbf{f}}(\mathrm{B})] \cup [\mathrm{N} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})] \\ &= [(\mathrm{K} - \mathrm{N}) \cap \overset{-1}{\mathbf{g}}(\mathrm{B})] \cup [\mathrm{N} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})] \\ &= \big[(\mathrm{K} - \mathrm{N}) \cap \big(\mathrm{K} \cap \overset{-1}{\mathbf{g}}(\mathrm{B})\big)\big] \cup [\mathrm{N} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})] \end{aligned}$$

that $\mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})$ is also measurable (being the union of two measurable sets).

IV.70, ℓ. 20, 21.

"... every continuous linear form on V is the restriction of a form $\mathbf{a}' \in \mathrm{F}'$"

The assertion is immediate from the Hahn–Banach theorem; it follows that when F is replaced by V, the condition a) continues to hold (with $\mathbf{g}$ as in the preceding note) for the function $\mathbf{g}_0 : \mathrm{X} \to \mathrm{V}$ defined by $x \mapsto \mathbf{g}(x)$ (as of course does condition b)).

The argument in the 'sufficiency' part of Prop. 10 then shows that the set $\mathrm{A} = \mathrm{K} \cap \overset{-1}{\mathbf{g}}_0(\mathrm{B})$ is measurable. But

$$\mathrm{A} = \{x \in \mathrm{K} : \ \mathbf{g}_0(x) \in \mathrm{B}\} = \{x \in \mathrm{K} : \ \mathbf{g}(x) \in \mathrm{B}\} = \mathrm{K} \cap \overset{-1}{\mathbf{g}}(\mathrm{B}),$$

thus $\mathrm{K} \cap \overset{-1}{\mathbf{g}}(\mathrm{B})$ is measurable, hence so is $\mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})$ as argued in the preceding note.

IV.70, ℓ. 21, 22.

"...the same reasoning as in Prop. 10 then shows that $\mathrm{K} \cap \overset{-1}{\mathbf{f}}(\mathrm{B})$ is measurable."

See the preceding note.

IV.70, ℓ. −10, −9.

"We may regard F as a subspace of a countable product $\prod_n \mathrm{E}_n$ of Banach spaces (TVS, II, §4, No. 3)"

As the proof of 'necessity' is immediate (see the note for **IV.69,** ℓ. −2), one is concerned here with the proof of 'sufficiency'.

The cited reference refers back to an earlier section (TVS, II, §1, No. 3); let us review the argument, which entails some subtleties.

If F is any locally convex space (but not necessarily Hausdorff), its topology can be generated by a family of continuous semi-norms (TVS, II, §4, No. 1, Cor. of Prop. 1), say $(p_\iota)_{\iota \in \mathrm{I}}$. For each ι, the set

$$\mathrm{N}_\iota = \{\mathbf{a} \in \mathrm{F} : \ p_\iota(\mathbf{a}) = 0\}$$

is a closed linear subspace of F, the quotient topology on $\mathrm{F}_\iota = \mathrm{F}/\mathrm{N}_\iota$ (TVS, II, §4, No. 4, *Example* I) coincides with the norm topology for the norm on F_ι deduced from p_ι (TVS, II, §1, No. 3); thus the quotient mapping $u_\iota : \mathrm{F} \to \mathrm{F}_\iota$ is a continuous linear mapping of F into the normed space F_ι. Writing $\prod_\iota \mathrm{F}_\iota$ for the product locally convex space (TVS, II, §4, No. 3), $u : \mathrm{F} \to \prod_\iota \mathrm{F}_\iota$ for the mapping $u(\mathbf{a}) = \big(u_\iota(\mathbf{a})\big)_{\iota \in \mathrm{I}}$ and $\mathrm{pr}_\iota : \prod_{\chi \in \mathrm{I}} \mathrm{F}_\chi \to \mathrm{F}_\iota$ for the ι'th projection mapping, so that $\mathrm{pr}_\iota \circ u = u_\iota$ for all ι, the continuity of the u_ι implies that of u (GT, I, §4, No. 1, Prop. 1).

For F to be Hausdorff, it is necessary and sufficient that the family (p_ι) be separating ($p_\iota(\mathbf{a}) = 0$ for all $\iota \Rightarrow \mathbf{a} = 0$), equivalently $\bigcap_\iota \mathrm{N}_\iota = \{0\}$, which is equivalent to the injectivity of u. Writing $\mathrm{M} = u(\mathrm{F})$, regarded as a topological subspace of $\prod_\iota \mathrm{F}_\iota$, one sees that u implements a topological vector space isomorphism $\mathrm{F} \to \mathrm{M}$ (TVS, II, §1, No. 3 and I, §1, No. 7, Prop. 7); the crux of the matter is that the topology of F is the coarsest locally convex topology that renders continuous the semi-norms p_ι (equivalently, the linear mappings u_ι; equivalently, the linear mapping u). {Thus the topology generated by a separating family of semi-norms is an 'initial topology' with respect to locally convex topologies (TVS, II, §4, No. 3, Prop. 4).}

Assuming F Hausdorff, let E_ι be the Banach space obtained by completing F_ι. Since $\prod_\iota F_\iota$ is a topological subspace of $\prod_\iota E_\iota$ (GT, I, §4, No. 1, Cor. of Prop. 3), M is also a topological subspace of $\prod_\iota E_\iota$ (transitivity of induced topologies); thus, every Hausdorff locally convex space is isomorphic as a topological vector space to a (topological) subspace of a product of Banach spaces (TVS, II, §1, No. 3, Prop. 3). {Since a continuous linear mapping of topological vector spaces is uniformly continuous (GT, III, §3, No. 1, Prop. 3) it follows that if F is Hausdorff then the uniform structures of F and M are isomorphic; if, moreover, F is complete, then so is M, consequently M is a closed linear subspace of $\prod_\iota F_\iota$ (GT, II, §3, No. 4, Prop. 8) hence also of $\prod_\iota E_\iota$ (TVS, II, §4, last paragraph of No. 3).

Consider now the space F of the present corollary. Since F is metrizable, it has a countable fundamental system (V_n) of neighborhoods of 0 (TVS, I, §3, No. 1), and since F is locally convex, the V_n may be taken to be closed, balanced and convex (TVS, II, §4, No. 1). If p_n is the gauge of V_n, the topology of F is generated by the family of semi-norms (p_n) (*loc. cit.*, Cor. of Prop. 1; see also the note for **III.39,** ℓ. 8–11). With notations as in the foregoing discussion, if $u : F \rightarrow \prod_n F_n$ is the mapping defined by $u(\mathbf{a}) = \big(u_n(\mathbf{a})\big)$, where $F_n = F/\overset{-1}{p_n}(0)$ (a normed space) and $u_n : F \rightarrow F_n$ is the quotient mapping, then u defines a topological vector space isomorphism of F onto $M = u(F)$ regarded as a topological subspace of $\prod_n F_n$. If E_n is the Banach space completion of F_n, then u may be regarded as linear mapping $F \rightarrow \prod_n E_n$ that defines a topological vector space isomorphism of F onto the topological subspace $M = u(F)$ of $\prod_n E_n$. Since F has a countable dense subset, so does the subspace $\mathrm{pr}_n M$ of E_n $(n = 1, 2, 3, \dots)$; replacing E_n by the closure of $\mathrm{pr}_n M$ in E_n, we can suppose that the E_n are separable Banach spaces. Identifying F with M, we have shown that F is a topological linear subspace of the product of a sequence of separable Banach spaces.

IV.70, ℓ. −8, −7.

"For every n, the mapping $\mathrm{pr}_n \circ \mathbf{f}$ is then measurable by Prop. 10"

With notations as in the preceding note, write $E = \prod_n E_n$ and regard F as a topological linear subspace of E.

Since $\mathbf{f} : X \rightarrow F$, $F \subset E$ and $\mathrm{pr}_n : E \rightarrow E_n$, the composition that is possible is $\mathrm{pr}_n \circ \theta \circ \mathbf{f}$, where $\theta : F \rightarrow E$ is the canonical injection; since $\mathrm{pr}_n \circ \theta = \mathrm{pr}_n | F$, it is the composition $(\mathrm{pr}_n | F) \circ \mathbf{f}$ that we are to show is measurable. Let us write $\mathbf{u}_n = \mathrm{pr}_n\big|\mathrm{F}$, which is a continuous linear mapping $\mathrm{F} \to \mathrm{E}_n$, so that $\mathbf{u}_n \circ \mathbf{f} : \mathrm{X} \to \mathrm{E}_n$. To show that $\mathbf{u}_n \circ \mathbf{f}$ is measurable, it suffices by Prop. 10 to show that, given any $\mathbf{a}'_n \in \mathrm{E}'_n$, the function

$$(*) \qquad x \mapsto \langle (\mathbf{u}_n \circ \mathbf{f})(x), \mathbf{a}'_n \rangle \qquad (x \in \mathrm{X})$$

is measurable. Now,

$$\langle (\mathbf{u}_n \circ \mathbf{f})(x), \mathbf{a}'_n \rangle = \langle \mathbf{u}_n(\mathbf{f}(x)), \mathbf{a}'_n \rangle = \langle \mathbf{f}(x), {}^t\mathbf{u}_n \mathbf{a}'_n \rangle = \langle \mathbf{f}(x), \mathbf{a}'_n \circ \mathbf{u}_n \rangle ,$$

where ${}^t\mathbf{u}_n : \mathrm{E}'_n \to \mathrm{F}'$ is the transpose of $\mathbf{u}_n$; since ${}^t\mathbf{u}_n \mathbf{a}'_n \in \mathrm{F}'$, by hypothesis the function

$$x \mapsto \langle \mathbf{f}(x), {}^t\mathbf{u}_n \mathbf{a}'_n \rangle \qquad (x \in \mathrm{X})$$

is measurable, in other words the function $(*)$ is indeed measurable.

To summarize, for every index n, the function $(\mathrm{pr}_n\big|\mathrm{F}) \circ \mathbf{f} : \mathrm{X} \to \mathrm{E}_n$ is measurable.

IV.70, ℓ. -7.

"... therefore $\mathbf{f}$ is measurable by No. 3, Th. 1."

Conserving the notations of the preceding note, we have a sequence of measurable functions $\mathbf{u}_n \circ \mathbf{f} : \mathrm{X} \to \mathrm{E}_n$, where $\mathbf{u}_n = \mathrm{pr}_n\big|\mathrm{F}$. For all $x \in \mathrm{X}$, $\mathbf{f}(x) \in \mathrm{F} \subset \mathrm{E} = \prod_n \mathrm{E}_n$, hence

$$\mathbf{f}(x) = \big(\mathrm{pr}_n(\mathbf{f}(x))\big) = \big((\mathrm{pr}_n\big|\mathrm{F})(\mathbf{f}(x))\big) = \big((\mathbf{u}_n \circ \mathbf{f})(x)\big);$$

it follows from the cited Th. 1 that, taking u to be the canonical injection $\mathbf{f}(\mathrm{X}) \to \mathrm{E}$, the mapping $u \circ \mathbf{f} : \mathrm{X} \to \mathrm{E}$ is measurable. But $\mathbf{f}$ takes its values in F, and we wish to show that $\mathbf{f} : \mathrm{X} \to \mathrm{F}$ is measurable.

Let $\mathrm{K} \subset \mathrm{X}$ be compact. By the measurability of $u \circ \mathbf{f}$, there exist a negligible set $\mathrm{N} \subset \mathrm{K}$ and a partition $(\mathrm{K}_i)_{i \in \mathrm{I}}$ of $\mathrm{K} - \mathrm{N}$ into compact sets K_i on each of which $u \circ \mathbf{f}$ is continuous, that is, the function

$$x \mapsto (u \circ \mathbf{f})(x) = u\big(\mathbf{f}(x)\big) = \mathbf{f}(x) \in \mathrm{F} \qquad (x \in \mathrm{K}_i)$$

is continuous, whence the measurability of $\mathbf{f}$.

IV.71, ℓ. $4, 5$.

"... the polar sets V_n° are equicontinuous and their union is all of F'."

The inclusion $\mathrm{V}_n^{\circ\circ} \supset \mathrm{V}_n$ shows that $\mathrm{V}_n^{\circ\circ}$ is a neighborhood of 0 in F, hence V_n° is equicontinuous (TVS, III, §3, No. 5, Prop. 7).

Let $\mathbf{a}' \in \mathrm{F}'$. Since $\mathbf{a}'$ is continuous at 0 and the V_n are basic, there exists an index m such that $\mathbf{a}'(\mathrm{V}_m) \subset \{\lambda : \ |\lambda| \leqslant 1\}$, that is, $|\mathbf{a}'(x)| \leqslant 1$ for all $x \in \mathrm{V}_m$; since V_m is circled, this means that $\mathbf{a}' \in \mathrm{V}_m^\circ$ (TVS, II, §8, No. 4).

IV.71, ℓ. 9.

"...each of the $\mathrm{S}_{\mathbf{y}}$ is measurable"

$\mathrm{S}_{\mathbf{y}}$ is the inverse image, under the measurable mapping $x \mapsto \langle \mathbf{y}, \mathbf{f}(x)\rangle$, of the closed set $\{c : \ |c| \leqslant 1\}$ in the field of scalars, hence is measurable by No. 5, Prop. 7.

IV.71, ℓ. 9, 10.

"X_n is the intersection of the countable family of $\mathrm{S}_{\mathbf{y}}$ for $\mathbf{y} \in \mathrm{D} \cap \mathrm{V}_n$."

Since $\overline{\mathrm{D}} = \mathrm{F}$ and V_n is open in F, one has $\mathrm{V}_n = \mathrm{V}_n \cap \overline{\mathrm{D}} \subset \overline{\mathrm{V}_n \cap \mathrm{D}}$ (GT, I, §1, No. 6, Prop. 5), whence $\overline{\mathrm{V}}_n = \overline{\mathrm{V}_n \cap \mathrm{D}}$. The following computation then depends on a formula for the polar of a balanced set (cf. TVS, II, §8, No. 4) and the continuity of the linear forms $\mathbf{f}(x)$:

$$\begin{aligned} x \in \mathrm{X}_n \quad &\Leftrightarrow \quad \mathbf{f}(x) \in \mathrm{V}_n^\circ \\ &\Leftrightarrow \quad |\langle \mathbf{y}, \mathbf{f}(x)\rangle| \leqslant 1 \ \text{ for all } \mathbf{y} \in \mathrm{V}_n \\ &\Leftrightarrow \quad |\langle \mathbf{y}, \mathbf{f}(x)\rangle| \leqslant 1 \ \text{ for all } \mathbf{y} \in \overline{\mathrm{V}}_n \\ &\Leftrightarrow \quad |\langle \mathbf{y}, \mathbf{f}(x)\rangle| \leqslant 1 \ \text{ for all } \mathbf{y} \in \overline{\mathrm{V}_n \cap \mathrm{D}} \\ &\Leftrightarrow \quad |\langle \mathbf{y}, \mathbf{f}(x)\rangle| \leqslant 1 \ \text{ for all } \mathbf{y} \in \mathrm{V}_n \cap \mathrm{D} \\ &\Leftrightarrow \quad x \in \mathrm{S}_{\mathbf{y}} \qquad\quad \text{ for all } \mathbf{y} \in \mathrm{V}_n \cap \mathrm{D}, \end{aligned}$$

that is, $\mathrm{X}_n = \bigcap_{\mathbf{y} \in \mathrm{V}_n \cap \mathrm{D}} \mathrm{S}_{\mathbf{y}}$.

IV.71, ℓ. 10–13.

"...for every compact subset K of X and every $\varepsilon > 0$, there exists an integer n such that $|\mu|\big(\mathrm{K} - (\mathrm{K} \cap \mathrm{X}_n)\big) \leqslant \varepsilon/4$, and then a compact subset K_1 of $\mathrm{K} \cap \mathrm{X}_n$ such that $|\mu|\big((\mathrm{K} \cap \mathrm{X}_n) - \mathrm{K}_1\big) \leqslant \varepsilon/4$"

Since the X_k are measurable, the $\mathrm{K} \cap \mathrm{X}_k$ form an increasing sequence of integrable (No. 1, Prop. 3) sets with union K, therefore

$$|\mu|\big(\mathrm{K} - (\mathrm{K} \cap \mathrm{X}_k)\big) \to 0$$

(§4, No. 5, Cor. of Prop. 7), whence the existence of n. The existence of K_1 then follows from the 'inner regularity' of $|\mu|$ (§4, No. 6, Cor. 1 of Th. 4).

IV.71, ℓ. 18, 19.

"...the restriction of $\mathbf{f}$ to K_2 is therefore continuous"

Let (x_j) be a directed family in K_2 such that $x_j \to x \in \mathrm{K}_2$; we are to show that $\mathbf{f}(x_j) \to \mathbf{f}(x)$ in F' for $\sigma(\mathrm{F}', \mathrm{F})$, that is, for the topology of pointwise convergence in F. We know that $\langle \mathbf{y}, \mathbf{f}(x_j)\rangle \to \langle \mathbf{y}, \mathbf{f}(x)\rangle$ for each $\mathbf{y} \in \mathrm{D}$, by the choice of K_2; thus $\mathbf{f}(x_j) \to \mathbf{f}(x)$ for the topology of pointwise convergence in D. But $\mathbf{f}(x_j)$ and $\mathbf{f}(x)$ belong to the equicontinuous set $\mathbf{f}(\mathrm{K}_2) \subset \mathrm{V}_n^\circ$, therefore $\mathbf{f}(x_j) \to \mathbf{f}(x)$ for the topology of pointwise convergence in F (GT, X, §2, No. 4, Th. 1).

IV.71, ℓ. 20, 21.

"If $\mathbf{z}'$ is a continuous linear form on F, its restriction $\mathbf{z}'_n$ to F_n is continuous"

This is where the hypothesis $\mathrm{F} = \varinjlim \mathrm{F}_n$ gets used: the canonical injection $i_n : \mathrm{F}_n \to \mathrm{F}$ is continuous (TVS, II, §4, No. 4, *Example* II), therefore the composite $\mathbf{z}' \circ i_n = \mathbf{z}' \big| \mathrm{F}_n$ is continuous. But it is not assumed that F_n is a topological subspace of F (TVS, *loc. cit.*, *Remark*).

IV.71, ℓ. 21–23.

"... the dual F′ of F may be identified (algebraically) with a linear subspace of the product $\prod_n \mathrm{F}'_n$, and then $\mathrm{pr}_n(\mathbf{z}') = \mathbf{z}'_n$."

Define a linear mapping $u : \mathrm{F}' \to \prod_n \mathrm{F}'_n$ by $u(\mathbf{z}') = (\mathbf{z}'_n)$, where $\mathbf{z}'_n = \mathbf{z}' \big| \mathrm{F}_n \in \mathrm{F}'_n$ (see the preceding note). The injectivity of u follows from $\bigcup_n \mathrm{F}_n = \mathrm{F}$. Identifying F′ with the linear subspace $u(\mathrm{F}')$ of the product space via u, u becomes the canonical injection $\mathrm{F}' \to \prod_n \mathrm{F}'_n$, $\mathbf{z}'$ becomes $(\mathbf{z}'_n)$, and $\mathrm{pr}_n \mathbf{z}' = \mathbf{z}'_n$.

IV.71, ℓ. 24, 25.

"... the topology $\sigma(\mathrm{F}', \mathrm{F})$ is none other than the topology induced by the product topology of the topologies $\sigma(\mathrm{F}'_n, \mathrm{F}_n)$."

Recalling that $\mathrm{F} = \bigcup_n \mathrm{F}_n$, if $(\mathbf{z}'_j)$ is a directed family in F'_n and if $\mathbf{z}' \in \mathrm{F}'$, then

$$
\begin{aligned}
\mathbf{z}'_j \to \mathbf{z}' \text{ in } \mathrm{F}' \text{ for } \sigma(\mathrm{F}',\mathrm{F}) \quad &\Leftrightarrow \quad \langle \mathbf{a}, \mathbf{z}'_j \rangle \to \langle \mathbf{a}, \mathbf{z}' \rangle \quad \text{for all } \mathbf{a} \in \mathrm{F} \\
&\Leftrightarrow \quad (\forall\, n) \;\; \langle \mathbf{a}, \mathbf{z}'_j \rangle \to \langle \mathbf{a}, \mathbf{z}' \rangle \text{ for all } \mathbf{a} \in \mathrm{F}_n \\
&\Leftrightarrow \quad (\forall\, n)\, (\forall\, \mathbf{a} \in \mathrm{F}_n) \quad \langle \mathbf{a}, (\mathbf{z}'_j)_n \rangle \to \langle \mathbf{a}, (\mathbf{z}')_n \rangle \\
&\Leftrightarrow \quad (\forall\, n) \;\; \mathrm{pr}_n \mathbf{z}'_j \to \mathrm{pr}_n \mathbf{z}' \quad \text{for } \sigma(\mathrm{F}'_n, \mathrm{F}_n) \\
&\Leftrightarrow \quad \mathbf{z}'_j \to \mathbf{z}' \quad \text{in } \prod_n \mathrm{F}'_n ,
\end{aligned}
$$

where $\prod_n \mathrm{F}'_n$ is equipped with the indicated product topology (GT, I, §7, No. 6, Cor. 1 of Prop. 10).

IV.71, ℓ. −11.

" $\langle \mathbf{a}_n, \mathrm{pr}_n \circ \mathbf{f} \rangle = \langle \mathbf{a}_n, \mathbf{f} \rangle$ "

For all $\mathbf{a}_n \in \mathrm{F}_n$ and $x \in \mathrm{X}$, one has

$$\begin{aligned} \langle \mathbf{a}_n, \mathrm{pr}_n \circ \mathbf{f}\rangle(x) &= \langle \mathbf{a}_n, (\mathrm{pr}_n \circ \mathbf{f})(x)\rangle \\ &= \langle \mathbf{a}_n, \mathrm{pr}_n\big(\mathbf{f}(x)\big)\rangle \\ &= \langle \mathbf{a}_n, \mathbf{f}(x)\big|\mathrm{F}_n\rangle \\ &= \langle \mathbf{a}_n, \mathbf{f}(x)\rangle = \langle \mathbf{a}_n, \mathbf{f}\rangle(x) \end{aligned}$$

by the definition of the notations.

IV.71, ℓ. −4 to −2.

"... if $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$ then there exists a sequence $(\mathbf{g}_n)$ of continuous functions with compact support that converges almost everywhere to $\mathbf{f}$ (§3, No. 4, Cor. 2 of Th. 3)"

Recall that $\mathscr{L}^p_{\mathrm{F}}$ is defined to be the closure of the subset $\mathscr{K}(\mathrm{X};\mathrm{F})$ of $\mathscr{F}^p_{\mathrm{F}}(\mathrm{X},\mu)$ for the topology of convergence in mean of order p (§3, No. 4, Def. 2).

IV.72, ℓ. −12, −11.

Corollary 1. — *For a set to be integrable, it is necessary and sufficient that it be measurable and have finite outer measure.*

By definition, A is measurable (resp. integrable) if and only if φ_{A} is measurable (resp. integrable); and $|\mu|^*(\mathrm{A}) = |\mu|^*(\varphi_{\mathrm{A}})$. Quote Th. 5.

IV.73, ℓ. −2.

" g' is upper semi-continuous on X "

It is the same to show that $-g'$ is lower semi-continuous. Let $k \in \mathbf{R}$ and let $\mathrm{A} = \{x \in \mathrm{X} : \ -g'(x) \leqslant k\} = \{x \in \mathrm{X} : \ g'(x) \geqslant -k\}$; the problem is to show that A is a closed subset of X (GT, IV, §6, No. 2, Prop. 1). If $-k \leqslant 0$ then $\mathrm{A} = \mathrm{X}$ (because $g' \geqslant 0$). If $-k > 0$ then, since $g' = 0$ on $\mathrm{X} - \mathrm{Y}$, one has $\mathrm{A} = \{y \in \mathrm{Y} : \ g(y) \geqslant -k\} \subset \mathrm{Supp}(g)$; since $g \in \mathscr{K}(\mathrm{Y})$, $\mathrm{Supp}(g)$ is compact, so its closed subset A is compact in Y, hence also in X, therefore A is closed in X (GT, I, §9, No. 3, Prop. 4).

Addendum. One observes that the proof, that g' is upper semi-continuous on X when $g \in \mathscr{K}_+(\mathrm{Y})$, does not require that Y be locally compact—it can be any subspace of X (and X can be any Hausdorff space).

When Y is locally compact, it contains enough compact sets for every point to have a fundamental system of compact neighborhoods, and $\mathscr{K}_+(\mathrm{Y})$ contains enough functions to separate the points of Y (III, §1, No. 2, Lemma 1). But if Y is an arbitrary subset of X, does it contain enough compact sets, and $\mathscr{K}(\mathrm{Y};\mathbf{C})$ enough functions, to be useful?

At any rate, $\mathscr{K}(\mathrm{Y};\mathbf{C})$ is, for arbitrary Y, a complex vector space closed under complex conjugation, and $\mathscr{K}(\mathrm{Y};\mathbf{R})$ is a Riesz space, so that every $g \in \mathscr{K}(\mathrm{Y};\mathbf{C})$ is a linear combination of four elements of $\mathscr{K}_+(\mathrm{Y})$.

More about such matters in the next note.

IV.74, ℓ. 3–6.

"... *denote by* μ_Y *or* $\mu|Y$, *the measure defined by the formula*

$$\int g\,d\mu_Y = \int g'\,d\mu \tag{1}$$

for every function $g \in \mathscr{K}(Y;\mathbf{C})$, *where* g' *denotes the function equal to* g *on* Y *and to* 0 *on* $X - Y$."

The mapping $\mathscr{K}(Y;\mathbf{C}) \to \mathfrak{F}(X;\mathbf{C})$ defined by $g \mapsto g'$ is linear and positive ($g \geqslant 0 \Rightarrow g' \geqslant 0$); writing $g = g_1 - g_2 + ig_3 - ig_4$ with $g_j \in \mathscr{K}_+(Y)$, it follows from the foregoing that $g' = g_1' - g_2' + ig_3' - ig_4'$ is μ-integrable, so that $g \mapsto g'$ is a linear mapping $\mathscr{K}(Y;\mathbf{C}) \to \mathscr{L}^1_{\mathbf{C}}(Y,\mu)$. One may therefore define a *linear form* $\mu_Y : \mathscr{K}(Y;\mathbf{C}) \to \mathbf{C}$ by $\mu_Y(g) = \int g'\,d\mu$.

If, moreoever, $\mu \geqslant 0$, then $g \mapsto \int g'\,d\mu$ defines a positive linear form on $\mathscr{K}(Y;\mathbf{R})$, that is, a positive measure on Y (III, §1, No. 5, Th. 1); we set aside, for the moment, the question of whether μ_Y is a (complex) measure.

What if μ is not positive? Of course $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ with the μ_j positive measures $\leqslant |\mu|$, but, absent an analysis of $\int d(\mu+\nu)$ (see the Note for V.10, ℓ. 13, 14), it is not clear how to exploit this expression for μ to show that μ_Y is a linear combination of (positive) measures. Instead, let us show that μ_Y is a measure by verifying the criterion of Ch. III, §1, No. 3: given any compact subset K of Y, there exists a constant $M_K \geqslant 0$ such that

$$|\mu_Y(g)| \leqslant M_K\,\|g\| \quad \text{for all } g \in \mathscr{K}(Y,K;\mathbf{C});$$

indeed, K is also a compact subset of X, its characteristic function φ_K is μ-integrable, and, for every $g \in \mathscr{K}(Y;\mathbf{C})$, one has $g' = \varphi_K\,g'$ and $|g'| \leqslant \|g\|\varphi_K$, whence (§4, No. 2, Prop. 2)

$$|\mu_Y(g)| = \left|\int \varphi_K\,g'\,d\mu\right| \leqslant \int |\varphi_K\,g'|\,d|\mu| \leqslant \int \varphi_K\,\|g\|\,d|\mu| = \|g\|\cdot|\mu|(K),$$

thus $M_K = |\mu|(K)$ meets the requirements (§4, No. 6, Cor. 1 of Prop. 10).

Addendum. Pursuing the addendum to the preceding note, suppose Y is an arbitrary subspace of X, and let $g \in \mathscr{K}_+(Y)$. As observed there, g' is lower semi-continuous on X; it is also bounded and has compact support. The argument in the preceding note shows that for every $k \in \mathbf{R}$, the set $\{x \in X :\ g'(x) \geqslant k\}$ is either equal to X or is a compact subset of X, hence is measurable for μ (No. 1, Cor. 1 of Prop. 3); therefore g' is measurable (No. 5, Prop. 8). Since every $g \in \mathscr{K}(Y;\mathbf{C})$ is a linear combination of four elements of $\mathscr{K}_+(Y)$, it follows from the linearity of $g \mapsto g'$ that g' is bounded, measurable, and has compact support; therefore $|\mu|^*(g') < +\infty$, g' is μ-integrable (No. 6, Th. 5), and one can define a linear form $\mu_Y : \mathscr{K}(Y;\mathbf{C}) \to \mathbf{C}$ by $\mu_Y(g) = \mu(g')$; and when $\mu \geqslant 0$, $g \mapsto \mu_Y(g')$ defines a positive linear form on $\mathscr{K}(Y;\mathbf{R})$.

But ... we can't call μ_Y a measure (even if $\mu \geqslant 0$), because the term 'measure' has been defined, and its properties developed, only for locally compact spaces. At any rate, the argument used for locally compact Y shows that for every compact set K in Y, there exists a constant $M_K \geqslant 0$ such that $|\mu_Y(g)| \leqslant M_K \|g\|$ for all $g \in \mathscr{K}(Y,K;\mathbf{C})$.

A theory of measures on Hausdorff spaces is developed in Chapter IX, permitting induced measures on measurable subsets (IX, §2, No. 1, Def. 1). The correspondence $g \mapsto g^0$ there seems to be the analogue of $g \mapsto g'$, but I have not explored further the relationship between the two contexts.

IV.74, $\ell.$ −13.

"(i) *For every compact* (resp. *open*) *subset* H *of* K, $\mu_K(H) = \mu(H)$."

We remark that this will imply that

$$(*) \qquad \mu_K(B) = \mu(B)$$

for every Borel set B in K, in other words, for every Borel set B in X such that $B \subset K$ (GT, IX, §6, No. 3, *Remark* 2).

For, writing $\mathfrak{B}(T)$ for the set of Borel sets in a topological space T (the tribe generated by its open sets, equivalently by its closed sets), we have

$$\mathfrak{B}(K) = \{B \in \mathfrak{B}(X) :\ B \subset K\} = \{B \cap K :\ B \in \mathfrak{B}(X)\},$$

concisely $\mathfrak{B}(K) = \mathfrak{B}(X) \cap K$. Every set in $\mathfrak{B}(X)$ (resp. $\mathfrak{B}(K)$) is μ-measurable (resp. μ_K-measurable) by No. 4, Cor. 3 of Th. 2, thus every set in $\mathfrak{B}(K)$ is both μ_K-measurable and μ-measurable; and since $\mu_K(K)$ and $\mu(K)$ are finite (moreover equal by (i)), every $B \in \mathfrak{B}(K)$ is both μ_K-integrable and μ-integrable (No. 6, Cor. 1 of Th. 5).

The set function $B \mapsto \mu_K(B)$ ($B \in \mathfrak{B}(K)$) is a 'finite measure' in the sense of Halmos (*Measure theory*, Van Nostrand, New York, 1950; reprinted by Springer-Verlag); in particular, if (B_n) is any sequence of pairwise disjoint sets in $\mathfrak{B}(K)$ then

$$\mu_K\Big(\bigcup_n B_n\Big) = \sum_n \mu_K(B_n)$$

(§4, No. 5, Prop. 9), and

$$\mu_K(A - B) = \mu_K(A - A \cap B) = \mu_K(A) - \mu_K(A \cap B)$$

for all $A, B \in \mathfrak{B}(K)$ (*loc. cit.*, Prop. 7). The same is true of the set function $B \mapsto \mu(B)$ ($B \in \mathfrak{B}(K)$). Let $\mathscr{C}$ be the set of all compact (i.e., closed) subsets of K. By (i), (∗) holds for every set in $\mathscr{C}$; since $\mathscr{C}$ is closed under finite unions and intersections, and since the tribe of subsets of K generated by $\mathscr{C}$ is $\mathfrak{B}(K)$, it follows that (∗) holds for every set in $\mathfrak{B}(K)$ (Th. 51.F in Halmos' book, p. 223; or S.K. Berberian, *Measure and integration*, Th. 2 on p. 185, Macmillan, New York, 1965, reprinted by Chelsea). The heart of the matter is that the clan generated by $\mathscr{C}$ is the set of all finite disjoint unions of sets $C - D$, where $C, D \in \mathscr{C}$ and $D \subset C$.

IV.74, ℓ. −9.

"(i) We can restrict ourselves to the case that H is compact."

Suppose the assertion proved for compact sets, so that, in particular, $\mu_K(K) = \mu(K)$, and let J be an open set in K (that is, a set of the form $U \cap K$, where U is an open set in X); then the set $H = K - J$ is compact, therefore by assumption

$$\mu_K(K - J) = \mu(K - J),$$

that is (§4, No. 5, Prop. 7),

$$\mu_K(K) - \mu_K(J) = \mu(K) - \mu(J),$$

whence $\mu_K(J) = \mu(J)$.

{It follows that $\mu_K(H) = \mu(H)$ for every Borel set H in K (see the preceding note).}

IV.74, ℓ. −5, −4.

"g'_α is upper semi-continuous"

The argument is given in the note for **IV.73,** ℓ. −2.

IV.75, ℓ. 1, 2.

"(ii) If N is μ-negligible then, for every $\varepsilon > 0$, there exists a relatively compact open neighborhood U of N in X such that $\mu(U) \leqslant \varepsilon$"

Since $\mu^*(N) = 0$, there exists an open set V in X such that $N \subset V$ and $\mu^*(V) \leqslant \varepsilon$ (§1, No. 4, Prop. 19). But $N \subset K$, where K is compact; let W be a relatively compact open set in X such that $K \subset W$. Set $U = V \cap W$; U is an open neighborhood of N, $\overline{U} \subset \overline{W}$ shows that it is relatively compact, and $\mu^*(U) \leqslant \mu^*(V) \leqslant \varepsilon$. Since U is integrable (§4, No. 6, Cor. 1 of Prop. 10), we may write $\mu(U)$ for $\mu^*(U)$.

IV.75, ℓ. 7, 8.

"(iii) For every open set U in K that intersects S, by hypothesis $\mu_K(U \cap S) \neq 0$"

Arguing contrapositively, assuming $\mu_K(U \cap S) = 0$ let us show that $U \cap S = \varnothing$. Since μ (hence also μ_K) is *positive*, $U \cap S$ is μ_K-negligible (§2, No. 2, Def. 2). On the other hand, since S is closed in K, $U \cap (K - S)$ is an open set in K that is contained in $K - S$, therefore $U \cap (K - S)$ is μ_K-negligible (*loc. cit.*, Prop. 5). It follows that the union

$$(U \cap S) \cup \big(U \cap (K - S)\big) = U \cap K = U$$

is a μ_K-negligible open set in K, consequently $U \subset K - S$ by the cited Prop. 5, that is, $U \cap S = \varnothing$.

IV.75, ℓ. 9.

" $\mu_S(U \cap S) \neq 0$ by (i)"

By the transitivity of induced topologies, $U \cap S$ is also an open set in the compact subset S of X, therefore $\mu_S(U \cap S) = \mu(U \cap S)$ by (i) with S playing the role of K.

IV.75, ℓ. 10.

"... this proves that $\mathrm{Supp}(\mu_S) = S$."

By the foregoing argument, if U is an open set in K such that $U \cap S \neq \varnothing$, one has $\mu_S(U \cap S) \neq 0$. The sets $U \cap S$ (U open in K) are the open sets in the subspace S of X (by the transitivity of induced topologies). Thus, the only μ_S-negligible open set in S is the empty set, therefore $S - \varnothing = S$ is the support of μ_S (§2, No. 2, Prop. 5).

IV.75, ℓ. 13–15.

"... by definition ..."

Ch. III, §1, No. 6, formula (12).

IV.75, ℓ. −13, −12.

"... let K be the support of f and let U be a compact neighborhood of K in X such that $|\mu|(U - K) \leqslant \varepsilon$"

Since $|\mu|(K) = |\mu|^*(K) < +\infty$ (§4, No. 6, Cor. 1 of Prop. 10), there exists an open set G such that $K \subset G$ and $|\mu|^*(G) \leqslant |\mu|(K) + \varepsilon$ (§1, No. 4, Prop. 19). For each $x \in K$, choose an open neighborhood V_x of x with $\overline{V}_x$ compact and $\overline{V}_x \subset G$; cover K by a finite number of such neighborhoods, and let V be their union. Thus $K \subset V \subset \overline{V} \subset G$; since all sets in sight are integrable (§4, No. 6, Prop. 10), $|\mu|(\overline{V} - K) \leqslant |\mu|(G - K) = |\mu|(G) - |\mu|(K) \leqslant \varepsilon$ (§4, No. 5, Prop. 7) and $U = \overline{V}$ meets the requirements.

IV.75, ℓ. −12 to −10.

"... by Urysohn's theorem, there exists a function $f_1 \in \mathscr{K}_+(X)$, extending f, with support contained in U and such that $\|f_1\| = \|f\|$."

It is straightforward to define a continuous function on X, with compact support contained in U, that agrees with f on K; the trick is to ensure the the extension agrees with f on all of Y, i.e., is equal to 0 on $\mathrm{Y} - \mathrm{K}$. The following argument is terribly long-winded. There must be a shortcut; I did not find it. (On the plus side, the argument works for any subspace Y of X.)

Theorem. Let X be a locally compact space, Y any subspace of X (not necessarily locally compact), and let $f \in \mathscr{K}(\mathrm{Y};\mathbf{R})$. Then f may be extended to a function $f_1 \in \mathscr{K}(\mathrm{X};\mathbf{R})$, such that if $[a,b]$ is the smallest closed interval in $\mathbf{R}$ containing 0 and the compact set $f(\mathrm{Y})$, then also $f_1(\mathrm{X}) \subset [a,b]$; consequently $\|f_1\| = \|f\|$, and if $f \geqslant 0$ then also $f_1 \geqslant 0$.

Moreover, if K is the (compact) support of f in Y, and U is a neighborhood of K in X, one can arrange that the support of f_1 is contained in U.

Proof. If $\mathrm{A} \subset \mathrm{Y}$, we write $\mathrm{Cl}_{\mathrm{Y}}(\mathrm{A})$ for the closure of A in Y; thus $\mathrm{Cl}_{\mathrm{Y}}(\mathrm{A}) = \overline{\mathrm{A}} \cap \mathrm{Y}$ (GT, I, §3, No. 1, Prop. 1). Since the support K of f is compact in Y, it is also compact in X, hence is closed in X.

From the local compactness of X, we know that the compact neighborhoods of K in X are basic, so we can suppose that U is compact. Note that $\mathrm{K} \subset \mathrm{U} \cap \mathrm{Y}$. Define a subset $\mathrm{Z} \subset \mathrm{X}$ by

$$\mathrm{Z} = \mathrm{K} \cup \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}}\,;$$

since $\mathrm{K} = \overline{\mathrm{K}}$, we have

$$\mathrm{Z} = \overline{\mathrm{K}} \cup \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}} = \overline{\mathrm{K} \cup (\mathrm{U} \cap \mathrm{Y} - \mathrm{K})} = \overline{\mathrm{U} \cap \mathrm{Y}} \subset \overline{\mathrm{U}} = \mathrm{U}\,,$$

thus $\mathrm{Z} = \overline{\mathrm{U} \cap \mathrm{Y}}$ is a compact subset of the compact subspace U of X.

Let us analyze the intersection of the two terms defining Z:

$$\begin{aligned} \mathrm{K} \cap \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}} &= (\mathrm{K} \cap \mathrm{Y}) \cap \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}} \\ &= \mathrm{K} \cap \big(\mathrm{Y} \cap \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}}\big) \\ &= \mathrm{K} \cap \mathrm{Cl}_{\mathrm{Y}}(\mathrm{U} \cap \mathrm{Y} - \mathrm{K})\,; \end{aligned}$$

since $f = 0$ on $\mathrm{Y} - \mathrm{K} \supset \mathrm{U} \cap \mathrm{Y} - \mathrm{K}$ and f is continuous on Y, it follows that $f = 0$ on $\mathrm{Cl}_{\mathrm{Y}}(\mathrm{U} \cap \mathrm{Y} - \mathrm{K})$, hence on $\mathrm{K} \cap \mathrm{Cl}_{\mathrm{Y}}(\mathrm{U} \cap \mathrm{Y} - \mathrm{K}) = \mathrm{K} \cap \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}}$, therefore we may define a function $f_0 : \mathrm{Z} \to \mathbf{R}$ by the formula

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{K} \\ 0 & \text{if } x \in \overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}}\,. \end{cases}$$

Since f_0 is continuous on each of its closed subsets K and $\overline{\mathrm{U} \cap \mathrm{Y} - \mathrm{K}}$, it is continuous on their union Z (GT, I, §3, No. 2, Prop. 4). One has

$K \subset Z \subset U$; the domain Z of f_0 neither visibly contains nor is contained in Y, so f_0 may not extend nor be extended by f—all we're sure of is that they agree on K.

From the definition of Z, it is clear that

$$Z - K \subset \overline{U \cap Y - K},$$

therefore $f_0 = 0$ on $Z - K$; one can therefore redescribe the continuous function $f_0 : Z \to \mathbf{R}$ by the formula

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in K \\ 0 & \text{if } x \in Z - K, \end{cases}$$

where $Z = \overline{U \cap Y}$, equipped with the topology induced by that of X (or of U).

The sets $f(Y)$ and $f_0(Z)$ can differ only by 0; for, they both contain $f(K)$ and both are contained in $f(K) \cup \{0\}$. Let $[a, b]$ be the smallest closed interval of $\mathbf{R}$ containing 0 and $f(K)$; then $[a, b]$ contains $f(Y)$ and $f_0(Z)$, $\|f\| = \|f_0\| = \max\{|a|, |b|\}$, and $f \geqslant 0 \Leftrightarrow f_0 \geqslant 0 \Leftrightarrow a \geqslant 0$. Let us regard f_0 as a function $Z \to [a, b]$.

Now, the compact space Z is a closed subset of the normal space U (GT, IX, §4, No. 1, Prop. 1); by the proof of Urysohn's theorem (*loc. cit.*, No. 2, Th. 2), there exists a continuous function $F_0 : U \to [a, b]$ such that $F_0|Z = f_0$ (a result also known as the 'Tietze extension theorem'). In particular, $F_0|K = f_0|K = f|K$.

Let V be an open set in X such that $K \subset V \subset U$. There exists a continuous function $h : X \to [0, 1]$ such that $h(x) = 1$ for $x \in K$, and $h(x) = 0$ for $x \in X - V$ (III, §1, No. 2, Lemma 1); the cited Lemma 1 provides such a function with compact support contained in V, but in any case $\mathrm{Supp}(h) \subset \overline{V} \subset \overline{U} = U$ also shows that h has compact support.

Define a function $f_1 : X \to \mathbf{R}$ by

$$f_1(x) = \begin{cases} h(x)F_0(x) & \text{if } x \in U \\ 0 & \text{if } x \in X - U. \end{cases}$$

Since $h = 0$ on $X - V \supset U - V$, we have $f_1 = 0$ on $U - V$, hence $f_1 = 0$ on

$$(X - U) \cup (U - V) = X - V;$$

thus $f_1|X - V = 0$ and $f_1|U = (h|U)F_0$ are continuous, where $X - V$ and U are closed sets in X with union X, therefore f_1 is continuous on X. Since $0 \in [a, b]$, $[a, b]$ contains the segment joining 0 with any of its points; it follows that $f_1(X) \subset [a, b]$.

Since $K \subset U$ and $h = 1$ on K, we have $f_1\big|K = F_0\big|K = f\big|K$; to show that $f_1\big|Y = f$, we need only show that $f_1 = 0$ on $Y - K$.

Let $x \in Y - K$. If $x \in X - U$, then $f_1(x) = 0$ by the definition of f_1; whereas if $x \in U$ then $x \in U \cap Y - K \subset Z$, therefore

$$f_1(x) = h(x)F_0(x) = h(x)f_0(x) = h(x) \cdot 0 = 0$$

by the definitions of f_1 and f_0.

IV.75, ℓ. −8.

"... then $h \in \mathscr{K}(Y; \mathbf{C})$ "

From $|h| \leqslant f$ one infers $\mathrm{Supp}(h) \subset \mathrm{Supp}(f) = K$, thus $\mathrm{Supp}(h)$ is a closed subset of the compact subset K of Y.

IV.75, ℓ. −8.

"... and $\mu(h_1) - \mu_Y(h) = \mu(h_1\varphi_{U-K})$ "

From $|h_1| \leqslant f_1$ one infers $\mathrm{Supp}(h_1) \subset \mathrm{Supp}(f_1) \subset U$, whence

$$h_1\varphi_{U-K} = h_1\varphi_U - h_1\varphi_K = h_1 - h_1\varphi_K\,,$$

where $h_1 \in \mathscr{K}(X; \mathbf{C})$ and $h_1\varphi_{U-K}$, $h_1\varphi_K$ are μ-integrable (No. 6, Cor. 3 of Th. 5); thus

$$\mu(h_1\varphi_{U-K}) = \mu(h_1) - \mu(h_1\varphi_K)\,,$$

where the first and third μ's are abbreviations for $\int d\mu$. The assertion is that $\mu(h_1\varphi_K) = \mu_Y(h)$. Recall that by definition $\mu_Y(h) = \mu(h')$, where $h'\big|Y = h$ and $h'\big|X - Y = 0$; thus it will suffice to show that $h_1\varphi_K = h'$. Indeed, since $f = 0$ on $Y - K$ and $|h| \leqslant f$, one has $h = 0$ on $Y - K$; therefore

$$\begin{aligned} x \in Y - K \quad &\Rightarrow \quad h_1(x)\varphi_K(x) = h(x) \cdot 0 = 0 \cdot 0 = h(x) \\ x \in K \quad &\Rightarrow \quad h_1(x)\varphi_K(x) = h(x) \cdot 1 = h(x) \\ x \in X - Y \quad &\Rightarrow \quad h_1(x)\varphi_K(x) = h_1(x) \cdot 0 = 0\,, \end{aligned}$$

thus $h_1\varphi_K$ is equal to h on $Y = (Y - K) \cup K$, and to 0 on $X - Y$, in other words $h_1\varphi_K = h'$.

IV.75, ℓ. −6.

" $|\mu(h_1) - \mu_Y(h)| \leqslant \|f\| \cdot |\mu|(U - K) \leqslant \varepsilon\,\|f\|$ "

Note that $|h_1\varphi_{U-K}| \leqslant |f_1|\,\varphi_{U-K} \leqslant \|f\|_1\,\varphi_{U-K} = \|f\|\,\varphi_{U-K}$, whence $|\mu(h_1) - \mu_Y(h)| = |\mu(h_1\varphi_{U-K})| \leqslant |\mu|(|h_1\,\varphi_{U-K}|) \leqslant \|f\| \cdot |\mu|(U - K) \leqslant \varepsilon\,\|f\|$.

IV.75, ℓ. −4.

" $|\mu|(f_1) - |\mu|_Y(f) = |\mu|(f_1\varphi_{U-K})$ and $\big||\mu|(f_1) - |\mu|_Y(f)\big| \leqslant \varepsilon\,\|f\|$ "

We know that $f_1 \in \mathscr{K}_+(\mathrm{X})$ and $\mathrm{Supp}(f_1) \subset \mathrm{U}$, therefore

$$f_1\varphi_{\mathrm{U}-\mathrm{K}} = f_1\varphi_{\mathrm{U}} - f_1\varphi_{\mathrm{K}} = f_1 - f_1\varphi_{\mathrm{K}}\,,$$

whence $|\mu|(f_1\varphi_{\mathrm{U}-\mathrm{K}}) = |\mu|(f_1) - |\mu|(f_1\varphi_{\mathrm{K}})$. To prove the first assertion, we need only show that $f_1\varphi_{\mathrm{K}} = f'$. Indeed, since $f_1\big|\mathrm{Y} = f$ and $f = 0$ on $\mathrm{Y} - \mathrm{K}$, one has

$$\begin{aligned}
x \in \mathrm{Y} - \mathrm{K} \quad &\Rightarrow \quad f_1(x)\varphi_{\mathrm{K}}(x) = f(x)\cdot 0 = 0 \cdot 0 = f(x)\\
x \in \mathrm{K} \quad &\Rightarrow \quad f_1(x)\varphi_{\mathrm{K}}(x) = f(x)\cdot 1 = f(x)\\
x \in \mathrm{X} - \mathrm{Y} \quad &\Rightarrow \quad f_1(x)\varphi_{\mathrm{K}}(x) = f_1(x)\cdot 0 = 0\,,
\end{aligned}$$

thus $f_1\varphi_{\mathrm{K}}$ is equal to f on Y, and to 0 on $\mathrm{X} - \mathrm{Y}$, in other words $f_1\varphi_{\mathrm{K}} = f'$.

The first assertion and $f_1\varphi_{\mathrm{U}-\mathrm{K}} \geqslant 0$ imply that $|\mu|(f_1) - |\mu|_{\mathrm{Y}}(f) \geqslant 0$; finally, from $0 \leqslant f_1\varphi_{\mathrm{U}-\mathrm{K}} \leqslant \|f_1\|\varphi_{\mathrm{U}-\mathrm{K}} = \|f\|\varphi_{\mathrm{U}-\mathrm{K}}$, we infer that

$$\big||\mu|(f_1) - |\mu|_{\mathrm{Y}}(f)\big| = |\mu|(f_1\varphi_{\mathrm{U}-\mathrm{K}}) \leqslant \|f\|\cdot|\mu|(\mathrm{U}-\mathrm{K}) \leqslant \varepsilon\,\|f\|\,.$$

IV.75, ℓ. −2.

“ $|\mu|_{\mathrm{Y}}(f) \leqslant |\mu_{\mathrm{Y}}(h)| + \varepsilon(1 + 2\|f\|) \leqslant |\mu_{\mathrm{Y}}|(f) + \varepsilon(1+2\|f\|)$ ”

In slow motion: the inequalities

$$\begin{gathered}
\big||\mu|(f_1) - |\mu|_{\mathrm{Y}}(f)\big| \leqslant \varepsilon\|f\|\,,\\
|\mu|(f_1) \leqslant |\mu(h_1)| + \varepsilon\,,\\
|\mu(h_1)| \leqslant |\mu(h_1) - \mu_{\mathrm{Y}}(h)| + |\mu_{\mathrm{Y}}(h)| \leqslant \varepsilon\,\|f\| + |\mu_{\mathrm{Y}}(h)|\,,\\
|\mu_{\mathrm{Y}}(h)| \leqslant |\mu_{\mathrm{Y}}|(|h|) \leqslant |\mu_{\mathrm{Y}}|(f)
\end{gathered}$$

yield, respectively,

$$\begin{aligned}
|\mu|_{\mathrm{Y}}(f) \leqslant |\mu|(f_1) + \varepsilon\|f\| &\leqslant |\mu|(h_1) + \varepsilon + \varepsilon\|f\|\\
&\leqslant (\varepsilon\|f\| + |\mu_{\mathrm{Y}}(h)|) + \varepsilon + \varepsilon\|f\|\\
&= |\mu_{\mathrm{Y}}(h)| + \varepsilon(1 + 2\|f\|)\\
&\leqslant |\mu_{\mathrm{Y}}|(f) + \varepsilon(1+2\|f\|)\,.
\end{aligned}$$

IV.76, ℓ. 18.

“It is immediate (No. 2, Prop. 5) that *d*) implies *a*) ”

Assuming *d*), we are to show that for a set $\mathrm{B} \subset \mathrm{A}$,

$$\mathrm{B} \text{ is locally negligible} \quad \Leftrightarrow \quad \mathrm{B} \cap \mathrm{K} \text{ is negligible for every } \mathrm{K} \in \mathfrak{K}\,.$$

Proof of ⇒: Indeed, if B is *any* locally negligible set in X, then $B \cap K$ is negligible for *every* compact set K in X by the cited Prop. 5.

Proof of ⇐: Assume B has the indicated property; according to the criterion of the cited Prop. 5, we are to show that $B \cap K$ is negligible for *every* compact set K in X.

Suppose first that K is a compact subset of A. By *d*) we may write $K = N \cup \bigcup_n H_n$ with N negligible and (H_n) a sequence of sets in $\mathfrak{K}$; then

$$B \cap K = (B \cap N) \cup \bigcup_n (B \cap H_n)\,,$$

where $B \cap N$ is negligible and the $B \cap H_n$ are negligible by the assumption on B, therefore $B \cap K$ is negligible.

Now let K be *any* compact set in X. Since A is measurable, one can write $A \cap K = N \cup \bigcup_n K_n$ with N negligible and (K_n) a sequence of compact sets (see the *Lemma* in the note for **IV.67,** $\ell.\ -1$); since $B \subset A$, one has

$$B \cap K = (B \cap A) \cap K = B \cap (A \cap K) = (B \cap N) \cup \bigcup_n (B \cap K_n)\,,$$

where the $B \cap K_n$ are negligible by the preceding paragraph, therefore $B \cap K$ is negligible.

IV.76, $\ell.\ -19, -18$.

"...one defines recursively a sequence (H_p) of sets of $\mathfrak{K}$ such that $H_{n+1} \subset B - \bigcup_{p \leqslant n} H_p$ and $|\mu|(B - \bigcup_{p \leqslant n} H_p) \leqslant 1/n$ (§4, No. 6, Th. 4)."

By *b*), choose $H_1 \in \mathfrak{K}$ with $H_1 \subset B$ and $|\mu|(B - H_1) \leqslant 1$. Since $B - H_1$ is integrable, there exists a compact set $B_1 \subset B - H_1$ such that $|\mu|\big((B - H_1) - B_1\big) \leqslant 1/4$ by Cor. 1 of the cited Th. 4; if, by *b*), $H_2 \in \mathfrak{K}$ is chosen so that $H_2 \subset B_1$ and $|\mu|(B_1 - H_2) \leqslant 1/4$, then since $H_2 \subset B_1 \subset B - H_1$, so that

$$(B - H_1) - H_2 = \big((B - H_1) - B_1\big) \cup (B_1 - H_2)\,,$$

it follows that

$$|\mu|\big((B - H_1) - H_2\big) \leqslant 1/4 + 1/4\,,$$

that is, $|\mu|\big(B - (H_1 \cup H_2)\big) \leqslant 1/2$.

Suppose $H_1, H_2, \ldots, H_n$ in $\mathfrak{K}$ already constructed satisfying the relevant inclusions and inequalities, in particular

$$H_n \subset B - \bigcup_{p \leqslant n-1} H_p \quad \text{and} \quad |\mu|\Big(B - \bigcup_{p \leqslant n} H_p\Big) \leqslant \frac{1}{n}\,.$$

Then $\mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p$ is integrable; choose a compact set $\mathrm{B}_n \subset \mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p$ such that

$$|\mu|\Big(\big(\mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p\big) - \mathrm{B}_n\Big) \leqslant \frac{1}{2(n+1)},$$

then, by b), a set $\mathrm{H}_{n+1} \in \mathfrak{K}$ such that $\mathrm{H}_{n+1} \subset \mathrm{B}_n$ and

$$|\mu|(\mathrm{B}_n - \mathrm{H}_{n+1}) \leqslant \frac{1}{2(n+1)}\,.$$

Then $\mathrm{H}_{n+1} \subset \mathrm{B}_n \subset \mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p$, thus

$$\big(\mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p\big) - \mathrm{H}_{n+1} = \Big(\big(\mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p\big) - \mathrm{B}_n\Big) \cup \big(\mathrm{B}_n - \mathrm{H}_{n+1}\big),$$

therefore

$$|\mu|\Big(\big(\mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p\big) - \mathrm{H}_{n+1}\Big) \leqslant \frac{1}{2(n+1)} + \frac{1}{2(n+1)}\,,$$

that is, $|\mu|\big(\mathrm{B} - \bigcup_{p \leqslant n+1} \mathrm{H}_p\big) \leqslant \frac{1}{n+1}$, which completes the induction.

Set $\mathrm{N} = \mathrm{B} - \bigcup_{p=1}^{\infty} \mathrm{H}_p$; then $\mathrm{N} \subset \mathrm{B} - \bigcup_{p \leqslant n} \mathrm{H}_p$ for all n, therefore $|\mu|^*(\mathrm{N}) \leqslant 1/n$ for all n, thus N is negligible.

IV.76, ℓ. −17.

"It remains to prove that a) implies b)."

In view of $|\mu|(\mathrm{K}_0 - \mathrm{K}) = |\mu|(\mathrm{K}_0) - |\mu|(\mathrm{K})$ (§4, No. 5, Prop. 7), the meaning of b) is that, for every compact subset K_0 of A,

$$|\mu|(\mathrm{K}_0) = \sup_{\mathrm{K} \in \mathfrak{K},\, \mathrm{K} \subset \mathrm{K}_0} |\mu|(\mathrm{K})\,.$$

The proof by contradiction ultimately rests on the fact that the measure (for $|\mu|$) of an integrable set is the supremum of the measures of its compact subsets (§4, No. 6, Cor. 1 of Th. 4).

IV.76, ℓ. −13, −12.

" B is integrable and $|\mu|(\mathrm{B}) = \alpha$ "

Because (L_n) is an increasing sequence of integrable sets such that the sequence $|\mu|(\mathrm{L}_n)$ has a finite supremum α (§4, No. 5, Prop. 8).

IV.76, ℓ. −11, −10.

"$|\mu|\big(K \cap (K_0 - B)\big) = 0$, which, by virtue of *a*), will imply a contradiction."

It will imply, by *a*), that $K_0 - B$ is locally negligible; but $|\mu|^*(K_0 - B)$ is finite, therefore $K_0 - B$ is negligible (No. 2, Cor. 1 of Prop. 5).

IV.77, ℓ. 1.

"The set of *all* compact subsets of A is μ-dense in A."

The set $\mathfrak{K}$ of all compact subsets of A clearly satisfies (PL_I) and (PL_{II}); let us verify that $\mathfrak{K}$ satisfies the condition *c*) of Prop. 12. Since A is measurable, that is, φ_A is a measurable function on X (No. 1, Def. 2), given any compact subset K of X there exists a sequence (K_n) of pairwise disjoint compact subsets of K such that the set $N = K - \bigcup_n K_n$ is negligible (No. 1, Def. 1; the continuity properties of φ_A are of no interest here); if, in particular, $K \subset A$, that is, $K \in \mathfrak{K}$, then also $K_n \in \mathfrak{K}$ for all n.

In this sense, a measurable subspace of a locally compact space does have an abundance of compact subsets (cf. the *Addendum* to the note for IV.73, ℓ. −2).

IV.77, ℓ. 3, 4.

"If $X - A$ is locally μ-negligible, then every set of compact subsets of A that is μ-dense in A is also μ-dense in X."

Since $X - A$ is measurable (see the note for **IV.61,** ℓ. −4), so is A. Let $\mathfrak{K}$ be a set of compact subsets of A that is μ-dense in A; in particular, the condition *a*) of Prop. 12 is valid for A. We will show that $\mathfrak{K}$ is μ-dense in X by verifying that X satisfies the criterion *a*) of Prop. 12: given any subset B of X such that $B \cap K$ is negligible for every $K \in \mathfrak{K}$, we are to show that B is locally negligible. Now,

$$B = (B - A) \cup (B \cap A)\,;$$

we show that both terms of the union are locally negligible. Since $X - A$ is locally negligible, so is its subset $B - A$. On the other hand, for every $K \in \mathfrak{K}$, the set

$$(B \cap A) \cap K = B \cap (A \cap K) = B \cap K$$

is by assumption negligible, therefore $B \cap A$ is locally negligible by the μ-density of $\mathfrak{K}$ in A and its property *a*) in Prop. 12.

IV.77, ℓ. 10–12.

"If K is a compact subset of X, it comes to the same to say that a set of compact subsets of K is μ-dense in K or that it is μ_K-*dense* in K; this follows from Lemmas 2 and 3 of No. 7 and condition *b*) of Prop. 12."

If $K_0 \subset K$ is compact in the subspace K of X (hence is compact in X) and if $J \in \mathfrak{K}$, $J \subset K_0$, then

$$\begin{aligned} |\mu|(K_0 - J) &= |\mu|(K_0) - |\mu|(J) \\ &= |\mu|_K(K_0) - |\mu|_K(J) \qquad \text{(Lemma 2, (i))} \\ &= |\mu_K|(K_0) - |\mu_K|(J) \qquad \text{(Lemma 3)} \\ &= |\mu_K|(K_0 - J)\,; \end{aligned}$$

in particular, $|\mu|(K_0 - J) \leqslant \varepsilon \Leftrightarrow |\mu_K|(K_0 - J) \leqslant \varepsilon$ and the assertion is now clear from the criterion *b*) of Lemma 12.

IV.77, ℓ. -8 to -6.

"Def. 7 shows that the *union* of a locally countable set of μ-measurable (resp. locally μ-negligible) subsets of a locally compact space is μ-measurable (resp. locally μ-negligible) (No. 1, Prop. 3 and No. 2, Prop. 5)."

Let $\mathfrak{A}$ be a locally countable set of subsets of the locally compact space X equipped with a measure μ; present $\mathfrak{A}$ as a family $(A_\alpha)_{\alpha\in I}$, faithfully indexed by I, that is, $\alpha \mapsto A_\alpha$ is an injection $I \to \mathfrak{P}(X)$ with range $\mathfrak{A}$. Let $A = \bigcup_{\alpha\in I} A_\alpha$.

For each $x \in X$, choose an open neighborhood U_x of x in X such that the set

$$I_x = \{\alpha \in I :\ U_x \cap A_\alpha \neq \varnothing\}$$

is countable; in particular, $U_x \cap A_\alpha = \varnothing$ when $\alpha \notin I_x$, thus

$$U_x \cap A = U_x \cap \bigcup_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} U_x \cap A_\alpha = \bigcup_{\alpha\in I_x} U_x \cap A_\alpha\,,$$

where the index set I_x is countable.

Assuming the A_α are measurable (resp. locally negligible), we are to show that A is measurable (resp. locally negligible). Note that when the A_α are locally negligible, they are also measurable (see the note for IV.61, ℓ. -4).

Given any compact subset K of X, it will suffice by No. 1, Prop. 3 (resp. No. 2, Prop. 5) to show that $K \cap A$ is integrable (resp. negligible).

With the open sets U_x chosen as above, the U_x $(x \in K)$ form an open covering of K, hence

$$K \subset U_{x_1} \cup \cdots \cup U_{x_n}$$

for a suitable finite set of points $x_1, \dots, x_n$ of K. Then $K = \bigcup_{i=1}^{n} K \cap U_{x_i}$, therefore

$$K \cap A = \bigcup_{i=1}^{n} K \cap U_{x_i} \cap A\,.$$

For each i the set,

$$\mathrm{K} \cap \mathrm{U}_{x_i} \cap \mathrm{A} = \mathrm{K} \cap \bigcup_{\alpha \in \mathrm{I}_{x_i}} \mathrm{A}_\alpha = \bigcup_{\alpha \in \mathrm{I}_{x_i}} \mathrm{K} \cap \mathrm{A}_\alpha$$

is a countable union of measurable sets, hence is measurable; being contained in K, it is integrable (No. 6, Cor. 1 of Th. 5). Thus $\mathrm{K} \cap \mathrm{A}$ is a finite union of integrable sets, hence is integrable. This shows that A is measurable.

If, moreover, the A_α are locally negligible, then every $\mathrm{K} \cap \mathrm{A}_\alpha$ is negligible, hence each $\mathrm{K} \cap \mathrm{U}_{x_i} \cap \mathrm{A}$ is a countable union of negligible sets, therefore so is $\mathrm{K} \cap \mathrm{A}$, thus $\mathrm{K} \cap \mathrm{A}$ is negligible. This shows that A is locally negligible.

IV.77, ℓ. −5 to −1.

"PROPOSITION 14."

For an application, see *Théories spectrales*, Ch. II, §3, No. 3, Lemma 3 (p. 146, item 4)) and Prop. 5, (iii).

A nontrivial special case ($\mathrm{A} = \mathrm{X}$ and $\mathfrak{K}$ the set of all compact subsets of X; cf. IV.77, ℓ. 1): There exists a locally countable set $\mathfrak{H}$ of pairwise disjoint compact sets such that $\mathrm{X} - \bigcup_{\mathrm{K} \in \mathfrak{H}} \mathrm{K}$ is locally negligible.

IV.78, ℓ. 13.

" $|\mu|(\mathrm{K} \cap \mathrm{V}) > 0$ for every $\mathrm{K} \in \mathfrak{H}_\mathrm{V}$ "

The set $\mathrm{K} \cap \mathrm{V}$ is a nonempty open set in the subspace K of X, and $\mathrm{Supp}(\mu_\mathrm{K}) = \mathrm{K}$, therefore $|\mu_\mathrm{K}|(\mathrm{K} \cap \mathrm{V}) > 0$ by §2, No. 2, Prop. 5; and $|\mu_\mathrm{K}|(\mathrm{K} \cap \mathrm{V}) = |\mu|_\mathrm{K}(\mathrm{K} \cap \mathrm{V}) = |\mu|(\mathrm{K} \cap \mathrm{V})$ by Lemmas 3 and 2 of No. 7.

IV.78, ℓ. 15.

" N is μ-measurable"

The set $\bigcup_{\mathrm{K} \in \mathfrak{H}} \mathrm{K}$ is measurable (see the note for IV.77, ℓ. −8 to −6), and the difference of two measurable sets is measurable.

IV.78, ℓ. 15, 16.

"If N were not locally negligible, it would contain a non-negligible compact set L_0 "

If N is not locally negligible then there exists a compact set K in X such that $\mathrm{K} \cap \mathrm{N}$ is not negligible (No. 2, Prop. 5). Since $\mathrm{K} \cap \mathrm{N}$ is measurable and has finite exterior measure ($\leqslant |\mu|(\mathrm{K})$), it follows that $\mathrm{K} \cap \mathrm{N}$ is integrable (No. 6, Cor. 1 of Th. 5); thus $|\mu|(\mathrm{K} \cap \mathrm{N}) = |\mu|^*(\mathrm{K} \cap \mathrm{N}) > 0$, therefore there exists a compact set $\mathrm{L}_0 \subset \mathrm{K} \cap \mathrm{N}$ such that $|\mu|(\mathrm{L}_0) > 0$ by 'inner regularity' of $|\mu|$ (§4, No. 6, Cor. 1 of Th. 4).

IV.79, ℓ. 3.

" *d*) implies *c*)"

Suppose $f : \mathrm{A} \to \mathrm{F}$ satisfies d); let $y \in \mathrm{F}$ be a point such that the function $\overline{f} : \mathrm{X} \to \mathrm{F}$ defined by

$$\overline{f}(x) = \begin{cases} f(x) & \text{for } x \in \mathrm{A} \\ y & \text{for } x \in \mathrm{X} - \mathrm{A} \end{cases}$$

is μ-measurable. (By hypothesis, y can be any point of F; the existence of a single such point is sufficient to prove that c) is satisfied.)

Let $\mathrm{G} = \mathrm{F}$, $g = \overline{f} : \mathrm{X} \to \mathrm{F} = \mathrm{G}$, and let j be the identity homeomorphism $\mathrm{F} \to \mathrm{F} = \mathrm{G}$. For every $x \in \mathrm{A}$,

$$g(x) = \overline{f}(x) = f(x) = j\big(f(x)\big) = (j \circ f)(x)\,,$$

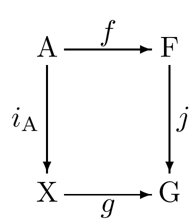
thus $g\big|\mathrm{A} = j \circ f$ as mappings $\mathrm{A} \to \mathrm{G}$.

IV.79, ℓ. 3, 4.

"The fact that c) implies a) follows from condition c) of Prop. 12 of No. 8."

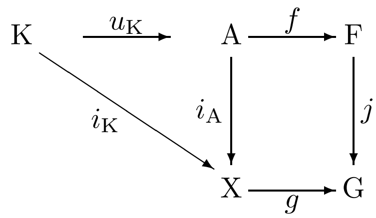
One assumes that there exist a topological space G, 'a homeomorphism j of F onto a subspace of G,' and a μ-measurable mapping $g : \mathrm{X} \to \mathrm{G}$ such that $g\big|\mathrm{A} = j \circ f$.

What the author means by the phrase in quotes is that $j : \mathrm{F} \to \mathrm{G}$ is a mapping such that, if the range $j(\mathrm{F})$ of j is equipped with the topology induced by G, and if $j_0 : \mathrm{F} \to j(\mathrm{F})$ is the function having the graph of j, then j_0 is a homeomorphism. It follows that j is injective (obvious) and that j is continuous; for, if $i : f(\mathrm{F}) \to \mathrm{G}$ is the canonical injection, then i is continuous ($j(\mathrm{F})$ has the initial topology for i) hence so is the composite function $j = i \circ j_0$. Since j_0 is a homeomorphism, it is trivial that F has the initial topology for j_0; it then follows that F has the initial topology for $j = i \circ j_0$ ('transitivity of initial topologies', GT, I, §2, No. 3, Prop. 5). Indeed, for a mapping $j : \mathrm{F} \to \mathrm{G}$ to have the property that j_0 is a homeomorphism, it is necessary and sufficient that j *be injective and that* F *have the initial topology for* j; for, if the condition in italics holds then j_0 is bijective and, by the cited Prop. 5, F has the initial topology for j_0, consequently j_0 is a homeomorphism. The further assumption that $g\big|\mathrm{A} = j \circ f$ is that the diagram



is commutative, where $i_A : A \to X$ is the canonical injection.

Consider now any subset K of A, and let $u_K : K \to A$ and $i_K : K \to X$ be the canonical injections, so that $i_K = i_A \circ u_K$ as mappings $K \to X$. Then the diagram



is commutative; in particular,

$$(j \circ f) \circ u_K = (g \circ i_A) \circ u_K = g \circ (i_A \circ u_K) = g \circ i_K .$$

Note that if K is given the topology induced by X (equivalently by A), then

$$f\big|K \text{ is continuous} \quad \Leftrightarrow \quad g\big|K \text{ is continuous;}$$

for, since F has the initial topology for j, the following conditions are equivalent:

$$f\big|K = f \circ u_K \quad \text{is a continuous mapping } K \to F$$
$$j \circ (f \circ u_K) \quad \text{is a continuous mapping } K \to G$$

and since $j \circ (f \circ u_K) = (j \circ f) \circ u_K = g \circ i_K = g\big|K$, the assertion is proved. {One observes that the injectivity of j is not needed for this argument; it is sufficient that F have the initial topology for j.}

Now let $\mathfrak{H}$ be the set of all compact subsets K of A such that $f\big|K$ is continuous (equivalently, $g\big|K$ is continuous); we are to show that $\mathfrak{H}$ is μ-dense in A.

It is clear the $\mathfrak{H}$ is closed under finite unions (GT, I, §3, No. 2, Prop. 4), thus satisfies (PL_{II}) of Prop. 12, and (PL_I) is obviously satisfied; to complete the proof, it will suffice to show that $\mathfrak{H}$ satisfies the condition c) of Prop. 12 (No. 8, Def. 6) stated for $\mathfrak{H}$ instead of $\mathfrak{K}$.

To that end, let B be a compact subset of A. Since $g : X \to G$ is μ-measurable, one can write

$$B = N \cup \bigcup_n K_n ,$$

where (K_n) is a sequence of pairwise disjoint compact sets such that $N = B - \bigcup_n K_n$ is μ-negligible and $g\big|K_n$ is continuous for all n (No. 1, Def. 1); therefore $f|K_n$ is continuous for all n, thus $K_n \in \mathfrak{H}$ for all n, which completes the proof that $\mathfrak{H}$ is μ-dense in A, and hence that the condition a) of Prop. 15 is implied by its condition c).

{One observes that the condition a) of Prop. 15 is implied by the following condition weaker than c):

c') *There exist a topological space* G*, a mapping* $j : F \to G$ *such that* F *has the initial topology for* j*, and a* μ*-measurable mapping* g *of* X *into* G*, such that* $g|A = j \circ f$.

In particular, condition a) of Prop. 15 is implied by the following special case of c), namely, that f may be extended to a μ-measurable mapping $X \to F$:

c'') *There exists a* μ*-measurable mapping* $g : X \to F$ *such that* $g|A = f$.

For, if c'') is satisfied, then c') is satisfied with $G = F$ and $j : F \to F$ the identity mapping.

Thus, after the proof of Prop. 15 is completed, one can add c') and c'') to the list of equivalent conditions. See also the last paragraph of the note for IV.79, ℓ. 7 for a condition d') weaker than d) that can be added to the list.}

IV.79, ℓ. 5.

"b) implies a)"

By assumption, $\mathfrak{K}$ is a set of compact subsets of A that is μ-dense in A, such that for every $K \in \mathfrak{K}$, the function $f|K$ is μ_K-measurable.

Let $\mathfrak{H}$ be the set of all compact subsets H of A such that $f|H$ is continuous; we are to show that $\mathfrak{H}$ is μ-dense in A. We know that $\mathfrak{H}$ satisfies (PL_I) and (PL_{II}) (cf. the preceding note); by Prop. 13, it will suffice to show that for every $K \in \mathfrak{K}$, the set $\mathfrak{H}_K = \{H \in \mathfrak{H} : H \subset K\}$ is μ_K-dense in K. (Incidentally, one sees easily that $\mathfrak{H}_K = \{H \cap K : H \in \mathfrak{H}\}$.)

Let $K \in \mathfrak{K}$. Since $\mathfrak{H}$ satisfies the conditions (PL_I) and (PL_{II}), it is immediate that $\mathfrak{H}_K$ also satisfies them. Consider Prop. 12 with K playing the role of both X and A, and μ_K playing the role of μ; in this context, let us show that $\mathfrak{H}_K$ satisfies the property c) of Prop. 12 (hence is μ_K-dense in K). To this end, let B be any compact subset of K. Since $K \in \mathfrak{K}$, $f|K$ is by assumption μ_K-measurable, therefore (No. 1, Def. 1) one can write

$$B = N \cup \bigcup_n H_n ,$$

where (H_n) is a sequence of pairwise disjoint compact sets in B such that $f|H_n = (f|K)|H_n$ is continuous for every n, and $N = B - \bigcup_n H_n$ is μ_K-negligible; in particular, $H_n \in \mathfrak{H}_K$, thus the desired property c) is verified.

IV.79, ℓ. 7.

"a) implies d)."

Assuming that the set $\mathfrak{H} = \{K \subset A : \ K \text{ compact}, \ f|K \text{ continuous}\}$ is μ-dense in A, we are to show that, given any $y \in F$, the extension of f to X by y, that is, the function $g : X \to F$ defined by

$$g(x) = \begin{cases} f(x) & \text{for } x \in A \\ y & \text{for } x \in X - A, \end{cases}$$

is μ-measurable. By No. 1, Prop. 1 it will suffice to show that, given any compact set $L \subset X$ and any $\varepsilon > 0$, there exists a compact set $K \subset L$ such that $|\mu|(L - K) \leqslant \varepsilon$ and $g|K$ is continuous.

The sets $L \cap A$ and $L \cap (X - A)$ are μ-integrable, since the intersection of an integrable set and a measurable set is integrable (No. 6, Cor. 3 of Th. 5). Therefore there exist compact sets $P \subset L \cap A$ and $Q \subset L \cap (X - A)$ such that

$$|\mu|\big((L \cap A) - P\big) \leqslant \varepsilon/4 \quad \text{and} \quad |\mu|\Big(\big(L \cap (X - A)\big) - Q\Big) \leqslant \varepsilon/4$$

(§4, No. 6, Cor. 1 of Th. 4). Since P is a compact subset of A, and $\mathfrak{H}$ is μ-dense in A, by b) of Prop. 12 there exists a set $H \in \mathfrak{H}$ such that $H \subset P$ and

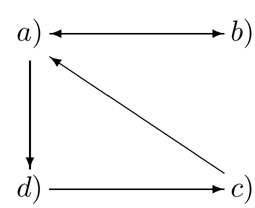
$$|\mu|(P - H) \leqslant \varepsilon/2\,.$$

Set $K = H \cup Q$, which is a compact subset of L; since $g|H = (g|A)|H = f|H$ is continuous (because $H \in \mathfrak{H}$) and $g|Q$ is continuous (even constant), it follows that $g|K$ is continuous (GT, I, §3, No. 2, Prop. 4). Finally, one has

$$\begin{aligned} L - K &= \big[(L \cap A) \cup \big(L \cap (X - A)\big)\big] - (H \cup Q) \\ &\subset (L \cap A - H) \cup \Big(\big(L \cap (X - A)\big) - Q\Big) \\ &= (L \cap A - P) \cup (P - H) \cup \Big(\big(L \cap (X - A)\big) - Q\Big), \end{aligned}$$

whence $|\mu|(L - K) \leqslant \varepsilon/4 + \varepsilon/2 + \varepsilon/4$ by the subadditivity of $|\mu|$ (§4, No. 5, Prop. 6).

An outline of the proof:



(where the arrows signify implication). The argument for ' $d) \Rightarrow c)$ ' shows that if there exists at least one element $y \in \mathrm{F}$ such that the extension g of f to X by y is measurable, then c) holds, whence a) and therefore d) hold; thus, the existence of an element $y \in \mathrm{F}$ for which g is measurable implies that for *every* $y' \in \mathrm{F}$ the extension g' of f to X by y' is measurable. {This is obvious when F is a topological vector space, since $g' - g$ is then a measurable step function (No. 5) and $g' = (g' - g) + g$ (No. 3, Cor. 3 of Th. 1), but it is not obvious when F is merely a topological space.} Thus, with the proof of Prop. 15 completed, one can add to its list of equivalent conditions the following:

d′) There exists an element $y_0 \in \mathrm{F}$ *such that the extension of* f *to* X *by* y_0 *is* μ*-measurable.*

IV.79, ℓ. −17, −16.

"If A is locally μ-negligible, then *every* mapping of A into F is therefore μ-measurable."

Let $f : \mathrm{A} \to \mathrm{F}$, where A is locally μ-negligible. Then A is μ-measurable (see the note for IV.61, ℓ. −4), and it will suffice to show that the condition a) of Prop. 15 is satisfied. Let $\mathfrak{H}$ be the set of all compact subsets K of A such that $f\big|\mathrm{K}$ is continuous (for example, every finite subset of A belongs to $\mathfrak{H}$). One knows that $\mathfrak{H}$ satisfies $(\mathrm{PL_I})$ and $(\mathrm{PL_{II}})$ of Prop. 12; to show that $\mathfrak{H}$ is μ-dense in A, it will suffice to verify that $\mathfrak{H}$ satisfies the condition c) of Prop. 12. Let B be a compact subset of A; then $\mathrm{B} = \mathrm{A} \cap \mathrm{B}$ is negligible (No. 2, Prop. 5), so setting $\mathrm{N} = \mathrm{B}$ and $\mathrm{K}_1 = \varnothing$ one has a partition $\mathrm{B} = \mathrm{N} \cup \mathrm{K}_1$ with $\mathrm{K}_1 \in \mathfrak{H}$ trivially. (The same argument shows that if $\mathrm{A} \subset \mathrm{X}$ is locally μ-negligible then the set $\mathfrak{H}_0 = \{\varnothing\}$ is μ-dense in A, so *a fortiori* $\mathfrak{H}$ is μ-dense in A.)

IV.79, ℓ. −4.

"... the conclusion therefore follows from Prop. 12 of No. 8."

Recall that $\mathfrak{H} = \{\mathrm{K} \in \mathfrak{K} : \ \mathrm{K} \subset \mathrm{A} \text{ and } f\big|\mathrm{K} \text{ is continuous}\}$; the conclusion in question is that $\mathfrak{H}$ is μ-dense in A. Clearly $\mathfrak{H}$ satisfies $(\mathrm{PL_I})$ and $(\mathrm{PL_{II}})$ of Prop. 12; it will suffice to show that it satisfies the condition b) of that proposition. {The proof requires only that $\mathfrak{K}$ be μ-dense in A.}

Thus, given any compact set $\mathrm{K}_0 \subset \mathrm{A}$ and any $\varepsilon > 0$, we seek a set $\mathrm{K} \in \mathfrak{H}$ such that $\mathrm{K} \subset \mathrm{K}_0$ and $|\mu|(\mathrm{K}_0 - \mathrm{K}) \leqslant \varepsilon$. Since $\mathfrak{K}$ is μ-dense in A, by b) of Prop. 12 there exists a set $\mathrm{K}_1 \in \mathfrak{K}$ such that $\mathrm{K}_1 \subset \mathrm{K}_0$ and $|\mu|(\mathrm{K}_0 - \mathrm{K}_1) \leqslant \varepsilon/2$.

On the other hand since $f : \mathrm{A} \to \mathrm{F}$ is measurable, by Def. 8 and d) of Prop. 15, the function $g : \mathrm{X} \to \mathrm{F}$ obtained by extending f to X by an (any) element of F is measurable. Then (No. 1, Prop. 1) there exists a compact set $\mathrm{K} \subset \mathrm{K}_1$ such that $|\mu|(\mathrm{K}_1 - \mathrm{K}) \leqslant \varepsilon/2$ and $g\big|\mathrm{K}$ is continuous;

since $K \subset K_1 \in \mathfrak{K}$ one has $K \in \mathfrak{K}$, and since $f|K = (g|A)|K = g|K$ is continuous, one has $K \in \mathfrak{H}$. Finally, since $K \subset K_1 \subset K_0$ one infers from $K_0 - K = (K_0 - K_1) \cup (K_1 - K)$ that $|\mu|(K_0 - K) \leqslant \varepsilon/2 + \varepsilon/2$.

IV.80, ℓ. 1, 2.

"In view of Lemma 2 of No. 7, this follows at once from Prop. 1 of No. 1 and condition *a*) of Prop. 15."

Let $\mathfrak{H}$ be the set of compact subsets H of K such that $f|H$ is continuous. One knows that

$$\mathfrak{H} \text{ is } \mu\text{-dense in } K \quad \Leftrightarrow \quad \mathfrak{H} \text{ is } \mu_K\text{-dense in } K$$

(see the note for IV.77, ℓ. 10–12). We are to show that

$$f \text{ is } \mu\text{-measurable} \quad \Leftrightarrow \quad f \text{ is } \mu_K\text{-measurable}$$

(the statement on the left, in the sense of Def. 8; the statement on the right, in the sense of No. 1, Def. 1 applied in the compact space K).

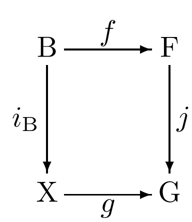
Proof of $\Rightarrow$: Suppose f is μ-measurable. By criterion *a*) of Prop. 15, we know that $\mathfrak{H}$ is μ-dense in K, hence is μ_K-dense in K; from the characterization of density in *b*) of Prop. 12 (applied to the measurable subset K of the compact space K), it follows that f is μ_K-measurable (No. 1, Prop. 1, applied in the space K).

Proof of $\Leftarrow$: Suppose f is μ_K-measurable. By No. 1, Prop. 1 and *b*) of Prop. 12 (both applied in the space K) we see that $\mathfrak{H}$ is μ_K-dense in K, hence μ-dense in K, therefore f is μ-measurable by *a*) of Prop. 15.

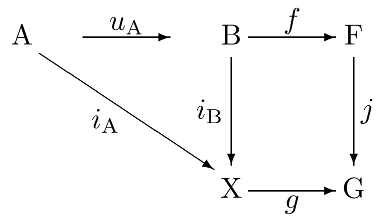
IV.80, ℓ. 7, 8.

"The condition being obviously necessary . . . "

Suppose $f : B \to F$ is μ-measurable (in the sense of Def. 8). Fix $A \in \mathfrak{A}$; we are to show that $f|A$ is μ-measurable. Applying to B and f the criterion *c*) of Prop. 15, there exist a topological space G, an injective mapping $j : F \to G$ such that F has the initial topology for j, and a μ-measurable mapping $g : X \to G$ such that $g|B = j \circ f$ (see the note for IV.79, ℓ. 3,4); thus the diagram



is commutative, where $i_{\mathrm{B}} : \mathrm{B} \to \mathrm{X}$ is the canonical injection. It follows that the diagram



is also commutative, where u_{A} and i_{A} are the canonical injections $\mathrm{A} \to \mathrm{B}$ and $\mathrm{A} \to \mathrm{X}$. In particular,

$$g\big|\mathrm{A} = g \circ i_{\mathrm{A}} = j \circ f \circ u_{\mathrm{A}} = j \circ (f\big|\mathrm{A})\,,$$

therefore $f\big|\mathrm{A}$ is μ-measurable by criterion c) of Prop. 15 applied to A and $f\big|\mathrm{A}$ in place of B and f.

IV.80, ℓ. 9,10.

"By hypothesis, there exists a sequence (A_n) of sets belonging to $\mathfrak{A}$ such that the $\mathrm{K} \cap \mathrm{A}_n$ form a covering of K."

Since $\mathrm{K} \subset \mathrm{B}$ one has

$$\mathrm{K} = \mathrm{K} \cap \mathrm{B} = \bigcup_{\mathrm{A} \in \mathfrak{A}} \mathrm{K} \cap \mathrm{A}\,,$$

and $\mathrm{K} \cap \mathrm{A} = \varnothing$ for all but countably many A in $\mathfrak{A}$ (No. 9, sentence after Def. 7).

IV.80, ℓ. 14.

"... the restriction of f to C_n is μ-measurable "

Since $f\big|\mathrm{A}_n$ is μ-measurable by hypothesis, and $\mathrm{C}_n \subset \mathrm{A}_n$ is measurable, $f\big|\mathrm{C}_n = (f\big|\mathrm{A}_n)\big|\mathrm{C}_n$ is μ-measurable by the argument in the proof of "necessity" (see the note for ℓ. 7,8).

IV.80, ℓ. 14–16.

"... there exists a partition of C_n formed by a μ-negligible set N_n and a sequence $(\mathrm{L}_{mn})_{m \geqslant 0}$ of compact sets such that $f\big|\mathrm{L}_{mn}$ is continuous."

Recall that K is a compact subset of the μ-measurable set B, and that C_n is a μ-integrable (hence μ-measurable) subset of K.

Let $\mathfrak{K}_n$ be the set of compact sets $\mathrm{H} \subset \mathrm{C}_n$ such that $(f\big|\mathrm{C}_n)\big|\mathrm{H} = f\big|\mathrm{H}$ is continuous; since $f\big|\mathrm{C}_n$ is μ-measurable, $\mathfrak{K}_n$ is μ-dense in C_n by a) of Prop. 15. Then by No. 9, Prop. 14, there exists a locally countable set $\mathfrak{H}_n \subset \mathfrak{K}_n$, consisting of pairwise disjoint (compact) subsets of C_n, such that the set $\mathrm{N}_n = \mathrm{C}_n - \bigcup_{\mathrm{H}\in\mathfrak{H}_n} \mathrm{H}$ is locally negligible. Since $\mathrm{N}_n \subset \mathrm{K}$, $\mathrm{N}_n = \mathrm{N}_n \cap \mathrm{K}$ is negligible (No. 2, Prop. 5). Since $\mathfrak{H}_n$ is locally countable (because $\mathfrak{K}_n$ is), $\mathrm{K}\cap\mathrm{H} = \varnothing$ for all but countably many H in $\mathfrak{H}_n$; that is, since $\mathrm{H}\subset\mathrm{C}_n\subset\mathrm{K}$ for all $\mathrm{H}\in\mathfrak{H}_n$, $\mathrm{H} = \mathrm{K}\cap\mathrm{H} = \varnothing$ for all but countably many H in $\mathfrak{H}_n$, in other words $\mathfrak{H}_n$ is countable.

IV.80, ℓ. 16–18.

"Since $\mathrm{N} = \bigcup_n \mathrm{N}_n$ is μ-negligible, we see that condition *a*) of Prop. 15 is satisfied, whence the proposition."

Note that

$$\mathrm{K} = \bigcup_n \mathrm{C}_n = \bigcup_n \Big(\mathrm{N}_n \cup \bigcup_m \mathrm{L}_{mn}\Big) = \mathrm{N} \cup \bigcup_n \Big(\bigcup_m \mathrm{L}_{mn}\Big)$$

where all unions are 'disjoint', therefore $\mathrm{K} - \mathrm{N} = \bigcup_{m,n} \mathrm{L}_{mn}$.

Thus it has been shown that, given any compact subset K of B, there exists a partition of K formed of a μ-negligible set N and a countable family (L_{mn}) of compact sets such that the $f\big|\mathrm{L}_{mn}$ are continuous. This shows that the set $\mathfrak{H}$ of all compact sets $\mathrm{H}\subset\mathrm{B}$ such that $f\big|\mathrm{H}$ is continuous satisfies condition *c*) of Prop. 12 (as well, of course, the conditions $(\mathrm{PL_I})$ and $(\mathrm{PL_{II}})$), therefore $\mathfrak{H}$ is μ-dense in B (No. 8, Def. 6); thus condition *a*) of Prop. 15 is satisfied, consequently $f : \mathrm{B}\to\mathrm{F}$ is indeed μ-measurable (Def. 8).

IV.80, ℓ. $-15, -14$.

"... these generalizations are left to the reader."

The proofs are sketched in the note after the next.

IV.80, ℓ. -14 to -11.

"... the principle of localization (No. 2, Prop. 4) remains valid when it is assumed that each of the functions g_x is only defined in V_x (or almost everywhere in V_x) and is measurable."

The key to interpreting this statement is the phrase "and is measurable". Two interpretations of 'measurable function' are available: Def. 1 or Def. 8; it is clearly Def. 8 that is intended here.

Let us call Prop. 4′ the proposed generalization of Prop. 4. One is given:

(i) a topological space F and a mapping $f : \mathrm{X}\to\mathrm{F}$;

(ii) for each $x\in\mathrm{X}$, a neighborhood V_x of x in X, a negligible subset N_x of V_x, and a function g_x, measurable in the sense of Def. 8, whose (measurable) domain of definition B_x contains the (integrable, hence measurable) set $\mathrm{A}_x = \mathrm{V}_x - \mathrm{N}_x$, such that $g_x\big|\mathrm{A}_x = f\big|\mathrm{A}_x$.

We are to prove that f is measurable. By d) of Prop. 15, the function $g_x : \mathrm{B}_x \to \mathrm{F}$ may be extended to a function $h_x : \mathrm{X} \to \mathrm{F}$ that is measurable in the sense of Def. 1. Since

$$h_x|\mathrm{A}_x = (h_x|\mathrm{B}_x)|\mathrm{A}_x = g_x|\mathrm{A}_x = f|\mathrm{A}_x\,,$$

that is, $h_x|(\mathrm{V}_x - \mathrm{N}_x) = f|(\mathrm{V}_x - \mathrm{N}_x)$, one has $h_x(y) = f(y)$ almost everywhere in V_x, therefore f is measurable by No. 2, Prop. 4.

IV.80, ℓ. −17 to −14.

"Property d) of Prop. 15 makes it possible to immediately generalize the properties of measurable functions defined on all of X, observed in Nos. 2 to 5, to measurable functions defined on a measurable subset A of X; these generalizations are left to the reader."

We write, for example, Th. 1′ for the contemplated generalization of Th. 1, skipping over results not susceptible to generalization via property d) of Prop. 15 (for instance, No. 2, Prop. 5, to which Prop. 15 can add nothing).

No. 2, Cor. 4′ of Prop. 5. Not amenable to generalization. If A is a measurable subset of X (equipped with the induced topology), and if $f : \mathrm{A} \to \mathrm{F}$ is a function such that the set

$$\mathrm{N} = \{x \in \mathrm{A} : \ f \text{ is not continuous at } x\}$$

is locally negligible, one would like to show that the extension f' of f to X by an element of F is measurable, presumably by applying Cor. 4 (without the prime) to f'. But the extension to f' might introduce a set of discontinuities not locally negligible, rendering Cor. 4 inapplicable.

For example, let $\mathrm{X} = \mathbf{R}$ equipped with Lebesgue measure, $\mathrm{F} = \mathbf{R}$, $\mathrm{A} = \mathbf{Q}$ equipped with the induced topology, and $f : \mathbf{Q} \to \mathbf{R}$ the constant function equal to 1. Then $\mathrm{N} = \varnothing$, the extension f' of f to $\mathbf{R}$ by 0 (the characteristic function of $\mathbf{Q}$ in $\mathbf{R}$) is a measurable function, but it is nowhere continuous, hence owes nothing to Cor. 4 for its measurability.

No. 2, Prop. 6′. Let $\mathrm{A} \subset \mathrm{X}$ be measurable, $f : \mathrm{A} \to \mathrm{F}$ measurable, and $g : \mathrm{A} \to \mathrm{F}$ a function such that $g = f$ locally almost everywhere in A; we are to show that g is measurable.

Let f', g' be the extensions of f and g to X by some point y_0 in F. Then f' is measurable by d) of Prop. 15, and $g' = f'$ locally almost everywhere in X, therefore g' is measurable by the original Prop. 6, and finally g is measurable by d) of Prop. 15.

No. 3, Th. 1′. With X, μ, (F_n) and F as in Th. 1, let A be a measurable subset of X, $f_n : \mathrm{A} \to \mathrm{F}_n$ a sequence of mappings μ-measurable in the sense of Def. 8, $f = (f_n) : \mathrm{A} \to \mathrm{F}$, and $u : f(\mathrm{A}) \to \mathrm{G}$ a continuous mapping into a topological space G. We are to show that the composite mapping $u \circ f : \mathrm{A} \to \mathrm{G}$ is μ-measurable.

Fix a point $a \in \mathrm{A}$, define $y_n = f_n(a) \in \mathrm{F}_n$ for all n, and let $y = (y_n) = \bigl(f_n(a)\bigr) = f(a) \in \mathrm{F}$. Write f' for the extension of f to X by y; clearly $f' = (f'_n)$, where, for each n, f'_n is the extension of f_n to X by y_n.

Note that $f'(\mathrm{X}) = f(\mathrm{A})$; for, $f'\big|\mathrm{A} = f$, whereas if $x \in \mathrm{X} - \mathrm{A}$ then $f'(x) = y = f(a) \in f(\mathrm{A})$. It follows that the composite function $u \circ f' : \mathrm{X} \to \mathrm{G}$ is defined. Moreover, $(u \circ f')\big|\mathrm{A} = u \circ (f'\big|\mathrm{A}) = u \circ f$, whereas if $x \in \mathrm{X} - \mathrm{A}$ then $(u \circ f')(x) = u\bigl(f'(x)\bigr) = u(y)$; thus $u \circ f'$ is the extension of $u \circ f$ to X by $u(y) = u\bigl(f(a)\bigr) = (u \circ f)(a)$.

So much for the notation. For each n, f_n is μ-measurable, therefore f'_n is μ-measurable by Def. 8 and d) of Prop. 15. Since $f' = (f'_n)$ and u is continuous on $f(\mathrm{A}) = f'(\mathrm{X})$, it follows from the original Th. 1 that $u \circ f'$ is μ-measurable, that is, the extension of $u \circ f$ to X by $u(y)$ is μ-measurable; thus $u \circ f$ satisfies the condition d') at the end of the note for IV.79, ℓ. 7 hence is μ-measurable by Def. 8.

In the present situation, it was convenient to extend the f_n to X 'uniformly' by the particular elements $f_n(a)$ of $f_n(\mathrm{A})$ rather than by general elements of the F_n; this led to an extension of $u \circ f$ to X by a particular element of G, whence the utility of adding the weaker condition d') to the list in Prop. 15.

To avoid tedious repetitions, henceforth a function $f : \mathrm{A} \to \mathrm{F}$ will be said to be *μ-measurable in the sense of Def. 8* if it satisfies the conditions of Prop. 15 to which have been added the equivalent conditions d'), c'), c'') described at the end of the notes for IV.79, ℓ. 7 and IV.79, ℓ. 3,4. When there is only a single measure μ in the picture, '*μ-measurable*' may be abbreviated to '*measurable*'.

No. 3, Cor. 1′ of Th. 1. Suppose A is a measurable subset of X, and $f, g : \mathrm{A} \to \overline{\mathbf{R}}$ are functions μ-measurable in the sense of Def. 8; we are to show that $\sup(f, g)$ and $\inf(f, g)$ are also such functions.

With $'$ signifying 'extension to X by 0', we know from Def. 8 that f', g' are measurable functions $\mathrm{X} \to \overline{\mathbf{R}}$; since

$$\bigl(\sup(f, g)\bigr)' = \sup(f', g') \quad \text{and} \quad \bigl(\inf(f, g)\bigr)' = \inf(f', g')$$

are measurable by the original Cor. 1, it follows from Def. 8 that $\sup(f, g)$ and $\inf(f, g)$ are measurable functions $\mathrm{A} \to \overline{\mathbf{R}}$.

No. 3, Cor. 2′ of Th. 1. Given a measurable subset A of X and a function $f : \mathrm{A} \to \overline{\mathbf{R}}$, we are to show that f is measurable if and only if f^+ and f^- are measurable. Writing $\mathbf{0}$ for the function on A (or on X) identically equal to $0 \in \overline{\mathbf{R}}$, and f' for the extension of f to X by 0, one has

$$(f^+)' = \bigl(\sup(f, \mathbf{0})\bigr)' = \sup(f', \mathbf{0}') = \sup(f', \mathbf{0}) = (f')^+ ,$$

so it is clear from Def. 8 that f^+ is measurable if and only if $(f')^+$ is measurable; similarly, f^- is measurable if and only if $(f')^-$ is measurable. Therefore

$$\begin{aligned} f \text{ is measurable } &\Leftrightarrow\ f' \text{ is measurable} \quad \text{(Def. 8)} \\ &\Leftrightarrow\ (f')^+ \text{ and } (f')^- \text{ are measurable} \quad \text{(Cor. 2)} \\ &\Leftrightarrow\ f^+ \text{ and } f^- \text{ are measurable.} \end{aligned}$$

No. 3, Cor. 3′ of Th. 1. Assuming A a measurable subset of X, and $\mathbf{f}, \mathbf{g}$ measurable mappings $\mathrm{A} \to \mathrm{F}$, we are to show that $\mathbf{f} + \mathbf{g}$ and $\alpha\mathbf{f}$ (α a scalar) are also measurable. If $\mathbf{f}', \mathbf{g}'$ are the extensions of $\mathbf{f}, \mathbf{g}$ to X by 0, then

$$(\mathbf{f} + \mathbf{g})' = \mathbf{f}' + \mathbf{g}' \quad \text{and} \quad (\alpha\mathbf{f})' = \alpha\mathbf{f}' ,$$

therefore $\mathbf{f} + \mathbf{g}$ and $\alpha\mathbf{f}$ are measurable by Def. 8 and the original Cor. 3.

No. 3, Cor. 4′ of Th. 1. With notations as in Cor. 4, assume that the f_k and $\mathbf{f}$ are defined on a measurable subset A of X. Then $\mathbf{f}' = \sum_{k=1}^{n} \mathbf{e}_k f_k'$, where $\mathbf{f}'$ and f_k' are the extensions to X by $0 \in \mathbf{K}^n$ and $0 \in \mathbf{K}$, respectively ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$), whence

$$\begin{aligned} \mathbf{f} \text{ is measurable } &\Leftrightarrow\ \mathbf{f}' \text{ is measurable} \quad \text{(Def. 8)} \\ &\Leftrightarrow\ f_1', \ldots, f_n' \text{ are measurable} \quad \text{(original Cor. 4)} \\ &\Leftrightarrow\ f_1, \ldots, f_n \text{ are measurable} \quad \text{(Def. 8)}. \end{aligned}$$

No. 3, Cor. 5′ of Th. 1. With F, G, H and $(u, v) \mapsto [u \cdot v]$ as in the original Cor. 5, suppose $\mathbf{f} : \mathrm{A} \to \mathrm{F}$ and $\mathbf{g} : \mathrm{A} \to \mathrm{G}$ are measurable mappings; we are to show that the mapping $[\mathbf{f} \cdot \mathbf{g}] : \mathrm{A} \to \mathrm{H}$ defined by $x \mapsto [\mathbf{f}(x), \mathbf{g}(x)]$ $(x \in \mathrm{A})$ is measurable.

To simplify (at least visually) the notation, let us write $h(u, v) = [u \cdot v]$ $(u \in \mathrm{F}, v \in \mathrm{G})$; by hypothesis, $h : \mathrm{F} \times \mathrm{G} \to \mathrm{H}$ is a continuous mapping, but it need not be bilinear (see the note for IV.64, ℓ. 10–13). We are to show that the mapping $k = h \circ (\mathbf{f}, \mathbf{g}) : \mathrm{A} \to \mathrm{H}$, defined by $k(x) = h\bigl(\mathbf{f}(x), \mathbf{g}(x)\bigr)$ $(x \in \mathrm{A})$ is measurable.

Let $\mathbf{f}'$ and $\mathbf{g}'$ be the extensions of $\mathbf{f}$ and $\mathbf{g}$ to X by $0 \in \mathrm{F}$ and $0 \in \mathrm{G}$, respectively, and let k' be the extension of k to X by the element

$h(0,0)$ of H. (Of course if h is bilinear, then $h(0,0) = 0 \in \mathrm{H}$.) Then $k' = h \circ (\mathbf{f}', \mathbf{g}')$, that is,

$$k'(x) = h\big(\mathbf{f}'(x), \mathbf{g}'(x)\big) \quad \text{for all } x \in \mathrm{X}.$$

Since $\mathbf{f}'$ and $\mathbf{g}'$ are measurable (Def. 8), k' is measurable by the original Cor. 5, therefore k is measurable (Def. 8).

No. 3, Cor. 6′ of Th. 1. If $\mathbf{f} : \mathrm{A} \to \mathrm{F}$ and $\mathbf{f}'$ is the extension of $\mathbf{f}$ to X by 0, then $|\mathbf{f}|' = |\mathbf{f}'|$, etc.

No. 4, Th. 2′ (Egoroff). With X, μ, A, $\mathfrak{F}$ as in the original Th. 2, let M be a measurable subset of X, and $(f_\alpha)_{\alpha\in\mathrm{A}}$ a family measurable mappings of M into a metrizable space F. (Why “M”? Our favorite letter A has been pre-empted by the index set, and the letters B, C figure in the proof of Th. 2.) Assume that there exists a locally negligible set $\mathrm{N} \subset \mathrm{M}$ such that $\lim_{\mathfrak{F}} f_\alpha(x)$ exists in F for every $x \in \mathrm{M} - \mathrm{N}$, and let $f : \mathrm{M} \to \mathrm{F}$ be any function such that

$$\lim{}_{\mathfrak{F}} f_\alpha(x) = f(x) \quad \text{for all } x \in \mathrm{M} - \mathrm{N}.$$

We are to show that:

1° f is measurable; and

2° for every compact subset K of M and every $\varepsilon > 0$, there exists a compact set $\mathrm{K}_1 \subset \mathrm{K}$ such that $|\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant \varepsilon$ and such that the $f_\alpha\big|\mathrm{K}_1$ are continuous and converge uniformly to $f\big|\mathrm{K}_1$.

Fix a point $y_0 \in \mathrm{F}$ and let f'_α, f' be the extensions of f_α, f to X by y_0. The f'_α are measurable by Def. 8, and $f'_\alpha \to f'(x)$ for all $x \in \mathrm{X} - \mathrm{N}$; for, if $x \in \mathrm{M} - \mathrm{N}$ then $f'_\alpha(x) = f_\alpha(x) \to f(x) = f'(x)$, and if $x \in \mathrm{X} - \mathrm{M}$ then $f'_\alpha(x) = y_0 \to y_0 = f'(x)$.

Thus the f'_α and f' satisfy the hypotheses of the original Th. 2, hence satisfy the corresponding conditions 1° and 2°: f' is measurable—hence so is f, by Def. 8—and f'_α, f' satisfy 2° for every compact set $\mathrm{K} \subset \mathrm{X}$, in particular for every compact set $\mathrm{K} \subset \mathrm{M}$. But this is not the end of the story: given a compact set $\mathrm{K} \subset \mathrm{M}$ and an $\varepsilon > 0$, we must find a compact set $\mathrm{K}_1 \subset \mathrm{K}$ that avoids the points of N, i.e., is contained in $\mathrm{K} - \mathrm{N}$, so that $f'_\alpha\big|\mathrm{K}_1 = f_\alpha\big|\mathrm{K}_1$ and $f'\big|\mathrm{K}_1 = f\big|\mathrm{K}_1$.

Let K be a compact set in M and let $\varepsilon > 0$. Since N is measurable (IV.61, ℓ. −4), the set $\mathrm{K} - \mathrm{N}$ is integrable, hence there exists a compact set $\mathrm{K}' \subset \mathrm{K} - \mathrm{N}$ such that $|\mu|\big((\mathrm{K} - \mathrm{N}) - \mathrm{K}'\big) \leqslant \varepsilon/2$ (§4, No. 6, Cor. 1 of Th. 4). Note that

$$\mathrm{K} - \mathrm{K}' \subset \big((\mathrm{K} - \mathrm{N}) - \mathrm{K}'\big) \cup (\mathrm{K} \cap \mathrm{N}),$$

where $K \cap N$ is negligible (No. 2, Prop. 5), therefore $|\mu|(K - K') \leqslant \varepsilon/2$. Now choose (by 2°) a compact set $K_1 \subset K'$ such that $|\mu|(K' - K_1) \leqslant \varepsilon/2$ and such that the $f'_\alpha|K_1$ are continuous and converge uniformly to $f'|K_1$. Finally, from $K_1 \subset K' \subset K - N$ we infer that the $f'_\alpha|K_1 = f_\alpha|K_1$ and $f'|K_1 = f|K_1$, and from $K_1 \subset K' \subset K$ we infer that $|\mu|(K - K_1) \leqslant \varepsilon/2 + \varepsilon/2$.

No. 4, Cor. 1′ *of Th. 2.* Let $A \subset X$ be measurable, and let (f_n) be a sequence of measurable functions $A \to \overline{\mathbf{R}}$. Writing f'_n for the extension of f_n to X by 0, one has $\left(\sup_n f_n\right)' = \sup_n f'_n$, etc.

No. 5, Th. 3′. With X, μ and F as in Th. 3 (in particular, F is metrizable), let A be a measurable subset of X. A function $g : A \to F$ will be called a *measurable step function* if g has only finitely many values, each assumed on a measurable subset of X. It follows that g is measurable in the sense of Def. 8; for, $X - A$ is a measurable set, so if $f' : X \to F$ is the extension of f to X by some element $y_0 \in F$, then f' is a measurable step function in the sense of the first paragraph of No. 5, hence is measurable (as noted there). {There is no harm if y_0 is already a value of g; in this case $\overset{-1}{g'}(y_0)$ is the disjoint union $\overset{-1}{g}(y_0) \cup (X - A)$, so that $\overset{-1}{g}(y_0) = \overset{-1}{g'}(y_0) - (X - A) = \overset{-1}{g'}(y_0) \cap A$ will be measurable if and only if $\overset{-1}{g'}(y_0)$ is measurable.} Note that if $g : A \to F$ is a measurable step function, then for every measurable set $B \subset A$, the restriction $g|B$ is a measurable step function on B because, for $y \in F$, $(g|B)^{-1}(y) = B \cap \overset{-1}{g}(y)$ is the intersection of measurable sets.

The proposed Th. 3′ can then be stated as follows.

If $f : A \to F$ is a function such that, for every compact set $K \subset A$, there exists a sequence $g_n : A \to F$ of measurable step functions on A such that $g_n(x) \to f(x)$ for almost every x in K, then f is measurable.

It may be helpful to outline the long-winded proof. We will show:

1) For every compact set $K \subset A$, the restriction $f|K$ is measurable (proof based on Th. 2′ above).

2) For every compact set $K \subset A$, the set $\mathfrak{H}_K$ of all compact sets $H \subset K$ such that $f|H$ is continuous is μ-dense in A (proved by verifying criterion *c*) of Prop. 12 in the context of the measurable subset K of X).

3) The set $\mathfrak{H}$ of all compact sets $H \subset A$ such that $f|H$ is continuous is μ-dense in A (proof based on Prop. 13, with $\mathfrak{K}$ the set of all compact subsets of A).

4) f is measurable (proof by verifying that $\mathfrak{H}$ satisfies criterion *a*) of Prop. 15).

The details are as follows.

1) Let $K \subset A$ be compact. By assumption there exist a sequence $g_n : A \to F$ of measurable step functions and a negligible set $N \subset K$ such that $g_n(x) \to f(x)$ for all $x \in K - N$. Then $(g_n|K)$ is a sequence of measurable step functions on K such that $g_n|K \to f|K$ pointwise on $K - N$, therefore $f|K$ is measurable by Th. 2′ (the generalization of Egoroff's theorem proved above; metrizability of F is needed here).

2) Let $K \subset A$ be compact, and let $\mathfrak{H}_K$ be the set of all compact sets $H \subset K$ such that $f|H$ is continuous. We know that $\mathfrak{H}_K$ satisfies (PL_I) and (PL_{II}) of Prop. 12 (in the context of the measurable subset K of X); it also satisfies criterion *c*) of Prop. 12, for, if B is a compact subset of K (hence of A), then $f|B$ is measurable by the preceding paragraph, therefore $f|B$ has a measurable extension $(f|B)'$ to X. By No. 1, Def. 1, there exists a sequence (H_n) of pairwise disjoint compact subsets of B such that $(f|B)'|H_n$ is continuous for all n, and such that $B - \bigcup_n H_n$ is negligible. But $(f|B)'|H_n = (f|B)|H_n = f|H_n$, therefore $H_n \in \mathfrak{H}_K$. Thus $\mathfrak{H}_K$ is μ-dense in K (No. 8, Def. 6).

3) Let $\mathfrak{K}$ be the set of all compact sets $K \subset A$, and let $\mathfrak{H}$ be the set of all compact sets $H \subset A$ such that $f|H$ is continuous; thus, in the notation of 2), $\mathfrak{H}_K = \{H \in \mathfrak{H} : H \subset K\}$ for every $K \in \mathfrak{K}$. We know that $\mathfrak{H}$ satisfies (PL_I) and (PL_{II}) of Prop. 12. By the preceding paragraph, for every $K \in \mathfrak{K}$ the set $\mathfrak{H}_K$ is μ-dense in K; since $\mathfrak{K}$ is μ-dense in A (IV.77, ℓ. 1), it follows from Prop. 13 that $\mathfrak{H}$ is μ-dense in A.

4) By the preceding paragraph, $\mathfrak{H}$ satisfies condition *a*) of Prop. 15, therefore f is measurable by Def. 8.

A simplification of the above argument yields an extension of the Principle of localization (No. 2, Prop. 4):

Proposition. *Let* X *be a locally compact space,* μ *a measure on* X, A *a measurable subset of* X, *and* $f : A \to F$ *a mapping of* A *into a topological space* F (not necessarily metrizable). *Let* $\mathfrak{K}$ *be the set of all compact subsets of* A. *Then*

$$f \textit{ is measurable} \quad \Leftrightarrow \quad f|K \textit{ is measurable for all } K \in \mathfrak{K}.$$

Proof. $\Rightarrow$: Fix $y_0 \in F$ and let f' be the extension of f to X by y_0. Then f' is measurable by Prop. 15, therefore $f'|B$ is measurable for every measurable set B in X (because $f'|B$ has a measurable extension to X, namely f'). In particular, if $K \in \mathfrak{K}$, then $f|K = (f'|A)|K = f'|K$ is measurable.

$\Leftarrow$: Let $K \in \mathfrak{K}$ and let $\mathfrak{H}_K$ be the set of all $H \in \mathfrak{K}$ such that $f|H$ is continuous. By assumption, $f|K$ is measurable, so by a part of the argument in 2) above, $\mathfrak{H}_{\mathrm{K}}$ is μ-dense in K . By the argument in 3) above, the set $\mathfrak{H} = \{\mathrm{H} \in \mathfrak{K} : f\big|\mathrm{H}$ is continuous$\}$ is μ-dense in A, therefore f is μ-measurable by the argument in 4) above.

Remark. A simpler result in this vein: If A, B are measurable sets in X such that $\mathrm{B} \subset \mathrm{A}$, and if $f : \mathrm{A} \to \mathrm{F}$ is measurable (F any topological space), then $f\big|\mathrm{B}$ is also measurable. For, f has a measurable extension to X, therefore so does $f\big|\mathrm{B}$.

No. 5, Cor. 1′ of Th. 3. As in the original Cor. 1, Let X be a locally compact space equipped with a measure, and let F be a Banach space. Let A be a measurable subset of X and let $\mathbf{f} : \mathrm{A} \to \mathrm{F}$ be a mapping measurable in the sense of Def. 8; we are to show that, given any compact set $\mathrm{K} \subset \mathrm{A}$, there exists a sequence $\mathbf{g}_n : \mathrm{A} \to \mathrm{F}$ of measurable step functions (as defined in the note for Th. 3′) such that (i) $\mathrm{Supp}(\mathbf{g}_n) \subset \mathrm{K}$, (ii) $|\mathbf{g}_n| \leqslant |\mathbf{f}|$ on A, and (iii) $\mathbf{g}_n(x) \to \mathbf{f}(x)$ for almost every x in K.

Let K be a compact subset of A. By Def. 8, $\mathbf{f}$ may be extended to a measurable mapping $\mathbf{h} : \mathrm{A} \to \mathrm{F}$, so by the original Cor. 1 there exists a sequence $\mathbf{h}_n : \mathrm{X} \to \mathrm{F}$ of measurable step functions such that $\mathrm{Supp}(\mathbf{h}_n) \subset \mathrm{K}$, $|\mathbf{h}_n| \leqslant |\mathbf{h}|$ on X, and $\mathbf{h}_n(x) \to \mathbf{h}(x)$ for almost every $x \in \mathrm{K}$. The restrictions $\mathbf{g}_n = \mathbf{h}_n\big|\mathrm{A}$ meet all the requirements.

No. 5, Cor. 2′ of Th. 3. As in the original Cor. 2, let X be a locally compact space countable at infinity, equipped with a measure, and let F be a metrizable space. Let A be a measurable subset of X, and $f : \mathrm{A} \to \mathrm{F}$ a mapping measurable in the sense of Def. 8; we are to show that there exists a sequence $g_n : \mathrm{A} \to \mathrm{F}$ of measurable step functions (as defined in the note for Th. 3′) such that $g_n(x) \to f(x)$ for almost every $x \in \mathrm{A}$.

By Def. 8, f may be extended to a measurable mapping $h : \mathrm{X} \to \mathrm{F}$, so by the original Cor. 2 there exists a sequence of measurable step functions $h_n : \mathrm{X} \to \mathrm{F}$ such that $h_n(x) \to h(x)$ for almost every $x \in \mathrm{X}$. Then the restrictions $g_n = h_n\big|\mathrm{A}$ are measurable step functions on A such that $g_n(x) \to f(x)$ for almost every $x \in \mathrm{A}$.

No. 5, Prop. 7′. Let X be a locally compact space equipped with a measure (say μ), and F any topological space. By the original Prop. 7, if $h : \mathrm{X} \to \mathrm{F}$ is measurable and U is an open set in F, then $\overset{-1}{h}(\mathrm{U})$ is a measurable set. Since the measurable subsets of X form a tribe (No. 4, Cor. 2 of Th. 2), it follows that $\overset{-1}{h}(\mathrm{B})$ is measurable for every Borel set B in F (the set $\mathscr{S}$ of sets $\mathrm{S} \subset \mathrm{F}$ such that $\overset{-1}{h}(\mathrm{S})$ is measurable in X is a tribe that contains every open set, hence $\mathscr{S}$ contains every Borel set). Now, if $B \subset F$ then

$$A \cap \overset{-1}{h}(B) = \{x \in A : \ h(x) \in B\} = \{x \in A : \ f(x) \in B\} = \overset{-1}{f}(B),$$

thus if B is a Borel set in F then $\overset{-1}{f}(B)$ is the intersection of two measurable sets, hence is measurable.

In particular, if B is any open set or closed set in F, then $\overset{-1}{f}(B)$ is a measurable set of X that is contained in A, that is, a μ-measurable subset of A.

Remark. The term 'measurable subset of A' was avoided, as it begs the question of whether A is a locally compact subspace of X equipped with a measure of its own.

Suppose, indeed, that A is a locally compact subspace of X, hence is a measurable subset of X (No. 7, first paragraph). One has the measure $\mu_A = \mu|A$ induced on A (No. 7, Def. 4), defined by $\mu_A(g) = \int g'\,d\mu$ for $g \in \mathscr{K}(A; \mathbf{C})$, where g' is the extension of g to X by 0. Borrowing from the future (Ch. V, §7, No. 1, Cor. of Prop. 2), we know that a set $C \subset A$ is μ_A-measurable if and only if it is μ-measurable. And (*loc. cit.*, Prop. 2), a mapping $f : A \to F$ is μ_A-measurable if and only if it is μ-measurable in the sense of Def. 8, hence has a μ-measurable extension $h : X \to F$; in this case, if C is a Borel set in F, then $\overset{-1}{f}(C) = A \cap \overset{-1}{h}(C)$ is a μ-measurable set contained in A by the earlier discussion, hence (by the foregoing) is a μ_A-measurable subset of A.

The bottom line: when A is a locally compact subspace of X, the term 'measurable subset of A' is unambiguous.

No. 5, Th. 4′. As in the original Th. 4, let X be a locally compact space equipped with a measure μ, F a metrizable topological space, and d a metric on F compatible with its topology.

The original Th. 4 asserts that a mapping $h : X \to F$ is measurable if and only if it satisfies the following two conditions (slightly reworded):

a) For every closed ball B in F, $\overset{-1}{h}(B)$ is a measurable set in X;

b) for every compact set $K \subset X$, there exist a countable set $H \subset F$ and a negligible set $N \subset K$ such that $h(K - N) \subset \overline{H}$.

An equivalent formulation of *b*) is that for every compact set $K \subset X$, there exists a negligible set $N \subset K$ such that the topological subspace $h(K - N)$ of F has a countable dense subset, equivalently (GT, IX, §2, No. 8, Prop. 12), has a countable base for open sets, in other words (*loc. cit.*, Def. 4) is a separable metrizable space. For, writing $S = h(K - N)$, if there exists a countable set $H \subset F$ such that $S \subset \overline{H}$, then the subspace $\overline{H}$ of F has a countable base for open sets, therefore so does the subspace S, therefore S has a countable subset H_0 whose closure in S is equal to S, that is (GT, I, §3, No. 1, Prop. 1), $S \cap \overline{H}_0 = S$, in other words $S \subset \overline{H}_0$. (The same argument shows that in a separable metrizable space, every subspace is separable.) Condition *b*) can be rephrased succinctly:

b) for every compact set $K \subset X$, there exists a negligible set $N \subset K$ such that $h(K - N)$ is a separable (topological) subspace of F.

The proposed Th. 4′ can then be formulated as follows: If A is a measurable subset of X then a mapping $f : A \to F$ is measurable in the sense of Def. 8 if and only if it satisfies the following two conditions:

a′) For every closed ball B in F, the set $\overset{-1}{f}(B) \subset A$ is measurable;

b′) for every compact set $K \subset A$, there exists a negligible set $N \subset K$ such that $f(K - N)$ is a separable subspace of F.

For, suppose f is measurable. By Def. 8, f has an extension $h : X \to F$ that is measurable. Given any closed ball B in F and any compact set $K \subset A$, we know from the original Th. 4 that $\overset{-1}{h}(B)$ is measurable and that there exists a negligible set $N \subset K$ such that $h(K - N)$ is a separable subspace of F. Then

$$A \cap \overset{-1}{h}(B) = \{x \in A : \ h(x) \in B\} = \{x \in A : \ f(x) \in B\} = \overset{-1}{f}(B),$$

thus $\overset{-1}{f}(B)$ is the intersection of two measurable sets, whence *a′*) is satisfied. And

$$f(K - N) = (h|A)(K - N) \subset h(K - N);$$

thus $f(K - N)$ is a subspace of the separable subspace $h(K - N)$, whence *b′*) is satisfied.

Conversely, suppose f satisfies *a′*) and *b′*). Fix a point $y_0 \in F$ and let $h : X \to F$ be the extension of f to X by y_0; to prove that f is measurable, it will suffice by Def. 8 to show that h is measurable, and by the original Th. 4 we need only verify that h satisfies the conditions *a*) and *b*).

a) If B is a closed ball in F, then

$$\overset{-1}{h}(B) = \begin{cases} \overset{-1}{f}(B) & \text{if } y_0 \notin B \\ \overset{-1}{f}(B) \cup (X - A) & \text{if } y_0 \in B; \end{cases}$$

since $\overset{-1}{f}(B)$ is measurable by *a′*), we see that $\overset{-1}{h}(B)$ is also measurable.

b) Let $K \subset X$ be compact; we seek a negligible set $N \subset K$ such that $h(K - N)$ is separable. Since A is measurable, $K \cap A$ is integrable (No. 2, Prop. 3), hence there exists a sequence (K_n) of pairwise disjoint compact subsets of $K \cap A$ such that the set $N' = K \cap A - \bigcup_n K_n$ is negligible (§4, No. 6, Cor. 2, 2° of Th. 4). For each n, since $K_n \subset A$ is compact, by $b')$ there exists a negligible set $N_n \subset K_n$ such that $f(K_n - N_n)$ is separable. The set $N = N' \cup \bigcup_n N_n$ is negligible, $N \subset K \cap A \subset K$, and

$$K \cap A = N' \cup \bigcup_n K_n = N' \cup \bigcup_n \left[N_n \cup (K_n - N_n)\right] = N \cup \bigcup_n (K_n - N_n),$$

therefore $K \cap A - N = \bigcup_n (K_n - N_n)$; then

$$f(K \cap A - N) = \bigcup_n f(K_n - N_n),$$

thus $f(K \cap A - N)$ is the union of a sequence of separable subspaces of F, therefore

$$(*) \qquad f(K \cap A - N) \quad \text{is separable.}$$

We assert that $h(K - N)$ is separable. For,

$$K - N = [(K - N) \cap (X - A)] \cup [(K - N) \cap A] \subset (X - A) \cup [(K \cap A - N)],$$

whence

$$h(K - N) \subset h(X - A) \cup h(K \cap A - N) = \{y_0\} \cup f(K \cap A - N);$$

in view of $(*)$, $h(K - N)$ is a subspace of a separable subspace of F, hence is separable. Thus h satisfies condition b), which completes the proof that f is measurable.

No. 5, Prop. 8'. Let X be a locally compact space equipped with a measure, A a measurable subset of X, $f : X \to \overline{\mathbf{R}}$, and D a countable dense subset of $\mathbf{R}$. Then:

1) If f is measurable, then the set $\{x \in A : \ f(x) \geqslant a\}$ is measurable for every $a \in \overline{\mathbf{R}}$.

2) If, for every $a \in D$, the set $\{x \in A : \ f(x) \geqslant a\}$ is measurable, then f is measurable.

Proof. Let $h : X \to \overline{\mathbf{R}}$ be the extension of f to X by 0.

1) If f is measurable, then h is measurable (Def. 8). For every $a \in \overline{\mathbf{R}}$, $[a, +\infty]$ is a closed subset of $\overline{\mathbf{R}}$, therefore the set

$$\{x \in A : \ h(x) \geqslant a\} = \overset{-1}{h}([a, +\infty])$$

is measurable by Prop. 7; consequently the set

$$\{x \in \mathrm{A} :\ f(x) \geqslant a\} = \{x \in \mathrm{A} :\ h(x) \geqslant a\} = \mathrm{A} \cap \overset{-1}{h}([a, +\infty])$$

is measurable.

2) For every $a \in \mathrm{D}$, the set

$$\{x \in \mathrm{X} :\ h(x) \geqslant a\} = \begin{cases} \{x \in \mathrm{A} :\ f(x) \geqslant a\} & \text{if } a > 0 \\ \{x \in \mathrm{A} :\ f(x) \geqslant a\} \cup (\mathrm{X} - \mathrm{A}) & \text{if } a \leqslant 0 \end{cases}$$

is measurable, therefore h is measurable by the original Prop. 8, and so f is measurable by Def. 8.

It may be useful to review here the proof of Prop. 8. Suppose $h : \mathrm{X} \to \overline{\mathbf{R}}$ is a function such that $\{x \in \mathrm{X} :\ h(x) \geqslant a\}$ is measurable for every $a \in \mathrm{D}$. As observed in the note for IV.69, ℓ. 10, the topological space $\overline{\mathbf{R}}$ can be metrized in such a way that the closed balls are precisely its closed intervals $[a, b]$. Since $\overline{\mathbf{R}}$ has a countable dense subset, condition $b)$ of Th. 4 is trivially satisfied by h, so to prove that h is measurable, it suffices to show that $\overset{-1}{h}([a, b])$ is measurable for every closed interval $[a, b]$ of $\overline{\mathbf{R}}$.

For every $a \in \overline{\mathbf{R}}$, the set $\{x \in \mathrm{X} :\ h(x) \geqslant a\}$ is measurable: this is trivial if $a = -\infty$, and if $a > -\infty$ then there exists a sequence $a_n \in \mathrm{D}$ such that $a_n < a$ and $a_n \to a$, whence

$$\{x \in \mathrm{X} :\ h(x) \geqslant a\} = \bigcap_n \{x \in \mathrm{X} :\ h(x) \geqslant a_n\}$$

is the intersection of countably many sets that are measurable by hypothesis. Its complement $\{x \in \mathrm{X} :\ h(x) < a\}$ is therefore also measurable (empty when $a = -\infty$).

For every $b \in \overline{\mathbf{R}}$ the set $\{x \in \mathrm{X} :\ h(x) \leqslant b\}$ is measurable. This is trivial if $b = +\infty$. Assuming $b < +\infty$, there exists a sequence $a_n \in \mathrm{D}$ such that $b < a_n$ and $a_n \to b$. Then

$$[-\infty, b] = \bigcap_n [-\infty, a_n[= \bigcap \complement [a_n, +\infty],$$

therefore $\{x \in \mathrm{X} :\ h(x) \leqslant b = \overset{-1}{h}([-\infty, b]) = \bigcap_n \complement \overset{-1}{h}([a_n, +\infty])$ is the intersection of a sequence of measurable sets.

Finally, if $[a, b]$ is any closed interval in $\overline{\mathbf{R}}$ then the set

$$\overset{-1}{h}([a,b]) = \{x \in \mathrm{X} : \ h(x) \geqslant a\} \cap \{x \in \mathrm{X} : \ h(x) \leqslant b\}$$

is measurable, therefore h is measurable by Th. 4.

Incidentally, since the measurable subsets of X form a tribe (No. 4, Cor. 2 of Th. 2), it is clear that the set $\mathscr{S} = \{\mathrm{S} \subset \overline{\mathbf{R}} : \ \overset{-1}{h}(\mathrm{S}) \text{ is measurable}\}$ is also a tribe, and since $\mathscr{S}$ contains all closed intervals, it contains the tribe they generate, namely, the tribe of Borel sets. Thus $\overset{-1}{h}(\mathrm{B})$ is measurable for every Borel set B in $\overline{\mathbf{R}}$.

No. 5, Prop. 9′. Let X be a locally compact space equipped with a measure, F a metrizable compact space, A a measurable subset of X, and $f : \mathrm{A} \to \mathrm{F}$ a mapping measurable in the sense of Def. 8. Then f is the uniform limit of a sequence $g_n : \mathrm{A} \to \mathrm{F}$ of measurable step functions.

See the earliler note for Th. 3′ for the definition of 'measurable step function'. Let $h : \mathrm{X} \to \mathrm{F}$ be a measurable extension of f (Def. 8). By the original Prop. 9, there exists a sequence of measurable step functions $h_n : \mathrm{X} \to \mathrm{F}$ such that $h_n \to h$ uniformly in X; the sequence of restrictions $g_n = h_n | \mathrm{A}$ meets the requirements.

No. 5, Prop. 10′. Let X be a locally compact space equipped with a measure, F a separable Banach space (over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$), F′ its dual, and let $(\mathbf{a}'_n)$ be a weakly dense sequence in the unit ball of F′.

Let A be a measurable subset of X, $\mathbf{f} : \mathrm{A} \to \mathrm{F}$ a mapping. Then the following conditions are equivalent:

a) $\mathbf{f}$ is measurable.

b) For every n, the scalar-valued function $x \mapsto \langle \mathbf{f}(x), \mathbf{a}'_n \rangle$ $(x \in \mathrm{A})$ is measurable.

Write $f_n : \mathrm{A} \to \mathbf{K}$ for the function $f_n(x) = \langle \mathbf{f}(x), \mathbf{a}'_n \rangle$ $(x \in \mathrm{A})$. Let $\mathbf{h} : \mathrm{X} \to \mathrm{F}$ be the extension of $\mathbf{f}$ to X by $0 \in \mathrm{F}$, and, for each n, let $h_n : \mathrm{X} \to \mathrm{F}$ be the extension of f_n to X by $0 \in \mathbf{K}$. Note that

$$(*) \qquad \langle \mathbf{h}(x), \mathbf{a}'_n \rangle = h_n(x) \quad \text{for all } x \in \mathrm{X}.$$

Then:

$$\begin{aligned} \mathbf{f} \text{ is measurable} \;&\Leftrightarrow\; \mathbf{h} \text{ is measurable} \\ &\Leftrightarrow\; h_n \text{ is measurable for all } n \\ &\Leftrightarrow\; f_n \text{ is measurable for all } n \end{aligned}$$

(the first equivalence by Def. 8; the second by $(*)$ and the original Prop. 10; the third by Def. 8), thus *a*) ⇔ *b*).

No. 5, Cor. 1′ of Prop. 10. Let X be a locally compact space equipped with a measure, F any Banach space, and A a measurable subset of X. In order that a mapping $\mathbf{f} : A \to F$ be measurable, it is necessary and sufficient that it satisfy the following two conditions:

a′) For every $\mathbf{a}' \in F'$, the scalar function $x \mapsto \langle \mathbf{f}(x), \mathbf{a}' \rangle$ $(x \in A)$ is measurable;

b′) for every compact set $K \subset A$, there exist a countable set $H \subset F$ and a negligible set $N \subset K$ such that $f(K - N) \subset \overline{H}$.

"Necessity": Assuming **f** measurable, let $\mathbf{h} : X \to F$ be a measurable extension of **f** to X (Def. 8). By the original Cor. 1, **h** satisfies conditions *a*) and *b*) of that corollary; then $\mathbf{f} = \mathbf{h}|A$ satisfies *a′*) by *a*) and Def. 8. If K is a compact subset of A, and if $H \subset F$ and $N \subset K$ are chosen as in *b*) of Cor. 1, then

$$\mathbf{f}(K - N) = \mathbf{h}(K - N) \subset \overline{H},$$

thus **f** satisfies *b′*).

"Sufficiency": Suppose **f** satisfies *a′*) and *b′*). Fix a point $\mathbf{y}_0 \in F$ and let **h** be the extension of **f** to X by $\mathbf{y}_0$. To show that **f** is measurable, equivalently (Def. 8) that **h** is measurable, it will suffice to show that **h** satisfies the conditions *a*) and *b*) of Cor. 1.

Now, **h** satisfies *b*) of Cor. 1 by the argument for the proof of *b*) in the earlier proof of Th. 4′. Also, **h** satisfies *a*) of Cor. 1: for, if $\mathbf{a}' \in F'$ then the function $x \mapsto \langle \mathbf{h}(x), \mathbf{a}' \rangle$ $(x \in X)$ is the extension to X of the function $x \mapsto \langle \mathbf{f}(x), \mathbf{a}' \rangle$ $(x \in A)$ by $\langle \mathbf{y}_0, \mathbf{a}' \rangle$, hence is measurable by *a′*) and Def. 8.

No. 5, Cor. 2′ of Prop. 10. Let X and F be as in the original Cor. 2, and let A be a measurable subset of X. In order that a mapping $\mathbf{f} : A \to F$ be measurable, it is necessary and sufficient that, for every $\mathbf{a}' \in F'$, the scalar function $x \mapsto \langle \mathbf{f}(x), \mathbf{a}' \rangle$ $(x \in A)$ be measurable.

For, let $\mathbf{h} : X \to F$ be the extension of **f** to X by $0 \in F$. For each $\mathbf{a}' \in F'$, the function

$$x \mapsto \langle \mathbf{h}(x), \mathbf{a}' \rangle \quad (x \in X) \tag{1}$$

is the extension to X by 0 of the function

$$x \mapsto \langle \mathbf{f}(x), \mathbf{a}' \rangle \quad (x \in A). \tag{2}$$

Then:

$$\begin{aligned} \mathbf{f} \text{ is measurable} \quad &\Leftrightarrow \quad \mathbf{h} \text{ is measurable} \\ &\Leftrightarrow \quad \text{the functions (1) are all measurable} \\ &\Leftrightarrow \quad \text{the functions (2) are all measurable} \end{aligned}$$

(the first equivalence, by Def. 8; the second, by the original Cor. 2; the third, by Def. 8).

No. 5, Prop. 11′. Statement and proof as in the preceding, with "Cor. 2" replaced by Prop. 11" (but note that the conditions on F are different in the two statements).

IV.81, ℓ. 2, 3.

"...if $\mathrm{V}' \subset \mathrm{V}$, $\mathrm{B}' \supset \mathrm{B}$ and $\delta' \leqslant \delta$, then

$$\boldsymbol{W}(\mathrm{V}', \mathrm{B}', \delta') \subset \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta) \text{ "}$$

Let $(f, g) \in$ left side, and write

$$\mathrm{M} = \{x \in \mathrm{B} : \big(f(x), g(x)\big) \notin \mathrm{V}\}, \quad \mathrm{M}' = \{x \in \mathrm{B}' : \big(f(x), g(x)\big) \notin \mathrm{V}'\};$$

the assumptions $\mathrm{B} \subset \mathrm{B}'$, $\mathrm{V} \supset \mathrm{V}'$ clearly imply $\mathrm{M} \subset \mathrm{M}'$, therefore

$$|\mu|^*(\mathrm{M}) \leqslant |\mu|^*(\mathrm{M}') \leqslant \delta' \leqslant \delta,$$

whence $(f, g) \in$ right side.

Remark. $\big(f(x), g(x)\big) \in \mathrm{V}$ means that $f(x)$ and $g(x)$ are 'near of order V ' (or ' V-close'). Then $(f, g) \in \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$ means, so to speak, that f and g are uniformly within V of each other on the complement, relative to B, of a subset of B of exterior measure $\leqslant \delta$. Since A has an abundance of integrable (indeed, compact) subsets B, as B expands and V and δ shrink, it is clear that one has here the makings of a concept of uniform convergence for measurable functions $f : \mathrm{A} \to \mathrm{F}$.

IV.81, ℓ. 4.

"...it therefore suffices to verify the axiom $(\mathrm{U}'_{\mathrm{III}})$ (GT, II, §1, No. 1)."

One disposes of $(\mathrm{U}'_{\mathrm{I}})$ and $(\mathrm{U}'_{\mathrm{II}})$ as follows. Since every entourage V for F contains the diagonal of $\mathrm{F} \times \mathrm{F}$, the set

$$\mathrm{M} = \{x \in \mathrm{B} : \big(f(x), f(x)\big) \notin \mathrm{V}\} = \varnothing,$$

thus $|\mu|^*(\mathrm{M}) = 0 < \delta$, whence $(f, f) \in \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$ for all functions $f : \mathrm{A} \to \mathrm{F}$. And $\big(f(x), g(x)\big) \in \mathrm{V} \Leftrightarrow \big(g(x), f(x)\big) \in \overset{-1}{\mathrm{V}}$, therefore

$$(f, g) \in \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta) \quad \Leftrightarrow \quad (g, f) \in \boldsymbol{W}(\overset{-1}{\mathrm{V}}, \mathrm{B}, \delta).$$

Writing $\mathrm{W} = \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$ and $\mathrm{W}' = \boldsymbol{W}(\overset{-1}{\mathrm{V}}, \mathrm{B}, \delta)$, we then have $\mathrm{W}' = \overset{-1}{\mathrm{W}}$.

IV.81, ℓ. 4–6.

"Now, if V' is an entourage such that $\overset{2}{V'} \subset V$, then

$$\boldsymbol{W}(V', B, \delta/2) \circ \boldsymbol{W}(V', B, \delta/2) \subset \boldsymbol{W}(V, B, \delta)\,."$$

If δ_1, δ_2 are > 0, then

$$\boldsymbol{W}(V', B, \delta_1) \circ \boldsymbol{W}(V', B, \delta_2) \subset \boldsymbol{W}(V, B, \delta_1 + \delta_2)\,.$$

For, let $(f, g) \in$ left side; then, for a suitable function h, one has $(f, h) \in$ second factor and $(h, g) \in$ first factor (GT, II, §1, No. 1, footnote). Since

$$\bigl(f(x), h(x)\bigr) \in V' \;\;\&\;\; \bigl(h(x), g(x)\bigr) \in V' \quad\Rightarrow\quad \bigl(f(x), g(x)\bigr) \in V' \circ V' \subset V\,,$$

one has

$$\bigl(f(x), g(x)\bigr) \notin V \quad\Rightarrow\quad \bigl(h(x), g(x)\bigr) \notin V' \;\text{ or }\; \bigl(f(x), h(x)\bigr) \notin V'\,;$$

thus, writing

$$\begin{aligned} M &= \{x \in B :\ \bigl(f(x), g(x)\bigr) \notin V\,\} \\ M_1 &= \{x \in B :\ \bigl(h(x), g(x)\bigr) \notin V'\,\} \\ M_2 &= \{x \in B :\ \bigl(f(x), h(x)\bigr) \notin V'\,\}\,, \end{aligned}$$

we see that $M \subset M_1 \cup M_2$, therefore

$$|\mu|^*(M) \leqslant |\mu|^*(M_1) + |\mu|^*(M_2) \leqslant \delta_1 + \delta_2\,,$$

whence $(f, g) \in$ right side.

IV.81, ℓ. 9–11.

"... there exists a compact set $K \in \mathfrak{K}$ contained in B such that $|\mu|(B - K) \leqslant \delta$, and therefore $\boldsymbol{W}(V, K, \delta) \subset \boldsymbol{W}(V, B, 2\delta)$."

Since B is integrable, there exists a compact set $K_1 \subset B$ such that $|\mu|(B - K_1) \leqslant \delta/2$ (§4, No. 6, Cor. 1 of Th. 4). In turn, since $\mathfrak{K}$ is μ-dense in A (such a $\mathfrak{K}$ exists—cf. IV.77, ℓ. 1), K_1 has a subset $K \in \mathfrak{K}$ such that $|\mu|(K_1 - K) \leqslant \delta/2$ (by *b*) of Prop. 12), therefore $|\mu|(B - K) \leqslant \delta/2 + \delta/2$.

Suppose $(f, g) \in \boldsymbol{W}(V, K, \delta)$. Write

$$\begin{aligned} M_1 &= \{x \in K :\ \bigl(f(x), g(x)\bigr) \notin V\,\} \\ M_2 &= \{x \in B :\ \bigl(f(x), g(x)\bigr) \notin V\,\}\,. \end{aligned}$$

By assumption $|\mu|^*(M_1) \leqslant \delta$; we wish to show that $|\mu|^*(M_2) \leqslant 2\delta$.

From $K \subset B$ we see that $M_1 \subset M_2$. Moreover, $M_2 - M_1 \subset B - K$: for, if $x \in M_2 - M_1$ then $x \in M_2 \subset B$, therefore $\bigl(f(x), g(x)\bigr) \notin V$; if one had $x \in K$, then from $x \notin M_1$ we would infer that $\bigl(f(x), g(x)\bigr) \in V$, a contradiction. From $M_2 - M_1 \subset B - K$ we infer that

$$M_2 \subset M_1 \cup (B - K)\,,$$

whence $|\mu|^*(M_2) \leqslant |\mu|^*(M_1) + |\mu|^*(B - K) \leqslant \delta + \delta$.

IV.81, ℓ. −14 to −11.

"... the set M of $x \in \mathrm{B}$ such that $f(x) \neq g(x)$ is μ-integrable, because it is the inverse image, under the μ-measurable mapping $x \mapsto (f(x), g(x))$, of the complement of the diagonal in $\mathrm{F} \times \mathrm{F}$, which is open (No. 5, Prop. 7)"

The mapping is measurable by Th. 1′ (the generalization of No. 3, Th. 1 proved in the note for IV.80, ℓ. −17 to −14). The measurability of M then follows from Prop. 7′ (the generalization of No. 5, Prop. 7 proved in the same note), and M is then integrable by No. 6, Cor. 1 of Th. 5.

IV.81, ℓ. −11 to −9.

"... if $|\mu|(\mathrm{M}) = \alpha > 0$, there exists a compact subset $\mathrm{K} \subset \mathrm{M}$ such that $|\mu|(\mathrm{M} - \mathrm{K}) < \alpha/2$ and such that the restrictions of f and g to K are continuous"

Since M is integrable, there exists a compact set $\mathrm{K}_0 \subset \mathrm{M}$ such that $|\mu|(\mathrm{M} - \mathrm{K}_0) < \alpha/4$ (§4, No. 6, Cor. 1 of Th. 4).

The set $\mathfrak{K}$ of compact sets $\mathrm{K} \subset \mathrm{M}$ such that $(f, g)\big|\mathrm{K}$ is continuous (equivalently, both $f\big|\mathrm{K}$ and $g\big|\mathrm{K}$ are continuous) is μ-dense in M by the condition *a*) of No. 10, Prop. 15; applying *b*) of No. 8, Prop. 12 (with M playing the role of A), there exists a set $\mathrm{K} \in \mathfrak{K}$ such that $\mathrm{K} \subset \mathrm{K}_0$ and $|\mu|(\mathrm{K}_0 - \mathrm{K}) < \alpha/4$.

Finally, $|\mu|(\mathrm{M} - \mathrm{K}) \leqslant |\mu|(\mathrm{M} - \mathrm{K}_0) + |\mu|(\mathrm{K}_0 - \mathrm{K}) < \alpha/4 + \alpha/4$.

IV.81, ℓ. −9, −8.

"... there exists an entourage V_0 of F such that $(f(x), g(x)) \notin \mathrm{V}_0$ for all $x \in \mathrm{K}$"

The mapping $x \mapsto (f(x), g(x))$ $(x \in \mathrm{K})$ is a continuous mapping of the compact space K onto a compact subset S of $\mathrm{F} \times \mathrm{F}$ such that $\mathrm{S} \cap \Delta = \varnothing$, where Δ is the diagonal of $\mathrm{F} \times \mathrm{F}$ (recall that $\mathrm{K} \subset \mathrm{M}$ and $f(x) \neq g(x)$ for all $x \in \mathrm{M}$). The closed entourages form a fundamental system of entourages for F (GT, II, §1, No. 2, Cor. 2 of Prop. 2); since F is Hausdorff, it follows that $\bigcap \mathrm{V} = \Delta$, where V runs over the set of all *closed* entourages for F (*loc. cit.*, Prop. 3). Thus, since $\mathrm{S} \cap \Delta = \varnothing$, for every $p \in \mathrm{S}$ there exists a closed entourage V_p such that $p \notin \mathrm{V}_p$, that is, $p \in \complement \mathrm{V}_p$; the sets $\complement \mathrm{V}_p$ thus form an open covering of the compact set S. Let $p_1, \ldots, p_n$ be a finite set of points of S such that

$$\mathrm{S} \subset \complement \mathrm{V}_{p_1} \cup \cdots \cup \complement \mathrm{V}_{p_n} = \complement (\mathrm{V}_{p_1} \cap \cdots \cap \mathrm{V}_{p_n});$$

then $\mathrm{V}_0 = \mathrm{V}_{p_1} \cap \cdots \cap \mathrm{V}_{p_n}$ is a closed entourage such that $\mathrm{S} \cap \mathrm{V}_0 = \varnothing$, therefore

$$x \in \mathrm{K} \quad \Rightarrow \quad \big(f(x), g(x)\big) \in \mathrm{S} \quad \Rightarrow \quad \big(f(x), g(x)\big) \notin \mathrm{V}_0,$$

thus V_0 meets the requirements.

IV.81, ℓ. -8.

"... consequently $(f,g) \notin \boldsymbol{W}(\mathrm{V}_0, \mathrm{B}, \alpha/2)$."

Let $\mathrm{M}_0 = \{x \in \mathrm{B} : \big(f(x), g(x)\big) \notin \mathrm{V}_0\}$; we are to show that $|\mu|^*(\mathrm{M}_0) > \alpha/2$.

Clearly $\mathrm{K} \subset \mathrm{M}_0$ by the construction of V_0, and $\mathrm{M}_0 \subset \mathrm{M}$ because V_0 contains the diagonal Δ of $\mathrm{F} \times \mathrm{F}$ (if $x \in \mathrm{M}_0$ then $\big(f(x), g(x)\big) \notin \mathrm{V}_0$, hence $\big(f(x), g(x)\big) \notin \Delta$, that is, $f(x) \neq g(x)$, and so $x \in \mathrm{M}$); thus $\mathrm{K} \subset \mathrm{M}_0 \subset \mathrm{M}$. Then

$$\alpha/2 > |\mu|(\mathrm{M} - \mathrm{K}) = |\mu|(\mathrm{M}) - |\mu|(\mathrm{K}) = \alpha - |\mu|(\mathrm{K}),$$

whence $|\mu|(\mathrm{K}) > \alpha - \alpha/2 = \alpha/2$ and finally $|\mu|^*(\mathrm{M}_0) \geqslant |\mu|(\mathrm{K}) > \alpha/2$.

To summarize: *Write* $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ (F a Hausdorff uniform space) *and let* $\mathfrak{B}$ *be a fundamental system of entourages for* F. *For every integrable set* $\mathrm{B} \subset \mathrm{A}$,

$$\{(f,g) \in \mathscr{S} \times \mathscr{S} : \ f = g \ \text{ a.e. } \textit{in } \mathrm{B}\} = \bigcap_{\mathrm{V} \in \mathfrak{B},\, \delta > 0} \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta).$$

Proof of $\subset$: Suppose $(f,g) \in$ left side, so that the set

$$\mathrm{N} = \{x \in \mathrm{B} : \ f(x) \neq g(x)\}$$

is negligible. Let V be any entourage for F. Since V contains the diagonal Δ of $\mathrm{F} \times \mathrm{F}$, one has $\{x \in \mathrm{B} : \ \big(f(x), g(x)\big) \notin \mathrm{V}\} \subset \mathrm{N}$, therefore

$$|\mu|^*(\{x \in \mathrm{B} : \ \big(f(x), g(x)\big) \notin \mathrm{V}\}) \leqslant |\mu|^*(\mathrm{N}) = 0 < \delta$$

for all $\delta > 0$; thus $(f,g) \in \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$ for every entourage V (in particular for every $\mathrm{V} \in \mathfrak{B}$) and every $\delta > 0$, whence $(f,g) \in$ right side.

Proof of $\supset$: Assuming $(f,g) \notin$ left side, we are to show that $(f,g) \notin$ right side.

Writing $\mathrm{M} = \{x \in \mathrm{B} : \ f(x) \neq g(x)\}$ (an integrable set), by assumption $|\mu|(\mathrm{M}) = \alpha > 0$, and the earlier argument shows that there exists a closed entourage V_0 such that $(f,g) \notin \boldsymbol{W}(\mathrm{V}_0, \mathrm{B}, \alpha/2)$. Choose $\mathrm{V} \in \mathfrak{B}$ so that $\mathrm{V} \subset \mathrm{V}_0$; then $\boldsymbol{W}(\mathrm{V}, \mathrm{B}, \alpha/2) \subset \boldsymbol{W}(\mathrm{V}_0, \mathrm{B}, \alpha/2)$, therefore $(f,g) \notin \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \alpha/2)$ and so $(f,g) \notin$ right side.

IV.81, ℓ. -7 to -5.

"... if F is Hausdorff, then the intersection of *all* the entourages of $\mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ is the set of pairs (f,g) such that $f(x) = g(x)$ *locally almost everywhere in* A."

(It is then a triviality that the same is true of the intersection of *any fundamental system* of entourages of $\mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$.)

Recall that the entourages of $\mathscr{S}(\mathrm{A},\mu;\mathrm{F})$ are the supersets of the sets $\boldsymbol{W}(\mathrm{V},\mathrm{B},\delta)$, where $\delta > 0$, V runs over the set of entourages for the uniformity of F, and B runs over the set of integrable subsets of A. A fundamental system of entourages is also given by the sets $\boldsymbol{W}(\mathrm{V},\mathrm{K},\delta)$, where V runs over a fundamental system of entourages for the uniformity of F, K runs over any set $\mathfrak{K}$ of compact subsets of A that is μ-dense in A, and δ runs over, say, the values $1/n$ ($n = 1,2,3,\dots$).

Write $\mathscr{S} = \mathscr{S}(\mathrm{A},\mu;\mathrm{F})$ and let $\mathscr{N}$ be the intersection of all the entourages for $\mathscr{S}$. To be specific, let $\mathfrak{K}$ be the set of all compact subsets of A, and let $\mathfrak{B}$ be a fundamental system of entourages for the uniformity of F, so that

$$\begin{aligned}\mathscr{N} &= \bigcap_{\mathrm{K}\in\mathfrak{K},\,\mathrm{V}\in\mathfrak{B},\,\delta>0} \boldsymbol{W}(\mathrm{V},\mathrm{K},\delta)\\ &= \bigcap_{\mathrm{K}\in\mathfrak{K}}\Big(\bigcap_{\mathrm{V}\in\mathfrak{B},\,\delta>0}\boldsymbol{W}(\mathrm{V},\mathrm{K},\delta)\Big)\\ &= \bigcap_{\mathrm{K}\in\mathfrak{K}}\{(f,g)\in\mathscr{S}\times\mathscr{S} :\ f=g \ \text{ a.e. in } \mathrm{K}\}\\ &= \{(f,g)\in\mathscr{S}\times\mathscr{S} :\ \text{for every compact set } \mathrm{K}\subset\mathrm{A},\ f=g \ \text{ a.e. in } \mathrm{K}\}\end{aligned}$$

(the third equality, by the formula proved in the preceding note). For each pair $(f,g)\in\mathscr{S}\times\mathscr{S}$, write

$$\mathrm{M}(f,g) = \{x\in\mathrm{A} :\ f(x)\neq g(x)\}\,;$$

$\mathrm{M}(f,g)$ is a measurable set (by the argument in the note for IV.81, $\ell.\ -14$ to -11). The assertion to be proved is that

$$(f,g)\in\mathscr{N} \quad\Leftrightarrow\quad \mathrm{M}(f,g) \text{ is locally negligible,}$$

in other words (No. 2, Prop. 5) that

$$(f,g)\in\mathscr{N} \quad\Leftrightarrow\quad \mathrm{K}\cap\mathrm{M}(f,g) \text{ is negligible for every compact set } \mathrm{K}\subset\mathrm{X}.$$

Proof of $\Leftarrow$: In particular, for every compact set $\mathrm{K}\subset\mathrm{A}$, the set $\mathrm{K}\cap\mathrm{M}(f,g)$ is negligible; since

$$\mathrm{K} - \mathrm{K}\cap\mathrm{M}(f,g) = \{x\in\mathrm{K} :\ f(x) = g(x)\}\,,$$

this means that $f = g$ almost everywhere in K. Thus $(f,g)\in\mathscr{N}$ by the formula for $\mathscr{N}$ proved above.

Proof of $\Rightarrow$: Assume $(f,g) \in \mathscr{N}$ and let $\mathrm{K} \subset \mathrm{X}$ be compact; we are to show that $\mathrm{K} \cap \mathrm{M}(f,g)$ is negligible. Since $\mathrm{M}(f,g)$ is measurable, there exists a partition

$$(*) \qquad \mathrm{K} \cap \mathrm{M}(f,g) = \mathrm{N} \cup \bigcup_n \mathrm{K}_n$$

with N negligible and (K_n) a sequence of compact sets (see the note for IV.67, ℓ. -1).

Now, $\mathrm{K}_n \subset \mathrm{M}(f,g) \subset \mathrm{A}$. Since $(f,g) \in \mathscr{N}$, we know that $f = g$ almost everywhere in K_n (by the formula for $\mathscr{N}$ proved above), that is, the set

$$\{x \in \mathrm{K}_n : \ f(x) \neq g(x)\} = \mathrm{K}_n \cap \mathrm{M}(f,g) = \mathrm{K}_n$$

is negligible. It then follows from $(*)$ that $\mathrm{K} \cap \mathrm{M}(f,g)$ is negligible.

IV.81, ℓ. -5 to -1.

"The Hausdorff uniform space associated with $\mathscr{S}(\mathrm{A},\mu;\mathrm{F})$, which we shall denote $\mathrm{S}(\mathrm{A},\mu;\mathrm{F})$ ... therefore consists of the *equivalence classes* for the relation « $f(x) = g(x)$ locally almost everywhere in A » in the set $\mathscr{S}(\mathrm{A},\mu;\mathrm{F})$."

It is implicit in the wording that the notation $\mathrm{S}(\mathrm{A},\mu;\mathrm{F})$ is reserved for the case that F is Hausdorff, as the following argument will show.

Consider the uniform space $\mathscr{S} = \mathscr{S}(\mathrm{A},\mu;\mathrm{F})$, with F not necessarily Hausdorff. The associated Hausdorff space S can be obtained as follows. As in the preceding note, let $\mathscr{N}$ be the intersection of all the entourages for $\mathscr{S}$. The condition « $(f,g) \in \mathscr{N}$ » clearly defines an equivalence relation $f \sim g$ in $\mathscr{S}$; write $\dot{f}$ for the equivalence class of f, and

$$\mathrm{S} = \mathscr{S}/\!\sim\, = \{\dot{f} : \ f \in \mathscr{S}\}$$

for the quotient set. For each entourage W for $\mathscr{S}$, write

$$\dot{\mathrm{W}} = \{(\dot{f},\dot{g}) : \ (f,g) \in \mathrm{W}\};$$

it is routine to check that the sets $\dot{\mathrm{W}}$ form a fundamental system of entourages for a uniform structure on S, and this structure is Hausdorff since

$$\bigcap_{\mathrm{W}} \dot{\mathrm{W}} = \bigcap_{\mathrm{W}} \{(\dot{f},\dot{g}) : \ (f,g) \in \mathrm{W}\} = \{(\dot{f},\dot{g}) : \ (f,g) \in \mathrm{W} \text{ for all } \mathrm{W}\}$$
$$= \{(\dot{f},\dot{g}) : \ (f,g) \in \mathscr{N} = \bigcap \mathrm{W}\} = \{(\dot{f},\dot{g}) : \ \dot{f} = \dot{g}\}$$

(GT, II, §1, No. 2, Prop. 3).

In brief, S is derived from $\mathscr{S}$ by passing to quotients for the equivalence relation « $(f,g) \in \mathscr{N}$ ». For this to be the same relation as

« $f(x) = g(x)$ locally almost everywhere in A »,

one requires the formula for $\mathscr{N}$ in the preceding note, whose proof assumes that F is Hausdorff.

IV.82, ℓ. 7.

"It follows from No. 10, Prop. 16 that ψ is *bijective*."

It is tacit that F is Hausdorff (see the preceding note). Let $A_0 = \bigcup_{\lambda\in L} A_\lambda$ and $N = A - A_0$; thus A_0 is measurable (IV.77, ℓ. −8 to −6) and N is locally negligible.

ψ is *injective*. Suppose $f, g \in \mathscr{S}(A, \mu; F)$ with $\psi(\dot f) = \psi(\dot g)$, that is, $\dot f_\lambda = \dot g_\lambda$ for all λ, in other words $f|A_\lambda = g|A_\lambda$ locally almost everywhere for every λ; thus, for each λ there exists a locally negligible set $N_\lambda \subset A_\lambda$ such that $f = g$ on $A_\lambda - N_\lambda$. Then (pairwise disjointness)

$$f = g \quad \text{on} \quad \bigcup_\lambda (A_\lambda - N_\lambda) = \bigcup_\lambda A_\lambda - \bigcup_\lambda N_\lambda = A_0 - N_0\,,$$

where $N_0 = \bigcup_\lambda N_\lambda$; since the A_λ are locally countable, so are the N_λ, therefore N_0 is also locally negligible (*loc. cit.*).

Since $N \cup N_0$ is locally negligible and $A - (N\cup N_0) = (A - N) - N_0 = A_0 - N_0$, it follows that $f = g$ locally almost everywhere (on A), that is, $\dot f = \dot g$.

ψ is *surjective*. Let $u = (u_\lambda) \in \prod_\lambda S(A_\lambda, \mu; F)$; we seek a function $f \in \mathscr{S}(A, \mu; F)$ such that $\psi(\dot f) = u$, that is, $\dot f_\lambda = u_\lambda$ for all λ.

For each λ, let g^λ be any mapping in $\mathscr{S}(A_\lambda, \mu; F)$ whose equivalence class is u_λ (it is not assumed that g^λ is the restriction to A_λ of a mapping g on A). Let $f_0 = A_0 \to F$ be the mapping such that $f_0|A_\lambda = g^\lambda$ for all λ (disjointness); since g^λ is μ-measurable in A_λ for every λ, it follows that f_0 is μ-measurable in A_0 (No. 10, Prop. 16). Let $f_1 : X \to F$ be a measurable extension of f_0 to X (No. 10, Def. 8), and define $f = f_1|A$; then f is μ-measurable (Def. 8), that is, $f \in \mathscr{S}(A, \mu; F)$, and, for all λ,

$$f_\lambda = f|A_\lambda = (f_1|A)|A_\lambda = f_1|A_\lambda = (f_1|A_0)|A_\lambda = f_0|A_\lambda = g^\lambda$$

(the first equality defines f_λ, as in ℓ. 4, 5), whence $\dot f_\lambda = u_\lambda$ for all λ.

IV.82, ℓ. 7–9.

"Consider an entourage T of $S(A, \mu; F)$ that is the canonical image of a $\boldsymbol{W}(V, B, \delta)$, where B is a *compact* subset of A"

The sets $W = \boldsymbol{W}(V, B, \delta)$ form a fundamental system of entourages for $\mathscr{S}(A, \mu; F)$ (see the note for IV.81, ℓ. 9–11), therefore the sets $T = \dot W = \{(\dot f, \dot g) : (f, g) \in W\}$ form a fundamental system of entourages for $S(A, \mu; F)$ (see the Note for IV.81, ℓ. −5 to −1).

IV.82, ℓ. 9, 10.

"... the set J of $\lambda \in \mathrm{L}$ such that $\mathrm{B} \cap \mathrm{A}_\lambda \neq \varnothing$ is countable (No. 9), and $|\mu|(\mathrm{B}) = \sum_{\lambda \in \mathrm{J}} |\mu|(\mathrm{B} \cap \mathrm{A}_\lambda)$ "

Since the family $(\mathrm{A}_\lambda)_{\lambda \in \mathrm{J}}$ is locally countable, J is countable (paragraph following No. 9, Def. 7). With notations for A_0 and N as in the note for ℓ. 7, one then has

$$\mathrm{B} = \mathrm{B} \cap \mathrm{A} = (\mathrm{B} \cap \mathrm{A}_0) \cup (\mathrm{B} \cap \mathrm{N}) = \Big(\bigcup_{\lambda \in \mathrm{J}} \mathrm{B} \cap \mathrm{A}_\lambda \Big) \cup (\mathrm{B} \cap \mathrm{N}) \,,$$

where $\mathrm{B} \cap \mathrm{N}$ is negligible (No. 2, Prop. 5), whence the formula for $|\mu|(\mathrm{B})$ (§4, No. 5, Prop. 9).

IV.82, ℓ. 13–16.

"The image of T under $\psi \times \psi$ is ... which proves the proposition."

Let us pause to review product uniform spaces. Let $(\mathrm{X}_\lambda)_{\lambda \in \mathrm{L}}$ be a family of uniform spaces, $\mathrm{X} = \prod_{\lambda \in \mathrm{L}} \mathrm{X}_\lambda$ the product set, and $\mathrm{pr}_\lambda : \mathrm{X} \rightarrow \mathrm{X}_\lambda$ $(\lambda \in \mathrm{L})$ the family of projection mappings. The product uniformity of X is the initial uniform structure for the family $(\mathrm{pr}_\lambda)_{\lambda \in \mathrm{L}}$, that is, the coarsest uniform structure on X that renders uniformly continuous every pr_λ (GT, II, §2, No. 6, Def. 4). If $\lambda_0 \in \mathrm{L}$ and $\mathrm{W}_{\lambda_0} \subset \mathrm{X}_{\lambda_0} \times \mathrm{X}_{\lambda_0}$, then

$$(\mathrm{pr}_{\lambda_0} \times \mathrm{pr}_{\lambda_0})^{-1}(\mathrm{W}_{\lambda_0}) = \mathrm{W}_{\lambda_0} \times \prod_{\lambda \neq \lambda_0} (\mathrm{X}_\lambda \times \mathrm{X}_\lambda) \,.$$

More generally, if H is a finite subset of L, and if, for each $\lambda \in \mathrm{H}$, W_λ is a subset of $\mathrm{X}_\lambda \times \mathrm{X}_\lambda$, then

$$(*) \qquad \bigcap_{\lambda \in \mathrm{H}} (\mathrm{pr}_\lambda \times \mathrm{pr}_\lambda)^{-1}(\mathrm{W}_\lambda) = \prod_{\lambda \in \mathrm{H}} \mathrm{W}_\lambda \times \prod_{\lambda \in \mathrm{L} - \mathrm{H}} (\mathrm{X}_\lambda \times \mathrm{X}_\lambda) \,.$$

If H runs over the set of finite subsets of L and if, for each $\lambda \in \mathrm{H}$, W_λ runs over a fundamental system of entourages for X_λ, then the sets $(*)$ form a fundamental system of entourages for the product uniformity on X (GT, *loc. cit.*, No. 3, Prop. 4). {See also J.L. Kelley, *General topology* (Van Nostrand, New York, 1955), p. 182.}

ψ^{-1} *is uniformly continuous.* Given a basic entourage W for $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$, so that the set

$$\dot{\mathrm{W}} = \{ (\dot{f}, \dot{g}) : \ (f, g) \in \mathrm{W} \}$$

is a basic entourage for $S = S(A, \mu; F)$ (see the note for IV.81, ℓ. −5 to −1), it will suffice to show that $(\psi \times \psi)(\dot{W}) \supset \mathscr{V}$ for some entourage $\mathscr{V}$ for $\prod_{\lambda \in L} S_\lambda$, where $S_\lambda = S(A_\lambda, \mu; F)$ for all $\lambda \in L$; for then, writing

$$\Phi = \psi^{-1} \times \psi^{-1} = (\psi \times \psi)^{-1} : \prod_{\lambda \in L} S_\lambda \times \prod_{\lambda \in L} S_\lambda \to S \times S,$$

one has $\Phi^{-1}(\dot{W}) = (\psi \times \psi)(\dot{W}) \supset \mathscr{V}$, thus $\Phi^{-1}(\dot{W})$ contains—hence is—an entourage for $\prod_\lambda S_\lambda$, which will show that ψ^{-1} is uniformly continuous (GT, II, §2, No. 1, remarks following Def. 1).

For the basic entourage W for $\mathscr{S}$, take $W = \boldsymbol{W}(V, B, \delta)$, with V an entourage for F, B a compact subset of A, and $\delta > 0$, so that $\dot{W}$ is the set T of the text; we seek an entourage $\mathscr{V}$ for $\prod_\lambda S_\lambda$ such that $(\psi \times \psi)(\dot{W}) \supset \mathscr{V}$.

Let $H \subset J \subset L$ (J countable, H finite) be chosen as in the text and let m be the number of elements of H. For $\lambda \in H$ write $W_\lambda = \boldsymbol{W}(V, A_\lambda, \delta/2m)$, and

$$\dot{W}_\lambda = \{(\dot{f}^\lambda, \dot{g}^\lambda) : \ f^\lambda, g^\lambda \in \mathscr{S}_\lambda = \mathscr{S}(A_\lambda, \mu; F)\} = \{(u_\lambda, v_\lambda) : \ u_\lambda, v_\lambda \in S_\lambda\};$$

then $\dot{W}_\lambda$ is an entourage for S_λ, and the set

$$\mathscr{V} = \Big(\prod_{\lambda \in H} \dot{W}_\lambda\Big) \times \Big(\prod_{\lambda \in L - H} (S_\lambda \times S_\lambda)\Big) \subset \prod_{\lambda \in L} S_\lambda \times \prod_{\lambda \in L} S_\lambda$$

is an entourage for the product uniformity on $\prod_{\lambda \in L} S_\lambda$. Let us show that $\mathscr{V} \subset (\psi \times \psi)(\dot{W})$.

Let $(u, v) \in \mathscr{V}$. Say $u = (u_\lambda)_{\lambda \in L}$, $v = (v_\lambda)_{\lambda \in L}$, so that $(u_\lambda, v_\lambda) \in \dot{W}_\lambda$ for all $\lambda \in H$ (but no restriction on the $u_\lambda, v_\lambda \in S_\lambda$ for $\lambda \in L - H$). Since $\psi : S \to \prod_{\lambda \in L} S_\lambda$ is bijective, there exist functions $f, g \in \mathscr{S}$ such that $\psi(\dot{f}) = u$ and $\psi(\dot{g}) = v$, and so $(\psi \times \psi)\big((\dot{f}, \dot{g})\big) = (u, v)$. Thus, for every $\lambda \in L$, u_λ (resp. v_λ) is the equivalence class of $f_\lambda = f\big|A_\lambda$ (resp. $g_\lambda = g\big|A_\lambda$). By the definition of $\mathscr{V}$, for every $\lambda \in H$ one has $(u_\lambda, v_\lambda) \in \dot{W}_\lambda$ and so

$$|\mu|^*(\{x \in A_\lambda : \ \big(f(x), g(x)\big) \notin V\} \leqslant \delta/2m.$$

Write

$$M_\lambda = \{x \in A_\lambda : \ \big(f(x), g(x)\big) \notin V\} \quad \text{for } \lambda \in H,$$
$$M = \{x \in B : \ \big(f(x), g(x)\big) \notin V\};$$

for all $\lambda \in \mathrm{H}$ we have $\mathrm{M} \cap \mathrm{A}_\lambda \subset \mathrm{M}_\lambda$ and $|\mu|^*(\mathrm{M}_\lambda) \leqslant \delta/2m$.

If we can show that $|\mu|^*(\mathrm{M}) \leqslant \delta$, it will follow that $(f, g) \in \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$, whence $(\dot{f}, \dot{g}) \in \dot{\mathrm{W}}$, and so

$$(u, v) = (\psi \times \psi)\big((\dot{f}, \dot{g})\big) \in (\psi \times \psi)(\dot{\mathrm{W}}),$$

as we wish to show.

Now, $\mathrm{M} \subset \mathrm{B} \subset \mathrm{A}$, and we have a partition $\mathrm{A} = \mathrm{N} \cup \bigcup_{\lambda \in \mathrm{L}} \mathrm{A}_\lambda$ with N locally negligible. Then

$$\begin{aligned} \mathrm{M} = \mathrm{M} \cap \mathrm{B} \cap \mathrm{A} &= \mathrm{M} \cap \mathrm{B} \cap \Big(\mathrm{N} \cup \bigcup_{\lambda \in \mathrm{L}} \mathrm{A}_\lambda\Big) \\ &= (\mathrm{M} \cap \mathrm{B} \cap \mathrm{N}) \cup \Big[\mathrm{M} \cap \Big(\bigcup_{\lambda \in \mathrm{L}} \mathrm{B} \cap \mathrm{A}_\lambda\Big)\Big] \\ &= (\mathrm{M} \cap \mathrm{N}) \cup \Big[\mathrm{M} \cap \Big(\bigcup_{\lambda \in \mathrm{J}} \mathrm{B} \cap \mathrm{A}_\lambda\Big)\Big] \\ &= (\mathrm{M} \cap \mathrm{N}) \cup \Big[\bigcup_{\lambda \in \mathrm{J}} \mathrm{M} \cap \mathrm{A}_\lambda\Big] \\ &= (\mathrm{M} \cap \mathrm{N}) \cup \Big[\bigcup_{\lambda \in \mathrm{H}} \mathrm{M} \cap \mathrm{A}_\lambda\Big] \cup \Big[\bigcup_{\lambda \in \mathrm{J} - \mathrm{H}} \mathrm{M} \cap \mathrm{A}_\lambda\Big]; \end{aligned}$$

but $\mathrm{M} \cap \mathrm{A}_\lambda \subset \mathrm{M}_\lambda$ for all $\lambda \in \mathrm{H}$, and $\mathrm{M} \cap \mathrm{A}_\lambda \subset \mathrm{B} \cap \mathrm{A}_\lambda$ for all $\lambda \in \mathrm{J} - \mathrm{H}$, therefore

$$\mathrm{M} \subset (\mathrm{M} \cap \mathrm{N}) \cup \Big[\bigcup_{\lambda \in \mathrm{H}} \mathrm{M}_\lambda\Big] \cup \Big[\bigcup_{\lambda \in \mathrm{J} - \mathrm{H}} \mathrm{B} \cap \mathrm{A}_\lambda\Big].$$

The first term on the right side is negligible; in the middle term, $|\mu|^*(\mathrm{M}_\lambda) \leqslant \delta/2m$ for all $\lambda \in \mathrm{H}$; and the third term has outer measure $\leqslant \delta/2$ by the choice of H; therefore

$$|\mu|^*(\mathrm{M}) \leqslant 0 + m \cdot (\delta/2m) + \delta/2 = \delta,$$

as we wished to show.

ψ is uniformly continuous. Given a basic entourage $\mathscr{V}$ for the product uniformity on $\prod_{\lambda \in \mathrm{L}} \mathrm{S}_\lambda$, it will suffice to find an entourage W for $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ such that $(\psi \times \psi)(\dot{\mathrm{W}}) \subset \mathscr{V}$ (GT, *loc. cit.*).

We can suppose that

$$\mathscr{V} = \prod_{\lambda \in \mathrm{H}} \dot{\mathrm{W}}_\lambda \times \prod_{\lambda \in \mathrm{L} - \mathrm{H}} \mathrm{S}_\lambda \times \mathrm{S}_\lambda,$$

where H is a finite subset of L and, for each $\lambda \in \mathrm{H}$, $\mathrm{W}_\lambda = \boldsymbol{W}(\mathrm{V}, \mathrm{B}_\lambda, \delta)$ in the context of $\mathscr{S}_\lambda = \mathscr{S}(\mathrm{A}_\lambda, \mu; \mathrm{F})$, with V a basic entourage for F, B_λ a compact subset of A_λ, and $\delta > 0$; that is, $\mathscr{V}$ is the set of pairs of families $\big((u_\lambda), (v_\lambda)\big) \in \prod_{\lambda \in \mathrm{L}} \mathrm{S}_\lambda \times \prod_{\lambda \in \mathrm{L}} \mathrm{S}_\lambda$ such that, for $\lambda \in \mathrm{H}$, $(u_\lambda, v_\lambda) \in \dot{\mathrm{W}}_\lambda$.

Construct W as follows: let $\mathrm{B} = \bigcup_{\lambda \in \mathrm{H}} \mathrm{B}_\lambda$ (a compact subset of A such that $\mathrm{B} \cap \mathrm{A}_\lambda = \mathrm{B}_\lambda$ for $\lambda \in \mathrm{H}$, and $\mathrm{B} \cap \mathrm{A}_\lambda = \varnothing$ for $\lambda \in \mathrm{L} - \mathrm{H}$), and form the entourage $\mathrm{W} = \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$ for $\mathscr{S}$; let us show that the entourage $\dot{\mathrm{W}}$ for $\mathrm{S} = \mathrm{S}(\mathrm{A}, \mu; \mathrm{F})$ satisfies $(\psi \times \psi)(\dot{\mathrm{W}}) \subset \mathscr{V}$.

Let $(f, g) \in \mathrm{W}$. Then $(\psi \times \psi)\big((\dot{f}, \dot{g})\big) = \big((u_\lambda), (v_\lambda)\big)$, where u_λ (resp. v_λ) is the equivalence class of the function $f_\lambda = f\big|\mathrm{A}_\lambda$ (resp. $g_\lambda = g\big|\mathrm{A}_\lambda$), so that $(u_\lambda, v_\lambda) = (\dot{f}_\lambda, \dot{g}_\lambda)$; to show that $(\psi \times \psi)\big((\dot{f}, \dot{g})\big) \in \mathscr{V}$ we need only show that $(u_\lambda, v_\lambda) \in \dot{\mathrm{W}}_\lambda$ for all $\lambda \in \mathrm{H}$.

Let $\lambda \in \mathrm{H}$. Then

$$\{x \in \mathrm{B}_\lambda : \big(f_\lambda(x), g_\lambda(x)\big) \notin \mathrm{V}\} \subset \{x \in \mathrm{B} : \big(f(x), g(x)\big) \notin \mathrm{V}\};$$

since the outer measure of the right side is $\leqslant \delta$ (because $(f, g) \in \mathrm{W}$), the outer measure of the left side is also $\leqslant \delta$, in other words, $(f_\lambda, g_\lambda) \in \mathrm{W}_\lambda$, and so $(u_\lambda, v_\lambda) \in \dot{\mathrm{W}}_\lambda$.

IV.82, ℓ. −16 to −14.

"Since each A_n is the union of a negligible set and a sequence of compact sets, we can suppose that the A_n are already *compact* and pairwise disjoint."

Say $\mathrm{A} = \mathrm{N} \cup \bigcup_n \mathrm{A}_n$ with N locally negligible; replacing N by $\mathrm{N} - \bigcup_n \mathrm{A}_n$ we can suppose that $\mathrm{N} \cap \bigcup_n \mathrm{A}_n = \varnothing$. Then, replacing A_n by $\mathrm{A}_n - \bigcup_{k<n} \mathrm{A}_k$ (§4, No. 5) we can suppose that the A_n are pairwise disjoint. For each n, there exists a partition $\mathrm{A}_n = \mathrm{N}_n \cup \bigcup_k \mathrm{A}_{nk}$ with N_n negligible and (A_{nk}) a sequence of pairwise disjoint compact sets (§4, No. 6, Cor. 2 of Th. 4). Then

$$\mathrm{A} = \Big(\mathrm{N} \cup \bigcup_n \mathrm{N}_n\Big) \cup \bigcup_{n,k} \mathrm{A}_{nk},$$

where $\mathrm{N} \cup \bigcup_n \mathrm{N}_n$ is locally negligible, disjoint from $\bigcup_{n,k} \mathrm{A}_{nk}$, and the A_{nk} are pairwise disjoint compact sets.

IV.82, ℓ. −14.

"Prop. 17 then allows us to suppose that A is compact."

Writing $\mathrm{S} = \mathrm{S}(\mathrm{A}, \mu; \mathrm{F})$ and $\mathrm{S}_n = \mathrm{S}(\mathrm{A}_n, \mu; \mathrm{F})$, we know from Prop. 17 that the mapping $\psi : \mathrm{S} \to \prod_n \mathrm{S}_n$ is an isomorphism of uniform spaces. If the compact case is established, so that every S_n is metrizable, it will follow that $\prod_n \mathrm{S}_n$, hence S, is metrizable (GT, IX, §2, No. 4, Cor. 2 of Th. 1).

IV.82, ℓ. −14 to −11.

"If (V_n) is a countable fundamental system of entourages of F, it is clear that the $\boldsymbol{W}(\mathrm{V}_n, \mathrm{A}, 1/n)$ form a fundamental system of entourages of $\mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ as n runs over $\mathbf{N}$ "

Our objective is to prove that $\mathrm{S} = \mathrm{S}(\mathrm{A}, \mu; \mathrm{F})$ has a countable fundamental system of entourages (GT, IX, §2, No. 4, Th. 1), and to this end we are free to choose any countable fundamental system of entourages (V_n) for F; replacing V_n by $\mathrm{V}_1 \cap \cdots \cap \mathrm{V}_n$, we can suppose that the sequence (V_n) is decreasing. It will suffice to show that $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ has a countable fundamental system of entourages.

Let W be any entourage for $\mathscr{S}$. We know (IV.81, ℓ. 7–9) that there exist an entourage V of F, a compact subset K of A, and a number $\delta > 0$ such that $\boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta) \subset \mathrm{W}$. Choose an index m with $\mathrm{V}_m \subset \mathrm{V}$; since the sequence (V_n) is decreasing, we can choose m as large as we like, so we can suppose that $1/m < \delta$. Then (IV.81, ℓ. 1–3)

$$\boldsymbol{W}(\mathrm{V}_m, \mathrm{A}, 1/m) \subset \boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta) \subset \mathrm{W},$$

thus the entourages $\boldsymbol{W}(\mathrm{V}_n, \mathrm{A}, 1/n)$ form a fundamental system of entourages for $\mathscr{S}$.

IV.82, ℓ. −11.

"... whence the proposition."

GT, IX, §2, No. 4, Th. 1.

IV.83, ℓ. 1, 2.

"... the possibility of such a definition follows from the fact that (f_n) is a Cauchy sequence in $\mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$."

The crux of the matter is the following construction: If $(\delta_k)_{k \geqslant 0}$ is a decreasing sequence of numbers > 0 such that $\delta_k \to 0$, then there exists a subsequence $(f_{n_k})_{k \geqslant 0}$ (also Cauchy) of (f_n) such that

$$|\mu|\big(\{x \in \mathrm{B} : \ d\big(f_{n_k}(x), f_{n_{k+1}}(x)\big) > \delta_k \}\big) \leqslant \delta_k \quad (k = 0, 1, 2, \ldots).$$

For, set

$$\mathrm{V}_k = \{(y, y') \in \mathrm{F} \times \mathrm{F} : \ d(y, y') \leqslant \delta_k \} \quad (k = 0, 1, 2, \ldots)$$

(since d is compatible with the uniformity of F, the V_k form a fundamental system of entourages for F). Since B is integrable, we may form the sets

$$\mathrm{W}_k = \boldsymbol{W}(\mathrm{V}_k, \mathrm{B}, \delta_k);$$

W_k is the set of all pairs $(f,g) \in \mathscr{S} = \mathscr{S}(\mathrm{A},\mu;\mathrm{F})$ such that

$$|\mu|\big(\{x \in \mathrm{B} :\ \big(f(x),g(x)\big) \notin \mathrm{V}_k\,\}\big) \leqslant \delta_k\,,$$

that is,

$$|\mu|\big(\{x \in \mathrm{B} :\ d\big(f(x),g(x)\big) > \delta_k\,\}\big) \leqslant \delta_k\,.$$

It is clear that the W_k form a fundamental system of entourages for the uniformity of $\mathscr{S}$ (IV.81, ℓ. 1–3). Construct, by induction on k, a sequence $(n_k)_{k\geqslant 0}$ of indices $n_0 < n_1 < n_2 < \ldots$ such that

$$(*) \qquad n, n' \geqslant n_k \quad \Rightarrow \quad (f_n, f_{n'}) \in \mathrm{W}_k\,,$$

as follows: since (f_n) is Cauchy in $\mathscr{S}$, we may choose n_0 so that

$$n, n' \geqslant n_0 \quad \Rightarrow \quad (f_n, f_{n'}) \in \mathrm{W}_0$$

(GT, II, §3, No. 1, second paragraph following Def. 2). Having chosen $n_0 < \ldots < n_{r-1}$ satisfying $(*)$ for $k = 0,\ldots,r-1$, choose an index n_r such that $(*)$ is satisfied for $k = r$; since $(*)$ continues to hold if n_r is replaced by a larger integer, we can suppose that $n_r > n_{r-1}$, which completes the induction. In particular, for every k one has $n_{k+1} > n_k \geqslant n_k$ and so $(f_{n_k}, f_{n_{k+1}}) \in \mathrm{W}_k$, thus

$$(**) \qquad |\mu|\big(\{x \in \mathrm{B} :\ d\big(f_{n_k}(x), f_{n_{k+1}}(x)\big) > \delta_k\,\}\big) \leqslant \delta_k$$

by the definition of W_k.

For each $m \geqslant 0$, the sequence $(f_{mn})_{n\geqslant 0}$ is constructed inductively, as follows.

$m = 0$: Already defined is $f_{0n} = f_n$ $(n = 0,1,2,\ldots)$.

$m = 1$: Set $\delta_{1,k} = 1/2^{m+k+1} = 1/2^{1+k+1}$. Since (f_n) is Cauchy in $\mathscr{S}$, the argument for $(**)$ shows that (f_n) has a subsequence $(f_{1n})_{n\geqslant 0}$ (of course Cauchy) such that, writing

$$\mathrm{M}_{1n} = \{x \in \mathrm{B} :\ d\big(f_{1n}(x), f_{1,n+1}(x)\big) > 1/2^{1+n+1}\,\}$$

(a measurable set, by No. 3, Th. 1 and No. 5, Prop. 7), one has $|\mu|(\mathrm{M}_{1n}) \leqslant 1/2^{1+n+1}$. Note that

$$\sum_{n\geqslant 0} |\mu|(\mathrm{M}_{1n}) \leqslant \sum_{n\geqslant 0} \frac{1}{2^{1+n+1}} = \frac{1}{2}\sum_{n\geqslant 0} \frac{1}{2^{n+1}} = \frac{1}{2}\cdot 1 = 1/2^1\,.$$

$m > 1$: Assuming the (Cauchy) sequence $(f_{m-1,n})_{n\geqslant 0}$ already defined, set $\delta_{m,k} = 1/2^{m+k+1}$ and, by the argument for $(**)$, choose a subsequence $(f_{mn})_{n\geqslant 0}$ of $(f_{m-1,n})_{n\geqslant 0}$ such that, writing

$$\mathrm{M}_{mn} = \{x \in \mathrm{B} : \ d\big(f_{mn}(x), f_{m,n+1}(x)\big) > 1/2^{m+n+1}\},$$

one has $|\mu|(\mathrm{M}_{mn}) \leqslant 1/2^{m+n+1}$ $(n = 0, 1, 2, \dots)$, which completes the induction. Note that

$$(***) \qquad \sum_{n\geqslant 0} |\mu|(\mathrm{M}_{mn}) \leqslant \frac{1}{2^m} \sum_{n\geqslant 0} \frac{1}{2^{n+1}} = 1/2^m \qquad (m = 1, 2, 3, \dots)$$

For notational completeness, and convenience later in the proof, we define $\mathrm{M}_{0n} = \mathrm{B}$ for all $n \geqslant 0$.

IV.83, ℓ. 2, 3.

"Set $\mathrm{M}_m = \bigcup_{n\geqslant 0} \mathrm{M}_{mn}$; then

$$|\mu|(\mathrm{M}_m) \leqslant \sum_{n=0}^{\infty} |\mu|(\mathrm{M}_{mn}) \leqslant 1/2^m \text{ "}$$

It is intended here that $m \geqslant 1$; the inequality then follows from $(***)$ of the preceding note.

By the convention established at the end of the preceding note, $\mathrm{M}_0 = \mathrm{B}$; the above inequalities assure that the set $\mathrm{N} = \bigcap_{m\geqslant 0} \mathrm{M}_m$ is negligible, irrespective of the value of $|\mu|(\mathrm{M}_0)$.

IV.83, ℓ. 4, 5.

"...and, for every $x \in \mathrm{B} - \mathrm{M}_m$, we have $d(f_{mn}(x), f_{m,n+p}(x)) \leqslant 1/2^{m+n}$ for all $n \geqslant 0$ and all $p > 0$"

It is assumed here that $m \geqslant 1$. Thus,

$$x \in \mathrm{B} - \mathrm{M}_m = \mathrm{B} - \bigcup_{n\geqslant 0} \mathrm{M}_{mn} = \bigcap_{n\geqslant 0} (\mathrm{B} - \mathrm{M}_{mn})$$

means that $d\big(f_{mn}(x), f_{m,n+1}(x)\big) \leqslant 1/2^{m+n+1}$ for all $n \geqslant 0$. It then follows from the triangle inequality that

$$d\big(f_{mn}(x), f_{m,n+p}(x)\big) \leqslant \sum_{k=0}^{p-1} d\big(f_{m,n+k}(x), f_{m,n+k+1}(x)\big) \leqslant \sum_{k=0}^{p-1} \frac{1}{2^{m+n+k+1}}$$
$$= \frac{1}{2^{m+n}} \sum_{k=0}^{p-1} \frac{1}{2^{k+1}} < \frac{1}{2^{m+n}} \cdot 1 .$$

IV.83, ℓ. 8–10.

"...if $x \in B - N$, there exists an index m such that $x \notin M_m$, which proves that the sequence $(g_n(x))$ is a Cauchy sequence in F."

Since $x \notin N$ we know that there exists an index $m \geqslant 0$ such that $x \notin M_m$. But $x \in B = M_0$, so $m \geqslant 1$; by the preceding note, $(f_{mn}(x))_{n\geqslant 0}$ is Cauchy, hence so its its subsequence $(g_n(x))_{n\geqslant m}$.

IV.83, ℓ. 11–15.

"If now B is the union of a sequence (B_m) ... where P_m is negligible."

The notations for $m = 0$ can be handled as follows. We can suppose that $B_0 = \varnothing$; setting $P_0 = \varnothing$ and $g_{0n} = f_n$ for all $n \geqslant 0$, the sequence $(g_{n0})_{n\geqslant 0}$ is vacuously pointwise Cauchy in $B_0 - P_0$. The recursive definition then proceeds as in the text.

IV.83, ℓ. $-11, -10$.

"...we are thus reduced to proving the proposition when A is *integrable*"

A product of nonempty uniform spaces is complete if and only if every factor is complete (GT, II, §3, No. 5, Prop. 10).

IV.83, ℓ. $-9, -8$.

"...for every Cauchy sequence (f_n) in $\mathscr{S}(A, \mu; F)$ there exists a subsequence (f_{n_k}) that is convergent in $A - N$, where N is negligible"

This is immediate from Lemma 4 and the completeness of F.

But... where does the Cauchy sequence (f_n) come from? We are to show that $S = S(A, \mu; F)$ is complete, thus one begins with a Cauchy sequence (u_n) in S. Say $f_n \in \mathscr{S} = \mathscr{S}(A, \mu; F)$ with $u_n = \dot{f}_n$ (in the notation of Prop. 17); we need to know that (f_n) is Cauchy in $\mathscr{S}$.

Lemma. Let A be any measurable subset of X, and $W = \boldsymbol{W}(V, B, \delta)$, where V is an entourage for F, B is an integrable subset of A, and $\delta > 0$, and let

$$\dot{W} = \{(\dot{f}, \dot{g}) : (f, g) \in W\}.$$

Then, for a pair of functions f, g in $\mathscr{S}$, one has

$$(f, g) \in W \quad \Leftrightarrow \quad (\dot{f}, \dot{g}) \in \dot{W}.$$

$\Rightarrow$: By the definition of $\dot{W}$.

$\Leftarrow$: Suppose $(\dot{f}, \dot{g}) \in \dot{W}$. This means that there exists a pair $(h, k) \in W$ such that $(\dot{f}, \dot{g}) = (\dot{h}, \dot{k})$, that is, locally almost everywhere, $f = h$ and $g = k$. Let M be a locally negligible subset of A such that $f = h$ and $g = k$ everywhere in $A - M$. Since B is integrable, $B \cap M$ is negligible (No. 2, Cor. 1 of Prop. 5), and $f = h$ and $g = k$ everywhere in $\mathrm{B} \cap (\mathrm{A} - \mathrm{M}) = \mathrm{B} - \mathrm{B} \cap \mathrm{M}$; it follows that the sets

$$\{x \in \mathrm{B} : \ \big(f(x), g(x)\big) \notin \mathrm{V}\,\} \quad \text{and} \quad \{x \in \mathrm{B} : \ \big(h(x), k(x)\big) \notin \mathrm{V}\,\}$$

differ at most by the negligible set $\mathrm{B} \cap \mathrm{M}$, therefore

$$|\mu|^*\big(\{x \in \mathrm{B} : \ \big(f(x), g(x)\big) \notin \mathrm{V}\,\}\big) = |\mu|^*\big(\{x \in \mathrm{B} : \ \big(h(x), k(x)\big) \notin \mathrm{V}\,\}\big).$$

Since $(h, k) \in \mathrm{W}$, we know that $|\mu|^*\big(\{x \in \mathrm{B} : \ \big(h(x), k(x)\big) \notin \mathrm{V}\,\}\big) \leqslant \delta$, therefore $|\mu|^*\big(\{x \in \mathrm{B} : \ \big(f(x), g(x)\big) \notin \mathrm{V}\,\}\big) \leqslant \delta$ and so $(f, g) \in \mathrm{W}$. ♢

With notations as in the Lemma, if (u_n) is a sequence in S and if $u_n = \dot{f}_n$ with $f_n \in \mathscr{S}$, then

$$(u_n) \text{ is Cauchy in S} \quad \Leftrightarrow \quad (f_n) \text{ is Cauchy in } \mathscr{S}\,.$$

For, if $\mathrm{W} = \boldsymbol{W}(\mathrm{V}, \mathrm{B}, \delta)$ as in the Lemma, so that the sets W (resp. $\dot{\mathrm{W}}$) form a fundamental system of entourages for $\mathscr{S}$ (resp. S), one has

$$(u_m, u_n) \in \dot{\mathrm{W}} \quad \Leftrightarrow \quad (f_m, f_n) \in \mathrm{W}\,,$$

thus the assertion follows from GT, II, §3, No. 1, second paragraph following Def. 2.

IV.83, $\ell.\ -7, -6$.

"...is then μ-measurable "

By 1° of the generalization of Egoroff's Theorem proved in the note for IV.80, $\ell.\ -17$ to -14 (item "*No. 4, Th. 2′* in that note).

IV.83, $\ell.\ -6, -5$.

"...it follows from the extension of Egoroff's theorem mentioned in No. 10 that the sequence (f_{n_k}) *converges in measure* to f in A."

By 2° of the generalization cited in the preceding note, if $\mathrm{K} \subset \mathrm{A}$ is compact and $\delta > 0$, there exists a compact set $\mathrm{K}_1 \subset \mathrm{K}$ such that $|\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant \delta$ and $f_{n_k} \to f$ uniformly on K_1 (we do not need the continuity of the restrictions to K_1 established there). We are to show that (f_{n_k}) converges to f in the uniform space $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$.

Given a basic entourage W for $\mathscr{S}$, it will suffice to find an index k_0 such that

$$k \geqslant k_0 \quad \Rightarrow \quad (f_{n_k}, f) \in \mathrm{W}$$

(GT, II, §3, No. 1, second paragraph following Def. 2). We can suppose that

$$\mathrm{W} = \boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta)\,,$$

where $\delta > 0$, $\mathrm{V} = \{(y,y') \in \mathrm{F} \times \mathrm{F} : d(y,y') \leqslant \varepsilon\}$ (d a metric on F compatible with its uniformity, and $\varepsilon > 0$), and K is a compact subset of A (IV.81, ℓ. 7–9); thus W is the set of all pairs $(g,h) \in \mathscr{S} \times \mathscr{S}$ such that

$$|\mu|\big(\{x \in \mathrm{K} : d\big(g(x),h(x)\big) > \varepsilon\}\big) \leqslant \delta .$$

Let $\mathrm{K}_1 \subset \mathrm{K}$ be a compact set such that $|\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant \delta$ and $f_{n_k} \to f$ uniformly on K_1, and choose an index k_0 such that

$$k \geqslant k_0 \quad \Rightarrow \quad d\big(f_{n_k}(x), f(x)\big) \leqslant \varepsilon \text{ for all } x \in \mathrm{K}_1 ;$$

it follows that if $k \geqslant k_0$ then

$$\{x \in \mathrm{K} : d\big(f_{n_k}(x), f(x)\big) > \varepsilon\} \subset \mathrm{K} - \mathrm{K}_1 ,$$

whence

$$|\mu|\big(\{x \in \mathrm{K} : d\big(f_{n_k}(x), f(x)\big) > \varepsilon\}\big) \leqslant |\mu|(\mathrm{K} - \mathrm{K}_1) \leqslant \delta$$

and so $(f_{n_k}, f) \in \mathrm{W}$ by the preceding paragraph.

Note that, with W as above, $\mathrm{W}(f) = \{g \in \mathscr{S} : (g,f) \in \mathrm{W}\}$ is a basic neighborhood of f in $\mathscr{S}$, and the foregoing shows that for every index n, there exists an index k such that $n_k \geqslant n$ and $f_{n_k} \in \mathrm{W}(f)$; thus f is a cluster point of the sequence (f_n) (GT, I, §7, No. 3, second paragraph of *Example* 1).

IV.83, ℓ. -4.

"... f is a cluster point of the sequence (f_n) in $\mathscr{S}(\mathrm{A},\mu;\mathrm{F})$ "

Observed at the end of the preceding note.

IV.83, ℓ. $-3, -2$.

"...since the sequence (f_n) is by hypothesis a Cauchy sequence, it converges to f."

GT, II, §3, No. 2, Cor. 2 of Prop. 5.

To recapitulate the argument: After reducing to the case that A is integrable, one shows that every Cauchy sequence (f_n) in the uniform space $\mathscr{S} = \mathscr{S}(\mathrm{A},\mu;\mathrm{F})$ is convergent (in measure) to an $f \in \mathscr{S}$. If, then, (u_n) is a Cauchy sequence in the (metrizable) uniform space $\mathrm{S} = \mathrm{S}(\mathrm{A},\mu;\mathrm{F})$, say $u_n = \dot{f}_n$ with $f_n \in \mathscr{S}$, one knows that (f_n) is Cauchy in $\mathscr{S}$ (see the note for ℓ. $-9, -8$) hence is convergent in measure to an $f \in \mathscr{S}$; as $(f_n, f) \in \mathrm{W}$ is equivalent to $(\dot{f}_n, \dot{f}) \in \dot{\mathrm{W}}$ for an entourage $\mathrm{W} = \boldsymbol{W}(\mathrm{V},\mathrm{B},\delta)$ (*loc. cit.*), it follows that (u_n) converges to $u = \dot{f}$ in S.

IV.84, ℓ. 1.

"COROLLARY."

This is a corollary of Lemma 4, using parts of the proof of Prop. 19.

IV.84, ℓ. 10, 11.

"The assertion follows at once from the extension of Egoroff's theorem mentioned in No. 10."

One is assuming that there exists a locally negligible set $\mathrm{N} \subset \mathrm{A}$ such that $f_n(x) \to f(x)$ for all $x \in \mathrm{A} - \mathrm{N}$. The argument that f is measurable is given in the note for IV.83, ℓ. $-7, -6$ (completeness of F is not needed since the limits $f(x)$ for $x \in \mathrm{A} - \mathrm{N}$ are assumed to exist); and the argument that $f_n \to f$ in the uniform space $\mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ (that is, in measure) is given in the note for IV.83, ℓ. $-6, -5$ (valid for A measurable but not necessarily integrable; the role of integrability arises in the proof of Prop. 19 because of the citation of Lemma 4 in order to obtain a limit function f, whereas f is given here in advance).

IV.84, ℓ. 15.

"... f' is a μ-measurable mapping of $\mathrm{B} - \mathrm{N}$ into $\widehat{\mathrm{F}}$"

Write $g_k = f_{n_k}\big|\mathrm{B} - \mathrm{N}$; we know that $g_k : \mathrm{B} - \mathrm{N} \to \mathrm{F}$ is μ-measurable (f_{n_k} has a measurable extension to X, therefore so does g_k; see item c'') in the note for IV.79, ℓ. 3,4). Let $i : \mathrm{F} \to \widehat{\mathrm{F}}$ be the canonical injection, and $g_k' = i \circ g_k : \mathrm{B} - \mathrm{N} \to \widehat{\mathrm{F}}$; since i is continuous, g_k' is μ-measurable by the generalization of No. 3, Th. 1 (item *No. 3, Th. 1′* in the note for IV.80, ℓ. -17 to -14). One has $g_k' \to f'$ pointwise (on all of $\mathrm{B} - \mathrm{N}$), therefore f' is μ-measurable by 1° of the generalization of Egoroff's theorem (*loc. cit.*, item *No. 4, Th. 2′*).

IV.84, ℓ. 15, 16.

"... the sequence (f_n) converges in measure to f' in $\mathrm{B} - \mathrm{N}$ by (i)"

Lemma. If C is a measurable subset of A, then the mapping $f \mapsto f\big|\mathrm{C}$ of $\mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ into $\mathscr{S}(\mathrm{C}, \mu; \mathrm{F})$ is uniformly continuous.

For, $f\big|\mathrm{C}$ is measurable (since f has a measurable extension to X, so does $f\big|\mathrm{C}$). If K is a compact subset of C then it is also a compact subset of A; let V be a basic entourage for F, let $\delta > 0$, and write

$$\mathrm{W}_{\mathrm{A}} = \boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta) \quad \text{and} \quad \mathrm{W}_{\mathrm{C}} = \boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta)$$

for the corresponding entourages for $\mathscr{S}_{\mathrm{A}} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$ and $\mathscr{S}_{\mathrm{C}} = \mathscr{S}(\mathrm{C}, \mu; \mathrm{F})$. Then, for $f, g \in \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$,

$$\{x \in \mathrm{K} : \ \big(f(x), g(x)\big) \notin \mathrm{V}\} = \{x \in \mathrm{K} : \ \big((f\big|\mathrm{C})(x), (g\big|\mathrm{C})(x)\big) \notin \mathrm{V}\},$$

whence obviously

$$(f,g) \in W_A \quad \Leftrightarrow \quad (f\big|C, g\big|C) \in W_C\,.$$

Since W_C describes a basic entourage for $\mathscr{S}_C$, and W_A is an entourage for $\mathscr{S}_A$ that is mapped into W_C by $f \mapsto f\big|C$, the mapping is indeed uniformly continuous.

In particular, if the sequence (f_n) is Cauchy (in measure) in the uniform space $\mathscr{S}_A$, then $(f_n\big|C)$ is Cauchy in $\mathscr{S}_C$; and if $f_n \to f$ in measure then $f_n\big|C \to f\big|C$ in measure. ♢

Consider now the given convergent (in measure) sequence $f_n \to f$ in $\mathscr{S}_A$, the 'countably integrable' set $B \subset A$, the negligible set $N \subset B$, and the subsequence (f_{n_k}) of (f_n) such that $f_{n_k}(x) \to f'(x) \in \widehat{F}$ for all $x \in B - N$. Write $C = B - N$.

Writing $i : F \to \widehat{F}$ for the canonical injection, and $f'_{n_k} = i \circ (f_{n_k}\big|C)$, we have

$$f'_{n_k}(x) \to f'(x) \quad \text{for all } x \in C = B - N\,;$$

by (i) of the present Corollary, $f'_{n_k} \to f'$ (in measure) in $\mathscr{S}'_C = \mathscr{S}(C, \mu; \widehat{F})$.

On the other hand, since $f_n \to f$ in measure, hence $f_{n_k} \to f$ in measure, we know that $f_{n_k}\big|C \to f\big|C$ in measure by the Lemma; and since $i : F \to \widehat{F}$ is uniformly continuous, it follows that $i \circ (f_{n_k}\big|C) \to i \circ (f\big|C)$ in measure, that is, $f'_{n_k} \to i \circ (f\big|C)$ in $\mathscr{S}'_C$.

Thus, if $u_k \in S'_C = S(C, \mu; \widehat{F})$ is the equivalence class of f'_{n_k}, u is the class of $i \circ (f\big|C)$, and u' is the class of f', we have $u_k \to u$ and $u_k \to u'$ in S'_C; since S'_C is Hausdorff (IV.81, ℓ. −7 to −1), it follows that $u = u'$, thus $i \circ (f\big|C) = f'$ locally almost everywhere in C, and since C is a countable union of integrable sets, in fact $i \circ (f\big|C) = f'$ almost everywhere in C. But $f_{n_k}(x) \to f(x)$ for every $x \in C$, and $f'(x) = f(x)$ for almost every $x \in C$, therefore $f_{n_k}(x) \to f(x)$ for almost every $x \in C$. Since $B = C \cup N$ and N is negligible, we conclude that $f_{n_k}(x) \to f(x)$ for almost every $x \in B$; this proves the assertion (ii) of the Corollary.

As for the original assertion, since $f_n \to f$ in measure, the argument in the paragraph before the last shows that $i \circ (f_n\big|C) \to i \circ (f\big|C)$ in measure; but we now know that $i \circ (f\big|C) = f'$ almost everywhere in C, therefore

$$i \circ (f_n\big|C) \to f' \quad \text{in measure.}$$

This is the sense in which the original assertion holds, avoiding the abuse of notation where the f_n take values in F whereas f' takes its values in $\widehat{F}$.

IV.84, ℓ. 16, 17.

"... f' is therefore equal to f almost everywhere in B."

Writing $\mathrm{C} = \mathrm{B} - \mathrm{N}$ as in the preceding note, we know that $f' : \mathrm{C} \to \widehat{\mathrm{F}}$ is measurable (IV.84, ℓ. 15); its extension to X by a point $y_0 \in \widehat{\mathrm{F}}$ is then a measurable function on X (No. 10, Def. 8), whose restriction to A is also measurable (No. 10, Def. 8); denote this function $\mathrm{A} \to \widehat{\mathrm{F}}$ also by f', so that $f' \in \mathscr{S}(\mathrm{A}, \mu; \widehat{\mathrm{F}})$. Since $f'(x) = f(x)$ for almost every $x \in \mathrm{C}$ (by the next-to-last paragraph of the preceding note) and N is negligible, it follows that $f'(x) = (i \circ f)(x)$ for almost every $x \in \mathrm{B} = \mathrm{C} \cup \mathrm{N}$, that is, the functions $f', i \circ f$ of $\mathscr{S}(\mathrm{A}, \mu; \widehat{\mathrm{F}})$ are equal almost everywhere in B.

The assertion (ii) of the Corollary is proved in the next-to-last paragraph of the preceding note.

One observes that the statement of the Corollary makes no completeness assumptions; the complete space $\widehat{\mathrm{F}}$ is merely auxiliary to the proof. Assuming F to be complete would simplify the proof of the Corollary, but would needlessly restrict its generality.

IV.84, ℓ. -14 to -11.

"... if V_δ is the entourage of F formed by the pairs $(\mathbf{y}, \mathbf{z})$ such that $|\mathbf{y} - \mathbf{z}| \leqslant \delta$, the entourage $\boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta)$ is the set of pairs $(\mathbf{f}, \mathbf{g})$ of measurable mappings of A into F such that $\mathbf{f} - \mathbf{g} \in \boldsymbol{T}(\mathrm{B}, \delta)$."

Write $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$. By definition, $\boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta)$ is the set of pairs $(\mathbf{f}, \mathbf{g}) \in \mathscr{S} \times \mathscr{S}$ such that

$$|\mu|\big(\{x \in \mathrm{B} :\ \big(\mathbf{f}(x), \mathbf{g}(x)\big) \notin \mathrm{V}_\delta\}\big) \leqslant \delta\,,$$

whereas the condition $\mathbf{f} - \mathbf{g} \in \boldsymbol{T}(\mathrm{B}, \delta)$ means that

$$|\mu|\big(\{x \in \mathrm{B} :\ |(\mathbf{f} - \mathbf{g})(x)| \geqslant \delta\}\big) \leqslant \delta\,;$$

so the assertion will be true if

$$(\dagger) \qquad \big((\mathbf{f}(x), \mathbf{g}(x)\big) \notin \mathrm{V}_\delta \quad \Leftrightarrow \quad |\mathbf{f}(x) - \mathbf{g}(x)| \geqslant \delta\,.$$

We have a choice: if the definition of $\boldsymbol{T}(\mathrm{B}, \delta)$ in the text is retained, then one should define

$$(*) \qquad \mathrm{V}_\delta = \{(\mathbf{y}, \mathbf{z}) \in \mathrm{F} \times \mathrm{F} :\ |\mathbf{y} - \mathbf{z}| < \delta\}\,.$$

On the other hand, if the proposed definition of V_δ (with $\leqslant$ instead of $<$) is retained, then $\boldsymbol{T}(\mathrm{B}, \delta)$ should have been defined as the set of $\mathbf{f} \in \mathscr{S}$ such that

$$|\mu|\big(\{x \in \mathrm{B} :\ |\mathbf{f}(x)| > \delta\}\big) \leqslant \delta\,.$$

I propose to leave the definitions of $\boldsymbol{T}(\mathrm{B},\delta)$ and $\boldsymbol{W}(\mathrm{V}_\delta,\mathrm{B},\delta)$ as in the text, and to make them compatible by redefining V_δ as in (∗).

Note that entourages of the form $\boldsymbol{W}(\mathrm{V}_\delta,\mathrm{B},\delta)$ are basic for the uniformity of $\mathscr{S}$; for, if V is any entourage for F, B is any integrable subset of A, and δ is any number > 0, one can choose $\delta' > 0$ sufficiently small that both $\mathrm{V}_{\delta'} \subset \mathrm{V}$ and $\delta' < \delta$, and so $\boldsymbol{W}(\mathrm{V}_{\delta'},\mathrm{B},\delta') \subset \boldsymbol{W}(\mathrm{V},\mathrm{B},\delta)$. Note also that since the V_δ are symmetric, so are the $\boldsymbol{W}(\mathrm{V},\mathrm{B},\delta)$, that is,

$$(\mathbf{f},\mathbf{g}) \in \boldsymbol{W}(\mathrm{V}_\delta,\mathrm{B},\delta) \quad\Leftrightarrow\quad (\mathbf{g},\mathbf{f}) \in \boldsymbol{W}(\mathrm{V}_\delta,\mathrm{B},\delta).$$

IV.84, ℓ. −10.

" $\boldsymbol{T}(\mathrm{B},\delta) + \boldsymbol{T}(\mathrm{B},\delta) \subset \boldsymbol{T}(\mathrm{B},2\delta)$ "

Let $\mathbf{f},\mathbf{g} \in \mathscr{S}(\mathrm{A},\mu;\mathrm{F})$. One has

$$\{x \in \mathrm{B} : \ |(\mathbf{f}+\mathbf{g})(x)| \geqslant 2\delta\} \subset \{x \in \mathrm{B} : \ |\mathbf{f}(x)| \geqslant \delta\} \cup \{x \in \mathrm{B} : \ |\mathbf{g}(x)| \geqslant \delta\}$$

by the triangle inequality for the norm on F; if $\mathbf{f},\mathbf{g} \in \boldsymbol{T}(\mathrm{B},\delta)$ then the $|\mu|$-measure of each set on the right is $\leqslant \delta$, so the set on the left has measure $\leqslant 2\delta$, therefore $\mathbf{f}+\mathbf{g} \in \boldsymbol{T}(\mathrm{B},2\delta)$.

IV.84, ℓ. −10, −9.

" $\boldsymbol{T}(\mathrm{B},|\alpha|\delta) \subset \alpha\boldsymbol{T}(\mathrm{B},\delta)$ for every nonzero scalar α such that $|\alpha| \leqslant 1$ "

$\mathbf{f} \in \boldsymbol{T}(\mathrm{B},|\alpha|\delta)$ means that

$$(*) \qquad |\mu|\big(\{x \in \mathrm{B} : \ |\mathbf{f}(x)| \geqslant |\alpha|\delta\}\big) \leqslant |\alpha|\delta,$$

whereas $\mathbf{f} \in \alpha\boldsymbol{T}(\mathrm{B},\delta)$ means that $\alpha^{-1}\mathbf{f} \in \boldsymbol{T}(\mathrm{B},\delta)$, that is,

$$|\mu|\big(\{x \in \mathrm{B} : \ |\alpha^{-1}\mathbf{f}(x)| \geqslant \delta\}\big) \leqslant \delta,$$

in other words

$$(**) \qquad |\mu|\big(\{x \in \mathrm{B} : \ |\mathbf{f}(x)| \geqslant |\alpha|\delta\}\big) \leqslant \delta;$$

since $|\alpha| \leqslant 1$, (∗) implies (∗∗), whence the asserted inclusion.

This shows that $\alpha\boldsymbol{T}(\mathrm{B},\delta)$ is a neighborhood of $\mathbf{0}$ in $\mathscr{S}(\mathrm{A},\mu;\mathrm{F})$ when $0 < |\alpha| \leqslant 1$; a subtler argument is needed for arbitrary $\alpha \neq 0$ (see the note for ℓ. −2, −1).

IV.84, ℓ. −6.

" No. 3, Cor. 6 of Th. 1 "

More precisely, its generalization (item *No. 3, Cor. 6′ of Th. 1* in the note for IV.80, ℓ. −17 to −14).

IV.84, ℓ. $-2, -1$.

"... which completes the proof of assertion (i)."

Write $\mathscr{S} = \mathscr{S}(\mathrm{A}, \mu; \mathrm{F})$. Recall that

$$\boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta) = \{(\mathbf{f}, \mathbf{g}) \in \mathscr{S} \times \mathscr{S} : \ \mathbf{f} - \mathbf{g} \in \boldsymbol{T}(\mathrm{B}, \delta)\}\,.$$

Since the sets $\boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta)$ form a fundamental system of entourages for the uniformity of $\mathscr{S}$, the sets

$$\{\mathbf{f} \in \mathscr{S} : \ (\mathbf{f}, \mathbf{g}) \in \boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta)\}$$

form a fundamental system of neighborhoods of $\mathbf{g} \in \mathscr{S}$ for the topology of convergence in measure. In particular, the sets

$$\{\mathbf{f} \in \mathscr{S} : \ (\mathbf{f}, \mathbf{0}) \in \boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta)\} = \{\mathbf{f} \in \mathscr{S} : \ \mathbf{f} - \mathbf{0} \in \boldsymbol{T}(\mathrm{B}, \delta)\} = \boldsymbol{T}(\mathrm{B}, \delta)$$

form a fundamental system of neighborhoods of $\mathbf{0}$ in $\mathscr{S}$; and their translates

$$\mathbf{g} + \boldsymbol{T}(\mathrm{B}, \delta) = \{\mathbf{f} \in \mathscr{S} : \ \mathbf{f} - \mathbf{g} \in \boldsymbol{T}(\mathrm{B}, \delta)\} = \{\mathbf{f} \in \mathscr{S} : \ (\mathbf{f}, \mathbf{g}) \in \boldsymbol{W}(\mathrm{V}_\delta, \mathrm{B}, \delta)\}$$

form a fundamental system of neighborhoods of $\mathbf{g}$ in $\mathscr{S}$.

Let $\mathfrak{V}$ be the set of all *balanced* neighborhoods of $\mathbf{0}$ in $\mathscr{S}$, that is, the set of all balanced subsets of $\mathscr{S}$ that contain some set $\boldsymbol{T}(\mathrm{B}, \delta)$; it will suffice to show that $\mathfrak{V}$ satisfies the conditions $(\mathrm{EV_I})$, $(\mathrm{EV_{II}})$, $(\mathrm{EV_{III}})$ of TVS, I, §1, No. 5, Prop. 4.

$(\mathrm{EV_I})$: Every $\boldsymbol{T}(\mathrm{B}, \delta)$ is balanced. For, if $\mathbf{f} \in \boldsymbol{T}(\mathrm{B}, \delta)$ and $|\alpha| \leqslant 1$ then

$$\{x \in \mathrm{B} : \ |\alpha \mathbf{f}(x)| \geqslant \delta\} \subset \{x \in \mathrm{B} : \ |\mathbf{f}(x)| \geqslant \delta\}\,;$$

since the $|\mu|$-measure of the set on the right is $\leqslant \delta$, so is that of the set on the left, in other words $\alpha \mathbf{f} \in \boldsymbol{T}(\mathrm{B}, \delta)$. Thus every $\boldsymbol{T}(\mathrm{B}, \delta)$ belongs to $\mathfrak{V}$.

Every $\boldsymbol{T}(\mathrm{B}, \delta)$ is absorbent. For, given any $\mathbf{f} \in \mathscr{S}$, the argument in the text shows that there exists a positive number $\lambda = n^2$ such that $\mathbf{f} \in \lambda \boldsymbol{T}(\mathrm{B}, \delta)$, and since $\boldsymbol{T}(\mathrm{B}, \delta)$ is balanced, it is absorbent (TVS, *loc. cit.*, remarks following Def. 4).

Every set in $\mathfrak{V}$ is by assumption balanced, and since it contains some $\boldsymbol{T}(\mathrm{B}, \delta)$ it is absorbent. Thus $\mathfrak{V}$ satisfies $(\mathrm{EV_I})$.

$(\mathrm{EV_{III}})$: For every $\boldsymbol{T}(\mathrm{B}, \delta)$ one has $\boldsymbol{T}(\mathrm{B}, \delta/2) + \boldsymbol{T}(\mathrm{B}, \delta/2) \subset \boldsymbol{T}(\mathrm{B}, \delta)$.

$(\mathrm{EV_{II}})$: This follows from $(\mathrm{EV_I})$ and $(\mathrm{EV_{III}})$ (TVS, *loc. cit.*, *Remark* 2). For, let $\mathrm{U} \in \mathfrak{V}$ and let λ be a scalar $\neq 0$; we are to show that $\lambda \mathrm{U} \in \mathfrak{V}$. At any rate, since U is balanced, so is $\lambda \mathrm{U}$; we must show that $\boldsymbol{T}(\mathrm{B}, \delta') \subset \lambda \mathrm{U}$

for some integrable set $B \subset A$ and some number $\delta' > 0$. Say $\boldsymbol{T}(B,\delta) \subset U$. Then

$$2\,\boldsymbol{T}(B,\delta/2) \subset \boldsymbol{T}(B,\delta/2) + \boldsymbol{T}(B,\delta/2) \subset \boldsymbol{T}(B,\delta)\,;$$

thus $2\,\boldsymbol{T}(B,\delta/2) \subset \boldsymbol{T}(B,\delta)$ for every $\delta > 0$, whence, by induction,

$$2^n\,\boldsymbol{T}(B,\delta/2^n) \subset \boldsymbol{T}(B,\delta) \quad \text{for every integer } n \geqslant 1\,.$$

Choose n so large that $2^n|\lambda| \geqslant 1$; since $\boldsymbol{T}(B,\delta/2^n)$ is balanced, one has $(2^n|\lambda|)^{-1}\,\boldsymbol{T}(B,\delta/2^n) \subset \boldsymbol{T}(B,\delta/2^n)$, whence

$$\boldsymbol{T}(B,\delta/2^n) \subset |\lambda|2^n\,\boldsymbol{T}(B,\delta/2^n) \subset |\lambda|\,\boldsymbol{T}(B,\delta) \subset |\lambda|U = \lambda U$$

($|\lambda|U = \lambda U$ because U is balanced, hence circled).

IV.85, ℓ. 1–4.

"The relation $\int |\mathbf{f}|^p\,d|\mu| \leqslant \delta^{p+1}$ implies that if C is the set of $x \in X$ such that $|\mathbf{f}(x)| \geqslant \delta$, then

$$\delta^p|\mu|^*(C) \leqslant \int |\mathbf{f}|^p\,d|\mu| \leqslant \delta^{p+1}\,,$$

whence $|\mu|^*(C) \leqslant \delta$, which proves (iii)."

Recall that every p-th power integrable function $\mathbf{f} : X \to F$ is measurable, that is, $\mathscr{L}^p_F(X,\mu) \subset \mathscr{S}(X,\mu;F)$ (No. 6, Th. 5). Let us abbreviate $\mathscr{L}^p = \mathscr{L}^p_F(X,\mu)$ and $\mathscr{S} = \mathscr{S}(X,\mu;F)$.

We are to show that the norm topology on $\mathscr{L}^p$ is finer than the topology on $\mathscr{L}^p$ induced by that of $\mathscr{S}$; that is, for p-th power integrable functions, convergence in mean of order p implies convergence in measure; in other words, the canonical injection $\mathscr{L}^p \to \mathscr{S}$ is continuous. Since both $\mathscr{L}^p$ and $\mathscr{S}$ are topological vector spaces and the mapping is linear, it suffices to establish continuity at $\mathbf{0}$.

Thus, given a basic neighborhood U of $\mathbf{0}$ in $\mathscr{S}$, it suffices to show that $U \cap \mathscr{L}^p$ is a neighborhood of $\mathbf{0}$ in $\mathscr{L}^p$. We can suppose that $U = \boldsymbol{T}(B,\delta)$, where B is an integrable (or even compact) set in X, and $\delta > 0$. It will suffice to show that $\boldsymbol{T}(B,\delta)$ contains a closed ball in $\mathscr{L}^p$ centered at $\mathbf{0}$; thus we seek a real number $\rho > 0$ such that

$$(*) \qquad \{\mathbf{f} \in \mathscr{L}^p :\ N_p(\mathbf{f}) \leqslant \rho\} \subset \boldsymbol{T}(B,\delta)\,.$$

Suppose, provisionally, that $0 < \rho < +\infty$ and $\mathbf{f} \in \mathscr{L}^p$ with $N_p(\mathbf{f}) \leqslant \rho$. For $\mathbf{f}$ to belong to $\boldsymbol{T}(B,\delta)$, one must have

$$|\mu|\big(\{x \in B :\ |\mathbf{f}(x)| \geqslant \delta\big) \leqslant \delta\,.$$

Let $\mathrm{C} = \{x \in \mathrm{X} : \ |\mathbf{f}(x)| \geqslant \delta\}$. Since $|\mathbf{f}|$ is measurable (No. 3, Cor. 6 of Th. 1), so is the set C (No. 5, Prop. 8). In fact, C is integrable; for, if φ_{C} is its characteristic function, we have $\delta^p \varphi_{\mathrm{C}} \leqslant |\mathbf{f}|^p \varphi_{\mathrm{C}} \leqslant |\mathbf{f}|^p$, whence

$$\delta^p |\mu|^*(\mathrm{C}) \leqslant \int |\mathbf{f}|^p \, d|\mu| \leqslant \rho^p < +\infty$$

and we may write $|\mu|(\mathrm{C})$. To assure that $\mathbf{f} \in \boldsymbol{T}(\mathrm{B}, \delta)$, in other words that $|\mu|(\mathrm{B} \cap \mathrm{C}) \leqslant \delta$, it suffices to assure that $|\mu|(\mathrm{C}) \leqslant \delta$. But $\delta^p |\mu|(\mathrm{C}) \leqslant \rho^p$, that is, $|\mu|(\mathrm{C}) \leqslant (\rho/\delta)^p$; so $\mathbf{f} \in \boldsymbol{T}(\mathrm{B}, \delta)$ is assured if $(\rho/\delta)^p = \delta$, that is, $\rho^p = \delta^{p+1}$, $\rho = \delta^{1+1/p}$.

To recapitulate: given a neighborhood $\boldsymbol{T}(\mathrm{B}, \delta)$ of $\mathbf{0}$ in $\mathscr{S}$, where $\mathrm{B} \subset \mathrm{X}$ is integrable and $\delta > 0$, setting $\rho = \delta^{1+1/p}$ assures that $(*)$ is satisfied.

IV.85, ℓ. 5–7.

"In view of (iii), it suffices to show for example that $\mathscr{L}^1_{\mathrm{F}}$ is dense in $\mathscr{S}_{\mathrm{F}}$, since by definition $\mathscr{K}(\mathrm{X}; \mathrm{F})$ is dense in $\mathscr{L}^1_{\mathrm{F}}$ for the topology of convergence in mean."

The argument is easily modified to cover the case of $\mathscr{L}^p_{\mathrm{F}}$; borrowing from the next note, let us assume already established that $\mathscr{L}^p_{\mathrm{F}}$ is dense in $\mathscr{S}_{\mathrm{F}}$.

Let U be a nonempty open set in $\mathscr{S}_{\mathrm{F}}$; we are to show that $\mathrm{U} \cap \mathscr{K}(\mathrm{X}; \mathrm{F}) \neq \varnothing$. By the foregoing assumption, $\mathrm{U} \cap \mathscr{L}^p_{\mathrm{F}}$ is nonempty, and by (iii) it is an open set in $\mathscr{L}^p_{\mathrm{F}}$, therefore

$$\mathrm{U} \cap \mathscr{K}(\mathrm{X}; \mathrm{F}) = \mathrm{U} \cap \big(\mathscr{L}^p_{\mathrm{F}} \cap \mathscr{K}(\mathrm{X}; \mathrm{F})\big) = (\mathrm{U} \cap \mathscr{L}^p_{\mathrm{F}}) \cap \mathscr{K}(\mathrm{X}; \mathrm{F}) \neq \varnothing$$

by the density of $\mathscr{K}(\mathrm{X}; \mathrm{F})$ in $\mathscr{L}^p_{\mathrm{F}}$ (§3, No. 4, Def. 2).

IV.85, ℓ. 7–12.

"Now, let $\mathbf{f}$ be any element of $\mathscr{S}_{\mathrm{F}}$ ... and obviously $\mathbf{f} - \mathbf{g} \in \boldsymbol{T}(\mathrm{B}, \delta)$."

The objective is to prove that $\mathscr{L}^p_{\mathrm{F}}$ is dense in $\mathscr{S}_{\mathrm{F}}$. Given $\mathbf{f} \in \mathscr{S}_{\mathrm{F}}$ and a set $\boldsymbol{T}(\mathrm{B}, \delta)$, so that $\mathbf{f} + \boldsymbol{T}(\mathrm{B}, \delta)$ is a basic neighborhood of $\mathbf{f}$ in $\mathscr{S}_{\mathrm{F}}$, we are to show that $\big(\mathbf{f} + \boldsymbol{T}(\mathrm{B}, \delta)\big) \cap \mathscr{L}^p_{\mathrm{F}} \neq \varnothing$, establishing that $\mathbf{f}$ belongs to the closure of $\mathscr{L}^p_{\mathrm{F}}$ in $\mathscr{S}_{\mathrm{F}}$; thus we seek a function $\mathbf{g} \in \mathscr{L}^p_{\mathrm{F}}$ such $\mathbf{g} \in \mathbf{f} + \boldsymbol{T}(\mathrm{B}, \delta)$.

As shown in the proof of (i), there exists a number $n > 0$ such that the (integrable) set $\mathrm{C} = \{x \in \mathrm{B} : \ |\mathbf{f}(x)| \geqslant n\}$ satisfies $|\mu|(\mathrm{C}) \leqslant \delta$. Then $|\mathbf{f}| \leqslant 1/n$ on $\mathrm{B} - \mathrm{C}$, thus the function $\mathbf{g} : \mathrm{X} \to \mathrm{F}$ defined by $\mathbf{g} = \varphi_{\mathrm{B} - \mathrm{C}} \mathbf{f}$ is measurable (No. 3, remarks following Cor. 5 of Th. 1) and $|\mathbf{g}| \leqslant (1/n) \varphi_{\mathrm{B} - \mathrm{C}}$. It follows that $\mathbf{g} \in \mathscr{L}^p_{\mathrm{F}}$, since

$$\mathrm{N}_p(\mathbf{g}) = \mathrm{N}_p(|\mathbf{g}|) \leqslant (1/n) \mathrm{N}_p(\varphi_{\mathrm{B} - \mathrm{C}}) = (1/n) \big(|\mu|(\mathrm{B} - \mathrm{C})\big)^{1/p} < +\infty$$

(No. 6, Th. 5).

Finally, $\mathbf{g} \in \mathbf{f} + \boldsymbol{T}(\mathrm{B}, \delta)$. For, $\mathbf{g} - \mathbf{f} = 0$ on $\mathrm{B} - \mathrm{C}$, therefore if $|\mathbf{g}(x) - \mathbf{f}(x)| \geqslant \delta$ then $x \in \mathrm{X} - (\mathrm{B} - \mathrm{C})$; if, moreover, $x \in \mathrm{B}$, then $x \in \mathrm{B} - (\mathrm{B} - \mathrm{C}) = \mathrm{C}$, thus $\{x \in \mathrm{B} : |\mathbf{g}(x) - \mathbf{f}(x)| \geqslant \delta\} \subset \mathrm{C}$, whence

$$|\mu|\big(\{x \in \mathrm{B} : |\mathbf{g}(x) - \mathbf{f}(x)| \geqslant \delta\}\big) \leqslant |\mu|(\mathrm{C}) \leqslant \delta$$

and so $\mathbf{g} - \mathbf{f} \in \boldsymbol{T}(\mathrm{B}, \delta)$.

IV.85, ℓ. 19–27.

"Definition 10."

The definition is self-explanatory and needs no notes—but to the reader familiar with measures as set functions (let us call them 'set-measures', reserving 'measures' for the Bourbaki concept) as in the book of P.R. Halmos (*Measure theory*, Springer), the definition invites interpretation; so to speak, the numerical measures $|\mathbf{f}|^p \cdot |\mu|$ to be defined shortly are (i) equi-'absolutely continuous' with respect to $|\mu|$ and (ii) equi-'inner regular'.

To get a feeling for the definition, we consider the case that H consists of a single function. The essence of the definition concerns a positive measure ($|\mu|$) and positive numerical functions ($|\mathbf{f}|^p$) that are integrable with respect to the measure; let us simplify the notation by assuming that μ is a positive measure and $\mathrm{H} = \{f\}$, where $f \in \mathscr{L}^1(\mu)$ and $f \geqslant 0$. The conditions then read:

(i) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for a μ-integrable set A,

$$\mu(\mathrm{A}) \leqslant \delta \quad \Rightarrow \quad \int \varphi_{\mathrm{A}} f \, d\mu \leqslant \varepsilon ;$$

(ii) for every $\varepsilon > 0$ there exists a compact set $\mathrm{K} \subset \mathrm{X}$ such that

$$\int \varphi_{\mathrm{X} - \mathrm{K}} f \, d\mu \leqslant \varepsilon .$$

Our goal is to establish these properties by interpreting the symbol $f \cdot \mu$ as a measure on X. We will in fact construct two candidate measures ν and ρ (both positive and bounded). The precedent of Ch. III, §1, No. 4 ($g \cdot \mu$, g continuous) suggests strongly that the notation $f \cdot \mu$ be awarded to ν, which has the easiest and most natural definition, but it is ρ that yields the properties needed for item (i). For item (ii) there is a direct proof not making use of any new measures (Prop. A below). The two measures will be shown to be equal if every compact set in X is a G_δ (the intersection of a sequence of open sets), as is the case when X is metrizable (GT, IX, §2, No. 5, Prop. 7). My guess is that $\nu = \rho$ always, but I have not succeeded in putting the pieces together.

Proposition A. Condition (ii) always holds.

Proof. Recall that $\varphi_{\mathrm{A}} f$ is μ-integrable for every μ-measurable set A (No. 6, Cor. 3 of Th. 5). Writing $\mathfrak{K}$ for the set of all compact subsets of X, the assertion of (ii) is equivalent to

$$\sup_{\mathrm{K}\in\mathfrak{K}} \int \varphi_{\mathrm{K}} f \, d\mu = \int f \, d\mu \, .$$

By §4, No. 4, Cor. of Th. 3, there exists an increasing sequence (g_n) of μ-integrable functions with compact support such that $g_n(x) \to f(x)$ for μ-almost every x. Consider the doubly indexed family of numbers

$$\int \varphi_{\mathrm{K}} g_n \, d\mu \, ;$$

by the 'associativity of sups' we may compute the supremum of the family in two ways (a special case of GT, IV, §5, No. 4, Prop. 8):

$$(*) \qquad \sup_{\mathrm{K}\in\mathfrak{K}} \Bigl(\sup_n \int \varphi_{\mathrm{K}} g_n \, d\mu \Bigr) = \sup_n \Bigl(\sup_{\mathrm{K}\in\mathfrak{K}} \int \varphi_{\mathrm{K}} g_n \, d\mu \Bigr).$$

Now, for each $\mathrm{K} \in \mathfrak{K}$,

$$\sup_n \int \varphi_{\mathrm{K}} g_n \, d\mu = \int \varphi_{\mathrm{K}} f \, d\mu$$

because $\varphi_{\mathrm{K}} g_n \uparrow \varphi_{\mathrm{K}} f$ μ-almost everywhere (§3, No. 6, Th. 5; or §4, No. 3, Prop. 4 or Th. 2); whereas for each n,

$$\sup_{\mathrm{K}\in\mathfrak{K}} \int \varphi_{\mathrm{K}} g_n \, d\mu = \int g_n \, d\mu$$

since $\varphi_{\mathrm{K}} g_n = g_n$ for $\mathrm{K} = \operatorname{Supp} g_n$. Thus $(*)$ yields

$$\sup_{\mathrm{K}\in\mathfrak{K}} \int \varphi_{\mathrm{K}} f \, d\mu = \sup_n \int g_n \, d\mu = \int f \, d\mu \, ,$$

whence (ii). {We will see later that this number is $\rho(\mathrm{X})$.} ♢

Definition of the measure $\nu = f \cdot \mu$.

For each $h \in \mathscr{K} = \mathscr{K}(\mathrm{X}; \mathbf{C})$ the function hf is μ-measurable (No. 3, remark following Cor. 5 of Th. 1) and $|hf| \leqslant \|h\| \, f$, whence

$$\mathrm{N}_1(hf) \leqslant \|h\| \, \mathrm{N}_1(f) < +\infty \, ,$$

thus hf is μ-integrable (No. 6, Th. 5); the formula

$$\nu(h) = \int hf\,d\mu \qquad (h \in \mathscr{K})$$

defines a linear form on $\mathscr{K}$ such that $h \geqslant 0 \Rightarrow \nu(h) \geqslant 0$, that is, a *positive* measure ν on X (Ch. III, §1, No. 5, Th. 1). We write $\nu = f\cdot\mu$, a notation consistent with Ch. III, §1, No. 4 when f is continuous. Moreover,

$$|\nu(h)| \leqslant \nu(|h|) \leqslant \|h\| \int f\,d\mu\,,$$

thus ν is a *bounded* measure on X (*loc. cit.*, No. 8, remark following Def. 3). It follows that X is ν-integrable (§4, No. 7, Prop. 12) and that $\|\nu\| = \nu(\mathrm{X}) = \sup\{\nu(\mathrm{K}) :\ \mathrm{K}\subset \mathrm{X}\ \text{compact}\}$ (§4, No. 6, Cor. 1 of Th. 4).

Note that a compact set K in X is a G_δ if and only if there exists a decreasing sequence (h_n) in $\mathscr{K}_+(\mathrm{X})$ such that $h_n \downarrow \varphi_{\mathrm{K}}$: for, if (U_n) is a sequence of open sets such that $\mathrm{K} = \bigcap_n \mathrm{U}_n$, let $k_n \in \mathscr{K}_+(\mathrm{X})$ with $k_n = 1$ on K and $\mathrm{Supp}(k_n) \subset \mathrm{U}_n$ (III, §1, No. 2, Lemma 1) and let $h_n = \inf(k_1,\dots,k_n)$; conversely, if $h_n \downarrow \varphi_{\mathrm{K}}$ then for each n the set $\mathrm{K}_n = \{x :\ h_n(x) \geqslant 1\} = \bigcap_{m=1}^{\infty}\{x :\ h_n(x) > 1 - 1/m\}$ is a compact G_δ, and $\bigcap_{n=1}^{\infty}\mathrm{K}_n = \mathrm{K}$.

Deferring any hypothesis on X, we observe that if K is a compact G_δ then $\nu(\mathrm{K}) = \int \varphi_{\mathrm{K}} f\,d\mu$; for, if (h_n) is a decreasing sequence in $\mathscr{K}_+(\mathrm{X})$ such that $h_n \downarrow \varphi_{\mathrm{K}}$, then $\nu(h_n) \downarrow \int \varphi_{\mathrm{K}}\,d\nu = \nu(\mathrm{K})$ (§4, No. 3, Prop. 4) whereas $h_n f \downarrow \varphi_{\mathrm{K}} f$ implies $\int h_n f\,d\mu \downarrow \int \varphi_{\mathrm{K}} f\,d\mu$, thus

$$\nu(\mathrm{K}) = \lim_n \nu(h_n) = \lim_n \int h_n f\,d\mu = \int \varphi_{\mathrm{K}} f\,d\mu\,.$$

This gives the clue for defining a second measure.

Definition of the measure ρ.

Let Φ be the set of all *μ-integrable* subsets of X, and define a positive real-valued function α on Φ by the formula

$$\alpha(\mathrm{A}) = \int \varphi_{\mathrm{A}} f\,d\mu \qquad (\mathrm{A}\ \ \mu\text{-integrable}).$$

We propose to use Th. 5 of §4, No. 11 to construct a second measure on X; let us verify that its hypotheses are satisfied.

$(\mathrm{PC_I})$: §4, No. 5, Props. 6 and 7.

$(\mathrm{PC_{II}})$: If $\mathrm{K} \subset \mathrm{U}$ with K compact and U open, then $\mathrm{M} = \mathrm{K} \in \Phi$ satisfies $\mathrm{K} \subset \mathrm{M} \subset \mathrm{U}$.

(PM_{I}): If M, N are sets in Φ with $M \subset N$, then $\varphi_M f \leqslant \varphi_N f$, whence $\alpha(M) \leqslant \alpha(N)$.

(PM_{II}): For any sets $M, N \in \Phi$ one has $\varphi_{M \cup N} \leqslant \varphi_M + \varphi_N$, whence $\alpha(M \cup N) \leqslant \alpha(M) + \alpha(N)$.

(PM_{III}): If $M, N \in \Phi$ are disjoint, then $\varphi_{M \cup N} = \varphi_M + \varphi_N$, whence $\alpha(M \cup N) = \alpha(M) + \alpha(N)$.

(PM_{IV}): Given $M \in \Phi$ and $\varepsilon > 0$, we seek a compact set $K \subset M$ and an open set $U \supset M$ such that, for every $N \in \Phi$ satisfying $K \subset N \subset U$, one has $|\alpha(N) - \alpha(M)| \leqslant \varepsilon$.

Since M is μ-integrable, there exist a sequence of compact sets (K_n) and a sequence (U_n) of μ-integrable open sets such that $K_n \subset M \subset U_n$ and $\mu(U_n - K_n) \to 0$ (§4, No. 6, Th. 4). We can suppose that (K_n) is increasing and (U_n) is decreasing (contemplate $K_1 \cup \cdots \cup K_n$ and $U_1 \cap \cdots \cap U_n$). Then $(U_n - K_n)$ is decreasing. Let $A = \bigcup_n K_n$ and $B = \bigcap_n U_n$, which are μ-integrable sets such that $K_n \subset A \subset M \subset B \subset U_n$ for all n. Since $U_n - K_n \downarrow B - A$, we have

$$\mu(B - A) = \lim_n \mu(U_n - K_n) = 0,$$

that is, $B - A$ is μ-negligible. Then

$$\alpha(U_n - K_n) = \int \varphi_{U_n - K_n} f \, d\mu \downarrow \int \varphi_{B - A} f \, d\mu = 0;$$

choose n so that $\alpha(U_n - K_n) \leqslant \varepsilon/2$, and set $U = U_n$, $K = K_n$. For any $N \in \Phi$ such that $K \subset N \subset U$, we have

$$0 \leqslant \alpha(N) - \alpha(K) = \alpha(N - K) \leqslant \alpha(U - K) \leqslant \varepsilon/2,$$

and in particular $0 \leqslant \alpha(M) - \alpha(K) \leqslant \varepsilon/2$, whence $|\alpha(M) - \alpha(N)| \leqslant \varepsilon$.

Thus the hypotheses of §4, Th. 5 are fulfilled: there exists a (unique) measure ρ on X such that, for every $A \in \Phi$, A is ρ-integrable and $\rho(A) = \alpha(A)$ for all $A \in \Phi$, that is,

$$\text{(1)} \qquad \rho(A) = \int \varphi_A f \, d\mu \quad \text{for every } \mu\text{-integrable set } A \subset X,$$

whence $\rho(A) \leqslant \int f \, d\mu < +\infty$ for all such A. In particular, $\rho(K) = \int \varphi_K f \, d\mu$ for every compact set $K \subset X$, whence $\rho(K) \leqslant \int f \, d\mu$. For every open set U in X we know that

$$\rho^*(U) = \sup\{\rho(K) : K \subset U, K \text{ compact}\}$$

(§4, No. 6, Cor. 4 of Th. 4). In particular,

$$\rho^*(X) \leqslant \int f\, d\mu < +\infty\,,$$

therefore X is ρ-integrable (*loc. cit.*, Prop. 10), that is, ρ is a bounded measure and

$$\|\rho\| = \rho(X) = \sup\{\rho(K) : \ K \subset X \text{ compact}\} \leqslant \int f\, d\mu$$

(§4, No. 7, Prop. 12); we therefore have

$$\int \varphi_X\, d\rho = \rho(X) = \sup\{\int \varphi_K f\, d\mu : \ K \subset X \ \text{compact}\}\,. \tag{2}$$

Thus the condition (ii) is clearly equivalent to $\rho(X) = \int f\, d\mu$. I do not see how to derive the condition (ii) from (2); but we established (ii) at the outset (Prop. A), therefore

$$\int \varphi_X d\rho = \rho(X) = \int f\, d\mu\,. \tag{3}$$

On the other hand, the condition (i) does follow from the properties of ρ:

Proposition B. — The condition (i) is satisfied without restrictions on the locally compact space X.

Proof. Every μ-negligible set A in X is ν-negligible; for, such a set A is μ-integrable, hence it is ρ-integrable and $\rho(A) = 0$ by (1). Since ρ is a finite measure, it follows that, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if A is a μ-integrable set with $\mu(A) \leqslant \delta$ then $\rho(A) \leqslant \varepsilon$; since $\rho(A) = \int \varphi_A f\, d\mu$, this will prove (i).

The argument for set-measures (cf. Halmos, *op. cit.*, p. 125, §30, Th. B) can be adapted as follows. Assume to the contrary that for some $\varepsilon > 0$, no such δ exists; that is, for every $\delta > 0$ there exists a μ-integrable set A such that $\mu(A) \leqslant \delta$ but $\rho(A) > \varepsilon$. For every positive integer n, choose a μ-integrable set A_n such that $\mu(A_n) \leqslant 1/2^n$ and $\rho(A_n) > \varepsilon$, and let A be the set of all $x \in X$ that belong to A_n for infinitely many n, that is,

$$A = \bigcap_{n=1}^{\infty} \Big(\bigcup_{k \geqslant n} A_k \Big),$$

which is a μ-measurable set. Writing $B_n = \bigcup_{k=n}^{\infty} A_k$, B_n is μ-measurable and

$$\mu^*(B_n) \leqslant \sum_{k=n}^{\infty} \mu(A_k) \leqslant \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}},$$

therefore B_n is μ-integrable (No. 6, Cor. 1 of Th. 5); since $B_n \supset A$, it follows that A is μ-integrable and $\mu(A) \leqslant \mu(B_n) \to 0$, whence A is μ-negligible, so by assumption $\rho(A) = 0$. But $B_n \downarrow A$ and ρ is finite, therefore $\rho(B_n) \to \rho(A)$; since $B_n \supset A_n$, one has $\rho(B_n) > \varepsilon$ for all n, whence $\rho(A) \geqslant \varepsilon > 0$, a contradiction. ♢

Finally,

Proposition C. If every compact set in X is a G_δ, then $\nu = \rho$.

Proof. As observed at the end of the discussion of $\nu = f \cdot \mu$,

$$\nu(K) = \int \varphi_K f \, d\mu \quad \text{for all compact sets } K \subset X. \tag{4}$$

In view of the property of ρ established in (1) above, we have $\nu(K) = \rho(K)$ for every compact set K in X, therefore $\nu = \rho$ (§4, No. 10, Cor. 3 of Prop. 19). ♢

When $\nu = \rho$ we gain information on ρ, the definition of ν being transformed into

$$\rho(h) = \int hf \, d\mu \qquad \text{for } h \in \mathscr{K}, \tag{5}$$

and on ν, the property (1) of ρ being transformed into

$$\nu(A) = \int \varphi_A f \, d\mu \quad \text{for all } \mu\text{-integrable sets } A \subset X; \tag{6}$$

in terms of the notation $f \cdot \mu = \nu = \rho$, we thus have

$$(f \cdot \mu)(h) = \int h \, d(f \cdot \mu) = \int hf \, d\mu,$$
$$(f \cdot \mu)(A) = \int \varphi_A \, d(f \cdot \mu) = \int \varphi_A f \, d\mu$$

for functions $h \in \mathscr{K}$ and μ-integrable sets A.

Remark. — Since every μ-negligible set is ρ-negligible, it follows that every μ-measurable set A is ρ-measurable (remarks following No. 1, Def. 1), hence is ρ-integrable by the boundedness of ρ, but it is conceivable that the formula of (1) may fail. In particular, we are not authorized to substitute $A = X$ in formula (1) as a means of obtaining (ii).

For a discussion of the relations with the book of Halmos (*op. cit.*), see the next note.

IV.85, ℓ. 19–27. Echo.

"DEFINITION 10."

The materials are at hand for a comparative anatomy of 'measure' as in *Integration* and in the book of Halmos (*Measure theory*, Springer; briefly, [MT]).

In brief, *measure* in the sense of Bourbaki corresponds to the *regular Borel measures* of [MT]. As in the preceding note, 'measures' in the sense of [MT] will be called *set-measures*. As vectorial matters are not treated in [MT], the discussion concerns only *numerical* measures, and for simplicity we restrict attention to *positive* measures.

To review some terminology of [MT], a set $\mathscr{S}$ of subsets of X is called: a *ring* ($=$clan) if it is closed under finite unions and differences, and an *algebra* if, moreover, it is closed under complementation (equivalently, $X \in \mathscr{S}$); a σ-*ring* (resp. σ-*algebra*) is a ring (resp. algebra) that is closed under countable unions. Thus $\mathscr{S}$ is a σ-algebra if and only if it is a tribe (in other words, a tribe is a clan that is closed under complementation and countable unions). The set $\mathscr{S}$ is said to be *hereditary* if it contains the subsets of any of its sets, that is, $B \subset A \in \mathscr{S} \Rightarrow B \in \mathscr{S}$.

Let μ be a positive measure on X, and write $\mathfrak{M}$ for the set of all μ-measurable sets $A \subset X$; $\mathfrak{M}$ is a tribe (No. 4, Cor. 2 of Th. 2), that is, a σ-algebra of subsets of X. Let $\lambda = \mu^*|\mathfrak{M}$ be the restriction of μ^* to $\mathfrak{M}$:

$$\lambda(A) = \mu^*(A) \qquad (A \in \mathfrak{M}).$$

Note that $\lambda(A) < +\infty \Leftrightarrow A$ is μ-integrable (No. 6, Cor. 1 of Th. 5), in which case $\lambda(A) = \mu(A)$.

Obviously $\lambda(\varnothing) = 0$ and $0 \leqslant \lambda(A) \leqslant +\infty$ for all $A \in \mathfrak{M}$, so to verify that λ is a set-measure, it remains only to check that it is countably additive [MT, §7, p. 30], established at the end of the following list of properties.

monotonicity: $A \subset B \Rightarrow \lambda(A) \leqslant \lambda(B)$ (§1, No. 4, Prop. 16).

finite additivity: $A \cap B = \varnothing \Rightarrow \lambda(A \cup B) = \lambda(A) + \lambda(B)$.

For, if A and B are both μ-integrable, the equality holds by §4, No. 5, Prop. 6, whereas if one of A, B is not integrable then $A \cup B$ is not integrable, whence both sides are equal to $+\infty$.

continuity from below: If $A = \bigcup_{n=1}^{\infty} A_n$, where (A_n) is an increasing sequence of sets in $\mathfrak{M}$, then $A \in \mathfrak{M}$ and

$$\lambda(A) = \sup \lambda(A_n)$$

(§1, No. 4, Prop. 17).

countable subadditivity: If (A_n) is any sequence of sets in $\mathfrak{M}$ and $\mathrm{A} = \bigcup_{n=1}^{\infty} \mathrm{A}_n$, then $\mathrm{A} \in \mathfrak{M}$ and

$$\lambda(\mathrm{A}) \leqslant \sum_{n=1}^{\infty} \lambda(\mathrm{A}_n)$$

(§1, No. 4, Prop. 18).

countable additivity: If $\mathrm{A} = \bigcup_{n=1}^{\infty} \mathrm{A}_n$, where (A_n) is a sequence of pairwise disjoint sets in $\mathfrak{M}$, then $\mathrm{A} \in \mathfrak{M}$ and

$$\lambda(\mathrm{A}) = \sum_{n=1}^{\infty} \lambda(\mathrm{A}_n) .$$

For, writing $\mathrm{B}_n = \bigcup_{k=1}^{n} \mathrm{A}_k$, one has

$$\lambda(\mathrm{A}) = \sup_n \lambda(\mathrm{B}_n) = \sup_n \sum_{k=1}^{n} \lambda(\mathrm{A}_k) = \sum_{n=1}^{\infty} \lambda(\mathrm{A}_n)$$

by continuity from below and finite additivity.

In particular, λ is a set-measure.

Let us write $\mathfrak{R}$ for the set of all μ-integrable sets $\mathrm{A} \subset \mathrm{X}$, that is,

$$\mathfrak{R} = \{\mathrm{A} \in \mathfrak{M} : \ \lambda(\mathrm{A}) < +\infty\} ;$$

$\mathfrak{R}$ is a *ring* (clan) of sets: $\mathrm{A}, \mathrm{B} \in \mathfrak{R} \Rightarrow \mathrm{A} \cup \mathrm{B}, \mathrm{A} - \mathrm{B} \in \mathfrak{R}$ (§4, No. 5, Prop. 7). If $\mathrm{A} = \bigcup_{n=1}^{\infty} \mathrm{A}_n$, where (A_n) is an increasing sequence of sets in $\mathfrak{R}$, then

$$\mathrm{A} \in \mathfrak{R} \quad \Leftrightarrow \quad \sup_n \lambda(\mathrm{A}_n) + \infty ,$$

in which case $\lambda(\mathrm{A}) = \sup_n \lambda(\mathrm{A}_n)$ (§4, No. 5, Prop. 8). The restriction of λ to $\mathfrak{R}$ is a *finite* set-measure on the ring $\mathfrak{R}$ [MT, §5, pp. 30, 31].

Borel sets and H-*Borel sets.*

Let $\mathfrak{B}$ be the tribe (σ-algebra) of Borel sets in X, that is, the tribe generated by the closed subsets of X (GT, IX, §6, No. 3, Def. 4). Of course $\mathfrak{B} \subset \mathfrak{M}$ (No. 4, Cor. 3 of Th. 2), and the restriction $\lambda|\mathfrak{B}$ is a set-measure on $\mathfrak{B}$. However, the set of 'Borel sets' in the sense of Halmos [MT, §51,

p. 219] is the *σ-ring* generated by the *compact* subsets of X; let us denote this σ-ring by $\mathfrak{B}_1$, and call its elements H-*Borel sets*.

Of course $\mathfrak{B}_1 \subset \mathfrak{B}$. In order that $\mathfrak{B}_1 = \mathfrak{B}$ it is necessary and sufficient that X be countable at infinity (i.e., σ-compact), that is, $X = \bigcup_n K_n$ for some sequence (K_n) of compact sets (in such a space, every closed set C is the union of the sequence of compact sets $C \cap K_n$), in which case $\mathfrak{B}_1$ is a tribe ($X \in \mathfrak{B}_1$). Indeed, $\mathfrak{B}_1$ is a tribe if and only if X is countable at infinity. {For, every element of $\mathfrak{B}_1$ is contained in a countable union of compact sets, since the hereditary σ-ring [MT, §10, p. 41] generated by a collection of sets must contain the σ-ring they generate; whence $X \in \mathfrak{B}_1$ if and only if X is σ-compact}.

Let us write $\lambda_1 = \lambda\big|\mathfrak{B}_1$, a set-measure on the σ-ring of H-Borel subsets of X. Since $\lambda_1(K) = \mu(K) < +\infty$ for all compact sets $K \subset X$, λ_1 is a *Borel measure* in the sense of MT, §52, p. 223. In fact, λ_1 is a *regular Borel measure* [MT, §52, p. 224] in the sense that, for every $A \in \mathfrak{B}_1$,

$$\lambda(A) = \sup\{\lambda(K) : K \subset A, \ K \text{ compact}\}$$

(*inner regularity*) and

$$\lambda(A) = \inf\{\lambda(U) : A \subset U, \ U \text{ open and } U \in \mathfrak{B}_1\}$$

(*outer regularity*). It suffices [MT, §52, Th. F, p. 228] to show that if $U \in \mathfrak{B}_1$ is an open set and $U \subset K_1$ for some compact set K_1, then

$$\lambda(U) = \sup\{\lambda(K) : K \subset U, \ K \text{ compact}\},$$

and this follows from the μ-integrability of U (§4, No. 6, Cor. 1 of Th. 4). {Incidentally, if $U \subset K_1$ with U open and K_1 compact, then necessarily $U \in \mathfrak{B}_1$; indeed, it suffices that U can be covered by a sequence of compact sets [MT, §51, Th. A, p. 219].}

To summarize:

Proposition A. If μ is a positive measure on the locally compact space X, then the restriction of μ^* to the σ-ring generated by the compact subsets of X is a regular Borel measure on X in the sense of Halmos [MT].

Conversely:

Proposition B. If λ is a regular Borel measure in the sense of [MT] (defined on the σ-ring generated by the compact subsets of X), then there exists a unique measure μ on X (in the sense of *Integration*) such that $\lambda(K) = \mu(K)$ for every compact set K in X.

This follows from §4, No. 11, Th. 5, with Φ the set of all compact sets $K \subset X$, and $\alpha = \lambda|\Phi$: for, the condition (PM_{IV}) of Th. 5 follows from the regularity of λ, and the other conditions are obvious from the properties of λ.

In effect, the cited Th. 5 corresponds to the theory of *regular contents* in the book of Halmos [MT, §54].

IV.85, ℓ. −6 to −4.

"*Remark.* — Suppose μ is bounded. For every $a > 0$, the set of measurable mappings of X into F such that $|\mathbf{f}(x)| \leqslant a$ almost everywhere is equi-integrable of order p, and this is true for any $p \in [1, +\infty[$."

Let H be the set of all such $\mathbf{f} \in \mathscr{L}^p_F$. For each $\mathbf{f} \in H$, write $\rho_{\mathbf{f}}$ for the bounded measure on X determined by the condition that

$$(*) \qquad \rho_{\mathbf{f}}(A) = \int \varphi_A |\mathbf{f}|^p \, d|\mu|$$

for every μ-integrable set $A \subset X$ (see the first note for ℓ. 19-27). Write $\mathfrak{M}$ for the tribe of all μ-measurable sets $A \subset X$, and $\mathfrak{K}$ for the set of all compact sets $K \subset X$; since μ is assumed to be bounded, every $A \in \mathfrak{M}$ is μ-integrable, hence (∗) holds for all $A \in \mathfrak{M}$. Moreover, since $|\mathbf{f}| \leqslant a$ μ-almost everywhere for each $\mathbf{f} \in H$, one has

$$\rho_{\mathbf{f}}(A) \leqslant a^p |\mu|(A) \leqslant a^p |\mu|(X) = a^p \sup_{K \in \mathfrak{K}} |\mu|(K)$$

for every $\mathbf{f} \in H$ and $A \in \mathfrak{M}$.

Given any $\varepsilon > 0$, choose $K \in \mathfrak{K}$ so that $a^p |\mu|(X - K) \leqslant \varepsilon$ and set $\delta = \varepsilon / a^p$. Then, for every $A \in \mathfrak{M}$ with $|\mu|(A) \leqslant \delta$ one has

$$\int |\mathbf{f}|^p \varphi_A \, d|\mu| = \rho_{\mathbf{f}}(A) \leqslant a^p |\mu|(A) \leqslant a^p \delta = \varepsilon ,$$

and

$$\int |\mathbf{f}|^p \varphi_{X - K} \, d|\mu| = \rho_{\mathbf{f}}(X - K) \leqslant a^p |\mu|(X - K) \leqslant \varepsilon,$$

thus H satisfies the conditions (i) and (ii) of Def. 10.

IV.86, ℓ. 2–4.

"Let $\mathbf{f}, \mathbf{g}$ in H be such that

$$|\mathbf{f}(x) - \mathbf{g}(x)| \leqslant \left(\frac{\varepsilon}{|\mu|(K)} \right)^{1/p}$$

for $x \in K$, except on a set M of measure $\leqslant \delta$."

Note first of all that only compact sets K of measure > 0 are in play: assuming $\mathrm{N}_p(\mathbf{f}) > 0$, for ε sufficiently small the K in property (ii) of Def. 10 will have to be non-negligible.

Where are we going? In the notations of Prop. 20, we have

$$\mathrm{H} \subset \mathscr{L}^p_{\mathrm{F}}(\mathrm{X}, \mu) \subset \mathscr{S}(\mathrm{X}, \mu; \mathrm{F})\,;$$

by (iii) of Prop. 20, the uniformity $\mathfrak{U}$ of $\mathscr{S} = \mathscr{S}(\mathrm{X}, \mu; \mathrm{F})$ (the uniformity of convergence in measure) induces on $\mathscr{L}^p_{\mathrm{F}} = \mathscr{L}^p_{\mathrm{F}}(\mathrm{X}, \mu)$ a uniformity coarser than the uniformity $\mathfrak{U}_p$ of convergence in mean of order p, in other words, for functions in $\mathscr{L}^p_{\mathrm{F}}$, convergence in mean of order p implies convergence in measure. Succinctly, the trace of $\mathfrak{U}$ on $\mathscr{L}^p_{\mathrm{F}} \times \mathscr{L}^p_{\mathrm{F}}$ is contained in $\mathfrak{U}_p$:

$$\mathfrak{U} \cap (\mathscr{L}^p_{\mathrm{F}} \times \mathscr{L}^p_{\mathrm{F}}) \subset \mathfrak{U}_p\,,$$

whence $\mathfrak{U} \cap (\mathrm{H} \times \mathrm{H}) \subset \mathfrak{U}_p \cap (\mathrm{H} \times \mathrm{H})$. We are embarked on a proof of the reverse inclusion.

Thus, given a basic entourage $\mathrm{U} \in \mathfrak{U}_p$, we seek an entourage $\mathrm{W} \in \mathfrak{U}$ such that

$$(*) \qquad \mathrm{W} \cap (\mathrm{H} \times \mathrm{H}) \subset \mathrm{U} \cap (\mathrm{H} \times \mathrm{H})\,,$$

which will show that $\mathrm{U} \cap (\mathrm{H} \times \mathrm{H}) \in \mathfrak{U} \cap (\mathrm{H} \times \mathrm{H})$ and so $\mathfrak{U}_p \cap (\mathrm{H} \times \mathrm{H}) \subset \mathfrak{U} \cap (\mathrm{H} \times \mathrm{H})$.

We can suppose that

$$\mathrm{U} = \{(\mathbf{f}, \mathbf{g}) \in \mathscr{L}^p_{\mathrm{F}} \times \mathscr{L}^p_{\mathrm{F}} :\ \mathrm{N}_p(\mathbf{f} - \mathbf{g}) \leqslant [(2^{p+1} + 1)\varepsilon]^{1/p}\}\,.$$

Let V be the entourage for F defined by

$$\mathrm{V} = \{(\mathbf{y}, \mathbf{z}) \in \mathrm{F} \times \mathrm{F} :\ |\mathbf{y} - \mathbf{z}| \leqslant \big(\varepsilon/|\mu|(\mathrm{K})\big)^{1/p}\}\,,$$

and form the entourage $\mathrm{W} \in \mathfrak{U}$ defined by

$$\mathrm{W} = \boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta)\,,$$

consisting of all pairs $(\mathbf{f}, \mathbf{g}) \in \mathscr{S} \times \mathscr{S}$ such that

$$|\mu|^*\big(\{x \in \mathrm{K} :\ \big(\mathbf{f}(x), \mathbf{g}(x)\big) \notin \mathrm{V}\,\}\big) \leqslant \delta\,.$$

Writing, for any pair $(\mathbf{f}, \mathbf{g}) \in \mathscr{S} \times \mathscr{S}$,

$$\begin{aligned} \mathrm{M}(\mathbf{f}, \mathbf{g}) &= \{x \in \mathrm{K} :\ \big(\mathbf{f}(x), \mathbf{g}(x)\big) \notin \mathrm{V}\,\} \\ &= \{x \in \mathrm{K} :\ |\mathbf{f}(x) - \mathbf{g}(x)| > \big(\varepsilon/|\mu|(\mathrm{K})\big)^{1/p}\}\,, \end{aligned}$$

we thus have

$$\mathrm{W} = \{(\mathbf{f},\mathbf{g}) \in \mathscr{S} \times \mathscr{S} :\ |\mu|^*\big(\mathrm{M}(\mathbf{f},\mathbf{g})\big) \leqslant \delta\,\}\,.$$

To establish $(*)$, we wish to show that if $(\mathbf{f},\mathbf{g}) \in \mathrm{W} \cap (\mathrm{H}\times\mathrm{H})$, that is, if $\mathbf{f},\mathbf{g} \in \mathrm{H}$ are such that $|\mu|^*\big(\mathrm{M}(\mathbf{f},\mathbf{g})\big) \leqslant \delta$, then $(\mathbf{f},\mathbf{g}) \in \mathrm{U}$, that is, $\mathrm{N}_p(\mathbf{f}-\mathbf{g}) \leqslant [(2^{p+1}+1)\varepsilon]^{1/p}$.

The statement of ℓ. 2–4 starts the ball rolling by saying: let $(\mathbf{f},\mathbf{g}) \in \boldsymbol{W}(\mathrm{V},\mathrm{K},\delta) \cap (\mathrm{H}\times\mathrm{H})$. The rest of the computation is straightforward.

IV.86, ℓ. 5, 6.

Based on property (ii) in Def. 10.

IV.86, ℓ. 8.

Based on property (i) in Def. 10.

IV.86, ℓ. 11.

Follows from ℓ. 10 because

$$|\mathbf{f}-\mathbf{g}|^p \leqslant \frac{\varepsilon}{|\mu|(\mathrm{K})} \quad \text{on } \mathrm{K} - \mathrm{M}$$

by the definition of M in ℓ. 4.

IV.86, ℓ. −12 to −6.

"*Lemma* 5."

One notes that the condition (i) reveals the Lemma as an analog, in locally compact spaces, of Lebesgue's criterion for Riemann-integrability (see, for example, §11.4, pp. 211–214 of my book *A first course in real analysis*, Springer). Specialized to the case of a bounded function $f : [a,b] \to \mathbf{R}$ and Lebesgue measure μ on $[a,b]$, the lemma says (assuming Lebesgue's criterion) that f is Riemann-integrable if and only if, for every $\varepsilon > 0$, there exist continuous functions $g,h : [a,b] \to \mathbf{R}$, with $h \geqslant 0$, such that

$$|f-g| \leqslant h \leqslant 2\sup|f| \quad \text{and} \quad \int h\,d\mu \leqslant \varepsilon\,.$$

In effect, approximation of f by step functions (say, in the sense of *op. cit.*, p. 174, Th. 9.8.2) is replaced by an approximation by continuous functions.

IV.87, ℓ. 1–4.

"Multiplying $g_1,\ldots,g_n$ by a suitable same element of $\mathscr{K}(\mathrm{X})$, we can suppose in addition that

$$|g_1\mathbf{a}_1 + \cdots + g_n\mathbf{a}_n| \leqslant \mathrm{M} = \sup|\mathbf{f}|$$

on X, whence $k \leqslant 2M$."

I am indebted to Professor Dixmier for crucial help with the proof of Lemma 5.

For technical reasons, let us assume the g_i and $\mathbf{a}_i$ so chosen that, on writing $k = |\mathbf{f} - g_1\mathbf{a}_1 - \cdots - g_n\mathbf{a}_n|$, one has $\int k\,d\mu \leqslant \varepsilon/4$ (instead of $\varepsilon/2$). When the g_i are replaced by suitable functions g_i', the function $k' = |\mathbf{f} - g_i'\mathbf{a}_1 - \cdots - g_n'\mathbf{a}_n|$ will satisfy $\int k'\,d\mu \leqslant \varepsilon/2$.

Write $\mathbf{p} = g_1\mathbf{a}_1 + \cdots + g_n\mathbf{a}_n$. Thus $\mathbf{p} : X \to F$, $k = |\mathbf{f} - \mathbf{p}|$ and $\int k\,d\mu \leqslant \varepsilon/4$. We seek a continuous function $\varphi : X \to \mathbf{R}$, $0 \leqslant \varphi \leqslant 1$, such that, on defining

$$g_i' = \varphi g_i\,, \quad \mathbf{p}' = \varphi\mathbf{p} = g_1'\mathbf{a}_1 + \cdots + g_n'\mathbf{a}_n\,, \quad k' = |\mathbf{f} - \mathbf{p}'|\,,$$

one has $|\mathbf{p}'| \leqslant M$ and $\int k'\,d\mu \leqslant \varepsilon/2$. We define

$$\varphi(x) = \begin{cases} \dfrac{M}{|\mathbf{p}(x)|} & \text{if } |\mathbf{p}(x)| > M \\ 1 & \text{if } |\mathbf{p}(x)| \leqslant M\,. \end{cases}$$

The continuity of φ is a special case of the following proposition (with $g = |\mathbf{p}|$):

Proposition. Let X be any topological space, $g : X \to \mathbf{R}$ a continuous function $\geqslant 0$, and M a real number > 0. Let

$$U = \{x \in X : \ g(x) > M\}$$

and define $\varphi : X \to [0,1]$ by

$$\varphi(x) = \begin{cases} \dfrac{M}{g(x)} & \text{if } x \in U \\ 1 & \text{if } x \in X - U\,. \end{cases}$$

Then φ is continuous.

Proof. By the continuity of g, U is open and $g \geqslant M$ on $\overline{U}$. On the other hand, $X - U = \{x \in X : \ g(x) \leqslant M\}$, therefore

$$(*) \qquad g(x) = M \text{ on } \overline{U} \cap (X - U) = \overline{U} - U\,.$$

(Incidentally, $\overline{U} - U = \overline{U} \cap \overline{X - U} = \partial U$ (the boundary of U.)

Now, $\overline{U}$, $X - U$ is a closed covering of X, so it will suffice to show that the restrictions $\varphi|\overline{U}$ and $\varphi|X - U$ are continuous (GT, I, §3, No. 2, Prop. 4). Indeed, $\varphi|X - U$ is the constant function 1. On the other hand,

$$\varphi(x) = \frac{M}{g(x)} \quad \text{on} \quad \overline{U} = U \cup (\overline{U} - U)\,;$$

for, if $x \in \mathrm{U}$ then $\varphi(x) = \mathrm{M}/g(x)$ by definition, whereas if $x \in \overline{\mathrm{U}} - \mathrm{U} = \overline{\mathrm{U}} \cap (\mathrm{X} - \mathrm{U})$ then $g(x) = \mathrm{M}$ (by $(*)$) and $\varphi(x) = 1$ (by definition), so again $\varphi(x) = \mathrm{M}/g(x)$. ♢

Returning to the proof of Lemma 5, with $\mathrm{U} = \{x \in \mathrm{X} : |\mathbf{p}(x)| > \mathrm{M}\}$, $g_i' = \varphi g_i$ and $\mathbf{p}' = \varphi \mathbf{p}$, we have

$$|\mathbf{p}'(x)| = \varphi(x)|\mathbf{p}(x)| = \begin{cases} \dfrac{\mathrm{M}}{|\mathbf{p}(x)|} \cdot |\mathbf{p}(x)| = \mathrm{M} & \text{if } x \in \mathrm{U} \\ 1 \cdot |\mathbf{p}(x)| \leqslant \mathrm{M} & \text{if } x \in \mathrm{X} - \mathrm{U}, \end{cases}$$

thus $|\mathbf{p}'(x)| \leqslant \mathrm{M}$ on X.

claim: $|\mathbf{f} - \mathbf{p}'| \leqslant 2|\mathbf{f} - \mathbf{p}|$.

For, if $x \in \mathrm{X} - \mathrm{U}$, that is, if $|\mathbf{p}(x)| \leqslant \mathrm{M}$, then $\varphi(x) = 1$ and so $\mathbf{p}'(x) = \mathbf{p}(x)$, whence

$$|\mathbf{f}(x) - \mathbf{p}'(x)| = |\mathbf{f}(x) - \mathbf{p}(x)| \leqslant 2|\mathbf{f}(x) - \mathbf{p}(x)|.$$

On the other hand, if $x \in \mathrm{U}$, that is, if $|\mathbf{p}(x)| > \mathrm{M}$, then

$$\varphi(x) = \frac{\mathrm{M}}{|\mathbf{p}(x)|} < 1 \quad \text{and} \quad \mathbf{p}'(x) = \varphi(x)\mathbf{p}(x) = \frac{\mathrm{M}}{|\mathbf{p}(x)|} \cdot \mathbf{p}(x),$$

thus

$$\begin{aligned} |\mathbf{p}(x) - \mathbf{p}'(x)| &= \Big|\mathbf{p}(x) - \frac{\mathrm{M}}{|\mathbf{p}(x)|}\mathbf{p}(x)\Big| \\ &= \Big|\big(1 - \frac{\mathrm{M}}{|\mathbf{p}(x)|}\big)\mathbf{p}(x)\Big| = \big(1 - \frac{\mathrm{M}}{|\mathbf{p}(x)|}\big)|\mathbf{p}(x)| \\ &= |\mathbf{p}(x)| - \mathrm{M} = |[\mathbf{p}(x) - \mathbf{f}(x)] + \mathbf{f}(x)| - \mathrm{M} \\ &\leqslant |\mathbf{p}(x) - \mathbf{f}(x)| + |\mathbf{f}(x)| - \mathrm{M} \\ &\leqslant |\mathbf{p}(x) - \mathbf{f}(x)| + \mathrm{M} - \mathrm{M} = |\mathbf{p}(x) - \mathbf{f}(x)|, \end{aligned}$$

therefore

$$\begin{aligned} |\mathbf{f}(x) - \mathbf{p}'(x)| &\leqslant |\mathbf{f}(x) - \mathbf{p}(x)| + |\mathbf{p}(x) - \mathbf{p}'(x)| \\ &\leqslant |\mathbf{f}(x) - \mathbf{p}(x)| + |\mathbf{p}(x) - \mathbf{f}(x)| = 2|\mathbf{f}(x) - \mathbf{p}(x)|, \end{aligned}$$

and the claimed inequality is established.

{For the purposes of Lemma 5, the continuous function φ is adequate; but the assertion promises a φ with compact support. This is easily arranged: let $g \in \mathscr{K}(\mathrm{X})$ be such that $0 \leqslant g \leqslant 1$ and $g = 1$ on the union of the supports of the g_i; then $gg_i = g_i$ so $(\varphi g)g_i = \varphi g_i = g_i'$ and $(\varphi g)\mathbf{p} = \varphi \mathbf{p} = \mathbf{p}'$, thus, replacing φ by $\varphi g \in \mathscr{K}(\mathrm{X})$, the functions g_i'

and $\mathbf{p}'$ are unchanged. In particular, the inequality $|\mathbf{f} - \mathbf{p}'| \leqslant 2|\mathbf{f} - \mathbf{p}|$ is invariant under the modification of φ (no need to reconsider its proof with a modified φ). From this point on, the argument makes no reference to φ, hence is valid whether or not φ was modified.}

Writing $k' = |\mathbf{f} - \mathbf{p}'|$, we thus have $k' \leqslant 2|\mathbf{f} - \mathbf{p}| = 2k$, and so

$$\int k'\,d\mu \leqslant 2\int k\,d\mu \leqslant 2(\varepsilon/4) = \varepsilon/2\,;$$

and of course $|\mathbf{p}'| \leqslant \mathrm{M}$.

Thus, replacing the notations $g_i', \mathbf{p}', k'$ by $g_i, \mathbf{p}, k$, we can suppose that $|\mathbf{p}| \leqslant \mathrm{M}$; moreover, $k \leqslant 2\mathrm{M}$ since

$$k(x) = |\mathbf{f}(x) - \mathbf{p}(x)| \leqslant |\mathbf{f}(x)| + |\mathbf{p}(x)| \leqslant \mathrm{M} + \mathrm{M}$$

for all $x \in \mathrm{X}$.

IV.87, ℓ. 6.

"Then $2\mathrm{M} \geqslant l \geqslant k$ on X, and $l = k$ on $\mathrm{X} - \mathrm{N}'$ "

We know that $k : \mathrm{X} \to [0, 2\mathrm{M}]$. By definition (GT, IV, §5, No. 6)

$$l(x) = \inf_{\mathrm{V}} \Big(\sup_{y \in \mathrm{V}} k(y) \Big),$$

where V runs over the filter of neighborhoods of x (or any base thereof), therefore $l : \mathrm{X} \to [0, 2\mathrm{M}]$, thus $l \leqslant 2\mathrm{M}$; and $l(x) \geqslant k(x)$ because $x \in \mathrm{V}$ for all V.

If $x \in \mathrm{X} - \mathrm{N}'$ then k is continuous at x, hence is upper semi-continuous at x (GT, IV, §6, No. 2, comment following Def. 1), hence

$$k(x) = \limsup_{y \to x} k(y) = l(x)$$

(*loc. cit.*, 'dual' of Prop. 3). The function l is upper semi-continuous (*loc. cit.*, the 'dual' of Prop. 4), and is called the *upper semi-continuous regularization* of k.

IV.87, ℓ. 7.

"... therefore $\int l\,d\mu \leqslant \varepsilon/2$."

Since $l = k$ almost everywhere, so that $\mu^*(l) = \mu^*(k) < +\infty$, it follows that $l \in \mathscr{F}^1_{\mathbf{R}}$ (§3, No. 3), hence $l \in \mathscr{L}^1_{\mathbf{R}}$ (§3, No. 4, third paragraph following Def. 2) and $\int l\,d\mu = \int k\,d\mu \leqslant \varepsilon/2$.

IV.87, ℓ. 7–9.

" l is bounded and upper semi-continuous, hence is the lower envelope of the set of bounded continuous functions $\geqslant l$."

Of course $0 \leqslant l \leqslant 2\mathrm{M}$, and the upper semi-continuity of l was observed in the note before the last. Then $2\mathrm{M} - l$ is $\geqslant 0$ and lower semi-continuous, that is, in the notation of §1, No. 1, $2\mathrm{M} - l \in \mathscr{I}_+$, therefore $2\mathrm{M} - l$ is the upper envelope of the functions $g \in \mathscr{K}_+$ such that $g \leqslant 2\mathrm{M} - l$ (*loc. cit.*, Lemma); consequently $l - 2\mathrm{M}$ is the lower envelope of the functions $-g$, and so

$$l = (l - 2\mathrm{M}) + 2\mathrm{M}$$

is the lower envelope of the functions $-g + 2\mathrm{M} = 2\mathrm{M} - g$, which are indeed bounded and continuous. Write

$$\mathrm{H} = \{h \in \mathscr{C}(\mathrm{X}) : \ h \text{ is bounded and } l \leqslant h\}.$$

Of course $l \leqslant \inf \mathrm{H}$, where $(\inf \mathrm{H})(x) = \inf\{h(x) : \ h \in \mathrm{H}\}$; but we have just seen that the functions $2\mathrm{M} - g$ described above form a subset H' of H such that $l = \inf \mathrm{H}'$, therefore

$$l \leqslant \inf \mathrm{H} \leqslant \inf \mathrm{H}' = l,$$

whence $l = \inf \mathrm{H}$ as claimed.

IV.87, ℓ. 9–11.

"Therefore there exists a bounded continuous function $h \geqslant l$ on X such that $h \leqslant 2\mathrm{M}$ and $\int h\,d\mu \leqslant \int l\,d\mu + \varepsilon/2$ (§4, No. 4, Cor. 2 of Prop. 5)."

Maintaining the notations of the preceding note, since the set of functions $g \in \mathscr{K}_+(\mathrm{X})$ with $g \leqslant 2\mathrm{M} - l$ is directed upward, with upper envelope $2\mathrm{M} - l$, the set H' of functions $2\mathrm{M} - g$ is directed downward, with lower envelope l, therefore

$$\int l\,d\mu = \inf_{h' \in \mathrm{H}'} \int h'\,d\mu$$

by the cited Cor. 2. Since the $h' \in \mathrm{H}'$ satisfy $0 \leqslant l \leqslant h' \leqslant 2\mathrm{M}$, therefore $\mathrm{H}' \subset \mathrm{H}$, it follows from the preceding formula that there exists an $h \in \mathrm{H}$ such that $h \leqslant 2\mathrm{M}$ and

$$\int h\,d\mu \leqslant \int l\,d\mu + \varepsilon/2 = \int k\,d\mu + \varepsilon/2 \leqslant \varepsilon/2 + \varepsilon/2\,;$$

moreover,

$$|\mathbf{f} - g_1\mathbf{a}_1 - \cdots - g_n\mathbf{a}_n| = |\mathbf{f} - \mathbf{p}| = k \leqslant l \leqslant h \leqslant 2\mathrm{M},$$

thus the proof of (i) $\Rightarrow$ (ii) is complete.

IV.87, ℓ. 14–15.

"For every $x \in X$, $\omega(x)$ is the oscillation of $\mathbf{f} - g_1\mathbf{a}_1 - \cdots - g_n\mathbf{a}_n$ at x, therefore $\omega(x) \leqslant 2h(x)$."

Let us write $\omega_{\mathbf{f}}(x) = \omega(x)$ for the oscillation of $\mathbf{f}$ at x, that is (GT, IX, §2, No. 3),

$$\omega_{\mathbf{f}}(x) = \inf_{V} \delta\big(\mathbf{f}(V)\big),$$

where V runs over the filter of neighborhoods of x, and

$$\delta\big(\mathbf{f}(V)\big) = \sup_{y,y' \in V} |\mathbf{f}(y) - \mathbf{f}(y')|$$

is the diameter of $\mathbf{f}(V)$. Note that $V \mapsto \delta\big(\mathbf{f}(V)\big)$ is a decreasing function of V, and, since $|\mathbf{f}| \leqslant M$, one has $0 \leqslant \omega_{\mathbf{f}} \leqslant 2M$.

Why are we considering ω?; because $\mathbf{f}$ is continuous at x if and only if $\omega(x) = 0$ (GT, *loc. cit.*, comment following the proof of Prop. 4), therefore

$$\{x \in X : \ \omega(x) > 0\} = N$$

(the set of discontinuities of $\mathbf{f}$). Our objective is to show that $\mu^*(N) = 0$, that is, $\omega(x) = 0$ for almost every x; since $\omega \geqslant 0$, it will suffice to show that ω is integrable and $\int \omega \, d\mu = 0$ (§2, No. 3, Th. 1).

Thus, given any $\varepsilon > 0$ (the $\mathbf{a}_i$, g_i and h being chosen as in (ii)), it will suffice to show that ω is integrable and $\int \omega \, d\mu \leqslant 2\varepsilon$. The proof exploits the invariance of $\omega_{\mathbf{f}}$ under perturbation of $\mathbf{f}$ by a continuous function:

Proposition. (1) For any functions $\mathbf{f}, \mathbf{g} : X \to F$, one has $\omega_{\mathbf{f}+\mathbf{g}} \leqslant \omega_{\mathbf{f}} + \omega_{\mathbf{g}}$.

(2) If, moreover, $\mathbf{g}$ is continuous, then $\omega_{\mathbf{f}+\mathbf{g}} = \omega_{\mathbf{f}}$.

(3) $\omega_{\mathbf{f}}$ is upper semi-continuous.

Proof. (1) Let V be a neighborhood of $x \in X$. For all $y, y' \in V$ one has

$$\begin{aligned} |(\mathbf{f}+\mathbf{g})(y) - (\mathbf{f}+\mathbf{g})(y')| &= \big|[\mathbf{f}(y) - \mathbf{f}(y')] + [\mathbf{g}(y) - \mathbf{g}(y')]\big| \\ &\leqslant |\mathbf{f}(y) - \mathbf{f}(y')| + |\mathbf{g}(y) - \mathbf{g}(y')| \leqslant \delta\big(\mathbf{f}(V)\big) + \delta\big(\mathbf{g}(V)\big), \end{aligned}$$

whence

$$(*) \qquad \delta\big((\mathbf{f}+\mathbf{g})(V)\big) \leqslant \delta\big(\mathbf{f}(V)\big) + \delta\big(\mathbf{g}(V)\big).$$

We are to show that $\omega_{\mathbf{f}+\mathbf{g}}(x) \leqslant \omega_{\mathbf{f}}(x) + \omega_{\mathbf{g}}(x)$. This is trivial if either of $\omega_{\mathbf{f}}(x)$ or $\omega_{\mathbf{g}}(x)$ is infinite; if both are finite, then $(*)$ shows that 'ultimately' (with respect to the neighborhood filter at x) all terms of $(*)$ are finite, and since they are decreasing functions of V , passage to the limit yields the desired inequality.

(2) If, moreover, $\mathbf{g}$ is continuous, then $\omega_{\mathbf{g}} = 0$ and so $\omega_{\mathbf{f}+\mathbf{g}} \leqslant \omega_{\mathbf{f}}$; similarly $\omega_{\mathbf{f}} = \omega_{(\mathbf{f}+\mathbf{g})+(-\mathbf{g})} \leqslant \omega_{\mathbf{f}+\mathbf{g}}$.

(3) See GT, *loc. cit.*, Prop. 4. ♢

Returning to the proof of (ii) ⇒ (i), writing $\mathbf{g} = -g_1\mathbf{a}_1 - \cdots - g_n\mathbf{a}_n$ and $\mathbf{w} = \mathbf{f} + \mathbf{g}$, we have $|\mathbf{w}| \leqslant h$. It follows that $\omega_{\mathbf{w}} \leqslant 2h$. For, if V is a neighborhood of $x \in \mathrm{X}$ then, for all $y, y' \in \mathrm{V}$ one has

$$|\mathbf{w}(y) - \mathbf{w}(y')| \leqslant |\mathbf{w}(y)| + |\mathbf{w}(y')| \leqslant h(y) + h(y') \,;$$

but

$$h(y) = [h(y) - h(x)] + h(x) \leqslant |h(y) - h(x)| + h(x) \leqslant \delta\big(h(\mathrm{V})\big) + h(x)$$

and similarly $h(y') \leqslant \delta\big(h(\mathrm{V})\big) + h(x)$. Thus

$$|\mathbf{w}(y) - \mathbf{w}(y')| \leqslant \big[\delta\big(h(\mathrm{V})\big) + h(x)\big] + \big[\delta\big(h(\mathrm{V})\big) + h(x)\big]$$

for all $y, y' \in \mathrm{V}$, therefore

$$\delta\big(\mathbf{w}(\mathrm{V})\big) \leqslant 2\,\delta\big(h(\mathrm{V})\big) + 2\,h(x)\,;$$

since h is continuous at x , passing to the limit with respect to the neighborhood filter at x , one has

$$\omega_{\mathbf{w}}(x) \leqslant 2 \cdot \omega_h(x) + 2\,h(x) = 2 \cdot 0 + 2\,h(x)\,,$$

thus $\omega_{\mathbf{w}} \leqslant 2h$.

Finally, $\omega_{\mathbf{f}} = \omega_{\mathbf{f}+\mathbf{g}} = \omega_{\mathbf{w}} \leqslant 2h$; since $\omega_{\mathbf{f}}$ is upper semi-continuous, hence measurable (No. 5, Cor. of Prop. 8), and h is integrable, it follows that $\omega_{\mathbf{f}}$ is integrable and

$$\int \omega_{\mathbf{f}}\, d\mu \leqslant 2 \int h\, d\mu \leqslant 2\varepsilon\,.$$

This completes the proof of Lemma 5.

IV.87, ℓ. −3.

"... we can suppose in addition that $h' \geqslant h$."

In slow motion: If K′ is a compact neighborhood of K (GT, I, §9, No. 7, Prop. 10) and V is the interior of K′ , there exists a function $f \in \mathscr{K}(\mathrm{X})$, with $\operatorname{Supp} f \subset \mathrm{V} \subset \mathrm{K}'$, such that $0 \leqslant f \leqslant 1$ and $f = 1$ on K (Ch. III, §1, No. 2, *Lemma* 1); in particular, $f = 0$ on $\mathrm{X} - \mathrm{K}'$. The continuous function $k = 2\mathrm{M}(1 - f)$ satisfies

$$0 \leqslant k \leqslant 2\mathrm{M}\,, \quad k = 0 \text{ on } \mathrm{K}\,, \quad k = 2\mathrm{M} \text{ on } \mathrm{X} - \mathrm{K}'\,.$$

Setting $h' = \sup(h, k)$, it follows that

$$0 \leqslant h \leqslant h' \leqslant 2\mathrm{M}\,, \quad 0 \leqslant h' - h \leqslant 2\mathrm{M} - h \leqslant 2\mathrm{M}\,,$$

$h' = h$ on K, and $h' = 2\mathrm{M}$ on $\mathrm{X} - \mathrm{K}'$.

IV.87, ℓ. $-3, -2$.

"Then $\int (h' - h)\,d\mu \leqslant 2\mathrm{M}\mu^*(\mathrm{X} - \mathrm{K}) \leqslant 2\mathrm{M}\varepsilon$."

With notations as in the preceding note,

$$0 \leqslant h' - h = (h' - h)\varphi_{\mathrm{X}-\mathrm{K}} \leqslant 2\mathrm{M} \cdot \varphi_{\mathrm{X}-\mathrm{K}}\,,$$

whence $0 \leqslant \int (h' - h)\,d\mu \leqslant 2\mathrm{M}\mu(\mathrm{X} - \mathrm{K}) \leqslant 2\mathrm{M}\varepsilon$.

IV.87, ℓ. $-2, -1$.

"$\ldots\ h' = h_1 + 2\mathrm{M}$, where $h_1 \in \mathscr{K}(\mathrm{X})$."

With the foregoing notations, set $h_1 = h' - 2\mathrm{M}$; since $h' - 2\mathrm{M} = 0$ on $\mathrm{X} - \mathrm{K}'$, one has $h_1 \in \mathscr{K}(\mathrm{X})$ (and $h_1 \leqslant 0$).

IV.88, ℓ. $1, 2$.

"$\int h'\,d\nu = \int h_1\,d\nu + 2\mathrm{M}\|\nu\|$ tends to $\int h_1\,d\mu + 2\mathrm{M}\|\mu\| = \int h'\,d\mu$ with respect to $\mathfrak{B}$."

$\int h_1\,d\nu \to \int h_1\,d\mu$ because $h_1 \in \mathscr{K}(\mathrm{X})$ and $\nu \to \mu$ vaguely, whereas $\|\nu\| \to \|\mu\|$ by hypothesis, and

$$\int h'\,d\nu = \int h_1\,d\nu + 2\mathrm{M}\int 1\,d\nu = \int h_1\,d\nu + 2\mathrm{M}\|\nu\|$$

by §4, No. 7, Prop. 12.

IV.88, ℓ. 2–4.

"There then exists an $\mathrm{A} \in \mathfrak{B}$ such that, for every $\nu \in \mathrm{A}$,

$$\Big|\int (g_1\mathbf{a}_1 + \cdots + g_n\mathbf{a}_n)\,d\nu - \int (g_1\mathbf{a}_1 + \cdots + g_n\mathbf{a}_n)\,d\mu\Big| \leqslant \varepsilon\,,$$

$$\int h\,d\nu \leqslant \int h'\,d\nu \leqslant \int h'\,d\mu + \varepsilon \leqslant \int h\,d\mu + 2\mathrm{M}\varepsilon + \varepsilon \leqslant 2(\mathrm{M} + 1)\varepsilon\,."$$

By Ch. III, §3, No. 1, *Example* 1 following Def. 1,

$$\Big|\int(g_1\mathbf{a}_1+\cdots+g_n\mathbf{a}_n)\,d\nu-\int(g_1\mathbf{a}_1+\cdots+g_n\mathbf{a}_n)\,d\mu\Big|$$
$$=\Big|\Big(\int g_1\,d\nu-\int g_1\,d\mu\Big)\mathbf{a}_1+\cdots+\Big(\int g_n\,d\nu-\int g_n\,d\mu\Big)\mathbf{a}_n\Big|\,;$$

since $\nu\to\mu$ vaguely, the coefficients of the $\mathbf{a}_i$ tend to 0, hence there exists an $\mathrm{A}_1\in\mathfrak{B}$ such that the inequality in the first row holds for every $\nu\in\mathrm{A}_1$.

In the second row of the display, the crucial inequality is the second one: since $\int h'\,d\nu\to\int h'\,d\mu$, there exists an $\mathrm{A}_2\in\mathfrak{B}$ such that for every $\nu\in\mathrm{A}_2$ one has

$$\int h'\,d\nu\leqslant\int h'\,d\mu+\varepsilon\,.$$

The third inequality restates $\int(h'-h)\,d\mu\leqslant 2\mathrm{M}\varepsilon$, and the fourth restates $\int h\,d\mu\leqslant\varepsilon$.

Finally, choose $\mathrm{A}\in\mathfrak{B}$ such that $\mathrm{A}\subset\mathrm{A}_1\cap\mathrm{A}_2$.

IV.88, ℓ. 5–8.

"These inequalities imply

$$\Big|\int\mathbf{f}\,d\nu-\int\mathbf{f}\,d\mu\Big|\leqslant$$
$$\int h\,d\nu+\Big|\int(g_1\mathbf{a}_1+\cdots+g_n\mathbf{a}_n)\,d\nu-\int(g_1\mathbf{a}_1+\cdots+g_n\mathbf{a}_n)\,d\mu\Big|+\int h\,d\mu$$
$$\leqslant 2(\mathrm{M}+2)\varepsilon\text{ ''}$$

Writing $\mathbf{p}=g_1\mathbf{a}_1+\cdots+g_n\mathbf{a}_n\in\mathscr{K}(\mathrm{X};\mathrm{F})$, one has

$$\int\mathbf{f}\,d\nu-\int\mathbf{f}\,d\mu=\int(\mathbf{f}-\mathbf{p})\,d\nu+\int\mathbf{p}\,d\nu-\int\mathbf{p}\,d\mu-\int(\mathbf{f}-\mathbf{p})\,d\mu\,,$$

where $|\mathbf{f}-\mathbf{p}|\leqslant h$ and $\Big|\int\mathbf{p}\,d\nu-\int\mathbf{p}\,d\mu\Big|\leqslant\varepsilon$ by the preceding note, therefore

$$\Big|\int\mathbf{f}\,d\nu-\int\mathbf{f}\,d\mu\Big|\leqslant\int h\,d\nu+\varepsilon+\int h\,d\mu\leqslant 2(\mathrm{M}+1)\varepsilon+\varepsilon+\varepsilon\,.$$

IV.88, ℓ. 10, 11.

"*Remark.*"

By §4, No. 7, Cor. of Prop. 13, $\mathbf{f}$ is integrable with respect to every bounded measure.

IV.89, ℓ. 3.

" $\mu_n(\mathrm{A})$ tends to $\mu(\mathrm{A})$."

One proposes to apply Prop. 22 to the function $f = \varphi_{\mathrm{A}}$. The space X is the disjoint union of the interior, exterior and boundary ('frontier') of A,

$$\mathrm{X} = \overset{\circ}{\mathrm{A}} \cup (\complement \mathrm{A})^{\circ} \cup \partial \mathrm{A}$$

(GT, I, §1, No. 6). Since f is constant on each of the open sets $\overset{\circ}{\mathrm{A}}$ and $(\complement \mathrm{A})^{\circ}$, it is continuous at those points, whereas $\partial\mathrm{A} = \overline{\mathrm{A}} \cap \overline{\complement \mathrm{A}}$ shows that it is discontinuous at the points of $\partial\mathrm{A}$, therefore the set of discontinuities of f is precisely $\partial\mathrm{A}$, which is by assumption negligible. Thus f satisfies the conditions of Prop. 22, whence $\mu_n(\mathrm{A}) = \int f\,d\mu_n \to \int f\,d\mu = \mu(\mathrm{A})$.

IV.89, ℓ. 4, 5.

" if p_n denotes the number of integers $k \in [0, n]$ such that $e^{2i\pi k\theta} \in \mathrm{A}$, then $n^{-1}p_n$ tends to $\mu(\mathrm{A})$ as n tends to $+\infty$."

For every integer $k \geqslant 0$,

$$\nu_k(\mathrm{A}) = \int \varphi_{\mathrm{A}}\,d\nu_k = \begin{cases} 1 & \text{if } e^{2i\pi k\theta} \in \mathrm{A} \\ 0 & \text{otherwise,} \end{cases}$$

therefore

$$\mu_n(\mathrm{A}) = \frac{1}{n+1}\big(\nu_0(\mathrm{A}) + \cdots + \nu_n(\mathrm{A})\big) = \frac{p_n}{n+1}\,,$$

thus

$$\frac{p_n}{n} = \frac{n+1}{n} \cdot \frac{p_n}{n+1} = \frac{n+1}{n} \cdot \mu_n(\mathrm{A}) \to \mu(\mathrm{A})\,.$$

A stunning application of Prop. 22.

§6. CONVEXITY INEQUALITIES

IV.89, ℓ. -4 to -1.

"COROLLARY."

A variation on the argument shifts the burden from the measure to the function:

Proposition. Let μ be a positive measure, $\mathbf{f} : \mathrm{X} \to \mathrm{F}$ an integrable function that is equal to zero outside some integrable subset A of X. If C is the closed convex envelope of $\mathbf{f}(\mathrm{X})$, then $\int \mathbf{f}\,d\mu \in \mu(\mathrm{A}) \cdot \mathrm{C}$.

Proof. If $\mu(\mathrm{A}) = 0$ then $\mathbf{f}$ is negligible and $\int \mathbf{f}\,d\mu = \mathbf{0} \in 0 \cdot \mathrm{C}$.

If $\mu(\mathrm{A}) > 0$ then $g = \varphi_{\mathrm{A}}$ meets the requirements of Th. 1, $\mathbf{f}g = \mathbf{f}$, and

$$\frac{\int \mathbf{f}\,d\mu}{\mu(\mathrm{A})} = \frac{\int \mathbf{f}g\,d\mu}{\int g\,d\mu} \in \mathrm{C}\,. \ \diamondsuit$$

The Corollary is a special case: If μ is a bounded positive measure and $\mathbf{f} : \mathrm{X} \to \mathrm{F}$ is any integrable function, then X qualifies to play the role of A, thus $\int \mathbf{f}\,d\mu \in \mu(\mathrm{X}) \cdot \mathrm{C}$.

For generalizations with F a Hausdorff locally convex space, see Ch. III, §3, No. 2, Prop. 4, and, below, §7, No. 1, Cor. of Prop. 1.

IV.90, $\ell.$ 13, 14.

"For every $\alpha > \mathrm{M}_\infty(f)$, the set of $x \in \mathrm{X}$ such that $f(x) > \alpha$ is locally negligible "

By the definition of $\mathrm{M}_\infty(f)$ as the infimum of all "locally almost everywhere majorants" of f, there exists a number β such that $\mathrm{M}_\infty(f) \leqslant \beta < \alpha$ and $f(x) \leqslant \beta$ locally almost everywhere; the set $\{x : \ f(x) > \beta\}$ is thus locally negligible, therefore so is its subset $\{x : \ f(x) > \alpha\}$.

IV.90, $\ell.$ 18.

" $m_\infty(f) \leqslant \mathrm{M}_\infty(f)$ if the measure μ is nonzero"

It suffices that $m_\infty(f) \leqslant f(x) \leqslant \mathrm{M}_\infty(f)$ for some point $x \in \mathrm{X}$. We know that there exist locally negligible sets A and B such that $f \leqslant \mathrm{M}_\infty(f)$ on $\mathrm{X} - \mathrm{A}$ and $f \geqslant m_\infty(f)$ on $\mathrm{X} - \mathrm{B}$; if no such x exists, then

$$(\mathrm{X} - \mathrm{A}) \cap (\mathrm{X} - \mathrm{B}) = \varnothing\,,$$

so $\mathrm{X} = \mathrm{A} \cup \mathrm{B}$ is locally negligible. Then $\mathrm{K} = \mathrm{K} \cap \mathrm{X}$ is negligible for all compact sets K (§5, No. 2, Prop. 5), therefore $|\mu|^*(\mathrm{X}) = 0$ (§4, No. 6, Cor. 4 of Th. 4), and so $\|\mu\| = 0$ (p. IV.4, $\ell.$ −7, −6).

IV.90, $\ell.$ −8.

". . . provided the second member is defined "

That is, $\mathrm{M}_\infty(f)$ and $\mathrm{M}_\infty(g)$ are not infinite of opposite signs. For locally almost every x, $f(x) + g(x)$ is defined, $f(x) \leqslant \mathrm{M}_\infty(f)$ and $g(x) \leqslant \mathrm{M}_\infty(g)$, and the desired inequality

$$f(x) + g(x) \leqslant \mathrm{M}_\infty(f) + \mathrm{M}_\infty(g)$$

follows from a property of addition in $\overline{\mathbf{R}}$ (GT, IV, §4, No. 3): if $\alpha \leqslant \beta$, $\gamma \leqslant \delta$ and $\alpha + \gamma$, $\beta + \delta$ are both defined, then $\alpha + \gamma \leqslant \beta + \delta$.

{If both members of the inequality are finite, then all numbers in sight are finite, and one is reduced to a property of addition in $\mathbf{R}$. If $\alpha + \gamma = -\infty$

or $\beta+\delta=+\infty$, the inequality is trivial. If $\alpha+\gamma=+\infty$ then one of α, γ is equal to $+\infty$, whence one of β, δ is equal to $+\infty$, therefore $\beta+\delta=+\infty$. One argues similarly if $\beta+\delta=-\infty$.}

IV.90, ℓ. −4.

"... provided the second member is defined."

For multiplication in $\overline{\mathbf{R}}$, only the four products $0\cdot(\pm\infty)$ and $(\pm\infty)\cdot 0$ are forbidden (GT, IV, §4, No. 3).

However, if one makes the convention $0\cdot(+\infty)=(+\infty)\cdot 0=0$, then all products in $\overline{\mathbf{R}}_+$ are admitted and

$$\alpha\leqslant\beta \ \ \& \ \ \gamma\leqslant\delta \quad\Rightarrow\quad \alpha\gamma\leqslant\beta\delta\,.$$

For, if one of α,γ is 0 then $\alpha\gamma=0\leqslant\beta\delta$. Suppose $\alpha>0$ and $\gamma>0$: if one of β,δ is infinite then $\alpha\gamma\leqslant+\infty=\beta\delta$, whereas if they are both finite, one is reduced to a property of multiplication in $\mathbf{R}_+$.

With this convention, the formula (2) is then valid: for locally almost every x,

$$0\leqslant f(x)\leqslant \mathrm{M}_\infty(f) \ \text{ and } \ 0\leqslant g(x)\leqslant \mathrm{M}_\infty(g)\,,$$

whence $f(x)g(x)\leqslant \mathrm{M}_\infty(f)\,\mathrm{M}_\infty(g)$.

IV.91, ℓ. 3–10.

"Proposition 1 (Inequality of the mean)."

As the proof entails arithmetical operations on functions not necessarily everywhere defined, nor necessarily finite where they are defined, it is helpful to review the definitions and make some preliminary simplifications.

To say that g is integrable means that there exists a function $h\in\mathscr{L}^1_{\mathbf{R}}$ such that $g=h$ almost everywhere (§4, No. 1, last sentence). Let D be the set of $x\in\mathrm{X}$ such that $g(x)$ is defined, finite and $\geqslant 0$, and define $g':\mathrm{X}\to\mathbf{R}$ to be equal to g at the points of D, and to 0 at the points of $\mathrm{X}-\mathrm{D}$. Then $\mathrm{X}-\mathrm{D}$ is negligible, so $g'=g$ almost everywhere; and $g'=h$ almost everywhere, so g' is integrable (§4, *loc. cit.*). The sets $\{x:\ g'(x)\neq 0\}$ and $\{x:\ g(x)\neq 0\}$ differ at most by a negligible set; replacing g by g', we can suppose that g is everywhere defined, finite and $\geqslant 0$: none of the conclusions of Prop. 1 is thereby affected.

To say that f is bounded in measure implies that there exists a negligible set N such that, for every $x\in\mathrm{X}-\mathrm{N}$, $f(x)$ is defined and finite. In particular, f need not be defined on all of X. Let A be the domain of f; thus $\mathrm{X}-\mathrm{N}\subset\mathrm{A}\subset\mathrm{X}$. Then the set $\mathrm{X}-\mathrm{A}\subset\mathrm{N}$ is negligible, hence measurable, therefore A is measurable; to say that f is measurable means that the extension f' of f to X by 0 is measurable (§5, No. 10, Def. 8). Let $f'':\mathrm{X}\to\mathbf{R}$ be the extension by 0 of $f|\mathrm{X}-\mathrm{N}$ to X, so that $f''=f'$

except possibly on the negligible set $\mathrm{A} - (\mathrm{X} - \mathrm{N}) = \mathrm{A} \cap \mathrm{N}$; thus $f'' = f'$ almost everywhere and so f'' is measurable (§5, No. 2, Prop. 6) and everywhere finite-valued. Moreover, $f'' = f$ almost everywhere; it follows that f'' is also bounded in measure, with same M_∞ and m_∞ as f. Moreover, $f''g = fg$ almost everywhere. To say that fg is integrable means that there exists a $k \in \mathscr{L}^1_{\mathbf{R}}$ such that $fg = k$ almost everywhere; then $f''g = k$ almost everywhere, hence $f''g$ is also integrable. Replacing f by f'' alters none of the conclusions of Prop. 1.

To summarize: we can suppose that $g : \mathrm{X} \to \mathbf{R}_+$ and that $f : \mathrm{X} \to \mathbf{R}$.

IV.91, ℓ. 13–15.

"... because the set of points $x \in \mathrm{X}$ where $g(x) \neq 0$ is a countable union of integrable sets (§5, No. 6, Lemma 1)."

And a locally negligible integrable set is negligible (§5, No. 2, Cor. 1 of Prop. 5).

IV.91, ℓ. 15.

"It follows that fg is integrable (§5, No. 6, Th. 5) "

For, fg is measurable (§5, No. 3, remark following Cor. 5) and, writing $c = \max\{|m_\infty(f)|, |\mathrm{M}_\infty(f)|\}$, one has $-cg \leqslant fg \leqslant cg$ almost everywhere; thus $|fg| \leqslant cg$ almost everywhere, whence $\mathrm{N}_1(fg) \leqslant c \cdot \mathrm{N}_1(g) < +\infty$.

IV.91, ℓ. −9 to −5.

"Definition 2."

Note that $|\mathbf{f}|$ is defined and finite everywere on X, thus, for a function $f : \mathrm{X} \to \mathbf{R}$, the terminology here is consistent with that of No. 2.

IV.91, ℓ. −4, −3.

"A function $\mathbf{f}$ in $\mathscr{L}^\infty_{\mathrm{F}}$ may thus be characterized by the fact that there exists a *bounded measurable* function equal locally almost everywhere to $\mathbf{f}$."

Suppose $\mathbf{f} \in \mathscr{L}^\infty_{\mathrm{F}}$. In particular, $\mathrm{M}_\infty(|\mathbf{f}|) < +\infty$, and the set

$$\mathrm{N} = \{x \in \mathrm{X} : \ |\mathbf{f}(x)| > \mathrm{M}_\infty(|\mathbf{f}|)\}$$

is locally negligible. Define $\mathbf{g} : \mathrm{X} \to \mathrm{F}$ by $\mathbf{g} = \varphi_{\complement \mathrm{N}}\mathbf{f}$. Since $\mathbf{g} = \mathbf{f}$ locally almost everywhere, $\mathbf{g}$ is measurable (§5, No. 2, Prop. 6). Moreover, $|\mathbf{g}| \leqslant \mathrm{M}_\infty(|\mathbf{f}|)$ everywhere on X, thus $\mathbf{g}$ is bounded.

Conversely, suppose $\mathbf{g} : \mathrm{X} \to \mathrm{F}$ is a bounded measurable function such that $\mathbf{f} = \mathbf{g}$ locally almost everywhere; say $|\mathbf{g}(x)| \leqslant \mathrm{M} < +\infty$ for all $x \in \mathrm{X}$. Then $\mathbf{f}$ is measurable (§5, No. 2, Prop. 6) and $\mathrm{N}_\infty(\mathbf{f}) = \mathrm{M}_\infty(|\mathbf{f}|) = \mathrm{M}_\infty(|\mathbf{g}|) \leqslant \mathrm{M} < +\infty$, therefore $\mathbf{f} \in \mathscr{L}^\infty_{\mathrm{F}}$.

IV.92, ℓ. 5–7.

"... for every integer m, there exist a locally negligible set H_m and an integer n_0 such that $|\mathbf{f}(x) - \mathbf{f}_n(x)| \leqslant 1/m$ for every integer $n \geqslant n_0$ and every $x \notin H_m$ "

For each m, let $n_0(m)$ be an index such that

$$n \geqslant n_0(m) \quad \Rightarrow \quad N_\infty(\mathbf{f} - \mathbf{f}_n) \leqslant 1/m\,;$$

thus, for each $n \geqslant n_0(m)$, there exists a locally negligible set H_{mn} such that

$$|\mathbf{f}(x) - \mathbf{f}_n(x)| \leqslant M_\infty(|\mathbf{f} - \mathbf{f}_n|) \leqslant 1/m \ \text{ for all } x \in X - H_{mn}\,.$$

The set $H_m = \bigcup_{n \geqslant n_0(m)} H_{mn}$ is then locally negligible and

$$n \geqslant n_0(m) \quad \Rightarrow \quad |\mathbf{f}(x) - \mathbf{f}_n(x)| \leqslant 1/m \ \text{ for all } x \in X - H_m\,.$$

IV.92, ℓ. 10.

"... the converse is immediate."

That is, if there exists a locally negligible set H such that $\mathbf{f}_n \to \mathbf{f}$ uniformly on $X - H$, then $N_\infty(\mathbf{f}_n - \mathbf{f}) \to 0$. For,

$$N_\infty(\mathbf{f}_n - \mathbf{f}) = M_\infty(|\mathbf{f}_n - \mathbf{f}|) \leqslant \sup_{x \in X - H} |\mathbf{f}_n(x) - \mathbf{f}(x)| \to 0\,.$$

IV.92, ℓ. 16, 17.

"... its topology is defined by the *norm* deduced from N_∞ by passage to the quotient "

TVS, II, §1, No. 3 (p. TVS II.5, ℓ. 5–6).

IV.92, ℓ. $-9, -8$.

" ... $\mathbf{g}$ is bounded on the set of $x \in X$ where $|\mathbf{g}_{k_1}(x)| \leqslant N_\infty(\mathbf{g}_{k_1})$ "

Since $\mathbf{g} = 0$ on A, it is the points of $X - A$ in which we are interested. One has

$$X - A \subset X - A_1 = \bigcap_{r,s \geqslant k_1} (X - A_{rs})\,.$$

By definition, $\mathbf{g}_n = \mathbf{f}_n$ on $X - A$ for all n, so for all $n \geqslant k_1$ one has

$$\mathbf{g}_n - \mathbf{g}_{k_1} = \mathbf{f}_n - \mathbf{f}_{k_1} \quad \text{on} \quad X - A \subset X - A_{nk_1}\,;$$

therefore, by the definition of A_{nk_1},

$$|\mathbf{g}_n(x) - \mathbf{g}_{k_1}(x)| = |\mathbf{f}_n(x) - \mathbf{f}_{k_1}(x)| \leqslant 1 \quad \text{on } X - A$$

for all $n \geqslant k_1$.

Suppose now that $x \in \mathrm{X} - \mathrm{A}$ and $|\mathbf{g}_{k_1}(x)| \leqslant \mathrm{N}_\infty(\mathbf{g}_{k_1})$. Then, for all $n \geqslant k_1$, one has

$$\begin{aligned}|\mathbf{g}(x)| &\leqslant |\mathbf{g}(x) - \mathbf{g}_n(x)| + |\mathbf{g}_n(x) - \mathbf{g}_{k_1}(x)| + |\mathbf{g}_{k_1}(x)| \\ &\leqslant |\mathbf{g}(x) - \mathbf{g}_n(x)| + 1 + \mathrm{N}_\infty(\mathbf{g}_{k_1}),\end{aligned}$$

and passage to the limit yields $|\mathbf{g}(x)| \leqslant 0+1+\mathrm{N}_\infty(\mathbf{g}_{k_1})$. Thus $\mathbf{g}$ is bounded by $1 + \mathrm{N}_\infty(\mathbf{g}_{k_1})$ on the indicated set.

IV.92, ℓ. −7.

"... $\mathbf{g}$ belongs to $\mathscr{L}^\infty_\mathrm{F}$."

The set $\mathrm{N} = \{x \in \mathrm{X} : |\mathbf{g}_{k_1}(x)| > \mathrm{N}_\infty(\mathbf{g}_{k_1})\}$ is locally negligible, so $\mathbf{g} = \varphi_{\complement \mathrm{N}}\mathbf{g}$ locally almost everywhere, and the measurable function $\varphi_{\complement \mathrm{N}}\mathbf{g}$ is bounded by the preceding note.

IV.93, ℓ. 2–4.

"... if there exists a continuous function $\mathbf{f}$ with negligible compact support and not identically zero"

Such a function exists if and only if $\mathrm{Supp}\,\mu \neq \mathrm{X}$:

If: Suppose $\mathrm{Supp}\,\mu \neq \mathrm{X}$. Then $\mathrm{U} = \mathrm{X} - \mathrm{Supp}\,\mu$ is a nonempty negligible open set (§2, No. 2, Prop. 5). Choose a nonzero function $f \in \mathscr{K}_+(\mathrm{X})$ such that $\mathrm{Supp}\, f \subset \mathrm{U}$, and a nonzero vector $\mathbf{a} \in \mathrm{F}$, and define $\mathbf{f} = f\mathbf{a}$; then $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{F})$, $\mathbf{f} \neq \mathbf{0}$, and the set $\mathrm{Supp}\,\mathbf{f} = \mathrm{Supp}\, f \subset \mathrm{U}$ is negligible. Moreover, $|\mathbf{f}| = |f| \cdot |\mathbf{a}| \leqslant \|f\| \cdot |\mathbf{a}| < +\infty$, thus $\mathbf{f} \in \mathscr{C}^b(\mathrm{X};\mathrm{F})$ (Ch. III, §1, No. 2).

Only if: Suppose $\mathbf{f} \in \mathscr{K}(\mathrm{X};\mathrm{F})$ with $\mathbf{f} \neq \mathbf{0}$ and $\mathrm{Supp}\,\mathbf{f}$ negligible. Then the open set $\mathrm{V} = \{x : \mathbf{f}(x) \neq \mathbf{0}\} \subset \mathrm{Supp}\,\mathbf{f}$ is nonempty and negligible, hence $\mathrm{V} \subset \mathrm{X} - \mathrm{Supp}\,\mu$ by the cited Prop. 5, thus $\mathrm{Supp}\,\mu \neq \mathrm{X}$. ◊

For such a function $\mathbf{f}$, one has $|\mathbf{f}| = 0$ almost everywhere, hence locally almost everywhere, therefore $\mathrm{M}_\infty(|\mathbf{f}|) = 0$; thus $\mathrm{N}_\infty(\mathbf{f}) = 0 < \|\mathbf{f}\|$. This proves the implication

$$\mathrm{N}_\infty(\mathbf{f}) = \|\mathbf{f}\| \text{ for all } \mathbf{f} \in \mathscr{C}^b(\mathrm{X};\mathrm{F}) \quad \Rightarrow \quad \mathrm{Supp}\,\mu = \mathrm{X}.$$

IV.93, ℓ. 7.

"... which shows that $\mathrm{N}_\infty(\mathbf{f}) = \|\mathbf{f}\|$."

We are assuming that $\mathrm{Supp}\,\mu = \mathrm{X}$ and $\mathbf{f} \in \mathscr{C}^b(\mathrm{X};\mathrm{F})$. At any rate, $\mathrm{N}_\infty(\mathbf{f}) \leqslant \|\mathbf{f}\|$. Supposing to the contrary that $\mathrm{M}_\infty(|\mathbf{f}|) = \mathrm{N}_\infty(\mathbf{f}) < \|\mathbf{f}\|$, choose α so that $\mathrm{M}_\infty(|\mathbf{f}|) < \alpha < \|\mathbf{f}\|$ and let $\mathrm{U} = \{x : |\mathbf{f}(x)| > \alpha\}$, a nonempty open set. We know from the definition of M_∞ that $|\mathbf{f}(x)| \leqslant \alpha$ locally almost everywhere, therefore U is locally negligible. Let V be a nonempty open set with $\overline{\mathrm{V}}$ compact and $\overline{\mathrm{V}} \subset \mathrm{U}$; then $\overline{\mathrm{V}} = \overline{\mathrm{V}} \cap \mathrm{U}$ is negligible (§5, No. 2, Prop. 5), thus V is a nonempty negligible open set, contradicting $X - \operatorname{Supp}\mu = \varnothing$ (§2, No. 2, Prop. 5).

IV.93, ℓ. 11–13.

"...but its canonical image in L^∞_F is a closed subspace of L^∞_F (which can moreover be identified with $\mathscr{C}^b(X;F)$ in the case contemplated)."

For any measure μ (with no asssumption about its support) every $\mathbf{f} \in \mathscr{C}^b(X;F)$ belongs to $\mathscr{L}^\infty_F$ and $N_\infty(\mathbf{f}) \leqslant \|\mathbf{f}\|$, thus the canonical injection $\mathscr{C}^b(X;F) \to \mathscr{L}^\infty_F$ is continuous for the norm topology on $\mathscr{C}^b(X;F)$ and the topology on $\mathscr{L}^\infty_F$ defined by the semi-norm N_∞. In turn, the quotient mapping $\mathscr{L}^\infty_F \to L^\infty_F$ defined by $\mathbf{f} \mapsto \dot{\mathbf{f}}$ is continuous, where L^∞_F is a Banach space equipped with the norm $\|\dot{\mathbf{f}}\|_\infty = N_\infty(\mathbf{f})$ (also written $N_\infty(\dot{\mathbf{f}})$) (Prop. 2). The composite mapping $\mathscr{C}^b(X;F) \to L^\infty_F$, defined by $\mathbf{f} \mapsto \dot{\mathbf{f}}$, satisfies $\|\dot{\mathbf{f}}\|_\infty \leqslant \|\mathbf{f}\|$ hence is continuous but in general not injective. Its image in L^∞_F may not be complete: if $\mathbf{f}_n \in \mathscr{C}^b(X;F)$ is a sequence such that $(\dot{\mathbf{f}}_n)$ is Cauchy in L^∞_F, there exists an $\mathbf{f} \in \mathscr{L}^\infty_F$ with $\|\dot{\mathbf{f}}_n - \dot{\mathbf{f}}\|_\infty \to 0$, but it is not assured that there exists a $\mathbf{g} \in \mathscr{C}^b(X;F)$ such that $\mathbf{f} = \mathbf{g}$ locally almost everywhere.

However, when $\operatorname{Supp}\mu = X$ (the case contemplated), one has $\|\mathbf{f}\| = N_\infty(\mathbf{f}) = \|\dot{\mathbf{f}}\|_\infty$, therefore the image of $\mathscr{C}^b(X;F)$ in L^∞_F is a complete, hence closed, linear subspace of L^∞_F. Thus, the mapping $\mathscr{C}^b(X;F) \to L^\infty_F$ is a linear injection such that the norm (and topology) induced by L^∞_F on the image coincide with those of $\mathscr{C}^b(X;F)$, whence the proposed identification $\mathscr{C}^b(X;F) \subset L^\infty_F$.

IV.93, ℓ. 15–17.

"This implies that the space $\mathscr{K}(X;F)$ of mappings of X into F, continuous with compact support, is in general *not dense in* L^∞_F"

If the space $\mathscr{K}(X;F) \subset \mathscr{C}^b(X;F)$ were dense in L^∞_F then, since $\mathscr{C}^b(X;F)$ is closed in L^∞_F, it would follow that $\mathscr{C}^b(X;F) = L^\infty_F$.

IV.93, ℓ. 19, 20.

"...the topology defined by the semi-norm N_∞ is finer than the topology induced on $\mathscr{L}^\infty_F$ by the topology of convergence in measure"

Let $\mathscr{S} = \mathscr{S}(X,\mu;F)$ be the set of all μ-measurable functions $\mathbf{f} : X \to F$, equipped with the uniform structure of convergence in measure (§5, No. 11). Then $\mathscr{L}^\infty_F \subset \mathscr{S}$; if $\boldsymbol{W}$ is a basic entourage for $\mathscr{S}$, then the set

$$\mathscr{W} = (\mathscr{L}^\infty_F \times \mathscr{L}^\infty_F) \cap \boldsymbol{W}$$

is a basic entourge for the induced uniformity on $\mathscr{L}^\infty_F$. It will suffice to show that $\mathscr{W}$ is also an entourage for the uniformity on $\mathscr{L}^\infty_F$ defined by the semi-norm N_∞; it is enough to exhibit an entourage $\mathscr{V}$ for the N_∞-uniformity such that $\mathscr{V} \subset \mathscr{W}$.

We can suppose that

$$\boldsymbol{W} = \boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta)\,,$$

where K is a compact subset of X, $\delta > 0$, and, for some $\varepsilon > 0$, V is the entourage of F given by

$$\mathrm{V} = \{(\mathbf{a}, \mathbf{b}) \in \mathrm{F} \times \mathrm{F} :\ |\mathbf{a} - \mathbf{b}| \leqslant \varepsilon\}$$

(§5, No. 11, paragraph preceding Def. 9). Recall that

$$\begin{aligned}\boldsymbol{W}(\mathrm{V}, \mathrm{K}, \delta) &= \bigl\{(\mathbf{f}, \mathbf{g}) \in \mathscr{S} \times \mathscr{S} :\ |\mu|^*\bigl(\{x \in \mathrm{K} :\ (\mathbf{f}(x), \mathbf{g}(x)) \notin \mathrm{V}\}\bigr) \leqslant \delta\bigr\} \\ &= \bigl\{(\mathbf{f}, \mathbf{g}) \in \mathscr{S} \times \mathscr{S} :\ |\mu|^*\bigl(\{x \in \mathrm{K} :\ |\mathbf{f}(x) - \mathbf{g}(x)| > \varepsilon\}\bigr) \leqslant \delta\bigr\},\end{aligned}$$

therefore

$$\mathscr{W} = \bigl\{(\mathbf{f}, \mathbf{g}) \in \mathscr{L}^\infty_{\mathrm{F}} \times \mathscr{L}^\infty_{\mathrm{F}} :\ |\mu|^*\bigl(\{x \in \mathrm{K} :\ |\mathbf{f}(x) - \mathbf{g}(x)| > \varepsilon\}\bigr) \leqslant \delta\bigr\}.$$

Let

$$\mathscr{V} = \{(\mathbf{f}, \mathbf{g}) \in \mathscr{L}^\infty_{\mathrm{F}} \times \mathscr{L}^\infty_{\mathrm{F}} :\ \mathrm{N}_\infty(\mathbf{f} - \mathbf{g}) < \varepsilon\}$$

(the same ε); it suffices to show that $\mathscr{V} \subset \mathscr{W}$, that is, for $\mathbf{f}, \mathbf{g} \in \mathscr{L}^\infty_{\mathrm{F}}$,

$$\mathrm{N}_\infty(\mathbf{f} - \mathbf{g}) < \varepsilon \quad \Rightarrow \quad |\mu|^*\bigl(\{x \in \mathrm{K} :\ |\mathbf{f}(x) - \mathbf{g}(x)| > \varepsilon\}\bigr) \leqslant \delta\,.$$

Suppose $\mathrm{N}_\infty(\mathbf{f} - \mathbf{g}) < \varepsilon$, that is, $\mathrm{M}_\infty(|\mathbf{f} - \mathbf{g}|) < \varepsilon$; then $|\mathbf{f} - \mathbf{g}| \leqslant \varepsilon$ locally almost everywhere, that is, the set

$$\mathrm{A} = \{x \in \mathrm{X} :\ |\mathbf{f}(x) - \mathbf{g}(x)| > \varepsilon\}$$

is locally negligible (see the note for IV.90, ℓ. 13, 14), therefore the set

$$\mathrm{K} \cap \mathrm{A} = \{x \in \mathrm{K} :\ |\mathbf{f}(x) - \mathbf{g}(x)| > \varepsilon\}$$

is negligible (§5, No. 2, Prop. 5) and so $|\mu|^*(\mathrm{K} \cap \mathrm{A}) = 0 < \delta$.

IV.94, ℓ. 3–7.

"Corollary 1."

The hypothesis of bilinearity can be omitted.

IV.94, ℓ. 8.

"... $\Phi(\mathbf{f}, \mathbf{g})$ is measurable (§5, No. 3, Cor. 5 of Th. 1)"

As in the cited Cor. 5, the hypothesis of bilinearity can be omitted; it is in the *criteria* for continuity that bilinearity plays a role (GT, IX, §3, No. 5, Th. 1 or TVS, I, §1, No. 6, Prop. 5). This observation is useful for the proof of Cor. 3 given below.

IV.94, ℓ. −13.

"For, $|\langle \mathbf{z}, \mathbf{z}' \rangle| \leqslant |\mathbf{z}| \cdot |\mathbf{z}'|$."

The inequality, noted in the proof of TVS, IV, §1, No. 3, Prop. 8, assures the continuity of the bilinear form $(\mathbf{z}, \mathbf{z}') \mapsto \langle \mathbf{z}, \mathbf{z}' \rangle$ (GT, IX, §3, No. 5, Th. 1).

For the definition of $|\mathbf{z}'|$, see GT, X, §3, No. 2, discussion preceding Prop. 6; TVS, IV, §2, No. 4; or the final section of TVS ("Summary of some important properties of Banach spaces", TVS p. 355).

IV.94, ℓ. −9 to −5.

"Corollary 3. — *Let* μ *be a positive measure on* X, F *a real* (resp. *complex*) *Hilbert space. On the space* L^2_F, *the symmetric* (resp. *Hermitian*) *form*

$$(\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \mapsto \int \langle \mathbf{f}, \mathbf{g} \rangle \, d\mu$$

defines a Hilbert space structure, for which the norm is equal to $\|\widetilde{\mathbf{f}}\|_2$."

A minor technical point: one could suppose that μ is an arbitrary measure, provided the integral is taken with respect to $|\mu|$, indispensable for the form to be positive; for, $\mathscr{L}^2_F(X, \mu) = \mathscr{L}^2_F(X, |\mu|)$ (§3, No. 4).

In the complex case (the real case being an obvious simplification) the form $\Phi : F \times F \to \mathbf{C}$ defined by $\Phi(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle$ (the scalar product, or 'inner product' of $\mathbf{a}$ and $\mathbf{b}$) is sesquilinear ($\langle c\mathbf{a}, d\mathbf{b} \rangle = \overline{c}d\langle \mathbf{a}, \mathbf{b} \rangle$), so a slight adjustment is needed to make Cor. 1 applicable. Let $\overline{F}$ be the Hilbert space conjugate to F (TVS, V, §1, No. 3,): $\overline{F}$ is the set F equipped with its original additive structure; if scalar multiples in F are denoted $c\mathbf{a}$ ($c \in \mathbf{C}$, $\mathbf{a} \in F$), then scalar multiples $c \cdot \mathbf{a}$ in $\overline{F}$ are defined by $c \cdot \mathbf{a} = \overline{c}\mathbf{a}$; and the Hermitian form on $\overline{F}$ is the conjugate of that on F (we need not introduce a notation for it). The norm on $\overline{F}$ (hence also the norm topology on $\overline{F}$) is identical to that on F. Defining $\Psi : \overline{F} \times F \to \mathbf{C}$ by $\Psi(\mathbf{a}, \mathbf{b}) = \Phi(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle$, Ψ is a bilinear form on $\overline{F} \times F$ (for instance, $\Psi(c \cdot \mathbf{a}, \mathbf{b}) = \langle c \cdot \mathbf{a}, \mathbf{b} \rangle = \langle \overline{c}\mathbf{a}, \mathbf{b} \rangle = c\langle \mathbf{a}, \mathbf{b} \rangle = c\Psi(\mathbf{a}, \mathbf{b})$) such that

$$|\Psi(\mathbf{a}, \mathbf{b})| = |\langle \mathbf{a}, \mathbf{b} \rangle| \leqslant |\mathbf{a}| \cdot |\mathbf{b}| ,$$

hence Ψ is continuous (GT, IX, §3, No. 5, Th. 1), therefore so is Φ, and $|\Phi(\mathbf{a}, \mathbf{b})| \leqslant |\mathbf{a}| \cdot |\mathbf{b}|$. Since the proof of Cor. 1 does not require that Φ be bilinear, we conclude that if $\mathbf{f}, \mathbf{g} \in \mathscr{L}^2_F$ then $\Phi(\mathbf{f}, \mathbf{g}) \in \mathscr{L}^1_{\mathbf{C}}$ and

$$\Big| \int \Phi(\mathbf{f}, \mathbf{g}) \, d\mu \Big| \leqslant N_2(\mathbf{f}) N_2(\mathbf{g}) .$$

Now,

$$\big(\Phi(\mathbf{f},\mathbf{g})\big)(x) = \langle \mathbf{f},\mathbf{g}\rangle(x) = \langle \mathbf{f}(x),\mathbf{g}(x)\rangle\,;$$

and if $\widetilde{\mathbf{f}} \in \mathrm{L}^2_{\mathrm{F}}$ is the class of $\mathbf{f}$ for equality almost everywhere (§3, No. 4, Def. 2), then

$$\mathrm{N}_2(\mathbf{f}) = \mathrm{N}_2(\widetilde{\mathbf{f}}) = \Big(\int |\mathbf{f}(x)|^2\,d\mu\Big)^{1/2} = \|\widetilde{\mathbf{f}}\|_2\,.$$

Thus $\langle \mathbf{f},\mathbf{g}\rangle \in \mathscr{L}^1_{\mathbf{C}}$ and

$$(*)\qquad \Big|\int \langle \mathbf{f},\mathbf{g}\rangle\,d\mu\Big| \leqslant \|\widetilde{\mathbf{f}}\|_2 \cdot \|\widetilde{\mathbf{g}}\|_2\,.$$

The integral in $(*)$ depends only on the classes of $\mathbf{f}$ and $\mathbf{g}$, so we may define a function $\mathrm{L}^2_{\mathrm{F}} \times \mathrm{L}^2_{\mathrm{F}} \to \mathbf{C}$ by

$$(\widetilde{\mathbf{f}},\widetilde{\mathbf{g}}) \mapsto \int \langle \mathbf{f},\mathbf{g}\rangle\,d\mu$$

that is clearly sesquilinear, Hermitian and positive, by the properties of the inner product in F and the linearity of integration. Writing

$$\langle \widetilde{\mathbf{f}},\widetilde{\mathbf{g}}\rangle = \int \langle \mathbf{f},\mathbf{g}\rangle\,d\mu\,,$$

one has

$$\langle \widetilde{\mathbf{f}},\widetilde{\mathbf{f}}\rangle = \int \langle \mathbf{f},\mathbf{f}\rangle\,d\mu = \int |\mathbf{f}|^2\,d\mu = \big(\mathrm{N}_2(\mathbf{f})\big)^2 = \big(\|\widetilde{\mathbf{f}}\|_2\big)^2,$$

thus $\langle \widetilde{\mathbf{f}},\widetilde{\mathbf{f}}\rangle > 0$ when $\widetilde{\mathbf{f}} \neq \mathbf{0}$. The norm $\widetilde{\mathbf{f}} \mapsto \|\widetilde{\mathbf{f}}\|$ derived from this form is

$$\|\widetilde{\mathbf{f}}\| = \langle \widetilde{\mathbf{f}},\widetilde{\mathbf{f}}\rangle^{1/2} = \|\widetilde{\mathbf{f}}\|_2\,;$$

as $\mathrm{L}^2_{\mathrm{F}}$ is complete for this norm, we conclude that it is a Hilbert space for the inner product $\langle \widetilde{\mathbf{f}},\widetilde{\mathbf{g}}\rangle$.

A shortcut, based on the characterization of a Hilbert space as a Banach space whose norm satisfies the "parallelogram law":

By the parallelogram law in F, $|\mathbf{f}+\mathbf{g}|^2 + |\mathbf{f}-\mathbf{g}|^2 = 2|\mathbf{f}|^2 + 2|\mathbf{g}|^2$; integration term-by-term yields

$$(\|\widetilde{\mathbf{f}}+\widetilde{\mathbf{g}}\|_2)^2 + (\|\widetilde{\mathbf{f}}-\widetilde{\mathbf{g}}\|_2)^2 = 2(\|\widetilde{\mathbf{f}}\|_2)^2 + 2(\|\widetilde{\mathbf{g}}\|_2)^2\,,$$

thus the norm of the Banach space L^2_F satisfies the parallelogram law, hence L^2_F is a Hilbert space, with inner product

$$\begin{aligned}\langle\widetilde{\mathbf{f}},\widetilde{\mathbf{g}}\rangle &= \frac{1}{4}\{(\|\widetilde{\mathbf{f}}+\widetilde{\mathbf{g}}\|_2)^2 - (\|\widetilde{\mathbf{f}}-\widetilde{\mathbf{g}}\|_2)^2 + i(\|\widetilde{\mathbf{f}}+i\widetilde{\mathbf{g}}\|_2)^2 - i(\|\widetilde{\mathbf{f}}-i\widetilde{\mathbf{g}}\|_2)^2\} \\ &= \frac{1}{4}\Big\{\int|\mathbf{f}+\mathbf{g}|^2\,d\mu - \int|\mathbf{f}-\mathbf{g}|^2\,d\mu + i\int|\mathbf{f}+i\mathbf{g}|^2\,d\mu - i\int|\mathbf{f}-i\mathbf{g}|^2\,d\mu\Big\} \\ &= \int\frac{1}{4}\{|\mathbf{f}+\mathbf{g}|^2 - |\mathbf{f}-\mathbf{g}|^2 + i|\mathbf{f}+i\mathbf{g}|^2 - i|\mathbf{f}-i\mathbf{g}|^2\}d\mu \\ &= \int\langle\mathbf{f}(x),\mathbf{g}(x)\rangle\,d\mu(x) = \int\langle\mathbf{f},\mathbf{g}\rangle\,d\mu\,.\end{aligned}$$

IV.94, ℓ. -4 to -1.

"COROLLARY 4."

Immediate from Cor. 1, with $\Phi(\lambda,\mathbf{a}) = \lambda\mathbf{a}$ (λ scalar, $\mathbf{a}\in F$).

IV.95, ℓ. $7, 8$.

"... since the inequality (8) is true for upper integrals (Ch. I, No. 2, Cor. of Prop. 2)"

One can suppose that the f_i are everywhere finite and $\geqslant 0$ (§5, No. 6, Cor. 3 of Th. 5), as required by the cited Cor. from Ch. I.

IV.95, ℓ. -5 to -3.

"Suppose first that $\mathbf{f}$ is an integrable *step* function, $\mathbf{f} = \sum_{k=1}^{n}\mathbf{a}_k\varphi_{A_k}$, where the A_k are pairwise disjoint (§4, No. 9, Lemma)."

The term 'measurable step function' is defined in the first paragraph of §5, No. 5, but the term 'integrable step function' is nowhere explicitly defined; we can infer its meaning from an earlier approximation theorem (§4, No. 10, Cor. 1 of Prop. 19) cited later in the present proof. Proposed definition: An *integrable step function* is an element of $\mathscr{E}_F(\Phi)$ (§4, No. 9, Def. 4), where Φ is the clan (*loc. cit.*, Prop. 17) of integrable subsets of X (§4, No. 5, Props. 6 and 7).

Proposition. For a function $\mathbf{f} : X \to F$, the following conditions are equivalent:

a) $\mathbf{f}$ is an integrable step function (in the above sense);

b) $\mathbf{f}$ is integrable and has only finitely many values.

Proof. *a*) $\Rightarrow$ *b*): Obvious.

b) $\Rightarrow$ *a*): Let $\mathbf{a}_1,\dots,\mathbf{a}_n$ be the distinct nonzero values of $\mathbf{f}$ and, for each index k, let $A_k = \overset{-1}{\mathbf{f}}(\mathbf{a}_k)$. Then every A_k is measurable (§5, No. 5, Prop. 7) and

$$\mathbf{f} = \sum_{k=1}^{n} \mathbf{a}_k \varphi_{\mathrm{A}_k} \,.$$

The function $|\mathbf{f}| = \sum\limits_{k=1}^{n} |\mathbf{a}_k| \varphi_{\mathrm{A}_k}$ is also integrable (§4, No. 2, Cor. 1 of Prop. 1) and, for each k, $\varphi_{\mathrm{A}_k} \leqslant |\mathbf{a}_k|^{-1}|\mathbf{f}|$, whence A_k is integrable (§5, No. 6, Th. 5); thus $\mathbf{f}$ is a step function with respect to the clan Φ of integrable sets. ♢

IV.95, ℓ. -3 to **IV.96,** ℓ. 1.

"For every $\varepsilon > 0$, there exists (for every index k) a vector $\mathbf{a}'_k \in \mathrm{F}'$ such that $|\mathbf{a}'_k|^q = |\mathbf{a}_k|^p$ if $p > 1$ (resp. $|\mathbf{a}'_k| = 1$ if $p = 1$) and $\langle \mathbf{a}_k, \mathbf{a}'_k \rangle \geqslant (1-\varepsilon)|\mathbf{a}_k| \cdot |\mathbf{a}'_k|$ (TVS, IV, §1, No. 3, Prop. 8)."

Fix k. By the cited Prop. 8,

$$|\mathbf{a}_k| = \sup_{\mathbf{a}' \in \mathrm{F}',\, |\mathbf{a}'|=1} |\langle \mathbf{a}_k, \mathbf{a}' \rangle| \,,$$

so there exists an $\mathbf{a}'$ with $|\mathbf{a}'| = 1$ and $|\langle \mathbf{a}_k, \mathbf{a}' \rangle| \geqslant (1-\varepsilon)|\mathbf{a}_k|$; multiplying $\mathbf{a}'$ by a suitable scalar of absolute value 1, we can suppose that $|\langle \mathbf{a}_k, \mathbf{a}' \rangle| = \langle \mathbf{a}_k, \mathbf{a}' \rangle$ and so

$$(*) \qquad \langle \mathbf{a}_k, \mathbf{a}' \rangle \geqslant (1-\varepsilon)|\mathbf{a}_k| \,.$$

{Indeed, by the Hahn-Banach theorem, there exists an $\mathbf{a}' \in \mathrm{F}'$ such that $|\mathbf{a}'| = 1$ and $\langle \mathbf{a}_k, \mathbf{a}' \rangle = |\mathbf{a}_k|$ (TVS, II, §8, No. 3, Cor. 1 of Th. 1). However, in the proof of 2°, for F not reflexive, one must make do with the analog of $(*)$.}

If $p = 1$, define $\mathbf{a}'_k = \mathbf{a}'$; then $|\mathbf{a}'_k| = 1$ and $(*)$ is precisely the desired inequality.

If $p > 1$ (and $< +\infty$), define $\mathbf{a}'_k = |\mathbf{a}_k|^{p/q}\mathbf{a}'$; then $|\mathbf{a}'_k| = |\mathbf{a}_k|^{p/q}$, so $|\mathbf{a}'_k|^q = |\mathbf{a}_k|^p$; and multiplication of the inequality $(*)$ by $|\mathbf{a}_k|^{p/q}$ yields

$$\langle \mathbf{a}_k, |\mathbf{a}_k|^{p/q}\mathbf{a}' \rangle \geqslant (1-\varepsilon)|\mathbf{a}_k| \cdot |\mathbf{a}_k|^{p/q} \,,$$

that is, $\langle \mathbf{a}_k, \mathbf{a}'_k \rangle \geqslant (1-\varepsilon)|\mathbf{a}_k| \cdot |\mathbf{a}'_k|$ as desired.

IV.96, ℓ. 7.

"... which proves the relation (9) in this case."

The case is $1 \leqslant p < +\infty$, with the normalization $\mathrm{N}_p(\mathbf{f}) = 1$. The argument shows that

$$\sup_{\mathbf{g} \in \mathscr{L}^q_{\mathrm{F}'},\, \mathrm{N}_p(\mathbf{g}) \leqslant 1} \left| \int \langle \mathbf{f}, \mathbf{g} \rangle \, d\mu \right| \geqslant \mathrm{N}_p(\mathbf{f}) \,;$$

the reverse inequality is immediate from Hölder's inequality (Th. 2).

IV.96, ℓ. 12.

$$\int \langle \mathbf{f}_1, \mathbf{g} \rangle \, d\mu \geqslant \mathrm{N}_p(\mathbf{f}_1) - \varepsilon \geqslant 1 - 2\varepsilon \,.$$

$1 = \mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_p(\mathbf{f} - \mathbf{f}_1) + \mathrm{N}_p(\mathbf{f}_1) \leqslant \varepsilon + \mathrm{N}_p(\mathbf{f}_1)$, whence the second inequality of the display.

IV.96, ℓ. −10.

$$\left| \int \langle \mathbf{f}, \mathbf{g} \rangle \, d\mu \right| \geqslant 1 - 3\varepsilon$$

Abbreviate the relation

$$\int \langle \mathbf{f}, \mathbf{g} \rangle \, d\mu = \int \langle \mathbf{f}_1, \mathbf{g} \rangle \, d\mu + \int \langle \mathbf{f} - \mathbf{f}_1, \mathbf{g} \rangle \, d\mu$$

as $\alpha = \beta + \gamma$; we know that β is real, but α and γ need not be. Citing inequalities proved earlier, one has

$$\big||\alpha| - |\beta|\big| \leqslant |\alpha - \beta| = |\gamma| \leqslant \mathrm{N}_p(\mathbf{f} - \mathbf{f}_1)\, \mathrm{N}_q(\mathbf{g}) \leqslant \varepsilon \cdot 1 \,,$$

whence $-\varepsilon \leqslant |\alpha| - |\beta| \leqslant \varepsilon$ and so

$$|\alpha| \geqslant |\beta| - \varepsilon \geqslant \beta - \varepsilon \geqslant (1 - 2\varepsilon) - \varepsilon \,.$$

IV.96, ℓ. −6.

"... the set of $x \in \mathrm{X}$ such that $|\mathbf{f}(x)| > \alpha$ is measurable"

For, $\mathbf{f}$ is measurable (§5, No. 6, Th. 5), therefore so is $|\mathbf{f}|$ (§5, No. 3, Cor. 6 of Th. 1), therefore so is the set in question (§5, No. 5, Prop. 8).

IV.96, ℓ. −5.

"... therefore it contains a compact set K of measure > 0."

Let $\mathrm{B} = \{x \in \mathrm{X} : \ |\mathbf{f}(x)| > \alpha\}$. Since B is not locally negligible, there exists a compact set H such that $\mathrm{B} \cap \mathrm{H}$ is not negligible (§5, No. 2, Prop. 5). Since B is measurable, $\mathrm{B} \cap \mathrm{H}$ is integrable (§5, No. 6, Cor. 3 of Th. 5), and since $\mathrm{B} \cap \mathrm{H}$ is not negligible it contains a compact set K of measure > 0 (§4, No. 6, Cor. 1 of Th. 4).

IV.96, ℓ. −3 to −1.

"... for every $\varepsilon > 0$, there exists a partition of K_1 into a finite number of integrable sets, in each of which the oscillation of $\mathbf{f}$ is $\leqslant \varepsilon$"

Since $\mathbf{f}|\mathrm{K}_1$ is continuous, for every $x \in \mathrm{K}_1$ there exists an open neighborhood U_x of x in X such that $|\mathbf{f}(y) - \mathbf{f}(x)| \leqslant \varepsilon/2$ on $\mathrm{U}_x \cap \mathrm{K}_1$. Cover K_1

with a finite number of open sets $U_1, \dots, U_n$ in X such that $\mathbf{f}\big|K_1$ has oscillation $\leqslant \varepsilon$ on each $U_i \cap K_1$. The sets $U_i \cap K_1$ are integrable, and since the integrable sets form a clan, the sets

$$U_1 \cap K_1,\ U_i \cap K_1 - \bigcup_{j<i} U_j \cap K_1 \quad (i = 2, \dots, n)$$

form a partition of K_1 with the desired properties.

IV.97, ℓ. 3, 4.

"... the function $\mathbf{g} = \varphi_A \cdot \mathbf{a}'/\mu(A)$ is integrable and $N_1(\mathbf{g}) = 1$ "

Writing $g = \dfrac{1}{\mu(A)}\varphi_A$, one has $g \in \mathscr{L}^1_{\mathbf{R}}$, $\int g\,d\mu = 1$ and $\mathbf{g} = g \cdot \mathbf{a}'$; thus $\mathbf{g}$ is an integrable step function $X \to F'$ (see the note for IV.95, ℓ. −5 to −3) and $|\mathbf{g}| = |\mathbf{a}'| \cdot g = g$, whence $N_1(\mathbf{g}) = \int g\,d\mu = 1$.

IV.97, ℓ. 5.

$$\int \langle \mathbf{f}, \mathbf{g}\rangle\,d\mu = \frac{1}{\mu(A)}\int \langle \mathbf{f}, \mathbf{a}'\rangle \varphi_A\,d\mu\,.$$

Since $\mathbf{f} \in \mathscr{L}^\infty_F$ and $\mathbf{g} \in \mathscr{L}^1_{F'}$, we know from Cor. 2 of Th. 2 that $\langle \mathbf{f}, \mathbf{g}\rangle \in \mathscr{L}^1_{\mathbf{C}}$. Explicitly, writing $\mathbf{g} = g \cdot \mathbf{a}'$ as in the preceding note, $\langle \mathbf{f}, \mathbf{g}\rangle$ is the function

$$x \mapsto \langle \mathbf{f}(x), g(x) \cdot \mathbf{a}'\rangle = \langle \mathbf{f}(x), \mathbf{a}'\rangle g(x)\,,$$

where the function $x \mapsto \langle \mathbf{f}(x), \mathbf{a}'\rangle$ belongs to $\mathscr{L}^\infty$ and $g \in \mathscr{L}^1$.

IV.97, ℓ. 7.

$$\int \langle \mathbf{f}, \mathbf{a}'\rangle \varphi_A\,d\mu = \langle \mathbf{a}, \mathbf{a}'\rangle \mu(A) + \int \langle \mathbf{f} - \mathbf{a}, \mathbf{a}'\rangle \varphi_A\,d\mu$$

Here $\mathbf{a}$ also represents the constant function $X \to F$ with value $\mathbf{a}$. Thus $\mathbf{f} - \mathbf{a} \in \mathscr{L}^\infty_F$ whereas $\varphi_A \cdot \mathbf{a}' \in \mathscr{L}^1_{F'}$, and

$$\langle \mathbf{f}, \mathbf{a}'\rangle \varphi_A = \langle \mathbf{a} + (\mathbf{f} - \mathbf{a}), \mathbf{a}'\rangle \varphi_A = \langle \mathbf{a}, \mathbf{a}'\rangle \varphi_A + \langle \mathbf{f} - \mathbf{a}, \mathbf{a}'\rangle \varphi_A\ ,$$

whence the asserted equation. Note that $\langle \mathbf{f}, \mathbf{a}'\rangle \varphi_A = \mu(A)\langle \mathbf{f}, \mathbf{g}\rangle$.

IV.97, ℓ. 11.

$$\left|\int \langle \mathbf{f}, \mathbf{g}\rangle\,d\mu\right| \geqslant |\langle \mathbf{a}, \mathbf{a}'\rangle| - \varepsilon \geqslant |\mathbf{a}| - 2\varepsilon > \alpha - 2\varepsilon$$

Since $\langle \mathbf{f}, \mathbf{a}'\rangle \varphi_A = \mu(A)\langle \mathbf{f}, \mathbf{g}\rangle$, the equation of ℓ. 7 may be written

$$\mu(A)\int \langle \mathbf{f}, \mathbf{g}\rangle\,d\mu = \langle \mathbf{a}, \mathbf{a}'\rangle \mu(A) + \int \langle \mathbf{f} - \mathbf{a}, \mathbf{a}'\rangle \varphi_A\,d\mu\,;$$

abbreviate the equation as $a = b + c$. Then

$$\big||a| - |b|\big| \leqslant |a - b| = |c| \leqslant \int \varepsilon\,\varphi_{\mathrm{A}}\,d\mu = \varepsilon\,\mu(\mathrm{A})\,,$$

therefore $-\varepsilon\,\mu(\mathrm{A}) \leqslant |a| - |b| \leqslant \varepsilon\,\mu(\mathrm{A})$, whence

$$|a| \geqslant |b| - \varepsilon\,\mu(\mathrm{A}) = |\langle \mathbf{a}, \mathbf{a}' \rangle| \cdot \mu(\mathrm{A}) - \varepsilon\,\mu(\mathrm{A}) = (|\langle \mathbf{a}, \mathbf{a}' \rangle| - \varepsilon)\mu(\mathrm{A})\,,$$

and so

$$\left|\int \langle \mathbf{f}, \mathbf{g} \rangle\,d\mu\right| = \frac{|a|}{\mu(\mathrm{A})} \geqslant |\langle \mathbf{a}, \mathbf{a}' \rangle| - \varepsilon \geqslant (|\mathbf{a}| - \varepsilon) - \varepsilon > \alpha - 2\varepsilon\,.$$

IV.97, $\ell.$ 12, 13.

"...the relation (9) is also verified in this case."

The foregoing shows that given any α with $0 < \alpha < \mathrm{N}_\infty(\mathbf{f})$ and any $\varepsilon > 0$, there exists a function $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}'}$ such that $\mathrm{N}_1(\mathbf{g}) = 1$ and $\left|\int \langle \mathbf{f}, \mathbf{g} \rangle\,d\mu\right| \geqslant \alpha - 2\varepsilon$. This shows that the supremum in (9) is $\geqslant \alpha - 2\varepsilon$ hence (letting $\alpha \to \mathrm{N}_\infty(\mathbf{f})$ and $\varepsilon \to 0$) it is $\geqslant \mathrm{N}_\infty(\mathbf{f})$. The reverse inequality is immediate from Cor. 2 of Th. 2.

IV.97, $\ell.$ $-15, -14$.

"...it suffices to observe that the interior $\mathring{\mathrm{B}}$ of B is dense in B and that $\mathring{\mathrm{B}} \cap \mathscr{E}$ is dense in $\mathring{\mathrm{B}}$, since $\mathring{\mathrm{B}}$ is open."

It is not necessary that the dense subset $\mathscr{E}$ of $\mathscr{L}^q_{\mathrm{F}'}$ be a linear subspace. By GT, I, §1, No. 6, Prop. 5,

$$\overline{\mathring{\mathrm{B}} \cap \mathscr{E}} \supset \mathring{\mathrm{B}} \cap \overline{\mathscr{E}} = \mathring{\mathrm{B}}\,,$$

therefore $\overline{\mathring{\mathrm{B}} \cap \mathscr{E}} \supset \overline{\mathring{\mathrm{B}}} = \mathrm{B}$ (TVS, II, §2, No. 6, Cor. 1 of Prop. 16), whereas $\overline{\mathring{\mathrm{B}} \cap \mathscr{E}} \subset \overline{\mathrm{B}} = \mathrm{B}$, thus $\overline{\mathring{\mathrm{B}} \cap \mathscr{E}} = \mathrm{B}$, whence obviously $\overline{\mathrm{B} \cap \mathscr{E}} = \mathrm{B}$. Let $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$ and let

$$s = \sup_{\mathbf{g} \in \mathrm{B} \cap \mathscr{E}} |\langle \mathbf{f}, \mathbf{g} \rangle|\,;$$

we are to show that $s = \mathrm{N}_p(\mathbf{f})$. The inequality $s \leqslant \mathrm{N}_p(\mathbf{f})$ is immediate from Cor. 2 of Th. 2. To prove the reverse inequality it will suffice, by (9), to show that $|\langle \mathbf{f}, \mathbf{g} \rangle| \leqslant s$ for every $\mathbf{g} \in \mathrm{B}$. Given $\mathbf{g} \in \mathrm{B}$, choose a sequence $\mathbf{g}_n \in \mathrm{B} \cap \mathscr{E}$ such that $\mathrm{N}_q(\mathbf{g}_n - \mathbf{g}) \to 0$. Then $\langle \mathbf{f}, \mathbf{g}_n \rangle \to \langle \mathbf{f}, \mathbf{g} \rangle$, and since $|\langle \mathbf{f}, \mathbf{g}_n \rangle| \leqslant s$ for all n, it follows that $|\langle \mathbf{f}, \mathbf{g} \rangle| \leqslant s$.

IV.97, ℓ. −12, −11.

"But in this case, the formula (9) is true as $\mathbf{g}$ runs over $\mathrm{B}\cap\mathscr{K}(\mathrm{X};\mathrm{F}')$, even for $p=1$."

The strategy is to refine the argument in the first paragraph of (i) in the proof of Prop. 3 restricted to the case $p = 1$, $q = +\infty$; the second paragraph then completes the argument without change.

We are given an integrable step function $\mathbf{f}:\mathrm{X}\to\mathrm{F}$ with $\mathrm{N}_1(\mathbf{f}) = 1$. Let

$$\mathrm{B} = \{\mathbf{g}\in\mathscr{L}^{\infty}_{\mathrm{F}'} :\ \mathrm{N}_\infty(\mathbf{g})\leqslant 1\};$$

we know from Prop. 3 that

$$\sup_{\mathbf{g}\in\mathrm{B}}\left|\int\langle\mathbf{f},\mathbf{g}\rangle\,d\mu\right| = \mathrm{N}_1(\mathbf{f}) = 1\,.$$

Inspection of the proof shows that B can be replaced by its subset consisting of all integrable step functions $\mathbf{g}$ such that $|\mathbf{g}|\leqslant 1$. Of course $|\mathbf{g}|\leqslant 1 \Rightarrow \mathrm{N}_\infty(\mathbf{g})\leqslant 1$. In the reverse direction, if $\mathbf{g}:\mathrm{X}\to\mathrm{F}'$ is a measurable function such that $\mathrm{N}_\infty(\mathbf{g})\leqslant 1$ then $|g(x)|\leqslant 1$ on the complement of a locally negligible set N; the function $\mathbf{g}_1 = \varphi_{\complement\mathrm{N}}\mathbf{g}$ is then measurable, $|\mathbf{g}_1|\leqslant 1$ and $\mathbf{g}=\mathbf{g}_1$ locally almost everywhere. Since the set $\mathrm{A}=\{x:\ \mathbf{f}(x)\neq 0\}$ is integrable, its intersection with N is negligible; for, $\mathrm{A}\cap\mathrm{N}$ is integrable (§5, No. 6, Cor. 3 of Th. 5), and every compact set $\mathrm{K}\subset\mathrm{A}\cap\mathrm{N}$ is negligible (§5, No. 2, Prop. 5), therefore $\mathrm{A}\cap\mathrm{N}$ is negligible (§4, No. 6, Cor. 1 of Th. 4). It follows that $\langle\mathbf{f},\mathbf{g}\rangle = \langle\mathbf{f},\mathbf{g}_1\rangle$ almost everywhere; thus, writing B_1 for the set of measurable functions $\mathbf{g}:\mathrm{X}\to\mathrm{F}'$ such that $|\mathbf{g}|\leqslant 1$, that is, in the notation of §5, No. 11,

$$\mathrm{B}_1 = \{\mathbf{g}\in\mathscr{S}(\mathrm{X},\mu;\mathrm{F}') :\ |\mathbf{g}|\leqslant 1\},$$

one has

$$(*)\qquad \sup_{\mathbf{g}\in\mathrm{B}_1}\left|\int\langle\mathbf{f},\mathbf{g}\rangle\,d\mu\right| = \mathrm{N}_1(\mathbf{f})\,.$$

Consider now the set

$$\mathrm{B}_2 = \mathrm{B}\cap\mathscr{K}(\mathrm{X};\mathrm{F}') = \{\mathbf{g}\in\mathscr{K}(\mathrm{X};\mathrm{F}') :\ \mathrm{N}_\infty(\mathbf{g})\leqslant 1\};$$

we are to show that

$$(**)\qquad \sup_{\mathbf{g}\in\mathrm{B}_2}\left|\int\langle\mathbf{f},\mathbf{g}\rangle\,d\mu\right| = \mathrm{N}_1(\mathbf{f})\,.$$

In fact, defining

$$B_3 = \{\mathbf{g} \in \mathscr{K}(X;F') : |\mathbf{g}| \leqslant 1\}$$

(a subset of $B_1 \cap B_2$) the argument will show that

$$(***) \qquad \sup_{\mathbf{g}\in B_3} \left| \int \langle \mathbf{f}, \mathbf{g} \rangle \, d\mu \right| = N_1(\mathbf{f}) .$$

{Incidentally, if $\operatorname{Supp}\mu = X$ then $B_2 = B_3$. For, suppose $\mathbf{g} \in B_2$ and let $U = \{x \in X : |\mathbf{g}(x)| > 1\}$. Then U is open, locally negligible (because $N_\infty(\mathbf{g}) \leqslant 1$) and has compact closure, therefore $U = U \cap \overline{U}$ is negligible (§5, No. 2, Prop. 5), hence $U = \varnothing$ (§2, No. 2, Prop. 5); in other words $|\mathbf{g}| \leqslant 1$, that is, $\mathbf{g} \in B_3$.}

IV.97, ℓ. −8 to − 6.

"There exists a finite number of pairwise disjoint compact sets K_i such that $\mathbf{g}$ has a constant value $\mathbf{a}'_i$ on each K_i and such that, if K is the union of the K_i, then $\int |\mathbf{f}| \varphi_{\complement K} \, d\mu \leqslant \varepsilon$."

As in (i) of the proof of Prop. 3, we can suppose that $\mathbf{f} = \sum_{i=1}^{n} \mathbf{a}_i \varphi_{A_i}$, where the $\mathbf{a}_i$ are the distinct nonzero values of $\mathbf{f}$, and the A_i are pairwise disjoint integrable sets. Given $\varepsilon > 0$, one constructs elements $\mathbf{a}'_i \in F'$ such that $|\mathbf{a}'_i| = 1$ and $\langle \mathbf{a}_i, \mathbf{a}'_i \rangle \geqslant (1-\varepsilon)|\mathbf{a}_i|$, and one defines $\mathbf{g} = \sum_{i=1}^{n} \mathbf{a}'_i \varphi_{A_i}$.

For each i, choose a compact set $K_i \subset A_i$ such that $\mu(A_i - K_i) \leqslant \varepsilon/n|\mathbf{a}_i|$ (§4, No. 6, Cor. 1 of Th. 4) and let $K = \bigcup_{i=1}^{n} K_i$. Then $A_i \cap \complement K = A_i - A_i \cap K = A_i - K_i$, therefore $\mathbf{f}\varphi_{\complement K} = \sum_{i=1}^{n} \mathbf{a}_i \varphi_{A_i - K_i}$ and

$$\int |\mathbf{f}| \varphi_{\complement K} \, d\mu = \sum_{i=1}^{n} |\mathbf{a}_i| \mu(A_i - K_i) \leqslant \sum_{i=1}^{n} \varepsilon/n = \varepsilon .$$

IV.97, ℓ. −3.

"Setting $\mathbf{h} = \sum \mathbf{a}'_i h_i$ "

We require that $\mathbf{h}$, hence the h_i, have compact support: $K_i \subset V_i \subset H_i \subset U_i$ for suitable open V_i and compact H_i; a suitable h_i then exists by Lemma 1 of Ch. III, §1, No. 2.

IV.97, ℓ. −1.

$$\int |\langle \mathbf{f}, \mathbf{h} \rangle| \varphi_{\complement K} \, d\mu \leqslant \varepsilon$$

For, $\langle \mathbf{f}, \mathbf{h} \rangle \varphi_{\complement K} = \langle \mathbf{f} \varphi_{\complement K}, \mathbf{h} \rangle$, whence, by Cor. 2 of Th. 2,

$$\int |\langle \mathbf{f}, \mathbf{h} \rangle| \varphi_{\complement K} \, d\mu \leqslant N_1(\mathbf{f} \varphi_{\complement K}) N_\infty(\mathbf{h}) \leqslant \varepsilon \cdot 1 \, .$$

IV.98, ℓ. 1.

"...consequently $\left| \int \langle \mathbf{f}, \mathbf{h} \rangle \, d\mu \right| \geqslant 1 - 3\varepsilon$, which proves our assertion."

Abbreviate the equation

$$\int \langle \mathbf{f}, \mathbf{h} \rangle \, d\mu = \int \langle \mathbf{f}, \mathbf{h} \rangle \varphi_K \, d\mu + \int \langle \mathbf{f}, \mathbf{h} \rangle \varphi_{\complement K} \, d\mu$$

by $\alpha = \beta + \gamma$. Then

$$\big| |\alpha| - |\beta| \big| \leqslant |\alpha - \beta| = |\gamma| \leqslant \int |\langle \mathbf{f}, \mathbf{h} \rangle| \varphi_{\complement K} \, d\mu \leqslant \varepsilon \, ,$$

thus $-\varepsilon \leqslant |\alpha| - |\beta| \leqslant \varepsilon$, whence $|\alpha| \geqslant |\beta| - \varepsilon$. Now,

$$\langle \mathbf{f}, \mathbf{h} \rangle \varphi_K = \langle \mathbf{f}, \mathbf{h} \varphi_K \rangle = \langle \mathbf{f}, \mathbf{g} \varphi_K \rangle = \langle \mathbf{f}, \mathbf{g} \rangle \varphi_K = \langle \mathbf{f}, \mathbf{g} \rangle - \langle \mathbf{f}, \mathbf{g} \rangle \varphi_{\complement K} \, ,$$

thus

$$\beta = \int \langle \mathbf{f}, \mathbf{g} \rangle \, d\mu - \int \langle \mathbf{f}, \mathbf{g} \rangle \varphi_{\complement K} \, d\mu \, ;$$

abbreviate this equation as $\beta = \sigma - \tau$, noting that $|\sigma| \geqslant 1 - \varepsilon$ by the choice of $\mathbf{g}$. Then

$$\big| |\beta| - |\sigma| \big| \leqslant |\beta - \sigma| = |\tau| \leqslant N_1(\mathbf{f} \varphi_{\complement K}) N_\infty(\mathbf{g}) \leqslant \varepsilon \cdot 1 \, ,$$

therefore $-\varepsilon \leqslant |\beta| - |\sigma| \leqslant \varepsilon$, whence

$$|\beta| \geqslant |\sigma| - \varepsilon \geqslant (1 - \varepsilon) - \varepsilon \, ,$$

and finally $|\alpha| \geqslant |\beta| - \varepsilon \geqslant (1 - 2\varepsilon) - \varepsilon$, which verifies the asserted inequality.

As in the note for IV.97, ℓ. −12, −11, writing $B_3 = \{ \mathbf{k} \in \mathscr{K}(X; F') : |\mathbf{k}| \leqslant 1 \}$, the foregoing shows that $\mathbf{h} \in B_3$ and

$$\sup_{\mathbf{k} \in B_3} \left| \int \langle \mathbf{f}, \mathbf{k} \rangle \, d\mu \right| \geqslant \left| \int \langle \mathbf{f}, \mathbf{h} \rangle \, d\mu \right| = |\alpha| \geqslant 1 - 3\varepsilon \, .$$

Since ε is arbitrary,

$$\sup_{\mathbf{k} \in B_3} \left| \int \langle \mathbf{f}, \mathbf{k} \rangle \, d\mu \right| \geqslant 1 = N_1(\mathbf{f}) \, ,$$

and the reverse inequality is immediate from Cor. 2 of Th. 2.

IV.98, ℓ. 4.

" whose support is contained in a countable union of compact sets K_n."

Here, by "support" is meant the set $\{x : f(x) \neq 0\}$; it is not necessary that its closure be contained in the union.

IV.98, ℓ. 8.

" ... a special case of (9)"

That is, (9) as generalized in Remark 1.

IV.98, ℓ. 8, 9.

" f is then equivalent to a function in $\mathscr{L}^p$ (§5, No. 6, Th. 5)."

Since $\mathrm{N}_p(f) < +\infty$, the set $\mathrm{N} = \{x : f(x) = +\infty\}$ is negligible (§2, No. 3, Prop. 7). Then $f\varphi_{\complement \mathrm{N}}$ is a finite-valued measurable function such that $f\varphi_{\complement \mathrm{N}} = f$ almost everywhere, whence $\mathrm{N}_p(f\varphi_{\complement \mathrm{N}}) = \mathrm{N}_p(f) < +\infty$, thus $f\varphi_{\complement \mathrm{N}} \in \mathscr{L}^p$ by the cited Th. 5. In other words, f is p-th power integrable (§3, No. 4, second paragraph after Def. 2).

Incidentally, the set $\{x : (f\varphi_{\complement \mathrm{N}})(x) \neq 0\}$ is contained in the union of a negligible set and a sequence of compact sets (§5, No. 6, Lemma 1), hence the same is true of $\{x : f(x) \neq 0\}$; thus the assumption about the 'support' of f, not used in the case $\mathrm{N}_p(f) < +\infty$, is in a sense redundant.

IV.98, ℓ. 10.

"... set $f_n = \inf(n, f\varphi_{\mathrm{K}_n})$."

It is understood that if $f(x) = +\infty$ and $x \notin \mathrm{K}_n$ then $f_n(x) = \inf(n, 0) = 0$ by the convention $+\infty \cdot 0 = 0$. One can suppose that the sequence (K_n) is increasing.

Note that f_n is measurable and $0 \leqslant f_n \leqslant n\varphi_{\mathrm{K}_n}$, hence f_n is integrable. If $p < +\infty$ then $\mathrm{N}_p(f_n) \leqslant n\,\mathrm{N}_p(\varphi_{\mathrm{K}_n}) < +\infty$, whereas $\mathrm{N}_\infty(f_n) \leqslant n < +\infty$, therefore (11) holds for f_n for $1 \leqslant p \leqslant +\infty$.

The sequence (f_n) is increasing, with upper envelope equal to f, that is, $f_n(x) \uparrow f(x)$ for all $x \in \mathrm{X}$. For, since $\varphi_{\mathrm{K}_n} \leqslant \varphi_{\mathrm{K}_{n+1}}$, $f_n \leqslant f_{n+1}$ is assured by $f(x)\varphi_{\mathrm{K}_n}(x) \leqslant f(x)\varphi_{n+1}(x)$ (even if $f(x) = +\infty$ and $x \notin \mathrm{K}_n$). If $f(x) = +\infty$ then $x \in \mathrm{K}_m$ for some index m; then, for all $n \geqslant m$, $f(x)\varphi_{\mathrm{K}_n}(x) = +\infty$, therefore $f_n(x) = n$, thus $f_n(x) \uparrow +\infty = f(x)$. If $0 < f(x) < +\infty$ then $x \in \mathrm{K}_m$ for some m, and one can suppose that $m \geqslant f(x)$; then $f_n(x) = f(x)$ for all $n \geqslant m$, whence $f_n(x) \uparrow f(x)$. And if $f(x) = 0$ then $f_n(x) = 0$ for all n, $f_n(x) \uparrow 0 = f(x)$.

IV.98, ℓ. 12, 13.

"... passing to the limit (assuming, as we may, that the sequence (K_n) is increasing), we have $\sup \int^* |fg|\,d\mu = +\infty$ (§1, No. 3, Th. 3)."

By assumption, $\mathrm{N}_p(f) = +\infty$. We know (see the preceding note) that the sequence (f_n) is increasing, with upper envelope f, and that

$N_p(f_n) < +\infty$ for all n. Thus every f_n satisfies (11): writing $B_0 = \{g \in \mathscr{K}(X;\mathbf{R}) : N_q(g) \leqslant 1\}$, we have

$$N_p(f_n) = \sup_{g\in B_0} \int^* |f_n g|\, d\mu \leqslant \sup_{g\in B_0} \int^* |fg|\, d\mu \leqslant +\infty = N_p(f)\,;$$

to verify that f satisfies (11), it suffices to show that $\sup_n N_p(f_n) = N_p(f) = +\infty$. When $p < +\infty$, this follows from the cited Th. 3.

There remains the case that $p = +\infty$, $q = 1$, $B_0 = \{g \in \mathscr{K}(X;\mathbf{R}) : N_q(g) \leqslant 1\}$. We are to show that $\sup_n N_\infty(f_n) = +\infty$. Assume to the contrary that the sequence $N_\infty(f_n)$ is bounded, say $N_\infty(f_n) \leqslant \alpha < +\infty$ for all n. Then $f_n \leqslant \alpha$ locally almost everywhere; let M_n be a locally negligible set such that $f_n \leqslant \alpha$ on $X - M_n$. The set $M = \bigcup_{n=1}^{\infty} M_n$ is then locally negligible and, for $x \in \complement M$, $f_n(x) \leqslant \alpha$ for all n, therefore $f(x) \leqslant \alpha$. Thus $f(x) \leqslant \alpha$ locally almost everywhere; but then $N_\infty(f) \leqslant \alpha$, contrary to $N_\infty(f) = +\infty$.

IV.98, ℓ. 14–17.

"COROLLARY. — *Let* μ *be a positive measure on* X, F *a Banach space,* F′ *its strong dual, and* $\mathbf{g}$ *any function in* $\mathscr{L}^q_{F'}$. *Then, the linear form on* L^p_F, *deduced from the linear form* $\mathbf{f} \mapsto \int \langle \mathbf{f}, \mathbf{g}\rangle\, d\mu$ *on* $\mathscr{L}^p_F$ *by passage to the quotient, is continuous and has norm* $N_q(\mathbf{g})$."

When $1 \leqslant p < +\infty$, L^p_F is defined to be the set of elements $\widetilde{\mathbf{f}}$ ($\mathbf{f} \in \mathscr{L}^p_F$), where $\widetilde{\mathbf{f}}$ is the class of $\mathbf{f}$ for the relation of equality almost everywhere (§2, No. 6, §3, No. 4, Def. 2). But $q = +\infty$ when $p = 1$, and $L^\infty_{F'}$ is defined to be the set of elements $\dot{\mathbf{g}}$ ($\mathbf{g} \in \mathscr{L}^\infty_{F'}$), where $\dot{\mathbf{g}}$ is the class of $\mathbf{g}$ for the relation of equality locally almost everywhere (No. 3); we are then in the situation of "mixing apples and oranges". The purpose of this note is to clarify this situation (of which the case of $1 < p < +\infty$, $q = (p-1)/p$ is a straightforward simplification).

Lemma. If $\mathbf{f}, \mathbf{f}_1 \in \mathscr{L}^1_F$ and $\mathbf{g}, \mathbf{g}_1 : X \to F'$ are functions such that $\mathbf{f} = \mathbf{f}_1$ almost everywhere and $\mathbf{g} = \mathbf{g}_1$ locally almost everywhere, then $\langle \mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}_1, \mathbf{g}_1\rangle$ *almost everywhere.*

Proof. It suffice to show that $\langle \mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}_1, \mathbf{g}\rangle$ almost everywhere and $\langle \mathbf{f}_1, \mathbf{g}\rangle = \langle \mathbf{f}_1, \mathbf{g}_1\rangle$ almost everywhere. By bilinearity (the note for IV.94, ℓ. 8 is pertinent here!) this reduces to showing that, for $\mathbf{f} \in \mathscr{L}^1_F$ and $\mathbf{g} : X \to F'$,

$$\mathbf{f} = 0 \text{ a.e.} \quad \Rightarrow \quad \langle \mathbf{f}, \mathbf{g}\rangle = 0 \text{ a.e.}$$
$$\mathbf{g} = 0 \text{ l.a.e.} \quad \Rightarrow \quad \langle \mathbf{f}, \mathbf{g}\rangle = 0 \text{ a.e.}$$

The first implication is obvious. Suppose $\mathbf{g} = 0$ locally almost everywhere, and let $M = \{x : \mathbf{g}(x) \neq 0\}$, $A = \{x : \mathbf{f}(x) \neq 0\}$; then $\mathbf{f} = \mathbf{f}\varphi_A$, $\mathbf{g} = \mathbf{g}\varphi_M$. We know that M is locally negligible; and, since $\mathbf{f} \in \mathscr{L}^1_F$, $A \subset N \cup \bigcup_{n=1}^{\infty} K_n$ with N negligible and the K_n compact (§5, No. 6, Lemma 1). Then the set

$$A \cap M \subset (N \cap M) \cup \bigcup_{n=1}^{\infty} K_n \cap M$$

is negligible (§5, No. 2, Prop. 5), therefore $\varphi_A\varphi_M = \varphi_{A\cap M} = 0$ almost everywhere, thus $\langle \mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}\varphi_A, \mathbf{g}\varphi_M\rangle = \langle \mathbf{f}, \mathbf{g}\rangle\varphi_A\varphi_M = 0$ almost everywhere. ♢

Fix $\mathbf{g} \in \mathscr{L}^{\infty}_{F'}$. If $\mathbf{f} \in \mathscr{L}^1_F$ then by Hölder's inequality (Cor. 2 of Th. 2) $\langle \mathbf{f}, \mathbf{g}\rangle$ is integrable and

$$\left| \int \langle \mathbf{f}, \mathbf{g}\rangle \, d\mu \right| \leqslant N_1(\mathbf{f}) N_\infty(\mathbf{g}) .$$

By the Lemma, we may define $\lambda : L^1_F \to \mathbf{R}$ by

$$\lambda(\widetilde{\mathbf{f}}) = \int \langle \mathbf{f}, \mathbf{g}\rangle \, d\mu \qquad (\widetilde{\mathbf{f}} \in L^1_F) ;$$

λ is a linear form, and $|\lambda(\widetilde{\mathbf{f}})| \leqslant N_1(\mathbf{f})N_\infty(\mathbf{g})$ shows that it is continuous and that its norm, defined by

$$\|\lambda\| = \sup_{\widetilde{\mathbf{f}} \in L^1_F,\, N_1(\widetilde{\mathbf{f}}) \leqslant 1} \left| \int \langle \mathbf{f}, \mathbf{g}\rangle \, d\mu \right| = \sup_{\mathbf{f} \in \mathscr{L}^1_F,\, N_1(\mathbf{f}) \leqslant 1} \left| \int \langle \mathbf{f}, \mathbf{g}\rangle \, d\mu \right| ,$$

satisfies $\|\lambda\| \leqslant N_\infty(\mathbf{g})$. Indeed, $\|\lambda\| = N_\infty(\mathbf{g})$ by formula (10) of Prop. 3. Moreover, the Lemma shows that λ depends only on the class $\dot{\mathbf{g}}$ of $\mathbf{g}$. Thus, to each $\mathbf{u} = \dot{\mathbf{g}} \in L^{\infty}_{F'}$ there corresponds a continuous linear form $\lambda_{\mathbf{u}} \in \left(L^1_F\right)'$ such that $\|\lambda_{\mathbf{u}}\| = N_\infty(\mathbf{g}) = N_\infty(\mathbf{u})$; and $\mathbf{u} \mapsto \lambda_{\mathbf{u}}$ defines a linear isometry of $L^{\infty}_{F'}$ into the dual space $\left(L^1_F\right)'$ (shown in Ch. V, §5, No. 8, Th. 4 to be surjective when $F = \mathbf{R}$). The dual $\left(L^1_F\right)'$, for F a separable Banach space, is calculated in Ch. VI, §2, No. 6, Prop. 10.

The 'dual' result also holds: for fixed $\mathbf{f} \in \mathscr{L}^1_F$, the correspondence

$$\dot{\mathbf{g}} \mapsto \int \langle \mathbf{f}, \mathbf{g}\rangle \, d\mu \qquad (\mathbf{g} \in \mathscr{L}^{\infty}_{F'})$$

defines a continuous linear form on $L^{\infty}_{F'}$, with norm equal to $N_1(\mathbf{f})$. The proof is formally the same, citing the formula (9) of Prop. 3 in the calculation of the norm. Whence a linear isometry $\mathrm{L}^1_{\mathrm{F}} \to \big(\mathrm{L}^\infty_{\mathrm{F}'}\big)'$; but this mapping need not be surjective, even when $\mathrm{F} = \mathbf{R}$ (Ch. V, §5, No. 8, Prop. 14).

When $1 < p < +\infty$, one does business with $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{g}}$, obtaining linear isometries $\mathrm{L}^p_{\mathrm{F}} \to \big(\mathrm{L}^q_{\mathrm{F}'}\big)'$ and $\mathrm{L}^q_{\mathrm{F}} \to \big(\mathrm{L}^p_{\mathrm{F}'}\big)'$, both of which are surjective when $\mathrm{F} = \mathbf{R}$ (Ch. V, *loc. cit.*, Th. 4).

IV.98, ℓ. −10 to −7.

"We already know (Ch. I, No. 3, Prop. 5) that the set J of *finite* numbers $p \geqslant 1$ such that $\mathrm{N}_p(\mathbf{f}) < +\infty$ is either empty or is an interval, and that $\log \mathrm{N}_p(\mathbf{f})$ is a convex function of $1/p$ on J (when $\mathbf{f}$ is not negligible); this of course implies the continuity of $p \mapsto \mathrm{N}_p(\mathbf{f})$ on J."

The cited Prop. 5 contemplates $f \mapsto \mathrm{N}_p(f)$ $\big(f \in \mathscr{F}(\mathrm{X};\mathbf{R}\big)$ for $0 < p < +\infty$. In the present context, a measurable function $\mathbf{f} : \mathrm{X} \to \mathrm{F}$ is fixed, $|\mathbf{f}|$ plays the role of f, and we are interested in the function $p \mapsto \mathrm{N}_p(|\mathbf{f}|) = \Big(\int^* |\mathbf{f}|^p \, d|\mu|\Big)^{1/p}$ for $1 \leqslant p < +\infty$ (§3, No. 2, Def. 1); the definition can be stretched to $0 < p < +\infty$ by means of the same formula, as follows. One observes (§3, No. 1) that the function M defined on the set of all *positive* functions $f : \mathrm{X} \to \mathbf{R}$ by $\mathrm{M}(f) = \int^* f \, d|\mu|$ satisfies the axioms 1°, 2°, 3° of Ch. I, No. 1, Def. 1. In Ch. I, No. 3 one defines, for every p with $1 \leqslant p < +\infty$, the vector space $\mathscr{F}^p = \mathscr{F}^p(\mathrm{X}, \mathrm{M})$ to be the set of all functions $f : \mathrm{X} \to \mathbf{R}$ such that $\mathrm{M}(|f|^p) < +\infty$. Then, for every p with $0 < p < +\infty$, one defines

$$\mathrm{N}_p(f) = \big(\mathrm{M}(|f|^p)\big)^{1/p} \qquad \big(f \in \mathscr{F}(\mathrm{X};\mathbf{R})\big),$$

accepting that $\mathrm{N}_p(f) = +\infty$ when $\mathrm{M}(|f|^p) = +\infty$. The notation $\mathscr{F}^p$ remains reserved for the vector space obtained when $1 \leqslant p < +\infty$, in which case $\mathrm{N}_p \big| \mathscr{F}^p$ is a semi-norm. Thus, for $\mathbf{f} : \mathrm{X} \to \mathrm{F}$, the proposed definition

$$\mathrm{N}_p(\mathbf{f}) = \Big(\int^* |\mathbf{f}|^p \, d|\mu|\Big)^{1/p} \qquad (0 < p < +\infty)$$

is consistent with the notations of Ch. I, No. 3.

Fix a function $\mathbf{f} : \mathrm{X} \to \mathrm{F}$. Let us dispose of a trivial special case: Suppose $\mathbf{f}$ is negligible, by which we mean that $\mathrm{N}_1(\mathbf{f}) = \int^* |\mathbf{f}| \, d|\mu| = 0$. {This is the definition suggested by the paragraph before the *Remark* in §3, No. 3 (see §3, No. 2, Def. 1 and Prop. 3); equivalently, $\mathbf{f} = 0$ almost everywhere (§2, No. 3, Th. 1); equivalently, $|\mathbf{f}|^p = 0$ almost everywhere, for $0 < p < +\infty$; equivalently, $\mathrm{N}_p(\mathbf{f}) = 0$ for $0 < p < +\infty$.} Then also $\mathrm{N}_\infty(\mathbf{f}) = 0$: for every $\alpha > 0$, $|\mathbf{f}| \leqslant \alpha$ almost everywhere, hence locally almost everywhere, therefore $\mathrm{N}_\infty(\mathbf{f}) = \inf \alpha = 0$. Thus $\mathrm{I} = [1, +\infty]$

and the function $p \mapsto \mathrm{N}_p(\mathbf{f})$ $(p \in \mathrm{I})$ is identically 0, as its restriction to $\mathrm{J} = [1, +\infty[$.

Assume henceforth that $\mathbf{f}$ is not negligible. By Prop. 5 of Ch. I, No. 3, the set

$$\mathrm{S} = \{1/p : \ 0 < p < +\infty\,,\ \mathrm{N}_p(\mathbf{f}) < +\infty\,\}$$

is either empty or is an interval (possibly degenerate). {This set is called I in the cited Prop. 5, but the letter I has been preempted here for other use.} Now,

$$\mathrm{J} = \{p : \ 1 \leqslant p < +\infty,\ \mathrm{N}_p(\mathbf{f}) < +\infty\,\} = \{p : \ 1/p \in \mathrm{S} \text{ and } 1/p \leqslant 1\,\}\,;$$

thus J is the image of $\mathrm{S} \cap]0, 1]$ under the mapping $s \mapsto 1/s$, that is,

$$\mathrm{J} = \{s^{-1} : \ s \in \mathrm{S} \cap]0, 1]\ ,$$

consequently J is either empty or is an interval (possibly degenerate).

If $\mathrm{J} = \varnothing$ then $\mathrm{S} \cap]0, 1] = \varnothing$, therefore

$$0 < \alpha \leqslant 1 \quad \Rightarrow \quad \alpha \notin \mathrm{S} \quad \Rightarrow \quad \mathrm{N}_{1/\alpha}(\mathbf{f}) = +\infty\,,$$

thus $\mathrm{N}_p(\mathbf{f}) = +\infty$ for all $p \in [1, +\infty[$, whence the set

$$\mathrm{I} = \{p \in [1, +\infty] : \ \mathrm{N}_p(\mathbf{f}) < +\infty\,\}$$

is equal to either $\varnothing$ or $\{+\infty\}$, and the assertions of Prop. 4 hold trivially.

If $\mathrm{J} = \{p\}$ for some $1/p \in]0, 1]$, then $1 \leqslant p < +\infty$, $0 < \mathrm{N}_p(\mathbf{f}) < +\infty$, and the continuity condition is trivially satisfied.

Finally, suppose J is a nondegenerate interval. If $\alpha \in \mathrm{S}$, then $\alpha = 1/p$ with $0 < \mathrm{N}_p(\mathbf{f}) < +\infty$, in particular $\log \mathrm{N}_p(\mathbf{f}) \neq -\infty$ (i.e., $\mathrm{N}_p(f) \neq 0$); so by the cited Prop. 5, we know that the function

$$(*) \qquad \alpha \mapsto \log \mathrm{N}_{1/\alpha}(\mathbf{f}) \qquad (\alpha \in \mathrm{S})$$

is convex on S, hence is continuous at every interior point of S (FRV, I, §4, No. 3, Prop. 6). Indeed, writing $g(\alpha) = \log \mathrm{N}_{1/\alpha}(\mathbf{f})$ $(\alpha \in \mathrm{S})$, the cited Prop. 6 shows that at every interior point of S, g has finite left and right derivatives (whence continuity at the point); and the proof shows that if an endpoint of S belongs to S, then g has a finite 1-sided derivative at that point, hence is continuous on the right or on the left at the point, as the case may be. Thus the mapping $(*)$ is continuous everywhere on S. A fortiori, its restriction

$$(**) \qquad \alpha \mapsto \log \mathrm{N}_{1/\alpha}(\mathbf{f}) \qquad (\alpha \in \mathrm{J}^{-1} = \mathrm{S} \cap]0, 1])$$

is continuous; composing it with the mapping $p \mapsto 1/p$ $(p \in \mathrm{J})$, we see that the mapping

$$(***) \qquad p \mapsto \log \mathrm{N}_p(\mathbf{f}) \quad (p \in \mathrm{J})$$

is continuous; finally, composing with the exponential function, the desired mapping

$$(****) \qquad p \mapsto \mathrm{N}_p(\mathbf{f}) \quad (p \in \mathrm{J})$$

is continuous.

IV.98, ℓ. −3 to −1.

"If $s \in \mathrm{J}$ then, for every finite number $p > s$, $|\mathbf{f}|^p = |\mathbf{f}|^s|\mathbf{f}|^{p-s}$, and the inequality of the mean shows that

$$(12) \qquad \mathrm{N}_p(\mathbf{f}) \leqslant \big(\mathrm{N}_s(\mathbf{f})\big)^{s/p}\big(\mathrm{N}_\infty(\mathbf{f})\big)^{(p-s)/p}."$$

Since $s \in \mathrm{J}$ and $\mathbf{f}$ is not negligible, $0 < \big(\mathrm{N}_s(\mathbf{f})\big)^{s/p} < +\infty$. If $\mathrm{N}_\infty(\mathbf{f}) = +\infty$ the right side of (12) is equal to $+\infty$ and the inequalilty is trivial.

Suppose $\mathrm{N}_\infty(\mathbf{f}) < +\infty$, that is, $+\infty \in \mathrm{I}$, thus $\mathbf{f}$ is bounded in measure. Note that $\mathrm{N}_\infty(\mathbf{f}) > 0$. {For, $\mathrm{N}_\infty(\mathbf{f}) = 0$ would imply that the set $\mathrm{A} = \{x : \mathbf{f}(x) \neq 0\}$ is locally negligible; but $|\mathbf{f}|^p$ is integrable, so $\mathrm{A} \subset \mathrm{N} \cup \bigcup_{n=1}^{\infty} \mathrm{K}_n$ for a suitable negligible set N and compact sets K_n (§5, No. 6, Lemma 1), whence $\mathrm{A} = (\mathrm{A} \cap \mathrm{N}) \cup \bigcup_{n=1}^{\infty} \mathrm{A} \cap \mathrm{K}_n$ is negligible (§5, No. 2, Prop. 5), contrary to the assumption that $\mathbf{f}$ is not negligible.} Thus $0 < \mathrm{N}_\infty(\mathbf{f}) < +\infty$. Since $p - s > 0$, one has

$$\mathrm{N}_\infty(|\mathbf{f}|^{p-s}) = \mathrm{M}_\infty(|\mathbf{f}|^{p-s}) = \big(\mathrm{M}_\infty(|\mathbf{f}|)\big)^{p-s} = \big(\mathrm{N}_\infty(\mathbf{f})\big)^{p-s} < +\infty$$

thus the measurable function $|\mathbf{f}|^{p-s}$ is bounded in measure, whereas $|\mathbf{f}|^s \in \mathscr{L}^1$; by the inequality of the mean (No. 2, Prop. 1), $|\mathbf{f}|^p = |\mathbf{f}|^s|\mathbf{f}|^{p-s}$ is integrable—therefore $p \in \mathrm{J}$—and

$$\int |\mathbf{f}|^p \, d|\mu| \leqslant \mathrm{M}_\infty(|\mathbf{f}|^{p-s}) \int |\mathbf{f}|^s \, d|\mu| ,$$

that is, $\big(\mathrm{N}_p(\mathbf{f})\big)^p \leqslant \big(\mathrm{N}_\infty(\mathbf{f})\big)^{p-s}\big(\mathrm{N}_s(\mathbf{f})\big)^s$, whence the asserted inequality. The argument shows that $p \in \mathrm{J}$ for all $p \in (s, +\infty)$, therefore J must be of the form $]a, +\infty[$ or $[a, +\infty[$, and then $\mathrm{I} = \,]a, +\infty]$ or $[a, +\infty]$.

A corollary of the argument is that if J consists of a single point then $+\infty \notin \mathrm{I}$, therefore $\mathrm{I} = \mathrm{J}$.

For use in the next note, we observe that as $p \to +\infty$, $\big(\mathrm{N}_s(\mathbf{f})\big)^{s/p} \to 1$; and if $\mathrm{N}_\infty(\mathbf{f}) < +\infty$ then $\big(\mathrm{N}_\infty(\mathbf{f})\big)^{(p-s)/p} \to \mathrm{N}_\infty(\mathbf{f})$ as $p \to +\infty$.

IV.99, ℓ. 1, 2.

"Letting p tend to $+\infty$, it follows that

$$\limsup_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_\infty(\mathbf{f})\,.\text{”} \tag{13}$$

The presumption is that J is of the form $]a, +\infty[$ or $[a, +\infty[$, so that $p \to +\infty$ makes sense. If $\mathrm{N}_\infty(\mathbf{f}) = +\infty$, the inequality holds trivially. If $\mathrm{N}_\infty(\mathbf{f}) < +\infty$ then, applying Prop. 11 of GT, IV, §5, No. 6 to the inequality (12), one has

$$\begin{aligned}\limsup_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) &\leqslant \limsup_{p\to+\infty} \Big[\big(\mathrm{N}_s(\mathbf{f})\big)^{s/p}\big(\mathrm{N}_\infty(\mathbf{f})\big)^{(p-s)/p}\Big] \\ &= \lim_{p\to+\infty} \Big[\big(\mathrm{N}_s(\mathbf{f})\big)^{s/p}\big(\mathrm{N}_\infty(\mathbf{f})\big)^{(p-s)/p}\Big] = 1 \cdot \mathrm{N}_\infty(\mathbf{f})\end{aligned}$$

(by the preceding note).

If the right endpoint of J is $< +\infty$, then $\mathrm{N}_\infty(\mathbf{f}) = +\infty$ by the preceding note, so $+\infty \notin \mathrm{I}$ and $\mathrm{I} = \mathrm{J}$.

IV.99, ℓ. 4.

"... thus I is indeed an interval of $\overline{\mathbf{R}}$, and $\overline{\mathrm{I}} = \overline{\mathrm{J}}$."

If $+\infty \notin \mathrm{I}$ then $\mathrm{I} = \mathrm{J}$, whereas if $+\infty \in \mathrm{I}$ then, by the foregoing, J has right endpoint equal to $+\infty$ and $\mathrm{I} = \mathrm{J} \cup \{+\infty\}$; in either case, I is an interval of $\overline{\mathbf{R}}$ and $\overline{\mathrm{I}} = \overline{\mathrm{J}}$.

IV.99, ℓ. 4–6.

"The proposition will be proved if we show that $p \mapsto \mathrm{N}_p(\mathbf{f})$ is continuous on $\overline{\mathrm{J}}$, and it suffices to establish continuity at the end-points of J."

After continuity is proved, the conclusion about convexity on $\overline{\mathrm{J}}$ will be a trivial consequence of the convexity on J already established.

We already know that $p \mapsto \mathrm{N}_p(\mathbf{f})$ is continuous on J; in particular one-sided continuity holds at an endpoint that belongs to J. Assuming the interval J to be nondegenerate, let r, s be its endpoints, $1 \leqslant r < s \leqslant +\infty$. There are three situations where continuity in $\overline{\mathrm{J}}$ remains to be established: (i) $r \notin \mathrm{J}$; (ii) $s < +\infty$ and $s \notin \mathrm{J}$; (iii) $s = +\infty$. In the following analysis, p always signifies an element of J.

(i) If $r \notin \mathrm{J}$, then $\mathrm{N}_r(\mathbf{f}) = +\infty$ and the problem is to show that $\mathrm{N}_p(\mathbf{f}) \to +\infty$ as $p \downarrow r$. The function $1/p \mapsto \log \mathrm{N}_p(\mathbf{f})$ is already known to be convex on J^{-1} (whose endpoints are $1/s < 1/r$, where $1/s = 0$ if $s = +\infty$); with the convention that $\log(+\infty) = +\infty$, the addition of the point $(1/r, +\infty)$ to its graph creates no new chords between pairs of points to challenge convexity.

(ii) If $s < +\infty$ and $s \notin \mathrm{J}$, then $\mathrm{N}_s(f) = +\infty$ and the problem is to show that $\mathrm{N}_p(\mathbf{f}) \to +\infty$ as $p \uparrow s$, with the same conclusion concerning convexity.

(iii) When $s = +\infty$, of course $s \notin \mathrm{J}$; and J^{-1} has endpoints $0 < 1/r$. Whether $+\infty \in \mathrm{I}$ or $+\infty \notin \mathrm{I}$, that is, $\mathrm{N}_\infty(\mathbf{f}) < +\infty$ or $\mathrm{N}_\infty(\mathbf{f}) = +\infty$, the problem is to show that $\mathrm{N}_p(\mathbf{f}) \to \mathrm{N}_\infty(\mathbf{f})$ as $p \uparrow +\infty$. It has already been shown that

$$\limsup_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_\infty(\mathbf{f})\,;$$

showing that

$$\liminf_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) \geqslant \mathrm{N}_\infty(\mathbf{f})$$

will establish that

$$\mathrm{N}_\infty(\mathbf{f}) \leqslant \liminf_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) \leqslant \limsup_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_\infty(\mathbf{f})\,,$$

whence the continuity of $p \mapsto \mathrm{N}_p(\mathbf{f})$ at $+\infty$. There remains the question of convexity in this case. We have $\mathrm{J} =]r, +\infty[$ or $[r, +\infty[$, therefore $\mathrm{J}^{-1} =]0, 1/r[$ or $]0, 1/r]$. If $\mathrm{N}_\infty(\mathbf{f}) = +\infty$ then $\log \mathrm{N}_\infty(\mathbf{f}) = +\infty$ and the addition of the point $(0, +\infty)$ to the graph of the function $\sigma(1/p) = \log \mathrm{N}_p(\mathbf{f})$ ($1/p \in \mathrm{J}^{-1}$) produces no new chords, so all is well with convexity. Otherwise, $0 < \mathrm{N}_\infty(\mathbf{f}) < +\infty$ and one adds the 'finite' point $(0, \log \mathrm{N}_\infty(\mathbf{f}))$ to the graph of σ, which does introduce new chords, namely the ones with left endpoint $(0, \log \mathrm{N}_\infty(\mathbf{f}))$. Suppose $0 < 1/t < 1/p$ and consider the chord C joining $(0, \log \mathrm{N}_\infty(\mathbf{f}))$ and $(1/p, \log \mathrm{N}_p(\mathbf{f}))$; we wish to show that the point $(1/t, \log \mathrm{N}_t(\mathbf{f}))$ lies on or below C. Let t' be such that $0 < 1/t' < 1/t < 1/p$ and let C′ be the chord joining $(1/t', \log \mathrm{N}_{t'}(\mathbf{f}))$ and $(1/p, \log \mathrm{N}_p(\mathbf{f}))$. By the convexity of σ on J^{-1}, we know that $(1/t, \log \mathrm{N}_t(\mathbf{f}))$ lies on or below C′; as t' converges to t, the chord C′ 'converges' to C, so in the limit $(1/t, \log \mathrm{N}_t(\mathbf{f}))$ lies on or below C. (One can, tediously, say it with inequalities.)

To summarize (assuming J a nondegenerate interval and $\mathbf{f}$ non-negligible): once $\lim \mathrm{N}_p(\mathbf{f})$ has been calculated at the points of the set $\overline{\mathrm{J}} - \mathrm{J}$ (a set with at most two elements, at least one of them finite), the issue of convexity of the mapping $1/p \mapsto \log \mathrm{N}_p(\mathbf{f})$ ($p \in \overline{\mathrm{I}} = \overline{\mathrm{J}}$)—in other words the convexity of $t \mapsto \log \mathrm{N}_{1/t}(\mathbf{f})$ on the closure of J^{-1}—is straightforward, and adds to what we know about J only in the case that J has right endpoint $+\infty$ and $\mathrm{N}_\infty(\mathbf{f}) < +\infty$.

IV.99, ℓ. 14.

" $\int |\mathbf{f}|^p \varphi_A \, d|\mu|$ tends to $\int |\mathbf{f}|^r \varphi_A \, d|\mu|$ (§4, No. 3, Prop. 4)."

When $p > r$, $|\mathbf{f}|^r \varphi_A \leqslant |\mathbf{f}|^p \varphi_A \in \mathscr{L}^1$ assures that $|\mathbf{f}|^r \varphi_A \in \mathscr{L}^1$.

IV.99, ℓ. 15.

" Therefore $\int |\mathbf{f}|^p \, d|\mu|$ tends to $\int^* |\mathbf{f}|^r \, d|\mu|$ "

We need only consider the case that $r \notin J$, so that $\int^* |\mathbf{f}|^r \, d|\mu| = +\infty$. Then

$$\int |\mathbf{f}|^p \, d|\mu| = \int |\mathbf{f}|^p \varphi_A \, d|\mu| + \int |\mathbf{f}|^p \varphi_{\complement A} \, d|\mu| \;\to\; \int |\mathbf{f}|^r \varphi_A \, d|\mu| + \int^* |\mathbf{f}|^r \varphi_{\complement A} \, d|\mu|$$

by the preceding Note, Th. 3 of §1, No. 3, and the continuity of addition in $[0, +\infty]$ (GT, IV, §4, No. 3, Prop. 7), and $\int^* |\mathbf{f}|^r \varphi_{\complement A} \, d|\mu| = +\infty$ follows from the fact that $|\mathbf{f}|^r$ is not integrable.

IV.99, ℓ. 15, 16.

"... which proves the continuity of $p \mapsto N_p(\mathbf{f})$ at r."

When $r \in J$ this is not news; assuming $r \notin J$, so that $N_r(\mathbf{f}) = +\infty$, we are to show that $N_p(\mathbf{f}) \to +\infty$ as $p \to r$. To simplify the notation, let us write $a_p = \int |\mathbf{f}|^p \, d|\mu|$ $(p \in J)$. We know that $a_p \to \int^* |\mathbf{f}|^r \, d|\mu| = +\infty$. It follows that $\log a_p \to +\infty$ (given any real number $M > 0$, $a_p > e^M$ for p sufficiently near r). As $p \to r$ one has $1/p \to 1/r$, therefore

$$\frac{1}{p} \log a_p \to \frac{1}{r} \cdot (+\infty) = +\infty$$

(GT, IV, §4, No. 3, Prop. 8), whence $(a_p)^{1/p} \to +\infty$, that is, $N_p(\mathbf{f}) \to +\infty$.

IV.99, ℓ. 17.

"The same reasoning may be applied at the point s if $s < +\infty$."

In particular, the roles of A and $\complement A$ are reversed: when $p < s$, $|\mathbf{f}|^s \varphi_{\complement A} \leqslant |\mathbf{f}|^p \varphi_{\complement A} \in \mathscr{L}^1$ assures that $|\mathbf{f}|^s \varphi_{\complement A} \in \mathscr{L}^1$.

IV.99, ℓ. −12.

"... let a be a number such that $0 < a < N_\infty(\mathbf{f})$."

Since $J \neq \varnothing$ we know that $N_\infty(\mathbf{f}) > 0$ (see the note for IV.98, ℓ. −3 to −1). {It is not enough to note that $\mathbf{f}$ is not negligible; it might still be locally negligible—that is, $N_\infty(\mathbf{f}) = 0$—but not when $J \neq \varnothing$.}

IV.99, ℓ. −10.

"... non-negligible"

Since $N_\infty(\mathbf{f})$ is the infimum of the numbers $M > 0$ such that $|\mathbf{f}| \leqslant M$ locally almost everywhere, it is not the case that $|\mathbf{f}| < a$ locally almost everywhere. That is, the set $\mathrm{A} = \{f :\ |\mathbf{f}(x)| \geqslant a\}$ is not locally negligible, hence is not negligible.

IV.99, ℓ. −7.

"... which completes the proof."

Letting $a \to \mathrm{N}_\infty(\mathbf{f})$, one has $\liminf\limits_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) \geqslant \mathrm{N}_\infty(\mathbf{f})$. This completes the proof of $\lim\limits_{p\to+\infty} \mathrm{N}_p(\mathbf{f}) = \mathrm{N}_\infty(\mathbf{f})$ and hence of the Proposition (see the note for ℓ. 4–6).

IV.99, ℓ. −6 to −4.

"COROLLARY."

By assumption, $1 \leqslant r < p < s \leqslant +\infty$ and $\mathbf{f} \in \mathscr{L}^r_{\mathrm{F}} \cap \mathscr{L}^s_{\mathrm{F}}$; we are to show that $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$, and can suppose that $\mathbf{f}$ is not negligible.

From $\mathbf{f} \in \mathscr{L}^r_{\mathrm{F}}$ we know that $\mathrm{N}_r(\mathbf{f}) < +\infty$, so that r belongs to the interval J of the proof of Prop. 4; if $s < +\infty$ then similarly $s \in \mathrm{J}$, therefore $\mathrm{N}_p(\mathbf{f}) < +\infty$ for all $p \in [r, s]$. On the other hand, if $s = +\infty$ then $\mathbf{f} \in \mathscr{L}^\infty_{\mathrm{F}}$ is bounded in measure, $\mathrm{N}_\infty(\mathbf{f}) < +\infty$; since J is nonempty, the proof of Prop. 4 shows that J is an interval with right endpoint $+\infty$, consequently $\mathrm{N}_p(\mathbf{f}) < +\infty$ for every $p \in [r, +\infty[$.

IV.100, ℓ. 9, 10.

"This is an immediate consequence of Prop. 4 above and of the Cor. of Prop. 4 of Ch. I, No. 3."

We may write simply $\|\mu\| = |\mu|(\mathrm{X}) = \int d|\mu|$ (§4, No. 7, Prop. 12). Since μ is bounded, every measurable set in X is integrable (§5, No. 6, Cor. 1 of Th. 5). It follows that every locally negligible set A in X is negligible; for, A is measurable (§5, No. 2, sentence after Def. 3), hence integrable, therefore $\mathrm{A} = \mathrm{N} \cup \bigcup\limits_{n=1}^{\infty} \mathrm{K}_n$ with N negligible and the K_n compact (§4, No. 6, Cor. 2 of Th. 4), and the $\mathrm{K}_n = \mathrm{A} \cap \mathrm{K}_n$ are also negligible (§5, No. 2, Prop. 5). If $\mathrm{N}_\infty(\mathbf{f}) < +\infty$, then $\mathrm{N}_\infty(\mathbf{f})$ may be described as the infimum of all real numbers $\alpha > 0$ such that $|\mathbf{f}| \leqslant \alpha$ almost everywhere, and $\mathrm{N}_\infty(\mathbf{f}) = 0$ means that $\mathbf{f} = 0$ almost everywhere (Nos. 2, 3). If $\mathrm{N}_\infty(\mathbf{f}) < +\infty$ then, for all $p \in [1, +\infty[$, $|\mathbf{f}|^p \leqslant \big(\mathrm{N}_\infty(\mathbf{f})\big)^p$ almost everywhere,

$$\int |\mathbf{f}|^p \, d|\mu| \leqslant \big(\mathrm{N}_\infty(\mathbf{f})\big)^p \int d|\mu|\,,$$

whence $\mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_\infty(\mathbf{f}) \cdot \|\mu\|^{1/p} < +\infty$.

As in Ch. I, No. 1, write P for the set of all functions $f : \mathrm{X} \to \mathbf{R}$ such that $f \geqslant 0$, and define

$$M(f) = \|\mu\|^{-1} |\mu|^*(f) \qquad (f \in \mathrm{P})\,;$$

then M satisfies the conditions $1°, 2°, 3°$ of Ch. I, No. 1 (§1, No. 3, Props. 10, 11, 12), and $M(1) = \|\mu\|^{-1}|\mu|^*(1) = 1$. For $0 < p < +\infty$ and any function $\mathbf{f} : \mathrm{X} \to \mathrm{F}$, define

$$\mathrm{N}'_p(\mathbf{f}) = \big(M(|\mathbf{f}|^p)\big)^{1/p}$$

(cf. the note for IV.98, ℓ. -10 to -7); since $M(\alpha f) = \alpha M(f)$ for $f \in \mathrm{P}$ and scalars $\alpha > 0$, we have

$$(*) \qquad \mathrm{N}'_p(\mathbf{f}) = \big(\|\mu\|^{-1}|\mu|^*(|\mathbf{f}|^p)\big)^{1/p} = \|\mu\|^{-1/p}\mathrm{N}_p(\mathbf{f}),$$

which suggests the definition $\mathrm{N}'_\infty(\mathbf{f}) = \mathrm{N}_\infty(\mathbf{f})$ (formally 'let $p \to \infty$ in $(*)$'). Since $M(1) = 1$, for every function $\mathbf{f} : \mathrm{X} \to \mathrm{F}$ the mapping $p \mapsto \mathrm{N}'_p(\mathbf{f})$ $(0 < p < +\infty)$ is increasing (Ch. I, No. 3, Cor. of Prop. 4). {Review: The argument given in Ch. I shows that if $0 < r < p$ and $\mathrm{N}'_p(\mathbf{f}) < +\infty$, then $\mathrm{N}'_r(\mathbf{f}) \leqslant \mathrm{N}'_p(\mathbf{f})$, whence $p \mapsto \mathrm{N}'_p(\mathbf{f})$ is finite-valued and increasing on the interval $[r, p]$. It follows that if $0 < r < p < +\infty$ then $\mathrm{N}'_r(\mathbf{f}) \leqslant \mathrm{N}'_p(\mathbf{f})$: trivial if $\mathrm{N}'_p(\mathbf{f}) = +\infty$, and true by the foregoing if $\mathrm{N}'_p(\mathbf{f}) < +\infty$.}

Consider now the assertion of Prop. 5 (in particular, $\mathbf{f}$ is measurable): let I be the set of all $p \in [1, +\infty]$ such that $\mathrm{N}_p(\mathbf{f}) < +\infty$, equivalently, $\mathrm{N}'_p(\mathbf{f}) < +\infty$, and assume that $\mathrm{I} \neq \varnothing$; we are to show that I is an interval and the function $p \mapsto \mathrm{N}'_p(\mathbf{f})$ $(p \in \mathrm{I})$ is increasing. We can suppose that $\mathbf{f}$ is not negligible, equivalently, $\mathrm{N}_\infty(\mathbf{f}) > 0$.

If $\mathrm{N}_\infty(\mathbf{f}) < +\infty$, that is, $\mathbf{f} \in \mathscr{L}^\infty_\mathrm{F}$, then, as observed in the first paragraph of the note, $\mathbf{f} \in \mathscr{L}^p_\mathrm{F}$ for all $p \in [1, +\infty[$. Thus $\mathrm{I} = [1, +\infty]$. We know that $p \mapsto \mathrm{N}'_p(\mathbf{f})$ is finite-valued and increasing on $[1, +\infty[$; it remains to show that if $1 \leqslant p < +\infty$ then $\mathrm{N}'_p(\mathbf{f}) \leqslant \mathrm{N}'_\infty(\mathbf{f})$. Indeed, $|\mathbf{f}|^p \leqslant \big(\mathrm{N}_\infty(\mathbf{f})\big)^p \cdot 1$ almost everywhere, therefore

$$\int |\mathbf{f}|^p \, d|\mu| \leqslant \big(\mathrm{N}_\infty(\mathbf{f})\big)^p \cdot \|\mu\|,$$

whence $\mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_\infty(\mathbf{f}) \cdot \|\mu\|^{1/p}$ and so

$$\mathrm{N}'_p(\mathbf{f}) = \|\mu\|^{-1/p}\mathrm{N}_p(\mathbf{f}) \leqslant \mathrm{N}_\infty(\mathbf{f}) = \mathrm{N}'_\infty(\mathbf{f}).$$

Finally, suppose $\mathrm{N}_\infty(\mathbf{f}) = +\infty$, so that

$$\mathrm{I} = \{p \in [1, +\infty[: \ \mathrm{N}'_p(\mathbf{f}) < +\infty\}.$$

By assumption, $\mathrm{I} \neq \varnothing$; let $p \in \mathrm{I}$. We know from the discussion above that $[1, p[\subset \mathrm{I}$, thus I is an interval (it is the interval J of the proof of Prop. 4) with left endpoint 1, conceivably degenerating to the single point 1, on which the function $p \mapsto \mathrm{N}'_p(\mathbf{f})$ is increasing (Ch. I, No. 3, Cor. of Prop. 4).

IV.100, ℓ. 11–13.

"Corollary."

The corollary is valid for $1 \leqslant r < s \leqslant +\infty$, except that when $s = +\infty$ the topology on $\mathscr{L}^{\infty}_{\mathrm{F}}$ defined by the semi-norm N_{∞} is the topology of 'uniform convergence locally almost everywhere' (No. 3).

For a measurable function $\mathbf{f} : \mathrm{X} \rightarrow \mathrm{F}$ let us write $\mathrm{I}_{\mathbf{f}}$ for the set of all $p \in [1, +\infty]$ such that $\mathrm{N}_p(\mathbf{f}) < +\infty$. Suppose $\mathbf{f} \in \mathscr{L}^s_{\mathrm{F}}$, that is, $s \in \mathrm{I}_{\mathbf{f}}$. By Prop. 5, $\mathrm{I}_{\mathbf{f}}$ is an interval with left endpoint 1, therefore $[1, s] \subset \mathrm{I}_{\mathbf{f}}$ and in particular $r \in \mathrm{I}_{\mathbf{f}}$, that is, $\mathbf{f} \in \mathscr{L}^r_{\mathrm{F}}$. Thus $\mathscr{L}^s_{\mathrm{F}} \subset \mathscr{L}^r_{\mathrm{F}}$. Moreover, for a fixed measurable function $\mathbf{f}$, if $\mathrm{I}_{\mathbf{f}} \neq \varnothing$ then, with notation as in the preceding note, the function

$$p \mapsto \mathrm{N}'_p(\mathbf{f}) = \|\mu\|^{-1/p}\mathrm{N}_p(\mathbf{f}) \qquad (\, p \in \mathrm{I}_{\mathbf{f}}\,)$$

is increasing. In particular (for the given r and s), $\mathrm{N}'_r(\mathbf{f}) \leqslant \mathrm{N}'_s(\mathbf{f})$ when $\mathbf{f} \in \mathscr{L}^s_{\mathrm{F}}$, that is,

$$(*) \qquad \|\mu\|^{-1/r}\mathrm{N}_r(\mathbf{f}) \leqslant \|\mu\|^{-1/s}\mathrm{N}_s(\mathbf{f})$$

(when $s = +\infty$, the convention is that $1/s = 0$ and the inequality reads $\|\mu\|^{-1/r}\mathrm{N}_r(\mathbf{f}) \leqslant 1 \cdot \mathrm{N}_{\infty}(\mathbf{f}) = \mathrm{N}'_{\infty}(\mathbf{f})$). Writing the inequality $(*)$ as

$$\mathrm{N}_r(\mathbf{f}) \leqslant \left(\frac{\|\mu\|^{-1/s}}{\|\mu\|^{-1/r}}\right)\mathrm{N}_s(\mathbf{f}) \qquad \text{for all } \mathbf{f} \in \mathscr{L}^s_{\mathrm{F}}\,,$$

we see that the canonical injection $\mathscr{L}^s_{\mathrm{F}} \rightarrow \mathscr{L}^r_{\mathrm{F}}$ is continuous for the respective semi-norm topologies; in other words, the topology on $\mathscr{L}^s_{\mathrm{F}}$ induced by the N_r-topology on $\mathscr{L}^r_{\mathrm{F}}$ is coarser than the N_s-topology.

Incidentally, $\mathscr{K}_{\mathrm{F}} \subset \mathscr{L}^p_{\mathrm{F}}$ for all $p \in [1, +\infty]$, and the normed space $\mathrm{L}^p_{\mathrm{F}}$ derived from the semi-norm N_p is a Banach space (No. 3, Prop. 2 and §3, No. 4, Th. 2).

IV.100, ℓ. −13.

" $\mathrm{N}_{\infty}(\mathbf{f}) = \|\mathbf{f}\| = \sup_{x \in \mathrm{X}} |\mathbf{f}(x)|$ "

Every function $\mathbf{f} : \mathrm{X} \rightarrow \mathrm{F}$ is continuous, hence measurable, and the empty set is the only negligible set, hence the only locally negligible set.

IV.100, ℓ. −13 to −11.

"... if there exists a number $\alpha > 0$ such that $|\mathbf{f}(x)| \geqslant \alpha$ for infinitely many values of $x \in \mathrm{X}$, then $\mathrm{N}_p(\mathbf{f}) = +\infty$ for every finite p "

In this case $\mathrm{I} = \varnothing$ or $\{+\infty\}$ according as $|\mathbf{f}|$ is unbounded or bounded.

IV.100, ℓ. -11 to -8.

" in the contrary case, there exists an $x_0 \in \mathrm{X}$ such that $|\mathbf{f}(x_0)| = \|\mathbf{f}\|$, whence

$$\mathrm{N}_\infty(\mathbf{f}) = |\mathbf{f}(x_0)| \leqslant \mathrm{N}_p(\mathbf{f})$$

for every finite p."

The contrary case: for every $\alpha > 0$, $|\mathbf{f}(x)| < \alpha$ for all but finitely many x; since the compact sets are the finite sets, this means that the (continuous) function $\mathbf{f}$ "tends to $\mathbf{0}$ at infinity", that is, $\mathbf{f} \in \mathscr{C}^0(\mathrm{X};\mathrm{F})$ (cf. Ch. III, §1, No. 2, Prop. 3). In particular, $\mathbf{f}$ is bounded.

Choose $y \in \mathrm{X}$ with $\mathbf{f}(y) \neq 0$. The set $\mathrm{A} = \{x : |\mathbf{f}(x)| \geqslant |\mathbf{f}(y)|\}$ is finite; if $x_0 \in \mathrm{A}$ is chosen so that $|\mathbf{f}(x_0)| = \max\{|\mathbf{f}(x)| : x \in \mathrm{A}\}$, then $\mathrm{N}_\infty(\mathbf{f}) = |\mathbf{f}(x_0)|$.

For every finite $p \geqslant 1$, $\big(\mathrm{N}_\infty(\mathbf{f})\big)^p = |\mathbf{f}(x_0)|^p \leqslant \big(\mathrm{N}_p(\mathbf{f})\big)^p$, whence $\mathrm{N}_\infty(\mathbf{f}) \leqslant \mathrm{N}_p(\mathbf{f})$.

If $\mathrm{N}_p(\mathbf{f}) = +\infty$ for all finite $p \geqslant 1$, then $\mathrm{I} = \{+\infty\}$ (an example is given at the end of this note).

Otherwise, there exists a finite $p \geqslant 1$ such that $\mathrm{N}_p(\mathbf{f}) < +\infty$, in which case I is a nondegenerate interval with right end-point $+\infty$ and left end-point r, $1 \leqslant r < +\infty$ (Prop. 4). Then $\overline{\mathrm{I}} = [r, +\infty]$, and $\overline{\mathrm{I}}^{-1} = [0, 1/r]$. We know from Prop. 4 that the function

$$g(\alpha) = \log \mathrm{N}_{1/\alpha}(\mathbf{f}) \qquad (\, \alpha \in [0, 1/r]\,)$$

is convex. Note that $g(0) = \log \mathrm{N}_\infty(\mathbf{f}) < +\infty$, whereas $g(1/r) = \log \mathrm{N}_r(\mathbf{f})$ is finite or equal to $+\infty$ according as $r \in \mathrm{I}$ or $r \notin \mathrm{I}$.

(i) Consider first the case that $r \in \mathrm{I}$, so that $\mathrm{I} = \overline{\mathrm{I}} = [r, +\infty]$, and g is a finite-valued convex function on $[0, 1/r]$. We know that for $r \leqslant p < +\infty$, $0 < \mathrm{N}_\infty(\mathbf{f}) \leqslant \mathrm{N}_p(\mathbf{f})$, whence

$$g(0) = \log \mathrm{N}_\infty(\mathbf{f}) \leqslant \log \mathrm{N}_p(\mathbf{f}) = g(1/p)\,,$$

that is, $g(0) \leqslant g(\alpha)$ for all $\alpha \in \,]0, 1/r]$; thus g takes its smallest value at the left end-point 0. Let us show that g is an increasing function. For $0 \leqslant \alpha, \beta \leqslant 1/r$, $\alpha \neq \beta$, write

$$\mathrm{M}_{\alpha\beta} = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = \mathrm{M}_{\beta\alpha}$$

for the slope of the chord of the graph of g joining the points $\big(\alpha, g(\alpha)\big)$ and $\big(\beta, g(\beta)\big)$. Since g is convex, for every $\alpha \in [0, 1/r]$ the function

$$(*) \qquad \gamma \mapsto \mathrm{M}_{\alpha\gamma} \qquad (\, \gamma \in [0, 1/r] - \{\alpha\}\,)$$

is increasing (FRV, I, §4, No. 3, Prop. 5). Suppose $0 \leqslant \alpha < \beta \leqslant 1/r$. If $\alpha = 0$, then

$$\mathrm{M}_{0\beta} = \frac{g(\beta) - g(0)}{\beta} \geqslant 0$$

by the minimality of $g(0)$, thus $g(0) \leqslant g(\beta)$ for all $\beta \in \,]0, 1/r]$. Whereas, if $0 < \alpha < \beta \leqslant 1/r$, necessarily $g(\alpha) \leqslant g(\beta)$; otherwise, $g(\alpha) > g(\beta)$ would imply that $\mathrm{M}_{\alpha\beta} < 0$, which, along with $\mathrm{M}_{\alpha 0} \geqslant 0$, would contradict the fact that the mapping $(*)$ is increasing.

(ii) Consider now the general case. If $r \in \mathrm{I}$ then g is increasing on I^{-1} by the foregoing. Suppose $r \notin \mathrm{I}$, that is, $g(1/r) = +\infty$. Fix a number γ, $0 < \gamma < 1/r$; the restriction of g to $[0, \gamma]$ is convex, therefore g is increasing by the argument of case (i). But for every such γ, $g(\gamma) < +\infty = g(1/r)$; thus g is increasing on $[0, 1/r] = \bar{\mathrm{I}}^{-1}$.

We have shown that if $\mathrm{I} \neq \varnothing$ and $\mathrm{I} \neq \{+\infty\}$ (equivalently, there exists a finite $p \geqslant 1$ such that $\mathrm{N}_p(\mathbf{f}) < +\infty$; see the Proposition below), then $\mathbf{f} \to 0$ at infinity, I is a nondegenerate interval with right end-point equal to $+\infty$, $\bar{\mathrm{I}} = [r, +\infty]$ with $1 \leqslant r < +\infty$, and the function

$$\alpha \mapsto \log \mathrm{N}_{1/\alpha}(\mathbf{f}) \qquad (\,\alpha \in \bar{\mathrm{I}}^{-1} = [0, 1/r]\,)$$

(convex by Prop. 4) is increasing. In particular, as $p \in \bar{\mathrm{I}}$ increases, $\alpha = 1/p$ decreases, therefore $\log \mathrm{N}_p(\mathbf{f})$ decreases, therefore $\mathrm{N}_p(\mathbf{f})$ decreases; thus $\mathrm{N}_p(\mathbf{f})$ is a decreasing function of $p \in \bar{\mathrm{I}}$, attaining its minimum at $p = +\infty$. ◊

Proposition. Let X be a discrete space, μ a measure on X, F a Banach space, $\mathbf{f} : \mathrm{X} \to \mathrm{F}$, $\mathbf{f} \neq \mathbf{0}$, and I the set of all $p \in [1, +\infty]$ such that $\mathrm{N}_p(\mathbf{f}) < +\infty$. The following conditions are equivalent:

(a) $\mathrm{I} \neq \varnothing$ and $\mathrm{I} \neq \{+\infty\}$;

(b) there exists a finite $p \geqslant 1$ such that $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$;

(c) $\mathbf{f} \to \mathbf{0}$ at infinity and $\mathrm{I} \neq \{+\infty\}$.

Proof. (a) ⇒ (b): Obvious.

(b) ⇒ (c): At any rate $p \in \mathrm{I}$, so $\mathrm{I} \neq \{+\infty\}$. If one did not have $\mathbf{f} \to \mathbf{0}$ at infinity, there would exist an $\varepsilon > 0$ such that $|\mathbf{f}(x)| \geqslant \varepsilon$ for infinitely many values of x, whence $\sum_{x \in \mathrm{X}} |\mathbf{f}(x)|^p = +\infty$, contrary to $\mathbf{f} \in \mathscr{L}^p_{\mathrm{F}}$.

(c) ⇒ (a): Since $\mathbf{f} \to \mathbf{0}$ at infinity $|\mathbf{f}|$ is bounded, that is, $+\infty \in \mathrm{I}$; thus $\mathrm{I} \neq \varnothing$, whereas $\mathrm{I} \neq \{+\infty\}$ by assumption. ◊

The following example shows that the condition $\mathrm{I} \neq \{+\infty\}$ in (c) cannot be omitted:

A positive-term sequence (c_n) *such that* $c_n \to 0$ *but* (c_n) *is not p-summable for any finite* $p \geqslant 1$. Here X is the set of positive integers with the discrete topology, μ is the atomic (hence discrete) measure defined by a mass $+1$ at every point of X (Ch. III, §1, No. 3, *Example* I), and a function on X is displayed as a sequence.

Let $c_n = \dfrac{1}{\log n}$. If $0 < p < +\infty$, then $c_n^p > \dfrac{1}{n}$ for all sufficiently large n, therefore $\sum\limits_{i=1}^{\infty} c_n^p = +\infty$. Thus $\mathrm{I} = \{+\infty\}$ for this sequence. {This example is given in Konrad Knopp's *Infinite sequences and series* (Dover, New York, 1956), p. 60, Example 7 of 3.2.1; and *Theory and application of infinite series* (Hafner, New York, 1951), p. 119, Example g).}

A sequence for which $\mathrm{I} = \varnothing$: Interlace (c_n) with any unbounded sequence.

A sequence for which $\mathrm{I} = [1, +\infty]$: Any absolutely summable sequence.

IV.100, ℓ. -4 to -1.

"Corollary." (of Prop. 6)

The corollary is valid for $1 \leqslant r < s \leqslant +\infty$, except that when $s = +\infty$ the topology on $\mathscr{L}_{\mathrm{F}}^{\infty}$ defined by the semi-norm N_{∞} is the topology of uniform convergence.

Let $1 \leqslant r < s \leqslant +\infty$ and let $f \in \mathscr{L}_{\mathrm{F}}^{r}$. In the notation of Prop. 6, $r \in \mathrm{I}$. By Prop. 6, $[r, +\infty] \subset \mathrm{I}$, hence also $s \in \mathrm{I}$, that is, $\mathbf{f} \in \mathscr{L}_{\mathrm{F}}^{s}$; thus $\mathscr{L}_{\mathrm{F}}^{r} \subset \mathscr{L}_{\mathrm{F}}^{s}$. Moreover, $r < s$ implies by Prop. 6 that $\mathrm{N}_r(\mathbf{f}) \geqslant \mathrm{N}_s(\mathbf{f})$, therefore the canonical injection $\mathscr{L}_{\mathrm{F}}^{r} \to \mathscr{L}_{\mathrm{F}}^{s}$ is continuous for the respective semi-norm topologies; in other words, the topology on $\mathscr{L}_{\mathrm{F}}^{r}$ induced by the N_s-topology of $\mathscr{L}_{\mathrm{F}}^{s}$ is coarser than the N_r-topology.

§7. BARYCENTERS

IV.101, ℓ. 7, 8.

"... the integral $\int \mathbf{x}\, d\mu(\mathbf{x})$ is therefore defined and is an element of E'^* (Ch. III, §3, No. 1)."

Write $i_{\mathrm{K}} : \mathrm{K} \to \mathrm{E}$ for the canonical injection; then $i_{\mathrm{K}} \in \mathscr{K}(\mathrm{K}; \mathrm{E}) \subset \widetilde{\mathscr{K}}(\mathrm{K}; \mathrm{E})$ and, by definition, $\int i_{\mathrm{K}}\, d\mu = \int i_{\mathrm{K}}(\mathbf{x})\, d\mu(\mathbf{x}) = \int \mathbf{x}\, d\mu(\mathbf{x})$ is the unique element $\mathbf{z} \in \mathrm{E}'^*$ such that, for all $\mathbf{x}' \in \mathrm{E}'$,

$$\begin{aligned}\langle \mathbf{z}, \mathbf{x}' \rangle &= \int \langle i_{\mathrm{K}}(\mathbf{x}), \mathbf{x}' \rangle\, d\mu(\mathbf{x}) = \int \langle \mathbf{x}, \mathbf{x}' \rangle\, d\mu(\mathbf{x}) \\ &= \int (\mathbf{x}' | \mathrm{K})\, d\mu = \mu(\mathbf{x}' | \mathrm{K}) = \mu(\mathbf{x}' \circ i_{\mathrm{K}}).\end{aligned}$$

See also the note for III.33, ℓ. 6–11.

IV.101, ℓ. 8, 9.

"Moreover, on K, the topology induced by the weak topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ is identical with the original topology."

As $\mathrm{K} \subset \mathrm{E}$, the assertion entails the identification of E as a linear subspace of E'^*: for $\mathbf{x} \in \mathrm{E}$ write $\mathbf{x}^*$ for the linear form on E' defined by $\mathbf{x}^*(\mathbf{x}') = \mathbf{x}'(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}' \rangle$ for all $\mathbf{x}' \in \mathrm{E}'$, and define the mapping (linear and injective) $\theta : \mathrm{E} \to \mathrm{E}'^*$ by $\theta(\mathbf{x}) = \mathbf{x}^*$ $(\mathbf{x} \in \mathrm{E})$.

The topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$ on E'^* is the initial topology for the family of 'evaluation mappings'

$$\text{(i)} \qquad f \mapsto f(\mathbf{x}') \qquad (f \in \mathrm{E}'^*)$$

indexed by $\mathbf{x}' \in \mathrm{E}'$. The topology on $\theta(\mathrm{E})$ induced by $\sigma(\mathrm{E}'^*, \mathrm{E}')$ is the initial topology for the canonical injection $\theta(\mathrm{E}) \to \mathrm{E}'^*$, which is, by the 'transitivity of initial topologies' (GT, I, §2, No. 3, Prop. 5), the initial topology for the family of mappings

$$\text{(ii)} \qquad \mathbf{x}^* \mapsto \mathbf{x}^*(\mathbf{x}') = \langle \mathbf{x}, \mathbf{x}' \rangle \qquad (\mathbf{x} \in \mathrm{E})$$

indexed by $\mathbf{x}' \in \mathrm{E}'$—which is, in turn, the topology $\sigma\big(\theta(\mathrm{E}), \mathrm{E}'\big)$; concisely,

$$\sigma(\mathrm{E}'^*, \mathrm{E}') \cap \theta(\mathrm{E}) = \sigma\big(\theta(\mathrm{E}), \mathrm{E}'\big).$$

By 'topology on E induced by $\sigma(\mathrm{E}'^*, \mathrm{E}')$ is meant the initial topology for the mapping $\theta : \mathrm{E} \to \mathrm{E}'^*$, where E'^* is equipped with the topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$, that is (transitivity again), the initial topology for the family of composed mappings

$$\text{(iii)} \qquad \mathbf{x} \mapsto \mathbf{x}^* \mapsto \mathbf{x}^*(\mathbf{x}') = \langle \mathbf{x}, \mathbf{x}' \rangle \qquad (\mathbf{x} \in \mathrm{E})$$

indexed by $\mathbf{x}' \in \mathrm{E}'$, which is precisely the weakened topology $\sigma(\mathrm{E}, \mathrm{E}')$ on E; after identifying E and $\theta(\mathrm{E})$ as vector spaces, we may write

$$\sigma(\mathrm{E}'^*, \mathrm{E}') \cap \mathrm{E} = \sigma(\mathrm{E}, \mathrm{E}').$$

Thus the topology induced on E by $\sigma(\mathrm{E}'^*, \mathrm{E}')$ is coarser (i.e., 'weaker') than the original topology, therefore the topology on K induced by $\sigma(\mathrm{E}'^*, \mathrm{E}')$ is coarser than its original topology; since K is compact for its original topology, and Hausdorff for $\sigma(\mathrm{E}'^*, \mathrm{E}') \cap \mathrm{K}$, the two topologies are equal (GT, I, §9, No. 4, Cor. 3 of Th. 2).

{See also the note for III.38, ℓ. 8.}

IV.101, ℓ. 9–12.

"... if C is the closed convex envelope of K in E'^* equipped with $\sigma(E'^*, E')$, then $C \cap E$ is the closed convex envelope of K in E for the original topology (or for the weakened topology $\sigma(E, E')$)."

Write $\mathscr{T}$ for the original topology on E, and $\mathscr{T}^*$ for the topology $\sigma(E'^*, E')$. Let:

1) A be the convex envelope of K (in E), and

2) B the closure of A in E for $\mathscr{T}$; thus B is the closed convex envelope of K in E for $\mathscr{T}$ (TVS, II, §2, No. 6, Prop. 14), equivalently, for the weakened topology $\sigma(E, E')$ (TVS, IV, §1, No. 2, Prop. 2).

As shown in the preceding note, viewing E as a linear subspace of E'^*, the induced topology $\mathscr{T}^* \cap E$ coincides with $\sigma(E, E')$.

Now, C is the closure of A in E'^* for $\mathscr{T}^*$; we are to show that $C \cap E = B$. From $A \subset E \subset E'^*$, we know that the closure of A in E for $\mathscr{T}^* \cap E = \sigma(E, E')$ is equal to $C \cap E$ (GT, I, §3, No. 1, Prop. 1), that is, $B = C \cap E$.

IV.101, ℓ. −15 to −13.

"Let μ be a *discrete* measure on K, positive and of total mass 1; it is thus of the form $\mu = \sum_{i=1}^{n} \lambda_i \varepsilon_{\mathbf{x}_i}$, where $\mathbf{x}_i \in K$, and the λ_i are real numbers such that $\lambda_i \geqslant 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$."

Here, K can be any compact space (not necessarily a subspace of a Banach space); let us write $x \in K$ instead of $\mathbf{x} \in K$.

To say that μ is discrete means that it is atomic (Ch. III, §1, No. 3, *Example* I), given by a function $\alpha : K \to \mathbf{C}$ such that $\sum_{x \in K} |\alpha(x)| < +\infty$, and that, morever, the set

$$N = \{x \in K : \ \alpha(x) \neq 0\}$$

is finite (because K is compact and $N = N \cap K$). Say $N = \{x_1, \ldots, x_n\}$, with the x_i distinct. By the definition of μ, for $f \in \mathscr{K}(K; \mathbf{C}) = \mathscr{C}(K; \mathbf{C})$ one has

$$\mu(f) = \sum_{x \in K} \alpha(x) f(x) = \sum_{i=1}^{n} \alpha(x_i) f(x_i) = \sum_{i=1}^{n} \alpha(x_i) \varepsilon_{x_i}(f),$$

thus $\mu = \sum_{i=1}^{n} \lambda_i \varepsilon_{x_i}$, where $\lambda_i = \alpha(x_i)$.

There exist functions $f_i \in \mathscr{C}(K)$, $0 \leqslant f_i \leqslant 1$, such that $f_i(x_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$, whence $\mu(f_i) = \alpha(x_i)$. The assumption that $\mu \geqslant 0$

is clearly equivalent to the condition $\lambda_i \geqslant 0$ for all i (i.e., $\alpha(x) \geqslant 0$ for all $x \in \mathrm{K}$). Moreover, from

$$|\mu(f)| \leqslant \Big(\sum_{i=1}^{n} \lambda_i\Big)\|f\| \qquad \big(f \in \mathscr{C}(\mathrm{K})\big)$$

we know that μ is bounded, with $\|\mu\| \leqslant \sum_{i=1}^{n} \lambda_i$; since K is compact, we have in fact

$$\|\mu\| = |\mu|(1) = \mu(1) = \sum_{i=1}^{n} \lambda_i \varepsilon_{x_i}(1) = \sum_{i=1}^{n} \lambda_i$$

(Ch. III, §1, No. 8, Cor. 2 of Prop. 10), and to say that μ has total mass equal to 1 means that $\sum_{i=1}^{n} \lambda_i = 1$, so that μ is a convex combination of Dirac measures.

IV.101, ℓ. −12.

“ $\mathbf{b}_\mu = \int \mathbf{x}\, d\mu(\mathbf{x}) = \sum_i \lambda_i \mathbf{x}_i$.”

For all $\mathbf{x}' \in \mathrm{E}'$,

$$\begin{aligned}\langle \mathbf{b}_\mu, \mathbf{x}' \rangle &= \Big\langle \int \mathbf{x}\, d\mu(\mathbf{x}), \mathbf{x}' \Big\rangle = \int \langle \mathbf{x}, \mathbf{x}' \rangle\, d\mu(\mathbf{x}) = \mu(\mathbf{x}' \big| \mathrm{K}) \\ &= \sum_i \lambda_i \varepsilon_{\mathbf{x}_i}(\mathbf{x}' \big| \mathrm{K}) = \sum_i \lambda_i (\mathbf{x}' \big| \mathrm{K})(\mathbf{x}_i) \\ &= \sum_i \lambda_i \mathbf{x}'(\mathbf{x}_i) = \sum_i \lambda_i \langle \mathbf{x}_i, \mathbf{x}' \rangle = \Big\langle \sum_i \lambda_i \mathbf{x}_i, \mathbf{x}' \Big\rangle.\end{aligned}$$

IV.101, ℓ. −11.

“$\mathbf{x}$ is the barycenter of the measure $\varepsilon_{\mathbf{x}}$.”

The discrete measure defined by $\alpha = \varphi_{\{\mathbf{x}\}}$ is $\mu = 1 \cdot \varepsilon_{\mathbf{x}}$.

IV.101, ℓ. −6, −5.

“This is nothing more than Prop. 5 of Ch. III, §3, No. 2 applied to the canonical injection of K into E.”

As in the note for ℓ. 9–12, let us instead write B for the closed convex envelope of K in E for the original topology $\mathscr{T}$ on E, reserving the letter C for the closed convex envelope of K in E'^* for topology $\mathscr{T}^* = \sigma(\mathrm{E}'^*, \mathrm{E}')$ as in the sentence preceding Def. 1, where it is shown that $\mathrm{C} \cap \mathrm{E} = \mathrm{B}$ (with E canonically identified as a linear subspace of E'^*, as in the note for ℓ. 8,9).

Write $\mathscr{M}_1$ for the set positive measures on K of total mass 1, and set

$$\mathrm{S} = \{\mathbf{b}_\mu : \ \mu \in \mathscr{M}_1\},$$

which is a subset of E'^* (Def. 1); we are to show that $\mathrm{E} \cap \mathrm{S} = \mathrm{B}$.

Let $\mathbf{f} : \mathrm{K} \to \mathrm{E}$ be the canonical injection; E and K being equipped with their original topologies, we have $\mathbf{f} \in \mathscr{C}(\mathrm{K};\mathrm{E})$, $\mathbf{f}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathrm{K}$, and, by Def. 1,

$$\mathrm{S} = \{\textstyle\int \mathbf{f}(\mathbf{x})\, d\mu(\mathbf{x}) : \ \mu \in \mathscr{M}_1\}.$$

By the cited Prop. 5, S is the closed convex envelope of $\mathbf{f}(\mathrm{K}) = \mathrm{K}$ in E'^* for the topology $\mathscr{T}^*$, that is, $\mathrm{S} = \mathrm{C}$; thus $\mathrm{E} \cap \mathrm{S} = \mathrm{E} \cap \mathrm{C} = \mathrm{B}$.

IV.101, ℓ. −4 to −2.

"Corollary. — *If the closed convex envelope* C *of* K *in* E *is compact, then the barycenter of every positive measure of total mass* 1 *on* K *belongs to* E."

The statement can (should) be sharpened from "... *belongs to* E" to "... *belongs to* C". For, with notations and hypotheses of the Corollary as stated in the text, if μ is a positive measure on K of total mass 1, so that $\mathbf{b}_\mu \in \mathrm{E}$ by the Corollary, then $\mathbf{b}_\mu$ is a point of E that is the barycenter of at least one positive measure on K of total mass 1, therefore $\mathbf{b}_\mu$ belongs to C by Prop. 1.

IV.101, ℓ. −1 to **IV.102,** ℓ. 2.

"For, C is then also the closed convex envelope of K in E'^* equipped with the weak topology $\sigma(\mathrm{E}'^*, \mathrm{E}')$, and it suffices to apply, to the canonical injection of K into E, Prop. 4 of Ch. III, §3, No. 2."

As in the note for IV.101, ℓ. −6, −5, let us instead write B for the closed convex envelope of K in E for the original topology $\mathscr{T}$ on E, and let C and S have the meanings in that note. Thus, writing A for the convex envelope of K, B is the closure of A in E for $\mathscr{T}$, and C is the closure of A in E'^* for $\mathscr{T}^* = \sigma(\mathrm{E}'^*, \mathrm{E}')$.

We are to show that $\mathrm{B} = \mathrm{C}$ and $\mathrm{S} \subset \mathrm{E}$. (We show in fact that $\mathrm{S} = \mathrm{B}$.)

By assumption, B is compact for $\mathscr{T}$, hence it is compact for the coarser topology $\sigma(\mathrm{E}, \mathrm{E}')$; but $\sigma(\mathrm{E}, \mathrm{E}')$ coincides with the topology on E induced by $\mathscr{T}^*$ (see the note for IV.101, ℓ. 8–9), therefore B is a compact subset of E'^* for $\mathscr{T}^*$, and so B is closed in E'^*. Now, $\mathrm{A} \subset \mathrm{B}$ and C is the closure of A in E'^*, therefore $\mathrm{C} \subset \mathrm{B}$; but $\mathrm{B} = \mathrm{C} \cap \mathrm{E} \subset \mathrm{C}$, and so $\mathrm{B} = \mathrm{C}$. Finally, $\mathrm{S} = \mathrm{C}$ was shown in the preceding note, thus $\mathrm{S} = \mathrm{B} \subset \mathrm{E}$.

IV.102, ℓ. 4.

"... or when E is quasi-complete."

TVS, III, §1, No. 6, third paragraph after Def. 6.

IV.102, ℓ. 16.

"Therefore $\sup_\alpha \int h_\alpha(\mathbf{x})\,d\mu(\mathbf{x}) = \sup_\alpha h_\alpha(\mathbf{b}_\mu) = f(\mathbf{b}_\mu)$ "

In the expression $\int h_\alpha(\mathbf{x})\,d\mu(\mathbf{x})$, by h_α is meant the restriction of h_α to K; and the second equality holds because 1) $\sup_\alpha (h_\alpha|\mathrm{K}) = f$, and 2) $\mathbf{b}_\mu \in \mathrm{K}$ by the preceding *Remark*.

IV.102, ℓ. 17, 18.

"When μ is a *discrete positive* measure on K of total mass 1, Prop. 2 yields anew the inequality that defines the convex functions on K."

To say that $f : \mathrm{K} \to \mathbf{R}^+$ is convex means that if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are distinct elements of K, and $\mathbf{x} = \sum_i \lambda_i \mathbf{x}_i$ with $\lambda_i \geqslant 0$ and $\sum_i \lambda_i = 1$, then

$$f(\mathbf{x}) \leqslant \sum_i \lambda_i f(\mathbf{x}_i) = \Big(\sum_i \lambda_i \varepsilon_{\mathbf{x}_i}\Big)(f)\,;$$

one knows from the *Example* following Def. 1 that $\sum_i \lambda_i \varepsilon_{\mathbf{x}_i}$ is the general discrete positive measure on K of total mass 1, and that its barycenter is $\mathbf{x}$. To formulate this result in terms of Prop. 2 we will make use of the following:

Remark. *If* K *is a compact space and* μ *is a positive discrete measure on* K, *then every lower semi-continuous function* $f : \mathrm{K} \to \mathbf{R}^+$ *is integrable*; *and if* $\mu = \sum_{i=1}^n \lambda_i \varepsilon_{x_i}$ *as in the note for* IV.101, ℓ. −15 to −13, *then* $\int f\,d\mu = \sum_{i=1}^n \lambda_i f(x_i)$.

Proof. Since f is measurable for any measure (§5, No. 5, Cor. of Prop. 8) and $\mu^* = \sum_i \lambda_i (\varepsilon_{x_i})^*$ (§1, No. 3, Prop. 15), one is reduced to the case that $\mu = \varepsilon_x$ (§5, No. 6, Th. 5). Let H be the set of all functions $g \in \mathscr{C}_+(\mathrm{K})$ such that $g \leqslant f$; by the Lemma of §1, No. 1,

$$f(x) = \sup_{g\in\mathrm{H}} g(x) = \sup_{g\in\mathrm{H}} \varepsilon_x(g)\,,$$

thus $f(x) = (\varepsilon_x)^*(f)$ in the sense of *loc. cit.*, Def. 1, hence also in the sense of §1, No. 3, Def. 3. Thus $\int^* f\,d\varepsilon_x = f(x) < +\infty$, whence $\int f\,d\varepsilon_x$ exists (§4, No. 4, Cor. 1 of Prop. 5) and is equal to $f(x)$. ◇

Finally, with K as in Prop. 2, *for a lower semi-continuous function* $f \geqslant 0$ *to be convex, it is necessary and sufficient that it satisfy the inequality of* Prop. 2 *for every positive discrete measure* μ *on* K *of total mass* 1.

Proof. Necessity: A special case of Prop. 2.

Sufficiency: The assumption is that $f(\mathbf{b}_\mu) \leqslant \int^* f\,d\mu$ for every positive discrete measure μ on K of total mass 1. If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are distinct points of K and if $\mathbf{x} = \sum \lambda_i \mathbf{x}_i$ with $\lambda_i \geqslant 0$ and $\sum \lambda_i = 1$, then $\mathbf{x} = \mathbf{b}_\mu$ for the measure $\mu = \sum \lambda_i \varepsilon_{\mathbf{x}_i}$ by the *Example* following Def. 1, so by assumption

$$f(\mathbf{x}) = f(\mathbf{b}_\mu) \leqslant \int^* f\,d\mu\,,$$

whence, citing the above Remark, $f(\mathbf{x}) \leqslant \int f\,d\mu = \sum \lambda_i f(\mathbf{x}_i)$.

IV.102, ℓ. −13.

"It suffices to note that $\inf_{\mathbf{x}\in \mathrm{K}} g(\mathbf{x}) = a$ is finite and apply Prop. 2 to $g - a$."

The values of g are finite, and $a = g(\mathbf{y})$ for some $\mathbf{y} \in \mathrm{K}$ (GT, IV, §6, No. 2, Th. 3).

The function $f = g - a$ satisfies the hypotheses of Prop. 2, so $(g-a)(\mathbf{b}_\mu) \leqslant \int^*(g-a)\,d\mu$; moreover, $g - a$ is bounded and measurable, and μ is bounded, therefore $g - a$ is integrable (§5, No. 6, Th. 5) and

$$g(\mathbf{b}_\mu) - a = (g-a)(\mathbf{b}_\mu) \leqslant \int (g-a)\,d\mu = \int g\,d\mu - a\,.$$

IV.102, ℓ. −3 to **IV.103,** ℓ. 1.

"For every point $\mathbf{a} \in \mathrm{K}$, there exists a closed convex neighborhood $\mathrm{V}_\mathbf{a}$ of 0 in E such that

$$|f_i(\mathbf{y}) - f_i(\mathbf{a})| \leqslant \delta/2 \tag{2}$$

for $1 \leqslant i \leqslant p$ and for every $\mathbf{y} \in \mathrm{W}_\mathbf{a} = \mathrm{K} \cap (\mathbf{a} + \mathrm{V}_\mathbf{a})$."

Let $\mathscr{T}$ be the given topology on the locally convex space E. As V varies over the set of all closed convex neighborhoods of 0 in E, the sets $\mathrm{W} = \mathrm{K} \cap (\mathbf{a} + \mathrm{V})$ vary over a fundamental system of neighborhoods of $\mathbf{a}$ in K for the topology on K induced by $\mathscr{T}$ (TVS, II, §4, No. 1). Fix $\mathbf{a} \in \mathrm{K}$. For each $i = 1, \ldots, p$ there exists, by the continuity of f_i, a closed convex neighborhood V_i of 0 in E such that (2) holds for all $\mathbf{y} \in \mathrm{W}_i = \mathrm{K} \cap (\mathbf{a} + \mathrm{V}_i)$. Setting $\mathrm{V}_\mathbf{a} = \bigcap_{i=1}^{p} \mathrm{V}_i$ and

$$\mathrm{W}_\mathbf{a} = \bigcap_{i=1}^{p} \mathrm{W}_i = \mathrm{K} \cap \bigcap_{i=1}^{p} (\mathbf{a} + \mathrm{V}_i) = \mathrm{K} \cap \Big(\mathbf{a} + \bigcap_{i=1}^{p} \mathrm{V}_i\Big) = \mathrm{K} \cap (\mathbf{a} + \mathrm{V}_\mathbf{a})\,,$$

the inequality (2) holds for every i and every $\mathbf{y} \in \mathrm{W}_\mathbf{a}$.

IV.103, ℓ. 5–7.

"Each of the measures μ_j is positive, of total mass 1, and its support is contained in the compact convex set $\mathrm{W}_{\mathbf{a}_j}$ "

If $\alpha_j \neq 0$ then $\mu_j(1) = \alpha_j^{-1}\,\mu(g_j \cdot 1) = \alpha_j^{-1}\alpha_j = 1$, and, citing Ch. III, §2, No. 3, Prop. 10,

$$\begin{aligned}\mathrm{Supp}(\mu_j) &= \mathrm{Supp}(g_j \cdot \mu) = \overline{\{\mathbf{y} \in \mathrm{Supp}(\mu) :\ g_j(\mathbf{y}) \neq 0\}} \\ &\subset \mathrm{Supp}(\mu) \cap \mathrm{Supp}(g_j) \subset \mathrm{Supp}(g_j) \subset \mathrm{W}_{\mathbf{a}_j}\,;\end{aligned}$$

whereas if $\alpha_j = 0$ then $\mu_j(1) = \varepsilon_{\mathbf{a}_j}(1) = 1$ and

$$\mathrm{Supp}(\mu_j) = \mathrm{Supp}(\varepsilon_{\mathbf{a}_j}) = \{\mathbf{a}_j\} \subset \mathrm{W}_{\mathbf{a}_j}\,.$$

IV.103, ℓ. 7–9.

"... by definition,

$$\mu = \sum_{j=1}^{r} \alpha_j \mu_j \tag{3}$$

since $g_j \cdot \mu = 0$ if $\mu(g_j) = 0$ "

If $\alpha_j = 0$, that is, if $\mu(g_j) = 0$, then, for every $f \in \mathscr{C}_+(\mathrm{K})$ with $0 \leqslant f \leqslant 1$, one has $0 \leqslant g_j f \leqslant g_j$, therefore $0 \leqslant \mu(g_j f) \leqslant \mu(g_j) = 0$, thus $g_j \cdot \mu = 0$, whence $\alpha_j \mu_j = 0 \cdot \varepsilon_{\mathbf{a}_j} = 0 = g_j \cdot \mu$; whereas if $\alpha_j > 0$ then $\alpha_j \mu_j = g_j \cdot \mu$ by the definition of μ_j. Thus

$$g_j \cdot \mu = \alpha_j \mu_j \qquad (j = 1, \dots, r)\,.$$

Since $\sum_{j=1}^{r} g_j = 1$, for every $f \in \mathscr{C}(\mathrm{K}; \mathbf{C})$ one has $f = \sum_{j=1}^{r} g_j f$, therefore

$$\mu(f) = \sum_{j=1}^{r} \mu(g_j f) = \sum_{j=1}^{r} (g_j \cdot \mu)(f) = \sum_{j=1}^{r} (\alpha_j \cdot \mu_j)(f)\,,$$

whence (3).

IV.103, ℓ. 11.

"Let $\mathbf{x}_j$ be the barycenter of μ_j, which belongs to $\mathrm{W}_{\mathbf{a}_j}$ (No. 1, Prop. 1)"

At any rate, μ_j is a positive measure on K of total mass 1, its barycenter $\mathbf{x}_j$ is defined, and $\mathbf{x}_j \in \mathrm{K}$ by the corollary of the cited Prop. 1 (since the closed convex envelope of K in E is K itself).

Let $\mathrm{K}_j = \operatorname{Supp} \mu_j$. We know that $\mathrm{K}_j \subset \mathrm{W}_{\mathbf{a}_j} = \mathrm{K} \cap (\mathbf{a}_j + \mathrm{V}_{\mathbf{a}_j})$. Since K_j is a compact subset of K, the induced measure $\rho_j = (\mu_j)_{\mathrm{K}_j}$ on K_j is defined: if $h \in \mathscr{C}(\mathrm{K}_j; \mathbf{C})$ and h' is the extension by 0 of h to K, then h' is μ_j-integrable and one defines $\rho_j(h) = \int h' \, d\mu_j$. In particular, $1 \in \mathscr{C}(\mathrm{K}_j)$ and $1' = \varphi_{\mathrm{K}_j}$ (the characteristic function of K_j in K); since μ_j has total mass 1 and $\mathrm{K} - \mathrm{K}_j$ is μ_j-negligible, one has

$$\rho_j(1) = \int 1' \, d\mu_j = \int \varphi_{\mathrm{K}_j} \, d\mu_j = \mu_j(\mathrm{K}_j) = \mu_j(\mathrm{K}) = 1 \,,$$

thus ρ_j is a (positive) measure on K_j of total mass 1. The barycenter $\mathbf{b}_{\rho_j}$ of ρ_j is, a priori, an element of E'^*; but $\mathrm{K}_j \subset \mathrm{W}_{\mathbf{a}_j}$, where $\mathrm{W}_{\mathbf{a}_j}$ is a compact convex subset of E, thus the closed convex envelope C_j of K_j in E is a compact convex subset of $\mathrm{W}_{\mathbf{a}_j}$, hence of E. It then follows from the Cor. of Prop. 1 (with K_j playing the role of K) that $\mathbf{b}_{\rho_j} \in \mathrm{C}_j \subset \mathrm{W}_{\mathbf{a}_j}$, so it will suffice to show that $\mathbf{b}_{\rho_j} = \mathbf{x}_j$, that is,

$$\text{barycenter of } (\mu_j)_{\operatorname{Supp} \mu_j} \quad = \quad \text{barycenter of } \mu_j \,;$$

and for this, it suffices to show that

$$(*) \qquad \langle \mathbf{b}_{\rho_j}, \mathbf{z}' \rangle = \langle \mathbf{x}_j, \mathbf{z}' \rangle \quad \text{for all } \mathbf{z}' \in \mathrm{E}' \,.$$

Let $\mathbf{z}' \in \mathrm{E}'$. As the letter $\mathbf{x}$ is frozen in the statement of Prop. 3 (namely, $\mathbf{x} = \mathbf{b}_\mu$), we employ $\mathbf{y}$ for the variable of integration in the following:

$$\begin{aligned} \langle \mathbf{x}_j, \mathbf{z}' \rangle &= \Big\langle \int \mathbf{y} \, d\mu_j(\mathbf{y}), \mathbf{z}' \Big\rangle \\ &= \int \langle \mathbf{y}, \mathbf{z}' \rangle \, d\mu_j(\mathbf{y}) = \int \langle \mathbf{y}, \mathbf{z}' \rangle \varphi_{\mathrm{K}_j}(\mathbf{y}) \, d\mu_j(\mathbf{y}) \end{aligned}$$

(all integrations are over K), whereas if $\mathbf{y} \mapsto \langle \mathbf{y}, \mathbf{z}' \rangle'$ $(\mathbf{y} \in \mathrm{K})$ denotes the extension by 0 to K of the function $\mathbf{y} \mapsto \langle \mathbf{y}, \mathbf{z}' \rangle$ $(\mathbf{y} \in \mathrm{K}_j)$, then

$$\begin{aligned} \langle \mathbf{b}_{\rho_j}, \mathbf{z}' \rangle &= \Big\langle \int \mathbf{y} \, d\rho_j(\mathbf{y}), \mathbf{z}' \Big\rangle = \int \langle \mathbf{y}, \mathbf{z}' \rangle \, d\rho_j(\mathbf{y}) \\ &= \int \langle \mathbf{y}, \mathbf{z}' \rangle \, d(\mu_j)_{\mathrm{K}_j}(\mathbf{y}) = \int \langle \mathbf{y}, \mathbf{z}' \rangle' \, d\mu_j(\mathbf{y}) \\ &= \int \langle \mathbf{y}, \mathbf{z}' \rangle \varphi_{\mathrm{K}_j}(\mathbf{y}) \, d\mu_j(\mathbf{y}) \end{aligned}$$

(the first three integrations are over K_j , the last two over K), thus $(*)$ is verified.

IV.103, ℓ. 13, 14.

"...its barycenter is $\sum_{j=1}^{r} \alpha_j \mathbf{x}_j$, which is also the barycenter of μ by virtue of (3), thus is equal to $\mathbf{x}$."

For each $\mathbf{z}' \in \mathrm{E}'$, the function $\mathbf{z}'\big|\mathrm{K}$ belongs to $\mathscr{C}(\mathrm{K};\mathbf{C}) = \mathscr{K}(\mathrm{K};\mathbf{C})$, therefore, citing (3) at the third equality, we have

$$\begin{aligned} \langle \mathbf{b}_\mu, \mathbf{z}' \rangle &= \int \langle \mathbf{y}, \mathbf{z}' \rangle \, d\mu(\mathbf{y}) = \mu(\mathbf{z}'\big|\mathrm{K}) \\ &= \sum_{j=1}^{r} \alpha_j \mu_j(\mathbf{z}'\big|\mathrm{K}) = \sum_{j=1}^{r} \alpha_j \int \langle \mathbf{y}, \mathbf{z}' \rangle \, d\mu_j(\mathbf{y}) \\ &= \sum_{j=1}^{r} \alpha_j \langle \mathbf{b}_{\mu_j}, \mathbf{z}' \rangle = \Big\langle \sum_{j=1}^{r} \alpha_j \mathbf{b}_{\mu_j}, \mathbf{z}' \Big\rangle, \end{aligned}$$

whence $\mathbf{b}_\mu = \sum_{j=1}^{r} \alpha_j \mathbf{b}_{\mu_j}$. But $\mathbf{x} = \mathbf{b}_\mu$ and $\mathbf{x}_j = \mathbf{b}_{\mu_j}$ by definition, thus $\mathbf{x} = \sum_{j=1}^{r} \alpha_j \mathbf{x}_j$. On the other hand $\mathbf{b}_\nu = \sum_{j=1}^{r} \alpha_j \mathbf{x}_j$ by the *Example* following Def. 1 of No. 1, thus $\mathbf{b}_\nu = \mathbf{x} = \mathbf{b}_\mu$.

IV.103, ℓ. 15, 16.

"...since $\mathrm{Supp}(\mu_j) \subset \mathrm{W}_{\mathbf{a}_j}$, $|\mu_j(f_i) - f_i(\mathbf{a}_j)| \leqslant \delta/2$ for $1 \leqslant i \leqslant p$."

Writing $\mathrm{K}_j = \mathrm{Supp}(\mu_j)$, we have $\mathrm{K}_j \subset \mathrm{W}_{\mathbf{a}_j} \subset \mathrm{K}$, and $\mathrm{K} - \mathrm{K}_j$ is a μ_j-negligible open set in K. Let φ_{K_j} be the characteristic function of K_j in K. Since $\mathrm{K}_j \subset \mathrm{W}_{\mathbf{a}_j}$, we know from (2) that the function $f_i - f_i(\mathbf{a}_j) \cdot 1$ in $\mathscr{C}(\mathrm{K};\mathbf{C}) = \mathscr{K}(\mathrm{K};\mathbf{C})$ satisfies

$$\varphi_{\mathrm{K}_j} |f_i - f_i(\mathbf{a}_j) \cdot 1| \leqslant \delta/2 \,,$$

whereas $|f_i - f_i(\mathbf{a}_j) \cdot 1| = \varphi_{\mathrm{K}_j} |f_i - f(\mathbf{a}_j) \cdot 1|$ μ_j-almost everywhere in K, therefore

$$\begin{aligned} |\mu_j(f_i - f_i(\mathbf{a}_j) \cdot 1)| &\leqslant \mu_j(|f_i - f_i(\mathbf{a}_j) \cdot 1|) = \int |f_i - f_i(\mathbf{a}_j) \cdot 1| \, d\mu_j \\ &= \int \varphi_{\mathrm{K}_j} |f_i - f_i(\mathbf{a}_j) \cdot 1| \, d\mu_j \leqslant (\delta/2)\mu_j(1) = \delta/2 \,, \end{aligned}$$

that is, $|\mu_j(f_i) - f_i(\mathbf{a}_j)| \leqslant \delta/2$.

IV.103, ℓ. 19–21.

"Since the α_j are $\geqslant 0$ and have sum 1, it follows from (3) and the definition of ν that ν satisfies the inequality (1)."

Fix an index i $(1 \leqslant i \leqslant p)$. We know that $|\mu_j(f_i) - \varepsilon_{\mathbf{x}_j}(f_i)| \leqslant \delta$ for all $j = 1, \ldots, r$. Citing (3) and the definition of ν, we have

$$\begin{aligned}
|\mu(f_i) - \nu(f_i)| &= \Big|\sum_{j=1}^{r} \alpha_j \mu_j(f_i) - \sum_{j=1}^{r} \alpha_j \varepsilon_{\mathbf{x}_j}(f_i)\Big| \\
&= \Big|\sum_{j=1}^{r} \alpha_j [\mu_j(f_i) - \varepsilon_{\mathbf{x}_j}(f_i)]\Big| \\
&\leqslant \sum_{j=1}^{r} \alpha_j |\mu_j(f_i) - \varepsilon_{\mathbf{x}_j}(f_i)| \leqslant \sum_{j=1}^{r} \alpha_j \delta = \delta\,.
\end{aligned}$$

Thus ν satisfies the inequality (1).

Summarizing: If μ is a positive measure on K of total mass 1, and $\mathbf{x}$ is its barycenter, then every vague neighborhood of μ contains a discrete positive measure ν of total mass 1 and barycenter $\mathbf{x}$; thus Prop. 3 is verified.

IV.103, $\ell.\ -5$.

"Suppose $\mathbf{x}$ is an extremal point of K "

The supposition would have implied that $\mathbf{x}$ belongs to K′ even if this had not been assumed in the statement of the Corollary (TVS, II, §7, No. 1, Cor. of Prop. 2). However, the condition $\mathbf{x} \in \mathrm{K}'$ is essential to *stating* the Corollary: by $\varepsilon_{\mathbf{x}}$ is meant the measure on K′ defined by $f \mapsto f(\mathbf{x})$ for all $f \in \mathscr{C}(\mathrm{K}')$, whence the need for $\mathbf{x} \in \mathrm{K}'$; it then follows that the positive measure $\varepsilon_{\mathbf{x}}$ on K′ of total mass 1 has barycenter $\mathbf{x}$ (No. 1, *Example* following Def. 1).

IV.103, $\ell.\ -5$ to -2.

"...to prove that $\varepsilon_{\mathbf{x}}$ is the only positive measure on K′, of total mass 1, having $\mathbf{x}$ as barycenter, it suffices, by Prop. 3, to see that the set of *discrete* measures ν on K′ that are positive, of total mass 1, and have $\mathbf{x}$ as barycenter, reduces to $\varepsilon_{\mathbf{x}}$."

Let us write $\mathscr{M}(\mathbf{x})$ for the set of all positive measures on K′, of total mass 1, having barycenter $\mathbf{x}$; we know that $\varepsilon_{\mathbf{x}} \in \mathscr{M}(\mathbf{x})$ (No. 1, *Example* following Def. 1), and we are to show (assuming $\mathbf{x}$ is an extremal point of K) that $\mathscr{M}(\mathbf{x}) = \{\varepsilon_{\mathbf{x}}\}$.

Let $\mathscr{M}_d(\mathbf{x})$ be the set of measures in $\mathscr{M}(\mathbf{x})$ that are discrete (for example $\varepsilon_{\mathbf{x}} \in \mathscr{M}_d(\mathbf{x})$, as noted in the cited example). It will suffice to show that $\mathscr{M}_d(\mathbf{x}) \subset \{\varepsilon_{\mathbf{x}}\}$; for, since $\{\varepsilon_{\mathbf{x}}\}$ is vaguely closed, it will then follow from Prop. 3 that $\mathscr{M}(\mathbf{x}) \subset \overline{\mathscr{M}_d(\mathbf{x})} \subset \overline{\{\varepsilon_{\mathbf{x}}\}} = \{\varepsilon_{\mathbf{x}}\}$ (the overbar signifying vague closure in $\mathscr{M}(\mathrm{K}';\mathbf{C})$), whence $\mathscr{M}(\mathbf{x}) = \{\varepsilon_{\mathbf{x}}\}$.

IV.104, ℓ. 1, 2.

"... the hypothesis that $\mathbf{x}$ is the barycenter of ν may be written $\mathbf{x} = \sum_{i=1}^{r} \lambda_i \mathbf{x}_i$."

Recalling that the $\mathbf{x}_i$ belong to K', the computation in the note for IV.103, ℓ. 13, 14 yields $\mathbf{b}_\nu = \sum_{i=1}^{r} \lambda_i \mathbf{b}_{\varepsilon_{\mathbf{x}_i}}$, that is, $\mathbf{x} = \sum_{i=1}^{r} \lambda_i \mathbf{x}_i$.

IV.104, ℓ. 9.

"Then $\mathbf{x}$ is the barycenter of $\lambda\mu' + (1-\lambda)\mu''$."

Arguing as in the preceding note, $\mathbf{b}_{\lambda\mu'+(1-\lambda)\mu''} = \lambda\mathbf{b}_{\mu'} + (1-\lambda)\mathbf{b}_{\mu''} = \lambda\mathbf{x}' + (1-\lambda)\mathbf{x}'' = \mathbf{x}$.

IV.104, ℓ. 9, 10.

"Therefore $\lambda\mu' + (1-\lambda)\mu'' = \varepsilon_{\mathbf{x}}$."

By the assumption on $\varepsilon_{\mathbf{x}}$.

IV.104, ℓ. 10.

"Therefore μ' and μ'' are proportional to $\varepsilon_{\mathbf{x}}$ "

Note first that $\mathrm{Supp}(\varepsilon_{\mathbf{x}}) = \{\mathbf{x}\}$. {At any rate, $\mathrm{Supp}(\varepsilon_{\mathbf{x}}) \neq \varnothing$ since $\varepsilon_{\mathbf{x}} \neq 0$, so it suffices to show that $\mathrm{Supp}(\varepsilon_{\mathbf{x}}) \subset \{\mathbf{x}\}$, that is, $\mathrm{K}' - \{\mathbf{x}\} \subset \mathrm{K}' - \mathrm{Supp}(\varepsilon_{\mathbf{x}})$, in other words, the restriction of $\varepsilon_{\mathbf{x}}$ to the open subset $\mathrm{Y} = \mathrm{K}' - \{\mathbf{x}\}$ of K' is zero (Ch. III, §2, No. 2, Def. 1). Indeed, if $g \in \mathscr{K}(\mathrm{Y}; \mathbf{C})$ and g' is the extension by 0 of g to K' (i.e., by setting $g'(\mathbf{x}) = 0$), then $(\varepsilon_{\mathbf{x}})_{\mathrm{Y}}(g) = \varepsilon_{\mathbf{x}}(g') = g'(\mathbf{x}) = 0$.}

From $0 \leqslant \lambda\mu' \leqslant \varepsilon_{\mathbf{x}}$ one then infers (*loc. cit.*, Prop. 3)

$$\mathrm{Supp}(\mu') = \mathrm{Supp}(\lambda\mu') \subset \mathrm{Supp}(\varepsilon_{\mathbf{x}}) = \{\mathbf{x}\},$$

therefore μ' is proportional to $\varepsilon_{\mathbf{x}}$ (*loc. cit.*, No. 4, Prop. 12), hence so is μ'' . And from $\mu'(1) = \varepsilon_{\mathbf{x}}(1) = \mu''(1) = 1$ one infers that $\mu' = \varepsilon_{\mathbf{x}} = \mu''$.

IV.104, ℓ. 10, 11.

"... whence $\mathbf{x}' = \mathbf{x}'' = \mathbf{x}$ "

For, $\mathbf{x}' = \mathbf{b}_{\mu'} = \mathbf{b}_{\varepsilon_{\mathbf{x}}} = \mathbf{x}$ and similarly $\mathbf{x}'' = \mathbf{x}$.

IV.104, ℓ. 18–20.

"... the subset U of $\mathrm{K} \times \mathrm{K} \times \mathrm{I}$ formed by the triples $(\mathbf{x}, \mathbf{y}, \lambda)$ such that $\mathbf{x} \neq \mathbf{y}$ and $0 < \lambda < 1$ is open in $\mathrm{K} \times \mathrm{K} \times \mathrm{I}$ "

If Δ is the diagonal of $\mathrm{K} \times \mathrm{K}$ then $\mathrm{U} = (\mathrm{K} \times \mathrm{K} - \Delta) \times (0, 1)$.

IV.104, ℓ. −16.

" $\mathrm{K} - \mathrm{M} = q(\mathrm{U})$ "

Let $\mathbf{z} \in \mathrm{K}$. Then

$$\mathbf{z} \text{ is not extremal in } \mathrm{K} \quad \Leftrightarrow \quad \mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \text{ for some } (\mathbf{x}, \mathbf{y}, \lambda) \in \mathrm{U},$$

that is, $\mathbf{z} \notin \mathrm{M} \Leftrightarrow \mathbf{z} \in q(\mathrm{U})$.

IV.104, ℓ. −6 to −4.

"By the Hahn-Banach theorem, for $(\mathbf{a}, b)$ to belong to S, it is necessary and sufficient that $h(\mathbf{a}, b) \geqslant 0$ for every continuous affine linear function h on $\mathrm{E} \times \mathbf{R}$ such that $h(\mathbf{x}, u(\mathbf{x})) \geqslant 0$ for $\mathbf{x} \in \mathrm{K}$."

Review. A function $f : \mathrm{E} \to \mathbf{R}$ is affine linear if there exist a linear form f_0 on E and a real number α such that $f(\mathbf{x}) = f_0(\mathbf{x}) + \alpha$ for all $\mathbf{x} \in \mathrm{E}$ (A, II, §9, No. 4, Prop. 6); f is continuous when $f_0 \in \mathrm{E}'$.

Thus, a function $h : \mathrm{E} \times \mathbf{R} \to \mathbf{R}$ is a continuous affine linear function if there exist a continuous linear form $h_0 \in (\mathrm{E} \times \mathbf{R})'$ and an $\alpha \in \mathbf{R}$ such that $h(\mathbf{x}, t) = h_0(\mathbf{x}, t) + \alpha$ for all $(\mathbf{x}, t) \in \mathrm{E} \times \mathbf{R}$. But we can make the identifications $(\mathrm{E} \times \mathbf{R})' = \mathrm{E}' \oplus \mathbf{R}' = \mathrm{E}' \oplus \mathbf{R}$ (TVS, IV, §1, No. 5), so there exist $f_0 \in \mathrm{E}'$ and $\beta \in \mathbf{R}$ such that $h_0(\mathbf{x}, t) = f_0(\mathbf{x}) + \beta t$, whence

$$h(\mathbf{x}, t) = \big(f_0(\mathbf{x}) + \beta t\big) + \alpha = \big(f_0(\mathbf{x}) + \alpha\big) + \beta t\,;$$

thus $f(\mathbf{x}) = f_0(\mathbf{x}) + \alpha$ defines a continuous affine linear function on E such that $h(\mathbf{x}, t) = f(\mathbf{x}) + \beta t$ for all $(\mathbf{x}, t) \in \mathrm{E} \times \mathbf{R}$. ◊

Necessity. Suppose $(\mathbf{a}, b) \in \mathrm{S}$ (the closed convex envelope in $\mathrm{E} \times \mathbf{R}$ of the graph G of u). If h is a continuous affine linear function on $\mathrm{E} \times \mathbf{R}$ such that $h \geqslant 0$ on G, then clearly $h \geqslant 0$ on S; in particular, $h(\mathbf{a}, b) \geqslant 0$.

Sufficiency. Arguing contrapositively, assuming $(\mathbf{a}, b) \notin \mathrm{S}$ let us show that there exists a continuous affine linear function h on $\mathrm{E} \times \mathbf{R}$ such that $h(\mathbf{a}, b) < 0$.

Since S is the intersection of the closed half-spaces that contain it (TVS, II, §5, No. 3, Cor. 1 of Prop. 4), there exist a continuous linear form h_0 on $\mathrm{E} \times \mathbf{R}$ and a real number α such that $h_0(\mathbf{x}, t) \geqslant \alpha$ for $(\mathbf{x}, t) \in \mathrm{S}$ and $h_0(\mathbf{a}, b) < \alpha$. Then $h = h_0 - \alpha$ is a continuous affine linear function on $\mathrm{E} \times \mathbf{R}$ such that $h \geqslant 0$ on S (equivalently, $h(\mathbf{x}, u(\mathbf{x})) \geqslant 0$ for all $\mathbf{x} \in \mathrm{K}$) and $h(\mathbf{a}, b) < 0$.

IV.104, ℓ. −3 to **IV.105,** ℓ. 2.

"... the relation $(\mathbf{a}, b) \in \mathrm{S}$ is equivalent to the following property: the relation

$$(4) \qquad f(\mathbf{x}) \geqslant \lambda\, u(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathrm{K}$$

implies

$$(5) \qquad f(\mathbf{a}) \geqslant \lambda b\,."$$

Here f stands for any continuous affine linear function on E, and $\lambda \in \mathbf{R}$; thus $h(\mathbf{x}, t) = f(\mathbf{x}) - \lambda t$ defines the most general continuous affine linear function h on $\mathrm{E} \times \mathbf{R}$, (4) signifies $h(\mathbf{x}, u(\mathbf{x})) \geqslant 0$ for all $\mathbf{x} \in \mathrm{K}$, and (5) signifies $h(\mathbf{a}, b) \geqslant 0$. Thus the proposition

$$(\mathbf{a}, b) \in \mathrm{S} \quad \Leftrightarrow \quad \big((4) \;\Rightarrow\; (5)\big)$$

is a restatement of the assertion just proved.

IV.105, ℓ. 11.

"This proves the lemma . . . "

The organization of the proof is somewhat complicated. With the notations established in the lemma, we are to show:

$$(\mathbf{a}, b) \in \mathrm{S} \quad \Leftrightarrow \quad \mathbf{a} \in \mathrm{K} \text{ and } u(\mathbf{a}) \leqslant b \leqslant \overline{u}(\mathbf{a})\,.$$

Proof of $\Rightarrow$: Suppose $(\mathbf{a}, b) \in \mathrm{S}$. Since K is convex and closed, clearly $\mathbf{a} \in \mathrm{K}$. Let h be any continuous affine linear function on $\mathrm{E} \times \mathbf{R}$. We know that

$$(*) \qquad h \geqslant 0 \text{ on } \mathrm{G} \quad \Rightarrow \quad h(\mathbf{a}, b) \geqslant 0\,.$$

Say h is given by $h(\mathbf{x}, t) = f(\mathbf{x}) - \lambda t$, where f is a continuous affine linear function on E, and $\lambda \in \mathbf{R}$.

case 1. $\lambda = 0$.

We know from $(*)$ that

$$\text{(i)} \qquad f \geqslant 0 \text{ on } \mathrm{K} \quad \Rightarrow \quad f(\mathbf{a}) \geqslant 0\,.$$

Now, the closed half-spaces in E are the sets $\{\mathbf{x} : f(\mathbf{x}) \geqslant 0\}$, where f is a continuous affine linear function on E. Therefore the validity of (i) says that $\mathbf{a}$ belongs to every closed half-space that contains K, so $\mathbf{a}$ belongs to their intersection, which is equal to K (TVS, II, §5, No. 3, Cor. 1 of Prop. 4).

Conversely, if $\mathbf{a} \in \mathrm{K}$, then $\mathbf{a}$ belongs to every closed half-space containing K, thus (i) is satisfied and so $(*)$ holds for every h with $\lambda = 0$.

Conclusion: Assuming $(\mathbf{a}, b) \in \mathrm{S}$,

$$(*) \text{ holds when } \lambda = 0 \quad \Leftrightarrow \quad \mathbf{a} \in \mathrm{K}\,.$$

case 2. $\lambda = -1$.

For such h, the meaning of $(*)$ is

$$f(\mathbf{x}) + t \geqslant 0 \text{ on } \mathrm{G} \quad \Rightarrow \quad f(\mathbf{a}) + b \geqslant 0\,,$$

that is,

$$f(\mathbf{x})+u(\mathbf{x})\geqslant 0 \text{ on } \mathrm{K} \quad\Rightarrow\quad f(\mathbf{a})+b\geqslant 0\,,$$

that is,

$$-f(\mathbf{x})\leqslant u(\mathbf{x}) \text{ on } \mathrm{K} \quad\Rightarrow\quad -f(\mathbf{a})\leqslant b\,;$$

since $f\mapsto -f$ is a permutation of the set of all continuous affine linear functions on E, we may write this implication as

$$\text{(ii)}\qquad f\leqslant u \text{ on } \mathrm{K} \quad\Rightarrow\quad f(\mathbf{a})\leqslant b\,.$$

Now, u is the supremum of the $f\big|\mathrm{K}$ as f runs over the set of all continuous affine linear functions such that $f\big|\mathrm{K}\leqslant u$ (TVS, II, §5, No. 4, Prop. 5; see the *Remark* at the end of this note); by (ii), $f(\mathbf{a})\leqslant b$ for every such f, therefore $u(\mathbf{a})\leqslant b$.

Conversely, if $\mathbf{a}\in\mathrm{K}$ (as is the case when $(\mathbf{a},b)\in\mathrm{S}$, as noted above) and $u(\mathbf{a})\leqslant b$, then $f\leqslant u$ on K implies that $f(\mathbf{a})\leqslant u(\mathbf{a})\leqslant b$, so the implication (ii) holds.

Conclusion: Assuming $(\mathbf{a},b)\in\mathrm{S}$,

$$(*) \text{ holds when } \lambda=-1 \quad\Leftrightarrow\quad u(\mathbf{a})\leqslant b\,.$$

case 3. $\lambda=1$.

For such h, the meaning of $(*)$ is

$$f(\mathbf{x})-t\geqslant 0 \text{ on } \mathrm{G} \quad\Rightarrow\quad f(\mathbf{a})-b\geqslant 0\,,$$

that is,

$$f(\mathbf{x})-u(\mathbf{x})\geqslant 0 \text{ on } \mathrm{K} \quad\Rightarrow\quad f(\mathbf{a})-b\geqslant 0\,,$$

that is,

$$\text{(iii)}\qquad u(\mathbf{x})\leqslant f(\mathbf{x}) \text{ on } \mathrm{K} \quad\Rightarrow\quad b\leqslant f(\mathbf{a})\,.$$

Now, $\overline{u}$ is, by definition, the lower envelope of the f, as f runs over the set of all continuous affine linear functions such that $u\leqslant f\big|\mathrm{K}$; by (iii), $b\leqslant f(\mathbf{a})$ for every such f, therefore $b\leqslant\overline{u}(\mathbf{a})$.

Conversely, if $\mathbf{a}\in\mathrm{K}$ (as is the case when $(\mathbf{a},b)\in\mathrm{S}$) and $b\leqslant\overline{u}(\mathbf{a})$, then

$$u\leqslant f \text{ on } \mathrm{K} \quad\Rightarrow\quad \overline{u}\leqslant f \text{ on } \mathrm{K}$$

by the definition of $\overline{u}$; in particular, $b\leqslant\overline{u}(\mathbf{a})\leqslant f(\mathbf{a})$, thus the implication (iii) holds.

Conclusion: Assuming $(\mathbf{a}, b) \in \mathrm{S}$,

$$(*) \text{ holds when } \lambda = 1 \quad \Leftrightarrow \quad b \leqslant \overline{u}(\mathbf{a}) .$$

Summary. If $(\mathbf{a}, b) \in \mathrm{S}$ then $\mathbf{a} \in \mathrm{K}$ and $u(\mathbf{a}) \leqslant b \leqslant \overline{u}(\mathbf{a})$. This completes the "*Proof* of $\Rightarrow$". Moreover, the "Conversely" assertions show that if $\mathbf{a} \in \mathrm{K}$ and $u(\mathbf{a}) \leqslant b \leqslant \overline{u}(\mathbf{a})$, then the implications (i), (ii), (iii) hold.

Proof of $\Leftarrow$: Assuming $\mathbf{a} \in \mathrm{K}$ and $u(\mathbf{a}) \leqslant b \leqslant \overline{u}(\mathbf{a})$, we know from the foregoing "Summary" that the implication $(*)$ holds in the special cases that $\lambda = 0, -1$ or 1 in the formula for h. To infer that $(\mathbf{a}, b) \in \mathrm{S}$, it suffices to show that $(*)$ holds for every h (i.e., with no restriction on λ).

Let us write $\mathrm{H}(\lambda)$ for the set of all continuous affine linear functions h on $\mathrm{E} \times \mathbf{R}$ of the form $h(\mathbf{x}, t) = f(\mathbf{x}) - \lambda t$. Then

$$\mathrm{H}(\lambda) = \begin{cases} \lambda \cdot \mathrm{H}(1) & \text{when } \lambda > 0 \\ |\lambda| \cdot \mathrm{H}(-1) & \text{when } \lambda < 0 \end{cases}$$

(because, for $\lambda \neq 0$, $f \mapsto |\lambda| f$ is a permutation of the set of all continuous affine linear functions on E); thus the validity of $(*)$ for all $\lambda \in \mathbf{R}$ is a consequence of its validity for $\lambda = 0, 1$, and -1. ♢

The proof of the cited Prop. 5 from TVS, II, §5, No. 4 is based on the following:

Remark. If X is a topological space and $f : \mathrm{X} \to \overline{\mathbf{R}}$ is lower semi-continuous, then the function $\varphi : \mathrm{X} \times \mathbf{R} \to \overline{\mathbf{R}}$ defined by $\varphi(x, t) = f(x) - t$ is lower semi-continuous.

Proof. Given a point $(x_0, t_0) \in \mathrm{X} \times \mathbf{R}$ and a real number $h \in \mathbf{R}$ such that $\varphi(\mathbf{x}_0, t_0) > h$, we seek a neighborhood of (x_0, t_0), say of the form $\mathrm{U} \times \mathrm{V}$, such that $\varphi(x, t) > h$ on $\mathrm{U} \times \mathrm{V}$ (GT, IV, §6, No. 2, Def. 1).

We have $f(x_0) - t_0 = \varphi(x_0, t_0) > h$, that is, $f(x_0) - (t_0 + h) > 0$; choose ε so that $0 < \varepsilon < f(x_0) - (t_0 + h)$, thus $f(x_0) > t_0 + h + \varepsilon$. Since f is lower semi-continuous at x_0, there exists a neighborhood U of x_0 such that $f(x) > t_0 + h + \varepsilon$ for all $x \in \mathrm{U}$.

Let $\mathrm{V} =]t_0 - \varepsilon, t_0 + \varepsilon[$, so that for $t \in \mathrm{V}$ one has $t < t_0 + \varepsilon$, hence $-t > -t_0 - \varepsilon$.

Then $\mathrm{U} \times \mathrm{V}$ is a neighborhood of (x_0, t_0) and, for every $(x, t) \in \mathrm{U} \times \mathrm{V}$,

$$\varphi(x, t) = f(x) - t > (t_0 + h + \varepsilon) - t > (t_0 + h + \varepsilon) - t_0 - \varepsilon = h ,$$

that is, $\varphi > h$ on $\mathrm{U} \times \mathrm{V}$. ♢

{The argument shows that if f is lower semi-continuous at a point $a \in \mathrm{X}$, then φ is lower semi-continuous at (a, b) for every $b \in \mathbf{R}$.}

IV.105, ℓ. −14.

“ $\overline{u}(\mathbf{x}) \geqslant \big(\overline{u}(\mathbf{y}) + \overline{u}(\mathbf{z})\big)/2$ ”

Proof #1. If f is a continuous affine linear function on E such that $f \geqslant u$ on K, then $f \geqslant \overline{u}$ on K by the definition of $\overline{u}$, therefore

$$\frac{1}{2}\overline{u}(\mathbf{y}) + \frac{1}{2}\overline{u}(\mathbf{z}) \leqslant \frac{1}{2}f(\mathbf{y}) + \frac{1}{2}f(\mathbf{z}) = f(\mathbf{x}),$$

and the assertion follows on taking the infimum of $f(\mathbf{x})$ over all such f.

Proof #2. Since $\overline{u}\big|\mathrm{K}$ is the lower envelope of a family of concave (even affine-linear) functions $f\big|\mathrm{K}$ minorized by the finite-valued function u, it follows that $\overline{u}\big|\mathrm{K}$ is concave (TVS, II, §2, No. 9). That is, if $\mathbf{y}, \mathbf{z} \in \mathrm{K}$ and $\mathbf{x} = \alpha\mathbf{y} + (1-\alpha)\mathbf{z}$ with $0 \leqslant \alpha \leqslant 1$, then $\overline{u}(\mathbf{x}) \geqslant \alpha\overline{u}(\mathbf{y}) + (1-\alpha)\overline{u}(\mathbf{z})$.

IV.105, ℓ. −11, −10.

“...there exists, by Prop. 1 of No. 1, a positive measure ν on G, of total mass 1, having $(\mathbf{a}, \overline{u}(\mathbf{a}))$ as barycenter.”

Recall that S is the closed convex envelope of the compact set G; note that Prop. 1 does not require that the closed convex envelope be compact.

IV.105, ℓ. −6.

$$\mathbf{a} = \int \mathbf{x}\, d\mu(\mathbf{x}) \quad \text{and} \quad \overline{u}(\mathbf{a}) = \int u(\mathbf{x})\, d\mu(\mathbf{x}). \tag{6}$$

The mapping $\varphi : \mathrm{K} \to \mathrm{E} \times \mathbf{R}$ defined by $\varphi(\mathbf{x}) = (\mathbf{x}, u(\mathbf{x}))$ $(\mathbf{x} \in \mathrm{K})$ is continuous and injective, and $\varphi(\mathrm{K}) = \mathrm{G}$ (the graph of u); defining $p : \mathrm{G} \to \mathrm{K}$ by $p(\mathbf{x}, u(\mathbf{x})) = \mathbf{x}$ $(\mathbf{x} \in \mathrm{K})$ (that is, $p = \mathrm{pr}_1\big|\mathrm{G}$), one sees that φ effects a homeomorphism of K onto the subspace G of $\mathrm{E} \times \mathbf{R}$, with inverse mapping p.

If $g \in \mathscr{C}(\mathrm{K}; \mathbf{R})$, then $g \circ p \in \mathscr{C}(\mathrm{G}; \mathbf{R})$ and one defines $\mu(g) = \nu(g \circ p)$; formally,

$$\int g(\mathbf{x})\, d\mu(\mathbf{x}) = \int \big(g \circ p\big)(\mathbf{x}, u(\mathbf{x}))\, d\nu(\mathbf{x}, u(\mathbf{x})) = \int g(\mathbf{x})\, d\nu(\mathbf{x}, u(\mathbf{x})). \tag{$*$}$$

On the other hand, if F is any Hausdorff locally convex space over $\mathbf{R}$ and if $\mathbf{g} \in \mathscr{C}(\mathrm{K}; \mathrm{F})$, then

$$\int \mathbf{g}\, d\mu = \int (\mathbf{g} \circ p)\, d\nu \qquad (\text{as elements of } \mathrm{F}'^{*}); \tag{$**$}$$

for, if $\mathbf{w}' \in \mathrm{F}'$ then

$$\Big\langle \int \mathbf{g}\, d\mu, \mathbf{w}' \Big\rangle = \int \langle \mathbf{g}, \mathbf{w}' \rangle \, d\mu = \int (\mathbf{w}' \circ \mathbf{g})\, d\mu = \int \big((\mathbf{w}' \circ \mathbf{g}) \circ p\big) d\nu$$
$$= \int \big(\mathbf{w}' \circ (\mathbf{g} \circ p)\big) d\nu = \int \langle \mathbf{g} \circ p, \mathbf{w}' \rangle \, d\nu = \Big\langle \int (\mathbf{g} \circ p)\, d\nu, \mathbf{w}' \Big\rangle.$$

In particular, consider $\mathrm{F} = \mathrm{E} \times \mathbf{R}$ and let $\mathbf{g} : \mathrm{K} \to \mathrm{E} \times \mathbf{R}$ be the function defined by

$$\mathbf{g}(\mathbf{x}) = (\mathbf{x}, u(\mathbf{x})) \qquad (\mathbf{x} \in \mathrm{K})\,;$$

then $(\mathbf{g} \circ p)(\mathbf{x}, u(\mathbf{x})) = \mathbf{g}(\mathbf{x}) = (\mathbf{x}, u(\mathbf{x}))$, thus $\mathbf{g} \circ p$ is the canonical injection $\mathrm{G} \to \mathrm{E} \times \mathbf{R}$, and $(**)$ yields the formula

$$(\dagger) \qquad \int (\mathbf{x}, u(\mathbf{x}))\, d\mu(\mathbf{x}) = \int (\mathbf{x}, u(\mathbf{x}))\, d\nu(\mathbf{x}, u(\mathbf{x})) = \mathbf{b}_\nu = (\mathbf{a}, \overline{u}(\mathbf{a}))$$

by the choice of ν. Let us calculate the left side of $(\dagger)$ as an element of $(\mathrm{E} \times \mathbf{R})'^*$: for $\mathbf{z}' + \lambda \in (\mathrm{E} \times \mathbf{R})' = \mathrm{E}' \oplus \mathbf{R}$ one has

$$\Big\langle \int (\mathbf{x}, u(\mathbf{x}))\, d\mu(\mathbf{x}), \mathbf{z}' + \lambda \Big\rangle = \int \langle (\mathbf{x}, u(\mathbf{x})), \mathbf{z}' + \lambda \rangle \, d\mu(\mathbf{x})$$
$$\int \big[\langle \mathbf{x}, \mathbf{z}' \rangle + \lambda u(\mathbf{x})\big]\, d\mu(\mathbf{x})$$
$$= \int \langle \mathbf{x}, \mathbf{z}' \rangle \, d\mu(\mathbf{x}) + \lambda \int u(\mathbf{x})\, d\mu(\mathbf{x})$$
$$= \Big\langle \int \mathbf{x}\, d\mu(\mathbf{x}), \mathbf{z}' \Big\rangle + \lambda \int u(\mathbf{x})\, d\mu(\mathbf{x})$$
$$= \Big\langle \Big(\int \mathbf{x}\, d\mu(\mathbf{x}), \int u(\mathbf{x})\, d\mu(\mathbf{x}) \Big), \mathbf{z}' + \lambda \Big\rangle,$$

thus

$$(\dagger\dagger) \qquad \int (\mathbf{x}, u(\mathbf{x}))\, d\mu(\mathbf{x}) = \Big(\int \mathbf{x}\, d\mu(\mathbf{x}), \int u(\mathbf{x})\, d\mu(\mathbf{x}) \Big);$$

from $(\dagger)$ and $(\dagger\dagger)$ we have $\Big(\int \mathbf{x}\, d\mu(\mathbf{x}), \int u(\mathbf{x})\, d\mu(\mathbf{x}) \Big) = (\mathbf{a}, \overline{u}(\mathbf{a}))$, whence (6).

IV.105, $\ell.\ -5, -4.$

"The function $\overline{u}$ is upper semi-continuous and bounded on K"

If f is a continuous affine linear function on E such that $f \geqslant u$ on K, then $f\big|\mathrm{K} \geqslant \overline{u}\big|\mathrm{K} \geqslant u$, thus $\overline{u}\big|\mathrm{K}$ is sandwiched between two continuous functions on the compact space K, hence is bounded on K. Since the functions $f|\mathrm{K}$ are upper semi-continuous (even continuous) so is their lower envelope $\overline{u}|\mathrm{K}$ (GT, IV, §6, No. 2, Th. 4).

IV.105, $\ell.\ -4, -3.$

"... hence is μ-integrable (§4, No. 4, Cor. 1 of Prop. 5)"

Let $c = \inf_{\mathbf{x}\in \mathrm{K}} u(\mathbf{x})$. For every continuous affine linear function f on E such that $f \geqslant u$ on K, one has $0 \leqslant u - c1_{\mathrm{K}} \leqslant f|\mathrm{K} - c1_{\mathrm{K}}$, and $\overline{u}|\mathrm{K} - c1_{\mathrm{K}}$ is the lower envelope of the positive continuous functions $f|\mathrm{K} - c1_{\mathrm{K}}$ hence is upper semi-continuous (GT, IV, §6, No. 2, Th. 4); since $\overline{u}|\mathrm{K} - c1_{\mathrm{K}}$ is bounded and μ is a bounded measure, it follows that $\mu^*(\overline{u}|\mathrm{K} - c1_{\mathrm{K}}) < +\infty$ (§1, No. 3, Prop. 10), therefore $\overline{u}|\mathrm{K} - c1_{\mathrm{K}}$ is integrable (§4, No. 4, Cor. 1 of Prop. 5), hence so is $\overline{u}|\mathrm{K}$.

IV.105, $\ell.\ -3, -2.$

"... the function $-\overline{u}$ is by definition convex"

The convexity of $-\overline{u}|\mathrm{K}$ is immediate from the concavity of $\overline{u}|\mathrm{K}$ (see the note for $\ell.\ -14$).

IV.106, $\ell.\ 3, 4.$

" $u(\mathbf{x}) = \overline{u}(\mathbf{x})$ *almost everywhere for* μ."

From $u \leqslant \overline{u}|\mathrm{K}$ we have $\int u\,d\mu \leqslant \int(\overline{u}|\mathrm{K})\,d\mu$, whereas $\int u\,d\mu \geqslant \int(\overline{u}|\mathrm{K})\,d\mu$ by (8); thus $\overline{u}|\mathrm{K} - u \geqslant 0$ and $\int(\overline{u}|\mathrm{K} - u)\,d\mu = 0$, therefore $\overline{u}|\mathrm{K} - u = 0$ μ-almost everywhere in K (§2, No. 3, Th. 1).

IV.106, $\ell.\ 5.$

"Theorem 1 will be proved once the following lemma has been established:"

Let E and K be as in the statement of Th. 1 and assume Lemma 3 established, whose proof shows that there exists a *continuous* strictly convex function $u : \mathrm{K} \to \mathbf{R}$. Construct $\overline{u}|\mathrm{K}$ as in Lemma 1.

Fix any point $\mathbf{a} \in \mathrm{K}$ and construct the measure μ on K as in the discussion following the proof of Lemma 2; thus μ has barycenter $\mathbf{a}$, and $u = \overline{u}|\mathrm{K}$ μ-almost everywhere, that is, the set $\{\mathbf{x} \in \mathrm{K} :\ u(\mathbf{x}) < \overline{u}(\mathbf{x})\}$, which is μ-measurable (§5, No. 5, Prop. 8) hence μ-integrable, is μ-negligible:

$$\mu(\{\mathbf{x} \in \mathrm{K} :\ u(\mathbf{x}) < \overline{u}(\mathbf{x})\}) = 0\,.$$

But, if M is the set of extremal points of K, we know from Lemma 2 that $\mathrm{K} - \mathrm{M} \subset \{\mathbf{x} \in \mathrm{K} :\ u(\mathbf{x}) < \overline{u}(\mathbf{x})\}$, therefore $\mathrm{K} - \mathrm{M}$ is μ-negligible.

Summarizing: Given any $\mathbf{a} \in \mathrm{K}$, there exists a positive measure μ on K with total mass 1 and barycenter $\mathbf{a}$, such that $\mu(\mathrm{K} - \mathrm{M}) = 0$.

{In the language of Ch. V, §5, No. 7, Def. 4, μ is concentrated on the set M of extremal points of K.}

IV.106, ℓ. 10.

"... subspace $\mathscr{A}$ "

Clearly a *linear* subspace of $\mathscr{C}(\mathrm{K};\mathbf{R})$.

IV.106, ℓ. 18.

"... for this it suffices that $h_n(\mathbf{x}) \neq h_n(\mathbf{x}')$ "

Let h be an affine linear function on K, and let $\mathbf{x}, \mathbf{x}'$ be points of K such that $h(\mathbf{x}) \neq h(\mathbf{x}')$; we are to show that h^2 is strictly convex on the segment $[\mathbf{x}, \mathbf{x}'] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{x}' : 0 \leqslant \lambda \leqslant 1\}$. That h^2 is convex has already been noted (TVS, II, §2, No. 8, *Examples*), and the computation in the cited *Examples* show that if $0 < \lambda < 1$ then

$$h^2\big(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}'\big) < \lambda h^2(\mathbf{x}) + (1-\lambda)h^2(\mathbf{x}').$$

Suppose $\mathbf{y}, \mathbf{y}' \in [\mathbf{x}, \mathbf{x}']$ with $\mathbf{y} \neq \mathbf{y}'$, and let $0 < \lambda < 1$; to prove that $h^2\big(\lambda\mathbf{y} + (1-\lambda)\mathbf{y}'\big) < \lambda h^2(\mathbf{y}) + (1-\lambda)h^2(\mathbf{y}')$, it suffices by the foregoing to show that $h(\mathbf{y}) \neq h(\mathbf{y}')$. Say

$$\mathbf{y} = \rho\mathbf{x} + (1-\rho)\mathbf{x}', \quad \mathbf{y}' = \sigma\mathbf{x} + (1-\sigma)\mathbf{x}' \quad (0 \leqslant \rho, \sigma \leqslant 1).$$

Assuming to the contrary that $h(\mathbf{y}) = h(\mathbf{y}')$, one has

$$\rho h(\mathbf{x}) + (1-\rho)h(\mathbf{x}') = \sigma h(\mathbf{x}) + (1-\sigma)h(\mathbf{x}'),$$

whence $(\rho-\sigma)h(\mathbf{x}) = (\rho-\sigma)h(\mathbf{x}')$; since $h(\mathbf{x}) \neq h(\mathbf{x}')$, necessarily $\rho - \sigma = 0$, contrary to $\mathbf{y} \neq \mathbf{y}'$.

Implicit in the foregoing: Suppose E is a vector space over $\mathbf{R}$, $\mathbf{x}$ and $\mathbf{x}'$ are distinct elements of E, $[\mathbf{x}, \mathbf{x}'] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{x}' : 0 \leqslant \lambda \leqslant 1\}$, and $h : [\mathbf{x}, \mathbf{x}'] \to \mathbf{R}$ is affine-linear. Then $h(\mathbf{x}) = h(\mathbf{x}') \Leftrightarrow h$ is constant on $[\mathbf{x}, \mathbf{x}']$, and $h(\mathbf{x}) \neq h(\mathbf{x}') \Leftrightarrow h$ is injective.

IV.106, ℓ. −13 to ℓ. −7.

"COROLLARY."

Review of terminology. To say that C is a **cone** in E with **vertex** 0 means that for every $\mathbf{x} \in \mathrm{C}$, $\mathbf{x} \neq 0$, C contains the set $\mathbf{R}^+\mathbf{x} = \{r\mathbf{x} : r \geqslant 0\}$, called a **closed half-line originating at** 0 (TVS, II, §2, No. 4, paragraph following Def. 3); in other words, C is the union of a set of half-lines originating at 0. A *convex* cone C with vertex 0 is said to be **proper** if it contains no line passing through 0 (*loc. cit.*, sentence before Prop. 9), in other words, if $\mathbf{x} \in \mathrm{C}$, $\mathbf{x} \neq 0$, then $-\mathbf{x} \notin \mathrm{C}$. A convex cone C with vertex 0 is proper if and only if 0 is an extremal point of C (TVS, II, §7, No. 2); for, to say that 0 is internal to a segment with distinct end-points in C is equivalent to saying that C contains a line through 0.

Let C be a convex cone in E with vertex 0, and let $D \subset C$ be a closed half-line originating at 0. One says that D is an **extremal generator** of C if, whenever I is an open segment in C that does not contain 0 but intersects D, necessarily $I \subset D$.

Examples. (i) In $E = \mathbf{R}^3$ let

$$C_1 = \{(x, y, z) : \ x \in \mathbf{R},\ y \in \mathbf{R},\ z \leqslant 0\},$$
$$C_2 = \{(x, y, z) : \ x \in \mathbf{R},\ y \geqslant 0,\ z \leqslant 0\},$$
$$C_3 = \{(x, y, z) : \ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\},$$

respectively: the points on or below the xy-plane; the points that are on or below the xy-plane and are on or to the right of the xz-plane; the points of the 'first octant'.

C_1 is a convex cone with vertex 0 (not proper) with no extremal generators; C_2 is a convex cone with vertex 0 (not proper) whose only extremal generators are the positive x-axis and the negative x-axis; C_3 is a proper convex cone with vertex 0, whose three extremal generators are the positive coordinate axes, and whose only extremal point is $(0, 0, 0)$.

(ii) In a plane in $\mathbf{R}^3$ that does not pass through $(0, 0, 0)$, let K be a circular disk, and let C be the set of all closed half-lines originating at $(0, 0, 0)$ and passing through some point of K; then C is a proper convex cone with vertex $(0, 0, 0)$ whose only extremal generators are the half-lines originating at $(0, 0, 0)$ that pass through some point of the circumference of K (i.e., some extremal point of K), and whose only extremal point is $(0, 0, 0)$.

If instead K is a convex polygon, then C is *polyhedral* (TVS, II, §7, Exer. 24), its extremal generators are the closed half-lines originating at $(0, 0, 0)$ that pass through a vertex (i.e., extremal point) of the polygon, and its only extremal point is $(0, 0, 0)$.

Let C be a convex set in a Hausdorff topological vector space E. A nonempty *compact* convex subset A of C is called a **cap** of C if $C - K$ is convex (TVS, II, §7, No. 2, Def. 3).

If $\mathbf{a}$ is an extremal point of C (if there are any!) then $A = \{\mathbf{a}\}$ is a cap of C.

In the above examples: C_1 contains no caps. For, assume to the contrary that A is a cap of C_1. If A contained a point P of the xy-plane, then $C_1 - A$ would contain two points Q and R on the xy-plane such that P is internal to the segment QR, contradicting the convexity of $C_1 - A$. Thus A is contained in the interior $z < 0$ of C_1. Let P be any point of the xy-plane and let Q be a point of A. Prolong PQ to a point R in the interior of C_1 that does not belong to A. Then Q belongs to the segment PR in the convex set $C_1 - A$, contrary to $Q \in A$.

Similarly C_2 contains no caps.

Slice C_3 into two parts by a plane, not containing $(0,0,0)$, that intersects all three positive coordinate axes—a bounded closed convex part A and an unbounded convex part $C_3 - A$ (so to speak, A is a compact 'truncate' of C_3). Then A is a cap of C_3, and C_3 is obviously the union of all of its caps (see also TVS, II, §7, No. 2, Prop. 5). Other caps of C_3: the segments OP, where $O = (0,0,0)$ and P is a point on one of the positive coordinate axes (so to speak, a bounded truncate of a positive coordinate axis); a triangular area OPQ, where P and Q are points on different positive coordinate axes (a truncate of the 'first quadrant' of one of the coordinate planes).

IV.106, ℓ. −6.

"For, **x** belongs to a cap K of C (TVS, II, §7, No. 2, Prop. 5)"

Write $\tau_w = \sigma(E, E')$ for the weakened topology on E. The cited Prop. 5, applied in the weak space (E, τ_w), asserts that C is the union of its caps K, which are convex sets compact for τ_w. In particular, **x** belongs to such a K.

IV.106, ℓ. −5.

" $M \cap K$ contains the set of extremal points of K (*loc. cit.*, Cor. 1 of Prop. 4)."

Let B be the set of extremal points of K. Since C is proper, 0 is extremal in C, hence also in K; thus $0 \in B$. By the cited Cor. 1 (applied in the weak space (E, τ_w) of the preceding note),

$$B = \{0\} \cup \{\text{certain points of } M\};$$

since $0 \in M$, we see that $B \subset M$, thus $B \subset M \cap K$.

IV.106, ℓ. −4.

"It then suffices to apply Th. 1."

As noted above, K is a convex subset of C that is compact for the weakened topology $\tau_w = \sigma(E, E')$, and by assumption C is metrizable for the topology induced by τ_w, therefore so is K; applying Th. 1 to the metrizable compact convex subset K of (E, τ_w), there exists a positive measure λ on $(K, \tau_w \cap K)$, of total mass 1, such that **x** is the barycenter of λ and $\lambda(K - B) = 0$, where B is the set of extremal points of K. As observed in the preceding note, $B \subset M \cap K$, therefore also $\lambda^*(K - M \cap K) = 0$.

IV.107, ℓ. 9, 10.

"... the second assertion follows at once from the definitions and (9)"

At issue is the continuity of the mapping $i_{\mathscr{H}} : X \to \mathscr{H}'$ for the topology $\sigma(\mathscr{H}', \mathscr{H})$ on $\mathscr{H}'$. For every $f \in \mathscr{H}$, the mapping

$$x \mapsto \langle f, i_{\mathscr{H}}(x) \rangle = f(x)$$

is continuous, and $\sigma(\mathscr{H}', \mathscr{H})$ is the initial topology for the family of linear forms

$$\alpha \mapsto \alpha(f) = \langle f, \alpha \rangle \qquad (\alpha \in \mathscr{H}')$$

indexed by the functions $f \in \mathscr{H}$ (TVS, II, §6, No. 2, Def. 2), whence the assertion (GT, I, §2, No. 3, Prop. 4).

In more direct terms, given any $x_0 \in X$, the continuity of $i_{\mathscr{H}}$ at x_0 may be seen as follows. A basic neighborhood V of $i_{\mathscr{H}}(x_0)$ in $\mathscr{H}'$ is given by

$$V = \{\alpha \in \mathscr{H}' : \ |\alpha(f_k) - \big(i_{\mathscr{H}}(x_0)\big)(f_k)| < \varepsilon \ \text{ for } k = 1, \dots, n\},$$

where $f_1, \dots, f_n \in \mathscr{H}$ and $\varepsilon > 0$. Then

$$\begin{aligned} \overset{-1}{i_{\mathscr{H}}}(V) &= \{x \in X : \ i_{\mathscr{H}}(x) \in V\} \\ &= \{x \in X : \ |(i_{\mathscr{H}}(x))(f_k) - (i_{\mathscr{H}}(x_0))(f_k)| < \varepsilon \ \text{ for } k = 1, \dots, n\} \\ &= \{x \in X : \ |f_k(x) - f_k(x_0)| < \varepsilon \ \text{ for } k = 1, \dots, n\} \\ &= \bigcap_{k=1}^{n} \{x \in X : \ |f_k(x) - f_k(x_0)| < \varepsilon\}, \end{aligned}$$

which is a neighborhood of x_0 in X by the continuity of the f_k.

IV.107, ℓ. −11 to −9.

"... and the assertion follows from the fact that $\mathscr{H}'$, equipped with the weak topology $\sigma(\mathscr{H}', \mathscr{H})$, is *quasi-complete* (TVS, III, §4, No. 2, Cor. 5 of Th. 1)."

In such a space, the closed convex envelope of a compact set is compact (TVS, III, §1, No. 6, sentence before Prop. 9; see also TVS, IV, §5, No. 5, Th. 3). For a simpler proof of the compactness of C, one may cite the weak compactness of the closed unit ball of the dual of a normed space (TVS, III, §3, No. 4, Cor. 3 of Prop. 4).

IV.107, ℓ. −6.

"... the Dirac measure $\varepsilon_{i_{\mathscr{H}}(x)}$ arising from ε_x."

The generic function $g \in \mathscr{C}\big(i_{\mathscr{H}}(X)\big) = \mathscr{K}\big(i_{\mathscr{H}}(X)\big)$ has the form $g = f'$, where $f \in \mathscr{C}(X) = \mathscr{K}(X)$ and

$$f'\big(i_{\mathscr{H}}(z)\big) = f(z) \qquad \text{for all } z \in X.$$

Let λ be a measure on X. The corresponding measure μ on $i_{\mathscr{H}}(\mathrm{X})$ is defined by

$$\mu(f') = \lambda(f) \qquad \text{for all } f \in \mathscr{C}(\mathrm{X}).$$

In particular, if $\lambda = \varepsilon_x$ then, for all $f \in \mathscr{C}(\mathrm{X})$,

$$\mu(f') = \varepsilon_x(f) = f(x) = f'\big(i_{\mathscr{H}}(x)\big) = \varepsilon_{i_{\mathscr{H}}(x)}(f'),$$

thus $\mu = \varepsilon_{i_{\mathscr{H}}(x)}$.

IV.107, ℓ. -6 to -3.

"To say that μ admits $i_{\mathscr{H}}(x)$ as barycenter means, by definition, that

$$\int_{\mathrm{X}} \langle h, i_{\mathscr{H}}(z)\rangle \, d\lambda(z) = \langle h, i_{\mathscr{H}}(x)\rangle$$

for every function $h \in \mathscr{H}$."

I did not succeed in working through the assertion as stated, but have persuaded myself that the following reformulation is correct: assuming $\lambda \in \mathscr{M}(\mathrm{X})$ and $\mu \in \mathscr{M}\big(i_{\mathscr{H}}(\mathrm{X})\big)$ are paired as in the preceding notes,

$$(*) \qquad b_\mu = i_{\mathscr{H}}(x) \quad \Leftrightarrow \quad h(x) = \int h \, d\lambda \quad \text{for all } h \in \mathscr{H},$$

that is, μ has barycenter $i_{\mathscr{H}}(x)$ if and only if λ satisfies (10) for every $h \in \mathscr{H}$.

The setting for discussing the barycenter of μ is as follows. We have a Hausdorff locally convex space $\mathrm{E} = \mathscr{H}'$, where $\mathscr{H} \subset \mathscr{C}(\mathrm{X})$ is a normed space, and E is equipped with the weak topology $\sigma(\mathscr{H}', \mathscr{H})$; then $\mathrm{E}' = (\mathscr{H}')' = \mathscr{H}$ (TVS, II, §6, No. 2, Prop. 3). We have a compact subspace $\mathrm{K} = i_{\mathscr{H}}(\mathrm{X})$ of E, whose closed convex envelope C in E is compact by part (i) of the present Proposition. Thus the measure μ on K is eligible to have a barycenter $b_\mu \in \mathrm{E}'^* = (\mathscr{H}')'^* = \mathscr{H}^*$ (No. 1, Def. 1), and in fact $b_\mu \in \mathrm{C} \subset \mathrm{E} = \mathscr{H}'$ (No. 1, Prop. 1 and its Corollary), characterized by the property

$$b_\mu(h) = \langle b_\mu, h\rangle = \Big\langle \int_{\mathrm{K}} \alpha \, d\mu(\alpha), h \Big\rangle = \int_{\mathrm{K}} \langle \alpha, h\rangle \, d\mu(\alpha) = \int_{\mathrm{K}} \alpha(h) \, d\mu(\alpha)$$

for all $h \in \mathscr{H} = \mathrm{E}'$. Defining $\hat{h} : \mathrm{K} \to \mathbf{R}$ by $\hat{h}(\alpha) = \alpha(h)$ for all $\alpha \in \mathrm{K} \subset \mathrm{E} = \mathscr{H}'$, one has $\hat{h} \in \mathscr{C}(\mathrm{K})$ because $\alpha \mapsto \alpha(h) = \langle \alpha, h\rangle$ is the restriction to K of one of the linear forms defining the topology $\sigma(\mathscr{H}', \mathscr{H})$ on E. The corresponding function $\hat{h}' \in \mathscr{C}(\mathrm{X})$ is defined by

$$\hat{h}'(z) = \hat{h}\big(i_{\mathscr{H}}(z)\big) = \big(i_{\mathscr{H}}(z)\big)(h) = h(z) \qquad \text{(by (9))}$$

for all $z \in \mathrm{X}$. Continuing the above calculation,

$$\begin{aligned}\langle b_\mu, h\rangle &= \int_{\mathrm{K}} \alpha(h)\,d\mu(\alpha) = \int_{\mathrm{K}} \hat{h}(\alpha)\,d\mu(\alpha)\\ &= \int_{\mathrm{X}} \hat{h}'(z)\,d\lambda(z) = \int_{\mathrm{X}} h(z)\,d\lambda(z) = \int h\,d\lambda\,.\end{aligned}$$

Therefore

$$\begin{aligned} b_\mu = i_{\mathscr{H}}(x) \quad &\Leftrightarrow \quad \langle b_\mu, h\rangle = \langle i_{\mathscr{H}}(x), h\rangle \ \text{ for all } h \in \mathscr{H}\\ &\Leftrightarrow \quad \int h\,d\lambda = \big(i_{\mathscr{H}}(x)\big)(h) = h(x) \ \text{ for all } \ h \in \mathscr{H}\,,\end{aligned}$$

whence (∗).

While the notations are at hand, let us complete the proof of (ii). Consider the statements

(a) $i_{\mathscr{H}}(x)$ is an extremal point of of C;

(b) ε_x is the only positive measure λ on X satisfying the condition (10) $\big(\lambda(h) = h(x)\big)$ for all $h \in \mathscr{H}$.

We are to show that (a) ⇔ (b).

Proof of (a) ⇒ (b). The measure $\lambda = \varepsilon_x$ trivially satisfies (10) for every $h \in \mathscr{H}$. Assuming (a), suppose λ is any positive measure on X satisfying (10) for every $h \in \mathscr{H}$ (in particular, $1_{\mathrm{X}} \in \mathscr{H}$, and $\lambda(1_{\mathrm{X}}) = 1_{\mathrm{X}}(x) = 1$, so λ has total mass 1). Let μ be the corresponding measure on $i_{\mathscr{H}}(\mathrm{X})$, which is also positive and of total mass 1. We know from (∗) that $b_\mu = i_{\mathscr{H}}(x)$; but $i_{\mathscr{H}}(x)$ is extremal in C by the assumption (a), therefore $\mu = \varepsilon_{i_{\mathscr{H}}(x)}$ by the Corollary of Prop. 3 of No. 2. Since the measure on X corresponding to $\varepsilon_{i_{\mathscr{H}}(x)}$ is ε_x, we conclude that $\lambda = \varepsilon_x$, whence (b).

Proof of (b) ⇒ (a). Assume (b). To prove (a), it suffices to verify that $i_{\mathscr{H}}(x)$ satisfies the conditions of the above-cited Corollary. At any rate, $\varepsilon_{i_{\mathscr{H}}(x)}$ is a positive measure on $i_{\mathscr{H}}(\mathrm{X})$, of total mass 1, whose barycenter is $i_{\mathscr{H}}(x)$ (No. 1, *Example*). On the other hand, suppose μ is a positive measure on $i_{\mathscr{H}}(\mathrm{X})$, of total mass 1, such that $b_\mu = i_{\mathscr{H}}(x)$, and let λ be the corresponding measure on X. By (∗), λ satisfies (10) for every $h \in \mathscr{H}$, therefore, by the assumption (b), $\lambda = \varepsilon_x$. As the measure on $i_{\mathscr{H}}(\mathrm{X})$ corresponding to ε_x is $\varepsilon_{i_{\mathscr{H}}(x)}$, we conclude that $\mu = \varepsilon_{i_{\mathscr{H}}(x)}$. Thus the conditions of the Corollary are satisfied, whence (a).

Remarks. 1. If λ is any measure on the compact space X, its restriction to the normed space $\mathscr{H}$ is a continuous linear form, and $\|\lambda|\mathscr{H}\| \leqslant \|\lambda\|$. The positive measures on X are the positive linear forms on $\mathscr{K}(\mathrm{X}) = \mathscr{C}(\mathrm{X})$ (Ch. III, §1, No. 5, Th. 1), and $\|\lambda\| = \lambda(1)$ for all such measures λ (§4, No. 7, Prop. 12); since $1 \in \mathscr{H}$ and $\|1\| = 1$, one has

$$\|\lambda|\mathscr{H}\| \geqslant |(\lambda|\mathscr{H})(1)| = \lambda(1) = \|\lambda\|\,,$$

thus $\|\lambda|\mathscr{H}\| = \|\lambda\| = \lambda(1)$ when $\lambda \geqslant 0$.

Let us write $\Phi : \mathscr{M}(\mathrm{X}) \to \mathscr{H}'$ for the mapping defined by $\Phi(\lambda) = \lambda\big|\mathscr{H}$. From $\|\Phi(\lambda)\| \leqslant \|\lambda\|$ we see that Φ is a continuous linear mapping between Banach spaces, and $\|\Phi\| \leqslant 1$; in fact $\|\Phi\| = 1$ since, if λ is any nonzero positive measure on X of norm 1 (for example, any Dirac measure ε_x) then $\|\Phi\| \geqslant \|\Phi(\lambda)\| = \|\lambda\big|\mathscr{H}\| = \|\lambda\| = 1$.

2. If $x \in \mathrm{X}$, λ is a positive measure on X of total mass 1, and μ is the corresponding measure on $i_{\mathscr{H}}(\mathrm{X})$, the equivalence $(*)$ can be expressed as

$$\lambda\big|\mathscr{H} = \varepsilon_x\big|\mathscr{H} \quad \Leftrightarrow \quad b_\mu = i_{\mathscr{H}}(x)\,.$$

3. Writing $\mathscr{M}_+(\mathrm{X})$ for the set of all positive measures on X, the assertion (ii) of Prop. 4, for a point $x \in \mathrm{X}$, can be expressed as

(ii) $\{\lambda \in \mathscr{M}_+(\mathrm{X}) :\ \Phi(\lambda) = \varepsilon_x\big|\mathscr{H}\} = \{\varepsilon_x\} \quad \Leftrightarrow \quad i_{\mathscr{H}}(x)$ is extremal in C.

If Φ is injective then the condition on the left in (ii) holds for every $x \in \mathrm{X}$, and so $i_{\mathscr{H}}(\mathrm{X})$ is precisely the set of all extremal points of C; this is the case if $\mathscr{H}$ is dense in $\mathscr{C}(\mathrm{X})$—for example, if $f \in \mathscr{H} \Rightarrow |f| \in \mathscr{H}$ (M.H. Stone's theorem, GT, X, §4, No. 1, Th. 2) or if $f, g \in \mathscr{H} \Rightarrow fg \in \mathscr{H}$ (*loc. cit.*, No. 4, Prop. 6).

The case that $\mathscr{H} = \mathscr{C}(\mathrm{X})$ has been taken up in §4, No. 8, Prop. 15 and is continued in Ch. VI, §1, No. 6, *Remark* 1). A (real) *character* of a commutative algebra $\mathscr{A}$ over $\mathbf{R}$ with unity is an algebra epimorphism $\mathscr{A} \to \mathbf{R}$; the characters of $\mathscr{C}(\mathrm{X})$ are precisely the ε_x (Gillman and Jerison, *Rings of continuous functions*, p. 57, item **4.9**, Van Nostrand, Princeton, N.J., 1960).

5. In a vague sense, $i_{\mathscr{H}}(\mathrm{X})$ is a 'linearization' of the compact space X, 'tailored' to the linear subspace $\mathscr{H}$ of $\mathscr{C}(\mathrm{X})$, placing X in a structurally richer context (topological vector spaces) than that of topological spaces. Prop. 4 reformulates a property of a point $x \in \mathrm{X}$ with respect to $\mathscr{H}$ in terms of measures on X; measures on $i_{\mathscr{H}}(\mathrm{X})$ play only an auxiliary role in the proof. The property is reformulated in Prop. 6 in topological terms, with measures playing a role only in the proof; this theme culminates in the theorem of Errett Bishop (No. 5, Th. 2), where measures are nowhere in sight. In Prop. 8 and in Choquet's theorem (No. 6, Th. 3), it is measures that are in the forefront.

All in all, Prop. 4 is a subtle, far-reaching result whose cunning remains a mystery to me.

IV.107, ℓ. -3 to -1.

"...the assertion (ii) is just the translation of the criterion of No. 2, Cor. of Prop. 3 for $i_{\mathscr{H}}(x)$ to be an extremal point of C."

See the preceding note.

IV.108, ℓ. 8, 9.

"the weakly closed hyperplane of $\mathscr{H}'$ with equation $\langle h, t'\rangle = \langle h, i_{\mathscr{H}}(x)\rangle$ is a *support hyperplane* of $i_{\mathscr{H}}(\mathrm{X})$."

Order of events: Fix $h \in \mathscr{H}$ and let x be a point of X where h attains its supremum. The linear form $t' \mapsto \langle h, t'\rangle$ $(t' \in \mathscr{H}')$ is continuous for $\sigma(\mathscr{H}', \mathscr{H})$; the set $\mathrm{H} = \{t' \in \mathscr{H}' : \langle h, t'\rangle = \langle h, i_{\mathscr{H}}(x)\rangle\}$ is a closed hyperplane in $\mathscr{H}'$, the relation

$$\text{« } \langle h, i_{\mathscr{H}}(z) \leqslant \langle h, i_{\mathscr{H}}(x)\rangle \quad \text{for all } z \in \mathrm{X} \text{ »}$$

expresses that the points of $i_{\mathscr{H}}(\mathrm{X})$ lie on the same side of H, and since H contains at least the point $i_{\mathscr{H}}(x)$ of $i_{\mathscr{H}}(\mathrm{X})$, it is a support hyperplane of $i_{\mathscr{H}}(\mathrm{X})$ (TVS, II, §5, No. 2, Def. 3). It follows that H is also a support hyperplane of the closed convex envelope C of $i_{\mathscr{H}}(\mathrm{X})$ in $\mathscr{H}'$, therefore H contains some extremal point t'_0 of C (*loc. cit.*, §7, No. 1, Cor. of Prop. 1). Necessarily $t'_0 \in i_{\mathscr{H}}(\mathrm{X})$ (*loc. cit.*, Cor. of Prop. 2), say $t'_0 = i_{\mathscr{H}}(y)$; then y is by definition $\mathscr{H}$-extremal, and

$$h(y) = \langle h, i_{\mathscr{H}}(y)\rangle = \langle h, t'_0\rangle = \langle h, i_{\mathscr{H}}(x)\rangle = h(x)$$

is the supremum of h (the third equality, because $t'_0 \in \mathrm{H}$).

IV.108, ℓ. 20–24.

"...it is known (TVS, II, §3, No. 1, Prop. 1) that the infimum of the numbers $\lambda(f)$, for all the positive measures on X such that $\lambda(h) = h(x)$ for every function $h \in \mathscr{H}$, is equal to the supremum of the numbers $h(x)$, where h runs over the set of functions $h \in \mathscr{H}$ such that $h \leqslant f$."

I should have said EVT instead of TVS (explanation below).

In the language of the cited Prop. 1, we are talking about the number α'; the notational 'table of concordance' between Prop. 1 and its application here is as follows:

$$\begin{aligned}
\mathrm{E} &\quad\leftrightarrow\quad \mathscr{C}(\mathrm{X}; \mathbf{R})\\
\mathrm{V} &\quad\leftrightarrow\quad \mathscr{H}\\
f &\quad\leftrightarrow\quad \varepsilon_x\big|\mathscr{H} \quad \text{(a positive linear form on } \mathscr{H}\text{)}\\
\mathrm{S}_f &\quad\leftrightarrow\quad \{\lambda \in \mathscr{M}_+(\mathrm{X}) : \ \lambda\big|\mathscr{H} = \varepsilon_x\big|\mathscr{H}\} \quad \text{(Ch. III, §1, No. 5, Th. 1)}\\
h \in \mathrm{S}_f &\quad\leftrightarrow\quad \lambda \in \mathscr{M}_+(\mathrm{X}) \text{ and } \lambda\big|\mathscr{H} = \varepsilon_x\big|\mathscr{H}\\
a \in \mathrm{E} &\quad\leftrightarrow\quad f \in \mathscr{C}(\mathrm{X}; \mathbf{R})\\
\alpha' &\quad\leftrightarrow\quad \sup_{h \in \mathscr{H},\, h \leqslant f} (\varepsilon_x\big|\mathscr{H})(h) = \sup_{h \in \mathscr{H},\, h \leqslant f} h(x)\\
\alpha'' &\quad\leftrightarrow\quad \inf_{h \in \mathscr{H},\, f \leqslant h} (\varepsilon_x\big|\mathscr{H})(h) = \inf_{h \in \mathscr{H},\, f \leqslant h} h(x)\,.
\end{aligned}$$

The assertion of the cited Prop. 1 in EVT (p. EVT II.22) is that

$$(*) \qquad \{h(a) :\ h \in \mathrm{S}_f\} = [\alpha', \alpha''],$$

whence

$$\alpha' = \inf\,[\alpha', \alpha''] = \inf\{h(a) :\ h \in \mathrm{S}_f\}.$$

Translated to the present context, this says that

$$\sup_{h\in\mathscr{H},\, h\leqslant f} h(x) = \inf\{\lambda(f) :\ \lambda \in \mathscr{M}_+(\mathrm{X}) \text{ and } \lambda\big|\mathscr{H} = \varepsilon_x\big|\mathscr{H}\},$$

as asserted.

Incidentally, the infimum in the assertion is attained: there exists a $\lambda' \in \mathrm{S}_{\varepsilon_x|\mathscr{H}}$ such that $\lambda'(f) = \sup_{h\in\mathscr{H},\, h\leqslant f} h(x)$, but λ' depends on f (because α' does) as well as on x. Similarly, there exists a $\lambda'' \in \mathrm{S}_{\varepsilon_x|\mathscr{H}}$ such that $\lambda''(f) = \inf_{h\in\mathscr{H},\, h\geqslant f} h(x)$.

{This argument was perfectly clear to me when I first studied it in 1974; it is lifted verbatim from my notes at the time. On rereading the argument when preparing these notes, with TVS at my elbow, I found that I no longer understood it. After several days of struggle, it occurred to me to look at the French original, and all was clear again.

The problem: In TVS, instead of the conclusion (∗), one finds the conclusion

$$\{h(a) :\ h \in \mathrm{S}_f\} \subset [\alpha', \alpha''],$$

which (i) follows at once from the fact that if $y, z \in \mathrm{V}$ with $z \leqslant a \leqslant y$, and $h \in \mathrm{S}_f$, then

$$f(z) = h(z) \leqslant h(a) \leqslant h(y) = f(y),$$

whence $\alpha' \leqslant h(a) \leqslant \alpha''$, and (ii) is of no help in proving (∗). The problem did not arise in 1974, as I was working from the 2nd French edition of Chs. I and II of EVT.}

IV.108, ℓ. 24–26.

"Suppose that x is $\mathscr{H}$-extremal; it then follows from Prop. 4, (ii) that for every function $f \in \mathscr{C}(\mathrm{X}; \mathbf{R})$,

$$(11) \qquad f(x) = \sup_{h\in\mathscr{H},\, h\leqslant f} h(x)."$$

Since x is $\mathscr{H}$-extremal, by the cited Prop. 4 the *only* positive measure λ on X that extends $\varepsilon_x\big|\mathscr{H}$ is $\lambda = \varepsilon_x$; thus, in the notation of the preceding note, $S_{\varepsilon_x|\mathscr{H}} = \{\varepsilon_x\}$. It follows that, for every $f \in \mathscr{C}(X;\mathbf{R})$, $[\alpha', \alpha''] = \{\varepsilon_x(f)\}$, whence $\alpha' = \alpha'' = f(x)$. Thus, if x is $\mathscr{H}$-extremal, then

$$f(x) = \sup_{h\in\mathscr{H},\, h\leqslant f} h(x) = \inf_{h\in\mathscr{H},\, h\geqslant f} h(x)$$

for every $f \in \mathscr{C}(X;\mathbf{R})$.

IV.108, ℓ. -1.

" $\lambda(\{x\}) = \inf_{U} \lambda(U) \geqslant 1-\varepsilon$ "

By §1, No. 4, Prop. 19.

IV.109, ℓ. 1.

"... therefore $\lambda(\{x\}) = 1$."

Obviously $\lambda(\{x\}) \geqslant 1$; but λ has total mass 1 since it satisfies (10), therefore $\lambda(\{x\}) \leqslant 1$.

IV.108, ℓ. 2.

"... necessarily $\lambda = \varepsilon_x$ "

From $\lambda(X - \{x\}) = \lambda(X) - \lambda(\{x\}) = 1-1 = 0$, one has $\mathrm{Supp}(\lambda) \subset \{x\}$ (§2, No. 2, Prop. 5), therefore $\mathrm{Supp}(\lambda) = \{x\}$ (because $\lambda \neq 0$); since $\lambda(1) = 1$, it follows that $\lambda = \varepsilon_x$ (Ch. III, §2, No. 4, Prop. 12).

IV.109, ℓ. 12, 13.

"The condition a) signifies that G contains the set of extremal points of C."

Recall that C is defined in Prop. 4 to be the closed convex envelope of $i_{\mathscr{H}}(X)$ in $\mathscr{H}'$ (the dual of the normed subspace $\mathscr{H}$ of $\mathscr{C}(X)$, equipped with the weak topology $\sigma(\mathscr{H}', \mathscr{H})$), and shown to be a compact subset of $\mathscr{H}'$. It follows that the set C_{ep} of extremal points of C is contained in the compact subset $i_{\mathscr{H}}(X)$ of $\mathscr{H}'$ (TVS, II, §7, No. 1, Cor. of Prop. 2).

By definition, $\mathrm{Ch}_{\mathscr{H}}(X)$ is the set of all $x \in X$ that satisfy the condition in (ii) of Prop. 4, that is, such that $i_{\mathscr{H}}(x) \in C_{ep}$; thus $\mathrm{Ch}_{\mathscr{H}}(X) = \overset{-1}{i_{\mathscr{H}}}(C_{ep})$. Since F is closed in X and $\check{S}_{\mathscr{H}}(X)$ is the closure of $\mathrm{Ch}_{\mathscr{H}}(X)$ in X, one has

$$F \supset \check{S}_{\mathscr{H}}(X) \quad\Leftrightarrow\quad F \supset \mathrm{Ch}_{\mathscr{H}}(X) \quad\Leftrightarrow\quad i_{\mathscr{H}}(F) \supset i_{\mathscr{H}}\big(\mathrm{Ch}_{\mathscr{H}}(X)\big),$$

that is, $\Leftrightarrow$ $G \supset C_{ep}$.

IV.109, ℓ. 13, 14.

"The condition b) signifies that G meets the intersection of $i_{\mathscr{H}}(X)$ with each of the closed support hyperplanes of $i_{\mathscr{H}}(X)$."

A (weakly) continuous linear form on $\mathscr{H}'$ is a function $\mathbf{z}' \mapsto \langle h, \mathbf{z}' \rangle$ ($\mathbf{z}' \in \mathscr{H}'$) defined by an element $h \in \mathscr{H}$, and if $\alpha \in \mathbf{R}$ the set

$$\{\mathbf{z}' \in \mathscr{H}' : \langle h, \mathbf{z}' \rangle = \alpha\}$$

is a typical closed hyperplane in $\mathscr{H}'$; since X is compact, there exists a point $y \in \mathrm{X}$ such that $\sup_{x \in \mathrm{X}} h(x) = h(y)$, that is,

$$\sup_{x \in \mathrm{X}} \langle h, i_{\mathscr{H}}(x) \rangle = \langle h, i_{\mathscr{H}}(y) \rangle ,$$

and since $\langle h, i_{\mathscr{H}}(x) \rangle \leqslant \langle h, i_{\mathscr{H}}(y) \rangle$ for all $x \in \mathrm{X}$, the set

$$(*) \qquad \mathrm{H} = \{\mathbf{z}' \in \mathscr{H}' : \langle h, \mathbf{z}' \rangle = \langle h, i_{\mathscr{H}}(y) \rangle\}$$

is a typical closed support hyperplane of $i_{\mathscr{H}}(\mathrm{X})$.

Note that a closed hyperplane H in $\mathscr{H}'$ is supporting for $i_{\mathscr{H}}(\mathrm{X})$ if and only if it is supporting for its closed convex envelope C. For, it is clear that $i_{\mathscr{H}}(\mathrm{X})$ lies to one side of H if and only if C does, so it suffices to observe that $\mathrm{H} \cap i_{\mathscr{H}}(\mathrm{X}) \neq \varnothing \Leftrightarrow \mathrm{H} \cap \mathrm{C} \neq \varnothing$, and this is clear from the fact that if H is supporting for C then $\mathrm{H} \cap \mathrm{C}$ contains at least one extremal point of C (TVS, II, §7, No. 1, Cor. of Prop. 1) and every extremal point of C belongs to $i_{\mathscr{H}}(\mathrm{X})$ (*loc. cit.*, Cor. of Prop. 2).

Condition *b*) says: If $h \in \mathscr{H}$ and if

$$\alpha = \sup_{x \in \mathrm{X}} \langle h, i_{\mathscr{H}}(x) \rangle = \sup_{x \in \mathrm{X}} h(x) ,$$

then there exists a point $y \in \mathrm{F}$ such that $\langle h, i_{\mathscr{H}}(y) \rangle = \alpha$; that is, every $h \in \mathscr{H}$ attains its supremum on $i_{\mathscr{H}}(\mathrm{X})$ at some point of $i_{\mathscr{H}}(\mathrm{F}) = \mathrm{G}$.

This means: If H is any closed support hyperplane of $i_{\mathscr{H}}(\mathrm{X})$, then there exists a point $y \in \mathrm{F}$ such that $i_{\mathscr{H}}(y) \in \mathrm{H}$, in other words $\mathrm{G} \cap \mathrm{H} \neq \varnothing$. Since $\mathrm{G} \subset i_{\mathscr{H}}(\mathrm{X})$, $\mathrm{G} \cap \mathrm{H} = \mathrm{G} \cap i_{\mathscr{H}}(\mathrm{X}) \cap \mathrm{H}$, so it is equivalent to say (as in the text) that G intersects $i_{\mathscr{H}}(\mathrm{X}) \cap \mathrm{H}$.

IV.109, ℓ. 15, 16.

"... the condition *c*) signifies that every point of $i_{\mathscr{H}}(\mathrm{X})$ is the barycenter of a measure with support contained in G"

Reviewing the notations employed in the proof of Prop. 4, let us write λ for a measure on the compact space X, and μ for the corresponding measure on the subspace $i_{\mathscr{H}}(\mathrm{X})$ of $\mathscr{H}'$ homeomorphic to X, defined by

$$\mu(f') = \lambda(f) \quad \text{for all } f \in \mathscr{C}(\mathrm{X}) ,$$

where $f'\big(i_{\mathscr{H}}(x)\big) = f(x)$ for all $x \in \mathrm{X}$ (see the note for p. IV.107, ℓ. -6). Writing $\mathrm{Z}(f) = \{x \in \mathrm{X} : f(x) = 0\}$ (the 'zero-set' of f), from

$$x \in \mathrm{Z}(f) \quad \Leftrightarrow \quad i_{\mathscr{H}}(x) \in \mathrm{Z}(f')$$

we see that $x \in \mathrm{Supp}(f) \Leftrightarrow i_{\mathscr{H}}(x) \in \mathrm{Supp}(f')$, whence $\mathrm{Supp}(f') = i_{\mathscr{H}}\big(\mathrm{Supp}(f)\big)$. It follows easily that $\mathrm{Supp}\,\mu = i_{\mathscr{H}}(\mathrm{Supp}\,\lambda)$ (remarks following Def. 1 of Ch. III, §2, No. 2).

Now, $i_{\mathscr{H}}(\mathrm{X})$ is a compact subset of the Hausdorff locally convex space $\mathscr{H}'$; if λ and μ are paired as above, the barycenter of μ is the element b_μ of $(\mathscr{H}')'^* = \mathscr{H}^*$ such that

$$(*) \qquad \langle b_\mu, h\rangle = \int h\,d\lambda \quad \text{for all } h \in \mathscr{H}$$

(see the note for p. IV.107, ℓ. -6 to -3).

For consistency with these notations, let us restate condition c) as follows (replace the letter λ by μ):

c^*) *For every point* $x \in \mathrm{X}$, *there exists a positive measure* λ *of total mass* 1 *on* X, *such that* $\mathrm{Supp}(\lambda) \subset \mathrm{F}$ *and* $h(x) = \int h\,d\lambda$ *for every function* $h \in \mathscr{H}$.

Then, if μ is the measure on $i_{\mathscr{H}}(\mathrm{X})$ corresponding to λ, the condition $\mathrm{Supp}\,\lambda \subset \mathrm{F}$ is equivalent to $i_{\mathscr{H}}(\mathrm{Supp}\,\lambda) \subset i_{\mathscr{H}}(\mathrm{F})$, that is, to $\mathrm{Supp}\,\mu \subset \mathrm{G}$. Thus, the condition c^*) says that for every point $x \in \mathrm{X}$, there exists a positive measure μ on $i_{\mathscr{H}}(\mathrm{X})$ of total mass 1, such that $\mathrm{Supp}\,\mu \subset \mathrm{G}$ and

$$\langle i_{\mathscr{H}}(x), h\rangle = \int h\,d\lambda \quad \text{for all } h \in \mathscr{H},$$

that is, in view of (*), $b_\mu = i_{\mathscr{H}}(x)$.

To summarize, condition c) is equivalent to the following condition: for every point $i_{\mathscr{H}}(x)$ of $i_{\mathscr{H}}(\mathrm{X})$, there exists a positive measure μ on $i_{\mathscr{H}}(\mathrm{X})$ of total mass 1, such that $\mathrm{Supp}\,\mu \subset \mathrm{G}$ and $b_\mu = i_{\mathscr{H}}(x)$.

IV.109, ℓ. 16–18.

"... by No. 1, Prop. 1, this is also equivalent to saying that the closed convex envelope of $i_{\mathscr{H}}(\mathrm{X})$ is equal to the closed convex envelope of G."

The closed convex envelope of $i_{\mathscr{H}}(\mathrm{X})$ in $\mathscr{H}'$ is C; denote by D the closed convex envelope of $\mathrm{G} = i_{\mathscr{H}}(\mathrm{F})$. Since $i_{\mathscr{H}}(\mathrm{X})$ is compact, C is equal to the set of all barycenters b_μ of positive measures μ on $i_{\mathscr{H}}(\mathrm{X})$ of total mass 1 (No. 1, Prop. 1); similarly, D is the set of all barycenters b_ν of positive measures ν on G of total mass 1. We are to show that

$$\text{condition } c) \text{ holds} \quad \Leftrightarrow \quad \mathrm{C} = \mathrm{D}.$$

Proof of $\Leftarrow$: Given a point $i_{\mathscr{H}}(x) \in i_{\mathscr{H}}(\mathrm{X})$ it suffices, by the last paragraph of the preceding note, to show that there exists a positive measure μ on $i_{\mathscr{H}}(\mathrm{X})$ of total mass 1, such that $\operatorname{Supp}\mu \subset \mathrm{G}$ and $b_\mu = i_{\mathscr{H}}(x)$.

Since $i_{\mathscr{H}}(x) \in \mathrm{C} = \mathrm{D}$, there exists a positive measure ν on G of total mass 1 such that $b_\nu = i_{\mathscr{H}}(x)$. Let μ be the measure on $i_{\mathscr{H}}(\mathrm{X})$ defined by

$$(\dagger) \qquad \mu(f) = \nu(f\big|\mathrm{G}) \quad \text{for } f \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X}))\,,$$

which is also positive and, since $\mu(1) = \nu(1\big|\mathrm{G}) = 1$, of total mass 1.

If $f \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X}))$ and $\operatorname{Supp}(f) \subset i_{\mathscr{H}}(\mathrm{X}) - \mathrm{G}$, then $f\big|\mathrm{G} = 0$, hence $\mu(f) = 0$; this is true in particular for the functions $f \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X}))$ that are extensions by 0 of functions $g \in \mathscr{K}(i_{\mathscr{H}}(\mathrm{X}) - \mathrm{G})$, therefore the restriction of μ to the open subset $i_{\mathscr{H}}(\mathrm{X}) - \mathrm{G}$ of $i_{\mathscr{H}}(\mathrm{X})$ is equal to 0 (Ch. III, §2, No. 1), and so

$$i_{\mathscr{H}}(\mathrm{X}) - \mathrm{G} \subset i_{\mathscr{H}}(\mathrm{X}) - \operatorname{Supp}\mu$$

(*loc. cit.*, No. 2, Def. 1), that is, $\operatorname{Supp}\mu \subset \mathrm{G}$.

It remains to show that $b_\mu = i_{\mathscr{H}}(x)$, that is, $b_\mu = b_\nu$; given $h \in \mathscr{H}$, it suffices to show that $\langle b_\mu, h\rangle = \langle b_\nu, h\rangle$.

Let $f \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X}))$ be the function

$$f(i_{\mathscr{H}}(y)) = \langle i_{\mathscr{H}}(y), h\rangle \qquad (y \in \mathrm{X})\,.$$

Then $f\big|\mathrm{G} \in \mathscr{C}(\mathrm{G})$ is the function

$$i_{\mathscr{H}}(y) \mapsto \langle i_{\mathscr{H}}(y), h\rangle \qquad (y \in \mathrm{F})\,.$$

By the definition of barycenter (No. 1, Def. 1),

$$\langle b_\mu, h\rangle = \int_{i_{\mathscr{H}}(\mathrm{X})} \langle \mathbf{z}', h\rangle\, d\mu(\mathbf{z}') = \mu(f)\,,$$

whereas

$$\langle b_\nu, h\rangle = \int_{\mathrm{G}} \langle \mathbf{z}', h\rangle\, d\nu(\mathbf{z}') = \nu(f\big|\mathrm{G})\,,$$

whence $\langle b_\mu, h\rangle = \mu(f) = \nu(f\big|\mathrm{G}) = \langle b_\nu, h\rangle$.

Proof of $\Rightarrow$: The foregoing computations show that if ν is a positive measure on G of total mass 1, then the measure μ on $i_{\mathscr{H}}(\mathrm{X})$ defined by ($\dagger$) is a positive measure of total mass 1 such that $b_\mu = b_\nu$; in particular, $b_\nu = b_\mu \in \mathrm{C}$ (No. 1, Prop. 1), and since D is equal to the set of all such b_ν (same Prop. 1), one has $\mathrm{D} \subset \mathrm{C}$.

Assuming that c) holds, we are to show that $\mathrm{C} \subset \mathrm{D}$. Since D is a closed convex set and C is the closed convex envelope of $i_{\mathscr{H}}(\mathrm{X})$, it will suffice to show that $i_{\mathscr{H}}(\mathrm{X}) \subset \mathrm{D}$.

Given $x \in \mathrm{X}$, we are to show that $i_{\mathscr{H}}(x) \in \mathrm{D}$. By c), $i_{\mathscr{H}}(x) = b_\mu$ for some positive measure μ on $i_{\mathscr{H}}(\mathrm{X})$ of total mass 1 such that $\mathrm{Supp}\,\mu \subset \mathrm{G}$. Let $\nu = \mu_{\mathrm{G}}$ be the restriction of μ to the compact subspace G of $i_{\mathscr{H}}(\mathrm{X})$, in the sense of §5, No. 7, Def. 4:

$$\int_{\mathrm{G}} g\, d\nu = \int_{i_{\mathscr{H}}(\mathrm{X})} g'\, d\mu \quad \text{for all } g \in \mathscr{C}(\mathrm{G}),$$

where g' is the extension by 0 of g to $i_{\mathscr{H}}(\mathrm{X})$. Note that $\mu(\mathrm{G}) = 1$; for,

$$i_{\mathscr{H}}(\mathrm{X}) - \mathrm{G} \subset i_{\mathscr{H}}(\mathrm{X}) - \mathrm{Supp}\,\mu,$$

thus the open subset $i_{\mathscr{H}}(\mathrm{X}) - \mathrm{G}$ of $i_{\mathscr{H}}(\mathrm{X})$ is μ-negligible, therefore $\mu(\mathrm{G}) = \mu(i_{\mathscr{H}}(\mathrm{X})) = 1$. It follows that if $g = 1$ (the function on G identically equal to 1) then $g' = \varphi_{\mathrm{G}}$, therefore

$$\nu(1) = \int_{i_{\mathscr{H}}(\mathrm{X})} \varphi_{\mathrm{G}}\, d\mu = \mu(\mathrm{G}) = 1,$$

thus ν is a positive measure on G of total mass 1; consequently $b_\nu \in \mathrm{D}$ (see the Note for No. 1, Cor. of Prop. 1).

To show that $b_\mu \in \mathrm{D}$, it will suffice to show that $b_\mu = b_\nu$ (the argument in the proof of $\mathrm{D} \subset \mathrm{C}$ does not apply here, for the present measure μ is not defined by the formula (†)); given $h \in \mathscr{H}$, it suffices to show that $\langle b_\mu, h\rangle = \langle b_\nu, h\rangle$. By the definition of b_μ,

$$\langle b_\mu, h\rangle = \int_{i_{\mathscr{H}}(\mathrm{X})} \langle \mathbf{z}', h\rangle\, d\mu(\mathbf{z}');$$

the integrand is the function $f \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X}))$ defined by

$$f(i_{\mathscr{H}}(y)) = \langle i_{\mathscr{H}}(y), h\rangle \quad \text{for } y \in \mathrm{X},$$

thus, since $\varphi_{\mathrm{G}} = 1$ μ-almost everywhere,

$$\langle b_\mu, h\rangle = \int_{i_{\mathscr{H}}(\mathrm{X})} f\, d\mu = \int_{i_{\mathscr{H}}(\mathrm{X})} \varphi_{\mathrm{G}} f\, d\mu.$$

But $(\varphi_{\mathrm{G}} f)(i_{\mathscr{H}}(y))$ is equal to $f(i_{\mathscr{H}}(y))$ when $i_{\mathscr{H}}(y) \in \mathrm{G}$, and to 0 when $i_{\mathscr{H}}(y) \notin \mathrm{G}$, thus $\varphi_{\mathrm{G}} f$ is the extension by 0 of the function $g \in \mathscr{C}(\mathrm{G})$ defined by

$$g(i_{\mathscr{H}}(y)) = \langle i_{\mathscr{H}}(y), h\rangle \quad \text{for } y \in \mathrm{F},$$

that is, $\varphi_{\mathrm{G}} f = g'$. Thus

$$\begin{aligned}\langle b_\mu, h\rangle &= \int_{i_{\mathscr{H}}(\mathrm{X})} \varphi_{\mathrm{G}} f\, d\mu = \int_{i_{\mathscr{H}}(\mathrm{X})} g'\, d\mu = \int_{\mathrm{G}} g\, d\nu \\ &= \int_{\mathrm{G}} \langle \mathbf{z}', h\rangle\, d\nu(\mathbf{z}') = \langle b_\nu, h\rangle\,.\end{aligned}$$

We extract from the foregoing argument the following observation (a similar argument will be employed below in the proof of Prop. 8):

If G *is any closed subset of* $i_{\mathscr{H}}(\mathrm{X})$, *and* μ *is a positive measure on* $i_{\mathscr{H}}(\mathrm{X})$ *of total mass* 1 *such that* $\operatorname{Supp}\mu \subset \mathrm{G}$, *then the restriction* $\nu = \mu_{\mathrm{G}}$ *of* ν *to* G *is also a positive measure of total mass* 1 *and* $b_\nu = b_\mu$.

IV.109, ℓ. 18, 19.

"The equivalence of the conditions a), b) and c) therefore follows from TVS, II, §7, No. 1, Cor. of Prop. 2."

In the cited Corollary, E is a Hausdorff locally convex space, A is a compact convex subset of E, and K is a compact subset of A. The dictionary for translating the Corollary to the present setting is as follows:

$$\begin{array}{rcll} \mathrm{E} & \leftrightarrow & \mathscr{H}' & \text{(equipped with } \sigma(\mathscr{H}', \mathscr{H})\text{)} \\ \mathrm{A} & \leftrightarrow & \mathrm{C} & \text{(the closed convex envelope of } i_{\mathscr{H}}(\mathrm{X})\text{)} \\ \mathrm{K} & \leftrightarrow & \mathrm{G} & (= i_{\mathscr{H}}(\mathrm{F}),\ \mathrm{F} \text{ a closed set in } \mathrm{X}) \\ \mathrm{K} \subset \mathrm{A} & \leftrightarrow & \mathrm{G} \subset \mathrm{C}\,. & \end{array}$$

Let us write a'), b'), c') for the (equivalent) conditions of the cited Corollary:

a') the closed convex envelope of K is equal to A;

b') K meets the intersection of A with any closed support hyperplane of A;

c') K contains every extremal point of A.

The translation of a'): the closed convex envelope of G is equal to C, i.e, in the notation of the preceding Notes, $\mathrm{D} = \mathrm{C}$; in other words, c) holds.

The translation of b'): G meets the intersection of C with any closed support hyperplane of C; in other words, b) holds.

The translation of c'): G contains every extremal point of C; equivalently, a) holds.

Thus, the equivalence of a'), b'), c') implies the equivalence of a), b), c), whence Prop. 7.

IV.109, ℓ. $-15, -14$.

"This is the translation of Th. 1 of No. 2, by transport of structure by means of the homeomorphism $x \mapsto i_{\mathscr{H}}(x)$, as in Prop. 5."

Probably intended: "as in Prop. 4."

The dictionary between Choquet's theorem (No. 2, Th. 1) and its application here is as follows:

$$\begin{array}{lcll} \mathrm{E} & \leftrightarrow & \mathscr{H}' & \text{(equipped with } \sigma(\mathscr{H}',\mathscr{H})\text{)} \\ \mathrm{K} & \leftrightarrow & \mathrm{C} & \text{(the closed convex envelope of } i_{\mathscr{H}}(\mathrm{X})\text{)} \\ \mathrm{M} & \leftrightarrow & \mathrm{C}_{\mathrm{ep}} & \text{(the set of extremal points of C)}. \end{array}$$

One knows that $i_{\mathscr{H}}\big(\mathrm{Ch}_{\mathscr{H}}(\mathrm{X})\big) = \mathrm{C}_{\mathrm{ep}} \subset i_{\mathscr{H}}(\mathrm{X}) \subset \mathrm{C}$ (see the note for IV.109, ℓ. 12, 13).

To apply Choquet's theorem, we must verify that C is a *metrizable* subspace of $\mathscr{H}'$. As noted in the proof of Prop. 4, (i), $i_{\mathscr{H}}(\mathrm{X})$ is contained in the set

$$\mathrm{B} = \{\mathbf{z}' \in \mathscr{H}' : \ \|\mathbf{z}'\| \leqslant 1\},$$

which is compact for $\sigma(\mathscr{H}',\mathscr{H})$ (TVS, III, §3, No. 4, Cor. 3 of Prop. 4), whence C is contained in B and is compact. Moreover, since the compact space X is assumed here to be metrizable, the Banach space $\mathscr{C}(\mathrm{X})$ is separable, i.e., of countable type (GT, X, §3, No. 3, Th. 1), therefore its linear subspace $\mathscr{H}$ is a normed space of countable type; it follows that B is metrizable for $\sigma(\mathscr{H}',\mathscr{H})$ (TVS, *loc. cit.*, Cor. 2 of Prop. 6, read "second axiom of countability"), therefore so is C.

For consistency with the notations in the proof of Prop. 4, it is useful to replace μ by λ in the statement of Prop. 8: *Given any* $x \in \mathrm{X}$, *we seek a positive measure* λ *on* X *of total mass* 1 *such that* $\lambda\big(\mathrm{X} - \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})\big) = 0$ *and* $h(x) = \int h\,d\lambda$ *for all* $h \in \mathscr{H}$.

By Choquet's theorem, C_{ep} is the intersection of a sequence of open sets in C (i.e., is a G_δ in C), hence is a Borel set in C (§5, No. 4, Cor. 3 of Th. 2); and, since $i_{\mathscr{H}}(x) \in \mathrm{C}$, there exists a positive measure ρ on C of total mass 1 such that $b_\rho = i_{\mathscr{H}}(x)$ and $\rho(\mathrm{C} - \mathrm{C}_{\mathrm{ep}}) = 0$. From

$$\mathrm{C}_{\mathrm{ep}} \subset i_{\mathscr{H}}(\mathrm{X}) \subset \mathrm{C}$$

we have $\rho\big(\mathrm{C} - i_{\mathscr{H}}(\mathrm{X})\big) \leqslant \rho(\mathrm{C} - \mathrm{C}_{\mathrm{ep}}) = 0$, thus the open set $\mathrm{C} - i_{\mathscr{H}}(\mathrm{X})$ in C is ρ-negligible, whence $\mathrm{C} - i_{\mathscr{H}}(\mathrm{X}) \subset \mathrm{C} - \mathrm{Supp}\,\rho$ (Ch. III, §2, No. 2, Def. 1), that is, $\mathrm{Supp}\,\rho \subset i_{\mathscr{H}}(\mathrm{X})$.

Let $\mu = \rho\big|i_{\mathscr{H}}(\mathrm{X})$ be the restriction of ρ to the compact subset $i_{\mathscr{H}}(\mathrm{X})$ of C; thus,

$$\mu(g) = \rho(g') \quad \text{for all } g \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X})),$$

where g' is the extension by 0 of g to C (§5, No. 7, Def. 4). In particular, if $g = 1$ then $g' = \varphi_{i_{\mathscr{H}}(\mathrm{X})}$ (the characteristic function of the subset $i_{\mathscr{H}}(\mathrm{X})$ of C), and since $g' = 1$ ρ-almost everywhere, one has

$$\mu(1) = \int_{\mathrm{C}} \varphi_{i_{\mathscr{H}}(\mathrm{X})}\,d\rho = \rho(1) = 1,$$

whence μ is a positive measure on $i_{\mathscr{H}}(\mathrm{X})$ of total mass 1. Note that, since $i_{\mathscr{H}}(\mathrm{X}) - \mathrm{C}_{\mathrm{ep}} \subset \mathrm{C} - \mathrm{C}_{\mathrm{ep}}$ is ρ-negligible, it is also μ-negligible (§5, No. 7, *Lemma* 2), that is, $\mu\big(i_{\mathscr{H}}(\mathrm{X}) - \mathrm{C}_{\mathrm{ep}}\big) = 0$.

We assert that $b_\mu = b_\rho$ $(= i_{\mathscr{H}}(x))$; given $h \in \mathscr{H}$, it suffices to show that $\langle b_\mu, h\rangle = \langle b_\rho, h\rangle$. By the definition of b_ρ,

$$\langle b_\rho, h\rangle = \int_{\mathrm{C}} \langle \mathbf{z}', h\rangle \, d\rho(\mathbf{z}') .$$

The integrand is the function $f \in \mathscr{C}(\mathrm{C})$ defined by

$$f(\mathbf{z}') = \langle \mathbf{z}', h\rangle \qquad (\mathbf{z}' \in \mathrm{C});$$

since $\varphi_{i_{\mathscr{H}}(\mathrm{X})} = 1$ ρ-almost everywhere, one has

$$\langle b_\rho, h\rangle = \int_{\mathrm{C}} f\, d\rho = \int_{\mathrm{C}} \varphi_{i_{\mathscr{H}}(\mathrm{X})} f \, d\rho ,$$

where $\big(\varphi_{i_{\mathscr{H}}(\mathrm{X})} f\big)(\mathbf{z}')$ is equal to $f(\mathbf{z}') = \langle \mathbf{z}', h\rangle$ if $\mathbf{z}' \in i_{\mathscr{H}}(\mathrm{X})$, and to 0 if $\mathbf{z}' \notin i_{\mathscr{H}}(\mathrm{X})$. On the other hand,

$$\langle b_\mu, h\rangle = \int_{i_{\mathscr{H}}(\mathrm{X})} \langle \mathbf{z}', h\rangle \, d\mu(\mathbf{z}') = \int_{i_{\mathscr{H}}(\mathrm{X})} g \, d\mu ,$$

where $g \in \mathscr{C}(i_{\mathscr{H}}(\mathrm{X}))$ is defined by

$$g(i_{\mathscr{H}}(y)) = \langle i_{\mathscr{H}}(y), h\rangle = h(y) \qquad \text{for } y \in \mathrm{X};$$

since g' is the ρ-integrable function such that $g'(\mathbf{z}')$ is equal to $\langle \mathbf{z}', h\rangle$ if $\mathbf{z}' \in i_{\mathscr{H}}(\mathrm{X})$, and to 0 if $\mathbf{z}' \notin i_{\mathscr{H}}(\mathrm{X})$, one has $g' = \varphi_{i_{\mathscr{H}}(\mathrm{X})} f$. Therefore

$$\langle b_\rho, h\rangle = \int_{\mathrm{C}} \varphi_{i_{\mathscr{H}}(\mathrm{X})} f \, d\rho = \int_{\mathrm{C}} g' \, d\rho = \int_{i_{\mathscr{H}}(\mathrm{X})} g \, d\mu = \langle b_\mu, h\rangle .$$

Thus $b_\mu = b_\rho = i_{\mathscr{H}}(x)$.

Let λ be the positive measure on X of total mass 1 that is paired with μ as in the proof of Prop. 4, (ii), so that μ is derived from λ via the homoemorphism $i_{\mathscr{H}}$ of X onto $i_{\mathscr{H}}(\mathrm{X})$. Since $b_\mu = i_{\mathscr{H}}(x)$, we know that

$$h(x) = \int_{\mathrm{X}} h \, d\lambda \qquad \text{for all } h \in \mathscr{H}$$

(see the note for IV.107, ℓ. -6 to -3). It remains only to show that $\lambda\big(\mathrm{X} - \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})\big) = 0$, and this is immediate from $\mu\big(i_{\mathscr{H}}(\mathrm{X}) - \mathrm{C}_{\mathrm{ep}}\big) = 0$ and the fact that

$$i_{\mathscr{H}}\big(\mathrm{X} - \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})\big) = i_{\mathscr{H}}(\mathrm{X}) - i_{\mathscr{H}}\big(\mathrm{Ch}_{\mathscr{H}}(\mathrm{X})\big) = i_{\mathscr{H}}(\mathrm{X}) - \mathrm{C}_{\mathrm{ep}} .$$

IV.109, ℓ. $-5, -4$.

"It then follows easily from Props. 5 and 7 that $\mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) = \check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X}) = \mathbf{S}_2$."

The dictionary between Prop. 7 and its application here:

$$\begin{array}{rcl} \mathrm{E} & \leftrightarrow & \mathbf{R}^3 \\ \mathrm{X} & \leftrightarrow & \mathrm{X} = \{\mathbf{x} \in \mathbf{R}^3 : \ \|\mathbf{x}\| \leqslant 1\} \\ \mathrm{F} & \leftrightarrow & \mathbf{S}_2 = \{\mathbf{x} \in \mathbf{R}^3 : \ \|\mathbf{x}\| = 1\} \\ \mathscr{H} & \leftrightarrow & \mathscr{H} \subset \mathscr{C}(\mathrm{X}) \quad \text{as described here.} \end{array}$$

Recall that, on a finite-dimensional real vector space, all compatible topologies coincide (TVS, I, §2, No. 3, Th. 2) and all hyperplanes are closed (*loc. cit.*, Cor. 1 of Th. 2); in particular, the word "weak" can be omitted from the present discussion, and $(\mathbf{R}^3)'$ may be identified with $\mathbf{R}^3$ equipped with its norm topology. Thus X is a compact convex subset of $\mathbf{R}^3$, with boundary $\mathbf{S}_2$, and interior equal to $\{x \in \mathbf{R}^3 : \ \|\mathbf{x}\| < 1\}$.

{'Intuitively', $\mathbf{S}_2$ is the set of extremal points of X, but let us play by the rules—no pictures, and only internal references.}

For $h \in \mathscr{H}$, write $\mathrm{M}_h = \{\mathbf{x} \in \mathrm{X} : \ h(\mathbf{x}) = \sup_{\mathbf{y} \in \mathrm{X}} h(\mathbf{y})\}$. If h is nonconstant, then by hypothesis $\mathrm{M}_h \subset \mathbf{S}_2$, whereas if h is constant then $\mathrm{M}_h = \mathrm{X}$; in either case $\mathrm{M}_h \cap \mathbf{S}_2 \neq \varnothing$, therefore

$$(*) \qquad \check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X}) \subset \mathbf{S}_2$$

by Prop. 7 (specifically, $b) \Rightarrow a)$).

On the other hand, if $\mathbf{x} \in \mathbf{S}_2$ then there exists a support (i.e., 'tangent') hyperplane H of X that contains $\mathbf{x}$ (TVS, §5, No. 2, Prop. 3). Say

$$\mathrm{H} = \{\mathbf{y} \in \mathbf{R}^3 : \ f(\mathbf{y}) = c\},$$

where $c \in \mathbf{R}$ and $f \in (\mathbf{R}^3)' \subset \mathscr{C}(\mathbf{R}^3)$, and $f(\mathbf{y}) \leqslant c$ for all $\mathbf{y} \in \mathrm{X}$. In particular, $\mathbf{x} \in \mathrm{H} \cap \mathbf{S}_2$. In fact, $\mathrm{H} \cap \mathbf{S}_2 = \{\mathbf{x}\}$. {For, if also $\mathbf{y} \in \mathrm{H} \cap \mathbf{S}_2$, then the point $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ also belongs to the convex set $\mathrm{H} \cap \mathrm{X}$. Since X lies to one side of H, $\mathrm{H} \cap \mathrm{X}$ cannot contain an interior point of X, therefore $\|\mathbf{z}\| = 1$. Then

$$1 = \|\mathbf{z}\|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\langle \mathbf{x} | \mathbf{y} \rangle,$$

whence $\langle \mathbf{x} | \mathbf{y} \rangle = 1 = \|\mathbf{x}\| \, \|\mathbf{y}\|$, and so $\mathbf{y} = \pm \mathbf{x}$. As $\mathbf{y} = -\mathbf{x}$ would imply $\mathbf{z} = 0$, we are left with $\mathbf{y} = \mathbf{x}$.}

By assumption, $\mathscr{H}$ contains the affine linear function $h = f\big|\mathrm{X}$, which attains its supremum c at $\mathbf{x}$; and since h can attain its supremum only at points of H, necessarily $\mathrm{M}_h = \mathrm{H} \cap \mathrm{X}$. By Prop. 5, $\mathrm{M}_h \cap \mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) \neq \varnothing$; but $\mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) \subset \check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X}) \subset \mathbf{S}_2$ by (*), therefore

$$\mathrm{M}_h \cap \mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) \subset \mathrm{H} \cap \mathbf{S}_2 = \{\mathbf{x}\},$$

whence $\mathrm{M}_h \cap \mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) = \{\mathbf{x}\}$ and so $\mathbf{x} \in \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})$. Thus $\mathbf{S}_2 \subset \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})$, and finally

$$\mathbf{S}_2 \subset \mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) \subset \check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X}) \subset \mathbf{S}_2,$$

whence equality throughout.

IV.110, ℓ. 13–15.

"The $\mathscr{H}_r$-extremal points in X are again called $\mathscr{H}$*-extremal*, the set of them is denoted $\mathrm{Ch}_{\mathscr{H}}(\mathrm{X})$, and the closure of the latter set is denoted $\check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X})$."

Thus $\mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) = \mathrm{Ch}_{\mathscr{H}_r}(\mathrm{X})$ and

$$\check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X}) = \overline{\mathrm{Ch}_{\mathscr{H}}(\mathrm{X})} = \overline{\mathrm{Ch}_{\mathscr{H}_r}(\mathrm{X})} = \check{\mathrm{S}}_{\mathscr{H}_r}(\mathrm{X}).$$

Note that the concept of $\mathscr{H}$-extremal point does not depend on the existence of measures, but is characterized in terms of measures in Prop. 4. If $x \in \mathrm{X}$ and if λ is a positive measure on X of total mass 1, the conditions

$$h(x) = \int h\,d\lambda \quad \text{for all } h \in \mathscr{H}_r \tag{10}$$

and

$$f(x) = \int f\,d\lambda \quad \text{for all } f \in \mathscr{H} \tag{10$'$}$$

are obviously equivalent, and if μ is the corresponding measure on $i_{\mathscr{H}_r}(\mathrm{X})$ then (see the note for IV.107, ℓ. −6 to −3)

$$\begin{aligned} b_\mu = i_{\mathscr{H}_r}(x) \quad &\Leftrightarrow \quad \text{the condition (10) holds} \\ &\Leftrightarrow \quad \text{the condition (10)$'$ holds,} \end{aligned}$$

and

$$\begin{aligned} x \in \mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) \quad &\Leftrightarrow \quad x \in \mathrm{Ch}_{\mathscr{H}_r}(\mathrm{X}) \quad \text{(by definition)} \\ &\Leftrightarrow \quad \text{(10) holds only for } \lambda = \varepsilon_x \quad \text{(Prop. 4)} \\ &\Leftrightarrow \quad \text{(10)$'$ holds only for } \lambda = \varepsilon_x\,. \end{aligned}$$

Inasmuch as the concept of $\mathscr{H}$-extremal point refers back to $\mathscr{H}_r$, it is not necessary here to define and develop the properties of $i_{\mathscr{H}}(\mathrm{X})$.

IV.110, ℓ. −12.

"... whence $|g(b)| = 1$ since $|g(b)| \leqslant 1$"

In fact, $g(b) = 1$; for, $1 = \mathscr{R}\big(g(b)\big) \leqslant |g(b)| \leqslant 1$, whence $\mathscr{I}\big(g(b)\big) = 0$.

The Proposition is remarkable in that $|f|$ is not required to belong to $\mathscr{H}$. In particular, with notations as in Prop. 5, if $h \in \mathscr{H}$ then $|h|$ attains its supremum at at least one $\mathscr{H}$-extremal point.

IV.111, ℓ. 3.

"The fact that *a*) implies *b*) follows from Prop. 9."

If $f \in \mathscr{H}$ then, by Prop. 9, $|f|$ attains its supremum at some point $b \in \mathrm{Ch}_{\mathscr{H}}(\mathrm{X}) \subset \check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X})$, and $b \in \mathrm{F}$ by *a*), therefore *b*) holds.

IV.111, ℓ. 4, 5.

"this is a matter of seeing that if *b*) is verified, then, for every $h \in \mathscr{H}_r$, F intersects the set of points where h attains its infimum in X."

Suppose established that every $h \in \mathscr{H}_r$ attains its infimum at some point of F. If $h \in \mathscr{H}_r$ then also $-h \in \mathscr{H}_r$; by supposition, $-h$ attains its infimum at some point $a \in \mathrm{F}$, whence

$$\sup h = -\inf(-h) = -\big((-h)(a)\big) = h(a)\,.$$

Then $\mathrm{F} \supset \check{\mathrm{S}}_{\mathscr{H}_r}(\mathrm{X}) = \check{\mathrm{S}}_{\mathscr{H}}(\mathrm{X})$ by "*b*) $\Rightarrow$ *a*)" of Prop. 7, thus *a*) of the present proposition holds.

IV.111, ℓ. 14, 15.

"Since $g - b \in \mathscr{H}$, the hypothesis on F implies that $|g - b| \leqslant b$"

By the hypothesis on F, $|g - b|$ attains its supremum at some point $a \in \mathrm{F}$; but $|g - b|$ is $\leqslant b$ at every point of F, therefore for every $x \in \mathrm{X}$ one has

$$|g(x) - b| \leqslant \sup|g - b| = |g(a) - b| \leqslant b\,,$$

that is, $|g - b| \leqslant b$.

Indeed, since $g - b \in \mathscr{H}_r$, one can replace *b*) by the weaker condition

$b)_r$ *For every* $g \in \mathscr{H}_r$, F *intersects the set of points of* X *where* $|g|$ *attains its supremum.*

IV.111, ℓ. −13 to −11.

"... a point where $|f|$ attains its supremum is a point where one of the functions $f, -f$ attains its supremum."

If f is a real function (continuous or not) such that $|f|$ attains its supremum at a point a, then either f or $-f$ attains its supremum at a. For, if $f(a) \geqslant 0$ then $\sup f = f(a)$ because, for all $x \in \mathrm{X}$,

$$f(x) \leqslant |f(x)| \leqslant |f|(a) = f(a);$$

whereas if $-f(a) \geqslant 0$ then, since $|-f| = |f|$ attains its supremum at a, the foregoing yields $\sup(-f) = (-f)(a)$.

On the other hand, if f attains its supremum at a, and $-f$ attains its supremum at b, then $|f|$ attains its supremum at one of these points. For, let $M = \sup f = f(a)$ and $m = \inf f = -\sup(-f) = -(-f)(b) = f(b)$. If f is constant, then $|f|$ assumes its supremum at every point. Assume f nonconstant, i.e., $m < M$.

If $m < M \leqslant 0$ then $\sup|f| = -m = -f(b)$.

If $m \leqslant 0 \leqslant \mathrm{M}$ then $\sup|f| = \max\{-m, M\} = \max\{-f(b), f(a)\}$.

If $0 \leqslant m < M$ then $\sup|f| = M = f(a)$.

In all cases, $\sup|f| = \max\{|m|, |M|\} = \max\{|f(b)|, |f(a)|\}$, thus $|f|$ attains its supremum at either a or b.

IV.111, ℓ. −11 to −9.

"For a vector space $\mathscr{H}$ of continuous *real* functions satisfying the hypotheses of No. 3, the Props. 9 and 10 are thus trivial corollaries of Props. 5 and 7, respectively."

Call 'Prop. 10_r' the result of stating Prop. 10 with $\mathscr{H}$ consisting of real functions, and denote its three conditions by $a)_r$, $b)_r$, $c)_r$. Let $a)$, $b)$, $c)$ have their meanings as in Prop. 7.

Condition $a)_r$ is identical to condition $a)$ of Prop. 7.

Condition $c)_r$ is identical to condition $c)$ of Prop. 7.

$b) \Rightarrow b)_r$: If $f \in \mathscr{H}$ then, by $b)$, f attains its supremum at some point of F; but also $-f \in \mathscr{H}$, so $-f$ attains its supremum at some point of F, therefore $|f|$ attains its supremum at one of these points (see the preceding note), whence $b)_r$.

$b)_r \Rightarrow b)$: As observed in the note for IV.111, ℓ. 14, 15, $b)_r$ implies $a)$, and $a)$ implies $b)$ by Prop. 7.

Thus $b) \Leftrightarrow b)_r$.

The equivalence of $a)$, $b)$, $c)$ (Prop. 7) then assures the equivalence of $a)_r$, $b)_r$, $c)_r$, thus Prop. 10_r is proved.

Call Prop. 9_r the result of stating Prop. 9 with $\mathscr{H}$ consisting of real functions. Let $f \in \mathscr{H}$. By Prop. 5, f attains its supremum at some point $a \in \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})$. Since also $-f \in \mathscr{H}$, $-f$ attains its supremum at some point $b \in \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})$. As noted above, $|f|$ attains its supremum at one of a, b, thus Prop. 9_r is proved.

IV.111, ℓ. −7 to −1.

"*Lemma* 4."

So to speak, if $a \in X$ is an "approximate peak point" and has a countable neighborhood base, then a is a "peak point" (relative to $\mathscr{H}$).

{For the terminology, cf. the book of G.M. Leibowitz (*Lectures on complex function algebras*, p. 54, Scott, Foresman, Glenview, IL, 1970).}

As indicated by the "resp. $\mathscr{C}(X;\mathbf{R})$" in the statement of the lemma, the proof works for either the real or the complex case.

IV.112, ℓ. 2, 3.

"... let $\lambda, \mu, \varepsilon$ be numbers such that

$$0 < \lambda < 1 , \quad 1 < \mu < \mu + \varepsilon \leqslant 1 + \lambda ."$$

The order of events: Fix λ with $0 < \lambda < 1$. Fix μ with $1 < \mu < 1+\lambda$. Then choose any $\varepsilon > 0$ (which will also remain fixed) such that $\mu+\varepsilon \leqslant 1+\lambda$, for example $\varepsilon = (1 + \lambda) - \mu$.

IV.112, ℓ. 4–6.

"We are going to define, by induction on n $(n \geqslant 1)$, a decreasing sequence (U_n) of open neighborhoods of a such that $U_n \subset V_n$ for all n, and a sequence (h_n) of functions in $\mathscr{H}$ "

To get a feeling for the intricate argument, I found it necessary to look at $n = 1, 2, 3$.

One is assuming that

$$(*) \qquad 0 < \lambda < 1 , \quad 1 < \mu < \mu + \varepsilon \leqslant 1 + \lambda ,$$

whence

$$(\dagger) \qquad 0 < \frac{\lambda}{\mu} < \frac{1}{\mu} < 1 .$$

The argument for $n = 1$. Define $U_1 = \overset{\circ}{V}_1$, set $U = U_1$, and apply the hypothesis to the inequalities (†), that is, with $c = \lambda/\mu$, $d = 1/\mu$ and $U = U_1$: there exists a function $f \in \mathscr{H}$ such that

$$(12_1) \qquad |f| \leqslant 1 , \quad |f(a)| \geqslant \frac{1}{\mu} , \quad |f(x)| \leqslant \frac{\lambda}{\mu} \ \text{ for all } x \in X - U_1 .$$

Define $h_1 = \dfrac{1}{f(a)} f$. Then

$$(13_1) \qquad |h_1| = \frac{|f|}{|f(a)|} \leqslant \frac{1}{|f(a)|} \leqslant \mu ;$$

$$(14_1) \qquad h_1(a) = \frac{f(a)}{f(a)} = 1\,;$$

$$(15_1) \qquad \text{for all } x \in \mathrm{X} - \mathrm{U}_1\,,\ |h_1(x)| = \frac{|f(x)|}{|f(a)|} \leqslant \frac{\lambda}{\mu} \cdot \frac{1}{|f(a)|} \leqslant \frac{\lambda}{\mu} \cdot \mu = \lambda\,;$$

$$(16_1) \qquad \text{for all } y \in \mathrm{X}\,,\ |\lambda^1 h_1(y)| = \lambda |h_1(y)| \leqslant \lambda\mu < \lambda(1+\lambda) = \sum_{j=1}^{2} \lambda^j\,.$$

With the convention $\mathrm{U}_0 = \mathrm{X}$, the condition $\mathrm{U}_1 \subset \mathrm{U}_0 \cap \mathrm{V}_1$ is fulfilled.

The argument for $n = 2$. The function λh_1 (formally equal to $\sum_{j=1}^{2-1} \lambda_j h_j$) is continuous and takes the value λ at a, therefore there exists an open neighborhood U_2 of a, which we can suppose to be contained in $\mathrm{U}_1 \cap \mathrm{V}_2$, such that

$$(17_2) \qquad |\lambda h_1(y)| < \lambda + \varepsilon\lambda^2 \qquad \text{for all } y \in \mathrm{U}_2\,.$$

Apply the hypothesis to (†) with $\mathrm{U} = \mathrm{U}_2$ to obtain a (new) function $f \in \mathscr{H}$ such that

$$(12_2) \qquad |f| \leqslant 1\,, \quad |f(a)| \geqslant \frac{1}{\mu}\,, \quad |f(x)| \leqslant \frac{\lambda}{\mu} \quad \text{for all } x \in \mathrm{X} - \mathrm{U}_2\,.$$

The function $h_2 = \dfrac{1}{f(a)} f$ then satisfies (13_2), (14_2) and (15_2). Set

$$g = \sum_{j=1}^{2} \lambda^j h_j = \lambda h_1 + \lambda^2 h_2\,;$$

we have $|g| \leqslant \lambda|h_1| + \lambda^2|h_2|$, and to prove (16_2) we must show that $|g(y)| < \sum_{j=1}^{3} \lambda^3$ for all $y \in \mathrm{X}$. At any rate, by (16_1) we know that

$$|g| < \sum_{j=1}^{2} \lambda^j + \lambda^2 |h_2| \qquad \text{on } \mathrm{X}\,;$$

if $y \in \mathrm{X} - \mathrm{U}_2$ then $|h_2(y)| \leqslant \lambda$ by (15_2), so

$$|g(y)| < \sum_{j=1}^{2} \lambda^j + \lambda^2 \cdot \lambda = \sum_{j=1}^{3} \lambda^j\,;$$

whereas if $y \in \mathrm{U}_2$ then $|h_2(y)| \leqslant \mu$ by (13_2) and, by (17_2), $|\lambda h_1(y)| < \lambda + \varepsilon\lambda^2$, therefore

$$\begin{aligned}|g(y)| &< (\lambda + \varepsilon\lambda^2) + \mu\lambda^2 \\ &= \lambda + (\mu + \varepsilon)\lambda^2 \\ &\leqslant \lambda + (1+\lambda)\lambda^2 = \sum_{j=1}^{3} \lambda^j \qquad \text{(by (*))}\end{aligned}$$

thus (16_2) is verified.

The argument for $n = 3$. The function $\sum_{j=1}^{2} \lambda^j h_j$ is continuous and takes the value $\sum_{j=1}^{2} \lambda^j$ at a, therefore there exists an open neighborhood U_3 of a, which we can suppose to be contained in $\mathrm{U}_2 \cap \mathrm{V}_3$, such that

$$(17_3) \qquad \Big|\sum_{j=1}^{2} \lambda^j h_j(y)\Big| < \lambda + \lambda^2 + \varepsilon\lambda^3 \qquad \text{for all } y \in \mathrm{U}_3 .$$

Apply the hypothesis to (†) with $\mathrm{U} = \mathrm{U}_3$ to obtain a function $f \in \mathscr{H}$ such that

$$(12_3) \qquad |f| \leqslant 1, \quad |f(a)| \geqslant \frac{1}{\mu}, \quad |f(x)| \leqslant \frac{\lambda}{\mu} \quad \text{for all } x \in \mathrm{X} - \mathrm{U}_3 .$$

The function $h_3 = \dfrac{1}{f(a)} f$ then satisfies (13_3), (14_3) and (15_3). Set

$$g = \sum_{j=1}^{3} \lambda^j h_j = \sum_{j=1}^{2} \lambda^j h_j + \lambda^3 h_3 .$$

To prove (16_3) we must show that $|g(y)| < \sum_{j=1}^{4} \lambda^j$ for all $y \in \mathrm{X}$. At any rate, by (16_2) we know that

$$|g| < \sum_{j=1}^{3} \lambda^j + \lambda^3 |h_3| \qquad \text{on X};$$

if $y \in \mathrm{X} - \mathrm{U}_3$ then $|h_3(y)| \leqslant \lambda$ by (15_3), so

$$|g(y)| < \sum_{j=1}^{3} \lambda^j + \lambda^3 \cdot \lambda = \sum_{j=1}^{4} \lambda^j ;$$

whereas if $y \in \mathrm{U}_3$ then $|h_3(y)| \leqslant \mu$ by (13_3) and, by (17_3),

$$\Big|\sum_{j=1}^{2} \lambda^j h_j(y)\Big| < \lambda + \lambda^2 + \varepsilon\lambda^3 \,,$$

therefore

$$\begin{aligned} |g(y)| &= (\lambda + \lambda^2 + \varepsilon\lambda^3) + \lambda^3\mu \\ &= \lambda + \lambda^2 + (\mu + \varepsilon)\lambda^3 \\ &< \lambda + \lambda^2 + (1 + \lambda)\lambda^3 = \sum_{j=1}^{4} \lambda^j \qquad \text{(by } (*)) \end{aligned}$$

thus (16_3) is verified.

IV.113, ℓ. 1.

"... for $x \in \mathrm{X} - \mathrm{U}_n$, we have $|h_p(x)| \leqslant \lambda$ for $1 \leqslant p \leqslant n$ "

A mirage, not justified by the n inequalities

$$(15_p) \qquad |h_p(x)| \leqslant \lambda \quad \text{for } x \in \mathrm{X} - \mathrm{U}_p$$

$(1 \leqslant p \leqslant n)$, the direction of the inclusions

$$\mathrm{X} - \mathrm{U}_n \supset \mathrm{X} - \mathrm{U}_{n-1} \supset \cdots \supset \mathrm{X} - \mathrm{U}_1$$

being unfavorable for $x \in \mathrm{X} - \mathrm{U}_n$. An alternate path to (16_n) is indicated in the preceding note.

IV.113, ℓ. 8, 9.

"... if $x \neq a$, there exists an integer n such that $x \notin \mathrm{U}_{n+1}$; therefore $|h_{n+k}(x)| \leqslant \lambda$ for all $k \geqslant 1$ "

Because $\bigcap_{j=1}^{\infty} \mathrm{U}_j \subset \bigcap_{j=1}^{\infty} \mathrm{V}_j = \{a\}$; and, for all $k \geqslant 1$,

$$x \in \mathrm{X} - \mathrm{U}_{n+1} \subset \mathrm{X} - \mathrm{U}_{n+k} \,,$$

whence $|h_{n+k}(x)| \leqslant \lambda$ by (15_{n+k}).

IV.113, ℓ. 13–20.

"Theorem 2"

With notations as in No. 4, the theorem says that, when $\mathscr{H}$ is a norm-closed subalgebra of $\mathscr{C}(\mathrm{X}; \mathbf{C})$ that contains the constants and separates the points of X, a is a 'peak point' of X (relative to $\mathscr{H}$) if and only if $a \in \mathrm{Ch}_{\mathscr{H}}(\mathrm{X})$ and a has a countable neighborhood base.

The analogous theorem for $\mathscr{H} \subset \mathscr{C}(\mathrm{X};\mathbf{R})$ is also true, as observed in subsequent notes.

{CAUTION: $\mathscr{H}_r$ need not be a subalgebra of $\mathscr{C}(\mathrm{X};\mathbf{C})$. For example, if X is the closed unit disk $\{z : |z| \leqslant 1\}$ of $\mathbf{C}$, and $\mathscr{H}$ is the norm-closure in $\mathscr{C}(\mathrm{X};\mathbf{C})$ of the algebra of complex polynomial functions on X, then every $f \in \mathscr{H}$ is differentiable in the open disk $\{z : |z| < 1\}$, hence the function $u(x,y) = \mathscr{R}f(x+iy)$ is harmonic there ($u_{xx} + u_{yy} = 0$); the function $z \mapsto x = \mathscr{R}(z)$ ($z \in \mathrm{X}$) belongs to $\mathscr{H}_r$, but its square $z \mapsto x^2$ cannot be the real part of a function in $\mathscr{H}$. This example of X and $\mathscr{H}$ is worked out in detail in the book of Leibowitz (pp. 55-64) cited in the note for IV.111, ℓ. −7 to −1.}

IV.113, ℓ. $-10, -9$.

"... by Prop. 9 of No. 4, a is an $\mathscr{A}$-extremal point."

Because a is the *only* point where $|f|$ attains its supremum.

{The cited Prop. 9 is also valid when $\mathscr{H} \subset \mathscr{C}(\mathrm{X};\mathbf{R})$ (No. 4, *Remark*).}

IV.113, ℓ. -3.

"... set $\varepsilon = \log d/\log c$"

Not to be confused with the ε in the *proof* of the cited lemma.

IV.113, ℓ. $-2, -1$.

"Since a is an $\mathscr{A}_r$-extremal point, there exists a function $g \in \mathscr{A}$ such that

$$\mathscr{R}(g) \geqslant 0\,, \quad \mathscr{R}\big(g(a)\big) \leqslant \varepsilon\,, \quad \mathscr{R}\big(g(x)\big) \geqslant 1 \ \text{ for } x \in \mathrm{X} - \mathrm{U}$$

(No. 3, Prop. 6, b))."

We are given $0 < c < d < 1$ and an open neighborhood U of a; the objective is to construct a function f satisfying the condition (12) of No. 5, Lemma 4.

The cited Prop. 6—which does not require $\mathscr{H}$ ($= \mathscr{A}_r$) to be norm-closed in $\mathscr{C}(\mathrm{X};\mathbf{R})$ nor that it be an algebra—proves the existence of a function $h \in \mathscr{A}_r$ having the indicated properties, and $h = \mathscr{R}g$ for some $g \in \mathscr{A}$.

{The argument remains valid when $\mathscr{A} \subset \mathscr{C}(\mathrm{X};\mathbf{R})$, with the obvious simplifications.}

IV.114, ℓ. $1, 2$.

"Set $f = c^g$; since f is the sum of the normally convergent series $\sum\limits_{n=0}^{\infty} (\log c)^n g^n/n!$, we have $f \in \mathscr{A}$"

Rather than explain the meaning of c^g, it is simpler to *define* the function f by the formula

$$f(x) = \exp[(\log c)g(x)] = \sum_{n=0}^{\infty} \frac{[(\log c)g(x)]^n}{n!}\,;$$

the series is normally convergent since $\|g\| < +\infty$, $|\log c| = |-\log \frac{1}{c}| = \log \frac{1}{c}$ and

$$\sum_{n=0}^{\infty} \frac{\big(|\log c|\,\|g\|\big)^n}{n!} = \sum_{n=0}^{\infty} \frac{\big[\big(\log \frac{1}{c}\big)\|g\|\big]^n}{n!} = e^{(\log \frac{1}{c})\,\|g\|} = \Big(\frac{1}{c}\Big)^{\|g\|} < +\infty,$$

and $f \in \mathscr{A}$ because the partial sums $\sum_{n=0}^{m} \frac{[(\log c)g]^n}{n!}$ belong to $\mathscr{A}$ and $\mathscr{A}$ is norm-closed.

IV.114, ℓ. 3.

"$|f| \leqslant 1\,,\quad |f(a)| \geqslant c^{\varepsilon} = d\,,\quad |f(x)| \leqslant c$ for $x \in \mathrm{X} - \mathrm{U}$."

Write $g = h + ik$, where $h, k \in \mathscr{A}_r$; thus $h = \mathscr{R}g \geqslant 0$. For every $x \in \mathrm{X}$,

$$\begin{aligned} f(x) &= \exp[(\log c)g(x)] = \exp[(\log c)h(x) + i(\log c)k(x)] \\ &= \exp[(\log c)h(x)] \cdot \exp[i(\log c)k(x)]\,; \end{aligned}$$

the first factor is positive and $\leqslant 1$ since $(\log c)h(x) \leqslant 0$, and the second factor has absolute value 1 since $(\log c)k(x)$ is real, therefore $|f(x)| \leqslant 1$.

Since $\mathscr{R}g \geqslant 0$ and $h(a) = \mathscr{R}g(a) \leqslant \varepsilon$, one has $0 \leqslant h(a) \leqslant \varepsilon$, whereas $\log c < 0$, therefore $(\log c)h(a) \geqslant (\log c)\varepsilon$; citing the formula for $f(x)$ as a product,

$$|f(a)| = \exp[(\log c)h(a)] \geqslant \exp[(\log c)\varepsilon] = \exp\Big[(\log c) \cdot \frac{\log d}{\log c}\Big] = d\,.$$

Finally, if $x \in \mathrm{X} - \mathrm{U}$ then $h(x) = \mathscr{R}g(x) \geqslant 1$, whereas $\log c < 0$, therefore $(\log c)h(x) \leqslant \log c$ and

$$|f(x)| = \exp[(\log c)h(x)] \leqslant \exp(\log c) = c\,.$$

{The argument remains valid when $\mathscr{A} \subset \mathscr{C}(\mathrm{X};\mathbf{R})$, with simplifications: $\mathscr{A}_r = \mathscr{A}$, $h = g$, k disappears from the stage, and $f(x) = \exp[(\log c)g(x)] = c^{g(x)}$. Thus Theorem 2 remains valid when $\mathscr{A} \subset \mathscr{C}(\mathrm{X};\mathbf{R})$.}

IV.114, ℓ. 9.

"*c) Let* $\mathfrak{M}$ *be the set of subsets* M *of* X *such that…*"

If $M \in \mathfrak{M}$ and $M' \supset M$ then obviously $M' \in \mathfrak{M}$; thus *minimal* sets M are of interest. By No. 4, Prop. 9, $\mathrm{Ch}_{\mathscr{A}}(X) \in \mathfrak{M}$. The implication *a)* ⇒ *c)* says that $\mathrm{Ch}_{\mathscr{A}}(X) \subset M$ for every $M \in \mathfrak{M}$, therefore $\mathrm{Ch}_{\mathscr{A}}(X) \subset \bigcap_{M\in\mathfrak{M}} M$, and the reverse inclusion is immediate from $\mathrm{Ch}_{\mathscr{A}}(X) \in \mathfrak{M}$; thus $\mathrm{Ch}_{\mathscr{A}}(X) = \bigcap_{M\in\mathfrak{M}} M$, $\mathrm{Ch}_{\mathscr{A}}(X)$ is the smallest element of $\mathfrak{M}$, and

$$\mathfrak{M} = \{A \subset X : A \supset \mathrm{Ch}_{\mathscr{A}}(X)\}.$$

In particular, $\check{S}_{\mathscr{A}}(X) \in \mathfrak{M}$.

IV.114, ℓ. 12.

"*d) Let* $\mathfrak{N}$ *be the set of subsets* N *of* X *such that* …"

Such a set N is called a *boundary* for the algebra $\mathscr{A}$ (cf. G.M. Leibowitz, *op. cit.*, p. 53, Exer. 4). If $N \in \mathfrak{N}$ and $N' \supset N$, then $N' \in \mathfrak{N}$. By Prop. 5 of No. 3 applied to $\mathscr{A}_r$, $\mathrm{Ch}_{\mathscr{A}}(X) = \mathrm{Ch}_{\mathscr{A}_r}(X) \in \mathfrak{N}$; arguing as in the preceding note, one sees that the meaning of the implication *a)* ⇒ *d)* is that $\mathrm{Ch}_{\mathscr{A}}(X) = \bigcap_{N\in\mathfrak{N}} N$, $\mathrm{Ch}_{\mathscr{A}}(X)$ is the smallest element of $\mathfrak{N}$, and

$$\mathfrak{N} = \{A \subset X : A \supset \mathrm{Ch}_{\mathscr{A}}(X)\},$$

which is also equal to $\mathfrak{M}$. In particular, $\check{S}_{\mathscr{A}}(X) \in \mathfrak{N}$.

One calls $\mathrm{Ch}_{\mathscr{A}}(X)$ the *Choquet boundary*, and $\check{S}_{\mathscr{A}}(X)$ the *Shilov boundary*, for $\mathscr{A}$ (Leibowitz, *op. cit.*, p. 49). Another valuable reference for §7 is the book of Robert R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, NJ, 1966.

IV.114, ℓ. −15, −14.

"…we can restrict ourselves to the case that X does not reduce to the single point *a*"

When $X = \{a\}$, each of *a)*, *c)*, *d)* (resp. *b)*) is trivially (resp. vacuously) true.

IV.115, ℓ. 8, 9.

"…the spaces X_1 and X_2 are *homeomorphic*, both being bounded convex sets in $\mathbf{R}^4$ with nonempty interior."

A *convex body* in a topological vector space over **R** or **C** is a closed convex set with nonempty interior. The theorem "*Any two compact convex bodies in* $\mathbf{R}^n$ *are homeomorphic*" is cited in the prerequisites of the book of E.H. Spanier (*Algebraic topology*, p. 10, McGraw-Hill, New York, 1966; reprinted by Springer-Verlag, New York); presumably a proof can be found in one of the 29 books listed there, but I have not tracked it down.

IV.115, ℓ. 12.

"... pointed convex cone in E."

With vertex 0 (TVS, II, §2, No. 4).

IV.115, ℓ. 12.

"One knows ..."

TVS, II, §2, No. 5, Prop. 13.

IV.115, ℓ. −9.

"The C_k are disjoint convex cones with union C."

The proof is delicate. Note that, since the f_λ are affine, hence convex, it follows that f is convex (FRV, I, §4, No. 2, Prop. 3). Since the f_λ are positively homogeneous, so is f: $f(tx) = tf(x)$ for $x \in \mathrm{C}$, $t > 0$.

For every $y \in \mathrm{C}$ there is at least one index j such that $f(y) = f_j(y)$, and $y \in \mathrm{C}_k$ if and only if k is the first such index; thus C is the union of the C_k, and the C_k to which y belongs is unique, whence disjointness.

Consider $\mathrm{C}_1 = \{y \in \mathrm{C}: \ f_1(y) = f(y)\}$. If $y, y' \in \mathrm{C}_1$ and $0 < t < 1$, we are to show that $ty + (1-t)y' \in \mathrm{C}_1$. At any rate, $ty + (1-t)y' \in \mathrm{C}$, and

$$\begin{aligned} f_1\big(ty + (1-t)y'\big) &= tf_1(y) + (1-t)f_1(y') && (f_1 \text{ is affine}) \\ &= tf(y) + (1-t)f(y') && (\text{because } y, y' \in \mathrm{C}_1). \end{aligned}$$

Assume to the contrary that $ty + (1-t)y' \in \mathrm{C}_k$ for some $k > 1$. Then, by the definition of C_k,

$$f\big(ty + (1-t)y'\big) > f_1\big(ty + (1-t)y'\big) = tf(y) + (1-t)f(y'),$$

which contradicts the convexity of f. Thus C_1 is convex; moreover, $f\big|\mathrm{C}_1 = f_1\big|\mathrm{C}_1$ is affine. Since $t\mathrm{C}_1 \subset \mathrm{C}_1$ for $t > 0$ by the positive homogeneity of f_1 and f, C_1 is a convex cone.

"At the other end", consider

$$\mathrm{C}_p = \{y \in \mathrm{C}: \ f_k(y) < f(y) \ \text{ for } 1 \leqslant k < p\}.$$

Of course $y \in \mathrm{C}_p \Rightarrow f_p(y) = f(y)$. Suppose $y, y' \in \mathrm{C}_p$ and $0 < t < 1$. If $ty + (1-t)y'$ did not belong to C_p, then $ty + (1-t)y' \in \mathrm{C}_k$ for some $k < p$, therefore

$$\begin{aligned} f\big(ty + (1-t)y'\big) &= f_k\big(ty + (1-t)y'\big) && (\text{because } ty + (1-t)y' \in \mathrm{C}_k) \\ &= tf_k(y) + (1-t)f_k(y') \\ &< tf_p(y) + (1-t)f_p(y') && (\text{because } y, y' \in \mathrm{C}_p \text{ and } k < p) \\ &= f_p\big(ty + (1-t)y'\big) \\ &\leqslant f\big(ty + (1-t)y'\big) && (\text{by the definition of } f), \end{aligned}$$

whence the absurdity $f\big(ty + (1-t)y'\big) < f\big(ty + (1-t)y'\big)$.

Thus C_p is convex, hence is a convex cone (positive homogeneity), and $f|C_p = f_p|C_p$ is affine.

"In between", consider $1 < m < p$ and

$$C_m = \{y \in C : \; f_k(y) < f(y) \text{ for } 1 \leqslant k < m\,, \text{ and } f_m(y) = f(y)\}\,.$$

If $y, y' \in C_m$ and $0 < t < 1$, then

$$(*) \qquad \begin{aligned} f_m\big(ty + (1-t)y'\big) &= tf_m(y) + (1-t)f_m(y') \\ &= tf(y) + (1-t)f(y') \qquad (\text{because } y, y' \in C_m). \end{aligned}$$

If $ty + (1-t)y'$ did not belong to C_m, one would have $ty + (1-t)y' \in C_k$ for some $k \neq m$, therefore

$$(**) \qquad f\big(ty + (1-t)y'\big) = f_k\big(ty + (1-t)y'\big) = tf_k(y) + (1-t)f_k(y')\,.$$

case 1. $k < m$. Then

$$\begin{aligned} f\big(ty+(1-t)y'\big) &= tf_k(y) + (1-t)f_k(y') && \big(\text{by } (**)\big) \\ &< tf(y) + (1-t)f(y') && (\text{because } y, y' \in C_m \text{ and } k < m\,) \\ &= f_m\big(ty + (1-t)y'\big) && \big(\text{by } (*)\big) \\ &\leqslant f\big(ty + (1-t)y'\big) && (\text{by the definition of } f\,)\,, \end{aligned}$$

whence the absurdity $f\big(ty + (1-t)y'\big) < f\big(ty + (1-t)y'\big)$.

case 2. $m < k < p$. Then

$$\begin{aligned} tf(y)+(1-t)f(y') &= f_m\big(ty + (1-t)y'\big) && \big(\text{by } (*)\big) \\ &< f_k\big(ty + (1-t)y'\big) && (\text{because } ty + (1-t)y' \in C_k \text{ and } m < k\,) \\ &= f\big(ty + (1-t)y'\big) && \big(\text{because } ty + (1-t)y' \in C_k\big)\,, \end{aligned}$$

contradicting the convexity of f.

Thus C_m is convex, indeed is a convex cone, and $f|C_m = f_m|C_m$ is affine.

IV.115, ℓ. -7.

"Then $y_1 + y_2 + \cdots + y_p = x$."

Recall that a convex cone with vertex 0 is closed under addition (TVS, II, §2, No. 4, Prop. 10); thus its operations are $x + y$ and tx ($t > 0$; or $t \geqslant 0$ when the cone is pointed). In particular, $y_k \in C$ for $k = 1, \ldots, p$. Thus

$$y_1 + \cdots + y_p = x_1 + \cdots + x_n$$

is immediate from the associativity theorem (A, I, §1, No. 3, Th. 1).

For use below, we note that a positively homogeneous affine function $g : \mathrm{C} \to \mathbf{R}$ is additive: $x + y = 2 \cdot \big(\frac{1}{2}x + \frac{1}{2}y\big)$, therefore

$$g(x+y) = 2g\Big(\frac{1}{2}x + \frac{1}{2}y\Big) = 2\Big[\frac{1}{2}g(x) + \frac{1}{2}g(y)\Big] = g(x) + g(y)\,.$$

In particular, the functions $f\big|\mathrm{C}_k = f_k\big|\mathrm{C}_k$ are *additive*.

IV.115, ℓ. $-7, -6$.

"Since f is affine on C_k, $f(y_1) + \cdots + f(y_p) = f(x_1) + \cdots + f(x_n)$."

In slow motion, let $\mathrm{I}_k = \big\{i \in \{1, \ldots, n\} : \ x_i \in \mathrm{C}_k\big\}$; then $y_k = \sum_{i \in \mathrm{I}_k} x_i$, and $f(y_k) = \sum_{i \in \mathrm{I}_k} f(x_i)$ by the additivity of $f\big|\mathrm{C}_k = f_k\big|\mathrm{C}_k$ (see the preceding note), and

$$\sum_{k=1}^{p} f(y_k) = \sum_{k=1}^{p} \Big(\sum_{i \in \mathrm{I}_k} f(x_i) \Big) = \sum_{i=1}^{n} f(x_i)$$

by the associativity theorem. Resist writing $f(x)$ for this sum; f is convex on C but need not be additive.

For use below, we note that the convexity and positive homogeneity of f implies that f is *subadditive*, that is, $f(x+y) \leqslant f(x) + f(y)$ for $x, y \in \mathrm{C}$; for,

$$f(x+y) = 2f\Big(\frac{1}{2}x + \frac{1}{2}y\Big) \leqslant 2\Big[\frac{1}{2}f(x) + \frac{1}{2}f(y)\Big] = f(x) + f(y)\,.$$

IV.115, ℓ. -5.

"(19) $$f(x) = \sup\big(f(y_1) + \cdots + f(y_p)\big),"$$

A misprint; read $\overline{f}(x)$ instead of $f(x)$.

IV.116, ℓ. 8.

"... therefore $\overline{f}$ is concave."

For $x, y \in \mathrm{C}$ and $0 < t < 1$,

$$\overline{f}\big(tx + (1-t)y\big) \geqslant \overline{f}(tx) + \overline{f}\big((1-t)y\big) = t\overline{f}(x) + (1-t)\overline{f}(y)\,.$$

IV.116, ℓ. 11.

"... weakly complete"

Regarding the product space $\mathbf{R} \times \mathrm{E}$ as a weak locally convex space.

IV.116, ℓ. 12, 13.

"By Lemma 5, this sum is equal to L."

Suppose $(t,x) \in \mathrm{L}$, that is, $x \in \mathrm{C}$ and $0 \leqslant t \leqslant \overline{f}(x)$. By Lemma 5 one can write $x = x_1 + \cdots + x_p$ with the x_i in C and $\overline{f}(x) = f(x_1) + \cdots + f(x_p)$. Since

$$0 \leqslant t \leqslant \overline{f}(x) = f(x_1) + \cdots + f(x_p)$$

and the $f(x_i)$ are $\geqslant 0$, one can write $t = t_1 + \cdots + t_p$ with $0 \leqslant t_i \leqslant f(x_i)$ for $i = 1, \ldots, p$: for, if $f(x_i) = 0$ for all i, so that $t = 0$, one sets $t_i = 0$ for all i; otherwise, one sets

$$t_i = \frac{f(x_i)}{f(x_1) + \cdots + f(x_p)} \cdot t = \frac{t}{f(x_1) + \cdots + f(x_p)} \cdot f(x_i) \leqslant 1 \cdot f(x_i) .$$

Then $(t_i, x_i) \in \mathrm{L}_i$ and $(t,x) = (t_1,x_1) + \cdots + (t_p,x_p)$; thus $\mathrm{L} \subset \mathrm{L}_1 + \cdots + \mathrm{L}_p$.

On the other hand, if $(t_1,x_1) \in \mathrm{L}_1, \ldots, (t_p,x_p) \in \mathrm{L}_p$ are given, so that $x_i \in \mathrm{C}$ and $0 \leqslant t_i \leqslant f(x_i)$ for all i, then

$$0 \leqslant t_1 + \cdots + t_p \leqslant f(x_1) + \cdots + f(x_p) \leqslant \overline{f}(x_1 + \cdots + x_p) ,$$

whence $(t_1 + \cdots + t_p, x_1 + \cdots + x_p) \in \mathrm{L}$, that is, $(t_1,x_1) + \cdots + (t_p,x_p) \in \mathrm{L}$. Thus $\mathrm{L}_1 + \cdots + \mathrm{L}_p \subset \mathrm{L}$.

Note that since the sets L_i and L are closed in $\mathbf{R}_+ \times \mathrm{C}$, which is complete hence closed in $\mathbf{R}_+ \times \mathrm{E}$, they are also closed in $\mathbf{R}_+ \times \mathrm{E}$.

IV.116, ℓ. 13, 14.

"... $\overline{f}$ is upper semi-continuous."

Equivalently, we are to show that $-\overline{f}$ is lower semi-continuous. Thus, given any $t \in \mathbf{R}$, it is to be shown that the set $\mathrm{A}_t = \{x \in \mathrm{C} : \overline{f}(x) \geqslant t\}$ is closed in C (GT, IV, §6, No. 2, Prop. 1). If $t < 0$ then $\mathrm{A}_t = \mathrm{C}$. If $t \geqslant 0$ then A_t is the inverse image, under the continuous mapping $\mathrm{C} \to \mathbf{R}_+ \times \mathrm{E}$ defined by $x \mapsto (t,x)$ $(x \in \mathrm{C})$, of the closed subset $\mathrm{L} = \{(s,x) : x \in \mathrm{C},\ 0 \leqslant s \leqslant \overline{f}(x)\}$ of $\mathbf{R}_+ \times \mathrm{E}$.

IV.116, ℓ. −11.

"... $\overline{f}$ is indeed convex."

The argument can be simplified slightly (no need to bring in ε). By Lemma 5 (with $x+y$ in the role of x) one can write

$$(*) \qquad x + y = z_1 + \cdots + z_p$$

with the z_i in C and $\overline{f}(x+y) = f(z_1) + \cdots + f(z_p)$. Applying the decomposition theorem to $(*)$ (A, VI, §1, No. 10, Th. 1) (with $q = 2$), $x+y$ is the sum of $2p$ elements z_{ij} of C $(i = 1, \ldots, p;\ j = 1, 2)$ such that

$$x = \sum_{i=1}^{p} z_{i1} , \quad y = \sum_{i=1}^{p} z_{i2} , \quad z_i = z_{i1} + z_{i2} \ \text{ for } i = 1, \ldots, p .$$

Setting $x_i = z_{i1}$ and $y_i = z_{i2}$ for $i = 1, \ldots, p$, one has

$$x = x_1 + \cdots + x_p\,, \quad y = y_1 + \cdots + y_p\,, \quad z_i = x_i + y_i \ \text{ for } i = 1, \ldots, p\,.$$

Since $f(z_i) \leqslant f(x_i) + f(y_i)$ (see the note for IV.115, ℓ. −7, −6) one has

$$\overline{f}(x+y) = \sum_{i=1}^{p} f(z_i) \leqslant \sum_{i=1}^{p} f(x_i) + \sum_{i=1}^{p} f(y_i) \leqslant \overline{f}(x) + \overline{f}(y)\,;$$

therefore, for $0 < t < 1$,

$$\overline{f}\big(tx + (1-t)y\big) \leqslant \overline{f}(tx) + \overline{f}\big((1-t)y\big) = t\overline{f}(x) + (1-t)\overline{f}(y)\,,$$

thus $\overline{f}$ is convex in C. Combined with concavity, this proves that $\overline{f}$ is affine. It follows directly from the inequalities established here that $\overline{f}(x+y) = \overline{f}(x) + \overline{f}(y)$ for all $x, y \in \mathrm{C}$ (see also the note for IV.115, ℓ. −7, −6).

IV.116, ℓ. −2.

"affine functions"

That is, the functions $z \mapsto \langle z, z' \rangle + c$ $(z \in \mathrm{E})$, were z' is a continuous linear form on E and c is a scalar; also called "affine linear functions" (A, II, §9, No. 4, Def. 3).

IV.117, ℓ. 1, 2.

"... where $f_1, \ldots, f_p$ belong to $\mathscr{A}$, $f_1 \geqslant 0, \ldots, f_p \geqslant 0$"

Read: where $f_1, \ldots, f_p$ belong to $\mathscr{A}$, are $\leqslant f$, and $f_1 \geqslant 0, \ldots, f_p \geqslant 0$.

IV.117, ℓ. 5.

"$\overline{f}(y) = f(y)$ if $y \in \mathrm{G}$."

By Lemma 5, there exist $y_1, \ldots, y_p \in \mathrm{C}$ with $y = y_1 + \cdots + y_p$ and $\overline{f}(y) = f(y_1) + \cdots + f(y_p)$. If $p = 1$ the assertion is trivial; assume $p > 1$. If $y = 0$ then $y_i = 0$ for all i (because $-y_i = \sum_{j \neq i} y_j \in \mathrm{C}$ and C is proper) and $\overline{f}(y) = 0 = f(y)$. Assume $y \neq 0$.

From $y - y_i = \sum_{j \neq i} y_i \in \mathrm{C}$ we know that $0 \leqslant y_i \leqslant y$ for the order relation $\leqslant$ on E with positive cone C, therefore $y_i = r_i y$ with $0 < r_i < 1$ (TVS, II, §7, No. 2, *Remark* 1), whence $\sum_{i=1}^{p} r_i = 1$ and

$$\overline{f}(y) = \sum_{i=1}^{p} f(y_i) = \sum_{i=1}^{p} f(r_i y) = \sum_{i=1}^{p} r_i f(y) = 1 \cdot f(y)\,.$$

IV.117, ℓ. 5, 6.

"Since $\lambda^*(\mathrm{K} - (\mathrm{K}\cap\mathrm{G})) = 0$, we have $\lambda(f|\mathrm{K}) = \lambda(\overline{f}|\mathrm{K})$."

We have just shown that $\overline{f} = f$ on G. Since $\mathrm{K} = (\mathrm{K}\cap\mathrm{G})\cup(\mathrm{K} - \mathrm{K}\cap\mathrm{G})$, where $\overline{f} = f$ on $\mathrm{K}\cap\mathrm{G}$ and $\lambda^*(\mathrm{K} - \mathrm{K}\cap\mathrm{G}) = 0$, we have $\overline{f}|\mathrm{K} = f|\mathrm{K}$ λ-almost everywhere, therefore $\lambda^*(\overline{f}|\mathrm{K}) = \lambda^*(f|\mathrm{K})$ (§2, No. 3, Prop. 6).

The function f (resp. $\overline{f}$) is lower (resp. upper) semi-continuous, by hypothesis (resp. Lemma 6), hence so is its restriction to K (GT, IV, §6, No. 2). It follows that $f|\mathrm{K}$ and $\overline{f}|\mathrm{K}$ are measurable with respect to any measure on K (§5, No. 5, Cor. of Prop. 8); being bounded, they are integrable with respect to the (bounded) measure λ (§5, No. 6, Th. 5), so one can drop the asterisks: $\lambda(f|\mathrm{K}) = \lambda(\overline{f}|\mathrm{K})$.

IV.117, ℓ. 9.

"Let $x \in \mathrm{K}$ be the barycenter of λ."

Since K is compact and convex, the barycenter of λ—a priori an element of E'^*—may be viewed as an element of K (No. 1, Cor. of Prop. 1).

IV.117, ℓ. 9, 10.

"If $g \in \mathscr{A}$ then $\lambda(g|\mathrm{K}) = g(x)$."

Recall that $x = b_\lambda = \int_{\mathrm{K}} y\, d\lambda(y) \in \mathrm{E}'^*$. If $g \in \mathscr{A}$ then $g|\mathrm{K} = z'|\mathrm{K}$ for some $z' \in \mathrm{E}'$, thus

$$g(x) = \langle x, z'\rangle = \int \langle y, z'\rangle\, d\lambda(y) = \int (g|\mathrm{K})(y)\, d\lambda(y) = \lambda(g|\mathrm{K})\,.$$

IV.117, ℓ. 10.

"Therefore $\lambda(\overline{f}|\mathrm{K}) = \overline{f}(x)$ (§4, No. 4, Cor. 2 of Prop. 5)."

Since $\lambda(g|\mathrm{K}) = g(x)$ for every $g \in \mathscr{A}$, and since $\lambda(1) = 1$, it is immediate that the equality holds for every $g \in \mathscr{A}'$. The functions in $\mathscr{A}'$ are continuous, $\overline{f}(\mathrm{K})$ is the lower envelope of a decreasing directed set $\mathscr{D}$ of functions $g|\mathrm{K}$ with $g \in \mathscr{A}'$, and the numbers $\lambda(g|\mathrm{K})$ $(= g(x))$ are bounded below (by $\overline{f}(x)$), therefore

$$\int (\overline{f}|\mathrm{K})\, d\lambda = \inf_{g\in\mathscr{D}} \int (g|\mathrm{K})\, d\lambda$$

by the cited Cor. 2, that is, $\lambda(\overline{f}|\mathrm{K}) = \inf_{g\in\mathscr{D}} g(x) = \overline{f}(x)$.

IV.117, ℓ. 13.

"... *admitting a compact sole* M"

This means (TVS, II, §7, No. 3) that $\mathrm{M} = \mathrm{C}\cap\mathrm{H}$, where H is a closed hyperplane in E that does not pass through the vertex 0 of C, the convex set M is compact, and M generates C in the sense that C is the smallest pointed cone with vertex 0 that contains M.

Example: Imagine in $\mathbf{R}^3$ an 'infinite pyramid' resting on its 'peak' at the origin $(0,0,0)$, extending indefinitely above the xy-plane. The extremal generators of C are its edges. A cross-section of C by a plane not passing through the origin is a convex polygonal area M that is a compact sole of C.

IV.117, ℓ. 16.

"... *such that* $\lambda^*(\mathrm{M} - (\mathrm{G} \cap \mathrm{M})) = 0$ "

In the language of Ch. V, §5, No. 7, Def. 4, λ is "concentrated on $\mathrm{G} \cap \mathrm{M}$", or " $\mathrm{G} \cap \mathrm{M}$ carries λ ". If G is closed (for example, if there are only finitely many extremal generators, as in the example of the preceding note) then $\mathrm{M} - \mathrm{G} \cap \mathrm{M}$ is open in M, so that $\mathrm{Supp}\,\lambda \subset \mathrm{G} \cap \mathrm{M}$. In the example of the preceding note, $\mathrm{G} \cap \mathrm{M}$ is the set of vertices of the polygonal boundary of M, and λ is a convex combination of the Dirac measures at the vertices (Ch. III, §2, No. 4, Prop. 12).

IV.117, ℓ. 17, 18.

"Replacing the topology of E by the weakened topology (which does not change the topology of M), ... "

Let $\mathscr{T}$ be the original topology on E, E′ the dual space of E, and $\mathscr{T}_{\mathrm{w}} = \sigma(\mathrm{E}, \mathrm{E}')$ the weakened topology on E. As M is compact for the induced topology $\mathscr{T} \cap \mathrm{M}$, and the coarser topology $\mathscr{T}_{\mathrm{w}} \cap \mathrm{M}$ is Hausdorff (by the Hahn–Banach theorem), the two topologies on M coincide (GT, I, §9, No. 4, Cor. 3 of Th. 2), whence the assertion. Thus M is a compact, hence closed, convex subset of E for $\mathscr{T}_{\mathrm{w}}$.

Also unchanged by the replacement: the set of measures on K, the continuous linear forms on E (i.e., the dual space E′), the closed hyperplanes in E, the closed convex sets in E (TVS, IV, §1, No. 2), the weak topology $\sigma(\mathrm{E}', \mathrm{E})$ on E′, the algebraic dual $(\mathrm{E}')^*$ of E′, and the concept of barycenter $b_\lambda \in (\mathrm{E}')^*$ of a positive measure λ on K of total mass 1.

IV.117, ℓ. 20, 21.

"... let h be a continuous linear form on E such that M is the intersection of C and the hyperplane with equation $h(x) = 1$."

See the note for ℓ. 13.

{" $h(y) = 1$ " would have been preferable, as the letter x is conscripted for duty as a barycenter.}

IV.117, ℓ. 21, 22.

"Let $\mathscr{S}$ be the subset of $\mathscr{C}(\mathrm{M})$ ... "

Clearly $\mathscr{S}$ is a pointed convex cone in $\mathscr{C}(\mathrm{M})$ with vertex 0, hence is closed under addition and under multiplication by scalars $\geqslant 0$; it follows that $\mathscr{S} - \mathscr{S}$ is a linear subspace of $\mathscr{C}(M)$. If $f, g \in \mathscr{S}$ then $f \cup g$ and $f \cap g$ (the upper and lower envelopes) also belong to $\mathscr{S}$. Other properties will be noted below.

{The order relation in $\mathscr{C}(M)$ (or in $\mathscr{S} - \mathscr{S}$) with positive cone $\mathscr{S}$ (TVS, II, §2, No. 5, Prop. 13) is not at issue here.}

IV.117, ℓ. −12.

"By Th. 3, $\lambda(f) = \lambda'(f)$ for every $f \in \mathscr{S}$."

Given $f \in \mathscr{S}$, let $f' : C \to \mathbf{R}$ be a continuous, positively homogeneous convex function such that $f'|M = f$. Write $\mathscr{M}_+^1(X)$ for the positive measures of total mass 1 on a compact space X.

The dictionary between Th. 3 and its Corollary is as follows:

$$\begin{array}{rcl} E & \leftrightarrow & E \\ C & \leftrightarrow & C \\ f & \leftrightarrow & f' \\ K & \leftrightarrow & M \\ f|K & \leftrightarrow & f'|M = f \\ \lambda, \lambda' \in \mathfrak{M}_+^1(K) & \leftrightarrow & \lambda, \lambda' \in \mathfrak{M}_+^1(M)\,. \end{array}$$

The assertion is now clear.

IV.117, ℓ. −11 to −9.

"If f_1, f_2, f_3, f_4 belong to $\mathscr{S}$, then

$$\begin{aligned} \sup(f_1 - f_2, f_3 - f_4) &= \sup(f_1 + f_4, f_3 + f_2) - (f_2 + f_4) \in \mathscr{S} - \mathscr{S} \\ \inf(f_1 - f_2, f_3 - f_4) &= -\sup(f_2 - f_1, f_4 - f_3) \in \mathscr{S} - \mathscr{S}\,." \end{aligned}$$

Regard $\mathscr{C}(M)$ as a Riesz space in the usual way, with $f \leqslant g$ the pointwise order relation, and with $f \cup g$ and $f \cap g$ as the pointwise supremum and infimum (Ch. III, §1, No. 5). As observed in the note for ℓ. 21, 22, $\mathscr{S}$ is closed under finite sups and infs. The first displayed relation is an application, in the Riesz space $\mathscr{C}(M)$, of the invariance of order under translation by the element $f_2 + f_4$ of $\mathscr{S}$ (Ch. II, §1, No. 1, formula (5)), and shows that $\mathscr{S} - \mathscr{S}$ is closed under finite sups; and the second displayed relation is a consequence of the first, showing that $\mathscr{S} - \mathscr{S}$ is closed under finite infs. (It follows that if $f \in \mathscr{S} - \mathscr{S}$ then $|f| = \sup(f, -f) \in \mathscr{S} - \mathscr{S}$.)

Conclusion: $\mathscr{S} - \mathscr{S}$ is itself a Riesz space for the order relation induced by that of $\mathscr{C}(M)$.

IV.117, ℓ. −8.

"Since $h|\mathrm{M} \in \mathscr{S}$, $\mathscr{S} - \mathscr{S}$ contains the constant functions."

The half-space $\mathrm{H}' = \{y \in \mathrm{E} : \ h(y) \geqslant 0\}$ is a pointed cone (albeit degenerate) with vertex 0, and $\mathrm{M} \subset \mathrm{H}'$ (since $h(y) = 1$ for $y \in \mathrm{M}$), therefore $\mathrm{C} \subset \mathrm{H}'$ (see the note for ℓ. 13), that is, $h \geqslant 0$ on C. Thus $h|\mathrm{M}$ qualifies for membership in $\mathscr{S}$, that is, the constant function 1_{M} belongs to $\mathscr{S}$, whence the assertion.

IV.117, ℓ. −6, −5.

"...this form is the difference of two continuous linear forms that are positive on C (TVS, II, §6, No. 8, Lemma 1)."

The weak completeness of C plays a role here.

IV.117, ℓ. −5 to −3.

"It follows from the foregoing that for α, β real, there exists $f \in \mathscr{S} - \mathscr{S}$ such that $f(x) = \alpha$, $f(y) = \beta$."

Say $k \in \mathrm{E}'$ with $k(x) \neq k(y)$ (Hahn–Banach), and write $k = k_1 - k_2$ with $k_i \in \mathrm{E}'$ and $k_i \geqslant 0$ on C for $i = 1, 2$ (see the preceding note). Then $k|\mathrm{M} = k_1|\mathrm{M} - k_2|\mathrm{M} \in \mathscr{S} - \mathscr{S}$, thus $\mathscr{S} - \mathscr{S}$ does contain a function that distinguishes between x and y; the passage to a function in the vector space $\mathscr{S} - \mathscr{S}$ (containing 1) taking on specified values at x and y is carried out in the proof of Stone's theorem (GT, X, §4, No. 1, Th. 2).

CHAPTER V

Integration of measures

§1. ESSENTIAL UPPER INTEGRAL

V.2, ℓ. 11, 12.

"... the condition $\mu^{\bullet}(f) = 0$ means that f is *locally negligible* (Ch. IV, §5, No. 2, Prop. 5)"

The condition $\mu^{\bullet}(f) = 0$ means that for each compact set K in T, $\mu^*(f\varphi_{\mathrm{K}}) = 0$, i.e., $f\varphi_{\mathrm{K}}$ is negligible (Ch. IV, §2, No. 2, Def. 1), i.e., the set $\{t : f(t)\varphi_{\mathrm{K}}(t) > 0\}$ is negligible (*loc. cit.*, No. 3, Th. 1), that is, writing $\mathrm{A} = \{t : f(t) > 0\}$, the set $\mathrm{A} \cap \mathrm{K}$ is negligible; this means that A is locally negligible (by the cited Prop. 5), in other words f is locally negligible (remarks following Cor. 4 of the cited Prop. 5).

V.2, ℓ. 14.

"The mapping $\mu^{\bullet}$ of $\mathscr{F}_+(\mathrm{T})$ into $\overline{\mathbf{R}}$ coincides with μ on $\mathscr{K}_+(\mathrm{T})$."

Let $f \in \mathscr{K}_+(\mathrm{T})$. By (1), $\mu^{\bullet}(f) \leqslant \mu^*(f)$; but $\mathrm{K} = \operatorname{Supp} f$ is compact and $f = f\varphi_{\mathrm{K}}$, therefore $\mu^*(f) = \mu^*(f\varphi_{\mathrm{K}}) \leqslant \mu^{\bullet}(f)$, whence $\mu^{\bullet}(f) = \mu^*(f) = \mu(f)$ (Ch. IV, §1, No. 1, remark following Def. 1).

V.2, ℓ. −6, −5.

"... a) from Proposition 6 of Ch. IV, §2, No. 3 and Proposition 5 of Ch. IV, §5, No. 2"

By assumption, the set $\mathrm{A} = \{t : f(t) \neq g(t)\}$ is locally negligible. For every compact set K in T, $\mathrm{A} \cap \mathrm{K}$ is negligible by the cited Prop. 5, thus $f\varphi_{\mathrm{K}} = g\varphi_{\mathrm{K}}$ almost everywhere, so $\mu^*(f\varphi_{\mathrm{K}}) = \mu^*(g\varphi_{\mathrm{K}})$ by the cited Prop. 6.

V.2, ℓ. −5, −4.

"... b), c), d) from Propositions 10, 11, 12 of Ch. IV, §1, No. 3."

As to *c*), the cited Prop. 11 assumes that α is finite. But *c*) is also true for $\alpha = +\infty$, as follows.

case 1. $\mu^\bullet(f) = 0$.

Then f is locally negligible, and the convention $(+\infty)\cdot 0$ in $\overline{\mathbf{R}}$ assures that $(+\infty)f$ is also locally negligible, whence $\mu^\bullet\big((+\infty)f\big) = 0 = (+\infty)\cdot 0 = (+\infty)\cdot\mu^\bullet(f)$.

case 2. $\mu^\bullet(f) > 0$.

Then $(+\infty)\cdot\mu^\bullet(f) = +\infty$, and $f(t) > 0 \Leftrightarrow (+\infty)\cdot f(t) = +\infty$ (GT, IV, §4, No. 3, Prop. 8). Write $\mathrm{A} = \{t : f(t) > 0\}$. By hypothesis, A is not locally negligible, hence there exists a compact set K in T such that $\mathrm{A}\cap\mathrm{K}$ is not negligible. But $(+\infty)f = +\infty$ on $\mathrm{A}\cap\mathrm{K}$, thus $(+\infty)f\cdot\varphi_{\mathrm{K}}$ is not almost everywhere finite, therefore $\mu^*\big((+\infty)f\cdot\varphi_{\mathrm{K}}\big) = +\infty$ (Ch. IV, §2, No. 3, Prop. 7), whence $\mu^\bullet\big((+\infty)f\big) = +\infty$ and the equality in c) reduces to $+\infty = +\infty$.

V.2, ℓ. $-2, -1$.

$$\begin{aligned}\lim_{n\to\infty}\mu^\bullet(f_n) &= \sup_{n\in\mathbf{N}}\ \sup_{\mathrm{K}\in\mathfrak{K}}\mu^*(f_n\varphi_{\mathrm{K}}) = \sup_{\mathrm{K}\in\mathfrak{K}}\ \sup_{n\in\mathbf{N}}\mu^*(f_n\varphi_{\mathrm{K}})\\ &= \sup_{\mathrm{K}\in\mathfrak{K}}\mu^*(f\varphi_{\mathrm{K}}) = \mu^\bullet(f)\,.\end{aligned}$$

By the theorem on monotone limits, $f = \sup_{n\in\mathbf{N}} f_n$ (GT, IV, §5, No. 2, Th. 2 applied with $\mathrm{X} = \overline{\mathbf{R}}$). In view of b), the first displayed equality says that $\lim_{n\to\infty}\mu^\bullet(f_n) = \sup_{n\in\mathbf{N}}\mu^\bullet(f_n)$ (the cited Th. 2, with $\mathrm{X} = \mathscr{F}_+(\mathrm{T})$ and $f = \mu^\bullet$).

The second displayed equality follows from the 'associativity of sups' (S, III, §1, No. 9, Cor. of Prop. 7).

The third displayed equality follows from Ch. IV, §1, No. 3, Th. 3.

V.3, ℓ. $6, 7$.

"One is immediately reduced to the proof of the analogous formula for the upper integral."

Write $\mathfrak{K}$ for the set of all compact subsets K of T. Assuming proven the analogous formula for $\int^*$, for each $\mathrm{K}\in\mathfrak{K}$ one has

$$\int^* f(g+h)\varphi_{\mathrm{K}}\,d\mu = \int^* (f\varphi_{\mathrm{K}})(g+h)\,d\mu = \int^* f\varphi_{\mathrm{K}}g\,d\mu + \int^* f\varphi_{\mathrm{K}}h\,d\mu\,;$$

as K varies over $\mathfrak{K}$, the supremum of the left-most member is by definition $\mu^\bullet\big(f(g+h)\big)$. Write $\alpha_{\mathrm{K}} = \int^* f\varphi_{\mathrm{K}}g\,d\mu$, $\beta_{\mathrm{K}} = \int^* f\varphi_{\mathrm{K}}h\,d\mu$, so that

$$\mu^\bullet\big(f(g+h)\big) = \sup_{\mathrm{K}\in\mathfrak{K}}\big(\alpha_{\mathrm{K}} + \beta_{\mathrm{K}}\big).$$

We know (GT, IV, §5, No. 7, formula (26)) that

$$\sup_{K_1,K_2\in\mathfrak{K}}\bigl(\alpha_{K_1}+\beta_{K_2}\bigr)=\sup_{K\in\mathfrak{K}}\alpha_K+\sup_{K\in\mathfrak{K}}\beta_K=\mu^\bullet(fg)+\mu^\bullet(fh)\,,$$

so it will suffice to show that

$$\sup_{K\in\mathfrak{K}}\bigl(\alpha_K+\beta_K\bigr)=\sup_{K_1,K_2\in\mathfrak{K}}\bigl(\alpha_{K_1}+\beta_{K_2}\bigr).$$

The left member γ is obviously $\leqslant$ the right member δ; on the other hand, for all $K_1, K_2 \in \mathfrak{K}$ one has

$$\alpha_{K_1}+\beta_{K_2}\leqslant\alpha_{K_1\cup K_2}+\beta_{K_1\cup K_2}\leqslant\gamma\,,$$

whence $\delta \leqslant \gamma$. {Perhaps this is a special case of GT, *loc. cit.*, Cor. 1 of Prop. 13; I have not checked the details.}

V.3, ℓ. 10.

"...it remains to establish the reverse inequality."

Let $\alpha = \int^* fg\,d\mu$, $\beta = \int^* fh\,d\mu$, $\gamma = \int^* f(g+h)\,d\mu$; we are to show that

$$(*)\qquad \alpha+\beta\leqslant\gamma\,.$$

At any rate, $0 \leqslant \alpha, \beta \leqslant \gamma$. If $\gamma = +\infty$ then $(*)$ holds trivially, so we can suppose that γ is finite; if one of α, β, γ is equal to 0 then $(*)$ holds trivially, so we can suppose that α, β, γ are all finite and > 0. It then follows that, at almost every point of T, the functions fg, fh and $f(g+h)$ are finite-valued (Ch. IV, §2, No. 3, Prop. 7), none of the sets

$$A=\{t:\ f(t)g(t)>0\},\ B=\{t:\ f(t)h(t)>0\},\ C=\{t: f(t)(g{+}h)(t)>0\}$$

is negligible, and $A \cup B = C$.

Let N be a negligible set such that fg, fh, and $f(g+h)$ are finite-valued on $T - N$. Replacement of f by $f\varphi_{T-N}$ does not change the numbers α, β, γ, so we can suppose that fg, gh, and $f(g+h)$ are finite at every point of T. This is useful for the next note.

Caution: The functions f and $g+h$ are permitted to have infinite values but, at a point where one of them is infinite, the other must be 0.

V.3, ℓ. 12, 13.

"...then $v \geqslant f$ and $u \geqslant v(g+h)$"

We can suppose, after excluding special cases and modifying f on a negligible set, that $f(g+h)$ is finite at every point of T (see the preceding note). It is tacit in the definition of v that if $(g+h)(t) = +\infty$ then $v(t) = 0$. Thus:

$$v(t) = \begin{cases} \dfrac{u(t)}{(g+h)(t)} & \text{if } 0 < (g+h)(t) < +\infty \\ 0 & \text{if } (g+h)(t) = +\infty \\ +\infty & \text{if } (g+h)(t) = 0\,. \end{cases}$$

Proof that $v \geqslant f$. If $(g+h)(t) = 0$ then $v(t) = +\infty \geqslant f(t)$; if $(g+h)(t) = +\infty$ then $f(t) = 0$ (by the finiteness of $f(g+h)$) and $v(t) = 0 = f(t)$; and if $0 < (g+h)(t) < +\infty$ then (since $u \geqslant f(g+h)$)

$$v(t) = \frac{u(t)}{(g+h)(t)} \geqslant \frac{f(t)(g+h)(t)}{(g+h)(t)} = f(t)\,.$$

Proof that $u \geqslant v(g+h)$. If $(g+h)(t) = 0$, the inequality holds trivially at t since $u \geqslant f(g+h) \geqslant 0$; if $(g+h)(t) = +\infty$ then $v(t) = 0$ and the inequality again holds trivially at t; and if $0 < (g+h)(t) < +\infty$ then $u(t) = v(t)(g+h)(t)$ by the definition of $v(t)$.

V.3, ℓ. 15.

"... v being measurable (Ch. IV, §5, No. 6, Cor. 4 of Th. 5)"

The cited Cor. 4 justifies the equality in the the display that follows, once the measurability of v is established. We assume that the definition of v has been fine-tuned as in the preceding note.

Let $\mathrm{A} = \{t :\ 0 < (g+h)(t) < +\infty\}$ and let $k = (g+h)\big|\mathrm{A}$ be the restriction of $g+h$ to A. Since $g+h$ is measurable (Ch. IV, §5, No. 3, Th. 1), the set A is measurable (*loc. cit.*, No. 5, Prop. 7), therefore the function $k : \mathrm{A} \to \mathbf{R}$ is measurable in the sense of Ch. IV, §5, No. 10, Def. 8 (see the criterion $c'')$ on p. IV.x127 above). Since $a \mapsto 1/a$ $(0 < a < +\infty)$ is continuous, the function $1/k : \mathrm{A} \to \mathbf{R}$ is also measurable (see the item *No.3, Th. 1′* on p. IV.x133). It follows that the extension by 0 of $1/k$ to T is measurable (Ch. IV, §5, No. 10, Prop. 15, criterion d)), that is, the function w defined by

$$w(t) = \begin{cases} \dfrac{1}{(g+h)(t)} & \text{for } t \in \mathrm{A} \\ 0 & \text{for } t \in \mathrm{T} - \mathrm{A} \end{cases}$$

is measurable. Since u is measurable (*loc. cit.*, No. 5, Cor. of Prop. 8), so is uw (see the note for IV.64, ℓ. 10–13), therefore $uw\big|\mathrm{A}$ is measurable; moreover, $uw\big|\mathrm{A} = v\big|\mathrm{A}$, and uw is the extension by 0 of $v\big|\mathrm{A}$ to T, briefly $(v\big|\mathrm{A})' = uw$. On the other hand, the sets

$$\mathrm{B} = \{t :\ (g+h)(t) = 0\}\,,\ \mathrm{C} = \{t :\ (g+h)(t) = +\infty\}$$

are also measurable (Ch. IV, §5, No. 5, Prop. 7), therefore the step function $(+\infty)\cdot\varphi_{\mathrm{B}}$ on T is measurable (*loc. cit.*), so the function

$$v = (v|\mathrm{A})' + (+\infty)\cdot\varphi_{\mathrm{B}} + 0\cdot\varphi_{\mathrm{C}} = uw + (+\infty)\cdot\varphi_{\mathrm{B}}$$

is the sum of measurable functions.

V.3, ℓ. $-7, -6$.

"The case of an infinite sequence may be deduced from this by means of Prop. 1, e)."

Writing $h_n = \sum_{k=0}^{n} g_k$, $h = \sum_{k\in\mathbf{N}} g_k$, one has $h_n \uparrow h$ pointwise, therefore $fh_n \uparrow fh$ pointwise (the convention $0\cdot(+\infty) = 0$ plays a role here), hence $\int^\bullet fh_n\,d\mu \uparrow \int^\bullet fh\,d\mu$ by the cited proposition.

V.4, ℓ. $1, 2$.

"The proof is immediate from the analogous statement in Ch. IV (§1, No. 3, Prop. 15)."

It is understood that the measures μ and ν are positive. In slow motion: for every $f \in \mathscr{F}_+(\mathrm{T})$, as K varies over the set of all compact subsets of T,

$$\begin{aligned}(\mu+\nu)^\bullet(f) &= \sup_{\mathrm{K}}(\mu+\nu)^*(f\varphi_{\mathrm{K}})\\ &= \sup_{\mathrm{K}}\left[\mu^*(f\varphi_{\mathrm{K}}) + \nu^*(f\varphi_{\mathrm{K}})\right]\\ &= \sup_{\mathrm{K}}\mu^*(f\varphi_{\mathrm{K}}) + \sup_{\mathrm{K}}\nu^*(f\varphi_{\mathrm{K}})\\ &= \mu^\bullet(f) + \nu^\bullet(f)\\ &= (\mu^\bullet + \nu^\bullet)(f)\,,\end{aligned}$$

where the second equality holds by the cited Prop. 15; the third, by the theorem on monotone limits (GT, IV, §5, No. 2, Th. 2) and the continuity of addition in $\overline{\mathbf{R}}_+$ (GT, IV, §4, No. 3); and the other three, by definitions.

V.4, ℓ. $6, 7$.

"It follows, by the definition of upper integral, that $\mu^*(f) \leqslant \mu^\bullet(f)$ "

In the notation of Ch. IV, §1, No. 1, $f \in \mathscr{I}_+(\mathrm{T})$, therefore $\mu^*(f) = \sup\mu(g)$ as g varies over the set of all functions in $\mathscr{K}_+(\mathrm{T})$ such that $g \leqslant f$ (*loc. cit.*, Def. 1). For any such g, write $\mathrm{K}(g)$ for its (compact) support; then $g = g\varphi_{\mathrm{K}(g)} \leqslant f\varphi_{\mathrm{K}(g)}$ and

$$\mu(g) = \mu^*(g) \leqslant \mu^*(f\varphi_{\mathrm{K}(g)}) \leqslant \mu^\bullet(f)\,,$$

whence the assertion.

V.4, ℓ. 20.

"Ch. IV, §1, No. 4, Prop. 19"

And Ch. IV, §4, No. 6, Prop. 10.

V.5, ℓ. 14.

"... f also has it."

By assumption, $f = \sum_{n\in\mathbf{N}} f_n$. What does the notation mean? For every $t \in \mathrm{T}$, the sum $\sum_{n\in\mathbf{N}} f_n(t)$ exists in $\overline{\mathbf{R}}_+$ as the supremum of the finite subsums (GT, IV, §7, No. 5, Prop. 2) and one defines a function $\sum_{n\in\mathbf{N}} f_n \in \mathscr{F}_+(\mathrm{T})$ by

$$\Big(\sum_{n\in\mathbf{N}} f_n\Big)(t) = \sum_{n\in\mathbf{N}} f_n(t)\,.$$

{I can't put my finger on where this notation is explicitly defined; it appears already in Ch. IV, §1, No. 1, Prop. 3 and No. 3, Prop. 13.}

Suppose that for each $n \in \mathbf{N}$ one has $f_n = \sum_{k\in\mathbf{N}} h_{nk}$, where $h_{nk} \in \mathscr{F}_+(\mathrm{T})$ $(k \in \mathbf{N})$, h_{n0} is μ-negligible, and, for every $k \geqslant 1$, there exists a compact set K_{nk} in T such that $h_{nk}\big|\mathrm{K}_{nk}$ is finite and continuous, and $h_{nk} = 0$ on $\mathrm{T} - \mathrm{K}_{nk}$. Thus, by assumption,

$$f = \sum_{n\in\mathbf{N}} f_n = \sum_{n\in\mathbf{N}} \Big(\sum_{k\in\mathbf{N}} h_{nk}\Big).$$

For every $t \in \mathrm{T}$, one has

$$\sum_{n\in\mathbf{N}} \Big(\sum_{k\in\mathbf{N}} h_{nk}(t)\Big) = \sum_{(n,k)\in\mathbf{N}\times\mathbf{N}} h_{nk}(t)\,;$$

for, all of the sums exist in $\overline{\mathbf{R}}_+$, and the equality holds by associativity (GT, IV, §7, No. 5, *Remark*). In other words,

$$\sum_{n\in\mathbf{N}} \Big(\sum_{k\in\mathbf{N}} h_{nk}\Big) = \sum_{(n,k)\in\mathbf{N}\times\mathbf{N}} h_{nk}\,,$$

that is,

$$f = \sum_{(n,k)\in\mathbf{N}\times\mathbf{N}} h_{nk}\,.$$

Another application of associativity yields

$$f = \sum_{n\in\mathbf{N},\,k\geqslant 1} h_{nk} + \sum_{n\in\mathbf{N}} h_{n0}\,.$$

Let $(h_n)_{n\geqslant 1}$ be any rearrangement of the h_{nk} $(n \in \mathbf{N}, k \geqslant 1)$, and $(\mathrm{K}_n)_{n\geqslant 1}$ the corresponding rearrangement of the K_{nk}, and define $h_0 = \sum_{n\in\mathbf{N}} h_{n0}$; then, by commutativity (GT, III, §5, No. 1, *Remarks* 2, 3),

$$f = \sum_{n\in\mathbf{N}} h_n \,,$$

where h_0 is negligible and, for $n \geqslant 1$, $h_n = 0$ on $\mathrm{T} - \mathrm{K}_n$ and $h_n \big| \mathrm{K}_n$ is finite-valued and continuous.

V.5, ℓ. 16–18.

"... since f is equal to the sum of the sequence (f_n), it will thus suffice to establish the proposition assuming f to be moderated and *bounded*."

For every n, $0 \leqslant \inf(f,n) \leqslant n$, in particular $\inf(f,0) = 0$ and $\inf(f,n)$ is bounded; and $0 \leqslant \inf(f,n) \leqslant f$, so $\inf(f,n)$ is moderated. Also, $\inf(f,n) \leqslant \inf(f,n+1)$. It follows that every f_n is $\geqslant 0$, bounded, measurable and moderated, and ('telescoping sum')

$$\sum_{k=0}^{n} f_k = \inf(f,n+1) - \inf(f,0) = \inf(f,n+1)\,.$$

Moreover, the sequence of functions $\inf(f,n+1)$ is increasing and has upper envelope f (even at points where $f(t) = +\infty$), whence $\sum_{n\in\mathbf{N}} f_n = f$.

V.5, ℓ. −15 to −12.

"Let $\mathfrak{K}$ be the set of compact subsets K of T such that $f \big| \mathrm{K}$ is continuous; since $\mathfrak{K}$ is μ-dense (Ch. IV, §5, No. 10, Prop. 15), L is the union of a negligible set N and a sequence $(\mathrm{K}_n)_{n\geqslant 1}$ of pairwise disjoint elements of $\mathfrak{K}$ (Ch. IV, §5, No. 8, Def. 6)."

The introduction of $\mathfrak{K}$ and the reference to the cited Prop. 15 are confusing, as that Proposition pertains to functions defined on a measurable subset of T and taking values in a topological space.

Our situation is much simpler. We have a bounded, measurable numerical function $f \geqslant 0$ defined on all of T, and we are given a compact set L in T such that $f = 0$ on $\mathrm{T} - \mathrm{L}$. Apply to f the definition of measurability (Ch. IV, §5, No. 1, Def. 1): there exist a negligible set $\mathrm{N} \subset \mathrm{L}$ and a countable family (K_n) of pairwise disjoint compact sets such that $\mathrm{L} - \mathrm{N} = \bigcup_n \mathrm{K}_n$ and $f \big| \mathrm{K}_n$ is continuous for every n. Then

$$f = f\varphi_{\mathrm{L}} = f\varphi_{\mathrm{N}} + \sum_n f\varphi_{\mathrm{K}_n} \,,$$

and the functions $h_0 = f\varphi_{\mathrm{N}}$, $h_n = f\varphi_{\mathrm{K}_n}$ meet the requirements.

V.5, $\ell.\ -2, -1$.

"The first assertion follows immediately from Lemma 1 of Ch. IV, §5, No. 6."

Assuming $\mu^*(f) < +\infty$ we are to show that f is μ-moderated. Since f is equal μ-almost everywhere to a finite-valued function (Ch. IV, §2, No. 3, Prop. 7), we can suppose that f is finite-valued (*Remark* 4 and Ch. IV, *loc. cit.*, Prop. 6), i.e., takes its values in the Banach space $\mathbf{R}$; then, by the cited Lemma 1, the set $\mathrm{A} = \{t : \ f(t) \neq 0\}$ satisfies the condition c) of Prop. 5, hence is μ-moderated, thus $f = 0$ on the complement of a μ-moderated set.

One notes that to say that f is μ-moderated, that is, f is zero on the complement of a μ-moderated set—in other words $\{t : \ f(t) \neq 0\} \subset \mathrm{M}$ for some moderated set M—is equivalent to saying that the set $\{t : \ f(t) \neq 0\}$ is moderated.

The following reformulation of 1) exhibits its parallelism with 3):

1′) *If* $\mu^*(f) < +\infty$ *then there exists a* μ*-moderated subset* A, *the union of a sequence of compact subsets of* T, *such that* $f = f\varphi_{\mathrm{A}}$ *almost everywhere.*

For, if $\mu^*(f) < +\infty$ we know from the foregoing remark that the set $\{t : \ f(t) \neq 0\}$ is moderated, hence

$$\{t : f(t) \neq 0\} \subset \mathrm{N} \cup \bigcup_n \mathrm{K}_n$$

with N negligible and (K_n) a sequence of compact sets. Let $\mathrm{A} = \bigcup_n \mathrm{K}_n$. If $f(t) \neq f(t)\varphi_{\mathrm{A}}(t)$ then $f(t) \neq 0$ and $t \notin \mathrm{A}$, whence $t \in \mathrm{N}$; thus $f = f\varphi_{\mathrm{A}}$ almost everywhere.

V.6, $\ell.\ 14$.

"...from which 3) follows."

$\mu^\bullet(f\varphi_{\complement \mathrm{A}}) = 0$ means that $f\varphi_{\complement \mathrm{A}}$ is locally negligible (V.2, $\ell.\ 11, 12$), thus the set

$$\mathrm{N} = \{t : \ f(t)\varphi_{\complement \mathrm{A}}(t) \neq 0\} = \complement \mathrm{A} \cap \{t : \ f(t) \neq 0\}$$

is locally negligible; on its complement $\complement \mathrm{N} = \mathrm{A} \cup \{t : \ f(t) = 0\}$ one clearly has $f = f\varphi_{\mathrm{A}}$.

V.6, $\ell.\ 15, 16$.

"Corollary 1."

The assertion is that $\mu^*(|f|) = 0 \Leftrightarrow \mu^\bullet(|f|) = 0$ and f is moderated.

$\Leftarrow$: Citing Prop. 7, 2), $\mu^*(|f|) = \mu^\bullet(|f|) = 0$.

$\Rightarrow$: The set $\mathrm{N} = \{t : \ f(t) \neq 0\}$ is negligible, hence (trivially) moderated; that is, f is moderated, whence $\mu^\bullet(|f|) = \mu^*(|f|) = 0$.

V.6, ℓ. 17, 18.

"Corollary 2."

All functions on T are μ-moderated, so $\mu^* = \mu^\bullet$ by Prop. 7, 2).

V.6, ℓ. −3 to −1.

"There exists a compact set K such that (Ch. IV, §4, No. 4, Cor. 1 of Prop. 5):

$$\mu^\bullet(h_0) - a \leqslant \mu^*(h_0\varphi_{\mathrm{K}}) = \mu(h_0\varphi_{\mathrm{K}}) \leqslant \mu^\bullet(h_0)\,.\text{"}$$

The existence of K satisfying $\mu^\bullet(h_0) - a \leqslant \mu^*(h_0\varphi_{\mathrm{K}}) \leqslant \mu^\bullet(h_0)$ follows from the definition of $\mu^\bullet$. Since h_0 and φ_{K} are upper semi-continuous, so is $h_0\varphi_{\mathrm{K}}$ (by the 'dual' of GT, IV, §6, No. 2, Prop. 2); but $h_0\varphi_{\mathrm{K}}$ is moderated, so $\mu^*(h_0\varphi_{\mathrm{K}}) = \mu^\bullet(h_0\varphi_{\mathrm{K}}) \leqslant \mu^\bullet(h_0) < +\infty$, therefore $h_0\varphi_{\mathrm{K}}$ is μ-integrable by the cited Cor. 1, whence $\mu^*(h_0\varphi_{\mathrm{K}}) = \mu(h_0\varphi_{\mathrm{K}})$ (Ch. IV, §4, No. 2, Prop. 1).

V.7, ℓ. 5.

"But (Ch. IV, §4, No. 4, Cor. 1 of Prop. 5) $\mu^\bullet(h_0\varphi_{\complement\mathrm{K}}) \leqslant a$ "

One has $h_0 = h_0\varphi_{\mathrm{K}} + h_0\varphi_{\complement\mathrm{K}}$, where φ_{K} and $\varphi_{\complement\mathrm{K}}$ are measurable and $h_0 \geqslant 0$; by No. 1, Prop. 2,

$$\mu^\bullet(h_0) = \mu^\bullet(h_0\varphi_{\mathrm{K}}) + \mu^\bullet(h_0\varphi_{\complement\mathrm{K}})\,;$$

but $\mu^\bullet(h_0\varphi_{\mathrm{K}}) = \mu^*(h_0\varphi_{\mathrm{K}}) = \mu(h_0\varphi_{\mathrm{K}})$ (preceding note), thus

$$\mu^\bullet(h_0\varphi_{\complement\mathrm{K}}) = \mu^\bullet(h_0) - \mu^\bullet(h_0\varphi_{\mathrm{K}}) = \mu^\bullet(h_0) - \mu(h_0\varphi_{\mathrm{K}}) \leqslant a\,.$$

V.7, ℓ. 8.

$$\text{"}\inf_{h\in\mathrm{H}} \mu^*(h\varphi_{\mathrm{K}}) \geqslant \inf_{h\in\mathrm{H}} \mu^\bullet(h) - a\,.\text{"}$$

For all $h \in \mathrm{H}$ with $h \leqslant h_0$, one has $\mu^\bullet(h\varphi_{\complement\mathrm{K}}) \leqslant \mu^\bullet(h_0\varphi_{\complement\mathrm{K}}) \leqslant a$ and so

$$\mu^\bullet(h) = \mu^\bullet(h\varphi_{\mathrm{K}}) + \mu^\bullet(h\varphi_{\complement\mathrm{K}}) \leqslant \mu^\bullet(h\varphi_{\mathrm{K}}) + a = \mu^*(h\varphi_{\mathrm{K}}) + a\,,$$

thus $\mu^*(h\varphi_{\mathrm{K}}) \geqslant \mu^\bullet(h) - a$, whence the asserted inequality.

V.7, ℓ. −8, −7.

"The closure of 0 for this topology is the space $\mathscr{N}_{\mathrm{F}}^{\infty}$ "

The set in question is $\{\mathbf{f} \in \overline{\mathscr{F}}_{\mathrm{F}}^{p} : \overline{\mathrm{N}}_p(\mathbf{f}) = 0\}$ (TVS, II, §1, No. 2, Prop. 2); to say that $\mathbf{f} : \mathrm{T} \to \mathrm{F}$ is locally negligible means that $\mu^\bullet(|\mathbf{f}|) = 0$ (V.2, ℓ. 11, 12). Thus the assertion is that

$$\overline{\mathrm{N}}_p(\mathbf{f}) = 0 \quad\Leftrightarrow\quad \mathbf{f} = \mathbf{0} \text{ locally almost everywhere.}$$

Given $\mathbf{f} \in \overline{\mathscr{F}}_{\mathrm{F}}^{p}$ let $\mathbf{g} \in \mathscr{F}_{\mathrm{F}}^{p}$ with $\mathbf{f} = \mathbf{g}$ locally almost everywhere. By the definitions, $\overline{\mathrm{N}}_p(\mathbf{f}) = \mathrm{N}_p(\mathbf{g}) = \big(\mu^*(|\mathbf{g}|^p)\big)^{1/p}$.

Proof of $\Rightarrow$: If $\overline{\mathrm{N}}_p(\mathbf{f}) = 0$, that is, $\mathrm{N}_p(\mathbf{g}) = 0$, then $\mathbf{g} = \mathbf{0}$ almost everywhere (Ch. IV, §3, No. 2, Prop. 3); but $\mathbf{f} = \mathbf{g}$ locally almost everywhere, therefore $\mathbf{f} = \mathbf{0}$ locally almost everywhere.

Proof of $\Leftarrow$: If $\mathbf{f} = \mathbf{0}$ locally almost everywhere, then also $\mathbf{g} = \mathbf{0}$ locally almost everywhere; but $\mathbf{g}$ is moderated, therefore $\mathbf{g} = \mathbf{0}$ almost everywhere by the *Lemma*, whence $\overline{\mathrm{N}}_p(\mathbf{f}) = \mathrm{N}_p(\mathbf{g}) = 0$.

V.7, ℓ. −6, −5.

"... the normed space $\overline{\mathscr{F}}{}^p_{\mathrm{F}}/\mathscr{N}^\infty_{\mathrm{F}}$ may be canonically identified with $\mathscr{F}^p_{\mathrm{F}}/\mathscr{N}_{\mathrm{F}}$ "

If $\mathbf{f} \in \overline{\mathscr{F}}{}^p_{\mathrm{F}}$, and $\mathbf{g} \in \mathscr{F}^p_{\mathrm{F}}$ with $\mathbf{f} = \mathbf{g}$ locally almost everywhere, the class $\widetilde{\mathbf{g}} = \mathbf{g} + \mathscr{N}_{\mathrm{F}}$ of $\mathbf{g}$ for equality almost everywhere depends only on the class $\mathbf{f} + \mathscr{N}^\infty_{\mathrm{F}}$ of $\mathbf{f}$ for equality locally almost everywhere; for, if $\mathbf{f}' \in \overline{\mathscr{L}}{}^p_{\mathrm{F}}$ with $\mathbf{f}' = \mathbf{f}$ locally almost everywhere, and if $\mathbf{g}' \in \mathscr{L}^p_{\mathrm{F}}$ with $\mathbf{f}' = \mathbf{g}'$ locally almost everywhere, then $\mathbf{g}' = \mathbf{g}$ locally almost everywhere, therefore $\mathbf{g} - \mathbf{g}' \in \mathscr{N}_{\mathrm{F}}$ by the *Lemma*. The correspondence $\mathbf{f} + \mathscr{N}^\infty_{\mathrm{F}} \mapsto \widetilde{\mathbf{g}} = \mathbf{g} + \mathscr{N}_{\mathrm{F}}$ (where $\mathbf{g} \in \mathscr{L}^p_{\mathrm{F}}$ and $\mathbf{f} = \mathbf{g}$ locally almost everywhere) is therefore a well-defined mapping

$$\overline{\mathscr{F}}{}^p_{\mathrm{F}}/\mathscr{N}^\infty_{\mathrm{F}} \to \mathscr{F}^p_{\mathrm{F}}/\mathscr{N}_{\mathrm{F}} ,$$

clearly linear and surjective. Moreover, by the definitions,

$$\overline{\mathrm{N}}_p(\mathbf{f}) = \mathrm{N}_p(\mathbf{g}) = \mathrm{N}_p(\widetilde{\mathbf{g}})$$

(cf. Ch. IV, §3, No. 2), thus the mapping is an isometry. Since $\mathscr{F}^p_{\mathrm{F}}$ is complete (*loc. cit.*, No. 3, Prop. 5), $\mathscr{F}^p_{\mathrm{F}}/\mathscr{N}_{\mathrm{F}}$ is a Banach space (given a Cauchy sequence $(\widetilde{\mathbf{g}}_n)$ in $\mathscr{F}^p_{\mathrm{F}}/\mathscr{N}_{\mathrm{F}}$, the sequence $(\mathbf{g}_n)$ is Cauchy in $\mathscr{F}^p_{\mathrm{F}}$, etc.), therefore so is $\overline{\mathscr{F}}{}^p_{\mathrm{F}}/\mathscr{N}^\infty_{\mathrm{F}}$, consequently $\overline{\mathscr{F}}{}^p_{\mathrm{F}}$ is complete.

Caution: Note the author's avoidance of the notation $\widetilde{\mathbf{f}}$ for the class $\mathbf{f} + \mathscr{N}^\infty_{\mathrm{F}}$ of $\mathbf{f}$ for equality locally almost everywhere, as this notation has been pre-empted for $\mathbf{f} + \mathscr{N}_{\mathrm{F}}$ in Ch. IV, §3, No. 2. {In another context, the notation $\dot{\mathbf{f}} = \mathbf{f} + \mathscr{N}^\infty_{\mathrm{F}}$ has been employed (Ch. IV, §6, No. 3).}

V.7, ℓ. −3, −2.

"We shall similarly denote by $\overline{\mathscr{L}}{}^p_{\mathrm{F}}(\mathrm{T}, \mu)$ (or $\overline{\mathscr{L}}{}^p_{\mathrm{F}}(\mu)$, or $\overline{\mathscr{L}}{}^p_{\mathrm{F}}$) the subspace $\mathscr{L}^p_{\mathrm{F}} + \mathscr{N}^\infty_{\mathrm{F}}$ of $\overline{\mathscr{F}}{}^p_{\mathrm{F}}$ "

As for the notation $\overline{\mathscr{L}}{}^p_{\mathrm{F}}$, note that $\mathscr{L}^p_{\mathrm{F}}$ is dense in $\overline{\mathscr{L}}{}^p_{\mathrm{F}}$, that is, $\overline{\mathscr{L}^p_{\mathrm{F}}} = \overline{\mathscr{L}}{}^p_{\mathrm{F}}$. For, $\{\mathbf{0}\} \subset \overline{\mathscr{L}}{}^p_{\mathrm{F}} \subset \overline{\mathscr{F}}{}^p_{\mathrm{F}}$ and the closure of $\{\mathbf{0}\}$ in $\overline{\mathscr{F}}{}^p_{\mathrm{F}}$ is $\mathscr{N}^\infty_{\mathrm{F}}$, therefore the closure of $\{\mathbf{0}\}$ in $\overline{\mathscr{L}}{}^p_{\mathrm{F}}$ is $\mathscr{N}^\infty_{\mathrm{F}} \cap \overline{\mathscr{L}}{}^p_{\mathrm{F}} = \mathscr{N}^\infty_{\mathrm{F}}$ (GT, I, §3, No. 1, Prop. 1); since $\mathscr{L}^p_{\mathrm{F}}$ contains $\{\mathbf{0}\}$, its closure in $\overline{\mathscr{L}}{}^p_{\mathrm{F}}$ is a linear subspace containing $\mathscr{L}^p_{\mathrm{F}}$ and $\mathscr{N}^\infty_{\mathrm{F}}$, hence containing $\mathscr{L}^p_{\mathrm{F}} + \mathscr{N}^\infty_{\mathrm{F}} = \overline{\mathscr{L}}{}^p_{\mathrm{F}}$.

Better yet, $\mathscr{K}_{\mathrm{F}}$ is dense in $\overline{\mathscr{L}}^{p}_{\mathrm{F}}$. For, if $\mathbf{f} \in \overline{\mathscr{L}}^{p}_{\mathrm{F}}$, say $\mathbf{f} = \mathbf{g}$ locally almost everywhere, where $\mathbf{g} \in \mathscr{L}^{p}_{\mathrm{F}}$, there exists a directed family $(\mathbf{g}_j)$ in $\mathscr{K}_{\mathrm{F}}$ such that $\mathrm{N}_p(\mathbf{g}_j - \mathbf{g}) \to 0$ (Ch. IV, §3, No. 4, Def. 2); since $\mathbf{g}_j - \mathbf{g} = \mathbf{g}_j - \mathbf{f}$ locally almost everywhere, one has

$$\overline{\mathrm{N}}_p(\mathbf{g}_j - \mathbf{f}) = \mathrm{N}_p(\mathbf{g}_j - \mathbf{g}) \to 0\,,$$

that is, $\mathbf{g}_j \to \mathbf{f}$ in $\overline{\mathscr{L}}^{p}_{\mathrm{F}}$.

V.7, ℓ. $-2, -1$.

"...one can also characterize $\overline{\mathscr{L}}^{p}_{\mathrm{F}}$ as the subspace of $\overline{\mathscr{F}}^{p}_{\mathrm{F}}$ constituted by the *measurable* mappings (Ch. IV, §5, No. 6, Th. 5)."

Note that every function in $\mathscr{N}^{\infty}_{\mathrm{F}}$ is measurable, since it is equal locally almost everywhere to the measurable function $\mathbf{0}$ (Ch. IV, §5, No. 2, Prop. 6). Let $\mathbf{f} \in \overline{\mathscr{F}}^{p}_{\mathrm{F}}$, say $\mathbf{f} = \mathbf{g}$ locally almost everywhere with $\mathbf{g} \in \mathscr{F}^{p}_{\mathrm{F}}$; then $\mathbf{f} - \mathbf{g} \in \mathscr{N}^{\infty}_{\mathrm{F}}$, so $\mathbf{f} - \mathbf{g}$ is measurable.

If $\mathbf{f}$ is measurable then so is $\mathbf{g} = \mathbf{f} - (\mathbf{f} - \mathbf{g})$; but $\mathbf{g} \in \mathscr{F}^{p}_{\mathrm{F}}$, therefore $\mathbf{g} \in \mathscr{L}^{p}_{\mathrm{F}}$ by the cited Th. 5, thus $\mathbf{f} = \mathbf{g} + (\mathbf{f} - \mathbf{g}) \in \mathscr{L}^{p}_{\mathrm{F}} + \mathscr{N}^{\infty}_{\mathrm{F}} = \overline{\mathscr{L}}^{p}_{\mathrm{F}}$.

Conversely, if $\mathbf{f} \in \mathscr{L}^{p}_{\mathrm{F}} + \mathscr{N}^{\infty}_{\mathrm{F}}$ then $\mathbf{f}$ is the sum of two measurable mappings, hence is measurable.

V.8, ℓ. $-15, -14$.

"Corollary."

Necessity: $\mathbf{f} \in \mathscr{L}^{p}_{\mathrm{F}} \subset \overline{\mathscr{L}}^{p}_{\mathrm{F}}$, and $|\mathbf{f}|^p$ (hence also $\mathbf{f}$) is moderated by Prop. 7, 1).

Sufficiency: If $\mathbf{f} \in \overline{\mathscr{L}}^{p}_{\mathrm{F}}$ is moderated then $\mu^*(|\mathbf{f}|^p) = \mu^\bullet(|\mathbf{f}|^p) < +\infty$ (Prop. 7, 2) and Prop. 9); moreover, $\mathbf{f}$ is measurable (Prop. 9) therefore $\mathbf{f} \in \mathscr{L}^{p}_{\mathrm{F}}$ (Ch. IV, §5, No. 6, Th. 5).

V.8, ℓ. -13 to -7.

"Definition 3."

In the present context, $\mathbf{f}$ denotes elements of $\overline{\mathscr{L}}^{1}_{\mathrm{F}}$; thus the composition indicated in Def. 3,

$$\overline{\mathscr{L}}^{1}_{\mathrm{F}} \to \mathrm{L}^{1}_{\mathrm{F}} \to \mathrm{F}$$

is the mapping $\mathbf{f} \mapsto \widetilde{\mathbf{g}} = \mathbf{g} + \mathscr{N}_{\mathrm{F}} \in \mathrm{L}^{1}_{\mathrm{F}}$ (where $\mathbf{g} \in \mathscr{L}^{1}_{\mathrm{F}}$ and $\mathbf{f} = \mathbf{g}$ locally almost everywhere), followed by the mapping $\widetilde{\mathbf{g}} \mapsto \mu(\mathbf{g})$. The continuity of the composite is immediate from the definitions:

$$\overline{\mathrm{N}}_1(\mathbf{f}) = \mathrm{N}_1(\mathbf{g}) = \mathrm{N}_1(\widetilde{\mathbf{g}}) = \|\widetilde{\mathbf{g}}\|_1 = \mu(|\mathbf{g}|)$$

(cf. Ch. IV, §3, No. 4) and $|\mu(\mathbf{g})| \leqslant \mu(|\mathbf{g}|)$ (Ch. IV, §4, No. 2, Prop. 2).

V.8, ℓ. $-6, -5$.

"Two essentially integrable functions that are equal locally almost everywhere have the same integral."

More precisely, if $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}$ and $\mathbf{f}' : \mathrm{T} \to \mathrm{F}$ is such that $\mathbf{f}' = \mathbf{f}$ locally almost everywhere, then also $\mathbf{f}' \in \overline{\mathscr{L}}^1_{\mathrm{F}}$ and $\int \mathbf{f}'\, d\mu = \int \mathbf{f}\, d\mu$. This is implicit in Def. 3. Indeed, $\mathbf{f}' - \mathbf{f} \in \mathscr{N}^\infty_{\mathrm{F}}$ and so $\mathbf{f}' = \mathbf{f} + (\mathbf{f}' - \mathbf{f}) \in \overline{\mathscr{L}}^1_{\mathrm{F}} + \mathscr{N}^\infty_{\mathrm{F}} = \overline{\mathscr{L}}^1_{\mathrm{F}}$; and if $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}$ with $\mathbf{f} = \mathbf{g}$ locally almost everywhere, one knows that the class $\widetilde{\mathbf{g}} = \mathbf{g} + \mathscr{N}_{\mathrm{F}}$ of $\mathbf{g}$ for equality almost everywhere (hence also $\int \mathbf{g}\, d\mu$) is independent of the class $\mathbf{f} + \mathscr{N}^\infty_{\mathrm{F}} = \mathbf{f}' + \mathscr{N}^\infty_{\mathrm{F}}$ of $\mathbf{f}$ for equality locally almost everywhere, whence $\int \mathbf{f}'\, d\mu = \int \mathbf{g}\, d\mu = \int \mathbf{f}\, d\mu$.

V.8, ℓ. $-5, -4$.

"For every function $f \geqslant 0$ that is finite and essentially integrable, $\int^\bullet f\, d\mu = \int f\, d\mu$."

Note that a subset of T is negligible if and only if it is moderated and locally negligible (No. 2, Cor. 1 of Prop. 7).

Let $g \in \mathscr{L}^1_{\mathbf{R}}$ with $f = g$ locally almost everywhere. Then $g \geqslant 0$ locally almost everywhere, thus the set $\mathrm{N} = \{t : g(t) < 0\}$ is locally negligible. On the other hand, g is moderated (No. 2, Prop. 7, 1)), thus the set $\mathrm{A} = \{t : g(t) \neq 0\}$ is moderated. But $\mathrm{N} \subset \mathrm{A}$, so $\mathrm{N} = \mathrm{N} \cap \mathrm{A}$ is both locally negligible and moderated, hence negligible, that is, $g \geqslant 0$ almost everywhere. Replacing g by $g\varphi_{\complement \mathrm{N}}$, we can suppose that $g \geqslant 0$ (everywhere). Then

$$\int^\bullet f\, d\mu = \int^\bullet g\, d\mu = \int g\, d\mu = \int f\, d\mu ,$$

the equalities holding, respectively, by No. 1, Prop. 1, a), No. 2, Prop. 7, 2), and Def. 3.

V.8, ℓ. -4 to -2.

"If A is a set whose characteristic function is essentially integrable, then A is said to be an *essentially μ-integrable set*"

For a subset A of T, the following conditions are equivalent:

a) A is essentially integrable;

b) there exists an integrable set C such that $\varphi_{\mathrm{A}} = \varphi_{\mathrm{C}}$ locally almost everywhere;

c) there exists an integrable set C such that the 'symmetric difference' of A and C, that is, the set $(\mathrm{A} - \mathrm{C}) \cup (\mathrm{C} - \mathrm{A})$, is locally negligible.

Proof. *b*) $\Leftrightarrow$ *c*): For, $(\mathrm{A} - \mathrm{C}) \cup (\mathrm{C} - \mathrm{A}) = \{t : \varphi_{\mathrm{A}}(t) \neq \varphi_{\mathrm{C}}(t)\}$.

b) $\Rightarrow$ *a*): Obvious from the definitions.

a) ⇒ *b*): By assumption, there exists a function $g \in \mathscr{L}^1_{\mathbf{R}}$ such that $\varphi_{\mathrm{A}} = g$ locally almost everywhere. Since $(\varphi_{\mathrm{A}})^2 = \varphi_{\mathrm{A}}$ we have $g^2 = g$ locally almost everywhere, hence $g^2 = g$ almost everywhere (*Lemma*). Let

$$\mathrm{B} = \{t : \ g(t)^2 = g(t)\} = \{t : \ g(t) = 0 \ \text{ or } 1\} ;$$

we know that $\complement \mathrm{B}$ is negligible. Let

$$\mathrm{C} = \{t : \ g(t) = 1\} = \overset{-1}{g}(1) = \{t \in \mathrm{B} : \ g(t) \neq 0\} ,$$

which is a measurable set (Ch. IV, §5, No. 5, Prop. 7). If $t \in \mathrm{C}$ then $g(t) = 1 = \varphi_{\mathrm{C}}(t)$; that is, $g = \varphi_{\mathrm{C}}$ on C. To show that $\varphi_{\mathrm{A}} = \varphi_{\mathrm{C}}$ locally almost everywhere, it will suffice to show that

$$\{t : \ g(t) \neq \varphi_{\mathrm{C}}(t)\} \subset \complement \mathrm{B}$$

(whence $g = \varphi_{\mathrm{C}}$ almost everywhere). Indeed, suppose $g(t) \neq \varphi_{\mathrm{C}}(t)$. By the foregoing, $t \notin \mathrm{C}$; therefore $\varphi_{\mathrm{C}}(t) = 0$ and so $g(t) \neq 0$. If one had $t \in \mathrm{B}$ it would follow that $g(t) = 1$, consequently $t \in \mathrm{C}$, a contradiction. Finally, since g is integrable and $\varphi_{\mathrm{C}} = g$ almost everywhere, one has $\mu^*(\varphi_{\mathrm{C}}) = \mu^*(g) < +\infty$, consequently φ_{C} is integrable (Ch. IV, §5, No. 6, Th. 5), that is, C is an integrable set.

V.9, *ℓ*. 12–16.

"We note for example the inequality

$$\left|\int \mathbf{f}\, d\mu\right| \leqslant \int |\mathbf{f}|\, d\mu , \tag{3}$$

valid for every essentially integrable function $\mathbf{f}$ with values in a Banach space."

Choose $\mathbf{f}_1 \in \mathscr{L}^1_{\mathrm{F}}$ with $\mathbf{f} = \mathbf{f}_1$ locally almost everywhere. Then $|\mathbf{f}_1| \in \mathscr{L}^1_{\mathbf{R}}$ (Ch. IV, §3, No. 5, Prop. 11) and $|\mathbf{f}| = |\mathbf{f}_1|$ locally almost everywhere, therefore, by definition, $\int \mathbf{f}\, d\mu = \int \mathbf{f}_1\, d\mu$ and $\int |\mathbf{f}|\, d\mu = \int |\mathbf{f}_1|\, d\mu$, thus (3) is immediate from the corresponding inequality for $\mathbf{f}_1$ (Ch. IV, §4, No. 2, Prop. 2).

Another example. The analogue of Th. 1 of Ch. IV, §4, No. 2: Suppose F, G are Banach spaces, $\mathbf{u} : \mathrm{F} \to \mathrm{G}$ is a continuous linear mapping, and $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}$. Choose $\mathbf{f}_1 \in \mathscr{L}^1_{\mathrm{F}}$ with $\mathbf{f} = \mathbf{f}_1$ locally almost everywhere. Obviously $\mathbf{u} \circ \mathbf{f} = \mathbf{u} \circ \mathbf{f}_1$ locally almost everywhere, therefore, by the cited Th. 1, $\mathbf{u} \circ \mathbf{f}_1 \in \mathscr{L}^1_{\mathrm{F}}$ and

$$\int (\mathbf{u} \circ \mathbf{f}_1)\, d\mu = \mathbf{u}\Big(\int \mathbf{f}_1\, d\mu\Big);$$

thus $\mathbf{u}\circ\mathbf{f}\in\overline{\mathscr{L}}^1_{\mathrm{F}}$ and

$$\int(\mathbf{u}\circ\mathbf{f})\,d\mu=\int(\mathbf{u}\circ\mathbf{f}_1)\,d\mu=\mathbf{u}\Big(\int\mathbf{f}_1\,d\mu\Big)=\mathbf{u}\Big(\int\mathbf{f}\,d\mu\Big).$$

V.9, ℓ. -9 to -7.

"To establish a), it suffices to show that for every compact subset L of T, $\int^* f\varphi_{\mathrm{L}}\,d\mu=\sup\limits_{\mathrm{K}}\int^* f\varphi_{\mathrm{K}}\,d\mu$, where K runs over the set of subsets of L belonging to $\mathfrak{K}$."

For, assuming this to be shown, it would follow that

$$\int^* f\varphi_{\mathrm{L}}\,d\mu\leqslant\sup_{\mathrm{K}\in\mathfrak{K}}\int^* f\varphi_{\mathrm{K}}\,d\mu\,,$$

whence, varying L, $\mu^\bullet(f)\leqslant\sup\limits_{\mathrm{K}\in\mathfrak{K}}\int^* f\varphi_{\mathrm{K}}\,d\mu$, whereas the reverse inequality follows trivially from the definition of $\mu^\bullet$.

V.10, ℓ. 2.

$$\text{“}\ \Big|\int\mathbf{f}\,d\mu-\int\mathbf{f}\varphi_{\mathrm{H}}\,d\mu\Big|\leqslant\int|\mathbf{f}|\varphi_{\complement\mathrm{H}}\,d\mu\text{”}$$

If $\mathbf{f}_1\in\mathscr{L}^1_{\mathrm{F}}$ with $\mathbf{f}=\mathbf{f}_1$ locally almost everywhere, then $\mathbf{f}_1\varphi_{\mathrm{H}}$, $\mathbf{f}_1\varphi_{\complement\mathrm{H}}$ belong to $\mathscr{L}^1_{\mathrm{F}}$ by Ch. IV, §5, No. 6, Th. 5, therefore

$$\int\mathbf{f}_1\,d\mu=\int\mathbf{f}_1\varphi_{\mathrm{H}}\,d\mu+\int\mathbf{f}_1\varphi_{\complement\mathrm{H}}\,d\mu$$

by the additivity implicit in Ch. IV, §4, No. 1, Def. 1.

V.10, ℓ. 5, 6.

"The Banach space $\overline{\mathscr{L}}^p_{\mathrm{F}}(\mathrm{T},\mu)$ may then be equipped with a natural complex Banach space structure"

The notation $\overline{\mathscr{L}}^p_{\mathrm{F}}(\mathrm{T},\mu)$ has only been defined for functions with values in a real Banach space, thus it is intended that, at the outset, F is regarded as a real Banach space by restriction to real scalars. However, if $\mathbf{f}\in\overline{\mathscr{L}}^p_{\mathrm{F}}$ and $c\in\mathbf{C}$, the function $c\mathbf{f}$ defined by $(c\mathbf{f})(t)=c\mathbf{f}(t)$ belongs to $\overline{\mathscr{L}}^p_{\mathrm{F}}$; for, $\mathbf{f}$ is measurable, hence so is $c\mathbf{f}$, and $\mu^\bullet(|c\mathbf{f}|^p)=|c|^p\mu^\bullet(|\mathbf{f}|^p)<+\infty$ (cf. Prop. 9). It is clear that $\overline{\mathscr{L}}^p_{\mathrm{F}}$ becomes, thereby, a complex vector space, on which $\mathbf{f}\mapsto\overline{\mathrm{N}}_p(\mathbf{f})=\big(\mu^\bullet(|\mathbf{f}|^p)\big)^{1/p}$ is a semi-norm that makes $\overline{\mathscr{L}}^p_{\mathrm{F}}$ a complete locally convex space over $\mathbf{C}$, and for which $\overline{\mathrm{N}}_p(\mathbf{f})=0$ if and only if $\mathbf{f}\in\mathscr{N}^\infty_{\mathrm{F}}$. The complex Banach space in question is the quotient space $\overline{\mathscr{L}}^p_{\mathrm{F}}/\mathscr{N}^\infty_{\mathrm{F}}$, equipped with the norm $\|\mathbf{f}+\mathscr{N}^\infty_{\mathrm{F}}\|_p=\overline{\mathrm{N}}_p(\mathbf{f})$.

To recapitulate: when F is a complex Banach space, $\overline{\mathscr{L}}_{F}^{p}$ is the same set of functions whether F is regarded as a real or a complex Banach space; all that changes is the field of scalars.

V.10, ℓ. 13, 14.

"Assertion b) of Prop. 10 then extends at once to complex measures."

Proof #1. Note that the inequality

$$(*) \qquad \left| \int \mathbf{f}\, d\theta \right| \leqslant \int |\mathbf{f}|\, d|\theta|\,,$$

valid for $\mathbf{f} \in \mathscr{L}_{F}^{1}(\theta) = \mathscr{L}_{F}^{1}(|\theta|)$ (Ch. IV, §4, No. 2, Prop. 2), generalizes at once for $\mathbf{f} \in \overline{\mathscr{L}}_{F}^{1}(\theta)$.

Let $\mathbf{f} \in \overline{\mathscr{L}}_{F}^{1}(\theta)$. Apply b) of Prop. 10 to $\mu = |\theta|$ and $|\mathbf{f}| \in \overline{\mathscr{L}}_{\mathbf{R}}^{1}(\mu)$ to obtain

$$\int |\mathbf{f}|\, d|\theta| = \lim_{\mathfrak{K}} \int |\mathbf{f}| \varphi_{K}\, d|\theta|\,,$$

that is,

$$\lim_{\mathfrak{K}} \int |\mathbf{f}| \varphi_{\complement K}\, d|\theta| = 0\,;$$

then, citing $(*)$, one has

$$\left| \int \mathbf{f} \varphi_{\complement K}\, d\theta \right| \leqslant \int |\mathbf{f}| \varphi_{\complement K}\, d|\theta|\,,$$

whence $\lim_{\mathfrak{K}} \int \mathbf{f} \varphi_{\complement K}\, d\theta = 0$, that is,

$$\int \mathbf{f}\, d\theta = \lim_{\mathfrak{K}} \int \mathbf{f} \varphi_{K}\, d\theta\,. \ \diamondsuit$$

Proof #2. The idea is to express θ as a linear combination of positive measures (Ch. III, §1, No. 5)

$$(**) \qquad \theta = \mu_1 - \mu_2 + i\mu_3 - i\mu_4\,, \quad 0 \leqslant \mu_k \leqslant |\theta| \quad (k = 1, \ldots, 4)$$

and apply b) of Prop. 10 to each of the μ_k. However, to put together the four limits, we need to know that the expression $\theta(\mathbf{f})$ is the corresponding linear combination of the $\mu_k(\mathbf{f})$; when $F = \mathbf{R}$ and $\mathbf{f} \in \mathscr{K}(T; \mathbf{R})$, this is immediate from the definition of the linear operations in $\mathscr{M}(T; \mathbf{R})$ (*loc. cit.*, No. 3), and the case of $\mathbf{f} \in \mathscr{K}(T; F)$, F a Banach space, is implicit in Ch. III, §3, No. 2, Prop. 1 and No. 3, Cor. 2 of Prop. 7, but for $\mathbf{f} \in \mathscr{L}_{F}^{1}(\theta) = \mathscr{L}_{F}^{1}(|\theta|)$, a different approach is required.

Recall that for $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$, $\int \mathbf{f}\,d\theta$ is defined as follows (Ch. IV, §4, No. 1, Def. 1): $\mathscr{K}_{\mathrm{F}}$ is dense in $\mathscr{L}^1_{\mathrm{F}}(\theta)$ for the semi-norm $\mathrm{N}_1 = |\theta|^*$; the linear mapping $\theta : \mathscr{K}_{\mathrm{F}} \to \mathrm{F}$ defined by $\theta(\mathbf{g}) = \int \mathbf{g}\,d\theta$ being continuous with respect to N_1, one extends θ by continuity to $\mathscr{L}^1_{\mathrm{F}}(\theta)$, and the value of the extension at $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$ is denoted $\theta(\mathbf{f})$ or $\int \mathbf{f}\,d\theta$.

Thus, given any $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$, we may choose a sequence $(\mathbf{g}_n)$ in $\mathscr{K}_{\mathrm{F}}$ such that $\mathrm{N}_1(\mathbf{f} - \mathbf{g}_n) \to 0$ (Ch. IV, §3, No. 4, Prop. 7), and we then have

$$\theta(\mathbf{f}) = \lim_n \theta(\mathbf{g}_n) \quad \text{in } \mathrm{F}.$$

If θ' is any measure on T such that $|\theta'| \leqslant |\theta|$, and if $\mathrm{N}'_1 = |\theta'|^*$, then (Ch. IV, §1, No. 3, Prop. 15)

$$\mathrm{N}'_1(\mathbf{f} - \mathbf{g}_n) \leqslant \mathrm{N}_1(\mathbf{f} - \mathbf{g}_n) \to 0,$$

consequently $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta')$ and

$$\theta'(\mathbf{f}) = \lim_n \theta'(\mathbf{g}_n) \quad \text{in } \mathrm{F}.$$

Suppose now that θ is decomposed as in $(**)$. The foregoing shows that, for $k = 1, \ldots, 4$, $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu_k)$ and

$$\mu_k(\mathbf{f}) = \lim_n \mu_k(\mathbf{g}_n) \quad \text{in } \mathrm{F},$$

therefore

$$\begin{aligned} \theta(\mathbf{f}) &= \lim_n \theta(\mathbf{g}_n) = \lim_n \big(\mu_1(\mathbf{g}_n) - \mu_2(\mathbf{g}_n) + i\mu_3(\mathbf{g}_n) - i\mu_4(\mathbf{g}_n)\big) \\ &= \lim_n \mu_1(\mathbf{g}_n) - \lim_n \mu_2(\mathbf{g}_n) + i \lim_n \mu_3(\mathbf{g}_n) - i \lim_n \mu_4(\mathbf{g}_n) \\ &= \mu_1(\mathbf{f}) - \mu_2(\mathbf{f}) + i\mu_3(\mathbf{f}) - i\mu_4(\mathbf{f}). \end{aligned}$$

Applying b) to each of the four terms on the right, one obtains the desired formula: for $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$,

$$\theta(\mathbf{f}) = \lim_{\mathfrak{K}} \mu_1(\mathbf{f}\varphi_{\mathrm{K}}) - \lim_{\mathfrak{K}} \mu_2(\mathbf{f}\varphi_{\mathrm{K}}) + i \lim_{\mathfrak{K}} \mu_3(\mathbf{f}\varphi_{\mathrm{K}}) - i \lim_{\mathfrak{K}} \mu_4(\mathbf{f}\varphi_{\mathrm{K}}) = \lim_{\mathfrak{K}} \theta(\mathbf{f}\varphi_{\mathrm{K}})$$

(the second equality, because $\theta(\mathbf{f}\varphi_{\mathrm{K}}) = \mu_1(\mathbf{f}\varphi_{\mathrm{K}}) - \mu_2(\mathbf{f}\varphi_{\mathrm{K}}) + i\mu_3(\mathbf{f}\varphi_{\mathrm{K}}) - i\mu_4(\mathbf{f}\varphi_{\mathrm{K}})$), that is,

$$\int \mathbf{f}\,d\theta = \lim_{\mathfrak{K}} \int \mathbf{f}\varphi_{\mathrm{K}}\,d\theta;$$

its extension to $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta)$ is then immediate. ◊

The definition of a 'universal' bilinear mapping $(\theta, \mathbf{f}) \mapsto \theta(\mathbf{f})$ remains elusive, as the eligible functions $\mathbf{f}$ depend on θ; but the foregoing arguments yield the following:

Lemma. If μ is any positive measure on T, and if $\mathscr{M}_\mu$ is the vector space of complex measures θ on T such that $|\theta| \leqslant c\mu$ for some scalar c (depending on θ), then $\theta(\mathbf{f})$ is defined for every $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu)$ and every $\theta \in \mathscr{M}_\mu$, and the mapping

$$(\theta, \mathbf{f}) \mapsto \theta(\mathbf{f}) \quad \big(\theta \in \mathscr{M}_\mu,\ \mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu)\big)$$

is bilinear.

Proof. It is clear that $\mathscr{M}_\mu$ is a vector space. If $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu)$ and $(\mathbf{g}_n)$ is a sequence in $\mathscr{K}_{\mathrm{F}}$ such that $\mu(\mathbf{f} - \mathbf{g}_n) \to 0$ then, for every $\theta \in \mathscr{M}_\mu$,

$$|\theta|^*(\mathbf{f} - \mathbf{g}_n) \to 0\,,$$

whence $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$ and $\theta(\mathbf{f}) = \lim_n \theta(\mathbf{g}_n)$ in F. Thus, if $\theta_1, \theta_2 \in \mathscr{M}_\mu$ then

$$\begin{aligned}(\theta_1 + \theta_2)(\mathbf{f}) &= \lim_n (\theta_1 + \theta_2)(\mathbf{g}_n) = \lim_n \big(\theta_1(\mathbf{g}_n) + \theta_2(\mathbf{g}_n)\big) \\ &= \lim_n \theta_1(\mathbf{g}_n) + \lim_n \theta_2(\mathbf{g}_n) = \theta_1(\mathbf{f}) + \theta_2(\mathbf{f})\,,\end{aligned}$$

and similarly $(a\theta)(\mathbf{f}) = a \cdot \theta(\mathbf{f})$ for all scalars a. ♢

It follows at once that if μ and μ' are positive measures on T and if $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu + \mu')$, then $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu) \cap \mathscr{L}^1_{\mathrm{F}}(\mu')$ and $(\mu + \mu')(\mathbf{f}) = \mu(\mathbf{f}) + \mu'(\mathbf{f})$. The converse is contained in the following:

Theorem. Let θ, θ' be any two complex measures on the locally compact space T, let F be a Banach space, and suppose $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta) \cap \mathscr{L}^1_{\mathrm{F}}(\theta')$. Then $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta + \theta')$ and $(\theta + \theta')(\mathbf{f}) = \theta(\mathbf{f}) + \theta'(\mathbf{f})$.

Proof. Since $\mathscr{L}^1_{\mathrm{F}}(\theta) = \mathscr{L}^1_{\mathrm{F}}(|\theta|)$ and $\mathscr{L}^1_{\mathrm{F}}(\theta') = \mathscr{L}^1_{\mathrm{F}}(|\theta'|)$ (Ch. IV, §3, No. 4, remark following Def. 2), one has $|\theta|^*(\mathbf{f}) < +\infty$ and $|\theta'|^*(\mathbf{f}) < +\infty$ (Ch. IV, §5, No. 6, Th. 5). Set $\mu = |\theta| + |\theta'|$. Then $\mu^* = |\theta|^* + |\theta'|^*$ (Ch. IV, §1, No. 3, Prop. 15), therefore

$$\mu^*(\mathbf{f}) = |\theta|^*(\mathbf{f}) + |\theta'|^*(\mathbf{f}) < +\infty\,;$$

by the cited Th. 5, to show that $\mathbf{f}$ is μ-integrable it will suffice to show that it is μ-measurable.

To this end, we employ the criterion of Ch. IV, §5, No. 1, Prop. 1. Let K be any compact subset of T, and let $\varepsilon > 0$. Since $\mathbf{f}$ is $|\theta|$-measurable, there exists a compact set $\mathrm{H} \subset \mathrm{K}$ such that

$$|\theta|^*(\mathrm{K} - \mathrm{H}) \leqslant \varepsilon/2 \quad \text{and } \mathbf{f}\big|\mathrm{H} \text{ is continuous}\,;$$

similarly, there exists a compact set $\mathrm{H}' \subset \mathrm{K}$ such that

$$|\theta'|^*(\mathrm{K} - \mathrm{H}') \leqslant \varepsilon/2 \quad \text{and } \mathbf{f}\big|\mathrm{H}' \text{ is continuous}.$$

Then $\mathrm{H} \cup \mathrm{H}'$ is a compact subset of K and

$$\mathrm{K} - (\mathrm{H} \cup \mathrm{H}') = (\mathrm{K} - \mathrm{H}) \cap (\mathrm{K} - \mathrm{H}'),$$

therefore

$$\begin{aligned}\mu^*\big(\mathrm{K} - (\mathrm{H} \cup \mathrm{H}')\big) &= |\theta|^*\big(\mathrm{K} - (\mathrm{H} \cup \mathrm{H}')\big) + |\theta'|^*\big(\mathrm{K} - (\mathrm{H} \cup \mathrm{H}')\big) \\ &\leqslant |\theta|^*(\mathrm{K} - \mathrm{H}) + |\theta'|^*(\mathrm{K} - \mathrm{H}') \leqslant \varepsilon/2 + \varepsilon/2;\end{aligned}$$

the coup de grace: $\mathbf{f}\big|(\mathrm{H} \cup \mathrm{H}')$ is continuous (GT, Ch. I, §3, No. 2, Prop. 4), thus $\mathbf{f}$ is μ-integrable.

Since $|\theta + \theta'| \leqslant |\theta| + |\theta'| = \mu$ (Ch. III, §1, No. 6) it follows from the Lemma that $\mathbf{f}$ is $(\theta + \theta')$-integrable and $(\theta + \theta')(\mathbf{f}) = \theta(\mathbf{f}) + \theta'(\mathbf{f})$. ♢

Remark. Recall that, for every measure ρ on T, the ρ-integrable sets form a clan (Ch. IV, §4, No. 9, *Example*) that contains every compact set (*loc. cit.*, No. 6, Cor. 1 of Prop. 10), hence contains the clan $\mathfrak{R}$ generated by the compact sets; thus every set in $\mathfrak{R}$ is 'universally integrable' in the sense that it is ρ-integrable for every measure ρ on T. Thus, in hindsight, one can omit the asterisks in the above computation (*loc. cit.*, No. 5, Def. 2). However, with the asterisks in place, the equality in the above display is justified by the known property $(|\theta| + |\theta'|)^* = |\theta|^* + |\theta'|^*$ of outer measures (Ch. IV, §1, No. 3, Prop. 15); to justify removing the asterisks would require verifying the special case of the Theorem being proved, for the measures $|\theta|, |\theta'|$ and the (universally integrable) numerical function $f = \varphi_{\mathrm{K} - (\mathrm{H} \cup \mathrm{H}')}$, which verification would bring back the asterisks.

The above *Theorem* extends, by induction, to finitely many complex measures; but the analogous proposition for $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta) \cap \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta')$, a special case of §2, No. 2, Prop. 3, is more complicated (see (i) of Cor. 3 below for a simpler proof).

Corollary 1. Let θ be any complex measure on T and suppose θ is decomposed as in $(**)$ (for example, in the canonical way). Then

$$\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta) \quad \Leftrightarrow \quad \mathbf{f} \in \bigcap_{k=1}^{4} \mathscr{L}^1_{\mathrm{F}}(\mu_{\mathrm{K}}),$$

in which case $\theta(\mathbf{f}) = \mu_1(\mathbf{f}) - \mu_2(\mathbf{f}) + i\mu_3(\mathbf{f}) - i\mu_4(\mathbf{f})$.

Proof. $\Rightarrow$: The argument for this is given in "Proof #2" above.

$\Leftarrow$: This follows at once from the above Theorem. ♢

Corollary 2. If θ is any *real* measure on T, then $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$ if and only if $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta^+) \cap \mathscr{L}^1_{\mathrm{F}}(\theta^-)$, in which case

$$|\theta|(\mathbf{f}) = \theta^+(\mathbf{f}) + \theta^-(\mathbf{f}), \quad \theta(\mathbf{f}) = \theta^+(\mathbf{f}) - \theta^-(\mathbf{f}).$$

Proof. By Cor. 1, $\mathscr{L}^1_{\mathrm{F}}(\theta) = \mathscr{L}^1_{\mathrm{F}}(\theta^+) \cap \mathscr{L}^1_{\mathrm{F}}(\theta^-)$ and the second formula holds for $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$; and, since $\mathscr{L}^1_{\mathrm{F}}(\theta) = \mathscr{L}^1_{\mathrm{F}}(|\theta|)$ and $|\theta| = \theta^+ + \theta^-$, the first formula is immediate from the Theorem. ♢

Corollary 3. Let $\theta_1, \ldots, \theta_n$ be complex measures on T.

(i) If $\mu = |\theta_1| + \cdots + |\theta_n|$ then

$$\bigcap_{k=1}^{n} \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta_k) = \overline{\mathscr{L}}^1_{\mathrm{F}}(\mu).$$

(ii) If $\theta = \theta_1 + \cdots + \theta_n$ then $\bigcap_{k=1}^{n} \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta_k) \subset \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta)$ and, for every $\mathbf{f} \in \bigcap_{k=1}^{n} \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta_k)$, one has $\int \mathbf{f}\, d\theta = \int \mathbf{f} d\theta_1 + \cdots + \int \mathbf{f} d\theta_n$.

(iii) $\bigcap_{k=1}^{n} \mathscr{L}^1_{\mathrm{F}}(\theta_k) = \mathscr{L}^1_{\mathrm{F}}(\mu) \subset \mathscr{L}^1_{\mathrm{F}}(\theta)$ and, for $\mathbf{f} \in \bigcap_{k=1}^{n} \mathscr{L}^1_{\mathrm{F}}(\theta_k)$, one has $\theta(\mathbf{f}) = \theta_1(\mathbf{f}) + \cdots + \theta_n(\mathbf{f})$.

Proof. {Recall that for any complex measure θ, $\overline{\mathscr{L}}^1_{\mathrm{F}}(\theta) = \overline{\mathscr{L}}^1_{\mathrm{F}}(|\theta|)$ by definition.}

(i) For a function $\mathbf{f} : \mathrm{T} \to \mathrm{F}$, $\mu^\bullet(|\mathbf{f}|) = |\theta_1|^\bullet(|\mathbf{f}|) + \cdots + |\theta_n|^\bullet(|\mathbf{f}|)$ by No. 1, Prop. 3, therefore $\mu^\bullet(|\mathbf{f}|) < +\infty$ if and only if $|\theta_k|^\bullet(|\mathbf{f}|) < +\infty$ for every k; in view of No. 3, Prop. 9, to show that $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\mu)$ if and only if $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta_k)$ for all k, it will suffice to show that

$$\mathbf{f} \text{ is } \mu\text{-measurable} \quad \Leftrightarrow \quad \mathbf{f} \text{ is } \theta_k\text{-measurable for } k = 1, \ldots, n.$$

To this end, recall that $\mu^* = |\theta_1|^* + \cdots + |\theta_n|^*$ (Ch. IV, §1, No. 3, Prop. 15).

$\Rightarrow$: If $\mathbf{f}$ is μ-measurable, it is clear from $|\theta_k|^* \leqslant \mu^*$ and the definition of measurability that $\mathbf{f}$ is θ_k-measurable.

$\Leftarrow$: Assuming $\mathbf{f}$ is θ_k-measurable for all k, we are to show that $\mathbf{f}$ is μ-measurable. As in the proof of the Theorem, we employ the criterion of Ch. IV, §5, No. 1, Prop. 1: if $\mathrm{K} \subset \mathrm{T}$ is compact and $\varepsilon > 0$, there exist compact sets $\mathrm{H}_k \subset \mathrm{K}$ $(k = 1, \ldots, n)$ such that $\mathbf{f}|\mathrm{H}_k$ is continuous and

$|\theta_k|^*(\mathrm{K} - \mathrm{H}_k) \leqslant \varepsilon/n$; then $\mathbf{f}\big|(\mathrm{H}_1 \cup \cdots \cup \mathrm{H}_n)$ is continuous and

$$\mu^*\big(\mathrm{K} - \bigcup_{j=1}^{n} \mathrm{H}_j\big) = \sum_{k=1}^{n} |\theta_k|^*\big(\mathrm{K} - \bigcup_{j=1}^{n} \mathrm{H}_j\big)$$
$$= \sum_{k=1}^{n} |\theta_k|^*\big(\bigcap_{j=1}^{n}(\mathrm{K} - \mathrm{H}_j)\big)$$
$$\leqslant \sum_{k=1}^{n} |\theta_k|^*(\mathrm{K} - \mathrm{H}_k) \leqslant \varepsilon\,,$$

thus $\mathbf{f}$ is μ-measurable by the cited Prop. 1.

(ii) Let $\mathbf{f} \in \bigcap_{k=1}^{n} \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta_k) = \overline{\mathscr{L}}^1_{\mathrm{F}}(\mu)$. Since $|\theta| \leqslant |\theta_1| + \cdots + |\theta_n| = \mu$, it follows that $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta)$; for, $\mathbf{f}$ is μ-measurable and $|\theta|^\bullet(|\mathbf{f}|) \leqslant \mu^\bullet(|\mathbf{f}|) < +\infty$ (Prop. 9), and from $|\theta|^* \leqslant \mu^*$ it follows from the definition of measurability that $\mathbf{f}$ is θ-measurable, consequently $\mathbf{f}$ is essentially θ-integrable (Prop. 9 again). Say $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\mu)$ with $\mathbf{f} = \mathbf{g}$ locally μ-almost everywhere, hence locally θ-almost everywhere. Since $|\theta_k|^\bullet \leqslant \mu^\bullet$, $\mathbf{f} = \mathbf{g}$ locally θ_k-almost everywhere for every k. Moreover, from $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\mu)$ and $|\theta_k| \leqslant \mu$ we know that $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\theta_k)$, thus $\mathbf{g} \in \bigcap_{k=1}^{n} \mathscr{L}^1_{\mathrm{F}}(\theta_k)$ and $\theta_k(\mathbf{f}) = \theta_k(\mathbf{g})$; by recursion on the *Theorem*, $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$ and

$$\theta(\mathbf{g}) = \theta_1(\mathbf{g}) + \cdots + \theta_n(\mathbf{g})\,,$$

whence $\theta(\mathbf{f}) = \theta_1(\mathbf{f}) + \cdots + \theta_n(\mathbf{f})$.

(iii) If $\mathbf{f} \in \bigcap_{k=1}^{n} \mathscr{L}^1_{\mathrm{F}}(\theta_k)$ then $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$ and $\theta(\mathbf{f}) = \sum_{k=1}^{n} \theta_k(\mathbf{f})$ follow recursively from the above *Proposition*. The inclusion $\mathscr{L}^1_{\mathrm{F}}(\theta) \subset \mathscr{L}^1_{\mathrm{F}}(\mu)$ follows from $|\theta| \leqslant \mu$ and the *Lemma* to the *Proposition*.

One has $\mathscr{L}^1_{\mathrm{F}}(\theta) = \mathscr{L}^1_{\mathrm{F}}(|\theta|)$ and similarly for the θ_k (Ch. IV, §3, No. 4, line following Def. 2). Let $\mathbf{f} : \mathrm{T} \to \mathrm{F}$ be any function. We know from $\mu^* = |\theta_1|^* + \cdots + |\theta_n|^*$ that $\mu^*(|\mathbf{f}|)$ is finite if and only if $|\theta_k|^*(|\mathbf{f}|)$ is finite for every k ; and, by the proof of (i), $\mathbf{f}$ is μ-measurable if and only if $\mathbf{f}$ is θ_k-measurable for every k ; thus $\mathbf{f}$ is μ-integrable if and only if it is θ_k-integrable for every k (Ch. IV, §5, No. 6, Th. 5), that is, $\bigcap_{k=1}^{n} \mathscr{L}^1_{\mathrm{F}}(\theta_k) = \mathscr{L}^1_{\mathrm{F}}(\mu)$.

Incidentally, since $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\theta)$ if and only if $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta)$ and $\mathbf{f}$ is θ-moderated (Cor. of Prop. 9), one can derive (iii) from (i) and (ii) based on the observation that $\mathbf{f}$ is μ-moderated if and only if it is θ_k-moderated for $k = 1, \ldots, n$; equivalently, writing $\mathrm{A} = \{t : \mathbf{f}(t) \neq 0\}$,

(†) A is μ-moderated $\Leftrightarrow$ A is θ_k-moderated for $k = 1, \ldots, n$.

⇒: Immediate from $|\theta_k|^* \leqslant \mu^*$ and criterion c) of No. 2, Prop. 5.

⇐: By the cited criterion c), for each k there exist a θ_k-negligible set N_k and a sequence (K_{ki}) of compact subsets of T such that

$$\mathrm{A} \subset \mathrm{N}_k \cup \bigcup_{i=1}^{\infty} \mathrm{K}_{ki}\,,$$

that is, $\mathrm{A} - \bigcup_{i=1}^{\infty} \mathrm{K}_{ki} \subset \mathrm{N}_k$; then the set $\mathrm{N} = \bigcap_{k=1}^{n} \mathrm{N}_k$ is θ_k-negligible for every k, hence is μ-negligible, and

$$\mathrm{A} - \bigcup_{k=1}^{n}\Big(\bigcup_{i=1}^{\infty}\mathrm{K}_{ki}\Big) = \bigcap_{k=1}^{n}\Big(\mathrm{A} - \bigcup_{i=1}^{\infty}\mathrm{K}_{ki}\Big) \subset \bigcap_{k=1}^{n}\mathrm{N}_k = \mathrm{N}\,,$$

thus $\mathrm{A} \subset \mathrm{N} \cup \bigcup_{k=1}^{n}\Big(\bigcup_{i=1}^{\infty}\mathrm{K}_{ki}\Big)$ and so A is μ-moderated.

Application of (†) to (iii): the equivalences

$$\begin{aligned}\mathbf{f} \in \bigcap_{k=1}^{n} \mathscr{L}^1_{\mathrm{F}}(\theta_k) \quad &\Leftrightarrow \quad \mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\theta_k) \text{ and } \mathbf{f} \text{ is } \theta_k\text{-moderated } (k = 1, \dots, n)\\ &\Leftrightarrow \quad \mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\mu) \text{ and } \mathbf{f} \text{ is } \mu\text{-moderated}\\ &\Leftrightarrow \quad \mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\mu)\end{aligned}$$

hold, respectively, by the Cor. of Prop. 9; by (i) and (†); and the Cor. of Prop. 9. ♢

V.10, ℓ. −7, −6.

" $\nu^*(f) \leqslant \nu(g) \leqslant \varepsilon$, or $\lambda^*_\alpha(f) \geqslant \lambda^*(f) - \varepsilon$ (Ch. IV, §1, No. 3, Prop. 15)."

Since $0 \leqslant f \leqslant g$, one has $\nu^*(f) \leqslant \nu^*(g) = \nu(g) = \lambda(g) - \lambda_\alpha(g) \leqslant \varepsilon$ by the choice of g. By the cited Prop. 15, $\lambda^* = (\lambda_\alpha + \nu)^* = \lambda^*_\alpha + \nu^*$, therefore $\lambda^*(f) - \lambda^*_\alpha(f) = \nu^*(f) \leqslant \varepsilon$, whence the second asserted inequality.

V.10, ℓ. −1.

" $\lambda^\bullet(f) = \sup_{n\in\mathbf{N}} \lambda^\bullet(f_n) = \sup_{n\in\mathbf{N}} \sup_{\alpha\in\mathrm{A}} \lambda^\bullet_\alpha(f_n) = \sup_{\alpha\in\mathrm{A}} \sup_{n\in\mathbf{N}} \lambda^\bullet_\alpha(f_n) = \sup_{\alpha\in\mathrm{A}} \lambda^\bullet_\alpha(f)$."

1st equality: No. 1, Prop. 1.
2nd equality: By the special case just proved.
3rd equality: Associativity of sups.
4th equality: No. 1, Prop. 1.

V.11, ℓ. 3, 4.

$$\lambda^\bullet(f) = \sup_{K\in\mathfrak{K}} \lambda^\bullet(f\varphi_K) = \sup_{K\in\mathfrak{K}} \sup_{\alpha\in A} \lambda^\bullet_\alpha(f\varphi_K)$$
$$= \sup_{\alpha\in A} \sup_{K\in\mathfrak{K}} \lambda^\bullet_\alpha(f\varphi_K) = \sup_{\alpha\in A} \lambda^\bullet_\alpha(f)\,.$$

1st equality: Definition of $\lambda^\bullet$, where $\lambda^\bullet(f\varphi_K) = \lambda^*(f\varphi_K)$ by No. 2, Prop. 7.

2nd equality: By the special case just proved.

3rd equality: Associativity of sups.

4th equality: Definition of $\lambda^\bullet_\alpha$.

V.11, ℓ. 5, 6.

"Corollary 1."

By Prop. 11 applied to $f = \varphi_N$, $\lambda^\bullet(N) = \sup_{\alpha\in A} \lambda^\bullet_\alpha(N)$. Thus (p. V.2, ℓ. 11, 12)

$$\begin{aligned} N \text{ locally } \lambda\text{-negligible} \quad &\Leftrightarrow \quad \lambda^\bullet(N) = 0 \\ &\Leftrightarrow \quad \lambda^\bullet_\alpha(N) = 0 \text{ for all } \alpha \\ &\Leftrightarrow \quad N \text{ locally } \lambda_\alpha\text{-negligible for all } \alpha\,. \end{aligned}$$

V.11, ℓ. 10, 11.

"The condition is obviously necessary, since $\lambda_\alpha \leqslant \lambda$ for every α (Ch. IV, §1, No. 3, Prop. 15)."

By the cited Prop. 15, $\lambda^*_\alpha \leqslant \lambda^*$, therefore

$$g \ \lambda\text{-negligible} \quad \Rightarrow \quad \lambda_\alpha\text{-negligible}\,,$$

whence g λ-measurable $\Rightarrow$ g λ_α-measurable (Ch. IV, §5, No. 1, comment following Def. 1).

V.11, ℓ. 13.

" . . . let L be a compact set such that . . . "

The assumption that L be compact is unnecessary and confusing.

Assume g is λ_α-measurable for all α, and let $\mathfrak{K}$ be the set of all compact subsets K of T such that $g|K$ is continuous. To show that g is λ-measurable, it suffices to show that $\mathfrak{K}$ is λ-dense (Ch. IV, §5, No. 10, Prop. 15, criterion *a*), with $A = X = T$); to this end, it will suffice to show that $\mathfrak{K}$ satisfies criterion *a*) of Ch. IV, §5, No. 8, Prop. 12. Thus, given an arbitrary subset L of T, we are to show that

$$L \text{ locally } \lambda\text{-negligible} \quad \Leftrightarrow \quad L\cap K \ \lambda\text{-negligible for every } K\in\mathfrak{K}\,.$$

⇒: Immediate from Ch. IV, §5, No. 2, Prop. 5.

⇐: By Cor. 1 it suffices to show that, given any α, L is locally λ_α-negligible. Since g is λ_α-measurable, $\mathfrak{K}$ is λ_α-dense by "$d) \Rightarrow a)$" of the cited Prop. 15. For every $\mathrm{K} \in \mathfrak{K}$, by assumption $\mathrm{L} \cap \mathrm{K}$ is λ-negligible, hence λ_α-negligible (because $\lambda_\alpha \leqslant \lambda$); since $\mathfrak{K}$ is λ_α-dense, it follows from criterion $a)$ of the cited Prop. 12 that L is locally λ_α-negligible.

§2. SUMMABLE FAMILIES OF POSITIVE MEASURES

V.12, ℓ. $-9, -8$.

"This follows at once from Remark 1, Prop. 11 of §1, No. 3 and Prop. 3 of §1, No. 1."

For every finite set $\mathrm{J} \subset \mathrm{A}$, write $\nu_{\mathrm{J}} = \sum_{\alpha \in \mathrm{J}} \lambda_\alpha$. By the cited Remark 1, $\nu = \sup_{\mathrm{J}} \nu_{\mathrm{J}}$ in $\mathscr{M}_+(\mathrm{T})$, in the sense that $\nu_{\mathrm{J}} \leqslant \nu$ for every J, and $\nu \leqslant \nu'$ for every measure ν' such that $\nu_{\mathrm{J}} \leqslant \nu'$ for all J; equivalently, $\nu(g) = \sup_{\mathrm{J}} \nu_{\mathrm{J}}(g)$ in $\mathbf{R}^+$ for every $g \in \mathscr{K}_+(\mathrm{T})$. In other words, $\nu = \sup_{\mathrm{J}} \nu_{\mathrm{J}}$ in the fully lattice-ordered Riesz space $\mathscr{M}(\mathrm{T}; \mathbf{R})$ (Ch. III, §1, No. 5, Th. 3).

Then, for every $f \in \mathscr{F}_+(\mathrm{T})$, $\nu^\bullet(f) = \sup_{\mathrm{J}} \nu^\bullet_{\mathrm{J}}(f)$ by the cited Prop. 11, where $\nu^\bullet_{\mathrm{J}}(f) = \sum_{\alpha \in \mathrm{J}} \lambda^\bullet_\alpha(f)$ by the cited Prop. 3, thus $\nu^\bullet(f) = \sum_\alpha \lambda^\bullet_\alpha(f)$.

V.12, ℓ. -7 to -5.

"Corollary 1"

For every subset M of T,

$$\nu^\bullet(\mathrm{M}) = \sum_{\alpha \in \mathrm{A}} \lambda^\bullet_\alpha(\mathrm{M})$$

by Prop. 1 (with $f = \varphi_{\mathrm{M}}$). If M is ν-integrable then, since $\lambda^*_\alpha \leqslant \nu^*$ (Ch. IV, §1, No. 3, Prop. 15), M is λ_α-integrable (Ch. IV, §4, No. 6, Cor. 1 of Th. 4) for every α; since M is ν-moderated (first sentence of §1, No. 3), $\nu^\bullet(\mathrm{M}) = \nu^*(\mathrm{M}) = \nu(\mathrm{M})$ (§1, No. 2, Prop. 7), and similarly for λ_α. Thus

$$\nu(\mathrm{M}) = \sum_{\alpha \in \mathrm{A}} \lambda_\alpha(\mathrm{M})$$

for every ν-integrable set M, and in particular for every compact set, or relatively compact open set (Ch. IV, §4, No. 6, Cor. 1 of Prop. 10).

V.12, ℓ. $-4, -3$.

"COROLLARY 2"

One knows that N is locally ν-negligible if and only if $\nu^\bullet(\mathrm{N}) = 0$ (p. V.2, ℓ. 11, 12), and $\nu^\bullet(\mathrm{N}) = \sum_{\alpha\in\mathrm{A}} \lambda_\alpha^\bullet(\mathrm{N})$ (put $f = \varphi_{\mathrm{N}}$ in Prop. 1), whence the assertion.

V.12, ℓ. $-2, -1$.

"COROLLARY 3"

If f is not ν-moderated then $\nu^*(f) = +\infty$ (No. 2, Prop. 7), so (4) holds trivially.

If f is ν-moderated then f is λ_α-moderated for every α. For, writing $\mathrm{A} = \{t : \ f(t) \neq 0\}$, we know that

$$\mathrm{A} \subset \mathrm{N} \cup \bigcup_n \mathrm{K}_n\,,$$

where (K_n) is a sequence of compact sets and $\nu^*(\mathrm{N}) = 0$ (No. 2, Def. 2); then, for every α, ${\lambda_\alpha}^* \leqslant \nu^*$, therefore $\lambda_\alpha^*(\mathrm{N}) = 0$ and so f is λ_α-moderated. By the cited Prop. 7, $\nu^\bullet(f) = \nu^*(f)$ and $\lambda_\alpha^\bullet(f) = \lambda_\alpha^*(f)$ for all α, so it follows from (3) that (4) holds with equality.

V.13, ℓ. 11.

"This follows at once from Cor. 2 of Prop. 11 of §1."

It does, provided one has verified the proposition for finite sums of the λ_α. The crux of the matter: If μ, μ' are positive measures on T, and $f : \mathrm{T} \to \mathrm{G}$, then

f is $(\mu + \mu')$-measurable $\;\Leftrightarrow\;$ f is measurable for μ and for μ'.

From $(\mu + \mu')^* = \mu^* + \mu'^*$ one sees that a subset N of T is negligible for $\mu + \mu'$ if and only if it is negligible for each of μ, μ'.

Proof of $\Rightarrow$: Immediate from the preceding remark and the definition of measurability (Ch. IV, §5, No. 1, Def. 1).

Proof of $\Leftarrow$: This is shown in the proof of the *Theorem* in the Note for p. V.10, ℓ. 13, 14 (see also Cor. 3 of the cited *Theorem*).

V.13, ℓ. $-10, -9$.

"The first part of the statement therefore follows at once from Props. 2 and 1."

At any rate,

$$(*) \qquad \sum_{\alpha\in\mathrm{A}} \lambda_\alpha^\bullet(|\mathbf{f}|) = \nu^\bullet(|\mathbf{f}|)$$

by Prop. 1.

Suppose $\mathbf{f}$ is essentially ν-integrable. Thus $\mathbf{f}$ is ν-measurable and $\nu^\bullet(|\mathbf{f}|) < +\infty$ (§1, No. 3, Prop. 9); then, for every $\alpha \in \mathrm{A}$, $\mathbf{f}$ is λ_α-measurable (Prop. 2) and $\lambda_\alpha^\bullet(|\mathbf{f}|) < +\infty$ by $(*)$, therefore $\mathbf{f}$ is essentially λ_α-integrable (§1, Prop. 9).

Conversely, suppose $\mathbf{f}$ is essentially λ_α-integrable for all α, and that $\sum_{\alpha\in\mathrm{A}} \lambda_\alpha^\bullet(|\mathbf{f}|) < +\infty$. Then, for every α, $\mathbf{f}$ is λ_α-measurable (§1, Prop. 9), therefore $\mathbf{f}$ is ν-measurable (Prop. 2) and $\nu^\bullet(|\mathbf{f}|) < +\infty$ by $(*)$, therefore $\mathbf{f}$ is essentially ν-integrable (§1, Prop. 9). We may then write $(*)$ as $\nu(|\mathbf{f}|) = \sum_{\alpha\in\mathrm{A}} \lambda_\alpha(|\mathbf{f}|)$, or

$$\int |\mathbf{f}|\, d\nu = \sum_{\alpha\in\mathrm{A}} \int |\mathbf{f}|\, d\lambda_\alpha$$

(§1, No. 3, Def. 3).

V.13, ℓ. −6 to −4.

"The set of $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\nu)$ that satisfy (6) is thus a closed linear subspace $\mathscr{H}$ of $\mathscr{L}^1_{\mathrm{F}}(\nu)$"

Note the absence of overbars; we are dealing with *integrable* functions. For $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\nu)$ write

$$\Phi(\mathbf{g}) = \int \mathbf{g}\, d\nu\,, \quad \Psi(\mathbf{g}) = \sum_{\alpha\in\mathrm{A}} \int \mathbf{g}\, d\lambda_\alpha\,,$$

both of which are defined and are elements of F; Φ and Ψ define linear mappings $\mathscr{L}^1_{\mathrm{F}}(\nu) \to \mathrm{F}$.

Φ is continuous for convergence in mean: for, by the definition of integral, it is the extension to $\mathscr{L}^1_{\mathrm{F}}(\nu)$ of the mapping $\mathbf{g} \mapsto \nu(\mathbf{g})$ $(\mathbf{g} \in \mathscr{K}_{\mathrm{F}}(\mathrm{T}))$ by continuity with respect to the semi-norm $\mathrm{N}_1 = \nu^*$ (Ch. IV, §4, No. 1, Def. 1).

Ψ is likewise continuous for convergence in mean, by virtue of the computation

$$\begin{aligned}|\Psi(\mathbf{g})| \leqslant \sum_{\alpha\in\mathrm{A}} \Big|\int \mathbf{g}\, d\lambda_\alpha\Big| \leqslant \sum_{\alpha\in\mathrm{A}} \int |\mathbf{g}|\, d\lambda_\alpha &= \sum_{\alpha\in\mathrm{A}} \lambda_\alpha(|\mathbf{g}|)\\ &= \sum_{\alpha\in\mathrm{A}} \lambda_\alpha^\bullet(|\mathbf{g}|) = \nu^\bullet(|\mathbf{g}|) = \nu(|\mathbf{g}|)\,.\end{aligned}$$

Since F is Hausdorff, it is elementary that the linear subspace $\mathscr{H} = \{\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\nu) : \Phi(\mathbf{g}) = \Psi(\mathbf{g})\}$ is closed in $\mathscr{L}^1_{\mathrm{F}}(\nu)$ (GT, I, §8, No. 1, Prop. 2).

V.13, ℓ. -4 to -2.

"...it contains the functions of the form $f \cdot \mathbf{a}$, where $\mathbf{a} \in \mathrm{F}$ and f denotes a finite integrable positive function (Prop. 1)."

Assuming $f \in \mathscr{L}^1_{\mathbf{R}}(\nu)$, $f \geqslant 0$, the assertion is that the function $\mathbf{f} = f \cdot \mathbf{a}$ satisfies (6).

One has $f \cdot \mathbf{a} \in \mathscr{L}^1_{\mathrm{F}}(\nu)$ and $\int (f \cdot \mathbf{a})\, d\nu = \left(\int f\, d\nu\right) \cdot \mathbf{a}$ (Ch. IV, §4, No. 2, Cor. 2 of Th. 1); from $\lambda_\alpha \leqslant \nu$ it follows that $f \in \mathscr{L}^1_{\mathbf{R}}(\lambda_\alpha)$, consequently $\int (f \cdot \mathbf{a})\, d\lambda_\alpha = \left(\int f\, d\lambda_\alpha\right) \cdot \mathbf{a}$. By Prop. 1,

$$\nu^\bullet(f) = \sum_{\alpha \in \mathrm{A}} \lambda^\bullet_\alpha(f)\,,$$

and since f is integrable (hence moderated) for ν and the λ_α, this may be written

$$\int f\, d\nu = \sum_{\alpha \in \mathrm{A}} \int f\, d\lambda_\alpha\,.$$

Then

$$\begin{aligned}\int (f \cdot \mathbf{a})\, d\nu &= \left(\int f\, d\nu\right) \cdot \mathbf{a} = \left(\sum_{\alpha \in \mathrm{A}} \int f\, d\lambda_\alpha\right) \cdot \mathbf{a} \\ &= \sum_{\alpha \in \mathrm{A}} \left(\int f\, d\lambda_\alpha\right) \cdot \mathbf{a} = \sum_{\alpha \in \mathrm{A}} \int (f \cdot \mathbf{a})\, d\lambda_\alpha\end{aligned}$$

(the third equality, by the continuity of scalar multiplication in F), thus $f \cdot \mathbf{a} \in \mathscr{H}$.

If $f \in \mathscr{K}_{\mathbf{R}}(\mathrm{T})$ then $f \cdot \mathbf{a} = f^+ \cdot \mathbf{a} - f^- \cdot \mathbf{a} \in \mathscr{H}$ by the foregoing; it follows that $\mathscr{H}$ is dense in $\mathscr{L}^1_{\mathrm{F}}(\nu)$ for the topology of convergence in mean (Ch. IV, §3, No. 5, Prop. 10), whence $\mathscr{L}^1_{\mathrm{F}}(\nu) = \overline{\mathscr{H}} = \mathscr{H}$ (the overbar signifying closure for that topology).

V.13, ℓ. $-2, -1$.

"Therefore $\mathscr{H} = \mathscr{L}^1_{\mathrm{F}}(\nu)$ and the proposition is established."

For the equality, see the preceding note. Given $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\nu)$, choose $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\nu)$ such that $\mathbf{f} = \mathbf{g}$ locally ν-almost everywhere; thus $\int \mathbf{f}\, d\nu = \int \mathbf{g}\, d\nu$. For each $\alpha \in \mathrm{A}$, one has $\lambda_\alpha \leqslant \nu$, therefore $\mathbf{g} \in \mathscr{L}^1_{\mathrm{F}}(\lambda_\alpha)$ (because $\lambda^*_\alpha \leqslant \nu^*$) and $\mathbf{f} = \mathbf{g}$ locally λ_α-almost everywhere (because $\lambda^\bullet_\alpha \leqslant \nu^\bullet$); therefore $\mathbf{f} \in \overline{\mathscr{L}}^1_{\mathrm{F}}(\lambda_\alpha)$ and $\int \mathbf{f}\, d\lambda_\alpha = \int \mathbf{g}\, d\lambda_\alpha$. Since $\mathscr{L}^1_{\mathrm{F}}(\nu) = \mathscr{H}$ we know that $\int \mathbf{g}\, d\nu = \sum_{\alpha \in \mathrm{A}} \int \mathbf{g}\, d\lambda_\alpha$, which may be written $\int \mathbf{f}\, d\nu = \sum_{\alpha \in \mathrm{A}} \mathbf{f}\, d\lambda_\alpha$, that is, $\mathbf{f}$ satisfies (6).

V.14, ℓ. 8–11.

"Conversely, if A is finite and if $\mathbf{f}$ is λ_α-integrable for all $\alpha \in \mathrm{A}$, then $\mathbf{f}$ is essentially ν-integrable by Prop. 3, and it suffices to verify that $\nu^*(|\mathbf{f}|) < +\infty$; this follows at once from the relation $\nu^* = \sum_{\alpha\in\mathrm{A}} \lambda_\alpha^*$ (Ch. IV, §1, No. 3, Prop. 15)."

The proof cites Prop. 3, which cites Prop. 2, which cites Cor. 2 of Prop. 11 of §1, whose proof employs the concept of a λ-dense set $\mathfrak{K}$ of compact sets.

For A finite, there is a simpler proof: in the Note for p. V.10, ℓ. 13, 14, see Cor. 3, (iii) of the Theorem proved there.

But for arbitrary A, the additivity property $\nu^\bullet = \sum_{\alpha\in\mathrm{A}} \lambda_\alpha^\bullet$ and the concept of λ-dense sets $\mathfrak{K}$ prove their worth.

V.14, ℓ. 12–17.

"Corollary 2"

See also the note for p. V.10, ℓ. 13, 14 (Cor. 1 of the Theorem proved there).

V.15, ℓ. 3, 4.

"... the linear form μ_α on $\mathscr{K}(\mathrm{T})$ is positive, therefore is a positive measure, with support contained in K_α."

That μ_α is a measure is shown in Ch. III, §1, No. 5, Th. 1.

To show that $\mathrm{Supp}(\mu_\alpha) \subset \mathrm{K}_\alpha$, it suffices to show that $\mu_\alpha | \complement \mathrm{K}_\alpha = 0$ (Ch. III, §2, No. 2, Def. 1 and the sentence before it). Indeed, if $f \in \mathscr{K}(\mathrm{T})$ and $\mathrm{Supp}(f) \subset \complement \mathrm{K}_\alpha$ then $f\varphi_{\mathrm{K}_\alpha} = 0$ and so $\mu_\alpha(f) = \mu(f\varphi_{\mathrm{K}_\alpha}) = 0$; thus $\mu_\alpha | \complement \mathrm{K}_\alpha = 0$ (*loc. cit.*, No. 1; for every $g \in \mathscr{K}(\complement \mathrm{K}_\alpha)$, the extension by 0 of g to T is such a function f).

{The earlier-cited Prop. 14 (Ch. IV, §5, No. 9) also shows that one can suppose that $\mathrm{Supp}(\mu_{\mathrm{K}_\alpha}) = \mathrm{K}_\alpha$; an alternate proof of Prop. 4 based on the measures μ_{K_α} is given below in the Note for V.15, ℓ. 11, 12.}

V.15, ℓ. 8, 9.

"... let A′ be the countable set formed by the $\alpha \in \mathrm{A}$ such that $\mathrm{S} \cap \mathrm{K}_\alpha \neq \varnothing$."

Remark following Ch. IV, §5, No. 9, Def. 7.

V.15, ℓ. 9, 10.

"... the set $\mathrm{N} \cap \mathrm{S}$ is μ-negligible"

Ch. IV, §5, No. 2, Prop. 5.

V.15, ℓ. 11, 12.

$$\text{“ } \mu(f) = \mu(f\varphi_{\mathrm{S}}) = \sum_{\alpha\in\mathrm{A}'} \mu(f\varphi_{\mathrm{S}\cap\mathrm{K}_\alpha}) = \sum_{\alpha\in\mathrm{A}'} \mu(f\varphi_{\mathrm{K}_\alpha})$$
$$= \sum_{\alpha\in\mathrm{A}} \mu(f\varphi_{\mathrm{K}_\alpha}) = \sum_{\alpha\in\mathrm{A}} \mu_\alpha(f)\,. \text{ ”}$$

1st equality: $f = f\varphi_{\mathrm{S}}$ by the definition of S.

2nd equality: One has

$$\mathrm{S} = \mathrm{S}\cap\mathrm{T} = \mathrm{S}\cap\Big(\mathrm{N}\cup\bigcup_{\alpha\in\mathrm{A}}\mathrm{K}_\alpha\Big)$$
$$= (\mathrm{S}\cap\mathrm{N})\cup\bigcup_{\alpha\in\mathrm{A}}(\mathrm{S}\cap\mathrm{K}_\alpha)$$
$$= (\mathrm{S}\cap\mathrm{N})\cup\bigcup_{\alpha\in\mathrm{A}'}(\mathrm{S}\cap\mathrm{K}_\alpha)$$

(because $\mathrm{S}\cap\mathrm{K}_\alpha = \varnothing$ when $\alpha\notin\mathrm{A}'$). It follows from the disjointness of the terms of the union that

$$f = f\varphi_{\mathrm{S}} = f\varphi_{\mathrm{S}\cap\mathrm{N}} + \sum_{\alpha\in\mathrm{A}'} f\varphi_{\mathrm{S}\cap\mathrm{K}_\alpha}\,;$$

since $\mathrm{S}\cap\mathrm{N}$ is negligible, so is $f\varphi_{\mathrm{S}\cap\mathrm{N}}$, thus all terms are μ-integrable, and every sum of finitely many terms on the right is positive and $\leqslant f$, therefore

$$\mu(f) = 0 + \sum_{\alpha\in\mathrm{A}'} \mu(f\varphi_{\mathrm{S}\cap\mathrm{K}_\alpha})$$

(Ch. IV, §1, No. 3, Th. 3 or Ch. IV, §4, No. 3, Cor. 2 of Th. 2).

3rd equality: $f\varphi_{\mathrm{S}\cap\mathrm{K}_\alpha} = (f\varphi_{\mathrm{S}})\varphi_{\mathrm{K}_\alpha} = f\varphi_{\mathrm{K}_\alpha}$.

4th equality: For $\alpha\in\mathrm{A}-\mathrm{A}'$ one has $\mathrm{S}\cap\mathrm{K}_\alpha = \varnothing$, whence $f\varphi_{\mathrm{K}_\alpha} = (f\varphi_{\mathrm{S}})\varphi_{\mathrm{K}_\alpha} = f\varphi_{\mathrm{S}\cap\mathrm{K}_\alpha} = f\cdot 0 = 0$.

Proof of Prop. 4 *based on the measures* μ_{K_α}. For each $\alpha\in\mathrm{A}$, the correspondence

$$f \mapsto \mu_{\mathrm{K}_\alpha}(f|\mathrm{K}_\alpha) \qquad (f\in\mathscr{K}(\mathrm{T}))$$

defines a positive linear form μ'_α on $\mathscr{K}(\mathrm{T})$, that is, a positive measure on T. Then $\mathrm{Supp}(\mu'_\alpha)\subset\mathrm{K}_\alpha$; for, if $f\in\mathscr{K}(\mathrm{T})$ and $\mathrm{Supp}(f)\subset \complement\,\mathrm{K}_\alpha$, then $f|\mathrm{K}_\alpha = 0$ and $\mu'_\alpha(f) = \mu_{\mathrm{K}_\alpha}(f|\mathrm{K}_\alpha) = 0$.

Fix $f\in\mathscr{K}_+(\mathrm{T})$ and define A' and S as above. One argues, as above, that

$$\mu(f) = \sum_{\alpha\in\mathrm{A}'} \mu(f\varphi_{\mathrm{K}_\alpha})\,;$$

since $f\varphi_{\mathrm{K}_\alpha} = (f|\mathrm{K}_\alpha)'$, where $(f|\mathrm{K}_\alpha)'$ is the extension by 0 of $f|\mathrm{K}_\alpha$ to T, $\mu\big((f|\mathrm{K}_\alpha)'\big) = \mu_{\mathrm{K}_\alpha}(f|\mathrm{K}_\alpha)$ by the definition of μ_{K_α} (Ch. IV, §5, No. 7, Def. 4), and $f|\mathrm{K}_\alpha = 0$ for $\alpha \in \mathrm{A} - \mathrm{A}'$, one has

$$\mu(f) = \sum_{\alpha\in\mathrm{A}'} \mu'_\alpha(f) = \sum_{\alpha\in\mathrm{A}} \mu'_\alpha(f)\,.$$

Varying $f \in \mathscr{K}_+(\mathrm{T})$ (and, along with it, A'), this proves that

$$\mu = \sum_{\alpha\in\mathrm{A}} \mu'_\alpha\,.$$

Finally, if $\alpha \in \mathrm{A}$ and $f \in \mathscr{K}_+(\mathrm{T})$ then

$$\mu'_\alpha(f) = \mu_{\mathrm{K}_\alpha}(f|\mathrm{K}_\alpha) = \mu\big((f|\mathrm{K}_\alpha)'\big) = \mu(f\varphi_{\mathrm{K}_\alpha}) = \mu_\alpha(f)\,,$$

therefore $\mu'_\alpha = \mu_\alpha$; the two procedures yield the same result.

V.15, ℓ. 15, 16.

"...let A' be the countable set of $\alpha \in \mathrm{A}$ such that K_α intersects one of the L_n."

For each n, let $\mathrm{A}'_n = \{\alpha \in \mathrm{A} : \mathrm{L}_n \cap \mathrm{K}_\alpha \neq \varnothing\}$; since L_n is compact and $(\mathrm{K}_\alpha)_{\alpha\in\mathrm{A}}$ is locally countable, A'_n is countable, therefore so is $\mathrm{A}' = \bigcup_n \mathrm{A}'_n$.

V.15, ℓ. 16.

"Then $\mu_\alpha = 0$ for $\alpha \notin \mathrm{A}'$ "

With A' defined as in the preceding note, suppose $\alpha \notin \mathrm{A}'$. Then $\mathrm{K}_\alpha \cap \mathrm{L}_n = \varnothing$ for all n, thus $\mathrm{K}_\alpha \subset \mathrm{T} - \bigcup_n \mathrm{L}_n$; but $\mathrm{T} - \bigcup_n \mathrm{L}_n$ is by construction μ-negligible, therefore so is K_α. Then, for all $f \in \mathscr{K}(\mathrm{T})$, the function $f\varphi_{\mathrm{K}_\alpha}$ is μ-negligible, and so $\mu_\alpha(f) = \mu\big(f\varphi_{\mathrm{K}_\alpha}\big) = 0$.

§3. INTEGRATION OF POSITIVE MEASURES

V.17, ℓ. 1–4.

"For, verifying that Λ is scalarly essentially integrable for a positive measure η on T comes down to verifying that $t \mapsto \lambda_t(g)$ is η-measurable and admits a finite essential upper integral, with respect to η, for every function $g \in \mathscr{K}_+(\mathrm{X})$."

Prop. 9 of §1, No. 3. In what follows, the role of η is played by μ and the μ_i.

V.17, ℓ. 4, 5.

"The proposition therefore follows at once from Prop. 11 of §1, No. 4 and its Corollary 2."

Suppose Λ is scalarly essentially μ-integrable. Given $g \in \mathscr{K}_+(\mathrm{X})$, let $\widehat{g} : \mathrm{T} \to \mathbf{R}_+$ be the function defined by $\widehat{g}(t) = \lambda_t(g)$; by assumption, $\widehat{g} \in \overline{\mathscr{L}}^1_{\mathbf{R}}(\mu)$, thus $\widehat{g}$ is μ-measurable and $\mu^\bullet(\widehat{g}) < +\infty$ (§1, No. 3, Prop. 9). Since $\mu = \sup_{i \in \mathrm{I}} \mu_i$, it follows from the cited Prop. 11 that

$$\sup_{i \in \mathrm{I}} \mu_i^\bullet(\widehat{g}) = \mu^\bullet(\widehat{g}) < +\infty\,.$$

Moreover, for every $i \in \mathrm{I}$, $\widehat{g}$ is μ_i-measurable by the cited Cor. 2, and since $\mu_i^\bullet(\widehat{g}) \leqslant \mu^\bullet(\widehat{g}) < +\infty$, $\widehat{g}$ is essentially μ_i-integrable by the cited Prop. 9; thus, since $g \in \mathscr{K}_+(\mathrm{X})$ is arbitrary, Λ is scalarly essentially μ_i-integrable. Writing

$$\nu = \int \lambda_t \, d\mu(t)\,, \quad \nu_i = \int \lambda_t \, d\mu_i(t)\,,$$

we have, for all $g \in \mathscr{K}_+(\mathrm{X})$,

$$\begin{aligned} \nu(g) &= \int \lambda_t(g)\, d\mu(t) = \int \widehat{g}\, d\mu \\ &= \mu^\bullet(\widehat{g}) = \sup_{i \in \mathrm{I}} \mu_i^\bullet(\widehat{g}) = \sup_{i \in \mathrm{I}} \int \widehat{g}\, d\mu_i \\ &= \sup_{i \in \mathrm{I}} \int \lambda_t(g)\, d\mu_i(t) = \sup_{i \in \mathrm{I}} \nu_i(g)\,; \end{aligned}$$

thus $\nu = \sup_{i \in \mathrm{I}} \nu_i$, which is the relation (2).

Conversely, suppose that for every $i \in \mathrm{I}$, Λ is scalarly essentially μ_i-integrable and, writing $\nu_i = \int \lambda_t \, d\mu_i(t)$, suppose that

$$\sup_{i \in \mathrm{I}} \nu_i(g) < +\infty \quad \text{for every } g \in \mathscr{K}_+(\mathrm{X})$$

(a condition equivalent to the existence of a measure $\rho \in \mathscr{M}(\mathrm{X})$ such that $\nu_i \leqslant \rho$ for all $i \in \mathrm{I}$). Let $g \in \mathscr{K}_+(\mathrm{X})$ and define $\widehat{g}(t) = \lambda_t(g)$ $(t \in \mathrm{T})$. By hypothesis, for every $i \in \mathrm{I}$, $\widehat{g}$ is essentially μ_i-integrable, hence $\widehat{g}$ is

μ_i-measurable and $\mu_i^\bullet(\widehat{g}) < +\infty$; therefore $\widehat{g}$ is μ-measurable by the cited Cor. 2, and since

$$\mu^\bullet(\widehat{g}) = \sup_{i\in \mathrm{I}} \mu_i^\bullet(\widehat{g}) = \sup_{i\in \mathrm{I}} \int \widehat{g}\, d\mu_i = \sup_{i\in \mathrm{I}} \nu_i(g) < +\infty \,,$$

$\widehat{g}$ is essentially μ-integrable. Since $g \in \mathscr{K}_+(\mathrm{X})$ is arbitrary, we conclude that Λ is scalarly essentially μ-integrable.

V.17, ℓ. 6–11.

"Corollary"

Consider first the case that $\mu = \sum_{\alpha\in \mathrm{A}} \mu_\alpha$ with A finite. Then $\overline{\mathscr{L}}^1(\mu) = \bigcap_{\alpha\in \mathrm{A}} \overline{\mathscr{L}}^1(\mu_\alpha)$ and

$$\mu^\bullet(f) = \sum_{\alpha\in \mathrm{A}} \mu_\alpha^\bullet(f) \quad \text{for all } f \in \overline{\mathscr{L}}^1(\mu)$$

(a special case of §2, No. 2, Prop. 3; cf. Cor. 3 of the Theorem in the Note for V.10, ℓ. 13–14). Given a mapping $\Lambda : t \mapsto \lambda_t \in \mathscr{M}_+(\mathrm{X})$, for $g \in \mathscr{K}_+(\mathrm{X})$ define $\widehat{g} : \mathrm{T} \to \mathbf{R}_+$ by $\widehat{g}(t) = \lambda_t(g)$. The following conditions are equivalent:

a) Λ is scalarly essentially μ-integrable;

b) $\widehat{g} \in \overline{\mathscr{L}}^1(\mu)$ for all $g \in \mathscr{K}_+(\mathrm{X})$;

c) $\widehat{g} \in \overline{\mathscr{L}}^1(\mu_\alpha)$ for all $\alpha \in \mathrm{A}$ and all $g \in \mathscr{K}_+(\mathrm{X})$;

d) Λ is scalarly essentially μ_α-integrable for all $\alpha \in \mathrm{A}$,

in which case the positive measures $\nu = \int \lambda_t \, d\mu(t)$ and $\nu_\alpha = \int \lambda_t \, d\mu_\alpha(t)$ are defined, where, for all $g \in \mathscr{K}_+(\mathrm{X})$,

$$\nu(g) = \int \lambda_t(g)\, d\mu(t) = \mu^\bullet(\widehat{g}) = \sum_{\alpha\in \mathrm{A}} \mu_\alpha^\bullet(\widehat{g}) = \sum_{\alpha\in \mathrm{A}} \int \lambda_t(g)\, d\mu_\alpha(t) = \sum_{\alpha\in \mathrm{A}} \nu_\alpha(g),$$

that is, $\nu = \sum_{\alpha\in \mathrm{A}} \nu_\alpha$. This proves the Corollary for the case that A is finite.

In the general case, for every finite set $\mathrm{J} \subset \mathrm{A}$ write $\mu_{\mathrm{J}} = \sum_{\alpha\in \mathrm{J}} \mu_\alpha$; then (μ_{J}) is an increasing directed family of positive measures, and we are assuming that the family admits μ as supremum. We know from the foregoing that Λ is scalarly essentially μ_α-integrable for all $\alpha \in \mathrm{A}$ if and only if it is scalarly essentially μ_{J}-integrable for all finite $\mathrm{J} \subset \mathrm{A}$, in which case $\nu_{\mathrm{J}} = \sum_{\alpha\in \mathrm{J}} \nu_\alpha$ for every J , that is,

$$\int \lambda_t \, d\mu_{\mathrm{J}}(t) = \sum_{\alpha\in \mathrm{J}} \int \lambda_t \, d\mu_\alpha(t) \,.$$

By Prop. 1, the following conditions are then equivalent:

(i) Λ is scalarly essentially μ-integrable;

(ii) Λ is scalarly essentially μ_{J}-integrable for every finite $\mathrm{J} \subset \mathrm{A}$, and the family $\left(\int \lambda_t \, d\mu_{\mathrm{J}}(t)\right) = \left(\sum_{\alpha\in \mathrm{J}} \int \lambda_t \, d\mu_\alpha(t)\right)$ indexed by the J's is bounded above in $\mathscr{M}_+(\mathrm{X})$;

(iii) Λ is scalarly essentially μ_α-integrable for all α, and the family $\left(\int \lambda_t \, d\mu_\alpha(t)\right)_{\alpha\in\mathrm{A}}$ is summable in $\mathscr{M}_+(\mathrm{X})$ in the sense of §2, No. 1,

in which case, for every $g \in \mathscr{K}_+(\mathrm{X})$,

$$\nu(g) = \int \lambda_t(g) \, d\mu(t) = \mu^\bullet(\widehat{g}) = \sum_{\alpha\in\mathrm{A}} \mu_\alpha^\bullet(\widehat{g}) = \sum_{\alpha\in\mathrm{A}} \int \lambda_t(g) \, d\mu_\alpha(t) = \sum_{\alpha\in\mathrm{A}} \nu_\alpha(g)$$

(the third equality by §2, No. 2, Prop. 1), whence the family $(\nu_\alpha)_{\alpha\in\mathrm{A}}$ is summable in $\mathscr{M}_+(\mathrm{X})$, with $\sum_{\alpha\in\mathrm{A}} \nu_\alpha = \nu$.

V.17, ℓ. 12, 13.

"It follows immediately that every scalarly essentially μ-integrable mapping is also scalarly essentially μ'-integrable for every measure $\mu' \leqslant \mu$."

Overkill. Writing $\mu'' = \mu - \mu'$, the assertion follows from $\mu = \mu' + \mu''$ and the case of the Corollary for A finite.

A more direct proof: Regard Λ as a mapping $\mathscr{F}_+(\mathrm{X}) \to \mathscr{F}_+(\mathrm{T})$, where, for $g \in \mathscr{F}_+(\mathrm{X})$, one writes $\Lambda(g)$ for the function $t \mapsto \lambda_t^\bullet(g)$ $(t \in \mathrm{T})$; in particular, when $g \in \mathscr{K}_+(\mathrm{X})$, $\Lambda(g)$ is the function $\widehat{g} \in \overline{\mathscr{L}}^1(\mu)$ defined by $\widehat{g}(t) = \lambda_t(g)$ as in the preceding note. To say that Λ is scalarly essentially μ-integrable means that

$$\Lambda\big(\mathscr{K}_+(\mathrm{X})\big) \subset \overline{\mathscr{L}}^1(\mu);$$

thus the assertion is immediate from the fact that

$$0 \leqslant \mu' \leqslant \mu \quad \Rightarrow \quad \overline{\mathscr{L}}^1(\mu) \subset \overline{\mathscr{L}}^1(\mu')$$

(see the Note for V.10, ℓ. 13, 14, proof of (i) in Cor. 3 of the Theorem).

V.17, ℓ. 17–25.

"Definition 1."

As in the preceding note, regard the scalarly essentially μ-integrable mapping Λ as a mapping $\mathscr{F}_+(\mathrm{X}) \to \mathscr{F}_+(\mathrm{T})$, let $\nu = \int \lambda_t \, d\mu(t)$ and, as in Ch. IV, §1, No. 1, write $\mathscr{I}_+(\mathrm{X})$ for the set of all lower semi-continuous functions $f \geqslant 0$ on X. If $f \in \mathscr{I}_+(\mathrm{X})$ then f is universally measurable (i.e., measurable for every measure on X) by Ch. IV, §5, No. 5, Cor. of

Prop. 8; moreover, $\nu^\bullet(f) = \nu^*(f)$ and $\lambda_t^\bullet(f) = \lambda_t^*(f)$ for every $t \in \mathrm{T}$ (§1, No. 1, Prop. 4), and

$$\big(\Lambda(f)(t)\big) = \lambda_t^\bullet(f) = \lambda_t^*(f) = \int^* f\, d\lambda_t\,.$$

Thus, to say that Λ is *μ-pre-adequate* means that for every $f \in \mathscr{I}_+(\mathrm{X})$ the function $\Lambda(f)$ is μ-measurable and

$$\nu^\bullet(f) = \mu^\bullet\big(\Lambda(f)\big),$$

which may also be written (when $f \in \mathscr{I}_+(\mathrm{X})$) as

$$\int^* f(x)\, d\nu(x) = \int^\bullet d\mu(t) \int^* f(x)\, d\lambda_t(x)\,.$$

It follows that, for every positive measure $\mu' \leqslant \mu$, Λ is scalarly essentially μ'-integrable (by the preceding note)—hence the measure $\nu' = \int \lambda_t\, d\mu'(t)$ is defined—and, for every $f \in \mathscr{I}_+(\mathrm{X})$, the function $t \mapsto \big(\Lambda(f)\big)(t) = \lambda_t^\bullet(f) = \lambda_t^*(f)$ is μ'-measurable (§2, No. 2, Prop. 2); thus, for the μ-pre-adequate mapping Λ to be *μ-adequate*, it is necessary and sufficient that for every positive measure $\mu' \leqslant \mu$ and every $f \in \mathscr{I}_+(\mathrm{X})$,

$$\int^\bullet f(x)\, d\nu'(x) = \int^\bullet d\mu'(t) \int^\bullet f(x)\, d\lambda_t(x)\,,$$

which may also be written (for $f \in \mathscr{I}_+(\mathrm{X})$) as

$$\int^* f(x)\, d\nu'(x) = \int^\bullet d\mu'(t) \int^* f(x)\, d\lambda_t(x)\,,$$

or $\nu'^*(f) = \mu'^\bullet\big(\Lambda(f)\big)$, where $\big(\Lambda(f)\big)(t) = \lambda_t^\bullet(f) = \lambda_t^*(f)$ for all $t \in \mathrm{T}$.

V.17, ℓ. -6 to -4.

"It can be shown that if Λ is μ-pre-adequate and if the measure $\nu = \int \lambda_t\, d\mu(t)$ is moderated—in particular if X is countable at infinity—then Λ is μ-adequate (Exer. 7)"

The exercise (p. V.97):

7) *a*) Let $\Lambda : t \mapsto \lambda_t$ be a scalarly essentially integrable mapping of T into $\mathscr{M}_+(\mathrm{X})$, and let $\nu = \int \lambda_t\, d\mu(t)$. Show that for every lower semi-continuous function $f \geqslant 0$ defined on X,

$$\int^\bullet f(x)\, d\nu(x) \leqslant \int^\bullet d\mu(t) \int^\bullet f(x)\, d\lambda_t(x)\,.$$

b) Suppose that Λ is μ-pre-adequate, denote by μ' a positive measure $\leqslant \mu$, and write $\mu = \mu' + \mu''$, $\nu' = \int \lambda_t \, d\mu'(t)$. Deduce from *a*) that, for every lower semi-continuous function $f \geqslant 0$ that is ν-*integrable*,

$$\int^{\bullet} f(x)\, d\nu'(x) = \int^{\bullet} d\mu'(t) \int^{\bullet} f(x)\, d\lambda_t(x)\,.$$

Extend this result to a lower semi-continuous function $f \geqslant 0$ that is ν-moderated. From this, deduce that if ν is a moderated measure (in particular, if X is countable at infinity), then Λ is μ-adequate.

Proof of a): Assuming $f \in \mathscr{I}_+(\mathrm{X})$, we are to show that

$$\nu^{\bullet}(f) \leqslant \mu^{\bullet}\big(\Lambda(f)\big),$$

where $\big(\Lambda(f)\big)(t) = \lambda_t^{\bullet}(f)$ for all $t \in \mathrm{T}$. Since f is lower semi-continuous, we know that f is measurable with respect to every measure (Ch. IV, §5, No. 5, Cor. of Prop. 8), and that $\nu^{\bullet}(f) = \nu^*(f)$ and $\lambda_t^{\bullet}(f) = \lambda_t^*(f)$ for every $t \in \mathrm{T}$ (§1, No. 1, Prop. 4). The key idea: since f is lower semi-continuous,

$$\nu^*(f) = \sup_{g \in \mathscr{K}_+(\mathrm{X}),\ g \leqslant f} \nu(g)$$

(Ch. IV, §1, No. 1, Def. 1 and the remark following Def. 3 of No. 3).

Let $g \in \mathscr{K}_+(\mathrm{X})$, $g \leqslant f$. Then, for every $t \in \mathrm{T}$, $\lambda_t^{\bullet}(g) \leqslant \lambda_t^{\bullet}(f)$; that is, $\Lambda(g) \leqslant \Lambda(f)$ pointwise on T, therefore $\mu^{\bullet}\big(\Lambda(g)\big) \leqslant \mu^{\bullet}\big(\Lambda(f)\big)$. On the other hand, by the definition of ν,

$$\nu(g) = \mu^{\bullet}(\widehat{g}) = \mu^{\bullet}\big(\Lambda(g)\big) \leqslant \mu^{\bullet}\big(\Lambda(f)\big)$$

(when $g \in \mathscr{K}_+(\mathrm{X})$ we may use the notation $\widehat{g}$ in place of $\Lambda(g)$) and finally

$$\nu^{\bullet}(f) = \nu^*(f) = \sup_{g \in \mathscr{K}_+(\mathrm{X}),\ g \leqslant f} \nu(g) \leqslant \mu^{\bullet}\big(\Lambda(f)\big).$$

Proof of b): Writing $\mu'' = \mu - \mu'$, from $\mu = \mu' + \mu''$ we know that Λ is scalarly essentially integrable for both μ' and μ'', so we can also define $\nu'' = \int \lambda_t \, d\mu''(t)$. The key observation is that

$$(\dagger) \qquad \nu' + \nu'' = \nu\,.$$

For, if $g \in \mathscr{K}_+(\mathrm{X})$ we know from $\mu = \mu' + \mu''$ that

$$\widehat{g} \in \overline{\mathscr{L}}^1(\mu) = \overline{\mathscr{L}}^1(\mu') \cap \overline{\mathscr{L}}^1(\mu'') \quad \text{and} \quad \mu^{\bullet}(\widehat{g}) = \mu'^{\bullet}(\widehat{g}) + \mu''^{\bullet}(\widehat{g})$$

(see, e.g., the Note for p. 17, ℓ. 6–11), therefore, citing the definition of ν, ν' and ν'',

$$\begin{aligned}\nu(g) &= \mu^\bullet(\widehat{g}) = (\mu' + \mu'')^\bullet(\widehat{g}) = (\mu'^\bullet + \mu''^\bullet)(\widehat{g}) \\ &= \mu'^\bullet(\widehat{g}) + \mu''^\bullet(\widehat{g}) = \nu'(g) + \nu''(g) = (\nu' + \nu'')(g)\,,\end{aligned}$$

whence (†). {A trivial special case of the Cor. of Prop. 1.}

Let $f \in \mathscr{I}_+(\mathrm{X})$. Since Λ is μ-pre-adequate, we know that

$$\nu^\bullet(f) = \mu^\bullet\big(\Lambda(f)\big).$$

Assuming in addition that f is ν-integrable—equivalently, essentially ν-integrable (§1, No. 1, Prop. 4)—hence also ν'-integrable and ν''-integrable, we are to show that

$$\nu'^\bullet(f) = \mu'^\bullet\big(\Lambda(f)\big).$$

At any rate, by Part a) we have

$$(\dagger\dagger) \qquad \nu'^\bullet(f) \leqslant \mu'^\bullet\big(\Lambda(f)\big) \quad \text{and} \quad \nu''^\bullet(f) \leqslant \mu''^\bullet\big(\Lambda(f)\big),$$

therefore

$$\begin{aligned}\nu^\bullet(f) &= \mu^\bullet\big(\Lambda(f)\big) = (\mu' + \mu'')^\bullet\big(\Lambda(f)\big) = (\mu'^\bullet + \mu''^\bullet)\big(\Lambda(f)\big) \\ &= \mu'^\bullet\big(\Lambda(f)\big) + \mu''^\bullet\big(\Lambda(f)\big) \\ &\geqslant \nu'^\bullet(f) + \nu''^\bullet(f) = (\nu'^\bullet + \nu''^\bullet)(f) \\ &= (\nu' + \nu'')^\bullet(f) = \nu^\bullet(f)\end{aligned}$$

(the inequality, by (††), and the last equality by (†)); we thus have equality throughout, and since $\nu^\bullet(f) = \nu^*(f) < +\infty$ the inequalities in (††) must in fact be equalities. This proves the first assertion of b).

Suppose now that f is merely a ν-moderated lower semi-continuous function $\geqslant 0$. With $\mu' \leqslant \mu$ and $\nu' = \int \lambda_t \, d\mu'(t)$ as above, we are to show that the equality $\nu'^\bullet(f) = \mu'^\bullet\big(\Lambda(f)\big)$ again holds. The key idea (§1, No. 2, Prop. 5, a)): every ν-moderated set is contained in the union of a sequence of μ-integrable open sets.

Let $\mathrm{A} = \{x \in \mathrm{X} : f(x) \neq 0\}$. By assumption there exists a ν-moderated set $\mathrm{B} \subset \mathrm{X}$ such that $f = 0$ on $\complement\mathrm{B}$, in other words $\mathrm{A} \subset \mathrm{B}$; since subsets of moderated sets are moderated, it is the same to say that A is ν-moderated. Let (U_n) be a sequence of ν-integrable open sets in X such that $\mathrm{A} \subset \mathrm{U} = \bigcup_n \mathrm{U}_n$; we can suppose that the sequence is increasing. The functions $f\varphi_{\mathrm{U}_n}$ are lower semi-continuous (GT, IV, §7, No. 2, Cor. of Prop. 1 and Prop. 2) and the sequence $(f\varphi_{\mathrm{U}_n})$ is increasing; if $x \in \mathrm{A}$

then $x \in \mathrm{U}$, so $\varphi_{\mathrm{U}_n}(x) = 1$ from some index onward, whereas if $x \notin \mathrm{A}$ then $(f\varphi_{\mathrm{U}_n})(x) = 0 = f(x)$ for all n, consequently the sequence has upper envelope f, concisely $f\varphi_{\mathrm{U}_n} \uparrow f$ pointwise on X. Define

$$f_n = \inf(n, f\varphi_{\mathrm{U}_n}),$$

which is also lower semi-continuous (GT, *loc. cit.*), and $f_n \uparrow f$ pointwise in X.

Being lower semi-continuous, f_n is ν-measurable, and, since $\nu^*(f_n) \leqslant n\nu^*(\mathrm{U}_n) < +\infty$, f_n is ν-integrable, therefore, by the preceding case,

$$(*) \qquad \nu'^{\bullet}(f_n) = \mu'^{\bullet}\big(\Lambda(f_n)\big) \quad (n = 1, 2, 3, \ldots);$$

but from $f_n \uparrow f$ it is immediate that $\lambda_t^{\bullet}(f_n) \uparrow \lambda_t^{\bullet}(f)$ for every $t \in \mathrm{T}$ (§1, No. 1, Prop. 1), that is, $\Lambda(f_n) \uparrow \Lambda(f)$ pointwise on T, and passage to the limit in $(*)$ yields $\nu'^{\bullet}(f) = \mu'^{\bullet}\big(\Lambda(f)\big)$.

Finally, suppose Λ is μ-pre-adequate, $\nu = \int \lambda_t \, d\mu(t)$, and let μ' be a positive measure $\leqslant \mu$; one knows that Λ is scalarly essentially μ'-integrable and, defining $\nu' = \int \lambda_t \, d\mu'(t)$, obviously $\nu' \leqslant \nu$. As observed in the Note for V.17, ℓ. 17–25, in order that Λ be μ'-pre-adequate, it is necessary and sufficient that $\nu'^{\bullet}(f) = \mu'^{\bullet}\big(\Lambda(f)\big)$ for every $f \in \mathscr{I}_+(\mathrm{X})$, a condition that is satisfied (as shown above) when every $f \in \mathscr{I}_+(\mathrm{X})$ is ν'-moderated. If ν is moderated, so that $\mathrm{X} = \mathrm{N} \cup \bigcup_n \mathrm{K}_n$ with (K_n) a sequence of compact sets and N ν-negligible, then N is also ν'-negligible, thus ν' is moderated and so every function on X is ν'-moderated. Conclusion: When Λ is μ-pre-adequate and $\nu = \int \lambda_t \, d\mu(t)$ is moderated, then Λ is μ'-pre-adequate for every positive measure $\mu' \leqslant \mu$, that is, Λ is μ-adequate.

V.17, ℓ. −4.

"... it is not known if these concepts are in general equivalent."

The question implicitly posed: Is every μ-pre-adequate mapping $\Lambda : t \mapsto \lambda_t$ automatically μ-adequate? {Perhaps the matter has been settled by now? (5-14-2007).}

Exercise 7 (see the preceding note) and Prop. B below give an affirmative answer for some special cases, and Exercise 8 (worked out later in this Note) effectively reduces the problem to the case that μ has compact support.

First, some simple preliminaries:

Proposition A. Let $\Lambda : t \mapsto \lambda_t$ be scalarly essentially μ-integrable, and suppose that $\mu = \sum_{i \in \mathrm{I}} \mu_i$ for some summable family $(\mu_i)_{i \in \mathrm{I}}$ in $\mathscr{M}_+(\mathrm{T})$. If Λ is μ_i-pre-adequate for all $i \in \mathrm{I}$, then Λ is μ-pre-adequate.

Proof. Since Λ is scalarly essentially μ-integrable, it is scalarly essentially μ_i-integrable for all i; write $\nu = \int \lambda_t \, d\mu(t)$, $\nu_i = \int \lambda_t \, d\mu_i(t)$, and, for every finite $\mathrm{J} \subset \mathrm{I}$, write $\mu_{\mathrm{J}} = \sum_{i \in \mathrm{J}} \mu_i$, $\nu_{\mathrm{J}} = \sum_{i \in \mathrm{J}} \nu_i$. Let $f \in \mathscr{I}_+(\mathrm{X})$ and write $\Lambda(f)$ for the function $t \mapsto \lambda_t^\bullet(f)$ $(t \in \mathrm{T})$. For every $i \in \mathrm{I}$, Λ is μ_i-pre-adequate; by Def. 1, $\Lambda(f)$ is μ_i-measurable and $\nu_i^\bullet(f) = \mu_i^\bullet\big(\Lambda(f)\big)$. It follows (§2, No. 2, Prop. 2) that $\Lambda(f)$ is μ-measurable and μ_{J}-measurable for all J, and $\nu_{\mathrm{J}}^\bullet(f) = \mu_{\mathrm{J}}^\bullet\big(\Lambda(f)\big) \leqslant \mu^\bullet\big(\Lambda(f)\big)$; in particular when $f \in \mathscr{K}_+(\mathrm{X})$, $\mu^\bullet\big(\Lambda(f)\big) < +\infty$ (since Λ is scalarly essentially μ-integrable), thus the family $\big(\nu_{\mathrm{J}}(f)\big)$ indexed by J is bounded, so the family $(\nu_i)_{i \in \mathrm{I}}$ is summable (§2, No. 1). By the Cor. of Prop. 1, $\nu = \sum_{i \in \mathrm{I}} \nu_i$. Then $\mu^\bullet = \sum_{i \in \mathrm{I}} \mu_i^\bullet$, $\nu^\bullet = \sum_{i \in \mathrm{I}} \nu_i^\bullet$ (§2, No. 2, Prop. 1) and, for every $f \in \mathscr{I}_+(\mathrm{X})$,

$$\mu^\bullet\big(\Lambda(f)\big) = \sum_{i \in \mathrm{I}} \mu_i^\bullet\big(\Lambda(f)\big) = \sum_{i \in \mathrm{I}} \nu_i^\bullet(f) = \nu^\bullet(f)\,,$$

thus Λ is μ-pre-adequate (Def. 1). ♢

Corollary. Suppose that Λ is scalarly essentially μ-integrable and that Λ is μ'-pre-adequate for every positive measure $\mu' \leqslant \mu$ with compact support. Then Λ is μ-adequate.

Proof. Given any positive measure $\mu' \leqslant \mu$ we are to show that Λ is μ'-pre-adequate. At any rate, Λ is scalarly essentially μ'-integrable (V.17, ℓ. 12, 13). By §2, No. 3, Prop. 4, $\mu' = \sum_{\alpha \in \mathrm{A}} \mu_\alpha$ for a summable family $(\mu_\alpha)_{\alpha \in \mathrm{A}}$ of positive measures on T with compact support. By hypothesis, Λ is μ_α-pre-adequate for all α, therefore Λ is μ'-pre-adequate by Prop. A. ♢

Proposition B. Let $(\mu_i)_{i \in \mathrm{I}}$ be an increasing directed family of positive measures on T that is bounded above in $\mathscr{M}_+(\mathrm{T})$, and let $\mu = \sup_{i \in \mathrm{I}} \mu_i$. Suppose that the mapping $\Lambda : t \mapsto \lambda_t$ is scalarly essentially μ-integrable, and that Λ is μ_i-pre-adequate for every $i \in \mathrm{I}$. Then Λ is μ-pre-adequate.

Proof. Let $\nu = \int \lambda_t \, d\mu(t)$. By Prop. 1 of No. 1 we know that Λ is scalarly essentially μ_i-integrable for all $i \in \mathrm{I}$, and, writing $\nu_i = \int \lambda_t \, d\mu_i(t)$ for $i \in \mathrm{I}$, $(\nu_i)_{i \in \mathrm{I}}$ is an increasing directed family for which ν serves as an upper bound in $\mathscr{M}_+(\mathrm{X})$, indeed $\nu = \sup_{i \in \mathrm{I}} \nu_i$.

We now make use of the assumption that Λ is μ_i-pre-adequate for all $i \in \mathrm{I}$. Let $f \in \mathscr{I}_+(\mathrm{X})$. For every $i \in \mathrm{I}$, the function $\Lambda(f) : t \mapsto \lambda_t^\bullet(f)$ is μ_i-measurable and $\nu_i^\bullet(f) = \mu_i^\bullet\big(\Lambda(f)\big)$, therefore f is μ-measurable (§1, No. 4, Cor. 2 of Prop. 11) and, by the cited Prop. 11,

$$\nu^\bullet(f) = \sup_{i \in \mathrm{I}} \nu_i^\bullet(f) = \sup_{i \in \mathrm{I}} \mu_i^\bullet\big(\Lambda(f)\big) = \mu^\bullet\big(\Lambda(f)\big),$$

thus Λ is μ-pre-adequate. ♢

Exercise 8 (p. V.97):

8) Let $\Lambda : t \mapsto \lambda_t$ be a mapping of T into $\mathscr{M}_+(\mathrm{X})$.

a) Let $\mu_1, \ldots, \mu_n$ be measures on T, and $\mu = \mu_1 + \cdots + \mu_n$. For Λ to be μ-adequate, it is necessary and sufficient that Λ be μ_i-adequate for every i (make use of the decomposition lemma).

b) Suppose that μ is the supremum of an increasing directed family $(\mu_i)_{i \in \mathrm{I}}$. For Λ to be μ-adequate, it is necessary and sufficient that Λ be μ_i-adequate for all $i \in \mathrm{I}$ and scalarly essentially μ-integrable.

c) Suppose that μ is the sum of a summable family $(\mu_j)_{j \in \mathrm{J}}$ of positive measures. For Λ to be μ-adequate, it is necessary and sufficient that Λ be μ_j-adequate for all $j \in \mathrm{J}$ and that Λ be scalarly essentially μ-integrable.

d) For Λ to be μ-adequate, it is necessary and sufficient that Λ be scalarly essentially μ-integrable and that Λ be μ'-pre-adequate for every measure $\mu' \leqslant \mu$ with compact support.

Proof of a): *Sufficiency.* Suppose that Λ is μ_i-adequate for all i $(1 \leqslant i \leqslant n)$. In particular, Λ is scalarly essentially μ_i-integrable for all i, therefore Λ is scalarly essentially μ-integrable by No. 1, Cor. of Prop. 1 for the case of finite sums.

Given a positive measure $\mu' \leqslant \mu$, we are to show that Λ is μ'-pre-adequate (No. 1, Def. 1). By the corollary of the "decomposition theorem" (A, VI, §1, No. 10, Cor. of Th. 1), one can write

$$\mu' = \mu'_1 + \cdots + \mu'_n\,, \quad \text{where } 0 \leqslant \mu'_i \leqslant \mu_i \qquad (1 \leqslant i \leqslant n)$$

(the term "decomposition lemma" is introduced at the end of Ch. II, §1, No. 1). For every i, Λ is μ'_i-pre-adequate (No. 1, Def. 1), therefore Λ is μ'-pre-adequate by the above Prop. A.

Necessity. If Λ is μ-adequate (in particular, scalarly essentially μ-integrable) then, for every positive measure $\mu' \leqslant \mu$, it is obvious from the definition of adequacy that Λ is also μ'-adequate. In particular, Λ is μ_i-adequate for every i.

Proof of b): *Sufficiency.* Suppose that Λ is scalarly essentially μ-integrable and μ_i-adequate for all $i \in \mathrm{I}$. Given any positive measure $\mu' \leqslant \mu$, we are to show that Λ is μ'-pre-adequate.

Define $\mu'_i = \inf(\mu', \mu_i)$ for all $i \in \mathrm{I}$; then $(\mu'_i)_{i \in \mathrm{I}}$ is an increasing directed family whose supremum in $\mathscr{M}_+(\mathrm{T})$ is equal to $\inf(\mu', \mu) = \mu'$ (A, Ch. VI, §1, No. 12, Prop. 13). Since $\mu'_i \leqslant \mu_i$, Λ is μ'_i-pre-adequate for every $i \in \mathrm{I}$, therefore Λ is μ'-pre-adequate by Prop. B.

Necessity. Same argument as in Part *a*).

Proof of c): *Sufficiency.* Suppose Λ is scalarly essentially μ-integrable and μ_j-adequate for all $j \in \mathrm{J}$. Let A be the set of all finite subsets

α of J and, for every $\alpha \in \mathrm{A}$, define $\mu_\alpha = \sum_{j\in\alpha} \mu_j$. By assumption, $\mu = \sup_{\alpha\in\mathrm{A}} \mu_\alpha$. By Part a), Λ is μ_α-adequate for every $\alpha \in \mathrm{A}$, therefore it is μ-adequate by Part b).

Necessity. Same argument as in Part a).

Proof of d): *Sufficiency.* This is the Corollary of Prop. A.

Necessity. If Λ is μ-adequate (in particular, scalarly essentially μ-integrable) then it is μ'-pre-adequate for every $\mu' \leqslant \mu$ regardless of support.

V.18, ℓ. 3–11.

"Proposition 2."

The ingenious argument in the text merges the proofs of Parts a) and b), at some cost in clarity of the structure of the argument. The following rearrangement highlights structure, at some cost in repetition.

Proof of a): We employ the notations $\Lambda(f)$ and $\widehat{g}$ as in the preceding notes (e.g., the Note for V.17, ℓ. -6 to -4); in particular, when $f \in \mathscr{I}_+(\mathrm{X})$, $\big(\Lambda(f)\big)(t) = \lambda_t^\bullet(f) = \lambda_t^*(f)$ for all $t \in \mathrm{T}$ (§1, No. 1, Prop. 4), and when $g \in \mathscr{K}_+(\mathrm{X})$, $\widehat{g}$ abbreviates $\Lambda(g)$, so that $\widehat{g}(t) = \lambda_t^\bullet(g) = \lambda_t(g)$ for all $t \in \mathrm{T}$. Since Λ is scalarly essentially μ-integrable, the measure $\nu = \int \lambda_t \, d\mu(t)$ is defined, by the formula $\nu(g) = \mu^\bullet(\widehat{g})$ for $g \in \mathscr{K}_+(\mathrm{X})$ (No. 1).

The vague continuity of $\Lambda : \mathrm{T} \to \mathscr{M}_+(\mathrm{T})$ means that for every $g \in \mathscr{K}_+(\mathrm{X})$, the function $t \mapsto \lambda_t(g)$ is continuous on T, that is, $\widehat{g}$ is continuous.

Let $f \in \mathscr{I}_+(\mathrm{X})$ and define $\mathrm{F} = \{g \in \mathscr{K}_+(\mathrm{X}) : \ g \leqslant f\}$. The functions $g \in \mathrm{F}$ form an increasing directed family of functions whose upper envelope is f (Ch. IV, §1, No. 1, Lemma), concisely $g \uparrow f$ (pointwise on X). By definition,

$$\nu^*(f) = \sup_{g\in\mathrm{F}} \nu(g)$$

(*loc. cit.*, Def. 1); similarly

$$\lambda_t^*(f) = \sup_{g\in\mathrm{F}} \lambda_t(g) = \sup_{g\in\mathrm{F}} \widehat{g}(t) \quad \text{for every } t \in \mathrm{T},$$

thus $\Lambda(f)$ is the upper envelope of the functions $\widehat{g}$ $(g \in \mathrm{F})$, and since the $\widehat{g}$ form an increasing directed family ($g \leqslant g' \Rightarrow \lambda_t(g) \leqslant \lambda_t(g')$), we may write $\widehat{g} \uparrow \Lambda(f)$ (pointwise on T). Since the $\widehat{g}$ are continuous, $\Lambda(t)$ is lower semi-continuous on T (GT, IV, §6, No. 2, Cor. of Th. 4).

To show that Λ is μ-adequate, it suffices to show that it is μ-pre-adequate; for, Λ is scalarly essentially μ'-integrable for every positive measure $\mu' \leqslant \mu$ and will therefore be μ'-pre-adequate by the same argument. We already know that $\Lambda(f)$ is lower semi-continuous, therefore μ-measurable

(Ch. IV, §5, No. 5, Cor. of Prop. 8), hence need only show that $\nu^\bullet(f) = \mu^\bullet\big(\Lambda(f)\big)$. Indeed, for all $g \in \mathrm{F}$ one has

$$\nu^\bullet(g) = \nu^*(g) = \nu(g) = \mu^\bullet(\widehat{g}) = \mu^*(\widehat{g}),$$

whence

$$\nu^\bullet(f) = \sup_{g\in\mathrm{F}} \nu^\bullet(g) = \sup_{g\in\mathrm{F}} \mu^*(\widehat{g}) = \mu^*\big(\Lambda(f)\big) = \mu^\bullet\big(\Lambda(f)\big)$$

(the 1st and 3rd equalities, by §1, No. 1, Prop. 4 and Ch. IV, §1, No. 1, Th. 1; the 2nd, by the preceding display; the 4th by §1, No. 1, Prop. 4).

Proof of b): Suppose Λ is scalarly essentially μ-integrable and vaguely μ-measurable. As in the text, let $\mathfrak{K}$ be the set, μ-dense in T, of all compact sets $\mathrm{K} \subset \mathrm{T}$ such that the mapping $\Lambda\big|\mathrm{K} : \mathrm{K} \to \mathscr{M}_+(\mathrm{X})$ is vaguely continuous (Ch. IV, §5, No. 10, Prop. 15), and let $(\mu_\alpha)_{\alpha\in\mathrm{A}}$ be a summable family of positive measures on T such that $\mu = \sum_{\alpha\in\mathrm{A}} \mu_\alpha$ and $\mathrm{Supp}\,\mu_\alpha \in \mathfrak{K}$ (§2, No. 3, Prop. 4). In particular, the restriction of Λ to $\mathrm{Supp}\,\mu_\alpha$ is vaguely continuous for all $\alpha \in \mathrm{A}$. To show that Λ is μ-adequate, it will suffice to show that it is μ_α-adequate for every $\alpha \in \mathrm{A}$ (see assertion *c*) of Exer. 8, proved in the preceding Note). We are thus reduced to proving the following: If Λ is scalarly essentially μ-integrable and its restriction to $\mathrm{S} = \mathrm{Supp}\,\mu$ is vaguely continuous, then Λ is μ-adequate; since, for every positive measure $\mu' \leqslant \mu$, Λ is μ'-scalarly essentially integrable and $\mathrm{Supp}\,\mu' \subset \mathrm{Supp}\,\mu$ (Ch. III, §2, No. 2, Prop. 3), it suffices to show that Λ is μ-pre-adequate. In the text, the argument for this is given in the proof of Part *a*), as follows (in slightly different notation). Let $\nu = \int \lambda_t \, d\mu(t)$; by definition, $\nu(g) = \mu^\bullet(\widehat{g})$ for $g \in \mathscr{K}_+(\mathrm{X})$ (§1, No. 1).

Let $f \in \mathscr{I}_+(\mathrm{X})$ and let $\mathrm{F} = \{g \in \mathscr{K}_+(\mathrm{X}) : \ g \leqslant f\}$. Given any $g \in \mathscr{K}_+(\mathrm{X})$, by the vague continuity assumption the restriction to S of the mapping $t \mapsto \widehat{g}(t) = \lambda_t(g)$ is continuous, that is, $\widehat{g}\big|\mathrm{S}$ is continuous; write $\overline{g}$ (instead of $\overline{h}_g$) for the function on T defined by

$$\overline{g}(t) = \begin{cases} \widehat{g}(t) & \text{for } t \in \mathrm{S} \\ +\infty & \text{for } t \in \mathrm{T} - \mathrm{S}. \end{cases}$$

Then $\overline{g}$ is lower semi-continuous on T; for, given any $k \in \mathbf{R}$, the set

$$\{t \in \mathrm{T} : \ \overline{g}(t) \leqslant k\} = \{t \in \mathrm{S} : \ \overline{g}(t) \leqslant k\} = \{t \in \mathrm{S} : \ \widehat{g}(t) \leqslant k\}$$

is closed in S (by the continuity of $\widehat{g}\big|\mathrm{S}$), hence is closed in T (because S is closed in T), whence the assertion (GT, IV, §6, No. 2, Prop. 1). Since

$\overline{g}(t) = \widehat{g}(t)$ for $t \in \mathrm{S}$, and $\mathrm{T} - \mathrm{S}$ is μ-negligible, one has $\overline{g} = \widehat{g}$ μ-almost everywhere, therefore

$$\mu^\bullet(\overline{g}) = \mu^\bullet(\widehat{g}) = \nu(g)$$

(§1, No. 1, Prop. 1, *a*), and the definition of ν).

The functions $\overline{g}$ $(g \in \mathrm{F})$ form an increasing directed family; let $\overline{f} = \sup_{g\in\mathrm{F}} \overline{g}$ be its upper envelope. Since the $\overline{g}$ are lower semi-continuous, so is $\overline{f}$, and for $t \in \mathrm{S}$ one has

$$\overline{f}(t) = \sup_{g\in\mathrm{F}} \overline{g}(t) = \sup_{g\in\mathrm{F}} \widehat{g}(t) = \sup_{g\in\mathrm{F}} \lambda_t(g) = \lambda_t^*(f) = \lambda_t^\bullet(f) = \big(\Lambda(f)\big)(t)\,.$$

{The 1st equality, by the definition of $\overline{f}$; the 2nd equality, because $t \in \mathrm{S}$; the 3rd, by the definition of $\widehat{g}$; the 4th, by Ch. IV, §1, No. 1, Def. 1; the 5th, by §1, No. 1, Prop. 4; the 6th, by the definition of $\Lambda(f)$}. Thus $\overline{f} = \Lambda(f)$ μ-almost everywhere; since $\overline{f}$ is μ-measurable (Ch. IV, §5, No. 5, Cor. of Prop. 8) so is $\Lambda(f)$ (*loc. cit.*, No. 2, Prop. 6), and

$$\mu^\bullet\big(\Lambda(f)\big) = \mu^\bullet(\overline{f}) = \mu^\bullet\big(\sup_{g\in\mathrm{F}} \overline{g}\big) = \sup_{g\in\mathrm{F}} \mu^\bullet(\overline{g}) = \sup_{g\in\mathrm{F}} \nu(g) = \nu^*(f) = \nu^\bullet(f)$$

(the 3rd equality, by §2, No. 2, Prop. 8; the 4th by the next-to-last display).

V.18, ℓ. 15–17.

"Similarly, set

$$h_f(t) = \lambda_t^*(f) = \lambda_t^\bullet(f) = \sup_{g\in\mathrm{F}} h_g(t)$$

(§1, No. 1, Prop. 4)."

In the notations used in earlier Notes (for example, the preceding one), $h_g(t) = \widehat{g}(t)$ for $g \in \mathscr{K}_+(\mathrm{X})$, $t \in \mathrm{T}$, and, for $f \in \mathscr{I}_+(\mathrm{X})$, $h_f(t) = \big(\Lambda(f)\big)(t) = \lambda_t^\bullet(f)$, where

$$\lambda_t^\bullet(f) = \lambda_t^*(f) = \sup_{g\in\mathrm{F}} \lambda_t(g) = \sup_{g\in\mathrm{F}} \widehat{g}(t) = \sup_{g\in\mathrm{F}} h_g(t)$$

(the 1st equality, by §1, No. 1, Prop. 4; the 2nd, by Ch. IV, §1, No. 1, Def. 1 and No. 3, Def. 3; the 3rd and 4th repeat definitions of the notations $\widehat{g}$ and h_g).

V.18, ℓ. −13.

"Set $\overline{h}_f = \sup_{g\in\mathrm{F}} \overline{h}_g$; then $\overline{h}_f = h_f$ on S."

If $t \in \mathrm{T} - \mathrm{S}$ then $\overline{h}_g(t) = +\infty$ for all $g \in \mathrm{F}$, so

$$\overline{h}_f(t) = \sup_{g\in\mathrm{F}} \overline{h}_g(t) = +\infty ,$$

whereas if $t \in \mathrm{S}$ then

$$\overline{h}_f(t) = \sup_{g\in\mathrm{F}} \overline{h}_g(t) = \sup_{g\in\mathrm{F}} h_g(t) = \sup_{g\in\mathrm{F}} \lambda_t(g) = \lambda_t^*(f) = h_f(t) ;$$

thus $\overline{h}_f$ is given by the same formula used for defining $\overline{h}_g$:

$$\overline{h}_f(t) = \begin{cases} h_f(t) & \text{for } t \in \mathrm{S} \\ +\infty & \text{for } t \in \mathrm{T} - \mathrm{S} . \end{cases}$$

V.18, ℓ. −13 to −11.

"For every $g \in \mathrm{F}$, the function $\overline{h}_g$ is lower semi-continuous; $\overline{h}_f$ is therefore lower semi-continuous . . . "

Given any $k \in \mathbf{R}$, the set

$$\{t \in \mathrm{T} : \ \overline{h}_g(t) \leqslant k\} = \{t \in \mathrm{S} : \ h_g(t) \leqslant k\} = \{t \in \mathrm{S} : \ \lambda_t(g) \leqslant k\}$$

is closed in S (by the vague continuity of $t \mapsto \lambda_t$ on S), hence is closed in T (because S is closed in T); thus $\overline{h}_g$ is lower semi-continuous on T (GT, §6, No. 2, Prop. 1), therefore so is $\overline{h}_f$ (*loc. cit.*, Th. 4).

V.18, ℓ. −10.

$$\text{“}\mu^*(\overline{h}_f) = \sup_{g\in\mathrm{F}} \mu^*(\overline{h}_g) = \sup_{g\in\mathrm{F}} \mu^*(h_g) = \sup_{g\in\mathrm{F}} \nu(g) = \nu^*(f) \text{”}$$

1st equality: Ch. IV, §1, No. 1, Th. 1.

2nd equality: Ch. IV, §2, No. 3, Prop. 6.

3rd equality: One has $\nu(g) = \mu^\bullet(h_g)$ by the definition of ν, and $\mu^\bullet(h_g) = \mu^\bullet(\overline{h}_g) = \mu^*(\overline{h}_g) = \mu^*(h_g)$ because $h_g = \overline{h}_g$ μ-almost everywhere and $\overline{h}_g$ is lower semi-continuous.

4th equality: Ch. IV, §1, No. 1, Def. 1.

V.19, ℓ. −12, −11.

". . . we identify $\mathscr{K}(\mathrm{X})$ with a subset of $\mathscr{C}(\mathrm{X}')$."

The open sets in X' are the open sets U of X together with the sets $(\mathrm{X} - \mathrm{K}) \cup \{\omega\}$, where K is a compact subset of X (GT, I, §9, No. 8); thus the open neighborhoods of ω are the sets $(\mathrm{X} - \mathrm{K}) \cup \{\omega\}$. For every $f \in \mathscr{K}(\mathrm{X})$, let f' be its extension to X' by defining $f'(\omega) = 0$. One sees easily that the functions f' are the functions in $\mathscr{K}(\mathrm{X}')$ that are equal to 0 on a neighborhood of ω.

V.19, ℓ. $-7, -6$.

"... let φ_n be a function in $\mathscr{K}_+(\mathrm{X})$ equal to 1 on $\overline{\mathrm{U}}_n$."

Ch. III, §1, No. 2, Lemma 1, with $n = 1$ and $\mathrm{V}_1 = \mathrm{X}$.

V.19, ℓ. $-6, -5$.

"... denote by S the countable set of elements of $\mathscr{K}(\mathrm{X})$ of the form $\varphi_n g$ ($n \in \mathbf{N}$, $g \in \mathrm{S}'$)."

The $g \in \mathrm{S}'$ belong to $\mathscr{C}(\mathrm{X}')$ and the φ_n to $\mathscr{K}(\mathrm{X})$, therefore the $\varphi_n g$ belong to $\mathscr{K}(\mathrm{X})$; thus $\mathrm{S} \subset \mathscr{K}(\mathrm{X})$. {$\mathscr{K}(\mathrm{X})$ is an ideal in the ring $\mathscr{C}(\mathrm{X}')$.}

V.19, ℓ. $-5, -4$.

"If $f \in \mathscr{K}(\mathrm{X})$, let (g_n) be a sequence of elements of S' that converges uniformly to f "

This approximation argument is the reason for embedding $\mathscr{K}(\mathrm{X})$ in $\mathscr{C}(\mathrm{X}')$; although f can be viewed as an element of $\mathscr{C}(\mathrm{X}')$, it is not assured that the g_n are 0 in a neighborhood of ω, that is, the functions $g_n|\mathrm{X}$ need not belong to $\mathscr{K}(\mathrm{X})$.

V.19, ℓ. $-2, -1$.

"The functions $f_n = \varphi_k g_n$ belong to S and satisfy the statement, with $\varphi = m\varphi_k$."

Note that $\varphi = (m \cdot 1_{\mathrm{X}'})\varphi_k$, where $m \cdot 1_{\mathrm{X}'} \in \mathrm{S}'$ by assumption, thus $\varphi \in \mathrm{S}$. Since $\varphi_k = 1$ on U_k, hence on $\operatorname{Supp} f$, one has $\varphi_k f = f$; thus, for every n,

$$|f_n - f| = |\varphi_k g_n - f| = |\varphi_k g_n - \varphi_k f| = \varphi_k |g_n - f| = \frac{1}{m}\varphi \cdot |g_n - f|,$$

where the vertical bars denote absolute value (not norm). Since $\|g_n - f\| \to 0$, given any $\varepsilon > 0$ there exists an index n_0 such that $|g_n - f| \leqslant m\varepsilon$ for all $n \geqslant n_0$, whence

$$|f_n - f| \leqslant m\varepsilon \cdot \Big(\frac{1}{m} \cdot \varphi\Big) = \varepsilon\varphi$$

for all $n \geqslant n_0$.

V.20, ℓ. 7–9.

"... the function $t \mapsto \lambda_t(f)$ is then the uniform limit on K of the continuous functions $t \mapsto \lambda_t(f_n)$; it is therefore continuous on K, and the proposition is proved."

Forget X': the functions f, f_n, φ belong to $\mathscr{K}(\mathrm{X})$; thus we may write $\widehat{f} = \lambda_t(f)$, etc., as in previous notes, and the problem is to show that $\|\widehat{f}_n - \widehat{f}\|_{\mathrm{K}} \to 0$, where

$$\|\widehat{f}_n - \widehat{f}\|_{\mathrm{K}} = \sup_{t \in \mathrm{K}} |\lambda_t(f_n) - \lambda_t(f)|.$$

Given $\varepsilon > 0$ choose n_0 so that $|f_n - f| \leqslant \varepsilon\varphi$ for all $n \geqslant n_0$; then

$$-\varepsilon\varphi \leqslant f_n - f \leqslant \varepsilon\varphi \quad \text{for all } n \geqslant n_0 \,,$$

therefore

$$-\varepsilon\lambda_t(\varphi) \leqslant \lambda_t(f_n) - \lambda_t(f) \leqslant \varepsilon\lambda_t(\varphi) \qquad \text{for all } n \geqslant n_0 \text{ and } t \in \mathrm{T}\,,$$

that is, $-\varepsilon\widehat{\varphi} \leqslant \widehat{f}_n - \widehat{f} \leqslant \varepsilon\widehat{\varphi}$, whence $|\widehat{f}_n - \widehat{f}| \leqslant \varepsilon\widehat{\varphi}$. We know that the restriction to K of the mapping $t \mapsto \big(\lambda_t(g)\big)_{g\in \mathrm{S}} \in \mathbf{R}^{\mathrm{S}}$ is continuous; therefore, for every 'coordinate' $g \in \mathrm{S}$ of the product space $\mathbf{R}^{\mathrm{S}}$, the restriction to K of the mapping $t \mapsto \lambda_t(g)$ is continuous, that is, $\widehat{g}\big|\mathrm{K}$ is continuous. In particular, since φ and the f_n belong to S, $\widehat{\varphi}\big|\mathrm{K}$ and the $\widehat{f}_n\big|\mathrm{K}$ are continuous, hence bounded; and from

$$\|\widehat{f}_n - \widehat{f}\|_{\mathrm{K}} \leqslant \varepsilon\, \|\widehat{\varphi}\|_{\mathrm{K}} \quad \text{for } n \geqslant n_0$$

we see that $\widehat{f}\big|\mathrm{K}$ is the uniform limit of the continuous functions $\widehat{f}_n\big|\mathrm{K}$, hence is continuous.

Since $f \in \mathscr{K}(\mathrm{X})$ is arbitrary, this shows that the restriction to K of the mapping $\Lambda : t \mapsto \lambda_t$ is vaguely continuous; and since $\mathrm{K} \in \mathfrak{K}$ is arbitrary and $\mathfrak{K}$ is μ-dense in T, Λ is indeed vaguely μ-measurable (Ch. IV, §5, No. 10, Prop. 15). A *tour de force.*

V.20, ℓ. -4 to -2.

"The first of the inequalities (6) then follows from the definition of $\int^* f(x)\, d\nu(x)$ (Ch. IV, §1, No. 3, Def. 3), and the second follows immediately from it."

The new actor in Prop. 3 is the function

$$(*) \qquad t \mapsto \int^* f(x)\, d\lambda_t(x) = \lambda_t^*(f) \qquad (t \in \mathrm{T})$$

for an arbitrary function $f \in \mathscr{F}_+(\mathrm{X})$. In analogy with the notation $\Lambda(f)$ for the function

$$t \mapsto \int^\bullet f(x)\, d\lambda_t(x) = \lambda_t^\bullet(f) \qquad (t \in \mathrm{T})$$

introduced in the above notes, it is an aid to grasping the formulas (and saves much scribbling) to write $\Lambda^*(f)$ for the function defined by $(*)$:

$$\big(\Lambda^*(f)\big)(t) = \lambda_t^*(f) \qquad (t \in \mathrm{T})\,;$$

the inequalities (6) may then be written succinctly as

$$\nu^*(f) \geqslant \mu^\bullet\big(\Lambda^*(f)\big) \geqslant \mu^\bullet\big(\Lambda(f)\big),$$

and the second inequality of (6) becomes obvious: $\lambda_t^*(f) \geqslant \lambda_t^\bullet(f)$ for all $t \in \mathrm{T}$ (§1, No. 1, formula (1)), thus $\Lambda^*(f) \geqslant \Lambda(f)$ pointwise on T, whence $\mu^\bullet\big(\Lambda^*(f)\big) \geqslant \mu^\bullet\big(\Lambda(f)\big)$ (*loc. cit.*, Prop. 1). As for the first inequality, since

$$\nu^*(f) = \inf_{g \geqslant f,\, g \in \mathscr{I}_+(\mathrm{X})} \nu^*(g)$$

(Ch. IV, §1, No. 3, Def. 3), it will suffice to show that

$$\nu^*(g) \geqslant \mu^\bullet\big(\Lambda^*(f)\big)$$

for every lower semi-continuous function $g \geqslant f$; for such a function g, one has (§1, No. 1, Prop. 4)

$$\lambda_t^\bullet(g) = \lambda_t^*(g) \geqslant \lambda_t^*(f) \qquad \text{for all } t \in \mathrm{T},$$

whence $\Lambda(g) = \Lambda^*(g) \geqslant \Lambda^*(f)$ pointwise on T, and since Λ is μ-pre-adequate,

$$\nu^*(g) = \nu^\bullet(g) = \mu^\bullet\big(\Lambda(g)\big) \geqslant \mu^\bullet\big(\Lambda^*(f)\big).$$

{Note that if, in addition, f is lower semi-continuous, then

$$\nu^\bullet(f) = \nu^*(f) \geqslant \mu^\bullet\big(\Lambda^*(f)\big) \geqslant \mu^\bullet\big(\Lambda(f)\big) = \nu^\bullet(f)$$

by the foregoing and the μ-pre-adequacy of Λ; thus for f not necessarily lower semi-continuous, the possible gap between the extreme members of (6) is interpolated by $\mu^\bullet\big(\Lambda^*(f)\big)$.}

V.20, $\ell.\ -2, -1$.

"The inequality (7) is proved in an analogous way if Λ is vaguely continuous, using (5) instead of (4)."

Writing $\Lambda^*(f)$ for the function $t \mapsto \lambda_t^*(f)$ we are to show that

$$\nu^*(f) \geqslant \mu^*\big(\Lambda^*(f)\big)\,,$$

and, as in the preceding note, it suffices to show that if $g \in \mathscr{I}_+(\mathrm{X})$ and $g \geqslant f$ then $\nu^*(g) \geqslant \mu^*\big(\Lambda(f)\big)$; indeed, by the monotonicity of outer measure one has $\Lambda^*(g) \geqslant \Lambda^*(f)$ pointwise on T, and, citing (5) applied to the lower semi-continuous function g,

$$\nu^*(g) = \mu^*\big(\Lambda^*(g)\big) \geqslant \mu^*\big(\Lambda^*(f)\big).$$

{Recall that if f is lower semi-continuous and if Λ is scalarly essentially μ-integrable and vaguely continuous, the function $\Lambda^*(f) = \Lambda(f)$ is lower semi-continuous and equality holds in (7) (Prop. 2, *a*)), whereas if f is lower semi-continuous and Λ is only assumed to be scalarly essentially μ-integrable then

$$\nu^*(f) \leqslant \mu^\bullet\big(\Lambda^*(f)\big)$$

by item *a*) of Exer. 7 (see the note for V.17, ℓ. −6 to −4); it is puzzling (intriguing) that in two attempts to generalize the equality (4) (namely, item *b*) of the present proposition, where $f \in \mathscr{F}_+(\mathrm{X})$ is arbitrary but Λ is μ-pre-adequate and vaguely continuous; and Exer. 7, where f is lower semi-continuous but Λ is only assumed to be scalarly essentially μ-integrable), the inequality in (7) "goes the other way" from the one in Exer. 7. That equality holds when both sets of hypotheses are assumed teaches us nothing, since (4) is then immediate from the definition of μ-pre-adequacy.}

V.21, ℓ. 1.

"The mapping $t \mapsto \lambda_t^*(1)$ is measurable"

For, Λ is μ-pre-adequate and the constant function 1 is lower semi-continuous, therefore the function $t \mapsto \lambda_t^\bullet(1)$ is μ-measurable and $\lambda_t^\bullet(1) = \lambda_t^*(1)$; with notations as in the preceding note, $\Lambda(1) = \Lambda^*(1)$ pointwise on T and $\nu^\bullet(1) = \mu^\bullet\big(\Lambda(1)\big)$, that is, $\nu^*(1) = \mu^\bullet\big(\Lambda^*(1)\big)$.

V.21, ℓ. 2–4.

"The set $\mathfrak{K}$ of compact subsets of T such the restriction of $t \mapsto \lambda_t^*(1)$ to K is finite and continuous is therefore μ-dense "

Write $h(t) = \lambda_t^*(1)$ for $t \in \mathrm{T}$ and let

$$\mathrm{A} = \{t \in \mathrm{T} :\ h(t) < +\infty\}\,.$$

By assumption, $\complement\mathrm{A}$ is locally μ-negligible; in particular, $\complement\mathrm{A}$ is μ-measurable (p. IV.61, ℓ. −4), therefore so is A. Since h is μ-measurable, it follows that the (finite-valued) function $h|\mathrm{A}$ is μ-measurable in the sense of Ch. IV, §5, No. 10, Def. 8 (see c'') in the Note for IV.79, ℓ. 3,4). To say that a compact set K belongs to $\mathfrak{K}$ signifies that $\mathrm{K} \subset \mathrm{A}$ and $h|\mathrm{K} = (h|\mathrm{A})|\mathrm{K}$ is continuous; since $h|\mathrm{A}$ is μ-measurable, it follows that $\mathfrak{K}$ is μ-dense in A (*loc. cit.*, Prop. 15, criterion *a*)), and since $\mathrm{T} - \mathrm{A}$ is locally μ-negligible, $\mathfrak{K}$ is μ-dense in T (p. IV.77 ℓ. 3, 4).

V.21, ℓ. 6.

"The mapping Λ is μ_α-adequate for every $\alpha \in \mathrm{A}$ "

Recall the remark following No. 1, Def. 1, to the effect that in Nos. 2 and 3, in "a,b,c propositions", the parts *a*) and *b*) will be valid for Λ μ-pre-adequate, whereas part *c*) will require that Λ be μ-adequate. The present proposition illustrates the principle and its logic: in part c) arguments, it is necessary to express μ as a sum of 'better-behaved' measures $\mu' \leqslant \mu$, whence the need for μ-adequacy.

V.21, ℓ. 8, 9.

"ν_α is a bounded measure (because $\lambda_t^*(1)$ is bounded on $\mathrm{Supp}(\mu_\alpha)$)."

Since the constant function 1 is lower semi-continuous and Λ is μ_α-pre-adequate, writing $h(t) = \lambda_t^\bullet(1) = \lambda_t^*(1)$ for $t \in \mathrm{T}$ one has

$$\nu_\alpha^*(1) = \mu_\alpha^\bullet(h)$$

by (4) of No. 1. Write $\mathrm{S}_\alpha = \mathrm{Supp}(\mu_\alpha)$, so that $\mathrm{S}_\alpha \in \mathfrak{K}$. By the definition of $\mathfrak{K}$, we know that the restriction of h to the compact space K_α is finite-valued and continuous, hence bounded, say $h\big|\mathrm{S}_\alpha \leqslant \mathrm{M} < +\infty$. Then $0 \leqslant h\varphi_{\mathrm{S}_\alpha} \leqslant \mathrm{M}\varphi_{\mathrm{S}_\alpha}$, whence

$$\mu_\alpha^\bullet(h\varphi_{\mathrm{S}_\alpha}) \leqslant \mathrm{M}\,\mu_\alpha^\bullet(\varphi_{\mathrm{S}_\alpha}) \leqslant \mathrm{M}\,\mu_\alpha^*(\varphi_{\mathrm{S}_\alpha}) = \mathrm{M}\,\mu_\alpha(\mathrm{S}_\alpha) < +\infty$$

(actually $\mu^\bullet(\varphi_{\mathrm{S}_\alpha}) = \mu^*(\varphi_{\mathrm{S}_\alpha})$ by §1, No. 2, Prop. 7); but $\mathrm{T} - \mathrm{S}_\alpha$ is μ_α-negligible, therefore $h = h\varphi_{\mathrm{S}_\alpha}$ μ_α-almost everywhere and so

$$\nu_\alpha^*(1) = \mu_\alpha^\bullet(h) = \mu_\alpha^\bullet(h\varphi_{\mathrm{S}_\alpha}) < +\infty\,,$$

thus ν_α is indeed bounded (Ch. IV, §4, No. 7, Prop. 12).

V.21, ℓ. 11–14.

"... then

$$\int^\bullet f(x)\,d\nu_\alpha(x) \geqslant \int^\bullet d\mu_\alpha(t) \int^* f(x)\,d\lambda_t(x) = \int^\bullet d\mu_\alpha(t) \int^\bullet f(x)\,d\lambda_t(x)$$

(the last equality due to the fact that λ_t is bounded locally almost everywhere, and Prop. 7 of § 1)."

As in earlier notes, let us write $\Lambda(f)$ for the function $t \mapsto \lambda_t^\bullet(f)$ and $\Lambda^*(f)$ for the function $t \mapsto \lambda_t^*(f)$, so that $\Lambda(f) \leqslant \Lambda^*(f)$ pointwise on T by (1) of §1, No. 1.

For locally μ-almost every $t \in \mathrm{T}$ one has $\lambda_t^*(1) = \lambda_t^\bullet(1) < +\infty$ and so λ_t is bounded (Ch. IV, §4, No. 7, Prop. 12); but if λ_t is bounded then it is moderated (§1, No. 2, *Remark* 2), therefore f is λ_t-moderated (*loc. cit.*, comment following Def. 2) and so $\lambda_t^*(f) = \lambda_t^\bullet(f)$ (*loc. cit.*, Prop. 7). It follows that $\Lambda^*(f) = \Lambda(f)$ locally μ-almost everywhere, therefore locally μ_α-almost everywhere, and so $\mu_\alpha^\bullet\big(\Lambda^*(f)\big) = \mu_\alpha^\bullet\big(\Lambda(f)\big)$.

To summarize: Assuming $\lambda_t^\bullet(1) < +\infty$ locally μ-almost everywhere, one has (by §1, No. 2, Prop. 7, and (6))

$$\nu_\alpha^\bullet(f) = \nu_\alpha^*(f) \geqslant \mu_\alpha^\bullet\bigl(\Lambda^*(f)\bigr) = \mu_\alpha^\bullet\bigl(\Lambda(f)\bigr)$$

for every α and for every function $f \geqslant 0$ on X, and summing over α yields (8):

$$\nu^\bullet(f) \geqslant \mu^\bullet\bigl(\Lambda^*(f)\bigr) = \mu^\bullet\bigl(\Lambda(f)\bigr)$$

(§2, No. 2, Prop. 1). Note that the first equality in (6) is sharpened in (8) (because $\int^* \geqslant \int^\bullet$), and the second inequality in (6) becomes an equality in (8).

V.21, ℓ. 18–24.

"Corollary 1"

{Cor. 1, Prop. 4 and Prop. 5 are variations on the same general pattern; as the variations are numerous and subtle, it is helpful to have the pattern in mind as one thrashes through the details.

One is given a mapping $\Lambda : t \mapsto \lambda_t$ that is scalarly essentially μ-integrable, so that the measure $\nu = \int \lambda_t \, d\mu(t)$ is defined; certain conditions are imposed on f and ν, and one shows that the same conditions are satisfied for "most" of the λ_t. The progression from *a*) to *c*) tends to be from general to special:

(i) In assertion *a*), Λ is assumed to be μ-pre-adequate, and "most" means "for locally μ-almost every t".

(ii) To the hypotheses of *a*), assertion *b*) adds the assumption that Λ is vaguely continuous, and "most" is sharpened to "for μ-almost every t".

(iii) In assertion *c*), Λ is assumed to be μ-adequate, and the function $t \mapsto \lambda_t^\bullet(1) = \lambda_t^*(1)$ is assumed to be finite-valued locally μ-almost everywhere, in other words, the measure λ_t is assumed to be bounded for locally μ-almost every $t \in \mathrm{T}$; "most" means "for locally μ-almost every t".}

Let $\Lambda : t \mapsto \lambda_t$ be scalarly essentially μ-integrable and let $\nu = \int \lambda_t \, d\mu(t)$. Assume in *a*) and *b*) that Λ is μ-pre-adequate, and in *c*) that Λ is μ-adequate. Let $f \in \mathscr{F}_+(\mathrm{X})$ and set

$$\mathrm{H} = \{t \in \mathrm{T} : \ \lambda_t^*(f) > 0\} = \{t \in \mathrm{T} : \ \bigl(\Lambda^*(f)\bigr)(t) > 0\}.$$

a) Assuming $\nu^*(f) = 0$, we are to show that $\mu^\bullet(\mathrm{H}) = 0$, that is, f is λ_t-negligible for locally μ-almost every $t \in \mathrm{T}$. Citing *a*) of Prop. 3, we have

$$0 = \nu^*(f) \geqslant \mu^\bullet\bigl(\Lambda^*(f)\bigr) \geqslant 0,$$

therefore $\mu^\bullet\bigl(\Lambda^*(f)\bigr) = 0$, that is, $\Lambda^*(f) = 0$ locally μ-almost everywhere in T; in other words, $\mu^\bullet(\mathrm{H}) = 0$.

b) Assuming $\nu^*(f) = 0$ and Λ vaguely continuous, we are to show that $\mu^*(\mathrm{H}) = 0$, that is, f is λ_t-negligible for μ-almost every $t \in \mathrm{T}$. Citing *b*) of Prop. 3, we have

$$0 = \nu^*(f) \geqslant \mu^*\big(\Lambda^*(f)\big) \geqslant 0\,,$$

therefore $\mu^*\big(\Lambda^*(f)\big) = 0$, that is, $\Lambda^*(f) = 0$ μ-almost everywhere in T; in other words, $\mu^*(\mathrm{H}) = 0$.

c) Assuming $\nu^\bullet(f) = 0$ and $\lambda_t^\bullet(1) < +\infty$ for locally μ-almost every $t \in \mathrm{T}$, we are to show that $\mu^\bullet(\mathrm{H}) = 0$, that is, f is λ_t-negligible for locally μ-almost every $t \in \mathrm{T}$. Citing *c*) of Prop. 3, we have

$$0 = \nu^\bullet(f) \geqslant \mu^\bullet\big(\Lambda^*(f)\big) \geqslant 0\,,$$

therefore $\mu^\bullet\big(\Lambda^*(f)\big) = 0$, that is, $\Lambda^*(f) = 0$ locally μ-almost everywhere in T; in other words, $\mu^\bullet(\mathrm{H}) = 0$.

V.21, ℓ. -8 to -6.

"f_0 is then λ_t-negligible except for t forming a set that is locally μ-negligible (and even μ-negligible, if Λ is vaguely continuous) by Cor. 1, and the statement then follows at once."

Write $\mathrm{N} = \{t \in \mathrm{T} :\ \lambda_t^*(f_0) > 0\}$ for the set of all $t \in \mathrm{T}$ such that f_0 is not λ_t-negligible. Since f_0 is ν-negligible, it follows from *a*) of Cor. 1 that $\mu^\bullet(\mathrm{N}) = 0$; and when Λ is vaguely continuous, $\mu^*(\mathrm{N}) = 0$ by *b*) of Cor. 1.

Let $\mathrm{M} = \{t \in \mathrm{T} :\ f \text{ is not } \lambda_t\text{-moderated}\}$; we are to show that $\mu^\bullet(\mathrm{M}) = 0$, and that $\mu^*(\mathrm{M}) = 0$ when Λ is vaguely continuous; it will suffice to show that $\mathrm{M} \subset \mathrm{N}$.

Assume $t \notin \mathrm{N}$ and let us show that $t \notin \mathrm{M}$, i.e., that f is λ_t-moderated; by criterion *c*) of §1, No. 2, Prop. 5, it suffices to show that the set $\{x \in \mathrm{X} :\ f(x) > 0\}$ is contained in the union of a λ_t-negligible set and a sequence of compact sets. Indeed, $f(x) > 0$ if and only if $f_n(x) > 0$ for some index $n \geqslant 0$, thus $\{x :\ f(x) > 0\}$ is equal to the set

$$\{x :\ f_0(x) > 0\} \cup \bigcup_{n \geqslant 1} \{x :\ f_n(x) > 0\} \subset \{x :\ f_0(x) > 0\} \cup \bigcup_{n \geqslant 1} \mathrm{K}_n,$$

where $\{x :\ f_0(x) > 0\}$ is λ_t-negligible (because $t \notin \mathrm{N}$).

V.22, ℓ. 6, 7.

"... the function f is constant on the complement B of a countable union of ν-integrable open sets."

Let A be a ν-moderated set such that f is constant on $\mathrm{X} - \mathrm{A}$, and let (U_r) be a sequence of ν-integrable sets such that $\mathrm{A} \subset \bigcup_r \mathrm{U}_r$ (§1, No. 2, Prop. 5, *a*)). Then $X - \bigcup_r U_r \subset X - A$ and the set $B = X - \bigcup_r U_r$ meets the requirements. The reason for choosing criterion *a*) of Prop. 5 is that the set B is then closed, hence is a Borel set, hence is measurable with respect to any measure.

V.22, ℓ. 7–9.

"There exists a partition of $X - B$ formed by a ν-negligible set N and a sequence (K_n) of compact sets such that the restriction of f to each K_n is continuous."

With $X - B = \bigcup_r U_r$ as in the preceding note, let (A_r) be a 'disjointification' of the U_r, for example $A_1 = U_1$, $A_{r+1} = U_{r+1} - \bigcup_{s \leqslant r} U_s$, so that $X - B$ is partitioned by the sequence (A_r) of ν-integrable sets. The sets A_r need no longer be open, but their union is, and B remains a Borel set.

Consider first the case of a single ν-integrable set A. By Ch. IV, §4, No. 6, Cor. 2 of Th. 4, there exists a countable partition of A,

$$A = N_0 \cup \bigcup_i C_i ,$$

with N_0 ν-negligible and the C_i compact. Then, since f is ν-measurable, for each i there exists a countable partition

$$C_i = N_i \cup \bigcup_j K_{ij}$$

with N_i ν-negligible, the K_{ij} compact and $f|K_{ij}$ continuous (Ch. IV, §5, No. 1, Def. 1); setting $N = N_0 \cup \bigcup_i N_i$ and enumerating the K_{ij} into a sequence (K_n), we have a partition

$$A = N \cup \bigcup_n K_n$$

of the desired sort.

With $X - B = \bigcup_r A_r$ as above, for each r let $A_r = N_r \cup \bigcup_j K_{rj}$ be such a partition of A_r. The negligible set $N = \bigcup_r N_r$ and an enumeration (K_n) of the compact sets K_{rj} then yield a partition

$$X - B = N \cup \bigcup_n K_n$$

that meets the requirements of the assertion.

V.22, ℓ. 9–11.

"Let S be the set of $t \in \mathrm{T}$ such that N is not λ_t-negligible: S is locally μ-negligible (resp. μ-negligible) by Cor. 1 of Prop. 3."

Part *a*) (resp. Part *b*)) of Cor. 1 is being applied to the ν-negligible function φ_{N}, the set S being denoted H there.

V.22, ℓ. 11–13.

"The sets $\mathrm{K}_n, \mathrm{B}, \mathrm{N}$ are measurable for every measure on X, and the restriction of f to each of them is λ_t-measurable for every $t \notin \mathrm{S}$."

Since B and the K_n are closed sets, all sets in the formula

$$\mathrm{N} = (\mathrm{X} - \mathrm{B}) - \bigcup_n \mathrm{K}_n$$

are Borel sets, hence are 'universally measurable' (Ch. IV, §5, No. 4, Cor. 3 of Th. 2).

By construction, $f\big|\mathrm{B}$ is a constant function, say equal to $y_0 \in \mathrm{G}$ at every point of B; its extension to X by y_0 is constant on X, hence continuous, hence 'universally measurable', in particular measurable for every λ_t $(t \notin \mathrm{S})$.

The function $f\big|\mathrm{K}_n$ is continuous. Let $\mathfrak{K}$ be the set of all compact subsets of K_n. Obviously $\mathfrak{K}$ is closed under finite unions, thus $\mathfrak{K}$ satisfies the conditions $(\mathrm{PL_I})$ and $(\mathrm{PL_{II}})$ of Ch. IV, §5, No. 8, Prop. 12; moreover, condition *b*) of the cited Prop. 12 is satisfied with $\mathrm{A} = \mathrm{K}_n$, $\mathrm{K} = \mathrm{K}_0$, and any measure on X, therefore $\mathfrak{K}$ is dense in K_n for every measure on X (*loc. cit.*, Def. 6). Thus $\mathfrak{K}$ and $f\big|\mathrm{K}_n$ satisfy condition *a*) of Ch. IV, §5, No. 10, Prop. 15 with $\mathrm{A} = \mathrm{K}_n$, for every measure on X, therefore $f\big|\mathrm{K}_n$ is measurable for every measure on X (*loc. cit.*, Def. 8) and in particular for every λ_t $(t \notin \mathrm{S})$.

Let $t \notin \mathrm{S}$. Then N is λ_t-negligible (hence also locally λ_t-negligible), therefore $f\big|\mathrm{N}$ is λ_t-measurable (see the Note for IV.79, ℓ. $-17, -16$).

Thus, for every $t \notin \mathrm{S}$, the restriction of f to B, N and every K_n is λ_t-measurable.

V.22, ℓ. 13–14.

"The function f is therefore λ_t-measurable for every $t \notin \mathrm{S}$ (Ch. IV, §5, No. 10, Prop. 16)."

Continuing the preceding note: since the set consisting of B, N and the K_n is countable, *a fortiori* locally countable, and since

$$\mathrm{X} = \mathrm{B} \cup \mathrm{N} \cup \bigcup_n \mathrm{K}_n ,$$

it follows that $f = f|\mathrm{X}$ is λ_t-measurable for every $t \notin \mathrm{S}$ (Ch. IV, §5, No. 10, Prop. 16).

V.22, ℓ. −4 to −2.

"The assertions concerning the set N have already been established (Prop. 4, and Cor. 2 of Prop. 3)."

Let

$$\mathrm{L} = \{t \in \mathrm{T} : f \text{ is not } \lambda_t\text{-moderated}\}$$
$$\mathrm{M} = \{t \in \mathrm{T} : f \text{ is not } \lambda_t\text{-measurable}\};$$

then $t \in \mathrm{N}$ means that $t \notin \complement \mathrm{L} \cap \complement \mathrm{M}$, thus

$$\mathrm{N} = \mathrm{L} \cup \mathrm{M}.$$

Since f is ν-measurable and ν-moderated, $\mu^\bullet(\mathrm{L}) = 0$ by the cited Cor. 2, and $\mu^\bullet(\mathrm{M}) = 0$ by a) of Prop. 4, therefore $\mu^\bullet(\mathrm{N}) = 0$; if, moreover, Λ is vaguely continuous, then $\mu^*(\mathrm{L}) = 0$ by Cor. 2, and $\mu^*(\mathrm{M}) = 0$ by b) of Prop. 4, therefore $\mu^*(\mathrm{N}) = 0$.

V.22, ℓ. −2 to **V.23,** ℓ. 3.

"By Prop. 6 of §1, No. 2, we may limit ourselves to proving a) (resp. b)) in each of the following special cases:

1) The function f is ν-negligible.

2) There exists a compact set K such that f is zero outside K and the restriction of f to K is continuous."

*The argument is intricate and repetitious. To lighten the burden of referencing, some basic results are bundled together here; any of the items in (α) will be referenced simply by "(α)", and similarly for (β).

Let ρ be a positive measure on a locally compact space, and let h and h_n ($n \in \mathbf{N}$) be numerical functions $\geqslant 0$ on that space.

(α) If $h_n \uparrow h$ pointwise then

$$\rho^*(h_n) \uparrow \rho^*(h) \quad \text{and} \quad \rho^\bullet(h_n) \uparrow \rho^\bullet(h)$$

(Ch. IV, §1, No. 3, Th. 3 and Ch. V, §1, No. 1, Prop. 1). If, moreover, the h_n are ρ-measurable, then so is h (Ch. IV, §5, No. 4, Cor. 1 of Th. 2). If the h_n are ρ-moderated, then so is h (Ch. V, §1, No. 2, *Remark* 3).

(β) If the h_n are ρ-measurable and $h = \sum_{n\in\mathbf{N}} h_n$, then h is ρ-measurable: the measurability of $h_0 + h_1$ follows from Ch. IV, §5, No. 3, Th. 1, applied to the mapping $x \mapsto \big(h_0(x), h_1(x)\big)$ (x in the given space), with $u(a,b) = a + b$ for $(a,b) \in \mathrm{A}' \times \mathrm{A}'$, where $\mathrm{A}' = \mathbf{R} \cup \{+\infty\}$ (GT, IV, §4, No. 3, Prop. 7); the measurability of the finite sums $h_0 + \cdots + h_n$ follows at once, and $h = \sup_{n \in \mathbf{N}} (h_0 + \cdots + h_n)$ is then measurable by (α). Moreover (the h_n still measurable)

$$\rho^*(h) = \sum_{n \in \mathbf{N}} \rho^*(h_n) \quad \text{and} \quad \rho^\bullet(h) = \sum_{n \in \mathbf{N}} \rho^\bullet(h_n)$$

by Ch. IV, §5, No. 6, Cor. 4 of Th. 5 and Ch. V, §1, No. 1, Cor. of Prop. 2). If the h_n are ρ-moderated, then so is h (Ch. V, §1, No. 3, *Remark* 3).$_*$

By the cited Prop. 6, there exists a sequence f_n $(n = 0, 1, 2, \ldots)$ of numerical functions $\geqslant 0$ on X such that f_0 satisfies 1), and the f_n for $n \geqslant 1$ satisfy 2) *and are finite-valued.* Moreover, the f_n for $n \geqslant 1$ are ν-measurable and ν-moderated: for f_0, this is obvious, and it is obvious the f_n for $n \geqslant 1$ are moderated (for any measure on X). It remains to check that if $n \geqslant 1$ then f_n is measurable. Let K_n be a compact set in X such that $f_n | \mathrm{K}_n$ is continuous and $f_n = 0$ outside K_n, let $a \in \mathbf{R}$ and let

$$\mathrm{A} = \{x \in \mathrm{X} : \ f_n(x) \geqslant a\};$$

if $a \leqslant 0$ then $\mathrm{A} = \mathrm{X}$, while if $a > 0$ then $\mathrm{A} = \{x \in \mathrm{K}_n : \ f_n(x) \geqslant a\}$ is a closed set in K_n by the continuity of $f_n | \mathrm{K}_n$, hence is closed in X. Thus f_n is upper semi-continuous (GT, IV, §6, No. 2, Prop. 1 applied to $-f_n$), hence is measurable for every measure on X (Ch. IV, §5, No. 5, Cor. of Prop. 8).

Assuming it has been shown that for every n, the function

$$t \mapsto \int^\bullet f_n(x)\, d\lambda_t(x)$$

(denoted $\Lambda(f_n)$ in earlier notes) is μ-measurable, and that

(i) $$\nu^\bullet(f_n) = \mu^\bullet\big(\Lambda(f_n)\big),$$

and, when Λ is vaguely continuous, that the function $\Lambda^*(f_n) : t \mapsto \lambda_t^*(f_n)$ is μ-measurable and μ-moderated, and satisfies

(ii) $$\nu^*(f_n) = \mu^*\big(\Lambda^*(f_n)\big),$$

we must verify that the same is true of f.

Verification of a) *for* f *assuming it has been verified for the* f_n *:* For every $n \in \mathbf{N}$, one shows as in the preceding Note that f_n is μ-measurable and μ-moderated for locally μ-almost every t; it follows that, for locally

μ-almost every t, f_n is μ-measurable and μ-moderated for every n. For every n, define

$$g_n = \sum_{k=0}^{n} f_k \,;$$

since $f_0, \ldots, f_n$ are λ_t-measurable and λ_t-moderated for locally μ-almost every t, the same is true of g_n and

$$\lambda_t^\bullet(g_n) = \sum_{k=0}^{n} \lambda_t^\bullet(f_k) \quad \text{for locally } \mu\text{-almost every } t \in \mathrm{T}$$

by (β). Thus $\Lambda(g_n) = \sum_{k=0}^{n} \Lambda(f_k)$ locally μ-almost everywhere on T, consequently

$$\text{(iii)} \qquad \mu^\bullet\big(\Lambda(g_n)\big) = \sum_{k=0}^{n} \mu^\bullet\big(\Lambda(f_k)\big) = \sum_{k=0}^{n} \nu^\bullet(f_k) = \nu^\bullet(g_n) \quad \text{for all } n$$

by (β) and (i). Since $g_n \uparrow f$ pointwise on T, for every $t \in \mathrm{T}$ one has $\lambda_t^\bullet(g_n) \uparrow \lambda_t^\bullet(f)$ by (α); thus $\Lambda(g_n) \uparrow \Lambda(f)$ pointwise on T, therefore

$$\text{(iv)} \qquad \mu^\bullet\big(\Lambda(f)\big) = \sup_n \mu^\bullet\big(\Lambda(g_n)\big)$$

by (α), and similarly $\nu^\bullet(f) = \sup_n \nu^\bullet(g_n)$. From (iii) and (iv) we see that

$$\nu^\bullet(f) = \sup_n \nu^\bullet(g_n) = \sup_n \mu^\bullet\big(\Lambda(g_n)\big) = \mu^\bullet\big(\Lambda(f)\big).$$

Since, by assumption, the $\Lambda(f_n)$ are μ-measurable, so are the $\sum_{k=0}^{n} \Lambda(f_k)$ by (β), therefore so are the $\Lambda(g_n)$ (Ch. IV, §5, No. 2, Prop. 6), therefore so is $\Lambda(f)$ by (α). And since every g_n is μ-measurable and μ-moderated, so is $f = \sup_n g_n$ by (α).

Verification of b) *for* f *assuming it has been verified for the* f_n*:* Assume in addition that Λ is vaguely continuous. For every n, one shows as in the preceding Note that for μ-almost every t, f_n is λ_t-measurable and λ_t-moderated; it follows that there exists a μ-negligible subset P of T such that f_n is λ_t-measurable and λ_t-moderated for all n and for all $t \in \mathrm{T} - \mathrm{P}$. In particular,

$$\lambda_t^*(g_n) = \sum_{k=0}^{n} \lambda_t^*(f_k) \quad \text{for all } n \text{ and for all } t \in \mathrm{T} - \mathrm{P}$$

by (β), thus $\Lambda^*(g_n) = \sum_{k=0}^{n} \Lambda^*(f_k)$ μ-almost everywhere for all n; since the $\Lambda^*(f_n)$ are by assumption μ-measurable and μ-moderated, it follows that $\sum_{k=0}^{n} \Lambda^*(f_k)$ is μ-measurable and μ-moderated by (β), therefore $\Lambda^*(g_n)$ is μ-measurable by Ch. IV, §5, No. 2, Prop. 6, and is obviously μ-moderated. Moreover,

$$\mu^*\big(\Lambda^*(g_n)\big) = \sum_{k=0}^{n} \mu^*\big(\Lambda^*(f_k)\big) \quad \text{for all } n$$

by (β); but

$$\sum_{k=0}^{n} \mu^*\big(\Lambda^*(f_k)\big) = \sum_{k=0}^{n} \nu^*(f_k) = \nu^*(g_n)$$

by (ii) and (β), therefore

(v) $$\mu^*\big(\Lambda^*(g_n)\big) = \nu^*(g_n) \quad \text{for all } n\,.$$

Since $g_n \uparrow f$ pointwise on T, it follows from (α) that

$$\lambda_t^*(g_n) \uparrow \lambda_t^*(f) \quad \text{for every } t \in \mathrm{T}\,,$$

thus $\Lambda^*(g_n) \uparrow \Lambda^*(f)$ pointwise on T; since the $\Lambda^*(g_n)$ are μ-measurable and μ-moderated, by (α) so is $\Lambda^*(f)$, and

(vi) $$\mu^*\big(\Lambda^*(f)\big) = \sup_n \mu^*\big(\Lambda^*(g_n)\big).$$

Finally,

$$\nu^*(f) = \sup_n \nu^*(g_n) = \sup_n \mu^*\big(\Lambda^*(g_n)\big) = \mu^*\big(\Lambda^*(f)\big)$$

by (α), (v) and (vi).

V.23, ℓ. 4.

"The special case 1) has already been treated (Cor. 1 of Prop. 3)."

Assuming f is ν-negligible (hence ν-measurable and ν-moderated), we are to show that *a*) and *b*) are satisfied. Let

$$\mathrm{N} = \{t \in \mathrm{T} : \ f \text{ is not both } \lambda_t\text{-measurable and } \lambda_t\text{-moderated}\}.$$

a) The assumption that f is ν-moderated is redundant. Let

$$\mathrm{H} = \{t \in \mathrm{T} : \ f \text{ is not } \lambda_t\text{-negligible}\};$$

by part *a*) of Cor. 1 of Prop. 3, H is locally μ-negligible; since a negligible function is both measurable and moderated, one has $\mathrm{N} \subset \mathrm{H}$, therefore N is also locally μ-negligible. {The information about N is redundant here, but N was not yet in the picture in the cited Cor. 1.}

If $t \notin \mathrm{H}$ then f is λ_t-negligible and, *a fortiori*, locally λ_t-negligible, thus $\lambda_t^*(f)$ and $\lambda_t^\bullet(f)$ are both equal to 0; the functions

$$\Lambda(f) : t \mapsto \lambda_t^\bullet(f) \quad \text{and} \quad \Lambda^*(f) : t \mapsto \lambda_t^*(f)$$

are therefore both locally μ-negligible, hence are μ-measurable, and

$$\mu^\bullet\bigl(\Lambda(f)\bigr) = 0 = \mu^\bullet\bigl(\Lambda^*(f)\bigr).$$

But f is ν-negligible, therefore $\nu^\bullet(f) = \nu^*(f) = 0$. In particular,

$$\nu^\bullet(f) = 0 = \mu^\bullet\bigl(\Lambda(f)\bigr),$$

thus f satisfies (9).

b) If, moreover, Λ is vaguely continuous, then the set H is μ-negligible by part *b*) of Cor. 1 of Prop. 3, hence so is N. Since f is λ_t-negligible for $t \notin \mathrm{H}$, $\Lambda^*(f) = 0$ μ-almost everywhere; thus $\Lambda^*(f)$ is μ-negligible, hence μ-moderated, and

$$\nu^*(f) = 0 = \mu^*\bigl(\Lambda^*(f)\bigr),$$

thus f satisfies (10).

V.23, ℓ. 8, 9.

"Moreover, f, g, h are ν-integrable."

Recall that f can be assumed to be finite-valued (§1, No. 2, Prop. 6). The functions f, g, h are all positive; being semi-continuous, they are (universally) measurable, and since $0 \leqslant f \leqslant h = \mathrm{M}\varphi_{\mathrm{G}}$,

$$\nu^*(f) \leqslant \mathrm{M}\,\nu^*(\mathrm{G}) < +\infty$$

shows that h and f are ν-integrable, therefore so is $g = h - f$.

V.23, ℓ. 11–13.

"By subtraction, we see that the function

$$t \mapsto \int^\bullet f(x)\,d\lambda_t(x) \quad \bigl(\text{resp. } \int^* f(x)\,d\lambda_t(x)\bigr)$$

is μ-measurable and that the formula (9) (resp. (10)) holds."

Verification of a). Since g and h are lower semi-continuous functions $\geqslant 0$, and Λ is μ-pre-adequate, it follows from Def. 1 that the functions

$$\Lambda(g) : t \mapsto \lambda_t^{\bullet}(g) \quad \text{and} \quad \Lambda(h) : t \mapsto \lambda_t^{\bullet}(h)$$

are μ-measurable and satisfy (4):

$$\nu^{\bullet}(g) = \mu^{\bullet}\big(\Lambda(g)\big) \quad \text{and} \quad \nu^{\bullet}(h) = \mu^{\bullet}\big(\Lambda(h)\big).$$

Since g is ν-integrable, $\mu^{\bullet}\big(\Lambda(g)\big) = \nu^*(g) < +\infty$, thus the μ-measurable function $\Lambda(g)$ is essentially μ-integrable, and similarly for $\Lambda(h)$. Write

$$\mathrm{N}_g = \{t \in \mathrm{T} : \ g \text{ is not both } \lambda_t\text{-measurable and } \mu\text{-moderated}\}.$$

As shown at the outset, $\mu^{\bullet}(\mathrm{N}_g) = 0$, and similarly $\mu^{\bullet}(\mathrm{N}_h) = 0$. For $t \in \complement(\mathrm{N}_g \cup \mathrm{N}_h)$, g and h are λ_t-measurable, hence so is $f = h - g$, therefore (§1, No. 1, Prop. 2)

$$\lambda_t^{\bullet}(h) = \lambda_t^{\bullet}(f+g) = \lambda_t^{\bullet}(f) + \lambda_t^{\bullet}(g),$$

thus $\Lambda(h) = \Lambda(f) + \Lambda(g)$ locally μ-almost everywhere. Since $\Lambda(g)$ and $\Lambda(h)$ are essentially μ-integrable, so is $\Lambda(f)$; for, if $u, v \in \mathscr{L}^1(\mu)$ are equal, locally μ-almost everywhere, to $\Lambda(g)$ and $\Lambda(h)$, respectively (§1, No. 3, Def. 3), then, for locally μ-almost every $t \in \mathrm{T}$,

$$v(t) = \lambda_t^{\bullet}(f) + u(t),$$

therefore $\Lambda(f) = v - u$ locally μ-almost everywhere, where $v - u \in \mathscr{L}^1(\mu)$, and so

$$\Lambda(f) = (v-u) + \big(\Lambda(f) - (v-u)\big) \in \mathscr{L}^1(\mu) + \mathscr{N}^{\infty} = \overline{\mathscr{L}}^1(\mu).$$

In particular $\Lambda(f), \Lambda(g), \Lambda(h)$ are μ-measurable functions $\geqslant 0$, therefore

$$\mu^{\bullet}\big(\Lambda(h)\big) = \mu^{\bullet}\big(\Lambda(f)\big) + \mu^{\bullet}\big(\Lambda(g)\big)$$

(§1, No. 1, Prop. 2), whence

$$\mu^{\bullet}\big(\Lambda(f)\big) = \mu^{\bullet}\big(\Lambda(h)\big) - \mu^{\bullet}\big(\Lambda(g)\big) = \nu^{\bullet}(h) - \nu^{\bullet}(g) = \nu(h) - \nu(g) = \nu(f),$$

thus f satisfies (9).

Verification of b). If, moreover, Λ is vaguely continuous, it follows from *a*) of No. 1, Prop. 2 that the functions

$$\Lambda^*(g) : t \mapsto \lambda_t^*(g) \quad \text{and} \quad \Lambda^*(h) : t \mapsto \lambda_t^*(h)$$

are μ-measurable and satisfy (5):

$$\nu^*(g) = \mu^*\big(\Lambda^*(g)\big) \quad \text{and} \quad \nu^*(h) = \mu^*\big(\Lambda^*(h)\big),$$

and the ν-integrability of g and h implies that that $\Lambda^*(g)$ and $\Lambda^*(h)$ are μ-integrable (Ch. IV, §5, No. 6, Th. 5).

As shown at the outset, g is λ_t-measurable and λ_t-moderated for μ-almost every t, and similarly for h, so the same is true of $f = h - g$, and

$$\lambda_t^*(h) = \lambda_t^*(f) + \lambda_t^*(g);$$

thus $\Lambda^*(h) = \Lambda^*(f) + \Lambda^*(g)$ μ-almost everywhere. Since $\Lambda^*(g)$ and $\Lambda^*(h)$ are μ-integrable, they are finite μ-almost everywhere (Ch. IV, §2, No. 3, Prop. 7), hence there exist functions $u, v \in \mathscr{L}^1(\mu)$ such that $\Lambda(g)$ and $\Lambda(h)$ are equal, μ-almost everywhere, to u and v, respectively. Then, for μ-almost every $t \in \mathrm{T}$,

$$v(t) = \lambda_t^*(f) + u(t),$$

therefore $\Lambda^*(f)$ is equal μ-almost everywhere to the μ-measurable function $v - u$, hence is μ-measurable (Ch. IV, §5, No. 2, Prop. 6). It follows that

$$\mu^*\big(\Lambda^*(h)\big) = \mu^*\big(\Lambda^*(f)\big) + \mu^*\big(\Lambda^*(g)\big)$$

(Ch. IV, §5, No. 6, Cor. 4 of Th. 5), therefore

$$\mu^*\big(\Lambda^*(f)\big) = \mu^*\big(\Lambda^*(h)\big) - \mu^*\big(\Lambda^*(g)\big) = \nu^*(h) - \nu^*(g) = \nu(f) < +\infty,$$

thus $\Lambda^*(f)$ is μ-integrable (Ch. IV, §5, No. 6, Th. 5), hence is μ-moderated (§1, No. 2, Prop. 7), and satisfies (10).

Thus parts *a*) and *b*) of Prop. 5 are proved in full generality.

V.23, ℓ. −12.

" f is ν_α-measurable and ν_α-moderated"

Because f is ν-measurable and $0 \leqslant \nu_\alpha \leqslant \nu$; and because ν_α is bounded (§1, No. 2, *Remark* 2).

V.23, ℓ. −9.

"It remains only to sum on α, applying Props. 1 and 2 of §2, No. 2."

By the cited Prop. 1, f satisfies (9).

By the cited Prop. 2, the function $\Lambda(f) : t \mapsto \lambda_t^\bullet(f)$ is μ-measurable.

By assertion *a*), N is locally μ_α-negligible for all α, therefore N is locally μ-negligible by Cor. 2 of the cited Prop. 1.

V.23, ℓ. $-2, -1$.

"This follows at once from Prop. 5 and the criterion for integrability (Ch. IV, §5, No. 6, Th. 5)."

Since $\mathbf{f}$ is ν-measurable and ν-moderated, so is $|\mathbf{f}|$, and $\nu^\bullet(|\mathbf{f}|) = \nu^*(|\mathbf{f}|)$ (§1, No. 2, Prop. 7). By Prop. 5, *a*), $\nu^\bullet(|\mathbf{f}|) = \mu^\bullet\big(\Lambda(|\mathbf{f}|)\big)$, where $\Lambda(|\mathbf{f}|)$ denotes the function $t \mapsto \lambda_t^\bullet(|\mathbf{f}|)$ on T. By the cited Th. 5,

$$\begin{aligned} \mathbf{f} \text{ is } \nu\text{-integrable} \quad &\Leftrightarrow \quad \nu^*(|\mathbf{f}|) < +\infty \\ &\Leftrightarrow \quad \nu^\bullet(|\mathbf{f}|) < +\infty \\ &\Leftrightarrow \quad \mu^\bullet\big(\Lambda(|\mathbf{f}|)\big) < +\infty\,, \end{aligned}$$

whence the Corollary.

Another way of packaging the Corollary:

COROLLARY$'$. — *Let* $\mathbf{f}$ *be a function defined on* X, *with values in a Banach space* F *or in* $\overline{\mathbf{R}}$, *suppose* $\Lambda : t \mapsto \lambda_t$ *is a* μ*-pre-adequate mapping* $\mathrm{T} \mapsto \mathscr{M}_+(\mathrm{X})$, *and let* $\nu = \int \lambda_t\, d\mu(t)$.

Then $\mathbf{f}$ *is* ν*-integrable if and only if it is* ν*-measurable,* ν*-moderated and satisfies* $\mu^\bullet\big(\Lambda(\mathbf{f})\big) < +\infty$.

V.24, ℓ. $13, 14$.

"This statement is true when $\mathbf{f}$ is a positive numerical function (Prop. 5)"

a) Assuming $\mathbf{f} \geqslant 0$ and ν-integrable, let

$$\mathrm{N} = \{t \in \mathrm{T} : \ \mathbf{f} \text{ is not both } \lambda_t\text{-measurable and } \lambda_t\text{-moderated}\}.$$

Since $\mathbf{f}$ is both ν-measurable and ν-moderated (§1, No. 2, Prop. 7, 1)), by part *a*) of Prop. 5, N is locally μ-negligible, the function $\Lambda(\mathbf{f}) : t \mapsto \lambda_t^\bullet(\mathbf{f})$ is μ-measurable, and

$$(*) \qquad \mu^\bullet\big(\Lambda(\mathbf{f})\big) = \nu^\bullet(\mathbf{f}) = \nu^*(\mathbf{f}) < +\infty\,,$$

therefore $\Lambda(\mathbf{f})$ is essentially μ-integrable (§1, No. 3, Prop. 9). It follows that $\Lambda(\mathbf{f})$ is finite locally μ-almost everywhere, that is, the set

$$\mathrm{P} = \{t \in \mathrm{T} : \ \lambda_t^\bullet(\mathbf{f}) = +\infty\}$$

is locally μ-negligible. If $t \notin \mathrm{P} \cup \mathrm{N}$ then $\mathbf{f}$ is λ_t-measurable, λ_t-moderated, and $\lambda_t^*(\mathbf{f}) = \lambda_t^\bullet(\mathbf{f}) < +\infty$, therefore $\mathbf{f}$ is λ_t-integrable, that is, $t \notin \mathrm{H}$. Thus $\mathrm{H} \subset \mathrm{P} \cup \mathrm{N}$, hence $\mu^\bullet(\mathrm{H}) = 0$.

Since $\Lambda(\mathbf{f})$ is essentially integrable, notational conventions permit writing $(*)$ as

$$\int \mathbf{f}(x)\,d\nu(x) = \int \Lambda(\mathbf{f})\,d\mu = \int d\mu(t) \int \mathbf{f}(x)\,d\lambda_t(x)$$

(§1, No. 3, Def. 3), that is, (11) holds.

Writing $\Lambda^*(\mathbf{f})$ for the function $t \mapsto \lambda_t^*(\mathbf{f})$, since $\mathbf{f}$ is λ_t-moderated for locally μ-almost every t, one has $\Lambda^*(\mathbf{f}) = \Lambda(\mathbf{f})$ locally μ-almost everywhere (§1, No. 2, Prop. 7), therefore $\Lambda^*(\mathbf{f})$ is also essentially μ-integrable, and

$$\mu^\bullet\big(\Lambda^*(\mathbf{f})\big) = \mu^\bullet\big(\Lambda(\mathbf{f})\big) = \nu^\bullet(\mathbf{f}) = \nu^*(\mathbf{f})\,.$$

b) Suppose, in addition, that Λ is vaguely continuous. Then by *b*) of Prop. 5, N is μ-negligible and $\Lambda^*(\mathbf{f})$ is μ-moderated; since $\Lambda^*(\mathbf{f})$ is essentially μ-integrable and μ-moderated, it is μ-integrable (§1, No. 3, Cor. of Prop. 9), hence is finite μ-almost everywhere (Ch. IV, §2, No. 3, Prop. 7). Let

$$\mathrm{P}^* = \{t \in \mathrm{T} :\ \lambda_t^*(\mathbf{f}) = +\infty\,\}\,,$$

which is a μ-negligible set (incidentally, the inclusion $\mathrm{P} \subset \mathrm{P}^*$ is useless here). If $t \notin \mathrm{P}^* \cup \mathrm{N}$ then $\mathbf{f}$ is λ_t-measurable and $\lambda_t^*(\mathbf{f}) < +\infty$, hence $\mathbf{f}$ is λ_t-integrable; therefore $\mathrm{H} \subset \mathrm{P}^* \cup \mathrm{N}$, whence H is μ-negligible.

Finally, the $\int$ signs in (11) may be interpreted as ordinary integrals (rather than essential integrals)—in the case of $\int \mathbf{f}(x) d\lambda_t(x)$, for μ-almost every t (namely, for $t \notin \mathrm{H}$).

V.24, ℓ. 16.

"...extends at once to $\mathbf{f}$ by subtraction."

At first glance 'obvious', it heralds a new phenomenon: heretofore, symbols such as

$$\Lambda(\mathbf{f}) : t \mapsto \lambda_t^\bullet(\mathbf{f})\,, \quad \Lambda^*(\mathbf{f}) : t \mapsto \lambda_t^*(\mathbf{f})\,, \quad \Lambda(\mathbf{f}) : t \mapsto \lambda_t(\mathbf{f})$$

have (in these notes) denoted numerical functions that are defined everywhere on T, either because $\mathbf{f} \geqslant 0$ or because $\mathbf{f} \in \mathscr{K}(\mathrm{X})$ (No. 1); even in the Cor. of Prop. 5 of No. 2, it is $\Lambda(|\mathbf{f}|)$ that figures in the proof, not $\Lambda(\mathbf{f})$. From now on, we have to deal with functions $t \mapsto \int \mathbf{f}\,d\lambda_t$ that are defined only on the set of $t \in \mathrm{T}$ such that $\mathbf{f} \in \overline{\mathscr{L}}^1(\lambda_t)$ (or $\mathscr{L}^1(\lambda_t)$), and which may have values in a Banach space.

A review of the core results on numerical functions that are involved:

(i) $\mathbf{f}$ is measurable $\Leftrightarrow$ $\mathbf{f}^+$ and $\mathbf{f}^-$ are measurable, in which case $|\mathbf{f}| = \mathbf{f}^+ + \mathbf{f}^-$ is measurable (Ch. IV, §5, No. 3, Cors. 2 and 3 of Th. 1).

(ii) $\mathbf{f}$ is integrable $\Leftrightarrow$ $\mathbf{f}$ is measurable and $|\mathbf{f}|$ is integrable (by (i) and *loc. cit.*, No. 6, Th. 5).

(iii) $\mathbf{f}$ is integrable $\Leftrightarrow$ $\mathbf{f}^+$ and $\mathbf{f}^-$ are integrable, in which case $\int \mathbf{f} = \int \mathbf{f}^+ - \int \mathbf{f}^-$.

The equivalence is clear from (i), (ii) and the fact that when $\mathbf{f}$ is measurable, $\int^* |\mathbf{f}| = \int^* \mathbf{f}^+ + \int^* \mathbf{f}^-$ (*loc. cit.*, No. 6, Cor. 4 of Th. 5). If $\mathbf{f}$ is integrable, by definition there exists a (finite-valued) function $\mathbf{g} \in \mathscr{L}^1$ such that $\mathbf{f} = \mathbf{g}$ almost everywhere, whence $\mathbf{f}^+ = \mathbf{g}^+$ and $\mathbf{f}^- = \mathbf{g}^-$ almost everywhere, and so $\int \mathbf{f} = \int \mathbf{g} = \int \mathbf{g}^+ - \int \mathbf{g}^- = \int \mathbf{f}^+ - \int \mathbf{f}^-$ by the linearity of integration on $\mathscr{L}^1$.

(iv) $\mathbf{f}$ is essentially integrable $\Leftrightarrow$ $\mathbf{f}$ is measurable and $|\mathbf{f}|$ is essentially integrable (§1, No. 3, Prop. 9).

(v) $\mathbf{f}$ is essentially integrable $\Leftrightarrow$ $\mathbf{f}^+$ and $\mathbf{f}^-$ are essentially integrable, in which case $\int \mathbf{f} = \int \mathbf{f}^+ - \int \mathbf{f}^-$.

The equivalence is clear from (i), (iv) and the fact that when $\mathbf{f}$ is measurable, $\int^\bullet |\mathbf{f}| = \int^\bullet \mathbf{f}^+ + \int^\bullet \mathbf{f}^-$ (§1, No. 1, Prop. 2). The formula for $\int \mathbf{f}$ is proved by the argument of (iii) with "almost everywhere" replaced by "locally almost everywhere".

Proof of a) for a numerical function $\mathbf{f}$. Assuming $\mathbf{f}$ ν-integrable, let

$$\begin{aligned} \mathrm{H}^+ &= \{t \in \mathrm{T} : \ \mathbf{f}^+ \text{ is not } \lambda_t\text{-integrable}\}, \\ \mathrm{H}^- &= \{t \in \mathrm{T} : \ \mathbf{f}^- \text{ is not } \lambda_t\text{-integrable}\}; \end{aligned}$$

since $\mathbf{f}$ is λ_t-integrable if and only if $\mathbf{f}^+$ and $\mathbf{f}^-$ are λ_t-integrable, one has

$$\begin{aligned} \complement \mathrm{H}^+ \cap \complement \mathrm{H}^- &= \{t \in \mathrm{T} : \ \mathbf{f}^+ \text{ and } \mathbf{f}^- \text{ are } \lambda_t\text{-integrable}\} \\ &= \{t \in \mathrm{T} : \ \mathbf{f} \text{ is } \lambda_t\text{-integrable}\} = \complement \mathrm{H}, \end{aligned}$$

whence $\mathrm{H} = \mathrm{H}^+ \cup \mathrm{H}^-$. Since $\mathbf{f}^+$ and $\mathbf{f}^-$ are ν-integrable, we know from the case of functions $\geqslant 0$ that H^+ and H^- are locally μ-negligible, hence so is H; that the functions

$$\Lambda(\mathbf{f}^+) : t \mapsto \lambda_t^\bullet(\mathbf{f}^+), \quad \Lambda(\mathbf{f}^-) : t \mapsto \lambda_t^\bullet(\mathbf{f}^-),$$

defined everywhere on T, are essentially μ-integrable; and that

$$\nu(\mathbf{f}^+) = \mu^\bullet\big(\Lambda(\mathbf{f}^+)\big), \quad \nu(\mathbf{f}^-) = \mu^\bullet\big(\Lambda(\mathbf{f}^-)\big).$$

With $\int$ denoting essential integral, define $\Lambda(\mathbf{f}) : \complement \mathrm{H} \to \mathbf{R}$ by

$$\big(\Lambda(\mathbf{f})\big)(t) = \int \mathbf{f}\, d\lambda_t = \int \mathbf{f}^+\, d\lambda_t - \int \mathbf{f}^-\, d\lambda_t = \lambda_t^\bullet(\mathbf{f}^+) - \lambda_t^\bullet(\mathbf{f}^-) \qquad (t \in \complement \mathrm{H}),$$

that is, $\Lambda(\mathbf{f}) = \Lambda(\mathbf{f}^+)\big|\complement\mathrm{H} - \Lambda(\mathbf{f}^-)\big|\complement\mathrm{H}$. We are to show that the function $\Lambda(\mathbf{f})$, defined locally μ-almost everywhere in T, is essentially μ-integrable in the sense of §1, No. 3, and that $\int \Lambda(\mathbf{f})\,d\mu = \nu(\mathbf{f})$.

Since $\Lambda(\mathbf{f}^+)$ and $\Lambda(\mathbf{f}^-)$ are essentially μ-integrable, there exist functions $h, k \in \mathscr{L}^1(\mu)$ such that $\Lambda(\mathbf{f}^+) = h$ and $\Lambda(\mathbf{f}^-) = k$ locally μ-almost everywhere (§1, No. 3, Def. 3); since H is locally μ-negligible, it follows that $\Lambda(\mathbf{f}) = h - k$ locally μ-almost everywhere, therefore $\Lambda(\mathbf{f})$ is essentially μ-integrable in the sense of §1, No. 3 (p. V.9, ℓ. 1–5) and

$$\begin{aligned}\int \Lambda(f)\,d\mu = \int (h-k)\,d\mu &= \int h\,d\mu - \int k\,d\mu \\ &= \int \Lambda(\mathbf{f}^+)\,d\mu - \int \Lambda(\mathbf{f}^-)\,d\mu \\ &= \nu(\mathbf{f}^+) - \nu(\mathbf{f}^-) = \nu(\mathbf{f}),\end{aligned}$$

thus (11) is satisfied.

{Incidentally, the functions $u = \varphi_{\complement\mathrm{H}}\Lambda(\mathbf{f}^+)$ and $v = \varphi_{\complement\mathrm{H}}\Lambda(\mathbf{f}^-)$ are the extensions by 0 of $\Lambda(\mathbf{f}^+)\big|\complement\mathrm{H}$ and $\Lambda(\mathbf{f}^-)\big|\complement\mathrm{H}$ to T; they are finite-valued, μ-measurable, and, locally μ-almost everywhere, $\Lambda(\mathbf{f}^+) = u$ and $\Lambda(\mathbf{f}^-) = v$, thus $u, v \in \overline{\mathscr{L}}^1(\mu)$ (§1, No. 3, Prop. 9). Moreover, the extension by 0 of $\Lambda(\mathbf{f})$ to T is equal pointwise to $u - v$. Thus, the extension by 0 of $\Lambda(\mathbf{f})$ to T belongs to $\overline{\mathscr{L}}^1(\mu)$ and may be substituted for $\Lambda(\mathbf{f})$ in (11).}

Proof of b) for a numerical function $\mathbf{f}$. Assuming in addition that Λ is vaguely continuous, the argument proceeds as in *a*), with "locally negligible" replaced by "negligible"; "locally almost everywhere" by "almost everywhere"; "essentially integrable" by "integrable"; and $\overline{\mathscr{L}}^1(\mu)$ by $\mathscr{L}^1(\mu)$.

V.24, ℓ. 19, 20.

"...the result pertaining to real functions implies at once the validity of the statement for the elements of $\mathscr{H}$."

By linearity it suffices to consider $\mathbf{f} \in \mathscr{H}$ of the form $\mathbf{f} = \mathbf{a}f$ (the function $x \mapsto f(x)\cdot\mathbf{a}$), where $f \in \mathscr{K}(\mathrm{X})$ and $\mathbf{a} \in \mathrm{F}$. One knows that $\mathbf{f}$ is integrable for every measure on X (Ch. IV, §3, No. 5, Cor. 2 of Th. 4); in particular, for every $t \in \mathrm{T}$, $\mathbf{f}$ is λ_t-integrable and

$$\int \mathbf{f}\,d\lambda_t = \left(\textstyle\int f\,d\lambda_t\right)\cdot\mathbf{a} = \lambda_t(f)\cdot\mathbf{a}$$

(Ch. IV, §4, No. 2, Cor. 2 of Th. 1). Thus, for such an $\mathbf{f}$, $\mathrm{H} = \varnothing$ and, writing $\Lambda(\mathbf{f})$ for the vector-valued function $t \mapsto \int \mathbf{f}\,d\lambda_t$ ($t \in \mathrm{T}$), one has

$$\Lambda(\mathbf{f}) = \mathbf{a}\Lambda(f)$$

pointwise on T, where $\Lambda(f)$ is the real-valued function $t \mapsto \lambda_t(f)$ $(t \in \mathrm{T})$. By the numerical case of *a*) (resp. *b*)) already treated, $\Lambda(f)$ is essentially μ-integrable (resp. μ-integrable, when Λ is vaguely continuous) and

$$\nu(f) = \int \Lambda(f)\, d\mu\,.$$

Being finite-valued, $\Lambda(f)$ belongs to $\overline{\mathscr{L}}^1_{\mathbf{R}}(\mu)$ (resp. $\mathscr{L}^1_{\mathbf{R}}(\mu)$). Let $h \in \mathscr{L}^1_{\mathbf{R}}(\mu)$ with $\Lambda(f) = h$ locally μ-almost everywhere (resp. μ-almost everywhere; when Λ is vaguely continuous, $h = \Lambda(f)$ will serve). Then $\int \Lambda(f)\, d\mu = \int h\, d\mu$, thus

$$\nu(f) = \int h\, d\mu\,.$$

Now, $\mathbf{a}h$ is μ-integrable and

$$\int \mathbf{a}h\, d\mu = \left(\textstyle\int h\, d\mu\right) \cdot \mathbf{a}$$

(*loc. cit.*, Cor. 2 of Th. 1); since $\Lambda(\mathbf{f}) = \mathbf{a}h$ locally μ-almost everywhere (resp. μ-almost everywhere), $\Lambda(\mathbf{f})$ is essentially μ-integrable (resp. μ-integrable) and

$$\int \Lambda(\mathbf{f})\, d\mu = \int \mathbf{a}h\, d\mu = \left(\textstyle\int h\, d\mu\right) \cdot \mathbf{a} = \nu(f) \cdot \mathbf{a} = \nu(\mathbf{f})\,,$$

that is, (11) holds for $\mathbf{f}$.

V.24, ℓ. 20, 21.

“ $\mathscr{H}$ is dense in $\mathscr{L}^1_{\mathrm{F}}(\nu)$ ”

Ch. IV, §3, No. 5, Prop. 10.

V.24, ℓ. 21–26.

“…for every $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\nu)$, there exists a sequence $(\mathbf{f}_n)$ of elements of $\mathscr{H}$ that has the following properties:

1) the sequence $(\mathbf{f}_n)$ converges to $\mathbf{f}$ in mean in $\mathscr{L}^1_{\mathrm{F}}(\nu)$, and ν-almost everywhere;

2) the function $g = |\mathbf{f}_0| + \sum_{n\in\mathbf{N}} |\mathbf{f}_{n+1} - \mathbf{f}_n|$ is such that $\nu^*(g) < +\infty$ (Ch. IV, §3, No. 4, Th. 3).”

From a sequence in $\mathscr{H}$ converging in mean to $\mathbf{f}$, by the cited Th. 3 select a subsequence $(\mathbf{f}_n)_{n\in\mathbf{N}}$ such that $\mathbf{f}_n$ converges to $\mathbf{f}$ ν-almost everywhere in X, the series $\sum_{n\in\mathbf{N}} \nu^*(|\mathbf{f}_{n+1} - \mathbf{f}_n|)$ is convergent and the series

$\sum_{n\in\mathbf{N}} |\mathbf{f}_{n+1}(x) - \mathbf{f}_n(x)|$ is convergent for ν-almost every $x \in \mathrm{X}$. Let g be the numerical function on X defined by

$$g(x) = |\mathbf{f}_0(x)| + \sum_{n\in\mathbf{N}} |\mathbf{f}_{n+1}(x) - \mathbf{f}_n(x)| \quad \text{for all } x \in \mathrm{X}\,,$$

that is, $g = |\mathbf{f}_0| + \sum_{n\in\mathbf{N}} |\mathbf{f}_{n+1} - \mathbf{f}_n|$. We know that g is finite for ν-almost every $x \in \mathrm{X}$; moreover, g is measurable for every measure on X, and

$$\nu^*(g) = \nu^*(|\mathbf{f}_0|) + \sum_{n\in\mathbf{N}} \nu^*(|\mathbf{f}_{n+1} - \mathbf{f}_n|) < +\infty$$

(see item (β) in the Note for p. V.22, ℓ. −2 to V.23, ℓ. 3), therefore g is ν-integrable.

V.24, ℓ. −11, −10.

"... N_1 is locally μ-negligible (resp. μ-negligible) by formula (6) (resp. (7))."

Proof #1. As observed in the preceding note, the function g is 'universally measurable' and ν-integrable. In particular, g is λ_t-measurable for all $t \in \mathrm{T}$, thus g is λ_t-integrable if and only if $\lambda_t^*(g) < +\infty$; in other words,

$$\mathrm{N}_1 = \{t \in \mathrm{T} : \ g \text{ is not } \lambda_t\text{-integrable}\},$$

therefore N_1 is locally μ-negligible (resp. μ-negligible) by *a*) (resp. *b*)) for the case of numerical functions. ♢

Lemma. *If h is a positive function on* T *such that $\mu^\bullet(h) < +\infty$, then h is finite locally μ-almost everywhere in* T.

Proof. By 3) of §1, No. 2, Prop. 7, there exists a μ-moderated positive function h' such that $h = h'$ locally μ-almost everywhere, whence $\mu^\bullet(h) = \mu^\bullet(h')$. Let A be a locally μ-negligible set such that $h = h'$ on $\complement\mathrm{A}$. By 2) of the cited Prop. 7,

$$\mu^*(h') = \mu^\bullet(h') = \mu^\bullet(h) < +\infty\,,$$

therefore h' is finite μ-almost everywhere (Ch. IV, §2, No. 3, Prop. 7); let B be a μ-negligible set such that h' is finite on $\complement\mathrm{B}$. If $t \in \complement(\mathrm{A}\cup\mathrm{B}) = \complement\mathrm{A} \cap \complement\mathrm{B}$, then $h(t) = h(t')$ and $h'(t)$ is finite, thus $h(t)$ is finite on the complement of the locally μ-negligible set $\mathrm{A}\cup\mathrm{B}$. ♢

Proof #2: By (6) (resp. (7)) of No. 2, Prop. 3,

$$\mu^\bullet\big(\Lambda^*(g)\big) \leqslant \nu^*(g) < +\infty \qquad (\text{resp. } \mu^*\big(\Lambda^*(g)\big) \leqslant \nu^*(g) < +\infty\,)\,,$$

therefore $\Lambda^*(g)$ is finite locally μ-almost everywhere (resp. μ-almost everywhere) by the Lemma (resp. Ch. IV, §2, No. 3, Prop. 7); thus $\mu^\bullet(\mathrm{N}_1) = 0$ (resp. $\mu^*(\mathrm{N}_1) = 0$). ♢

V.24, ℓ. −10 to −7.

"For $t \notin \mathrm{N}_1$, the $\mathbf{f}_n$ belong to $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$, the sequence $(\mathbf{f}_n)$ converges λ_t-almost everywhere, as well as for the topology of convergence in mean in $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ (Ch. IV, §3, No. 3, Prop. 6)."

Let us write

$$\mathbf{f}'_0 = \mathbf{f}_0\,, \quad \mathbf{f}'_n = \mathbf{f}_n - \mathbf{f}_{n-1} \quad \text{for } n = 1, 2, 3, \dots\,;$$

it is to the sequence $(\mathbf{f}'_n)_{n\in\mathbf{N}}$ and the measure λ_t that the cited Prop. 6 will be applied.

As observed in the Notes for ℓ. 19, 20 and 21–26, for every measure on X the $\mathbf{f}_n$ are integrable—hence so are the $\mathbf{f}'_n$ —and g is measurable. Since $t \notin \mathrm{N}_1$, $\lambda^*_t(g) < +\infty$, therefore g is λ_t-integrable and, by the argument in the latter Note,

$$\sum_{n\in\mathbf{N}} \lambda^*_t(|\mathbf{f}'_n|) = \lambda^*_t(g) < +\infty\,,$$

that is, the series with general term $\lambda^*_t(|\mathbf{f}'_n|)$ is summable. Therefore, by the cited Prop. 6, for λ_t-almost every $x \in \mathrm{X}$ the series with general term $\mathbf{f}'_n(x)$ is absolutely convergent in F; the n'th partial sum of the series is equal to $\mathbf{f}_n(x)$, thus, writing N_t for the set of all such x, we may define a function $\mathrm{X} \to \mathrm{F}$ by

$$\mathbf{f}_t(x) = \begin{cases} \lim\limits_{n\to\infty} \mathbf{f}_n(x) & \text{for } x \in \complement\,\mathrm{N}_t \\ 0 & \text{for } x \in \mathrm{N}_t\,. \end{cases}$$

We assert that $\mathbf{f}_t$ is λ_t-measurable. For, the $\mathbf{f}_n$ are universally measurable and $\complement\,\mathrm{N}_t$ is λ_t-measurable, therefore the functions $\varphi_{\complement \mathrm{N}_t}\mathbf{f}_n$ are λ_t-measurable for every n, hence so is their pointwise limit $\mathbf{f}_t$ (Ch. IV, §5, No. 4, Th. 2).

Also by the cited Prop. 6, $\mathbf{f}_t \in \mathscr{F}^1_{\mathrm{F}}(\lambda_t)$, that is, $\lambda^*_t(\mathbf{f}_t) < +\infty$; and $\mathbf{f}_n$ converges to $\mathbf{f}_t$ in mean for λ_t, that is, $\lambda^*_t(|\mathbf{f}_n - \mathbf{f}_t|) \to 0$. Since $\mathbf{f}_t$ is λ_t-measurable, we thus have $\mathbf{f}_t \in \mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ and $\mathbf{f}_n \to \mathbf{f}_t$ in the semi-normed space $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$, therefore

$$\int \mathbf{f}_t\, d\lambda_t = \lim_{n\to\infty} \int \mathbf{f}_n\, d\lambda_t$$

because the integral is by definition continuous on $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ (Ch. IV, §4, No. 1, Def. 1).

V.24, ℓ. -3 to -1.

"Suppose that t does not belong to $\mathrm{N}_1 \cup \mathrm{N}_2$; the sequence $(\mathbf{f}_n)$ converges in mean in $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$, and converges λ_t-almost everywhere to $\mathbf{f}$. Therefore $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ and $\int \mathbf{f}\, d\lambda_t = \lim_{n\to\infty} \int \mathbf{f}_n\, d\lambda_t$ (Ch. IV, §4, No. 1)."

Since $t \notin \mathrm{N}_1$ we may construct $\mathbf{f}_t \in \mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ as in the preceding Note; in particular, $\mathbf{f}_n \to \mathbf{f}_t$ λ_t-almost everywhere in X, and $\mathbf{f}_n \to \mathbf{f}_t$ in mean in $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$.

Since $t \notin \mathrm{N}_2$, M is λ_t-negligible, that is, $\mathbf{f}_n(x) \to \mathbf{f}(x)$ for λ_t-almost every $x \in \mathrm{X}$; but already $\mathbf{f}_n(x) \to \mathbf{f}_t(x)$ for λ_t-almost every $x \in \mathrm{X}$, therefore $\mathbf{f} = \mathbf{f}_t$ λ_t-almost everywhere. Since $\mathbf{f}_t \in \mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ we have also $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ and $\mathbf{f}_n \to \mathbf{f}$ in mean in $\mathscr{L}^1_{\mathrm{F}}(\lambda_t)$, whence $\int \mathbf{f}\, d\lambda_t = \lim_{n\to\infty} \int \mathbf{f}_n\, d\lambda_t$ (Ch. IV, §1, No. 1, Def. 1).

In particular, since $\mathbf{f}$ is λ_t-integrable we have $t \notin \mathrm{H}$, and we have shown that $\mathrm{H} \subset \mathrm{N}_1 \cup \mathrm{N}_2$; therefore H is locally μ-negligible (resp. μ-negligible).

V.25, ℓ. 2–4.

"... the function $t \mapsto \int \mathbf{f}\, d\lambda_t$ is equal locally μ-almost everywhere to the limit of a sequence of μ-measurable functions; it is therefore μ-measurable."

Since the $\mathbf{f}_n$ belong to $\mathscr{H}$ we know (see the Note for V.24, ℓ. 19, 20) that the functions $\Lambda(\mathbf{f}_n) : t \mapsto \int \mathbf{f}_n\, d\lambda_t$ are defined everywhere in T, are essentially μ-integrable (resp. μ-integrable), and satisfy $\nu(\mathbf{f}_n) = \int \Lambda(\mathbf{f}_n)\, d\mu$. Moreover (see the preceding Note) for every $t \in \complement(\mathrm{N}_1 \cup \mathrm{N}_2)$ one has $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}(\lambda_t)$ and $\int \mathbf{f}_n\, d\lambda_t \to \int \mathbf{f}\, d\lambda_t$.

Define a function $\Lambda(\mathbf{f}) : \mathrm{T} \to \mathrm{F}$ by the formula

$$(*) \qquad \big(\Lambda(\mathbf{f})\big)(t) = \begin{cases} \lim_{n\to\infty} \int \mathbf{f}_n\, d\lambda_t & \text{for } t \in \complement(\mathrm{N}_1 \cup \mathrm{N}_2) \\ 0 & \text{for } t \in \mathrm{N}_1 \cup \mathrm{N}_2 . \end{cases}$$

The functions $\varphi_{\complement(\mathrm{N}_1 \cup \mathrm{N}_2)} \Lambda(\mathbf{f}_n)$ are λ_t-measurable and converge pointwise on T to $\Lambda(\mathbf{f})$, therefore $\Lambda(\mathbf{f})$ is λ_t-measurable (Ch. IV, §5, No. 4, Th. 2).

Moreover, $\Lambda(\mathbf{f}_n) \to \Lambda(\mathbf{f})$ locally μ-almost everywhere (resp. μ-almost everywhere) in T by $(*)$.

V.25, ℓ. 8–11.

"Now, the function $t \mapsto \int^* g(x)\, d\lambda_t(x)$ is essentially μ-integrable (resp. μ-integrable) by Prop. 5. We may therefore apply Lebesgue's theorem, which yields

$$\int d\mu(t) \int \mathbf{f}(x)\, d\lambda_t(x) = \lim_{n\to\infty} \int d\mu(t) \int \mathbf{f}_n(x)\, d\lambda_t(x) = \lim_{n\to\infty} \int \mathbf{f}_n(x)\, d\nu(x). \text{"}$$

b) With notations as in the preceding Note, consider first the case that Λ is vaguely continuous, the $\Lambda(\mathbf{f}_n)$ are μ-integrable, and $\Lambda(\mathbf{f}_n) \to \Lambda(\mathbf{f})$ μ-almost everywhere in T. By the case of a positive numerical function, we know that the function $\Lambda^*(g)$ (defined everywhere on T) is μ-integrable; moreover, as just noted in the text, $|\Lambda(\mathbf{f}_n)| \leqslant \Lambda^*(g)$ on $\complement(\mathrm{N}_1 \cup \mathrm{N}_2)$, that is, μ-almost everywhere in T. It follows from Lebesgue's theorem (Ch. IV, §3, No. 7, Th. 6) that $\Lambda(\mathbf{f})$ is μ-integrable and $\Lambda(\mathbf{f}_n)$ converges to $\Lambda(\mathbf{f})$ in mean for μ, therefore $\int \Lambda(\mathbf{f}_n)\, d\mu \to \int \Lambda(\mathbf{f})\, d\mu$. Finally since, by construction, $\mathbf{f}_n \to \mathbf{f}$ in mean for ν, we have

$$\int \mathbf{f}\, d\nu = \lim_{n\to\infty} \int \mathbf{f}_n\, d\nu = \lim_{n\to\infty} \int \Lambda(\mathbf{f}_n)\, d\mu = \int \Lambda(\mathbf{f})\, d\mu\,,$$

thus $\mathbf{f}$ satisfies (11).

a) Assuming only that $\mathbf{f}$ is ν-integrable, we know that the (everywhere defined) functions $\Lambda(\mathbf{f}_n)$ are essentially μ-integrable, $\Lambda(\mathbf{f})$ is μ-measurable, and $\Lambda(\mathbf{f}_n) \to \Lambda(\mathbf{f})$ locally μ-almost everywhere in T. By the case of a positive numerical function, the function $\Lambda(g) : t \mapsto \lambda_t^\bullet(g)$ $(t \in \mathrm{T})$ is essentially μ-integrable and $\Lambda(g) = \Lambda^*(g)$ locally μ-almost everywhere in T. And $|\Lambda(\mathbf{f}_n)| \leqslant \Lambda^*(g)$ on $\complement(\mathrm{N}_1 \cup \mathrm{N}_2)$, that is, locally μ-almost everywhere in T.

Choose functions $h_n, k \in \mathscr{L}^1_{\mathrm{F}}(\mu)$ such that $\Lambda(\mathbf{f}_n) = h_n$ and $\Lambda^*(g) = k$ locally μ-almost everywhere in T (§1, No. 3, Def. 3). Then $h_n \to \Lambda(\mathbf{f})$ and $|h_n| \leqslant |k|$ locally μ-almost everywhere in T. Now, the sequence $(h_n(t))$ is Cauchy in F for locally μ-almost every $t \in \mathrm{T}$; but the set of $t \in \mathrm{T}$ such that $(h_n(t))$ is not Cauchy, namely, the set

$$\mathrm{Q} = \bigcup_{r=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{m,n \geqslant N,\ m \neq n} \{t \in \mathrm{T} :\ |h_m(t) - h_n(t)| > \frac{1}{r}\}\,,$$

is μ-moderated because the functions $h_m - h_n$ are μ-moderated; thus Q is both locally negligible and moderated for μ, hence is μ-negligible (§1, No. 2, Cor. 1 of Prop. 7), that is, the sequence $(h_n(t))$ is Cauchy in F for μ-almost every t in T. Defining

$$h(t) = \begin{cases} \lim\limits_{n\to\infty} h_n(t) & \text{for } t \in \mathrm{T} - \mathrm{Q} \\ 0 & \text{for } t \in \mathrm{Q}\,, \end{cases}$$

we have $h_n(t) \to h(t)$ μ-almost everywhere in T, therefore h is μ-measurable.

Similarly, since h_n and k are μ-moderated, we have $|h_n| \leqslant |k|$ μ-almost everywhere. It now follows from Lebesgue's theorem that h is μ-integrable, h_n converges to h in mean for μ, and so $\int h_n\, d\mu \to \int h\, d\mu$, that is, $\int \Lambda(\mathbf{f}_n)\,d\mu \to \int h\,d\mu$. Since $h_n \to \Lambda(\mathbf{f})$ locally μ-almost everywhere, we have $\Lambda(\mathbf{f}) = h$ locally μ-almost everywhere, therefore $\Lambda(\mathbf{f})$ is essentially μ-integrable and

$$\int \Lambda(\mathbf{f})\,d\mu = \int h\,d\mu = \lim_{n\to\infty} \int \Lambda(\mathbf{f}_n)\,d\mu = \lim_{n\to\infty} \int \mathbf{f}_n\,d\nu = \int \mathbf{f}\,d\nu\,,$$

thus $\mathbf{f}$ satisfies (11).

Remark. In case it is needed in the future: the foregoing argument shows that Lebesgue's theorem holds with "integrable" replaced by "essentially integrable", and "almost everywhere" by "locally almost everywhere".

V.25, ℓ. 17–19.

"Then $\mathbf{g} = \mathbf{f}$ almost everywhere for λ_t, except for t forming a locally μ-negligible set P (Cor. 1 c) of Prop. 3)."

We are assuming Λ is μ-adequate and $\lambda_t^\bullet(1) < +\infty$ for locally μ-almost every $t \in \mathrm{T}$. Given an essentially ν-integrable function $\mathbf{g}$ with values in F or in $\overline{\mathbf{R}}$, let

$$\mathrm{H}_\mathbf{g} = \{t \in \mathrm{T} :\ \mathbf{g} \text{ is not } \lambda_t\text{-integrable}\}$$

and write $\Lambda(\mathbf{g})$ for the function on T defined by

$$\big(\Lambda(\mathbf{g})\big)(t) = \begin{cases} \displaystyle\int \mathbf{g}\,d\lambda_t & \text{for } t \notin \mathrm{H}_\mathbf{g} \\ 0 & \text{for } t \in \mathrm{H}_\mathbf{g}\,; \end{cases}$$

we are to show that $\mathrm{H}_\mathbf{g}$ is locally μ-negligible, $\Lambda(\mathbf{g})$ is essentially μ-integrable, and $\int \Lambda(\mathbf{g})\,d\mu = \int \mathbf{g}\,d\nu$.

The letter $\mathbf{f}$ was reserved for a ν-integrable function such that $\mathbf{g} = \mathbf{f}$ locally ν-almost everywhere, so that $\int \mathbf{g}\,d\nu = \int \mathbf{f}\,d\nu$ (§1, No. 3, Def. 3), and to which part a) of the theorem is to be applied; we can suppose that if $\mathbf{g}$ is a numerical function then $\mathbf{f} \in \mathscr{L}^1_\mathbf{R}(\nu)$, that is, all values of $\mathbf{f}$ are finite (which facilitates subtraction). Writing

$$\mathrm{H} = \{t \in \mathrm{T} :\ \mathbf{f} \text{ is not } \lambda_t\text{-integrable}\}\,,$$

we know from part a) that H is locally μ-negligible, that the function $\Lambda(\mathbf{f})$ defined by

$$\big(\Lambda(\mathbf{f})\big)(t) = \begin{cases} \displaystyle\int \mathbf{f}\,d\lambda_t & \text{for } t \notin \mathrm{H} \\ 0 & \text{for } t \in \mathrm{H} \end{cases}$$

is essentially μ-integrable, and that $\int \Lambda(\mathbf{f})\,d\mu = \int \mathbf{f}\,d\nu$.

The numerical function $|\mathbf{g} - \mathbf{f}|$ is locally ν-negligible; by the cited Cor. 1 c) of Prop. 3, the set

$$\mathrm{P} = \{t \in \mathrm{T} :\ \lambda_t^*(|\mathbf{g} - \mathbf{f}|) > 0\}$$

is locally μ-negligible. Thus the set $\mathrm{P} \cup \mathrm{H}$ is locally μ-negligible.

Suppose $t \in \complement(\mathrm{P} \cup \mathrm{H})$; then $\mathbf{f}$ is λ_t-integrable (because $t \notin \mathrm{H}$) and $\mathbf{g} = \mathbf{f}$ λ_t-almost everywhere (because $t \notin \mathrm{P}$), therefore $\mathbf{g}$ is λ_t-integrable, i.e., $t \in \complement \mathrm{H}_{\mathbf{g}}$, and $\int \mathbf{g}\, d\lambda_t = \int \mathbf{f}\, d\lambda_t$. In particular, $\mathrm{H}_{\mathbf{g}} \subset \mathrm{P} \cup \mathrm{H}$, hence $\mathrm{H}_{\mathbf{g}}$ is locally μ-negligible. Moreover, $\Lambda(\mathbf{g}) = \Lambda(\mathbf{f})$ on $\complement(\mathrm{P} \cup \mathrm{H})$, thus $\Lambda(\mathbf{g}) = \Lambda(\mathbf{f})$ locally μ-almost everywhere, whence $\Lambda(\mathbf{g})$ is essentially μ-integrable and

$$\int \Lambda(\mathbf{g})\, d\mu = \int \Lambda(\mathbf{f})\, d\mu = \int \mathbf{f}\, d\nu = \int \mathbf{g}\, d\nu\,,$$

thus $\mathbf{g}$ satisfies (11).

V.25, ℓ. $-13, -12$.

"... it follows at once from the definitions that Λ' is also μ-adequate, and that Λ and Λ' have the same integral."

Given two functions $\Lambda : t \mapsto \lambda_t$, $\Lambda' : t \mapsto \lambda'_t$ of T into $\mathscr{M}_+(\mathrm{X})$, such that $\lambda_t = \lambda'_t$ for locally μ-almost every $t \in \mathrm{T}$:

(i) If Λ is scalarly essentially μ-integrable, with $\nu = \int \lambda_t\, d\mu(t)$, then Λ' is also scalarly essentially μ-integrable and $\int \lambda'_t\, d\mu(t) = \int \lambda_t\, d\mu(t)$, that is, $\nu' = \nu$.

For, if $f \in \mathscr{K}(\mathrm{X})$ then the functions

$$\Lambda(f) : t \mapsto \lambda_t(f)\,, \quad \Lambda'(f) : t \mapsto \lambda'_t(f)$$

are equal locally μ-almost everywhere in T; since $\Lambda(f)$ is essentially μ-integrable so is $\Lambda'(f)$ and $\nu'(f) = \mu^\bullet(\Lambda'(f)) = \mu^\bullet(\Lambda(f)) = \nu(f)$ for all $f \in \mathscr{K}(\mathrm{X})$, whence the assertion.

(ii) If Λ is μ-pre-adequate then so is Λ'.

For, for every lower semi-continuous function $f \geqslant 0$, $\Lambda'(f) = \Lambda(f)$ locally μ-almost everywhere in T and

$$\mu^\bullet(\Lambda'(f)) = \mu^\bullet(\Lambda(f)) = \nu^\bullet(f) = \nu'^\bullet(f)\,,$$

thus Λ' is μ-pre-adequate.

(iii) If Λ is μ-adequate then so is Λ'. For, given any positive measure $\rho \leqslant \mu$, $\Lambda = \Lambda'$ locally ρ-almost everywhere and Λ is ρ-pre-adequate, therefore so is Λ'.

V.25, ℓ. −6, −5.

"… the propositions proved in the preceding Nos. extend to μ-adequate functions defined locally μ-almost everywhere."

I don't see anything here except notation: if one comes across a function H of the indicated type, one should do business with a corresponding Λ, with the understanding that the measure denoted $\int \eta_t \, d\mu(t)$ is simply the measure $\nu = \int \lambda_t \, d\mu(t)$ and depends only on H, or on the equivalence class of Λ under the relation of equality locally μ-almost everywhere.

The following theme seems to me more interesting: Start with a function $\mathrm{H} : t \mapsto \eta_t \ (t \in \mathrm{A})$, where A is a subset of T such that $\mu^\bullet(\mathrm{T} - \mathrm{A}) = 0$ and $\eta_t \in \mathscr{M}_+(\mathrm{X})$ for all $t \in \mathrm{A}$. For each $f \in \mathscr{K}(\mathrm{X})$ we have a numerical function

$$\mathrm{H}(f) : t \mapsto \eta_t(f) \quad (t \in \mathrm{A}).$$

Suppose that, by some miracle, for every $f \in \mathscr{K}(\mathrm{X})$ the function $\mathrm{H}(f)$ is essentially μ-integrable, in other words that there exists a function g in $\mathscr{L}^1_{\mathbf{R}}(\mu)$ such that $\mathrm{H}(f) = g$ locally μ-almost everywhere in T (§1, No. 3, first paragraph on p. V.9). One can unambiguously define $\nu(f) = \int g \, d\mu$ and it is clear that ν is a positive linear form on $\mathscr{K}(\mathrm{X})$, i.e., is a positive measure on X. If $\Lambda : \mathrm{T} \to \mathscr{M}_+(\mathrm{X})$ is any extension of H to T with values in $\mathscr{M}_+(\mathrm{X})$, then, for every $f \in \mathscr{K}(\mathrm{X})$, the function

$$\Lambda(f) : t \mapsto \lambda_t(f) \quad (t \in \mathrm{T})$$

is equal locally μ-almost everywhere to $\mathrm{H}(f)$, hence to some $g \in \mathscr{L}^1_{\mathbf{R}}(\mu)$, thus $\Lambda(f)$ is essentially μ-integrable and $\int \Lambda(f) \, d\mu = \int g \, d\mu$; and since $\mathrm{H}(f) = g$ locally μ-almost everywhere, $\int g \, d\mu = \nu(f)$, whence

$$(*) \qquad \int \Lambda(f) \, d\mu = \nu(f).$$

Summarizing: Λ is scalarly essentially μ-integrable; $\int \lambda_t \, d\mu(t) = \nu$, because the left side of $(*)$ is equal to $\big(\int \lambda_t \, d\mu(t)\big)(f)$; and Λ has the contemplated relation to H.

Will any of this prove to be useful? (See also §4, No. 4, *Remark*.)

V.26, ℓ. −6, −5.

" N is a universally measurable set"

We are to show that the (numerical) function φ_{N} is universally measurable. For every $t \in \mathrm{T}$ let V_t be a compact (hence 'universally integrable') neighborhood of t in T. By the Principle of Localization (Ch. IV, §5, No. 2, Prop. 4) it suffices to show that for every $t \in \mathrm{T}$ there exists a universally measurable function $g_t : \mathrm{T} \to \mathbf{R}$ such that $\varphi_{\mathrm{N}}|\mathrm{V}_t = g_t|\mathrm{V}_t$. Now, $\varphi_{\mathrm{N}}|\mathrm{V}_t$ is equal to 1 on $\mathrm{V}_t \cap \mathrm{N}$ and to 0 on $\mathrm{V}_t - \mathrm{V}_t \cap \mathrm{N}$, therefore

$$\varphi_{\mathrm{N}}|\mathrm{V}_t = \varphi_{\mathrm{V}_t \cap \mathrm{N}}|\mathrm{V}_t\,;$$

since $\varphi_{\mathrm{V}_t \cap \mathrm{N}}$ is universally measurable (because $\mathrm{V}_t \cap \mathrm{N}$ is a Borel set), the functions $g_t = \varphi_{\mathrm{V}_t \cap \mathrm{N}}$ meet the requirements.

V.26, ℓ. −5.

"f' is a universally measurable function (Ch. IV, §5, No. 10, Prop. 16)."

The sets N and K_i $(i \in \mathrm{I})$ are universally measurable and form a locally countable family with union T, so it will suffice, by the cited Prop. 16, to show that the restriction of f' to each of these sets is universally measurable. The function $f'|\mathrm{N}$ is constant and the functions $f'|\mathrm{K}_i$ are continuous; the argument that they are all universally measurable is given in the Note for V.22, ℓ. 11–13.

V.26, ℓ. −2.

"Definition 3."

We record here some more-or-less immediate consequences of the definition.

(i) If $\Lambda : t \mapsto \lambda_t$ is pre-adequate for every positive measure on T with compact support, then Λ is a diffusion.

For, if μ is a positive measure with compact support then every positive measure $\mu' \leqslant \mu$ also has compact support (Ch. III, §2, No. 2, Prop. 3).

(ii) If μ is a positive measure on T and $\Lambda : t \mapsto \lambda_t$ is a mapping $\mathrm{T} \to \mathscr{M}_+(\mathrm{X})$ such that Λ is μ-pre-adequate, it is implicit that Λ is scalarly essentially μ-integrable (No. 1, Def. 1).

(iii) Let μ be any positive measure on T. One knows that μ is the sum of a family $(\mu_\alpha)_{\alpha \in \mathrm{A}}$ of positive measures on T with compact support (§2, No. 3, Prop. 4). Thus, if $\Lambda : t \mapsto \lambda_t$ is a diffusion then Λ is μ_α-adequate (hence scalarly essentially μ_α-integrable) for every α, but one does not know whether Λ is scalarly essentially μ-integrable.

However, if Λ is scalarly essentially μ-integrable and if μ is the sum of a family $(\mu_\alpha)_{\alpha \in \mathrm{A}}$ of positive measures such that Λ is μ_α-pre-adequate for every α, then Λ is μ-pre-adequate and (with $\int$ signifying essential integral)

$$\int \lambda_t \, d\mu(t) = \sum_{\alpha \in \mathrm{A}} \int \lambda_t \, d\mu_\alpha(t)$$

(see Proposition A in the Note for V.17, ℓ. −4).

(iv) Let μ be a positive measure on T. If a mapping $\Lambda : t \mapsto \lambda_t$ is scalarly essentially μ-integrable and if Λ is μ-pre-adequate for every positive measure $\mu' \leqslant \mu$ with compact support, then Λ is μ-adequate (see part d) of Exer. 8, worked out in the Note for V.17, ℓ. -4).

(v) Let $\Lambda : t \mapsto \lambda_t$ be a diffusion $\mathrm{T} \to \mathscr{M}_+(\mathrm{X})$ and let μ be a positive measure on T. In order that Λ be μ-adequate, it is necessary and sufficient that it be scalarly essentially μ-integrable. (This is equivalent to Prop. 11 below.)

Necessity: (ii).

Sufficiency: By (iv) and the definition of a diffusion.

(vi) If $\Lambda : t \mapsto \lambda_t$ is a diffusion $\mathrm{T} \to \mathscr{M}_+(\mathrm{X})$ and if $g \in \mathscr{K}_+(\mathrm{X})$, then the function $\Lambda(g) : t \mapsto \lambda_t(g)$ is universally measurable, and is integrable for every positive measure on T with compact support.

For, let μ be any positive measure on T, and write $\mu = \sum_{\alpha \in \mathrm{A}} \mu_\alpha$ with $(\mu_\alpha)_{\alpha \in \mathrm{A}}$ a family of positive measures on T with compact support. Since Λ is a diffusion, it is μ_α-adequate for every α, therefore $\Lambda(g)$ is μ_α-measurable for all α (No. 1, Def. 1), whence $\Lambda(g)$ is μ-measurable (§2, No. 2, Prop. 2).

Suppose now that μ is any positive measure with compact support. Then Λ is μ-adequate, and if $\nu = \int \lambda_t \, d\mu(t)$ and $g \in \mathscr{K}_+(\mathrm{X})$, one has

$$\mu^\bullet\big(\Lambda(g)\big) = \nu^\bullet(g) = \nu(g) < +\infty\,,$$

thus $\Lambda(g)$ is essentially μ-integrable (§1, No. 3, Prop. 9). But μ is bounded (Ch. III, §2, No. 3, Prop. 11), hence moderated (§1, No. 2, *Remark* 2), therefore every function on T is μ-moderated (*loc. cit.*, Def. 2), whence $\Lambda(g)$ is μ-integrable (§1, No. 3, Cor. of Prop. 9).

V.27, ℓ. $-14, -13$.

"…then u is not integrable for the measure $\mu = \sum_{n \geqslant 1} \frac{1}{n^2}\varepsilon_{t_n}$ with compact support"

(i) The family of measures $\frac{1}{n^2}\varepsilon_{t_n}$ $(n \geqslant 1)$ is summable: for, if $f \in \mathscr{K}_+(\mathrm{T})$ then for every finite set J of integers $\geqslant 1$ one has

$$\sum_{n \in \mathrm{J}} \big(\frac{1}{n^2}\varepsilon_{t_n}\big)(f) = \sum_{n \in \mathrm{J}} \frac{1}{n^2} f(t_n) \leqslant \|f\| \sum_{n \geqslant 1} \frac{1}{n^2} < +\infty\,,$$

where $\|f\|$ is the 'sup-norm' of f.

(ii) If h is a lower semi-continuous function $\geqslant 0$ on T then $\varepsilon_t^*(h) = h(t)$ for every $t \in \mathrm{T}$.

For, h is the upper envelope of the set $\mathscr{A}$ of all $f \in \mathscr{K}_+(\mathrm{T})$ such that $f \leqslant h$, and, by definition,

$$\varepsilon_t^*(h) = \sup_{f \in \mathscr{A}} \varepsilon_t(f) = \sup_{f \in \mathscr{A}} f(t) = h(t)\,.$$

{Since h is universally measurable (Ch. IV, §5, No. 5, Cor. of Prop. 8), it follows that h is ε_t-integrable $\Leftrightarrow h(t) < +\infty$ (*loc. cit.*, No. 6, Th. 5).}

(iii) $\mu^*(u) = +\infty$ (whence the assertion).

By definition, $\mu^*(u)$ is the infimum of $\mu^*(h)$ for all lower semi-continuous functions $h \geqslant u$, so it will suffice to show that $\mu^*(h) = +\infty$ for all such h. Indeed, by §2, No. 2, Cor. 3 of Prop. 1, one has

$$\mu^*(h) \geqslant \sum_n \big(\frac{1}{n^2}\varepsilon_{t_n}\big)^*(h) = \sum_n \frac{1}{n^2} h(t_n) \geqslant \sum_n \frac{1}{n^2} u(t_n) = +\infty$$

because $\dfrac{1}{n^2}u(t_n) \geqslant 1$. ◊

An example. The computation in (ii) throws light on the formula $\mu = \int \varepsilon_x \, d\mu(x)$ of Ch. III, §3, No. 1, *Example* 2, where it is restricted to functions in $\mathscr{K}(\mathrm{X})$ and is merely a notation.

In the present context, consider the mapping $\Lambda : x \mapsto \varepsilon_x$ of X into $\mathscr{M}_+(\mathrm{X})$, that is, $\mathrm{T} = \mathrm{X}$ and $\lambda_x = \varepsilon_x$ for all $x \in \mathrm{X}$. Since ε_x is bounded, $\varepsilon_x^\bullet = \varepsilon_x^*$ (§1, No. 2, Cor. 2 of Prop. 7), thus, for every function $f \geqslant 0$ on X, we have $\big(\Lambda(f)\big)(x) = \varepsilon_x^\bullet(f) = \varepsilon_x^*(f)$ for all $x \in \mathrm{X}$. In particular, if $f \in \mathscr{K}_+(\mathrm{X})$ the function

$$\Lambda(f) : x \mapsto \varepsilon_x^*(f) = \varepsilon_x(f) = f(x)$$

is 'universally integrable'; for every positive measure μ on X and every $f \in \mathscr{K}_+(\mathrm{X})$ one has

$$\mu^\bullet\big(\Lambda(f)\big) = \mu^\bullet(f) = \mu(f)\,,$$

thus Λ is scalarly essentially μ-integrable and $\nu = \mu$, that is, $\int \varepsilon_x \, d\mu(x) = \mu$. Moreover, the continuity of $\Lambda(f) = f$ for every $f \in \mathscr{K}_+(\mathrm{X})$ shows that Λ is vaguely continuous (better yet, see Ch. III, §1, No. 9, Prop. 13). {In fact, $\Lambda(f) = f$ for every function $f \geqslant 0$ on X; see item (v) later in this note.}

If h is any lower semi-continuous function $\geqslant 0$ on X, by (ii) above one has

$$\big(\Lambda(h)\big)(x) = \varepsilon_x^\bullet(h) = \varepsilon_x^*(h) = h(x)\,,$$

thus $\Lambda(h)$ is universally measurable (Ch. IV, §5, No. 5, Cor. of Prop. 8), and, for every positive measure μ on X,

$$\mu^\bullet\big(\Lambda(h)\big) = \mu^\bullet(h) = \nu^\bullet(h)\,;$$

thus Λ is μ-adequate for every positive measure μ on X, and in particular Λ is a diffusion, indeed, a bounded diffusion ($\|\varepsilon_x\| = 1$ for all x).

Fix a positive measure μ on X and consider a μ-integrable function $f \geqslant 0$. Let us apply No. 3, Th. 1, *b*): the set

$$\mathrm{H} = \{x \in \mathrm{X} : \ f \text{ is not } \varepsilon_x\text{-integrable}\}$$

is μ-negligible, the function

$$\Lambda(f) : x \mapsto \varepsilon_x^*(f)$$

is μ-integrable, and $\int f\, d\nu = \int \Lambda(f)\, d\mu$, that is

$$(*) \qquad \int f\, d\mu = \int \Lambda(f)\, d\mu\,.$$

As in the proof of the cited Th. 1, one extends this result, in particular $(*)$, to an arbitrary μ-integrable function f by considering $|f| = f^+ + f^-$ and $f = f^+ - f^-$. We shall show that the underlying reason for the equality $(*)$ is that $\Lambda(f) = f$ μ-almost everywhere; this entails a close look at the meaning of ε_x-integrability.

Fix a point $a \in \mathrm{X}$.

(i) The key fact is that $\mathrm{Supp}(\varepsilon_a) = \{a\}$. (See the Note for III.29, ℓ. 2–4.)

(ii) The negligible sets for ε_a are the subsets of $\mathrm{X} - \{a\}$, that is, the subsets of X that do not contain a. For, $\mathrm{X} - \{a\} = \complement\,\mathrm{Supp}(\varepsilon_x)$ is ε_a-negligible; whereas if $a \in \mathrm{A} \subset \mathrm{X}$ then $\varepsilon_a^*(\mathrm{A}) \geqslant \varepsilon_a^*(\{a\}) = 1$ because $\varphi_{\{a\}} = \varphi_{\mathrm{X}} = 1$ ε_a-almost everywhere, therefore A is not ε_a-negligible.

For something to happen ε_a-almost everywhere in X, it is necessary and sufficient that it happen at a; for, the set A where it does not happen is ε_a-negligible if and only if $a \notin \mathrm{A}$.

For example, if f, g are functions on X then $f = g$ ε_a-almost everywhere if and only if $f(a) = g(a)$.

(iii) Every function f on X is ε_a-measurable (obvious from Ch. IV, §5, No. 1, Def. 1, with $\mathrm{N} = \mathrm{K} - \{a\}$). It follows that f is ε_a-integrable if and only if $\varepsilon_a^*(|f|) < +\infty$ (*loc. cit.*, No. 6, Th. 5).

(iv) A numerical function f is ε_a-integrable $\Leftrightarrow$ $f(a)$ is finite, in which case $\int f\, d\varepsilon_a = f(a)$.

$\Rightarrow$: If f is ε_a-integrable, then the set $\mathrm{A} = \{x \in \mathrm{X} : \ f(x) = \pm\infty\}$ is negligible for ε_a (Ch. IV, §2, No. 3, Prop. 7), thus $a \notin \mathrm{A}$.

$\Leftarrow$: If $f(a)$ is finite, let $g \in \mathscr{K}(\mathrm{X})$ with $g(a) = f(a)$. Then g is ε_a-integrable and $f = g$ almost everywhere for ε_a, therefore f is ε_a-integrable and

$$\int f\, d\,\varepsilon_a = \int g\, d\varepsilon_a = \varepsilon_a(g) = g(a) = f(a)\,.$$

(v) Summarizing (iii) and (iv): f is ε_a-integrable $\Leftrightarrow$ $\varepsilon_a^*(|f|) < +\infty$ $\Leftrightarrow$ $f(a)$ is finite, in which case $\int f\, d\varepsilon_a = f(a)$.

If $f \geqslant 0$ then $\varepsilon_a^*(f) = f(a)$ always; for, if $f(a) = +\infty$ then f is not ε_a-integrable, so $\varepsilon_a^*(f) = +\infty = f(a)$. ♢

Returning to the discussion of a μ-integrable function f, in the context of Th. 1 of No. 3 we have

$$\begin{aligned} \mathrm{H} &= \{x \in \mathrm{X} :\ f \text{ is not } \varepsilon_a\text{-integrable}\} \\ &= \{x \in \mathrm{X} :\ |f| \text{ is not } \varepsilon_a\text{-integrable}\} \\ &= \{x \in \mathrm{X} :\ |f(x)| = +\infty\}\,, \end{aligned}$$

and we know from (v) above that for $x \notin \mathrm{H}$ one has

$$\big(\Lambda(f)\big)(x) = \int f\, d\varepsilon_x = f(x)\,;$$

thus $\Lambda(f) = f$ μ-almost everywhere, and the message of (∗) is that

$$\int f\, d\mu = \int f\, d\nu = \int f\, d\big(\textstyle\int \varepsilon_x\, d\mu(x)\big),$$

further justification for the notation $\mu = \int \varepsilon_x\, d\mu(x)$ of Ch. III.

V.27, $\ell.$ -13 to -11.

"...contrary to the hypothesis on Λ, which implies that $t \mapsto \lambda_t(g)$ is integrable for every positive measure with compact support."

See item (vi) of the Note for V.26, $\ell.$ -2.

V.27, $\ell.$ $-10, -9$.

"Conversely, the conditions 1) and 2) imply that Λ is scalarly essentially μ-integrable for every measure μ with compact support."

Conditions 1) and 3) imply that Λ is scalarly essentially μ-integrable for every positive measure with compact support, but stating 3) requires knowing that $\nu = \int \lambda_t\, d\mu(t)$ exists; thus we must infer from 1) and 2) that Λ is scalarly essentially μ-integrable.

Given $g \in \mathscr{K}_+(\mathrm{T})$, we must show that the mapping $\Lambda(g) : t \mapsto \lambda_t(g)$ is scalarly essentially μ-integrable; since it is μ-measurable by 1), it suffices to show that $\mu^*\big(\Lambda(g)\big) < +\infty$ (Ch. IV, §5, No. 6, Th. 5).

By assumption the set $\mathrm{K} = \mathrm{Supp}(\mu)$ is compact. By 2), every point of T has a neighborhood on which $\Lambda(g)$ is bounded; therefore $\Lambda(g)$ is bounded on K (cover K with a finite number of such neighborhoods). Say $\varphi_{\mathrm{K}}\Lambda(g) \leqslant M\varphi_{\mathrm{K}}$, where $M < +\infty$; since $\mu(\complement \mathrm{K}) = 0$, $\varphi_{\mathrm{K}} = 1$ μ-almost everywhere, therefore

$$\Lambda(g) = \varphi_{\mathrm{K}}\Lambda(g) \leqslant M\varphi_{\mathrm{K}} \quad \mu\text{-almost everywhere,}$$

whence $\mu^*\big(\Lambda(g)\big) \leqslant M\mu^*(\mathrm{K}) < +\infty$.

Remark. Note that the above proof cites 1) only for $f \in \mathscr{K}_+(\mathrm{X})$.

V.28, ℓ. 4.
" Λ is scalarly essentially μ-integrable"

See the *Remark* at the end of the preceding note. This property is required for the application of No. 1, Prop. 2.

V.28, ℓ. 10.
"... we shall denote by Λf the mapping $t \mapsto \lambda_t^\bullet(f)$."

The notation Λf is consistent with the notation $\Lambda(f)$ used earlier in these notes (see, e.g., the note for V.20, ℓ. -4 to -2), but is more restrictive. The restrictions: Λ must be a diffusion, and f must be a universally measurable function $\geqslant 0$.

V.28, ℓ. 10, 11.
"If μ is a positive measure on T such that Λ is scalarly essentially μ-integrable ... "

As is the case when Λ is μ-pre-adequate (No. 1, Def. 1), in particular when μ has compact support (Def. 3).

V.28, ℓ. 12–14.
"The definition of the integral then takes the form

$$\langle \mu\Lambda, f\rangle = \langle \mu, \Lambda f\rangle \quad \text{for } f \in \mathscr{K}_+(\mathrm{X})\text{. "}$$

By "the integral" is meant the measure $\mu\Lambda = \int \lambda_t \, d\mu(t)$. The displayed condition is

$$(\mu\Lambda)^\bullet(f) = \mu^\bullet(\Lambda f) \quad \text{for all } f \in \mathscr{K}_+(\mathrm{X})\,;$$

by employing the symbol $\mu\Lambda$, it presumes that the given diffusion Λ is scalarly essentially μ-integrable and recapitulates the definition of the measure $\nu = \int \lambda_t \, d\mu(t)$,

$$\nu(f) = \mu^\bullet\big(\Lambda f\big) \quad \text{for all } f \in \mathscr{K}_+(\mathrm{X}),$$

which is formula (1) of No. 1 written for a diffusion Λ.

{It can't *be* the definition of $\int \lambda_t \, d\mu(t)$, as the presence of the diffusion Λ requires (via the definition of adequacy) that one already know how to define the integral of a scalarly essentially μ-integrable mapping $t \mapsto \lambda_t$ $(t \in \mathrm{T})$. There is here a bit of potential incest that is to be avoided.}

V.28, ℓ. 16–18.

"...this amounts to saying (in view of Prop. 8) that Λ is scalarly essentially μ-integrable and $\langle \mu'\Lambda, f\rangle = \langle \mu', \Lambda f\rangle$ for every positive measure $\mu' \leqslant \mu$ and every lower semi-continuous positive function f."

(i) Suppose that the diffusion Λ is scalarly essentially μ-integrable and satisfies the stated condition. Then, for every positive measure $\mu' \leqslant \mu$, one knows that Λ is scalarly essentially μ'-integrable (see the Note for V.17, ℓ. 12, 13), so that the measure $\mu'\Lambda = \nu'$ is defined; and, for every lower semi-continuous function $f \geqslant 0$, Λf is universally measurable by 1) of Prop. 8, so the assumption that

$$\nu'^{\bullet}(f) = \mu'^{\bullet}(\Lambda f)$$

for all such f says that Λ is μ'-pre-adequate, and this, for every positive measure $\mu' \leqslant \mu$, thus Λ is μ-adequate (No. 1, Def. 1).

(ii) Conversely, if μ is a positive measure on T such that Λ is μ-adequate, then it is clear from No. 1, Def. 1 that Λ is scalarly essentially μ-integrable and satisfies the stated equality for every positive measure $\mu' \leqslant \mu$ and every lower semi-continuous function $f \geqslant 0$ (with the added spice, via 1) of Prop. 8, that Λf is universally measurable).

It turns out (Prop. 11 below) that the stated condition is superfluous: the diffusion Λ is μ-adequate for a measure $\mu \geqslant 0$ if and only if it is scalarly essentially μ-integrable. {See also item (v) in the Note for V.26, ℓ. -2.}

V.28, ℓ. 19–28.

"Proposition 10."

See the next Note.

V.28, ℓ. 28.

"See also the next proposition."

An excellent suggestion. Prop. 11 has already been verified in an earlier note (item (v) in the Note for V.26, ℓ. -2): the proposition says that if Λ is a diffusion and μ is a positive measure on T, then Λ is μ-adequate if and only if it is scalarly essentially μ-integrable. The proof of Prop. 10 can then be conducted as follows.

a) Since f and g are universally measurable, so are $f+g$ and af. For every $t \in \mathrm{T}$,

$$\lambda_t^\bullet(f+g) = \lambda_t^\bullet(f) + \lambda_t^\bullet(g) \qquad (\S1, \text{ No. 1, Prop. 2})$$
$$\lambda_t^\bullet(af) = a\lambda_t^\bullet(f) \qquad (\textit{loc. cit.}, \text{ Prop. 1})$$

thus

$$\Lambda(f+g) : t \mapsto (\Lambda f)(t) + (\Lambda g)(t) = (\Lambda f + \Lambda g)(t)$$
$$\Lambda(af) : t \mapsto a(\Lambda f)(t) = \big(a(\Lambda f)\big)(t)\,,$$

whence $\Lambda(f+g) = \Lambda f + \Lambda g$ and $\Lambda(af) = a(\Lambda f)$.

b) Assuming that Λ is μ-adequate and ν-adequate, so that, in particular, Λ is scalarly essentially integrable for both μ and ν, we are to show that Λ is $(\mu+\nu)$-adequate and that $(\mu+\nu)\Lambda = \mu\Lambda + \nu\Lambda$, that is,

$$\int \lambda_t \, d(\mu+\nu)(t) = \int \lambda_t \, d\mu(t) + \int \lambda_t \, d\nu(t)\,.$$

Let $f \in \mathscr{K}_+(\mathrm{X})$. Since Λ is scalarly essentially integrable for both μ and ν, we know that Λf is essentially integrable for both μ and ν, that is,

$$\Lambda f \in \overline{\mathscr{L}}^1(\mu) \cap \overline{\mathscr{L}}^1(\nu) = \overline{\mathscr{L}}^1(\mu+\nu)$$

(in the Note for V.10, ℓ. 13, 14, see item (i) of Cor. 3 of the *Theorem* there), therefore Λf is essentially $(\mu+\nu)$-integrable and

$$(\mu+\nu)^\bullet(\Lambda f) = \mu^\bullet(\Lambda f) + \nu^\bullet(\Lambda f)$$

(§1, No. 1, Prop. 3); thus Λ is $(\mu+\nu)$-scalarly essentially integrable—hence $(\mu+\nu)$-adequate by Prop. 11—and, for every $f \in \mathscr{K}_+(\mathrm{X})$,

$$\big((\mu+\nu)\Lambda\big)^\bullet(f) = (\mu+\nu)^\bullet(\Lambda f) = \mu^\bullet(\Lambda f) + \nu^\bullet(\Lambda f) = (\mu\Lambda)^\bullet(f) + (\nu\Lambda)^\bullet(f)$$

by the definition of adequacy (No. 1, Def. 1), that is,

$$\big((\mu+\nu)\Lambda\big)^\bullet(f) = (\mu\Lambda + \nu\Lambda)^\bullet(f)$$

(§1, No. 1, Prop. 3), whence $(\mu+\nu)\Lambda = \mu\Lambda + \nu\Lambda$ (§1, No. 1, comment preceding Prop. 1).

Similarly for $a\mu$.

V.28, ℓ. −11 to −9.

"Proposition 11."

See item (v) in the Note for V.26, ℓ. −2.

V.29, ℓ. 6, 7.

"COROLLARY 1."

Note that $\|\lambda_t\| = \lambda_t^*(1) < +\infty$ and $\|\mu\| = \mu^*(1) < +\infty$ (Ch. IV, §4, No. 7, Prop. 12); moreover, $\lambda_t^\bullet = \lambda_t^*$ and $\mu^\bullet = \mu^*$ because bounded measures are moderated (§1, No. 2, *Remark* 2 and Prop. 7).

To show that Λ is μ-adequate, it suffices (Prop. 11) to show that Λg is μ-integrable (hence essentially μ-integrable) for every $g \in \mathscr{K}_+(\mathrm{X})$. At any rate, Λg is universally measurable (Prop. 8). For all $t \in \mathrm{T}$ one has

$$0 \leqslant \lambda_t(g) \leqslant \|\lambda_t\| \, \|g\| \leqslant \|\Lambda\| \, \|g\| ,$$

thus Λg is bounded and

$$\mu^*(\Lambda g) \leqslant \|\Lambda\| \, \|g\| \, \mu^*(1) = \|\Lambda\| \, \|g\| \, \|\mu\| < +\infty ,$$

therefore Λg is μ-integrable (Ch. IV, §5, No. 6, Th. 5). Thus Λ is scalarly essentially μ-integrable and

$$(\mu\Lambda)(g) = \mu^*(\Lambda g) \leqslant \|\Lambda\| \, \|g\| \, \|\mu\| .$$

Then, for every $g \in \mathscr{K}(\mathrm{X})$, one has

$$|(\mu\Lambda)(g)| \leqslant (\mu\Lambda)(|g|) \leqslant \|\Lambda\| \, \|g\| \, \|\mu\| ,$$

thus the measure $\mu\Lambda$ is bounded and $\|\mu\Lambda\| \leqslant \|\Lambda\| \, \|\mu\|$.

V.29, ℓ. 12.

"It suffices to apply the Corollary of Prop. 1 of No. 1."

To set the stage slightly differently, let $\mu = \sum_{\alpha \in \mathrm{A}} \mu_\alpha$, where $(\mu_\alpha)_{\alpha \in \mathrm{A}}$ is any summable family of positive measures on T.

The cited Corollary says that Λ is scalarly essentially integrable for μ if and only if it is scalarly essentially integrable for every μ_α and the resulting measures $\mu_\alpha\Lambda$ are summable, in which case $\sum_{\alpha \in \mathrm{A}} \mu_\alpha\Lambda = \mu\Lambda$.

In the light of Prop. 11, one can replace "scalarly essentially integrable" by "adequate".

V.29, ℓ. 15–19.

"PROPOSITION 12."

Prop. 5 pertains to a mapping $\Lambda : t \mapsto \lambda_t$ and a measure μ such that Λ is μ-adequate in part *c*) but may be μ-pre-adequate in parts *a*) and *b*) (see the comments following No. 1, Def. 1); for simplicity, a global assumption of μ-adequacy is imposed on No. 2, where Prop. 5 appears.

In the context of a diffusion Λ, even if the statement of Prop. 5 only assumes that Λ is scalarly essentially μ-integrable, Prop. 11 ensures that it is μ-adequate. Prop. 5 remains more general than Prop. 12 in that f is only required to be measurable for $\nu = \mu\Lambda$; the reason that f is assumed to be universally measurable in Prop. 12 is that the notation Λf requires it.

At any rate, in the notations of Prop. 12, writing $\nu = \mu\Lambda = \int \lambda_t \, d\mu(t)$, formula (13) says that

$$\nu^\bullet(f) = \mu^\bullet(\Lambda f)\,;$$

when f is ν-moderated, this is the relation (9) given by part a) of Prop. 5, and when the measures λ_t are bounded it is the (same) relation (9) given by part c) of Prop. 5 (where the λ_t are only required to be bounded for locally μ-almost every $t \in \mathrm{T}$).

V.29, ℓ. −11 to −9.

"Corollary."

One is assuming, as in Prop. 12, that μ is a positive measure on T such that the diffusion Λ is μ-adequate, and that f is a universally measurable function $\geqslant 0$ on X.

If X is countable at infinity then every numerical function on X is moderated for every positive measure on X (in particular, f is moderated for $\mu\Lambda$). Thus the hypotheses in the Cor. fulfill the hypotheses in Prop. 12, consequently Λf is μ-measurable and (13) holds. But, by the definition of diffusion, the role of μ in Prop. 12 can be played by any positive measure on T with compact support, therefore Λf is universally measurable by Prop. 6 of No. 4.

From the foregoing we can extract the following, for application in No. 6:

Scholium. *Let* Λ *be a bounded diffusion of* T *in* X *and let* $f \geqslant 0$ *be a universally measurable function on* X. *Then*:

(i) Λf *is a universally measurable function on* T;

(ii) *if* μ *is a positive measure on* T *that belongs to the domain of* Λ, *then* $\langle \mu\Lambda, f\rangle = \langle \mu, \Lambda f\rangle$, *that is,* $(\mu\Lambda)^\bullet(f) = \mu^\bullet(\Lambda f)$.

(iii) *if* μ *is a bounded positive measure on* T, *then* μ *belongs to the domain of* Λ *(hence* $\langle \mu\Lambda, f\rangle = \langle \mu, \Lambda f\rangle$*),* $\mu\Lambda$ *is a bounded positive measure on* X, *and* $\|\mu\Lambda\| \leqslant \|\mu\|\,\|\Lambda\|$.

For, (i) and (ii) follow from the present Corollary, and (iii) is the conclusion of Cor. 1 of Prop. 11.

V.30, ℓ. 4.

"Set $\gamma_t = \lambda_t \mathrm{H}$ "

See item (iii) in the preceding Note: since H is a bounded diffusion of X in Y and λ_t is a bounded measure on X, λ_t belongs to the domain of H and $\|\lambda_t \mathrm{H}\| \leqslant \|\lambda_t\| \, \|\mathrm{H}\|$, whence $\|\lambda_t \mathrm{H}\| \leqslant \|\Lambda\| \, \|\mathrm{H}\|$; moreover, if f is any universally measurable function $\geqslant 0$ on Y, the function

$$\mathrm{H}f : x \mapsto \eta_x^\bullet(f) = \eta_x^*(f)$$

on X is universally measurable and $\langle \lambda_t \mathrm{H}, f \rangle = \langle \lambda_t, \mathrm{H}f \rangle$, that is,

$$\gamma_t^\bullet(f) = \lambda_t^\bullet(\mathrm{H}f)\,.$$

V.30, ℓ. 4–6.

"... we shall denote by Γ the mapping $\Lambda\mathrm{H}$ of T into $\mathscr{M}_+(\mathrm{Y})$, and by Γf the function $t \mapsto \langle \gamma_t, f \rangle$ (an abuse of notation, since we do not yet know whether Γ is a diffusion)."

This No. is about the composition of diffusions, an operation that is applied to mappings. How are we to regard a diffusion as a mapping? Let me count the ways ...

1° When mappings $\Lambda : t \mapsto \lambda_t$ were introduced in No. 1, clearly Λ was being regarded as a mapping $\mathrm{T} \to \mathscr{M}_+(\mathrm{X})$.

2° But Λ also induces a mapping $\mathscr{F}_+(\mathrm{X}) \to \mathscr{F}_+(\mathrm{T})$: if $f \in \mathscr{F}_+(\mathrm{X})$ the corresponding element of $\mathscr{F}_+(\mathrm{T})$ is the function

$$t \mapsto \lambda_t^\bullet(f) \qquad (t \in \mathrm{T})\,,$$

which we have been denoting in these notes by $\Lambda(f)$ (with parentheses) since the Note for V.17, ℓ. 12, 13; in this sense, $\Lambda : \mathscr{F}_+(\mathrm{X}) \to \mathscr{F}_+(\mathrm{T})$.

3° Finally, if Λ is a diffusion, we have a correspondence $\mu \mapsto \mu\Lambda$, where "μ is in the domain of Λ". It is time to introduce a notation for "the domain of Λ"; I propose writing

$$\mathscr{D}_\Lambda = \{\mu \in \mathscr{M}_+(\mathrm{T}) : \ \Lambda \text{ is } \mu\text{-adequate}\}\,.$$

Then Λ may be regarded as the mapping $\Lambda : \mathscr{D}_\Lambda \to \mathscr{M}_+(\mathrm{X})$ defined by $\mu \mapsto \mu\Lambda$ $(\mu \in \mathscr{D}_\Lambda)$. It is this perspective that is appropriate for the composition of diffusions: given diffusions Λ of T in X, and H of X in Y, if a measure μ on T belonging to $\mathscr{D}_\Lambda$ is such that the measure $\mu\Lambda$ on X belongs to $\mathscr{D}_\mathrm{H}$, then we may form the measure $(\mu\Lambda)\mathrm{H}$ on Y; such measures μ form the domain (in the set-theoretic sense) of the composite mapping $\Lambda \circ \mathrm{H}$ (resp. $\mathrm{H} \circ \Lambda$) according as mappings are written to the right (resp. left) of the elements on which they act.

A mildly instructive example is the bounded diffusion $t \mapsto \varepsilon_t$ of T in T considered in the Note for V.27, ℓ. $-14, -13$; let us denote it by E. As shown there, every positive measure μ on T is in the domain of E and

$\mu\mathrm{E} = \mu$; that is, $\mathscr{D}_{\mathrm{E}} = \mathscr{M}_+(\mathrm{T})$, and E as a mapping of measures is the identity mapping on $\mathscr{M}_+(\mathrm{T})$. As functions acting on the right,

$$\mathscr{D}_{\mathrm{E}\Lambda} = \{\mu \in \mathscr{M}_+(\mathrm{T}) :\ \mu\mathrm{E} = \mu \in \mathscr{D}_\Lambda\} = \mathscr{D}_\Lambda$$

and $\mu(\mathrm{E}\Lambda) = (\mu\mathrm{E})\Lambda = \mu\Lambda$; thus $\mathrm{E}\Lambda = \Lambda$. Similarly $\mathscr{D}_{\Lambda\mathrm{E}} = \mathscr{D}_\Lambda$ and $\Lambda\mathrm{E} = \Lambda$. Of course

$$\{\varepsilon_t :\ t \in \mathrm{T}\} \subset \mathscr{D}_\Lambda ,$$

and if $g \in \mathscr{K}_+(\mathrm{T})$ then the computation

$$\langle \varepsilon_t\Lambda, g\rangle = \langle \varepsilon_t, \Lambda g\rangle = \varepsilon_t^\bullet(\Lambda g) = (\Lambda g)(t) = \lambda_t(g)$$

shows that $\varepsilon_t\Lambda = \lambda_t$; thus, when Λ is restricted to this subset of $\mathscr{D}_\Lambda$ (vaguely homeomorphic to T), the result is in effect the presentation of Λ as a function $\mathrm{T} \to \mathscr{M}_+(\mathrm{X})$.

V.30, ℓ. 6–28.

"Then $\langle \gamma_t, f\rangle = \ldots$ in the course of the above proof."

The main goal is to prove that the mapping $\Gamma : t \mapsto \gamma_t$ of T into $\mathscr{M}_+(\mathrm{Y})$ is a (bounded) diffusion of T in Y ; the notation $\Gamma = \Lambda\mathrm{H}$ is proposed for it, but the relation of the notation with the set-theoretic composition of mappings remains to be developed (see item (viii) below).

The proof is essentially a repeated application of one or more parts of the Scholium in the Note for V.29, ℓ. −11 to −9 (referred to briefly as *Scholium*).

All numerical functions occurring in the proof will, one way or another, be universally measurable. It is helpful to 'color-code' the functions: the letter f is reserved for a general universally measurable function $\geqslant 0$ on Y ; g for an element of $\mathscr{K}_+(\mathrm{Y})$; and h for a lower semi-continuous function $\geqslant 0$ on Y ; they are eligible to be acted on by Γ or H , to produce functions on T or X . Until it is established that Γ is a diffusion, we provisionally employ the notation $\Gamma(f)$ (with parentheses) for the mapping $t \mapsto \gamma_t$ $(t \in \mathrm{T})$.

(i) By the *Scholium*, if $f \geqslant 0$ is a universally measurable function on Y then $\mathrm{H}f$ is a universally measurable function on X and, since the bounded measures λ_t belong to the domain of H , one has

$$\langle \lambda_t\mathrm{H}, f\rangle = \langle \lambda_t, \mathrm{H}f\rangle ,$$

that is, $\gamma_t^\bullet(f) = \lambda_t^\bullet(\mathrm{H}f)$ for all $t \in \mathrm{T}$; in other words,

$$(*) \qquad \Gamma(f) = \Lambda(\mathrm{H}f)$$

for every universally measurable function $f \geqslant 0$ on Y .

{The parentheses on the left side of $(*)$ are provisional, pending the status of Γ; on the right side, they group together the symbols that make up the universally measurable function on which Λ is acting.}

(ii) Let $f \geqslant 0$ be a universally measurable function on Y. By (i), $\mathrm{H}f$ is universally measurable on X. Since Λ is a bounded diffusion of X in Y, it follows from the *Scholium* that $\Lambda(\mathrm{H}f)$ is a universally measurable function on T and, for every measure $\mu \in \mathscr{D}_\Lambda$ (the domain of Λ),

$$\langle \mu\Lambda, \mathrm{H}f \rangle = \langle \mu, \Lambda(\mathrm{H}f) \rangle ,$$

that is, $(\mu\Lambda)^\bullet(\mathrm{H}f) = \mu^\bullet\big(\Lambda(\mathrm{H}f)\big)$. Combined with $(*)$, this shows that $\Gamma(f) = \Lambda(\mathrm{H}f)$ is a universally measurable function on T, and

$$(**) \qquad \langle \mu, \Gamma(f) \rangle = \langle \mu, \Lambda(\mathrm{H}f) \rangle = \langle \mu\Lambda, \mathrm{H}f \rangle$$

for every universally measurable function $f \geqslant 0$ on Y and every $\mu \in \mathscr{D}_\Lambda$.

(iii) If μ is any bounded (positive) measure on T, then the mapping $\Gamma : t \mapsto \gamma_t = \lambda_t \mathrm{H}$ $(t \in \mathrm{T})$ is scalarly essentially μ-integrable.

Given any $g \in \mathscr{K}_+(\mathrm{Y})$ we are to show that the function $\Gamma(g)$ is essentially μ-integrable; by (ii), $\Gamma(g) = \Lambda(\mathrm{H}g)$ is universally measurable, so it will suffice to show that $\mu^\bullet\big(\Gamma(g)\big) < +\infty$ (§1, No. 3, Prop. 9). The computation

$$(\mathrm{H}g)(x) = \eta_x(g) \leqslant \|\eta_x\| \, \|g\| \qquad (x \in \mathrm{X})$$

shows that $0 \leqslant \mathrm{H}g \leqslant \|\mathrm{H}\| \, \|g\|$, therefore

$$\lambda_t^*(\mathrm{H}g) \leqslant \|\mathrm{H}\| \, \|g\| \, \lambda_t^*(1) = \|\mathrm{H}\| \, \|g\| \, \|\lambda_t\| \leqslant \|\mathrm{H}\| \, \|g\| \, \|\Lambda\|$$

for all $t \in \mathrm{T}$, whence $0 \leqslant \Lambda(\mathrm{H}g) \leqslant \|\mathrm{H}\| \, \|g\| \, \|\Lambda\|$. Thus the universally measurable function $\Gamma(g) = \Lambda(\mathrm{H}g)$ is bounded, hence it is integrable for the bounded measure μ.

Anticipating that Λ will prove to be a diffusion, we denote by $\mu\Gamma$ the integral $\int \gamma_t \, d\mu(t)$ of Γ (without relinquishing control to auto-pilot).

(iv) If μ is a bounded measure on T, then $(\mu\Lambda)\mathrm{H} = \mu\Gamma$.

Here, $\mu\Gamma$ is the measure on Y defined in (iii). Since μ is a bounded measure on T, by the *Scholium* $\mu \in \mathscr{D}_\Lambda$ and $\mu\Lambda$ is a bounded measure on X, whence $\mu\Lambda \in \mathscr{D}_\mathrm{H}$ and $(\mu\Lambda)\mathrm{H}$ is a bounded measure on Y. For every $g \in \mathscr{K}_+(\mathrm{Y})$,

$$\langle \mu\Gamma, g \rangle = \langle \mu, \Gamma(g) \rangle = \langle \mu\Lambda, \mathrm{H}g \rangle = \langle (\mu\Lambda)\mathrm{H}, g \rangle$$

by the definition of $\mu\Gamma$, the equality $(**)$, and the fact that $\mu\Lambda \in \mathscr{D}_\mathrm{H}$; thus the measures $\mu\Gamma$ and $(\mu\Lambda)\mathrm{H}$ on Y are equal.

(v) Γ is a diffusion (hence is a bounded diffusion).

Assuming μ is a measure on T with compact support, we know from (iii) that Γ is scalarly essentially μ-integrable, and we write $\mu\Gamma$ for its integral $\int \gamma_t \, d\mu(t)$. We are to show that Γ is μ-adequate, and since the positive measures $\mu' \leqslant \mu$ also have compact support, it suffices to show that Γ is μ-pre-adequate.

Given any lower semi-continuous function $h \geqslant 0$ on Y, we must show that $\Gamma(h)$ is μ-measurable (already noted in (ii)) and that

$$(\mu\Gamma)^\bullet(h) = \mu^\bullet\big(\Gamma(h)\big),$$

that is, $\langle \mu\Gamma, h\rangle = \langle \mu, \Gamma(h)\rangle$. Indeed, since μ and $\mu\Lambda$ are bounded and h is universally measurable, one has

$$\langle \mu\Gamma, h\rangle = \langle (\mu\Lambda)\mathrm{H}, h\rangle = \langle \mu\Lambda, \mathrm{H}h\rangle = \langle \mu, \Lambda(\mathrm{H}h)\rangle = \langle \mu, \Gamma(h)\rangle$$

(the first equality by (iv), the second and third by the *Scholium*, and the fourth by (∗∗). Thus Γ is a diffusion.

Finally, $\|\gamma_t\| = \|\lambda_t \mathrm{H}\| \leqslant \|\lambda_t\| \, \|\mathrm{H}\| \leqslant \|\Lambda\| \, \|\mathrm{H}\| < +\infty$ by the *Scholium*, thus the diffusion Γ is bounded.

(vi) If $\mu \in \mathscr{D}_\Lambda$ and $\mu\Lambda \in \mathscr{D}_\mathrm{H}$ (so that $\mu\Lambda$ and $(\mu\Lambda)\mathrm{H}$ are defined) then $\mu \in \mathscr{D}_\Gamma$ and $\mu\Gamma = (\mu\Lambda)\mathrm{H}$.

Since Γ is a diffusion, it suffices by Prop. 11 to show that Γ is scalarly essentially μ-integrable. Given $g \in \mathscr{K}_+(\mathrm{Y})$ we are to show that Γg is essentially μ-integrable.

At any rate, Γg and $\mathrm{H}g$ are universally measurable (*Scholium*) and

$$\langle \mu, \Gamma g\rangle = \langle \mu, \Lambda(\mathrm{H}g)\rangle = \langle \mu\Lambda, \mathrm{H}g\rangle = \langle (\mu\Lambda)\mathrm{H}, g\rangle$$

(the first equality, by (∗∗); the second, by the *Scholium* applied to Λ; the third, because $\mu\Lambda \in \mathscr{D}_\mathrm{H}$), and the last expression on the right is the result of applying the measure $(\mu\Lambda)\mathrm{H}$ to $g \in \mathscr{K}_+(\mathrm{Y})$ hence is finite. Thus $\langle \mu, \Gamma g\rangle < +\infty$, that is, $\mu^\bullet(\Gamma g) < +\infty$, therefore Γg is essentially μ-integrable (§1, No. 3, Prop. 9).

Thus $\mu \in \mathscr{D}_\Gamma$. But then, for every $g \in \mathscr{K}_+(\mathrm{Y})$ one has

$$\langle \mu\Gamma, g\rangle = \langle \mu, \Gamma g\rangle = \langle (\mu\Lambda)\mathrm{H}, g\rangle \,,$$

consequently $\mu\Gamma = (\mu\Lambda)\mathrm{H}$.

(vii) By definition, $\Lambda\mathrm{H}$ is a *notation* for Γ; thus, assuming that $\mu \in \mathscr{D}_\Lambda$ and $\mu\Lambda \in \mathscr{D}_\mathrm{H}$, we know from (vi) that $\mu \in \mathscr{D}_{\Lambda\mathrm{H}}$; we are to show that the following equalities hold for every universally measurable function $f \geqslant 0$ on Y:

$$(15') \qquad \begin{aligned} &\langle \mu\Gamma, f\rangle = \langle \mu\Lambda, \mathrm{H}f\rangle = \langle \mu, \Gamma f\rangle\,; \\ &(\mu\Lambda)\mathrm{H} = \mu\Gamma\,; \quad \Lambda(\mathrm{H}f) = \Gamma f\,. \end{aligned}$$

In the first line of the display, the first equality holds by $(**)$, while $\langle \mu\Gamma, f\rangle = \langle \mu, \Gamma f\rangle$ holds by the *Scholium*.

In the second line, the first equality holds by (vi), the second by $(*)$.

(viii) Let $\Lambda : t \mapsto \lambda_t$ and $\mathrm{H} : x \mapsto \eta_x$ be bounded diffusions of T in X and of X in Y, respectively. If μ is any bounded positive measure on T then, by the *Scholium*, $\mu \in \mathscr{D}_\Lambda$ and $\mu\Lambda$ is bounded, hence $\mu\Lambda \in \mathscr{D}_\mathrm{H}$. Thus, viewing Λ and H as mappings

$$\Lambda : \mathscr{D}_\Lambda \to \mathscr{M}_+(\mathrm{X})\,, \quad \mathrm{H} : \mathscr{D}_\mathrm{H} \to \mathscr{M}_+(\mathrm{Y})\,,$$

the domain of the composite mapping (Λ followed by H) is nonempty. In this context, let us write diffusions on the right of elements of their domain, so that the composite in question is $\Lambda \circ \mathrm{H}$ (thus $\mu(\Lambda \circ \mathrm{H}) = (\mu\Lambda)\mathrm{H}$). We have shown in (vi) that

$$\mu \in \mathscr{D}_{\Lambda\circ\mathrm{H}} \quad \Rightarrow \quad \mu \in \mathscr{D}_\Gamma \quad \text{and} \quad \mu(\Lambda \circ \mathrm{H}) = (\mu\Lambda)\mathrm{H} = \mu\Gamma\,;$$

thus, identifying mappings with their graphs, we have the inclusion

$$\Lambda \circ \mathrm{H} \subset \Gamma\,,$$

that is, $\Lambda \circ \mathrm{H} \subset \Lambda\mathrm{H}$. The inclusion may be proper. {For example if $\eta_x = 0$ for all $x \in \mathrm{X}$, then $\mathrm{H}g = 0$ for all $g \in \mathscr{K}_+(\mathrm{Y})$, and so

$$\langle \gamma_t, g\rangle = \langle \lambda_t\mathrm{H}, g\rangle = \langle \lambda_t, \mathrm{H}g\rangle = 0 \quad \text{for all } t \in \mathrm{T}\,;$$

thus $\Gamma g = 0$ for all $g \in \mathscr{K}_+(\mathrm{Y})$. If μ is any positive measure on T then, for every $g \in \mathscr{K}_+(\mathrm{Y})$, $\Gamma g = 0$ is essentially μ-integrable, therefore $\mu \in \mathscr{D}_\Gamma$ (Prop. 11); thus $\mathscr{D}_\Gamma = \mathscr{M}_+(\mathrm{T})$, whereas $\mathscr{D}_{\Lambda\circ\mathrm{H}} \subset \mathscr{D}_\Lambda$, which may be a proper subset of $\mathscr{M}_+(\mathrm{T})$.}

Thus Γ is an *extension* of $\Lambda \circ \mathrm{H}$, of which, in a sense, it is the 'completion'.

V.30, $\ell.\ -2, -1$.

"It follows at once from Prop. 13 that

$$(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)\,."$$

Say $\Lambda_1 : x_1 \mapsto \lambda_{x_1} \in \mathscr{M}_+(\mathrm{X}_2)$ for $x_1 \in \mathrm{X}_1$. Let us write $\Gamma = \Lambda_1\Lambda_2$, a bounded diffusion of X_1 in X_3; for $x_1 \in \mathrm{X}_1$, λ_{x_1} is a bounded measure, hence belongs to the domain of Λ_2, and, writing $\gamma_{x_1} = \lambda_{x_1}\Lambda_2$, one has $\Gamma : x_1 \mapsto \gamma_{x_1}$ $(x_1 \in \mathrm{X}_1)$. Then

(a) $$(\Lambda_1\Lambda_2)\Lambda_3 = \Gamma\Lambda_3 : x_1 \mapsto \gamma_{x_1}\Lambda_3 = (\lambda_{x_1}\Lambda_2)\Lambda_3 \qquad (x_1 \in \mathrm{X}_1)\,.$$

On the other hand,

(b) $$\Lambda_1(\Lambda_2\Lambda_3) : x_1 \mapsto \lambda_{x_1}(\Lambda_2\Lambda_3) \qquad (x_1 \in \mathrm{X}_1)\,.$$

Since $(\lambda_{x_1}\Lambda_2)\Lambda_3 = \lambda_{x_1}(\Lambda_2\Lambda_3)$ by the equation in the southwest corner of the array (15), the mappings (a) and (b) are equal.

V.31, ℓ. 4–8.

"... one defines diffusions Λ and H by the formulas

$$\lambda_t = \varepsilon_{u(t)}\,, \quad \eta_x = \varepsilon_{v(x)}\,;$$

the diffusion $\Gamma = \Lambda\mathrm{H}$ is then given by

$$\gamma_t = \varepsilon_{(v\circ u)(t)}\,.\text{"}$$

If $f \in \mathscr{K}_+(\mathrm{X})$ then the function

$$t \mapsto \lambda_t(f) = \varepsilon_{u(t)}(f) = f\big(u(t)\big) = (f\circ u)(t)$$

is bounded and universally measurable, hence is integrable for every bounded measure on T, therefore:

(i) Λ is scalarly essentially μ-integrable for every bounded positive measure μ on T;

(ii) Λ is 'vaguely universally measurable', that is, vaguely measurable for every measure μ on T;

(iii) Λ is μ-adequate for every bounded positive measure μ on T, by (i), (ii) and Prop. 2, *b*) of No. 1; in particular,

(iv) Λ is μ-adequate for every positive measure μ on T with compact support, hence is a diffusion. And $\|\lambda_t\| = 1$ for all t, so Λ is a bounded diffusion of T in X.

Similarly H is a bounded diffusion of X in Y.

Then $\Gamma = \Lambda\mathrm{H}$ is defined by

$$\Gamma : t \mapsto \lambda_t\mathrm{H} = \varepsilon_{u(t)}\mathrm{H}\,;$$

but, for every $g \in \mathscr{K}_+(\mathrm{Y})$,

$$\langle \varepsilon_{u(t)}\mathrm{H}, g\rangle = \langle \varepsilon_{u(t)}, \mathrm{H}g\rangle = \big(\mathrm{H}g\big)\big(u(t)\big) = \eta_{u(t)}(g) = \varepsilon_{v(u(t))}(g)\,,$$

thus $\varepsilon_{u(t)}\mathrm{H} = \varepsilon_{(v\circ u)(t)}$ for all $t \in \mathrm{T}$, whence

$$\Gamma : t \mapsto \lambda_t\mathrm{H} = \varepsilon_{u(t)}\mathrm{H} = \varepsilon_{(v\circ u)(t)}\,.$$

§4. INTEGRATION OF POSITIVE POINT MEASURES

V.31, ℓ. -15.

"$\int^* f(x)\,d\lambda_t(x) = \int^\bullet f(x)\,d\lambda_t(x) = f\big(\pi(t)\big)g(t)$"

The assertion is that $\lambda_t^*(f) = \lambda_t^\bullet(f) = f\big(\pi(t)\big)g(t)$. Since λ_t is a bounded measure, $\lambda_t^\bullet = \lambda_t^*$ (§1, No. 2, Cor. 2 of Prop. 7).

If $g(t) = 0$ then $\lambda_t = 0$, so $\lambda_t^*(f) = 0$, and $f\big(\pi(t)\big)g(t) = 0$ (even if $f\big(\pi(t)\big) = +\infty$).

Otherwise, $0 < g(t) < +\infty$ and $\lambda_t = g(t)\varepsilon_{\pi(t)}$, whence

$$\lambda_t^*(f) = g(t)\varepsilon_{\pi(t)}^*(f) = g(t)f\big(\pi(t)\big)$$

by item (v) of the *Example* in the Note for V.27, ℓ. $-14, -13$.

V.31, ℓ. $-13, -12$.

"Every function (with values in a topological space) defined on X is λ_t-measurable for every $t \in \mathrm{T}$."

See item (iii) of the *Example* cited above.

V.31, ℓ. $-12, -11$.

"Every mapping $\mathbf{f}$ of X into a Banach space F is λ_t-integrable for all $t \in \mathrm{T}$, and $\int \mathbf{f}(x)\,d\lambda_t(x) = \mathbf{f}\big(\pi(t)\big)g(t)$."

If $g(t) = 0$ then $\lambda_t = 0$ and it is trivial that $\mathbf{f}$ is λ_t-integrable with integral 0. {The gory details are worked out at the end of this note.}

Otherwise $0 < g(t) < +\infty$ and $\lambda_t = g(t)\varepsilon_{\pi(t)}$. We know that $\mathbf{f}$ is measurable for λ_t; moreover,

$$\lambda_t^*(|\mathbf{f}|) = g(t)\varepsilon_{\pi(t)}^*(|\mathbf{f}|) = g(t)\cdot|\mathbf{f}|\big(\pi(t)\big) < +\infty$$

(the second equality by item (v) of the above-cited *Example*), therefore $\mathbf{f}$ is λ_t-integrable (Ch. IV, §5, No. 6, Th. 5).

It remains to show that

$$\int \mathbf{f}\,d\lambda_t = \mathbf{f}\big(\pi(t)\big)g(t) = g(t)\cdot(\mathbf{f}\circ\pi)(t)\,.$$

Given any continuous linear form u on F, it suffices to show that

$$(*) \qquad u\Big(\int \mathbf{f}\,d\lambda_t\Big) = g(t)\cdot(u\circ\mathbf{f})\big(\pi(t)\big).$$

By Th. 1 of Ch. IV, §4, No. 2, $u\circ\mathbf{f}$ is λ_t-integrable (see also the next Note) and

$$u\Big(\int \mathbf{f}\,d\lambda_t\Big) = \int (u\circ\mathbf{f})\,d\lambda_t\,;$$

but

$$\int (u\circ\mathbf{f})\,d\lambda_t = g(t)\int (u\circ\mathbf{f})\,d\varepsilon_{\pi(t)} = g(t)(u\circ\mathbf{f})\big(\pi(t)\big)$$

(the second equality, by item (iv) of the above-cited *Example*), whence the asserted equality.

{Write θ for the zero measure on X. One has $\theta^*(|\mathbf{f}|) = 0$; for, $\theta(g) = 0$ for every $g \in \mathscr{K}_+(\mathrm{X})$, therefore $\theta^*(h) = 0$ for every lower semi-continuous function $h \geqslant 0$ (Ch. IV, §1, No. 2, Def. 2), whence $\theta^*(f) = 0$ for every numerical function $f \geqslant 0$ (*loc. cit.*, No. 3, Def. 3). In particular $|\mathbf{f}|$ is θ-negligible for every function $\mathbf{f}$ with values in a Banach space F or in $\overline{\mathbf{R}}$, thus $\mathbf{f} = 0$ almost everywhere for θ, whence $\mathbf{f}$ is θ-integrable with integral 0 (p. IV.33, ℓ. −3).}

V.31, ℓ. −10 to −8.

"Finally, if f is an arbitrary numerical function defined on X, for f to be λ_t-integrable it is necessary and sufficient that $f\big(\pi(t)\big)g(t)$ be finite, in which case $\int f(x)\,d\lambda_t(x) = f\big(\pi(t)\big)g(t)$."

As shown in item (iv) of the above-cited *Example*, f is integrable for $\varepsilon_{\pi(t)}$ if and only if $f\big(\pi(t)\big)$ is finite, in which case $\int f\,d\varepsilon_{\pi(t)} = f\big(\pi(t)\big)$. Then,

$$\begin{aligned} f \text{ is } \lambda_t\text{-integrable} \quad &\Leftrightarrow \quad \lambda_t = 0 \text{ or } f \text{ is } \varepsilon_{\pi(t)}\text{-integrable} \\ &\Leftrightarrow \quad g(t) = 0 \text{ or } f\big(\pi(t)\big) \text{ is finite} \\ &\Leftrightarrow \quad g(t)f\big(\pi(t)\big) \text{ is finite}, \end{aligned}$$

in which case the asserted equality is evident from the preliminary remark.

V.31, ℓ. −4, −3.

" 2° *For every function* $f \in \mathscr{K}(\mathrm{X})$, *the mapping* $t \mapsto f\big(\pi(t)\big)g(t)$ *is essentially* μ*-integrable.*"

Defining $\lambda_t = g(t)\varepsilon_{\pi(t)}$ for $t \in \mathrm{T}$, the validity of 2° for every $f \in \mathscr{K}_+(\mathrm{X})$ says that the mapping $\Lambda : t \mapsto \lambda_t \in \mathscr{M}_+(\mathrm{X})$ is scalarly essentially μ-integrable (§3, No. 1).

Conversely, if the assertion in 2° holds for every $f \in \mathscr{K}_+(\mathrm{X})$, then it holds for every $f \in \mathscr{K}(\mathrm{X})$; for,

$$(f\circ\pi)g = \big((f^+ - f^-)\circ\pi\big)g = (f^+\circ\pi)g - (f^-\circ\pi)g$$

is then the difference of finite-valued essentially integrable functions, hence is essentially integrable.

It is important here that g is finite-valued; eventually (No. 4, *Remark* below), g may be defined only locally μ-almost everywhere and may have infinite values.

V.32, ℓ. 9–11.

"In the general case, the set of compact subsets K of T such that the restrictions of π and g to K are continuous is μ-dense (Ch. IV, §5, No. 10, Prop. 15)"

Proof #1: The mapping $f(t) = \big(\pi(t), g(t)\big)$ of T into $\mathrm{X} \times \overline{\mathbf{R}}$ is μ-measurable (Ch. IV, §5, No. 3, Th. 1, with u the identity mapping on $\mathrm{X}\times\overline{\mathbf{R}}$), and its compositions with the coordinate projection mappings are π and g; if K is a compact set in T such that $f\big|\mathrm{K}$ is continuous (the set of such K is μ-dense by the cited Prop. 15), then $\pi\big|\mathrm{K}$ and $g\big|\mathrm{K}$ are continuous.

Proof #2: Writing K for the generic compact subset of T, we know that the sets

$$\mathfrak{K}_\pi = \{\mathrm{K} : \ \pi\big|\mathrm{K} \text{ is continuous}\}, \quad \mathfrak{K}_g = \{\mathrm{K} : \ g\big|\mathrm{K} \text{ is continuous}\}$$

are μ-dense in T by criterion a) of the cited Prop. 15 (with $\mathrm{A} = \mathrm{T}$); it will clearly suffice to show that the set

$$\mathfrak{K} = \{\mathrm{K} \cap \mathrm{K}' : \ \mathrm{K} \in \mathfrak{K}_\pi \text{ and } \mathrm{K}' \in \mathfrak{K}_g\}$$

is also μ-dense. To this end, we verify that $\mathfrak{K}$ satisfies the conditions ($\mathrm{PL_I}$), ($\mathrm{PL_{II}}$) and b) of Ch. IV, §5, No. 8, Prop. 12.

($\mathrm{PL_I}$): If $\mathrm{K}\cap\mathrm{K}' \in \mathfrak{K}$ and B is a closed subset of $\mathrm{K}\cap\mathrm{K}'$, then $\mathrm{B}\subset\mathrm{K}$ and $\mathrm{B}\subset\mathrm{K}'$, whence $\mathrm{B} \in \mathfrak{K}_\pi \cap \mathfrak{K}_g$ and so $\mathrm{B} = \mathrm{B}\cap\mathrm{B} \in \mathfrak{K}$.

($\mathrm{PL_{II}}$): Suppose $\mathrm{K}_1 \cap \mathrm{K}'_1$ and $\mathrm{K}_2 \cap \mathrm{K}'_2$ belong to $\mathfrak{K}$. Then the set $\mathrm{B} = (\mathrm{K}_1 \cap \mathrm{K}'_1) \cup (\mathrm{K}_2 \cap \mathrm{K}'_2)$ is a closed subset of the set

$$(\mathrm{K}_1 \cup \mathrm{K}_2) \cap (\mathrm{K}'_1 \cup \mathrm{K}'_2) \in \mathfrak{K},$$

hence $\mathrm{B} \in \mathfrak{K}$ by the property ($\mathrm{PL_I}$) just proved.

Incidentally, $\mathfrak{K} = \mathfrak{K}_\pi \cap \mathfrak{K}_g$. For, $\mathfrak{K} \subset \mathfrak{K}_\pi \cap \mathfrak{K}_g$ since $\mathrm{K}\cap\mathrm{K}'$ is a closed subset of K and of K'; whereas if $\mathrm{K} \in \mathfrak{K}_\pi \cap \mathfrak{K}_g$ then $\mathrm{K} = \mathrm{K}\cap\mathrm{K} \in \mathfrak{K}$.

b): Let K_0 be a compact subset of T, let $\varepsilon > 0$, and choose $\mathrm{K} \in \mathfrak{K}_\pi$ and $\mathrm{K}' \in \mathfrak{K}_g$ so that $\mathrm{K} \subset \mathrm{K}_0$, $\mathrm{K}' \subset \mathrm{K}_0$ and

$$\mu(\mathrm{K}_0 - \mathrm{K}) \leqslant \varepsilon/2 , \quad \mu(\mathrm{K}_0 - \mathrm{K}') \leqslant \varepsilon/2 ;$$

then $K \cap K' \in \mathfrak{K}$ and $K_0 - K \cap K' = (K_0 - K) \cup (K_0 - K')$, whence $\mu\bigl(K_0 - K \cap K'\bigr) \leqslant \varepsilon$.

Remark. From here on, the proof that Λ is μ-adequate is straightforward.

Example. *If π and g are universally measurable, and g is bounded, then Λ is a bounded diffusion and $\|\Lambda\| = \|g\|$.*

For, if $f \in \mathscr{K}(X)$ then $t \mapsto g(t)f\bigl(\pi(t)\bigr)$ is universally measurable and bounded, hence μ-integrable for every bounded positive measure μ on T. By the part of Prop. 1 already proved, the mapping $\Lambda : t \mapsto \lambda_t = g(t)\varepsilon_{\pi(t)}$ is μ-adequate for every bounded positive measure μ, hence for every positive measure with compact support. Thus Λ is a diffusion (§3, No. 5, Def. 3) and $\|\lambda_t\| = g(t)$ for all $t \in T$.

V.32, ℓ. 18–20.

"The set $\mathfrak{K}$ of compact sets $K \subset S$ such that $g\big|K$ is continuous and $\Lambda\big|K$ is vaguely continuous is μ-dense in S (Ch. IV, §5, No. 10, Prop. 15)"

It is well to pause over the concept of vague measurability introduced in §3, No. 1 (whose definition is subtly different from that of vague continuity). To say that Λ is vaguely continuous means that for every $f \in \mathscr{K}(X)$, the function $\Lambda f : t \mapsto \lambda_t(f) = \langle \lambda_t, f \rangle$ is continuous (*loc. cit.*). Whereas:

To say that the mapping $\Lambda : T \to \mathscr{M}(X)$ is vaguely μ-measurable means that when $\mathscr{M}(X)$ is equipped with the vague topology, Λ is μ-measurable in the sense of Ch. IV, §5, No. 1, Def. 1; this is equivalent to the condition that the set $\mathfrak{K}_\Lambda$ of all compact sets K in T, such that the restriction $\Lambda\big|K : K \to \mathscr{M}(X)$ is continuous, is μ-dense in T (*loc. cit.*, No. 10, Prop. 15); and, for a compact set K in T, $K \in \mathfrak{K}_\Lambda$ means that for every $f \in \mathscr{K}(X)$, the numerical function $t \mapsto \lambda_t(f)$ $(t \in K)$ is continuous, that is, $\Lambda f\big|K$ is continuous.

{Caution. It is clear that if Λ is vaguely μ-measurable, then for every $f \in \mathscr{K}(X)$ the numerical function Λf is μ-measurable since, in the foregoing notations, $\mathfrak{K}_\Lambda \subset \mathfrak{K}_{\Lambda f}$, so that the μ-density of $\mathfrak{K}_\Lambda$ implies that of $\mathfrak{K}_{\Lambda f}$ (the set of all compact sets K such that $\Lambda f\big|K$ is continuous; clearly $\mathfrak{K}_{\Lambda f}$ satisfies (PL_I) and (PL_{II}) of Ch. IV, §5, No. 8, Prop. 12, and $\mathfrak{K}_\Lambda \subset \mathfrak{K}_{\Lambda f}$ ensures that it satisfies criterion *a*) of that Proposition). Problem: When is the converse true?}

Since S is a μ-measurable subset of T and the restriction $\Lambda\big|S$ has a (vaguely) μ-measurable extension to T (namely Λ itself), it follows that $\Lambda\big|S$ is μ-measurable (see criterion c'') in the Note for IV.79, ℓ. 3,4).

Similarly, $g\big|S$ is μ-measurable. {Alternate proof: the extension by 0 of $g\big|S$ to T is the μ-measurable function $g\varphi_S$, therefore $g\big|S$ is μ-measurable by criterion *d*) of the cited Prop. 15.}

With K denoting the generic compact subset of S, we know from criterion a) of the cited Prop. 15 that the sets

$$\mathfrak{K}_{\Lambda|\mathrm{S}} = \{\mathrm{K} : \ \Lambda\big|\mathrm{K} \text{ is continuous}\}, \quad \mathfrak{K}_{g|\mathrm{S}} = \{\mathrm{K} : \ g\big|\mathrm{K} \text{ is continuous}\}$$

are μ-dense in S. Obviously $\mathfrak{K}_{\Lambda|\mathrm{S}} \cap \mathfrak{K}_{g|\mathrm{S}} = \mathfrak{K}$ (the set in the assertion). One proves, as in the preceding Note, that the set

$$\{\mathrm{K} \cap \mathrm{K}' : \ \mathrm{K} \in \mathfrak{K}_{\Lambda|\mathrm{S}} \ \text{ and } \ \mathrm{K}' \in \mathfrak{K}_{g|\mathrm{S}}\}$$

is μ-dense in S and is equal to $\mathfrak{K}$ (with S playing the role of A in Ch. IV, §5, No. 8, Prop. 12).

V.32, ℓ. 21, 22.

"... and this implies the continuity of $\pi\big|\mathrm{K}$ (Ch. III, §1, No. 9, Prop. 13)."

By the cited Prop. 13, the mapping $x \mapsto \varepsilon_x$ $(x \in \mathrm{X})$ is a homeomorphism of X onto a subspace $\mathscr{E}$ of $\mathscr{M}(\mathrm{X})$, whence the continuity of the composite

$$t \mapsto \lambda_t \mapsto \frac{1}{g(t)}\lambda_t = \varepsilon_{\pi(t)} \mapsto \pi(t) \qquad (t \in \mathrm{K}),$$

where the last mapping is the restriction of the inverse homeomorphism $\mathscr{E} \to \mathrm{X}$ to the subspace $\{\varepsilon_{\pi(t)} : \ t \in \mathrm{K}\}$ of $\mathscr{E}$.

V.32, ℓ. −14 to −9.

"*Lemma.*"

The following graphics help to keep track of the numerous players in the otherwise elementary proof.

$$f(x) = \begin{cases} \inf\{g(t) : \ \pi(t) = x\} & \text{when } x \in \pi(\mathrm{T}) \\ +\infty & \text{when } x \in \mathrm{X} - \pi(\mathrm{T}). \end{cases}$$

$$\begin{array}{ccc} \mathrm{T} & \xrightarrow{\ \pi\ } & \mathrm{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{\mathbf{R}} & & \overline{\mathbf{R}} \end{array}$$

The range of π determines a partitioning of T into the sets of constancy of π: that is, the relation $\mathrm{R}\{t, t'\} \Leftrightarrow \pi(t) = \pi(t')$ defines a quotient set T/R whose elements are indexed by the elements of the range of π, thus there is a bijection $\pi(\mathrm{T}) \to \mathrm{T}/\mathrm{R}$, namely

$$x \mapsto \overset{-1}{\pi}(\{x\}) \qquad (x \in \pi(\mathrm{T})).$$

Then g defines a numerical function $g_{\mathrm{R}} : \mathrm{T}/\mathrm{R} \to \overline{\mathbf{R}}$ that assigns to an equivalence class the infimum of the restriction of g to that class:

$$g_{\mathrm{R}} : \overset{-1}{\pi}(\{x\}) \mapsto \inf g\big|\overset{-1}{\pi}(\{x\}) \qquad (x \in \pi(\mathrm{T}))\,;$$

the definition of the function $f : \mathrm{X} \to \overline{\mathbf{R}}$ then takes the form

$$f(x) = \begin{cases} g_{\mathrm{R}}\big(\overset{-1}{\pi}(\{x\})\big) & \text{when } x \in \pi(\mathrm{T}) \\ \quad +\infty & \text{when } x \in \mathrm{X} - \pi(\mathrm{T})\,. \end{cases}$$

V.32, ℓ. -1.

"... and nonempty"

Since $f(x) \leqslant a < +\infty$, $\overset{-1}{\pi}(x)$ is nonempty by the definition of f.

V.32, ℓ. -1 to **V.33,** ℓ. 1.

"... there exists a $t \in \overset{-1}{\pi}(x)$ such that ..."

Recall that the restriction of g to any subspace S of T is lower semi-continuous, since $\{t \in \mathrm{S} :\ g(t) \leqslant a\} = \mathrm{S} \cap \{t \in \mathrm{T} :\ g(t) \leqslant a\}$.

V.33, ℓ. 10, 11.

"*For every numerical function* $f \geqslant 0$ *defined on* X,

$$\text{(1)} \qquad \int^{\bullet} f(x)\, d\nu(x) = \int^{\bullet} f\big(\pi(t)\big) g(t)\, d\mu(t)\,. \text{"}$$

We know that the mapping $\Lambda : t \mapsto \lambda_t = g(t)\varepsilon_{\pi(t)}$ is μ-adequate (No. 1, Prop. 1), thus (1) is known to hold for every lower semi-continuous function $f \geqslant 0$ (§3, No. 1, Def. 1); the miracle of (π, g) is that (1) holds for *every* function $f \geqslant 0$.

V.33, ℓ. 13–15.

"By formula (4) of §3, No. 1, $\nu^{\bullet}(1) = \int_{\mathrm{K}} g(t)\, d\mu(t) < +\infty$, so that all of the measures that figure in formula (1) are bounded."

Since 1 is lower semi-continuous, by the cited formula (4) one has

$$(*) \qquad \nu^{\bullet}(1) = \int^{\bullet} d\mu(t) \int^{\bullet} 1\, d\lambda_t\,.$$

The crux of the matter is the calculation

$$\int^{\bullet} 1\, d\lambda_t = \lambda_t^{\bullet}(1) = \lambda_t^{*}(1) = \|\lambda_t\| = g(t)\|\varepsilon_{\pi(t)}\| = g(t) < +\infty$$

(the second equality by §1, No. 1, Prop. 4; or by the fact that λ_t is bounded, hence moderated, cf. §1, No. 2, Prop. 7). Thus $(*)$ may be written $\nu^*(1) = \mu^\bullet(g)$, or, since μ is bounded,

$$\nu^*(1) = \mu^*(g)\,.$$

Next, we note that g is μ-integrable. Since μ has support K, $\varphi_{\mathrm{K}} = 1$ μ-almost everywhere, therefore $g = \varphi_{\mathrm{K}} g$ μ-almost everywhere, so it suffices to show that $\varphi_{\mathrm{K}} g$ is μ-integrable. Indeed, $\varphi_{\mathrm{K}} g$ is μ-measurable and is bounded (because $g\big|\mathrm{K}$ is continuous), hence it is integrable with respect to the bounded measure μ; therefore g is μ-integrable and

$$\mu^*(g) = \int g\,d\mu = \int \varphi_{\mathrm{K}} g\,d\mu = \int_{\mathrm{K}} g\,d\mu$$

(the last equality is merely a notation). Finally

$$\nu^*(1) = \mu^*(g) = \int g\,d\mu < +\infty\,,$$

thus ν is bounded and $\|\nu\| = \int g\,d\mu$.

V.33, ℓ. -12.

"In view of formula (6) of §3, No. 2 ..."

The context here is the μ-adequate mapping $\Lambda : t \mapsto \lambda_t = g(t)\varepsilon_{\pi(t)}$, for which the cited formula (6) yields

$$\int^* f(x)\,d\nu(x) \geqslant \int^\bullet d\mu(t) \int^* f(x)\,d\lambda_t(x)\,,$$

where $\int^* f(x)\,d\lambda_t(x) = f\big(\pi(t)\big)g(t)$ (V.31, ℓ. -15) and $\mu^\bullet = \mu^*$ (preceding Note), thus

$$\int^* f(x)\,d\nu(x) \geqslant \int^* f\big(\pi(t)\big)g(t)\,d\mu(t)\,.$$

V.33, ℓ. $-5, -4$.

"... let u be the function $(h + \varepsilon)/g$, which is lower semi-continuous in K."

The function $u : \mathrm{T} \to \overline{\mathbf{R}}_+$ is defined by

$$u(t) = \begin{cases} \dfrac{h(t) + \varepsilon}{g(t)} & \text{when } g(t) > 0 \\ +\infty & \text{when } g(t) = 0\,. \end{cases}$$

The assertion is that $u\big|\mathrm{K}$ is lower semi-continuous; thus, given any $a \in \mathbf{R}$, we are to show that the set

$$\mathrm{K}_a = \{t \in \mathrm{K} : \ u(t) \leqslant a\,\}$$

is a closed set in K. From the definition of u, it is clear that

$$\mathrm{K}_a = \{t \in \mathrm{K} : \ g(t) > 0 \quad \text{and} \quad \frac{h(t)+\varepsilon}{g(t)} \leqslant a\,\}\,.$$

If $a < 0$ then $\mathrm{K}_a = \varnothing$ because $u \geqslant 0$. If $a = 0$ then

$$\mathrm{K}_0 = \{t \in \mathrm{K} : \ g(t) > 0 \quad \text{and} \quad \frac{h(t)+\varepsilon}{g(t)} \leqslant 0\,\} = \varnothing$$

because the fraction is > 0. So we can suppose that $a > 0$. Then

$$\begin{aligned}\mathrm{K}_a &= \{t \in \mathrm{K} : \ g(t) > 0 \quad \text{and} \ \ h(t)+\varepsilon \leqslant ag(t)\,\} \\ &= \{t \in \mathrm{K} : \ h(t)+\varepsilon \leqslant ag(t)\,\} \\ &= \{t \in \mathrm{K} : \ h(t)+\varepsilon - ag(t) \leqslant 0\,\}\,;\end{aligned}$$

since $h\big|\mathrm{K}$ is lower semi-continuous, and $(\varepsilon - ag)\big|\mathrm{K}$ is finite and continuous, their sum is lower semi-continuous (GT, IV, §6, No. 2, Prop. 2), therefore K_a is closed.

V.34, ℓ. 1, 2.

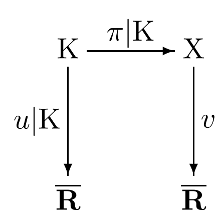
"for every $x \in \mathrm{X}$ let $v(x)$ be the infimum of $u(t)$ for $t \in \overset{-1}{\pi}(\{x\}) \cap \mathrm{K}$."

Note that $\overset{-1}{\pi}(\{x\}) \cap \mathrm{K} = (\pi\big|\mathrm{K})^{-1}(\{x\})$, thus $v(x) = +\infty$ when $x \notin \pi(\mathrm{K}) = \big(\pi\big|\mathrm{K}\big)(\mathrm{K})$ since $(\pi\big|\mathrm{K})^{-1}(\{x\}) = \varnothing$ and $\inf\limits_{t\in\varnothing} = +\infty$ by convention.

V.34, ℓ. 3, 4.

"...it is lower semi-continuous on X by the Lemma (applied to the restriction of π to K)"

The measure μ has compact support K, but it is a measure on T. The Lemma is purely topological, thus we are concerned here exclusively with topology. The picture is



The meaning of the parenthetical remark is that, in the Lemma, K plays the role of T, $\pi|K$ plays the role of π, $u|K$ plays the role of g, and v plays the role of f; thus the recipe for v is the one prescribed for f in the Lemma. Since u is lower semi-continuous, so is its restriction to K. Finally, it must be noted that the continuous mapping $\pi|K$ is proper; this follows from the compactness of K (GT, I, §10, No. 2, Cor. 2 of Th. 1). Thus the hypotheses of the Lemma are fulfilled.

V.34, ℓ. 4, 5.

"... and $v\big(\pi(t)\big)g(t) \leqslant h(t) + \varepsilon$ for all $t \in \mathrm{K}$ (recall that the first member is zero by convention if $g(t) = 0$)."

Let $t \in \mathrm{K}$. By the parenthetical remark, we can suppose that $g(t) > 0$. Set $x = \pi(t)$. By definition,

$$v(x) = \inf\{u(s) : \ s \in \overset{-1}{\pi}(\{x\}) \cap \mathrm{K}\} \leqslant u(t)$$

because $t \in \overset{-1}{\pi}(\{x\})$; then

$$v\big((\pi(t)\big)g(t) = v(x)g(t) \leqslant u(t)g(t) = h(t) + \varepsilon$$

by the definition of u.

V.34, ℓ. 7–9.

$$\begin{aligned}
\int^* f(x)\,d\nu(x) &\leqslant \int^* v(x)\,d\nu(x) \\
&= \int^* v\big(\pi(t)\big)g(t)\,d\mu(t) \leqslant \int_{\mathrm{K}}^* \big(h(t)+\varepsilon\big)\,d\mu(t) \\
&= \int^* h(t)\,d\mu(t) + \varepsilon\mu(1)\,.
\end{aligned} \tag{4}$$

{Incidentally, the desired inequality (3) is trivial if $\mu^*(h) = +\infty$, so we can suppose (to reduce the stress level) that h is μ-integrable.}

The first inequality follows from $v \geqslant f$, and the last equality, from $\varphi_{\mathrm{K}} = 1$ μ-almost everywhere (because $\operatorname{Supp}\mu \subset \mathrm{K}$). The second inequality results from

$$v\big((\pi(t)\big)g(t) \leqslant h(t) + \varepsilon \quad \text{ for } t \in \mathrm{K}\,,$$

equivalently $(v \circ \pi)g\varphi_{\mathrm{K}} \leqslant (h+\varepsilon)\varphi_{\mathrm{K}}$ everywhere on T, and the fact that $(v \circ \pi)g = (v \circ \pi)g\varphi_{\mathrm{K}}$ μ-almost everywhere.

It remains to verify the first equality. Since the mapping $\Lambda : t \mapsto \lambda_t$ is μ-adequate (No. 1, Prop. 1) and $v \geqslant 0$ is lower semi-continuous on X, formula (4) of §3, No. 1 is applicable and yields

$$\int^\bullet v(x)\, d\nu(x) = \int^\bullet d\mu(t) \int^\bullet v(x)\, d\lambda_t(x)\,,$$

that is, $\nu^\bullet(v) = \mu^\bullet\big((v\circ\pi)g\big)$ (p. V.31, ℓ. −15). Since v is lower semi-continuous and μ is bounded, this may be written $\nu^*(v) = \mu^*\big((v\circ\pi)g\big)$, which is the asserted equality.

V.34, ℓ. 13, 14.

"... the set $\mathfrak{K}$ of compact subsets K of T such that the restrictions of π and g to K are continuous is μ-dense (Ch. IV, §5, No. 10, Prop. 15)."

See also the Note for V.32, ℓ. 9–11.

V.34, ℓ. 16, 17.

"... the pair (π, g) being μ_α-adapted for every $\alpha \in \mathrm{A}$ "

If (π, g) is μ-adapted, then it is μ'-adapted for every positive measure $\mu' \leqslant \mu$:

1° π and g are μ'-measurable (because μ-negligible sets are μ'-negligible);

2° every essentially μ-integrable function is essentially μ'-integrable; indeed, writing $\mu'' = \mu - \mu'$, one has $\overline{\mathscr{L}}^1(\mu) = \overline{\mathscr{L}}^1(\mu') \cap \overline{\mathscr{L}}^1(\mu'')$ (see *Corollary* 3 in the Note for V.10, ℓ. 13, 14).

V.34, ℓ. −8 to −6.

"Corollary."

We propose to apply Th. 1 to the function $f = \varphi_{\mathrm{N}}$. Since $\varphi_{\mathrm{N}} \circ \pi = \varphi_{\pi^{-1}(\mathrm{N})}$, by Th. 1 one has

$$\nu^\bullet(\mathrm{N}) = \nu^\bullet(\varphi_{\mathrm{N}}) = \mu^\bullet\big((\varphi_{\mathrm{N}}\circ\pi)g\big) = \mu^\bullet\big(\varphi_{\pi^{-1}(\mathrm{N})}g\big);$$

thus N is locally negligible for ν if and only if the set

$$\{t \in \mathrm{T} :\ \big(\varphi_{\pi^{-1}(\mathrm{N})}g\big)(t) \neq 0\,\} = \overset{-1}{\pi}(\mathrm{N}) \cap \{t \in \mathrm{T} :\ g(t) \neq 0\,\}$$

is locally negligible for μ (§1, No. 1).

V.35, ℓ. 3, 4.

"... for every function $\psi \in \mathscr{K}(\mathrm{X})$, $\psi\circ\pi$ is continuous with compact support, since π is proper "

Let K be a compact set such that $\psi = 0$ on $\mathrm{X} - \mathrm{K}$. If $\psi\big(\pi(t)\big) \neq 0$ then $\pi(t) \in \mathrm{K}$, thus

$$\{t:\ (\psi \circ \pi)(t) \neq 0\} \subset \overset{-1}{\pi}(\mathrm{K})\,;$$

since π is proper, $\overset{-1}{\pi}(\mathrm{K})$ is compact (GT, I, §10, No. 2, Prop. 6), therefore

$$\mathrm{Supp}(\psi \circ \pi) = \overline{\{t:\ (\psi \circ \pi)(t) \neq 0\}} \subset \overset{-1}{\pi}(\mathrm{K})\,,$$

whence $\psi \circ \pi \in \mathscr{K}(\mathrm{T})$.

V.35, ℓ. 4, 5.

"... the pair (π, g) is therefore μ-adapted"

The functions π and g are universally measurable and, for every $\psi \in \mathscr{K}(\mathrm{X})$, the function $(\psi \circ \pi)g$ is continuous with compact support, hence is integrable with respect to every measure on T. Thus (π, g) is, so to speak, 'universally adapted', that is, μ-adapted for every positive measure μ on T.

Remark. It then follows from No. 1, Prop. 1 that the mapping $\Lambda : t \mapsto \lambda_t = g(t)\varepsilon_{\pi(t)}$ is μ-adequate for every positive measure μ on T, in particular for every positive measure with compact support, thus Λ is a diffusion of T in X (§3, No. 5, Def. 3) whose domain is all of $\mathscr{M}_+(\mathrm{T})$.

See also the Example in the Note for V.32, ℓ. 9–11.

V.35, ℓ. 5, 6.

"... the mapping $t \mapsto g(t)\varepsilon_{\pi(t)}$ is *vaguely continuous*."

For every $\psi \in \mathscr{K}(\mathrm{X})$, the mapping $t \mapsto g(t)\varepsilon_{\pi(t)}(\psi) = \psi\big((\pi(t)\big)g(t)$ is continuous (with compact support), whence the assertion (§3, No. 1).

Thus, for the diffusion $\Lambda : t \mapsto \lambda_t = g(t)\varepsilon_{\pi(t)}$ (see the preceding Note), $\Lambda\psi = (\psi \circ \pi)g \in \mathscr{K}_+(\mathrm{T})$ for all $\psi \in \mathscr{K}_+(\mathrm{X})$.

V.35, ℓ. −10.

"1° $\overline{f}(x) \geqslant f(x)$ for all $x \in \mathrm{X}$ (since $g(t) > 0$ for all $t \in \mathrm{T}$)."

If $x \notin \pi(\mathrm{T})$, then $\overline{f}(x) = +\infty$ and the inequality is trivial. If $x \in \pi(\mathrm{T})$ then $\overline{f}(x)$ is defined as an infimum over $\overset{-1}{\pi}(x)$; now, for every $t \in \overset{-1}{\pi}(x)$ one has

$$f(x) = f\big(\pi(t)\big) \leqslant \frac{h(t)}{g(t)}$$

by the choice of h, and taking the infimum over $t \in \overset{-1}{\pi}(x)$ yields $f(x) \leqslant \overline{f}(x)$.

V.35, ℓ. −9.

"2° $\overline{f}\big(\pi(t)\big)g(t) \leqslant h(t)$ for all $t \in \mathrm{T}$."

Fix $t \in \mathrm{T}$ and set $x = \pi(t)$. Thus $\overline{f}\big(\pi(t)\big) = \overline{f}(x)$ is defined as an infimum over $\overset{-1}{\pi}(x)$. Now, $\overset{-1}{\pi}(x)$ is compact since π is proper (GT, I, §10, No. 2, Th. 1), and h/g is lower semi-continuous on T, hence on $\overset{-1}{\pi}(x)$, therefore h/g attains its infimum on $\overset{-1}{\pi}(x)$ (GT, IV, §6, No. 2, Th. 3). Thus there exists a point $t_0 \in \overset{-1}{\pi}(x)$ such that

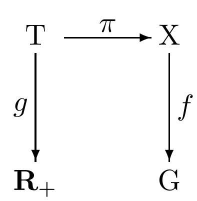
$$\overline{f}(x) = \frac{h(t_0)}{g(t_0)} \leqslant \frac{h(s)}{g(s)} \qquad \text{for all } s \in \overset{-1}{\pi}(x)\,,$$

and in particular $\overline{f}(x) \leqslant h(t)/g(t)$, whence $\overline{f}\big(\pi(t)\big)g(t) \leqslant h(t)$.

V.35, ℓ. −2 to **V.36,** ℓ. 4.

"Proposition 3."

The main players:



V.36, ℓ. 7.

"...is μ-dense in S (Ch. IV, §5, No. 10, Prop. 15)."

Since π is μ-measurable, its restriction to the μ-measurable set S is μ-measurable in the sense of *loc. cit.*, Def. 8 (because $\pi\big|\mathrm{S}$ has an extension to T that is μ-measurable, namely π itself; see the condition c'') in the Note for IV.79, ℓ. 3, 4). Therefore $\mathfrak{K}$ is μ-dense in S by the equivalent condition a) of the cited Prop. 15.

V.36, ℓ. 8–10.

"...it therefore suffices to prove that for every $\mathrm{K} \in \mathfrak{K}$, the set of compact subsets H of K, such that the restriction of $f \circ \pi$ to H is continuous, is μ-dense in K (Ch. IV, §5, No. 8, Prop. 13)."

Write $\mathfrak{H}$ for the set of all compact subsets H of S such that $(f \circ \pi)\big|\mathrm{H}$ is continuous. One sees easily that $\mathfrak{H}$ satisfies the conditions ($\mathrm{PL_I}$) and ($\mathrm{PL_{II}}$) of *loc. cit.*, Prop. 12, that is: if $\mathrm{H} \in \mathfrak{H}$ then every closed subset of H belongs to $\mathfrak{H}$; and if $\mathrm{H_1}, \mathrm{H_2} \in \mathfrak{H}$ then $\mathrm{H_1} \cup \mathrm{H_2} \in \mathfrak{H}$ (GT, I, §3, No. 2, Prop. 4).

Thus if, for every $\mathrm{K} \in \mathfrak{K}$, the set of all $\mathrm{H} \in \mathfrak{H}$ such that $\mathrm{H} \subset \mathrm{K}$ is shown to be μ-dense in K, it will follow from the cited Prop. 13 that $\mathfrak{H}$

is μ-dense in S, and this will establish that $(f \circ \pi)\big|\mathrm{S}$ is μ-measurable by criterion a) of Ch. IV, §5, No. 10, Prop. 15.

Review of the cited Prop. 13: With the notations $\mathfrak{K}$ and $\mathfrak{H}$ as in that Prop., given $\mathrm{K} \in \mathfrak{K}$ note that the set of $\mathrm{H} \in \mathfrak{H}$ such that $\mathrm{H} \subset \mathrm{K}$ is the *trace* $\mathfrak{H} \cap \mathrm{K}$ of $\mathfrak{H}$ on K, that is,

$$\{\mathrm{H} \in \mathfrak{H} : \ \mathrm{H} \subset \mathrm{K}\} = \{\mathrm{H} \cap \mathrm{K} : \ \mathrm{H} \in \mathfrak{H}\} .$$

For, the inclusion $\subset$ is immediate from $\mathrm{H} = \mathrm{H} \cap \mathrm{K}$; whereas if $\mathrm{H} \in \mathfrak{H}$ then $\mathrm{H} \cap \mathrm{K} \in \mathfrak{H}$ (because $\mathfrak{H}$ satisfies $(\mathrm{PL_I})$), whence the inclusion $\supset$.

The message of Prop. 13: If $\mathfrak{K}$ and $\mathfrak{H}$ are sets of compact subsets of the μ-measurable set A, such that (i) $\mathfrak{K}$ is μ-dense in A, (ii) $\mathfrak{H} \cap \mathrm{K}$ is μ-dense in K for every $\mathrm{K} \in \mathfrak{K}$, then $\mathfrak{H}$ is also μ-dense in A.

(A kind of 'transitivity' of μ-denseness.)

V.36, ℓ. 14, 15.

" $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{N})$ is μ-negligible by virtue of the Cor. of Th. 1 of No. 2"

By the cited Cor., the set $\mathrm{S} \cap \overset{-1}{\pi}(\mathrm{N})$ is locally μ-negligible, therefore $\mathrm{K} \cap \mathrm{S} \cap \overset{-1}{\pi}(\mathrm{N})$ is μ-negligible (Ch. IV, §5, No. 2, Prop. 5); and $\mathrm{K} \subset \mathrm{S}$.

V.36, ℓ. 15.

"... the sets $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n)$ are compact"

For, $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n) = (\pi\big|\mathrm{K})^{-1}(\mathrm{C}_n)$ and $\pi\big|\mathrm{K}$ is continuous.

V.36, ℓ. 16.

"... the restriction of $f \circ \pi$ to each of the latter sets is continuous"

Since $\pi\big|\mathrm{K}$ is continuous, so is $\pi\big|\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n)$. Thus π maps $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n)$ continuously into C_n; but $f\big|\mathrm{C}_n$ is continuous, therefore $f \circ \pi$ maps $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n)$ continuously into G.

V.36, ℓ. 17.

"... the restriction of $f \circ \pi$ to S is μ-measurable."

As in the Note for ℓ. 8–10, write $\mathfrak{H}$ for the set of all compact sets $\mathrm{H} \subset \mathrm{S}$ such that $(f \circ \pi)\big|\mathrm{H}$ is continuous. We know that $\mathfrak{H}$ satisfies $(\mathrm{PL_I})$ and $(\mathrm{PL_{II}})$ of Ch. IV, §5, No. 8, Prop. 12; to show that $(f \circ \pi)\big|\mathrm{S}$ is μ-measurable, it will suffice to show that $\mathfrak{H}$ is μ-dense in S (*loc. cit.*, No. 10, Prop. 15, criterion a)).

We know that the set $\mathfrak{K}$ of all compact subsets of S on which π is continuous is μ-dense in S. To show that $\mathfrak{H}$ is μ-dense in S, it will suffice to show that for every $\mathrm{K}' \in \mathfrak{K}$, the trace of $\mathfrak{H}$ on K', namely, the set

$$\mathfrak{H}_{\mathrm{K}'} = \mathfrak{H} \cap \mathrm{K}' = \{\mathrm{H} \in \mathfrak{H} : \ \mathrm{H} \subset \mathrm{K}'\} ,$$

is μ-dense in K' (*loc. cit.*, No. 8, Prop. 13, with T playing the role of X in Prop. 13, and S the role of A). Since $\mathfrak{H}$ satisfies $(\mathrm{PL_I})$ and $(\mathrm{PL_{II}})$, so does $\mathfrak{H}_{\mathrm{K}'}$.

Let $\mathrm{K}' \in \mathfrak{K}$. To show that $\mathfrak{H}_{\mathrm{K}'}$ is μ-dense in K', it will suffice to show that $\mathfrak{H}_{\mathrm{K}'}$ satisfies the criterion *c*) of *loc. cit.*, Prop. 12 (with T playing the role of X, K' the role of A, and $\mathfrak{H}_{\mathrm{K}'}$ the role $\mathfrak{K}$). Thus, let K be any compact subset of K'. Then $\mathrm{K} \in \mathfrak{K}$. Apply to K the arguments of the first part of the proof to produce a partition

$$\mathrm{K} = \big(\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{N})\big) \cup \bigcup_n \mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n),$$

where $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{N})$ is μ-negligible (ℓ. 14, 15) and $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n) \in \mathfrak{H}$ for all n (ℓ. 16); since $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n) \subset \mathrm{K} \subset \mathrm{K}'$, one has $\mathrm{K} \cap \overset{-1}{\pi}(\mathrm{C}_n) \in \mathfrak{H}_{\mathrm{K}'}$, and the criterion *c*) is verified.

V.36, ℓ. −17.

"...let us show that N is locally ν-negligible."

It will then follow from criterion *a*) of Ch. IV, §5, No. 8, Prop. 12 that $\mathfrak{L}$ is μ-dense (in X).

V.36, ℓ. −14.

"...is by hypothesis μ-dense in S (Ch. IV, §5, No. 10, Prop. 15)."

The argument for this is given in the Note for V.32, ℓ. 9–11.

V.36, ℓ. −13.

"It therefore suffices to prove that $\overset{-1}{\pi}(\mathrm{N}) \cap \mathrm{H}$ is μ-negligible for every $\mathrm{H} \in \mathfrak{H}$."

By criterion *a*) of Ch. IV, §5, No. 8, Prop. 12.

V.36, ℓ. −12 to −10.

" $\pi(\mathrm{H})$ is compact and may be identified with the quotient space of H by the equivalence relation $\pi(t) = \pi(t')$, π being identified with the canonical mapping of H onto this quotient space (GT, I, §5, No. 2, Prop. 3)."

Since $\pi|\mathrm{H}$ is a continuous mapping of the compact space H into the Hausdorff space X, it is a closed mapping hence is eligible for application of the cited Prop. 3. One has the diagram

$$\mathrm{H} \xrightarrow{\;p\;} \mathrm{H/R} \xrightarrow{\;h\;} \pi(\mathrm{H}) \xrightarrow{\;i\;} \mathrm{X}.$$

where R is the equivalence relation in H defined by $\pi(t) = \pi(t')$. The elements of H/R are the sets of constancy of $\pi|\mathrm{H}$. Let us write $\dot{t}$ for the equivalence class of $t \in \mathrm{H}$, that is,

$$\dot{t} = \mathrm{H} \cap \overset{-1}{\pi}\big(\pi(t)\big) = \big(\pi|\mathrm{H}\big)^{-1}\big(\pi(t)\big) \qquad (t \in \mathrm{H});$$

thus $p(t) = \dot{t}$ for $t \in \mathrm{H}$, $h(\dot{t}) = \pi(t)$ for $\dot{t} \in \mathrm{H/R}$, and i is the canonical injection of $\pi(\mathrm{H})$ into X.

By definition, $\mathrm{H/R}$ bears the quotient topology, that is, the final topology for the mapping p, and the topology of the subspace $\pi(\mathrm{H})$ is the initial topology for the mapping i. By the cited Prop. 3, h is a homeomorphism, therefore the topology on $\pi(\mathrm{H})$ is also the final topology for h; by transitivity (GT, I, §2, No. 4, Prop. 7) the topology on $\pi(\mathrm{H})$ is the final topology for the mapping $h \circ p : \mathrm{H} \to \pi(\mathrm{H})$, which is the mapping $\pi\big|\mathrm{H} : \mathrm{H} \to \pi(\mathrm{H})$.

Consequence: a mapping $\pi(\mathrm{H}) \to \mathrm{G}$ will be continuous if and only if its composition with $\pi\big|\mathrm{H}$ is continuous (*loc. cit.*, Prop. 6).

V.36, ℓ. -10 to -8.

"Since the restriction of $f \circ \pi$ to H is continuous, the restriction of f to $\pi(\mathrm{H})$ is therefore continuous"

The continuity of $(f \circ \pi)\big|\mathrm{H} = \big(f\big|\pi(\mathrm{H})\big) \circ (\pi\big|\mathrm{H})$ implies the continuity of $f\big|\pi(\mathrm{H})$ by the preceding Note.

V.36, ℓ. -2 to **V.37,** ℓ. 2.

"*Remark.*"

One is contemplating Prop. 3 with $\mathrm{G} = \mathrm{F}$ (a Banach space).

Suppose $(\mathbf{f} \circ \pi)\big|\mathrm{S}$ is μ-measurable. This is equivalent to the measurability of the function $\mathbf{h} = (\mathbf{f} \circ \pi)\varphi_{\mathrm{S}}$, because $\mathbf{h}$ is the extension by 0 of $(\mathbf{f} \circ \pi)\big|\mathrm{S}$ to T (Ch. IV, §5, No. 10, Prop. 15, criterion *d*)). Then $\mathbf{h}g$ is also measurable on T (*loc. cit.*, No. 3, Cor. 5 of Th. 1). Since $\mathrm{S} = \{t \in \mathrm{T} : \ g(t) \neq 0\}$ one has $g = \varphi_{\mathrm{S}} g$, therefore $(\mathbf{f} \circ \pi)g = (\mathbf{f} \circ \pi)\varphi_{\mathrm{S}} g = \mathbf{h}g$ is measurable.

Conversely, assume that $(\mathbf{f} \circ \pi)g$ is measurable. The function $g\big|\mathrm{S}$ is measurable (its extension by 0 to T is the measurable function g) and takes its values in $]0, +\infty[$; composing it with the continuous function $u(a) = 1/a$ on $]0, +\infty[$, it follows that the function

$$\frac{1}{g\big|\mathrm{S}} : t \mapsto 1/g(t) \qquad (t \in \mathrm{S})$$

is measurable by *loc. cit.*, No. 3, Th. 1 as generalized to measurability on S (see item *No.3, Th. 1′* in the Note for IV.80, ℓ. -17 to -14). Let k be the extension by 0 of $\dfrac{1}{g\big|\mathrm{S}}$ to T; since k is measurable on T, so is $(\mathbf{f} \circ \pi)gk$, therefore the function $\big((\mathbf{f} \circ \pi)gk\big)\big|\mathrm{S}$ is measurable. In other words, since $gk = 1$ on S, $(\mathbf{f} \circ \pi)\big|\mathrm{S}$ is measurable.

Note: The argument is valid for the case that $\mathrm{G} = \overline{\mathbf{R}}$; in particular, the equality of $\mathbf{h}$ and the extension by 0 of $(\mathbf{f} \circ \pi)\big|\mathrm{S}$ to T is assured by the convention $(\pm\infty) \cdot 0 = 0$.

V.37, ℓ. 15, 16.

"The measures μ and ν are then bounded"

Let $\mathrm{K} = \operatorname{Supp}\mu$. Then $\varphi_{\mathrm{K}} = 1$ μ-almost everywhere.

Boundedness of μ: $\|\mu\| = \mu^*(1) = \mu^*(\varphi_{\mathrm{K}}) = \mu^*(\mathrm{K}) < +\infty$.

Boundedness of ν: For every $h \in \mathscr{K}_+(\mathrm{X})$, one has $\nu(h) = \mu^\bullet\big((h\circ\pi)g\big)$, which can be written $\nu(h) = \mu^*\big((h\circ\pi)g\big)$ since μ is bounded (§1, No. 2, Prop. 7). By assumption, $g\big|\mathrm{K}$ is bounded, equivalently, $g\varphi_{\mathrm{K}}$ is bounded. Thus

$$(h\circ\pi)g = (h\circ\pi)g\varphi_{\mathrm{K}} \qquad \mu\text{-almost everywhere,}$$

where $h\circ\pi$ and $g\varphi_{\mathrm{K}}$ are bounded and μ-measurable; the bounded μ-measurable function $(h\circ\pi)g\varphi_{\mathrm{K}}$ is integrable for the bounded measure μ:

$$\mu^*\big((h\circ\pi)g\big) = \mu^*\big((h\circ\pi)g\varphi_{\mathrm{K}}\big) \leqslant \|h\|\,\|g\varphi_{\mathrm{K}}\|\,\mu^*(1) < +\infty\,,$$

whence $\nu(h) \leqslant \|h\|\,\|g\big|\mathrm{K}\|\,\|\mu\|$. Thus ν is bounded and $\|\nu\| \leqslant \|g\big|\mathrm{K}\|\,\|\mu\|$.

V.37, ℓ. 17, 18.

"... the function $\mathbf{f}\big(\pi(t)\big)g(t)$ is then μ-integrable, and the relation (9) is verified, by Th. 1 of §3, No. 3."

By No. 1, Prop. 1, the mapping $\Lambda : t \mapsto \lambda_t = g(t)\varepsilon_{\pi(t)}$ is μ-adequate, hence part *a*) of the cited Th. 1 is applicable.

V.37, ℓ. 19, 20.

" $\mathbf{f}$ is then ν-measurable (No. 3, Prop. 3 and *Remark*)"

Since $(\mathbf{f}\circ\pi)g$ is μ-integrable, it is μ-measurable; it follows from the cited Remark (or its analog for $\mathrm{G} = \overline{\mathbf{R}}$) that $(\mathbf{f}\circ\pi)\big|\mathrm{S}$ is μ-measurable, therefore $\mathbf{f}$ is ν-measurable by Prop. 3.

V.37, ℓ. −5, −4.

"The pair (g,π) is obviously μ_α-adapted for every $\alpha \in \mathrm{A}$"

See the Note for V.34, ℓ. 16, 17.

V.37, ℓ. −2, −1.

"Since the argument of A) may be applied to the measures μ_α, ν_α, the first part of the statement then follows from Prop. 3 of §2, No. 2."

Suppose $\mathbf{f}$ is essentially ν-integrable. By the cited Prop. 3 (and the boundedness of the ν_α), $\mathbf{f}$ is ν_α-integrable for every α, and

(i) $$\int \mathbf{f}\,d\nu = \sum_{\alpha\in\mathrm{A}} \int \mathbf{f}\,d\nu_\alpha \qquad \text{(absolutely summable)}.$$

By Part A), the function $\mathbf{h} = (\mathbf{f} \circ \pi)g$ is μ_α-integrable for every α, and

$$\text{(ii)} \qquad \int \mathbf{f}\, d\nu_\alpha = \int \mathbf{h}\, d\mu_\alpha\,.$$

But $|\mathbf{f}|$ is also essentially ν-integrable, and $(|\mathbf{f}| \circ \pi)g = |\mathbf{h}|$, therefore $|\mathbf{h}|$ is μ_α-integrable and

$$\text{(ii}^\bullet\text{)} \qquad \int^\bullet |\mathbf{f}|\, d\nu_\alpha = \int^\bullet |\mathbf{h}|\, d\mu_\alpha\,.$$

The equality

$$\text{(i}^\bullet\text{)} \qquad \int^\bullet |\mathbf{f}|\, d\nu = \sum_{\alpha \in \mathrm{A}} \int^\bullet |\mathbf{f}|\, d\nu_\alpha$$

holds by §2, No. 2, Prop. 1 (with no restrictions on $\mathbf{f}$). Now,

$$\sum_{\alpha \in \mathrm{A}} \int^\bullet |\mathbf{h}|\, d\mu_\alpha = \sum_{\alpha \in \mathrm{A}} \int |\mathbf{f}|\, d\nu_\alpha = \int^\bullet |\mathbf{f}|\, d\nu < +\infty$$

by (ii$^\bullet$), (i$^\bullet$) and the essential ν-integrability of $\mathbf{f}$; it then follows from the cited Prop. 3 that $\mathbf{h}$ is essentially μ-integrable and

$$\text{(iii)} \qquad \int \mathbf{h}\, d\mu = \sum_{\alpha \in \mathrm{A}} \int \mathbf{h}\, d\mu_\alpha \qquad \text{(absolutely summable)}\,.$$

Combining (i), (ii) and (iii), we see that

$$\int \mathbf{f}\, d\nu = \sum_{\alpha \in \mathrm{A}} \int \mathbf{f}\, d\nu_\alpha = \sum_{\alpha \in \mathrm{A}} \int \mathbf{h}\, d\mu_\alpha = \int \mathbf{h}\, d\mu\,,$$

thus (9) is verified

Conversely, suppose $\mathbf{h} = (\mathbf{f} \circ \pi)g$ is essentially μ-integrable. For every $\alpha \in \mathrm{A}$, $\mathbf{h}$ is μ_α-integrable, hence $\mathbf{f}$ is ν_α-integrable by Part A). It follows that $\mathbf{f}$ is ν-measurable (§2, No. 2, Prop. 2); and, since $|\mathbf{h}|$ is also essentially μ_α-integrable, the equality (ii$^\bullet$) holds for every α. Then

$$\int^\bullet |\mathbf{f}|\, d\nu = \sum_{\alpha \in \mathrm{A}} \int^\bullet |\mathbf{f}|\, d\nu_\alpha = \sum_{\alpha \in \mathrm{A}} \int^\bullet |\mathbf{h}|\, d\mu_\alpha = \int^\bullet |\mathbf{h}|\, d\mu < +\infty$$

(the 1st and 3rd equality, by §2, No. 2, Prop. 1, the 2nd equality by (ii$^\bullet$)), therefore $\mathbf{f}$ is essentially ν-integrable, and the relation (9) holds by the first part of the argument.

V.38, ℓ. 4, 5.

"The second part of the statement therefore follows from the first part and Proposition 2."

By the cited Prop. 2 and the inequality (1) of §1, No. 1, we have the diagram $\big($where $\mathbf{h} = (\mathbf{f} \circ \pi)g\big)$:

$$\begin{array}{ccc} \nu^*(|\mathbf{f}|) & \geqslant & \nu^\bullet(|\mathbf{f}|) \\ \| & & \vdots \\ \mu^*(|\mathbf{h}|) & \geqslant & \mu^\bullet(|\mathbf{h}|) \end{array}$$

If $\mathbf{f}$ is ν-integrable, then the (equal) elements of the first column are finite, $\mathbf{f}$ is ν-measurable, and $\nu^\bullet(|\mathbf{f}|) < +\infty$, therefore $\mathbf{f}$ is essentially ν-integrable (§1, No. 3, Prop. 9); whence, by the first part of the statement, $\mathbf{h}$ is μ-measurable (and more), and since $\mu^*(|\mathbf{h}|) = \nu^*(|\mathbf{f}|) < +\infty$, $\mathbf{h}$ is μ-integrable (Ch. IV, §5, No. 6, Th. 5). The argument is symmetric in $\mathbf{f}$ and $\mathbf{h}$: If $\mathbf{h}$ is μ-integrable, then the (equal) elements *etc.*

In this case, we have equality in the last column (by the first part of the statement) and, since integrable functions are moderated (§1, No. 3, Cor. of Prop. 9), equality in the top row and in the bottom row (*loc. cit.*, No. 2, Prop. 7), thus all four corners are equal.

V.38, ℓ. -15 to ℓ. -13.

"The statements of Ths. 1 and 2 and of Prop. 3 remain valid when π and g are only assumed to be defined locally almost everywhere."

With the notations (π, g) and (π', g') as in the immediately preceding text (π' and g' defined only locally μ-almost everywhere), write

$$\nu = \int \lambda_t \, d\mu(t) = \int g(t)\varepsilon_{\pi(t)} \, d\mu(t) \,.$$

We know that for $f \in \mathscr{K}(\mathrm{X})$, the function $(f \circ \pi)g$ is essentially μ-integrable and

$$(f \circ \pi')g' = (f \circ \pi)g \quad \text{locally } \mu\text{-almost everywhere,}$$

where $(f \circ \pi')g'$ is only known to be defined locally μ-almost everywhere. By definition (§1, No. 3, Def. 3) there exists a function $h \in \mathscr{L}^1(\mu)$ such that $(f \circ \pi)g = h$ locally μ-almost everywhere, and one writes

$$\int (f \circ \pi)g \, d\mu = \int h \, d\mu \,;$$

then also $(f\circ\pi')g' = h$ locally μ-almost everywhere, therefore, by the same Def. 3, $(f\circ\pi')g'$ is said to be essentially μ-integrable and one writes

$$\int (f\circ\pi')g'\,d\mu = \int h\,d\mu = \int (f\circ\pi)g\,d\mu\,.$$

This motivates defining the symbol $\int g'(t)\varepsilon_{\pi'(t)}\,d\mu(t)$ to be equal to the measure ν, with $\lambda'_t = g'(t)\varepsilon_{\pi'(t)}$ defined only locally μ-almost everywhere (cf. V.25, $\ell.\ -6,-5$).

Re *Theorem 1*: Assume $f \geqslant 0$ on T. The left side of (1) for (π', g') is, by the definition of $\int g'(t)\varepsilon_{\pi'(t)}\,d\mu(t)$, equal to $\nu^\bullet(f)$, thus is the same as for (π, g). On the right side of (1), for (π, g) we have $\mu^\bullet\big((f\circ\pi)g\big)$. The function $(f\circ\pi')g'$, defined only locally μ-almost everywhere, satisfies

$$(f\circ\pi')g' = (f\circ\pi)g \quad \text{locally } \mu\text{-almost everywhere};$$

by the extension of the notation $\mu^\bullet$ to functions $\geqslant 0$ defined only locally μ-almost everywhere, one therefore has

$$\mu^\bullet\big((f\circ\pi')g'\big) = \mu^\bullet\big((f\circ\pi)g\big),$$

that is, the right side of (1) for (π', g') is by definition equal to the right side for (π, g). Thus the extension of Th. 1 to (π', g') is simply a matter of notation.

Re *the first part of the statement of Theorem 2*: Simply a matter of notation.

Re *the second part of the statement of Theorem 2*: The terms "continuous" and "proper" are not appropriate for functions defined only locally almost everywhere. With (π, g) as in the statement, one assumes that π' and g' are functions, defined locally μ-almost everywhere in T, such that $\pi' = \pi$ and $g' = g$ locally μ-almost everywhere. The rest is simply notation.

{For the record: "proper" is defined *only* for continuous mappings (GT, I, §10, No. 1, Def. 1). Thus, the expression "continuous proper function" would be redundant; but the expression "proper continuous function" is not, since not every continuous function is proper.}

Re *Proposition 3*: The new element here is S. With (π', g') and (π, g) related as above, suppose (π, g) satisfies the hypotheses of Prop. 3. Let

$$\mathrm{S}' = \{t \in \mathrm{T} :\ g'(t) \text{ is defined and } 0 < g'(t) < +\infty\}\,.$$

Since $g = g'$ locally μ-almost everywhere, S and S' differ at most by a locally μ-negligible set, thus $\varphi_{\mathrm{S}} = \varphi_{\mathrm{S}'}$ locally μ-almost everywhere in T, whence S' is also μ-measurable. Moreover, $(f\circ\pi')\big|\mathrm{S}'$ and $(f\circ\pi)\big|\mathrm{S}$ are equal locally μ-almost everywhere; since their extensions by 0 to T are then equal locally μ-almost everywhere, it follows that $(f\circ\pi')\big|\mathrm{S}'$ is μ-measurable if and only if $(f\circ\pi)\big|\mathrm{S}$ is μ-measurable (Ch. IV, §5, No. 10, Prop. 15, criterion d)).

§5. MEASURES DEFINED BY NUMERICAL DENSITIES

V.38, ℓ. −10 to −3.

"PROPOSITION 1."

There is a problem here: each of the conditions a), b), c) implies that the function $\mathbf{g}$ is defined almost everywhere (not merely locally almost everywhere).

To set the stage, we note that, for a mapping $\mathbf{g}$ defined locally μ-almost everywhere in T and taking values in a topological space F, the domain of $\mathbf{g}$, being the complement of a locally negligible set, is measurable (Ch. IV, §5, No. 2, sentence following Def. 3); the following conditions are therefore equivalent:

(i) $\mathbf{g}$ is measurable in the sense of *loc. cit.*, No. 10, Def. 8;

(ii) $\mathbf{g}$ has a measurable extension to T (see the condition c'') in the Note for IV.79, ℓ. 3, 4);

(iii) every extension of $\mathbf{g}$ to T is measurable (Ch. IV, §5, No. 2, Prop. 6).

Re a): The assertion that $\mathbf{g}\varphi_{\mathrm{V}}$ is integrable means that the function $\mathbf{g}\varphi_{\mathrm{V}}$ (hence also $\mathbf{g}$) is defined almost everywhere, and is equal almost everywhere to an everywhere-defined integrable function (Ch. IV, §4, No. 1, last paragraph).

Re b): For the symbol $\int^* |\mathbf{g}|\varphi_{\mathrm{K}}\,d\mu$ to be defined, it is necessary that the function $|\mathbf{g}|\varphi_{\mathrm{K}}$ (hence also $\mathbf{g}$) be defined almost everywhere (Ch. IV, §3, No. 2, next-to-last paragraph).

Re c): Similar to a).

What is the problem? In Def. 1 below, a function $\mathbf{g}$ is defined to be *locally integrable* if it satisfies the conditions of Prop. 1. It follows from the foregoing that if $\mathbf{f}$ is a function defined locally almost everywhere but not almost everywhere, then $\mathbf{f}$ cannot be locally integrable—even if it is equal locally almost everywhere to a locally integrable function $\mathbf{g}$—contrary to the assertion immediately following Def. 1.

PROPOSED REMEDY: In the statement of Prop. 1, replace 'integrable' by 'essentially integrable', and $\int^*$ by $\int^\bullet$.

{So far, the essential upper integral $\int^\bullet$ has only been applied to everywhere-defined positive functions (§1, No. 1). For a function f defined locally almost everywhere in T and $\geqslant 0$ locally almost everywhere, define

$\int^\bullet f\,d\mu = \int^\bullet f^*\,d\mu$, where f^* is any function, defined and $\geqslant 0$ at every point of T, such that $f = f^*$ locally almost everywhere; for example, let f' be the extension by 0 of f to T and let $f^* = f'\varphi_{\complement \mathrm{N}}$, where N is the set (locally negligible) of points t where $f(t)$ is defined and < 0.}

Prop. 1 then takes the form:

PROPOSITION 1′. — *Let* $\mathbf{g}$ *be a function defined locally almost everywhere in* T *(for the positive measure* μ*), with values in a Banach space* F (resp. *in* $\overline{\mathbf{R}}$). *The following properties are equivalent*:

a′) *For every point* $t \in \mathrm{T}$, *there exists a neighborhood* V *of* t *such that the function* $\mathbf{g}\varphi_{\mathrm{V}}$ *is essentially* μ*-integrable.*

b′) *The function* $\mathbf{g}$ *is* μ*-measurable and, for every compact set* $\mathrm{K} \subset \mathrm{T}$, $\int^\bullet |\mathbf{g}|\varphi_{\mathrm{K}}\,d\mu < +\infty$.

c′) *For every numerical function* $h \in \mathscr{K}(\mathrm{T})$, $\mathbf{g}h$ *is essentially* μ*-integrable.*

Proof. a′) ⇒ b′): Assume that $\mathbf{g}$ satisfies a′), and let $\mathbf{g}'$ be its extension by 0 to T. To say that $\mathbf{g}\varphi_{\mathrm{V}}$ is essentially integrable (with respect to μ) means that there exists a function $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}$ (resp. $\mathbf{f} \in \mathscr{L}^1_{\overline{\mathbf{R}}}$ when the values of $\mathbf{g}$ are in $\overline{\mathbf{R}}$) such that $\mathbf{g}\varphi_{\mathrm{V}} = \mathbf{f}$ locally almost everywhere (§1, No. 3, first paragraph on INT V.9). Since $\mathbf{g} = \mathbf{g}'$ locally almost everywhere, one has $\mathbf{g}'\varphi_{\mathrm{V}} = \mathbf{f}$ locally almost everywhere, therefore $\mathbf{g}'\varphi_{\mathrm{V}}$ is also essentially integrable, and in particular measurable (Ch. IV, §5, No. 2, Prop. 6).

To show that $\mathbf{g}$ is measurable, it suffices to show that $\mathbf{g}'$ is measurable. For every $t \in \mathrm{T}$, by assumption there exists a neighborhood V_t of t such that $\mathbf{g}\varphi_{\mathrm{V}_t}$ is essentially integrable, and so $\mathbf{g}'\varphi_{\mathrm{V}_t}$ is essentially integrable, say $\mathbf{g}'\varphi_{\mathrm{V}_t} = \mathbf{f}_t$ locally almost everywhere, where $\mathbf{f}_t \in \mathscr{L}^1_{\mathrm{F}}$ (resp. $\mathbf{f}_t \in \mathscr{L}^1_{\overline{\mathbf{R}}}$). If W_t is a compact neighborhood of t such that $\mathrm{W}_t \subset \mathrm{V}_t$, then

$$\mathbf{g}'\varphi_{\mathrm{W}_t} = (\mathbf{g}'\varphi_{\mathrm{V}_t})\varphi_{\mathrm{W}_t} = \mathbf{f}_t\varphi_{\mathrm{W}_t}$$

locally almost everywhere, where $\mathbf{f}_t\varphi_{\mathrm{W}_t}$ is integrable (Ch. IV, §5, No. 6, Cor. 3 of Th. 5); replacing V_t by W_t, we can suppose that V_t is integrable, hence moderated (§1, No. 2, Def. 2). Moreover $\mathbf{g}'\varphi_{\mathrm{V}_t}$ is integrable, since it is essentially integrable and moderated (§1, No. 3, Cor. of Prop. 9). Define $\mathbf{g}_t = \mathbf{g}'\varphi_{\mathrm{V}_t}$. For every $t \in \mathrm{T}$, $\mathbf{g}_t$ is a measurable function such that $\mathbf{g}' = \mathbf{g}_t$ everywhere in V_t, consequently $\mathbf{g}'$ is measurable by the cited Principle of localization (Ch. IV, §5, No. 2, Prop. 4).

If, moreover, K is any compact set in T, then, by the argument of the text, the measurable function $\mathbf{g}'\varphi_{\mathrm{K}}$ satisfies $\int^* |\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu < +\infty$, hence $\mathbf{g}'\varphi_{\mathrm{K}}$ is integrable; since $\mathbf{g}\varphi_{\mathrm{K}} = \mathbf{g}'\varphi_{\mathrm{K}}$ locally almost everywhere, $\mathbf{g}\varphi_{\mathrm{K}}$ is essentially integrable and (by the convention about $\int^\bullet$ proposed above)

$$\int^\bullet |\mathbf{g}|\varphi_{\mathrm{K}}\,d\mu = \int^\bullet |\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu = \int^* |\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu < +\infty$$

(the second equality by §1, No. 2, Prop. 7, 2)).

b′) $\Rightarrow$ c′): Assume that $\mathbf{g}$ satisfies b′). Since $\mathbf{g}$ is measurable, its extension $\mathbf{g}'$ to T by 0 is measurable. As noted in the preceding paragraph, $\mathbf{g}'\varphi_{\mathrm{K}}$ is integrable for every compact set K in T, hence $\int^*|\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu < +\infty$. By the argument in the text, if $h \in \mathscr{K}(\mathrm{T})$ then $\mathbf{g}'h$ is integrable, and since $\mathbf{g}h = \mathbf{g}'h$ locally almost everywhere, it follows that $\mathbf{g}h$ is essentially integrable (§1, No. 3, second paragraph after Def. 3).

c′) $\Rightarrow$ a′): Assume that $\mathbf{g}$ satisfies c′), and let $\mathbf{g}'$ be its extension by 0 to T. Since $\mathbf{g}$ is defined locally almost everywhere, $\mathbf{g} = \mathbf{g}'$ locally almost everywhere. If $h \in \mathscr{K}(\mathrm{T})$, by assumption $\mathbf{g}h$ is essentially integrable, and since $\mathbf{g}'h = \mathbf{g}h$ locally almost everywhere, it follows that $\mathbf{g}'h$ is essentially integrable, and, being moderated, it is integrable. By the argument in the text, for every $t \in \mathrm{T}$ there exists a neighborhood V of t such that $\mathbf{g}'\varphi_{\mathrm{V}}$ is integrable, and since $\mathbf{g}\varphi_{\mathrm{V}} = \mathbf{g}'\varphi_{\mathrm{V}}$ locally almost everywhere, $\mathbf{g}\varphi_{\mathrm{V}}$ is essentially integrable. ♢

An equivalent form of Prop. 1′ is as follows (my guess is that this is what the author had in mind, and that the version in the text was intended as a more memorable paraphrase):

PROPOSITION 1″. — *Let* $\mathbf{g}$ *be a function defined locally almost everywhere in* T (*for the positive measure* μ), *with values in a Banach space* F (resp. *in* $\overline{\mathbf{R}}$), *and let* $\mathbf{g}'$ *be the extension of* $\mathbf{g}$ *to* T *by* 0. *The following properties are equivalent*:

a″) *For every point* $t \in \mathrm{T}$, *there exists a neighborhood* V *of* t *such that the function* $\mathbf{g}'\varphi_{\mathrm{V}}$ *is* μ*-integrable.*

b″) *The function* $\mathbf{g}$ *is* μ*-measurable and, for every compact set* $\mathrm{K} \subset \mathrm{T}$, $\int^*|\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu < +\infty$.

c″) *For every numerical function* $h \in \mathscr{K}(\mathrm{T})$, $\mathbf{g}'h$ *is* μ*-integrable.*

Proof. We will show: a′) $\Leftrightarrow$ a″); b′) $\Leftrightarrow$ b″); c′) $\Leftrightarrow$ c″). Since $\mathbf{g}$ is defined locally almost everywhere, we know that $\mathbf{g} = \mathbf{g}'$ locally almost everywhere.

a′) $\Leftrightarrow$ a″): Assuming $\mathbf{g}$ satisfies a′), it is shown in the proof of Prop. 1′ that for every $t \in \mathrm{T}$ there exists a neighborhood V of t such that $\mathbf{g}'\varphi_{\mathrm{V}}$ is integrable.

Conversely, if $\mathbf{g}'\varphi_{\mathrm{V}}$ is integrable then, since $\mathbf{g}\varphi_{\mathrm{V}} = \mathbf{g}'\varphi_{\mathrm{V}}$ locally almost everywhere, $\mathbf{g}\varphi_{\mathrm{V}}$ is essentially integrable.

b′) $\Leftrightarrow$ b″): At any rate, since $\mathbf{g}$ is defined on a measurable set, we know that $\mathbf{g}$ is measurable if and only if $\mathbf{g}'$ is measurable. Moreover, for every compact set $\mathrm{K} \subset \mathrm{T}$, one has $\mathbf{g}\varphi_{\mathrm{K}} = \mathbf{g}'\varphi_{\mathrm{K}} = (\mathbf{g}\varphi_{\mathrm{K}})'$ locally almost everywhere, whence

$$\int^{\bullet} |\mathbf{g}|\varphi_{\mathrm{K}}\,d\mu = \int^{\bullet} |\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu = \int^{*}|\mathbf{g}'|\varphi_{\mathrm{K}}\,d\mu\,,$$

the first equality by the definition of the left member, the second because $|\mathbf{g}'|\varphi_{\mathrm{K}}$ is moderated. Thus

$$\int^{\bullet} |\mathbf{g}|\varphi_{\mathrm{K}}\, d\mu < +\infty \quad \Leftrightarrow \quad \int^{*} |\mathbf{g}'|\varphi_{\mathrm{K}}\, d\mu < +\infty\,;$$

since $\mathbf{g}\varphi_{\mathrm{K}}$ is measurable if and only if $(\mathbf{g}\varphi_{\mathrm{K}})' = \mathbf{g}'\varphi_{\mathrm{K}}$ is measurable, this can also be expressed by saying that $\mathbf{g}\varphi_{\mathrm{K}}$ is essentially integrable if and only if $\mathbf{g}'\varphi_{\mathrm{K}}$ is integrable (§1, No. 3, Prop. 9).

c$'$) $\Leftrightarrow$ c$''$): Similarly, for $h \in \mathscr{K}(\mathrm{T})$,

$$\int^{\bullet} |\mathbf{g}h|\, d\mu = \int^{\bullet} |\mathbf{g}'h|\, d\mu = \int^{*} |\mathbf{g}'h|\, d\mu\,,$$

thus

$$\int^{\bullet} |\mathbf{g}h|\, d\mu < +\infty \quad \Leftrightarrow \quad \int^{*} |\mathbf{g}'h|\, d\mu < +\infty\,,$$

whence $\mathbf{g}h$ is essentially integrable if and only if $\mathbf{g}'h$ is integrable. ◊

It remains to see how the proposed modifications will play out in the rest of the text.

V.39, ℓ. 20, 21.

"If $\mathbf{g}$ is locally θ-integrable, then every function equal to $\mathbf{g}$ locally almost everywhere is locally integrable."

Let $\mathbf{f}$ and $\mathbf{g}$ be functions defined locally almost everywhere in T, with values in F (or $\overline{\mathbf{R}}$), such that $\mathbf{f} = \mathbf{g}$ locally almost everywhere, and let $\mathbf{f}'$ and $\mathbf{g}'$ be their extensions by 0 to T. Then $\mathbf{f}' = \mathbf{g}'$ locally almost everywhere and, for every numerical function $h \in \mathscr{K}(\mathrm{T})$, one has

$$\mathbf{f}'h = \mathbf{g}'h \quad \text{almost everywhere}$$

(because the set $\mathrm{K} = \operatorname{Supp} h$ is compact). In detail, the set

$$\mathrm{N} = \{t : \ \mathbf{f}'(t) \neq \mathbf{g}'(t)\,\}$$

is locally negligible, and, if $\mathbf{f}'(t)h(t) \neq \mathbf{g}'(t)h(t)$ then $h(t) = 1$ (by the convention $\pm\infty \cdot 0 = 0$) and $\mathbf{f}'(t) \neq \mathbf{g}'(t)$, whence $t \in \mathrm{N} \cap \mathrm{K}$, which is a negligible set (Ch. IV, §5, No. 2, Prop. 5).

It follows that $\mathbf{f}'h$ is integrable if and only if $\mathbf{g}'h$ is integrable (Ch. IV, §5, No. 6, Th. 5 and No. 2, Prop. 6, and §2, No. 3, Prop. 6); thus $\mathbf{f}$ is locally integrable if and only if $\mathbf{g}$ is locally integrable, by the criterion c$''$) of Prop. 1$''$ in the preceding Note.

V.39, ℓ. 21, 22.

"It is clear that the sum of two locally integrable functions is locally integrable."

See the next Note.

V.39, ℓ. 22–24.

"The functions with values in F, *everywhere defined* and locally integrable for θ, form a vector space denoted $\mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathrm{F})$ "

For, if $\mathbf{f},\mathbf{g}\in\mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathrm{F})$ and α is a scalar, then, for every $h\in\mathscr{K}(\mathrm{T})$, one has

$$(\mathbf{f}+\mathbf{g})h=\mathbf{f}h+\mathbf{g}h \quad\text{and}\quad (\alpha\mathbf{f})h=\alpha(\mathbf{f}h)\,,$$

and the assertion is immediate from criterion c$''$) of Prop. 1$''$ in the Note for Prop. 1.

The assertion of the preceding Note is an immediate consequence of the following:

Proposition. In order that a function $\mathbf{g}$ with values in a Banach space F (resp. in $\overline{\mathbf{R}}$) be locally integrable for a complex measure θ on T, it is necessary and sufficient that it be equal locally almost everywhere to a function $\mathbf{f}$ in $\mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathrm{F})$ (resp. $\mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathbf{R})$).

Proof. Sufficiency: Immediate from the Note for ℓ. 20, 21.

Necessity: Suppose $\mathbf{g}$ is locally integrable (in the sense of Def. 1), consider first the case that $\mathbf{g}$ takes its values in F, and let $\mathbf{g}'$ be its extension to T by 0. Then $\mathbf{g}'$ also takes its values in F, and since $\mathbf{g}'=\mathbf{g}$ locally almost everywhere, $\mathbf{g}'$ is locally integrable by the Note for ℓ. 20, 21, thus $\mathbf{f}=\mathbf{g}'$ meets the requirements.

Suppose now that the locally integrable function $\mathbf{g}$ takes its values in $\overline{\mathbf{R}}$. The foregoing argument is thwarted by the fact that $\mathbf{g}$ may have infinite values; let us show that $\mathbf{g}$ is finite locally almost everywhere.

Let A be the domain of $\mathbf{g}$, so that the set $\mathrm{N}=\complement\mathrm{A}$ is locally negligible, and let

$$\mathrm{M}=\{t\in\mathrm{A}:\ \mathbf{g}(t)=\pm\infty\}\,;$$

to show that M is locally negligible, it will suffice to show that for every compact set $\mathrm{K}\subset\mathrm{T}$, the set $\mathrm{K}\cap\mathrm{M}$ is negligible. By criterion b$''$) of Prop. 1$''$ in the Note for Prop. 1, $\int^*|\mathbf{g}'|\varphi_{\mathrm{K}}\,d|\theta|<+\infty$, therefore $|\mathbf{g}'|\varphi_{\mathrm{K}}$ is finite *almost everywhere* (Ch. IV, §2, No. 3, Prop. 7), thus the set

$$\{t:\ (\mathbf{g}'\varphi_{\mathrm{K}})(t)=\pm\infty\}=\{t\in\mathrm{K}:\ t\in\mathrm{A}\ \text{and}\ \mathbf{g}(t)=\pm\infty\}=\mathrm{K}\cap\mathrm{M}$$

is indeed negligible. Thus, $\mathbf{g}$ is finite on the set $\mathrm{A}-\mathrm{M}$, whose complement $\mathrm{N}\cup\mathrm{M}$ is locally negligible.

Then the function $\mathbf{f} = \mathbf{g}'\varphi_{A-M}$ on T is finite-valued and $\mathbf{f} = \mathbf{g}$ locally almost everywhere, therefore $\mathbf{f}$ is also locally integrable, whence $\mathbf{f} \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T}, \theta; \mathbf{R})$.

V.39, ℓ. 26, 27.

"... the topology defined by the semi-norms $\mathbf{g} \mapsto \int |\mathbf{g}\varphi_{\mathrm{K}}|\, d|\theta|$, where K runs over the set of compact subsets of T."

It may be useful to pause to get a feeling for the topology just defined. For the (locally convex) topology defined by a set of semi-norms, see TVS, II, §1, No. 2, or §37 of my book LFAOT.(1)

For each compact set $\mathrm{K} \subset \mathrm{T}$, let p_{K} be the semi-norm on $\mathscr{L}^1_{\mathrm{loc}} = \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T}, \theta; \mathrm{F})$ defined by $p_{\mathrm{K}}(\mathbf{g}) = \int |\mathbf{g}|\varphi_{\mathrm{K}}\, d|\theta|$. Equip $\mathscr{L}^1_{\mathrm{loc}}$ with the locally convex topology generated by the semi-norms p_{K}. Our objective is to prove the following:

Proposition. If E is a topological vector space and $u : \mathrm{E} \to \mathscr{L}^1_{\mathrm{loc}}$ is a linear mapping, the following conditions are equivalent:

a) u is continuous;

b) for every compact set K in T, the semi-norm $p_{\mathrm{K}} \circ u$ on E is continuous.

The proof is accomplished in two lemmas.

Lemma 1. Let Γ be a set of semi-norms on a vector space G and let $\tau(\Gamma)$ be the (locally convex) topology on G generated by Γ; a semi-norm p on G is continuous for $\tau(\Gamma)$ if and only if $p \leqslant p_1 + \cdots + p_n$ for a suitable finite set of semi-norms $p_i \in \Gamma$ (TVS, II, §1, or LFAOT, p. 151, 37.15 and 37.17).

In particular, a semi-norm p on $\mathscr{L}^1_{\mathrm{loc}}$ is continuous if and only if $p \leqslant cp_{\mathrm{K}}$ for some scalar $c > 0$ and some compact set $\mathrm{K} \subset \mathrm{T}$. For, if p is continuous, say $p \leqslant p_{\mathrm{K}_1} + \cdots + p_{\mathrm{K}_n}$ (Lemma 1), then $p \leqslant n \cdot p_{\mathrm{K}_1 \cup \cdots \cup \mathrm{K}_n}$; conversely, if $p \leqslant cp_{\mathrm{K}}$ then the continuity of p is immediate from Lemma 1.

The Proposition is a special case of the following:

Lemma 2 (LFAOT, p. 153, 37.24). Let G be a vector space equipped with the (locally convex) topology generated by a set Γ of semi-norms. If $u : \mathrm{E} \to \mathrm{G}$ is a linear mapping of a topological vector space E into G, following conditions are equivalent:

(i) u is continuous;

(ii) for every $p \in \Gamma$, the semi-norm $p \circ u$ on E is continuous.

Proof. (i) $\Rightarrow$ (ii): Because every $p \in \Gamma$ is continuous.

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(1) *Lectures on functional analysis and operator theory* (Springer, 1974), whose treatment of topological vector spaces was based on the fascicles of the first edition of EVT.

(ii) $\Rightarrow$ (i): Given any neighborhood V of 0 in G, it will suffice to show that $\overset{-1}{u}(\mathrm{V})$ is a neighborhood of 0 in E. We can suppose that V is convex, balanced and closed, since such neighborhoods are fundamental (TVS, II, §8, No. 2); there then exists a continuous semi-norm p on G such that $\mathrm{V} = \{\mathbf{y} \in \mathrm{G} : \ p(\mathbf{y}) \leqslant 1\}$ (TVS, II, §2, No. 11, Prop. 22, or LFAOT, p. 146, 37.4).

By Lemma 1, $p \leqslant p_1 + \cdots + p_n$ for suitable $p_i \in \Gamma$, whence $p \circ u \leqslant p_1 \circ u + \cdots + p_n \circ u$. Since the $p_i \circ u$ are, by assumption, continuous on E, the topology σ' they generate is coarser than the given topology σ on E; then $p \circ u$ is continuous for σ' (Lemma 1), hence for the finer topology σ. Then

$$\begin{aligned}\overset{-1}{u}(\mathrm{V}) &= \{\mathbf{x} \in \mathrm{E} : \ u(\mathbf{x}) \in \mathrm{V}\,\} \\ &= \{\mathbf{x} \in \mathrm{E} : \ p\big(u(\mathbf{x})\big) \leqslant 1\,\} \\ &= \{\mathbf{x} \in \mathrm{E} : \ (p \circ u)(\mathbf{x}) \leqslant 1\,\}\,,\end{aligned}$$

which is a neighborhood of 0 in E by the continuity of $p \circ u\}$. ♢

If G is a locally convex space, its topology is generated by the set of all continuous semi-norms on G (TVS, II, §4, No. 1, remarks following the Cor. of Prop. 1, or LFAOT, p. 151, 37.17); Lemma 2 then takes the form:

Remark 1. If E is a topological vector space, G is a locally convex space, and $u : \mathrm{E} \to \mathrm{G}$ is a linear mapping, then u is continuous if and only if, for every continuous semi-norm p on G, the semi-norm $p \circ u$ on E is continuous.

With notations as in Lemma 2, the topology τ on G generated by the set Γ of semi-norms is characterized by the validity of the equivalence (i) $\Leftrightarrow$ (ii):

Remark 2. Let G be a locally convex space whose topology τ is generated by a set Γ of semi-norms. If σ is a locally convex topology on G for which the equivalence (i) $\Leftrightarrow$ (ii) of Lemma 2 holds, then $\sigma = \tau$. The proof in three steps:

1) The semi-norms $p \in \Gamma$ are continuous for σ. For, consider the identity mapping $u : (\mathrm{G}, \sigma) \to (\mathrm{G}, \sigma)$; since u is continuous, by the assumption (i) $\Rightarrow$ (ii) for σ one knows that every $p \circ u = p$ $(p \in \Gamma)$ is continuous on G for σ.

2) It follows that for every $p \in \Gamma$, the (locally convex) topology τ_p on G generated by p satisfies $\tau_p \subset \sigma$ (LFAOT, p. 147, item (4) of 37.5), therefore $\tau \subset \sigma$.

3) Consider the identity mapping $v : (\mathrm{G}, \tau) \to (\mathrm{G}, \sigma)$. We know that for every $p \in \Gamma$, the semi-norm $p \circ v = p$ on G is continuous for τ (because $\tau_p \subset \tau$), therefore v is continuous by the assumption (ii) $\Rightarrow$ (i) for σ; that is, $\sigma \subset \tau$.

In particular, the locally convex topology on $\mathscr{L}^1_{\mathrm{loc}}$ generated by the semi-norms p_{K} is characterized by the validity of the equivalence $a) \Leftrightarrow b)$ of the Proposition.

V.39, ℓ. −12 to −10.

"The associated Hausdorff space, the quotient of $\mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathrm{F})$ by the subspace $\mathscr{N}^\infty_{\mathrm{F}}$ of mappings that are zero locally almost everywhere, is denoted $\mathrm{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathrm{F})$."

The space $\mathscr{N}^\infty_{\mathrm{F}}$ was defined in Ch. IV, §6, No. 3. As in the preceding Note, we write $p_{\mathrm{K}}(\mathbf{g}) = \int |\mathbf{g}|\varphi_{\mathrm{K}}\, d|\theta|$ for every $\mathbf{g} \in \mathscr{L}^1_{\mathrm{loc}} = \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathrm{F})$ and every compact set $\mathrm{K} \subset \mathrm{T}$.

Proof that $\mathscr{N}^\infty_{\mathrm{F}} \subset \mathscr{L}^1_{\mathrm{loc}}$: Let $\mathbf{g} \in \mathscr{N}^\infty_{\mathrm{F}}$. For every compact $\mathrm{K} \subset \mathrm{T}$, the function $\mathbf{g}\varphi_{\mathrm{K}}$ is locally negligible and moderated, hence negligible (§1, No. 2, Cor. 1 of Prop. 7), therefore integrable, whence $\mathbf{g} \in \mathscr{L}^1_{\mathrm{loc}}$ and $p_{\mathrm{K}}(\mathbf{g}) = 0$ for every K.

Conversely, if $\mathbf{g} \in \mathscr{L}^1_{\mathrm{loc}}$ and $p_{\mathrm{K}}(\mathbf{g}) = 0$ for every compact $\mathrm{K} \subset \mathrm{T}$, then the integrable function $|\mathbf{g}|\varphi_{\mathrm{K}}$ is negligible, whence $\mathbf{g}$ is locally negligible (Ch. IV, §5, No. 2, Prop. 5). Thus

$$\mathscr{N}^\infty_{\mathrm{F}} = \{\mathbf{g} \in \mathscr{L}^1_{\mathrm{loc}} :\ p_{\mathrm{K}}(\mathbf{g}) = 0 \text{ for every compact set } \mathrm{K} \subset \mathrm{T}\}.$$

Since the semi-norms p_{K} define the topology of $\mathscr{L}^1_{\mathrm{loc}}$, it follows that $\mathscr{N}^\infty_{\mathrm{F}}$ is the closure of $\{0\}$ in $\mathscr{L}^1_{\mathrm{loc}}$ (TVS, II, §1, No. 2, Prop. 2, (i)), therefore the quotient space $\mathscr{L}^1_{\mathrm{loc}}/\mathscr{N}^\infty_{\mathrm{F}}$ is the Hausdorff topological vector space associated with $\mathscr{L}^1_{\mathrm{loc}}$ (*loc. cit.*, No. 3). The space $\mathrm{L}^1_{\mathrm{loc}} = \mathscr{L}^1_{\mathrm{loc}}/\mathscr{N}^\infty_{\mathrm{F}}$ is also locally convex (TVS, II, §4, No. 4, *Example* I), its topology being defined by the family of semi-norms $\dot{\mathbf{g}} \mapsto p_{\mathrm{K}}(\mathbf{g})$ ($\dot{\mathbf{g}}$ the equivalence class of $\mathbf{g}$ in $\mathscr{L}^1_{\mathrm{loc}}$ for equality locally almost everywhere); the details are as follows.

As $\mathrm{L}^1_{\mathrm{loc}}$ bears the final topology for the quotient mapping $u : \mathscr{L}^1_{\mathrm{loc}} \to \mathrm{L}^1_{\mathrm{loc}}$ (*loc. cit.*), a semi-norm q on $\mathrm{L}^1_{\mathrm{loc}}$ is continuous if and only if the semi-norm $q \circ u : \mathbf{g} \mapsto q(\dot{\mathbf{g}})$ on $\mathscr{L}^1_{\mathrm{loc}}$ is continuous. Every continuous semi-norm p on $\mathscr{L}^1_{\mathrm{loc}}$ is zero on $\mathscr{N}^\infty_{\mathrm{F}}$ (because the p_{K} are), so the formula $q(\dot{\mathbf{g}}) = p(\mathbf{g})$ defines a semi-norm q on $\mathrm{L}^1_{\mathrm{loc}}$ such that $p = q \circ u$, and q is continuous by the foregoing. In particular, for every compact set $\mathrm{K} \subset \mathrm{T}$ the formula $q_{\mathrm{K}}(\dot{\mathbf{g}}) = p_{\mathrm{K}}(\mathbf{g})$ defines a continuous semi-norm on $\mathrm{L}^1_{\mathrm{loc}}$. On the other hand, if q is any continuous semi-norm on $\mathrm{L}^1_{\mathrm{loc}}$, then the semi-norm $q \circ u$ on $\mathscr{L}^1_{\mathrm{loc}}$ is continuous, therefore $q \circ u \leqslant p_{\mathrm{K}_1} + \cdots + p_{\mathrm{K}_n}$ for suitable $\mathrm{K}_1, \ldots, \mathrm{K}_n$ (Lemma 1 in the preceding Note), whence $q \leqslant q_{\mathrm{K}_1} + \cdots + q_{\mathrm{K}_n}$; it follows (Lemma 1 again) that the locally convex topology of $\mathrm{L}^1_{\mathrm{loc}}$ is generated by the semi-norms q_{K}.

{The argument carries over to the case that $\mathscr{L}^1_{\mathrm{loc}}$ and the p_{K} are replaced by a vector space E and a set Γ of semi-norms on E (cf., TVS, II, §1, No. 3, ℓ. 3–6 on p. TVS II.5).}

Remark. If T is countable at infinity, then L^1_{loc} is metrizable.

Proof. One knows (GT, I, §1, No. 9, Prop. 15) that T is the union of a sequence (U_n) of relatively compact open sets such that $\overline{U}_n \subset U_{n+1}$ for all n. Let $K_n = \overline{U}_n$; (K_n) is an increasing sequence of compact sets with union T, and each compact set K is contained in K_n for some n (*loc. cit.*, Cor. 1).

As in the preceding Note, write $q_K(\dot{\mathbf{g}}) = p_K(\mathbf{g}) = \int |\mathbf{g}|\varphi_K \, d|\theta|$ for the semi-norms q_K that generate the locally convex topology of L^1_{loc}. For every compact set K in T, one has $q_K \leqslant q_{K_n}$ for some n, consequently the topology of L^1_{loc} is generated by the sequence of semi-norms q_{K_n} (Lemma 1 in the preceding Note); it follows that the uniform structure of L^1_{loc} is metrizable (GT, IX, §2, No. 4, Cor. 1 of Th. 1). ♢

{In general, a Hausdorff locally convex space is metrizable if and only if its topology is generated by a sequence of semi-norms:[(1)] "If", by the cited Cor. 1; "Only if", because a metrizable locally convex space has a fundamental sequence of neighborhoods of 0, which one can suppose are closed, balanced and convex, whence the desired sequence of semi-norms (TVS, II, §2, No. 11, Prop. 22).}

V.39, ℓ. −9, −8.

"It can be shown that the topological vector spaces just defined are *complete* (Exer. 31)."

If T is countable at infinity, then L^1_{loc} is metrizable (*Remark* in the preceding Note), and the present assertion implies that L^1_{loc} is a Fréchet space (TVS, II, §8, No. 2, last sentence). The hint given in Exer. 31:

"Write $|\theta|$ in the form $\sum_\alpha \mu_\alpha$, where (μ_α) is a summable family of measures $\geqslant 0$ on T whose supports form a locally countable family of pairwise disjoint compact subsets. This defines a continuous mapping of $L^1_{loc}(T, \theta; F)$ into $\prod_\alpha L^1_F(T, \mu_\alpha)$. Let $\mathfrak{F}$ be a Cauchy filter on $L^1_{loc}(T, \theta; F)$. Its image in $\prod_\alpha L^1_F(T, \mu_\alpha)$ converges to an element $(\mathbf{f}_\alpha)$. Show that the $\mathbf{f}_\alpha$ define a θ-measurable function $\mathbf{f}$ on T, then that $\mathbf{f}$ is locally θ-integrable, and that $\dot{\mathbf{f}}$ is the limit of $\mathfrak{F}$ in $L^1_{loc}(T, \theta; F)$."

I did not succeed in showing that $\mathbf{f}$ is locally θ-integrable, but the following construction of a measurable function $\mathbf{f}$ may be a step in the right direction:

(1) Cf. H.H. Schaefer, *Topological vector spaces*, p. 48, Macmillan, 1966; W. Rudin, *Functional analysis*, p. 27, Remark 1.38, *c*), McGraw-Hill, 1973; J. Horvath, *Topological vector spaces and distributions. I.*, p. 113, Addison-Wesley, 1966; R.E. Edwards, *Functional analysis: Theory and applications*, p. 422, Holt, Rinehart and Winston, 1965.

Write $\mu = |\theta|$. By §2, No. 3, Prop. 4, $\mu = \sum_\alpha \mu_\alpha$, where (μ_α) is a summable family of positive measures on T whose supports $K_\alpha = \operatorname{Supp}\mu_\alpha$ are compact, pairwise disjoint, and form a locally countable family (K_α). The natural mapping is

$$u : L^1_{\text{loc}}(T, \mu; F) \to \prod_\alpha L^1_F(T, \mu_\alpha)$$

where $u(\dot{\mathbf{g}}) = \big((\mathbf{g}\varphi_{K_\alpha})^{\cdot}\big)$ for $\mathbf{g} \in \mathscr{L}^1_{\text{loc}}(T, \mu; F)$, characterized by

$$\text{pr}_\alpha \circ u : \dot{\mathbf{g}} \mapsto (\mathbf{g}\varphi_{K_\alpha})^{\cdot} \text{ for all } \alpha \text{ and } \mathbf{g}.$$

In detail:

(i) *Definition of* u. Let $\mathbf{g} \in \mathscr{L}^1_{\text{loc}}(T, \mu; F)$, and fix an index α. One knows that $\mathbf{g}$ is μ-measurable and $\mathbf{g}\varphi_{K_\alpha}$ is μ_α-integrable (Def. 1 and Prop. 1). Since $\mu_\alpha \leqslant \mu$, it follows that $\mathbf{g}$ is also μ_α-measurable (§1, No. 4, Cor. 2 of Prop. 11) and $\mathbf{g}\varphi_{K_\alpha}$ is μ_α-integrable. Since $\mu^*_\alpha(T - K_\alpha) = 0$ (Ch. IV, §2, No. 2, Prop. 5), it follows that $\varphi_{K_\alpha} = 1$ μ_α-almost everywhere, whence $\mathbf{g}\varphi_{K_\alpha} = \mathbf{g}$ μ_α-almost everywhere; thus $\mathbf{g} \in \mathscr{L}^1_F(T, \mu_\alpha)$. Moreover, if $\mathbf{g} = \mathbf{g}'$ locally μ-almost everywhere then $\mathbf{g} = \mathbf{g}'$ μ_α-almost everywhere, whence a well-defined mapping

$$u_\alpha : L^1_{\text{loc}}(T, \mu; F) \to L^1_F(T, \mu_\alpha)$$

such that $u_\alpha(\dot{\mathbf{g}}) = \dot{\mathbf{g}}$ for $\mathbf{g} \in \mathscr{L}^1_{\text{loc}}(T, \mu; F)$, where the over-dot signifies, in turn, the class of $\mathbf{g}$ in $\mathscr{L}^1_{\text{loc}}(T, \mu; F)$ for equality locally μ-almost everywhere and the class of $\mathbf{g}$ in $\mathscr{L}^1_F(T, \mu_\alpha)$ for equality μ_α-almost everywhere. Thus the proposed mapping u is defined by

$$u : \dot{\mathbf{g}} \mapsto (\dot{\mathbf{g}}) \quad \text{for } \mathbf{g} \in \mathscr{L}^1_{\text{loc}}(T, \mu; F),$$

where the $(\dot{\mathbf{g}})$ on the right side is the family, indexed by α, whose α-th coordinate is $\dot{\mathbf{g}} \in L^1_F(T, \mu_\alpha)$; in other words, $\text{pr}_\alpha \circ u = u_\alpha$.

(ii) *Continuity of* u. It suffices to show that for each α, the mapping $\text{pr}_\alpha \circ u = u_\alpha$ is continuous. The topology of $L^1_F(T, \mu_\alpha)$ is generated by the norm function q^α,

$$q^\alpha(\dot{\mathbf{g}}) = \int |\mathbf{g}|\, d\mu_\alpha = \int |\mathbf{g}|\varphi_{K_\alpha}\, d\mu_\alpha\,;$$

to prove that u_α is continuous, we need only show that the semi-norm $q^\alpha \circ u_\alpha$ on $L^1_{\text{loc}}(T, \mu; F)$ is continuous (Lemma 2 in the Note for V.39, ℓ. 26, 27). Indeed,

$$\begin{aligned}(q^\alpha \circ u_\alpha)(\dot{\mathbf{g}}) &= q^\alpha\big(u_\alpha(\dot{\mathbf{g}})\big) && \big(\dot{\mathbf{g}} \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{T}, \mu; \mathrm{F})\big) \\ &= q^\alpha(\dot{\mathbf{g}}) && \big(\dot{\mathbf{g}} \in \mathrm{L}^1_{\mathrm{F}}(\mathrm{T}, \mu_\alpha)\big) \\ &= \int |\mathbf{g}|\, d\mu_\alpha = \int |\mathbf{g}|\varphi_{\mathrm{K}_\alpha}\, d\mu_\alpha \\ &\leqslant \int |\mathbf{g}|\varphi_{\mathrm{K}_\alpha}\, d\mu = p_{\mathrm{K}_\alpha}(\mathbf{g}) = q_{\mathrm{K}_\alpha}(\dot{\mathbf{g}})\end{aligned}$$

(q_{K_α} is defined in the Note for V.39, ℓ. -12 to -10), that is, $q^\alpha \circ u_\alpha \leqslant q_{\mathrm{K}_\alpha}$; since q_{K_α} is a continuous semi-norm on $\mathrm{L}^1_{\mathrm{loc}}(\mathrm{T}, \mu; \mathrm{F})$, so is $q^\alpha \circ u_\alpha$.

(iii) Since u is linear and continuous, it is uniformly continuous (GT, III, §3, No. 1, Prop. 3); therefore, if $\mathfrak{F}$ is a Cauchy filter on $\mathrm{L}^1_{\mathrm{loc}}(\mathrm{T}, \mu; \mathrm{F})$, then the filter base $u(\mathfrak{F})$ is Cauchy on $\prod_\alpha \mathrm{L}^1_{\mathrm{F}}(\mathrm{T}, \mu_\alpha)$ (GT, II, §3, No. 1, Prop. 3). Now, $\prod_\alpha \mathrm{L}^1_{\mathrm{F}}(\mathrm{T}, \mu_\alpha)$ is locally convex for the product topology (TVS, II, §4, No. 3, Prop. 4), and, for the product uniformity (GT, II, §2, No. 6, Def. 4), it is complete (GT, II, §3, No. 5, Prop. 10), therefore the Cauchy filter base $u(\mathfrak{F})$ is convergent; say $u(\mathfrak{F}) \to (\dot{\mathbf{f}}_\alpha)$, where $\mathbf{f}_\alpha \in \mathscr{L}^1_{\mathrm{F}}(\mathrm{T}, \mu_\alpha)$ for all α.

We can suppose that the $\mathbf{f}_\alpha$ are *universally measurable*, i.e., measurable for every measure on T (§3, No. 4, Prop. 7).

(iv) *Definition of* $\mathbf{f}$. Let $(\mathbf{f}_\alpha)$ be the family of universally measurable functions on T constructed in (iii), with $\mathbf{f}_\alpha$ μ_α-integrable for all α. Define $\mathbf{f} : \mathrm{T} \to \mathrm{F}$ by

$$\mathbf{f}(t) = \begin{cases} \mathbf{f}_\alpha(t) & \text{for } t \in \mathrm{K}_\alpha \\ 0 & \text{for } t \in \mathrm{T} - \bigcup_\alpha \mathrm{K}_\alpha\,. \end{cases}$$

(iv) *Universal measurability of* $\mathbf{f}$. Since the family (K_α) is locally countable, the set $\mathrm{B} = \bigcup_\alpha \mathrm{K}_\alpha$, is universally measurable (Ch. IV, §5, No. 9); moreover, for every α, the function $\mathbf{f}\big|\mathrm{K}_\alpha = \mathbf{f}_\alpha\big|\mathrm{K}_\alpha$ is universally measurable (in the sense of *loc. cit.*, No. 10, Def. 8), therefore $\mathbf{f}\big|\mathrm{B}$ is universally measurable (*loc. cit.*, Prop. 16). Finally, $\mathbf{f}$ is the extension by 0 of $\mathbf{f}\big|\mathrm{B}$ to T, hence is universally measurable (*loc. cit.*, Prop. 15).

V.39, ℓ. $-7, -6$.

"Every measurable function $\mathbf{g}$, that is essentially bounded on every compact set, is locally integrable."

The term "essentially bounded" is not indexed and seems nowhere to be defined. My guess is that "$\mathbf{g}$ is essentially bounded" means the same thing as "$\mathbf{g}$ is bounded in measure" (Ch. IV, §6, No. 3, Def. 2), i.e., there exists a bounded function $\mathbf{h}$ such that $\mathbf{g} = \mathbf{h}$ locally almost everywhere, equivalently, $\mathrm{N}_\infty(\mathbf{g}) < +\infty$. If so, then a function $\mathbf{g} : \mathrm{T} \to \mathrm{F}$ is measurable and essentially bounded if and only if $\mathbf{g} \in \mathscr{L}^\infty_{\mathrm{F}}(\mathrm{T}, \theta)$ (*ibid.*); and, to say that $\mathbf{g}$ is essentially bounded on a compact set K means that $\mathbf{g}\varphi_{\mathrm{K}}$ is bounded almost everywhere (Ch. IV, §5, No. 2, Prop. 5), that is, $|\mathbf{g}|\varphi_{\mathrm{K}} \leqslant c$ almost everywhere for some constant $c < +\infty$. Thus, if $\mathbf{g}$ is measurable and is essentially bounded on a compact set K, then $\mathbf{g}\varphi_{\mathrm{K}}$ is integrable (*loc. cit.*, No. 6, Th. 5); and if this is true for every compact set K, then $\mathbf{g}$ is locally integrable.

V.39, ℓ. −2 to **V.40,** ℓ. 2.

"Let F, G, H be three Banach spaces, and $(\mathbf{u}, \mathbf{v}) \mapsto \Phi(\mathbf{u}, \mathbf{v})$ a continuous bilinear mapping of $\mathrm{F} \times \mathrm{G}$ into H. If $\mathbf{f}$ is locally integrable and takes its values in F, and if $\mathbf{g} \in \mathscr{L}^\infty_{\mathrm{G}}$, then $\Phi(\mathbf{f}, \mathbf{g})$ is locally integrable (Ch. IV, §6, No. 4, Cor. 1 of Th. 2)."

The rôle of bilinearity is to assure the existence of a real constant $c > 0$ such that $|\Phi(\mathbf{f}, \mathbf{g})| \leqslant c \cdot \mathrm{N}_1(\mathbf{f})\,\mathrm{N}_\infty(\mathbf{g})$ for $\mathbf{f} \in \mathscr{L}^1_{\mathrm{F}}$ and $\mathbf{g} \in \mathscr{L}^\infty_{\mathrm{G}}$ (GT, IX, §3, No. 5, Th. 1).

Suppose $\mathbf{f} \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T}, \theta; \mathrm{F})$ and $\mathbf{g} \in \mathscr{L}^\infty_{\mathrm{G}}(\mathrm{T}, \theta)$. For every compact set $\mathrm{K} \subset \mathrm{T}$, the function $\mathbf{f}\varphi_{\mathrm{K}}$ is integrable, therefore $\Phi(\mathbf{f}\varphi_{\mathrm{K}}, \mathbf{g}) \in \mathscr{L}^1_{\mathrm{H}}(\mathrm{T}, \theta)$ by the cited Cor. 1. But, for all $\in \mathrm{T}$, by definition

$$\begin{aligned}\big(\Phi(\mathbf{f}\varphi_{\mathrm{K}}, \mathbf{g})\big)(t) &= \Phi\big((\mathbf{f}\varphi_{\mathrm{K}})(t), \mathbf{g}(t)\big)\\ &= \Phi\big(\varphi_{\mathrm{K}}(t)\mathbf{f}(t), \mathbf{g}(t)\big)\\ &= \varphi_{\mathrm{K}}(t)\Phi\big(\mathbf{f}(t), \mathbf{g}(t)\big)\\ &= \big(\varphi_{\mathrm{K}}\Phi(\mathbf{f}, \mathbf{g})\big)(t)\,,\end{aligned}$$

thus $\varphi_{\mathrm{K}}\Phi(\mathbf{f}, \mathbf{g}) = \Phi(\mathbf{f}\varphi_{\mathrm{K}}, \mathbf{g})$ is integrable. Since K is arbitrary, $\Phi(\mathbf{f}, \mathbf{g})$ is locally integrable—and, in particular, measurable (Prop. 1).

V.40, ℓ. 5–9.

"... the set of t such that $g(t) = +\infty$ is then locally μ-negligible, because $g\varphi_{\mathrm{K}}$ is μ-integrable for every compact set K (Ch. IV, §2, No. 3, Prop. 7). Now let g' be a locally integrable function that is positive and *finite*, equal to g locally μ-almost everywhere"

See the Proposition in the Note for V.39, ℓ. 22–24. (Here, μ is a positive measure on T—for example, $\mu = |\theta|$ for some complex measure θ on T.)

V.40, ℓ. 9–11.

"The mapping $t \mapsto \lambda'_t$ of T into $\mathscr{M}_+(\mathrm{T})$ is vaguely μ-measurable and scalarly essentially integrable (or again, the pair (I, g'), where I is the identity mapping of T, is μ-adapted)"

The mapping $\Lambda : \mathrm{T} \to \mathscr{M}_+(\mathrm{T})$ defined by $\Lambda : t \mapsto \lambda'_t = g'(t)\varepsilon_t$ is scalarly essentially μ-integrable (§3, No. 1); for, for every $f \in \mathscr{K}(\mathrm{T})$, the function

$$\Lambda(f) : t \mapsto \lambda'_t(f) = g'(t)\varepsilon_t(f) = g'(t)f(t),$$

that is, the function $g'f$, is μ-integrable (Def. 1) hence essentially μ-integrable; one can therefore define the positive measure $\nu = \int \lambda'_t \, d\mu(t)$.

Since the functions I and g' are μ-measurable and, for every f in $\mathscr{K}(\mathrm{T})$, the function

$$t \mapsto f\big(I(t)\big)g'(t) = f(t)g'(t)$$

is μ-integrable, the pair (I, g') is μ-adapted (§4, No. 1, Def. 1); therefore Λ is vaguely measurable (and μ-adequate) (*loc. cit.*, Prop. 1).

The measure ν is defined by $\nu(f) = \int \lambda'_t(f)\, d\mu(t) = \int g'(t)f(t)\, d\mu(t)$ for all $f \in \mathscr{K}(\mathrm{T})$ (§3, No. 1); since $gf = g'f$ μ-almost everywhere, one is permitted to write

$$\nu(f) = \int gf \, d\mu = \int g(t)f(t)\, d\mu(t)$$

(Ch. IV, §4, No. 1, last paragraph), to replace Λ by the mapping $t \mapsto \lambda_t = g(t)\varepsilon_t$ (defined at the points in the domain of g where g is finite), and to write $\nu = \int \lambda_t \, d\mu(t)$ (§3, No. 3, *Remark*).

V.40, ℓ. -15 to -12.

"... one can write

$$(2) \qquad \begin{aligned} u &= g_1 - g_2 + i(g_3 - g_4) \\ \theta &= \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \end{aligned}$$

where $\mu_1 = (\mathscr{R}\theta)^+, \ldots$"

Understood, that when u takes its values in $\overline{\mathbf{R}}$, $g_3 = g_4 = 0$ and g_1, g_2 are not simultaneously equal to $+\infty$ at any point; when u takes its values in $\mathbf{C}$, the g_i are all finite-valued; when θ is a real measure, $\mu_3 = \mu_4 = 0$.

One need not take $\mu_1 = (\mathscr{R}\theta)^+$, etc., but it is essential that the g_i be θ-measurable, $0 \leqslant g_i \leqslant |g|$ and $0 \leqslant \mu_j \leqslant \mu$ for all i and j (the argument of §2, No. 2, Cor. 2 of Prop. 3 is then applicable); thus the local integrability of u for θ implies that g_i is locally μ_j-integrable for all i and j.

It is clear from the earlier discussion that one can suppose u to be defined and finite at every point of T, i.e., that $u \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T}, \theta; \mathbf{C})$.

{Conversely, if g_i is μ_j-integrable for all i and j, then each g_i is integrable for θ (§2, No. 2, Cor. 2 of Prop. 3), therefore so is u; it follows that if g_i is locally μ_j-integrable for all i and j, then u is locally θ-integrable.}

V.40, ℓ. −9 to −6.

"... the mapping

$$f \mapsto \int f(t)u(t)\,d\theta(t)$$

on $\mathscr{K}(\mathrm{T})$ is a complex measure."

By assumption, fu is $|\theta|$-integrable for every $f \in \mathscr{K}(\mathrm{T}) = \mathscr{K}(\mathrm{T};\mathbf{C})$ (No. 1, Def. 1). Regarding θ and the μ_j as linear forms on $\mathscr{K}(\mathrm{T})$, the second equation of (2) means that

$$\theta(f) = \mu_1(f) - \mu_2(f) + i\mu_3(f) - i\mu_4(f) \tag{i}$$

for all $f \in \mathscr{K}(\mathrm{T})$. It follows from the assumptions on the μ_j that every θ-integrable function h is μ_j-integrable for all j, and that

$$\int h\,d\theta = \int h\,d\mu_1 - \int h\,d\mu_2 + i\int h\,d\mu_3 - i\int h\,d\mu_4 \tag{ii}$$

(see the Note for V.10, ℓ. 13, 14). In particular, for every $f \in \mathscr{K}(\mathrm{T})$ one has

$$\int fu\,d\theta = \int fu\,d\mu_1 - \int fu\,d\mu_2 + i\int fu\,d\mu_3 - i\int fu\,d\mu_4\,. \tag{iii}$$

Also, for $f \in \mathscr{K}(\mathrm{T})$ one has

$$fu = fg_1 - fg_2 + ifg_3 - ifg_4\,;$$

substituting this expression for fu on the right side of (iii), it follows from the linearity of integration that the linear form $f \mapsto \int fu\,d\theta$ on $\mathscr{K}(\mathrm{T})$ is a linear combination of the (sixteen) linear forms

$$f \mapsto \int fg_i\,d\mu_j \qquad \big(f \in \mathscr{K}(\mathrm{T})\big), \tag{iv}$$

each of which is known to be a positive linear form on $\mathscr{K}(\mathrm{T};\mathbf{R})$, hence is a measure on T (Ch. III, §1, No. 5, Th. 1). Thus, the linear form $f \mapsto \int fu\,d\theta$ on $\mathscr{K}(\mathrm{T})$, being a linear combination of measures, is itself a measure, i.e., is continuous for the inductive limit topology (*loc. cit.*, No. 3, Def. 2).

V.41, ℓ. 6, 7.

"... if u_1 and u_2 are locally θ-integrable, then $(u_1+u_2)\cdot\theta = u_1\cdot\theta + u_2\cdot\theta$."

The function $u_1 + u_2$ is defined and finite locally θ-almost everywhere in T and, for every $f \in \mathscr{K}(\mathrm{T})$, $f \cdot (u_1 + u_2) = fu_1 + fu_2$ θ-almost everywhere; since fu_1 and fu_2 are θ-integrable, so is $f \cdot (u_1 + u_2)$, and

$$\int f \cdot (u_1 + u_2)\, d\theta = \int fu_1\, d\theta + \int fu_2\, d\theta\,,$$

whence the assertion. {Alternatively, one can suppose that $u_i \in \mathscr{L}^1_{\mathrm{loc}}$ etc.}

V.41, ℓ. 7–9.

"... if θ_1 and θ_2 are two measures on T, and if u is a function locally integrable for θ_1 and θ_2, then u is locally integrable for $\theta_1 + \theta_2$ and one has $u \cdot (\theta_1 + \theta_2) = u \cdot \theta_1 + u \cdot \theta_2$."

Let $\mathrm{A} = \{t \in \mathrm{T} : u(t) \text{ is defined and finite}\}$, and let $\mathrm{N} = \complement \mathrm{A}$. Since N is locally negligible for θ_1 and θ_2, it is locally negligible for $\theta_1 + \theta_2$ ($|\theta_1 + \theta_2|^\bullet \leqslant |\theta_1|^\bullet + |\theta_2|^\bullet$); we can therefore suppose that u is defined and finite everywhere on T.

By assumption, for every $f \in \mathscr{K}(\mathrm{T})$ one has $fu \in \mathscr{L}^1(\theta_1) \cap \mathscr{L}^1(\theta_2)$, therefore $fu \in \mathscr{L}^1(\theta_1 + \theta_2)$ and

$$\int fu\, d(\theta_1 + \theta_2) = \int fu\, d\theta_1 + \int fu\, d\theta_2$$

(see the *Theorem* in the Note for V.10, ℓ. 13, 14).

V.41, ℓ. −13.

$$\langle |\theta|, |f| \rangle = \sup_{c \in \mathscr{K}_1} |\langle \theta, cf \rangle| = \sup_{c \in \mathscr{B}_1} |\langle \theta, cf \rangle| \tag{4}$$

The brackets pertain to the duality between measures and integrable functions (discussed at length in the Note for V.10, ℓ. 13, 14). In the second member of (4), cf is integrable because f is essentially integrable and $c \in \mathscr{K}(\mathrm{T})$ is bounded, measurable and moderated (§1, No. 3, Cor. of Prop. 9), and so $\langle \theta, cf \rangle = \int (cf)\, d\theta$. However, in the first and third members, $|f|$ and cf are only known to be essentially integrable; nevertheless,

$$\langle |\theta|, |f| \rangle = \int |f|\, d|\theta|\,, \quad \langle \theta, cf \rangle = \int (cf)\, d\theta\,,$$

because $f = f'$ locally almost everywhere for some $f' \in \mathscr{L}^1_{\mathbf{C}}(\mathrm{T}, \theta)$ and, by definition (*loc. cit.*, Def. 3),

$$\int |f|\, d|\theta| = \int |f'|\, d|\theta|\,, \quad \int (cf)\, d\theta = \int (cf')\, d\theta$$

(the left members are 'essential integrals'—merely a notation for the right members, which are 'honest' integrals).

V.41, ℓ. −10 to −8.

"Obviously

$$\sup_{c\in\mathscr{K}_1} |\langle\theta, cf\rangle| \leqslant \sup_{c\in\mathscr{B}_1} |\langle\theta, cf\rangle| \leqslant \langle|\theta|, |f|\rangle$$

(Ch. IV, §4, No. 2, Prop. 2)."

The first inequality results from the fact that continuous functions are Borel (GT, IX, §6, No. 3, Prop. 10).[(1)] Note that if X is any locally compact space and Y is a metrizable space with a countable base for open sets (e.g., a separable Banach space), then every Borel mapping $c : \mathrm{X} \to \mathrm{Y}$ is universally measurable (§3, No. 4, Def. 2 and Ch. IV, §5, No. 5, *Remark* 2).

As for the second inequality: if $c \in \mathscr{B}_1$ then c is (universally) measurable and $|cf| \leqslant |f|$, whence

$$|\langle\theta, cf\rangle| \leqslant \langle|\theta|, |cf|\rangle \leqslant \langle|\theta|, |f|\rangle$$

by the generalization of the cited Prop. 2 to essentially integrable functions (signaled in the paragraph preceding §1, No. 3, Prop. 10).

V.41, ℓ. −8 to −4.

"...let g be an element of $\mathscr{K}(\mathrm{T};\mathbf{C})$ such that $|g| \leqslant |f|$; g is the uniform limit of a sequence (g_n) of elements of $\mathscr{K}(\mathrm{T};\mathbf{C})$ whose supports are contained in the open set U formed by the t such that $f(t) \neq 0$, and one may clearly suppose that $|g_n| \leqslant |f|$ for every n."

The condition $|g| \leqslant |f|$ looks forward to a supremum over all such g. For $n = 1, 2, 3, \ldots$ let $\mathrm{K}_n = \{t \in \mathrm{T} : |g(t)| \geqslant 1/n\}$; then K_n is a closed subset of the support of g, hence is compact, and $\mathrm{K}_n \subset \mathrm{U}$. For each n, choose $h_n \in \mathscr{K}_+(\mathrm{T})$ so that $0 \leqslant h_n \leqslant 1$, $h_n = 1$ on K_n, and $\operatorname{Supp} h_n \subset \mathrm{U}$ (Ch. III, §1, No. 2, Lemma 1). Then $h_n g \in \mathscr{K}(\mathrm{T};\mathbf{C})$, $|h_n g| \leqslant |g| \leqslant |f|$, and

$$\operatorname{Supp}(h_n g) \subset \operatorname{Supp} h_n \subset \mathrm{U};$$

moreover, $h_n g - g = 0$ on K_n, and, for $t \in \complement \mathrm{K}_n$,

$$|(h_n g - g)(t)| = \big(1 - h_n(t)\big) \cdot |g(t)| \leqslant |g(t)| < 1/n,$$

whence $\|h_n g - g\| \leqslant 1/n$. Thus $g_n = h_n g$ meets the requirements of the assertion (and, moreover, satisfies $|g_n| \leqslant |g|$).

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(1) The Borel sets of a topological space X are the elements of the tribe generated by the set of closed subsets of X (GT, IX, §6, No. 3, Def. 4). The proof of the cited Prop. 10, and Exer. 16 for GT, IX, §6, suggest that a mapping $f : \mathrm{X} \to \mathrm{Y}$ between topological spaces should be called a *Borel mapping* if the inverse image of every Borel set (equivalently, of every closed set) in Y is a Borel set in X, but GT does not pass to the act; TG passes to the act (TG, IX, §6, No. 3, Def. 5).

V.41, ℓ. −3 to **V.42,** ℓ. 2.

"... then $c_n \in \mathscr{K}_1$, $g = \lim_{n\to\infty} c_n f$, therefore $|\langle\theta, g\rangle| = \lim_{n\to\infty} |\langle\theta, c_n f\rangle|$, and finally

$$\sup_{|g|\leqslant|f|,\, g\in\mathscr{K}(\mathrm{T};\mathbf{C})} |\langle\theta, g\rangle| \leqslant \sup_{c\in\mathscr{K}_1} |\langle\theta, cf\rangle| .$$

One concludes by observing that the first member of this inequality is equal to $\langle|\theta|, |f|\rangle$ (Ch. III, §1, No. 6, formula (12))."

In slow motion,

$$c_n(t) = \begin{cases} \dfrac{g_n(t)}{f(t)} & \text{for } t \in \mathrm{U} \\ 0 & \text{for } t \in \complement\mathrm{U} . \end{cases}$$

From $\operatorname{Supp} c_n \subset \operatorname{Supp} g_n$ one knows that c_n has compact support; at issue is the continuity of c_n. From $\operatorname{Supp} g_n \subset \mathrm{U}$ one knows that the open sets U and $\complement(\operatorname{Supp} g_n)$ have union T, so it suffices to show that the restrictions of c_n to U and $\complement(\operatorname{Supp} g_n)$ are continuous. Indeed, from the definition of c_n, it is clear that $c_n\big|\mathrm{U}$ is continuous; whereas if $t \in \complement(\operatorname{Supp} g_n)$ then $c_n(t) = 0$ whether $t \in \mathrm{U}$ or $t \in \complement\mathrm{U}$, thus $c_n\big|\complement(\operatorname{Supp} g_n) = 0$.

Note that $c_n f = g_n$: the equality holds at the points of U by the definition of c_n, and both members are equal to 0 on the subset $\complement\mathrm{U}$ of $\complement(\operatorname{Supp} g_n)$. It follows that $\|g - c_n f\| = \|g - g_n\| \to 0$, thus $c_n f \to g$ uniformly. Let $\mathrm{K} = \overline{\mathrm{U}} = \operatorname{Supp} f$; then $\operatorname{Supp}(c_n f) \subset \operatorname{Supp} f = \mathrm{K}$ and $\operatorname{Supp} g \subset \operatorname{Supp} f = \mathrm{K}$ (because $|g| \leqslant |f|$), thus $c_n f$ and g belong to $\mathscr{K}(\mathrm{T},\mathrm{K};\mathbf{C})$, consequently $c_n f \to g$ in $\mathscr{K}(\mathrm{T},\mathrm{K};\mathbf{C})$. Since the norm topology on $\mathscr{K}(\mathrm{T},\mathrm{K};\mathbf{C})$ is equal to the topology induced by the (inductive limit) topology of $\mathscr{K}(\mathrm{T};\mathbf{C})$, it follows that $c_n f \to g$ in $\mathscr{K}(\mathrm{T};\mathbf{C})$; by the continuity of θ, $\theta(c_n f) \to \theta(g)$, whence

$$\lim_{n\to\infty} |\langle\theta, c_n f\rangle| = |\langle\theta, g\rangle| .$$

Passing to the limit in the inequality $|\langle\theta, c_n f\rangle| \leqslant \sup_{c\in\mathscr{K}_1} |\langle\theta, cf\rangle|$, one has

$$|\langle\theta, g\rangle| \leqslant \sup_{c\in\mathscr{K}_1} |\langle\theta, cf\rangle| ,$$

and the validity of this inequality for all g yields

$$\sup_{|g|\leqslant|f|,\, g\in\mathscr{K}(\mathrm{T};\mathbf{C})} |\langle\theta, g\rangle| \leqslant \sup_{c\in\mathscr{K}_1} |\langle\theta, cf\rangle| ,$$

in other words

$$\langle|\theta|, |f|\rangle \leqslant \sup_{c\in\mathscr{K}_1} |\langle\theta, cf\rangle|$$

by the cited formula (12). This implies that in the inequalities of ℓ. −9, equality holds throughout. Thus the equalities (4) are verified for the case that $f \in \mathscr{K}(\mathrm{T};\mathbf{C})$.

V.42, ℓ. 6.

"... the dense subspace $\mathscr{K}(\mathrm{T};\mathbf{C})$."

Note that the semi-norm $\overline{\mathrm{N}}_1$ on $\overline{\mathscr{L}}^1_{\mathbf{C}}$ coincides with N_1 on $\mathscr{L}^1_{\mathbf{C}}$ (§1, No. 2, Prop. 7, 2)).

The closure of $\mathscr{K}(\mathrm{T};\mathbf{C})$ in $\overline{\mathscr{L}}^1_{\mathbf{C}}$ is a linear subspace that contains $\mathscr{L}^1_{\mathbf{C}}$ (Ch. IV, §3, No. 4, Def. 2) and $\mathscr{N}^\infty_{\mathbf{C}}$ (§1, No. 3) hence it contains $\mathscr{L}^1_{\mathbf{C}} + \mathscr{N}^\infty_{\mathbf{C}} = \overline{\mathscr{L}}^1_{\mathbf{C}}$.

V.42, ℓ. 8, 9.

$$|\langle|\theta|,|f|\rangle - \langle|\theta|,|f'|\rangle| \leqslant \langle|\theta|,|f-f'|\rangle = \overline{\mathrm{N}}_1(f-f')$$
$$|\langle\theta,cf\rangle - \langle\theta,cf'\rangle| \leqslant \langle|\theta|,|c|\,|f-f'|\rangle \leqslant \overline{\mathrm{N}}_1(f-f')$$

If $g \in \mathscr{L}^1_{\mathbf{C}}$ with $f = g$ locally almost everywhere, then $cf = cg$ locally almost everywhere and

$$\langle\theta,cf\rangle = \int cf\,d\theta = \int cg\,d\theta = \langle\theta,cg\rangle$$

by the definitions, and similarly for the other five bracket expressions in the displayed relations; one can therefore suppose that f and f' belong to $\mathscr{L}^1_{\mathbf{C}}$, whence also $\overline{\mathrm{N}}_1(f-f') = \mathrm{N}_1(f-f')$.

The first line of the displayed relations then follows from the inequality $\big||f|-|f'|\big| \leqslant |f-f'|$ and the computation

$$\begin{aligned}\big|\langle|\theta|,|f|\rangle - \langle|\theta|,|f'|\rangle\big| &= \big|\langle|\theta|,|f|-|f'|\rangle\big| \\ &\leqslant \big\langle|\theta|,\big||f|-|f'|\big|\big\rangle \leqslant \langle|\theta|,|f-f'|\rangle = \mathrm{N}_1(f-f')\end{aligned}$$

(Ch. IV, §4, No. 2, Prop. 2); similarly for the second line of the display, taking into account that c is (universally) measurable and $|c| \leqslant 1$.

From the first line of the display, it is clear that the first member of (4) depends continuously on $f \in \overline{\mathscr{L}}^1_{\mathbf{C}}$. As to the second member, let us abbreviate by defining

$$\alpha(f) = \sup_{c\in\mathscr{K}_1} |\langle\theta,cf\rangle| \qquad (f \in \overline{\mathscr{L}}^1_{\mathbf{C}}).$$

For $f, f' \in \overline{\mathscr{L}}^1_{\mathbf{C}}$ and $c \in \mathscr{K}_1$, one has

$$\begin{aligned}|\langle\theta,cf\rangle| &\leqslant |\langle\theta,cf\rangle - \langle\theta,cf'\rangle| + |\langle\theta,cf'\rangle| \\ &\leqslant \overline{\mathrm{N}}_1(f-f') + |\langle\theta,cf'\rangle| \leqslant \overline{\mathrm{N}}_1(f-f') + \alpha(f')\end{aligned}$$

(the first $\leqslant$, by the triangle inequality in $\mathbf{C}$; the second $\leqslant$, by the second line of the display; the third $\leqslant$, by the definition of α). Varying $c \in \mathscr{K}_1$, one has

$$\alpha(f) \leqslant \overline{\mathrm{N}}_1(f - f') + \alpha(f')\,;$$

by symmetry in f and f', clearly

$$|\alpha(f) - \alpha(f')| \leqslant \overline{\mathrm{N}}_1(f - f')\,,$$

whence the continuity of the second member of (4) as a function of f, and a similar argument establishes the continuity of the third member of (4).

{For the purposes of Prop. 2, it is not necessary to introduce the parameter $c \in \mathscr{B}_1$; perhaps it is destined for a future application.}

V.42, ℓ. −6, −5.

"This follows at once from Prop. 2 and the formulas (6) of Ch. II, §1, No. 1."

The equality $\sup(g_1, g_2) = \frac{1}{2}(g_1 + g_2 + |g_1 - g_2|)$ holds on the set of points where g_1 and g_2 are both defined and finite—hence locally almost everywhere—therefore

$$\begin{aligned}
\sup(g_1, g_2)\cdot\mu &= \frac{1}{2}(g_1\cdot\mu + g_2\cdot\mu + |g_1 - g_2|\cdot\mu) \\
&= \frac{1}{2}(g_1\cdot\mu + g_2\cdot\mu + |(g_1 - g_2)\cdot\mu|) \\
&= \frac{1}{2}(g_1\cdot\mu + g_2\cdot\mu + |g_1\cdot\mu - g_2\cdot\mu|) \\
&= \sup(g_1\cdot\mu, g_2\cdot\mu)
\end{aligned}$$

(the second equality, by Prop. 2; the fourth, by the cited formula), and similarly

$$\inf(g_1, g_2)\cdot\mu = \inf(g_1\cdot\mu, g_2\cdot\mu)\,.$$

In particular, setting $g_1 = g$ and $g_2 = 0$, one has

$$\begin{aligned}
(g\cdot\mu)^+ &= \sup(g\cdot\mu, 0) = \sup(g, 0)\cdot\mu = g^+\cdot\mu \\
(g\cdot\mu)^- &= \big(-(g\cdot\mu)\big)^+ = \big((-g)\cdot\mu\big)^+ = (-g)^+\cdot\mu = g^-\cdot\mu\,.
\end{aligned}$$

V.42, ℓ. −3 to −1.

"*In the statements of this subsection, g denotes a positive numerical function, defined everywhere and locally μ-integrable, θ denotes a complex measure, and u a locally θ-integrable complex function.*"

Here μ denotes a positive measure on T. As announced in No. 2 (p. V.41, ℓ. 10, 11), henceforth all locally integrable functions are assumed to be defined everywhere on T. In particular, $u(t) \in \mathbf{C}$ for all $t \in \mathrm{T}$; the function $g \geqslant 0$ may take the value $+\infty$, but, being locally integrable, it is finite locally almost everywhere.

V.43, ℓ. 1–3.

"The remarks in No. 2 show that the results of §4 are applicable to the measure $\nu = g \cdot \mu = \int g(t)\varepsilon_t \, d\mu(t)$ (even though the measure $g(t)\varepsilon_t$ is not defined unless $g(t) \neq +\infty$)."

Since g is locally μ-integrable, there exists a (everywhere finite) function $g' \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T}, \mu; \mathbf{R})$ such that $g = g'$ locally μ-almost everywhere and $g'(t) \geqslant 0$ for all $t \in \mathrm{T}$. Defining $\lambda'_t = g'(t)\varepsilon_t$ for all $t \in \mathrm{T}$, the mapping $\Lambda' : t \mapsto \lambda'_t$ is scalarly essentially μ-integrable: for $f \in \mathscr{K}(\mathrm{T})$, the function $t \mapsto \lambda'_t(f) = g'(t)f(t)$ is the μ-integrable function $g'f$, and the formula

$$\nu(f) = \int g' f \, d\mu = \int \lambda'_t(f) \, d\mu(t)$$

defines a measure $\nu \geqslant 0$ on T (by the cited remarks). The functions gf and $g'f$ are defined everywhere in T and $gf = g'f$ μ-almost everywhere (f is μ-moderated), therefore gf is also μ-integrable, and one can also write $\nu(f) = \int gf \, d\mu$ (p. IV.34, ℓ. 1–5). By No. 2, Def. 2, $g \cdot \mu = \nu$.

Another perspective on ν: let $\mathrm{A} = \{t \in \mathrm{T} : \ g(t) \neq +\infty\}$ and consider the mapping

$$\Lambda : t \mapsto g(t)\varepsilon_t \qquad (t \in \mathrm{A}).$$

One knows (by the cited remarks) that (I, g') is μ-adapted, hence the mapping $\Lambda' : t \mapsto g'(t)\varepsilon_t$ $(t \in \mathrm{T})$ is μ-adequate (and vaguely μ-measurable) by §4, No. 1, Prop. 1. Since $\Lambda = \Lambda'$ locally μ-almost everywhere, we are authorized (§3, No. 3, *Remark*) to say that Λ is μ-adequate, and to write

$$\int g(t)\varepsilon_t \, d\mu(t) = \int g'(t)\varepsilon_t \, d\mu(t),$$

that is, $\int g(t)\varepsilon_t \, d\mu(t) = \nu$.

V.43, ℓ. 6.

"This follows from Th. 1 of §4, No. 2."

By assumption, g is defined and $\geqslant 0$ everywhere in T, but may have values equal to $+\infty$. With notations as in the preceding Note, (I, g') is μ-adapted; in particular, $g'(t)$ is finite and $\geqslant 0$ for every $t \in \mathrm{T}$, and one has $\nu = \int g'(t)\varepsilon_t \, d\mu(t)$.

If f is defined everywhere on T, the cited Th. 1 immediately yields $\nu^\bullet(f) = \mu^\bullet(fg')$. As fg is also everywhere-defined and $fg = fg'$ locally μ-almost everywhere, one has $\mu^\bullet(fg) = \mu^\bullet(fg')$, whence $\nu^\bullet(f) = \mu^\bullet(fg)$, that is,

$$(*) \qquad \int^\bullet f(t)\,d\nu(t) = \int^\bullet f(t)g(t)\,d\mu(t)\,.$$

When f is only defined and $\geqslant 0$ locally μ-almost everywhere, the same is true of fg and fg', and $fg = fg'$ locally μ-almost everywhere. Imitating p. V.9, ℓ. 1–5, if one defines $\int^\bullet f\,d\nu = \int^\bullet f'\,d\nu$, where f' is the extension by 0 of f to T, and similarly $\int^\bullet fg\,d\mu = \int^\bullet (fg)'\,d\mu = \int^\bullet f'g\,d\mu$ (the convention $0\cdot(+\infty) = 0$ is at work in the last equality), then

$$\int^\bullet f'(t)\,d\nu(t) = \int^\bullet f'(t)g(t)\,d\mu(t)$$

by (∗); in other words $\int^\bullet f(t)\,d\nu(t) = \int^\bullet f(t)g(t)\,d\mu(t)$, that is, f satisfies (∗) formally.

V.43, ℓ. 13.

"...the statement then follows at once from Prop. 3."

In Prop. 3, put $\mu = |\theta|$, $g = |u|$, $f = |\mathbf{f}|$; then $\nu = g\cdot\mu = |u|\cdot|\theta| = |u\cdot\theta|$ (No. 2, Prop. 2) and Prop. 3 yields

$$|u\cdot\theta|^\bullet(|\mathbf{f}|) = \int^\bullet |\mathbf{f}|\,d(|u\cdot\theta|) = \int^\bullet |\mathbf{f}|\,|u|\,d|\theta| = \int^\bullet |u\mathbf{f}|\,d|\theta| = |\theta|^\bullet(|u\mathbf{f}|)\,;$$

in particular, $|u\cdot\theta|^\bullet(|\mathbf{f}|) = 0 \Leftrightarrow |\theta|^\bullet(|u\mathbf{f}|) = 0$, thus $\mathbf{f}$ is locally negligible for $u\cdot\theta$ if and only if $u\mathbf{f}$ is locally negligible for θ (§1, No. 1).

V.43, ℓ. 17.

"One is immediately reduced to showing that ..."

Suppose, more generally, that u_1, u_2 are defined locally almost everywhere for θ, with values either in $\mathbf{C}$ or in $\overline{\mathbf{R}}$. For $i = 1, 2$ choose $v_i \in \mathscr{L}^1_{\text{loc}}(\mathrm{T}, \theta; \mathbf{C})$ with $u_i = v_i$ locally almost everywhere for θ. For every $f \in \mathscr{K}(\mathrm{T})$, fv_i is θ-integrable and $fu_i = fv_i$ $|\theta|$-almost everywhere (No. 1, Def. 1 and IV.34, ℓ. 1–5), whence $u_i\cdot\theta = v_i\cdot\theta$ (No. 2, Def. 2).

We can therefore suppose that u_1, u_2 are defined and complex-valued everywhere in T, whence $u_2\cdot\theta = (u_2 - u_1)\cdot\theta + u_1\cdot\theta$ (V.41, ℓ. 6, 7). Then $u_1\cdot\theta = u_2\cdot\theta \Leftrightarrow (u_2 - u_1)\cdot\theta = 0 \Leftrightarrow (u_2 - u_1)\mathbf{1}$ is locally negligible for θ (Cor. 1, with $\mathbf{1}$ the constant function equal to 1 on T), that is, if and only if $u_1 = u_2$ locally almost everywhere for θ.

V.43, ℓ. −9, −8.

"When u and θ are positive, this follows at once from Prop. 3 of §4, No. 3."

When $\theta \geqslant 0$ and the locally θ-integrable function u is $\geqslant 0$ everywhere in T then, by the first paragraph of No. 2, the pair (I, u) is θ-adapted and it follows from (1) that the measure $\nu = \int u(t)\, \varepsilon_t \, d\theta(t)$ is the measure $u \cdot \theta$ of Def. 2 (*loc. cit.*). Since $f \circ \mathrm{I} = f$, the assertion is indeed immediate from the cited Prop. 3.

V.43, ℓ. −8, −7.

"The result then extends to the case that u and θ are complex thanks to Prop. 2."

In slow motion,

$$\begin{aligned} f \text{ is measurable for } u\cdot\theta \quad &\Leftrightarrow \quad f \text{ is measurable for } |u\cdot\theta| \\ &\Leftrightarrow \quad f \text{ is measurable for } |u|\cdot|\theta| \\ &\Leftrightarrow \quad f\big|\mathrm{S} \text{ is measurable for } |\theta| \\ &\Leftrightarrow \quad f\big|\mathrm{S} \text{ is measurable for } \theta \end{aligned}$$

(the 1st and 4th equivalence, by Ch. IV, §5, No. 1, Def. 1), the 2nd by No. 2, Prop. 2, and the 3rd by the 'positive case' already considered). This settles the case that u is defined everywhere on T.

Suppose more generally that the locally θ-integrable function u (with values in $\mathbf{C}$ or in $\overline{\mathbf{R}}$) is defined only locally θ-almost everywhere, say with domain $\mathrm{A} \subset \mathrm{T}$, where $\mathrm{T} - \mathrm{A}$ is locally θ-negligible. Choose u' in $\mathscr{L}^1_{\mathrm{loc}}(\mathrm{T}, \theta; \mathbf{C})$ such that $u = u'$ locally θ-almost everywhere, and let N be a locally θ-negligible set such that $u(t)$ is defined and equal to $u'(t)$ on $\mathrm{T} - \mathrm{N}$, that is, $\mathrm{T} - \mathrm{N} \subset \mathrm{A}$ and $u = u'$ on $\mathrm{T} - \mathrm{N}$. By Cor. 2, $u\cdot\theta = u'\cdot\theta$. Let

$$\mathrm{S} = \{t \in \mathrm{A} : \ u(t) \neq 0\}, \quad \mathrm{S}' = \{t \in \mathrm{T} : \ u'(t) \neq 0\}.$$

By the foregoing, we know that $f\big|\mathrm{S}'$ is θ-measurable if and only if f is measurable for $u'\cdot\theta = u\cdot\theta$; to show that $f\big|\mathrm{S}$ is θ-measurable if and only if f is measurable for $u\cdot\theta$, we need only show that

$$(*) \qquad f\big|\mathrm{S} \text{ is } \theta\text{-measurable} \quad \Leftrightarrow \quad f\big|\mathrm{S}' \text{ is } \theta\text{-measurable.}$$

One knows that the set S' is θ-measurable (Ch. IV, §5, No. 5, Prop. 7), therefore so is $\mathrm{S}' \cap (\mathrm{T} - \mathrm{N})$. Note that

$$\mathrm{S} \cap (\mathrm{T} - \mathrm{N}) = \mathrm{S}' \cap (\mathrm{T} - \mathrm{N});$$

for, if $t \in \mathrm{T} - \mathrm{N}$ then $u(t) = u'(t)$, therefore $u(t) \neq 0 \Leftrightarrow u(t) \neq 0$, that is, $t \in \mathrm{S} \Leftrightarrow t \in \mathrm{S}'$. In other words, $\varphi_{\mathrm{S}}\varphi_{\complement \mathrm{N}} = \varphi_{\mathrm{S}'}\varphi_{\complement \mathrm{N}}$, whence $\varphi_{\mathrm{S}} = \varphi_{\mathrm{S}'}$ locally θ-almost everywhere, and so the θ-measurability of S' implies that of S.

Fix any point $c \in \mathrm{G}$ and let g (resp. g') be the extension by c of $f|\mathrm{S}$ (resp. $f|\mathrm{S}'$) to T. Observe that $g = g'$ locally θ-almost everywhere; indeed, on the set

$$\mathrm{T} - (\mathrm{S} \cup \mathrm{S}') = (\mathrm{T} - \mathrm{S}) \cap (\mathrm{T} - \mathrm{S}')$$

one has $g = g' = c$, whereas on the set

$$\begin{aligned}(\mathrm{S} \cup \mathrm{S}') \cap (\mathrm{T} - \mathrm{N}) &= [\mathrm{S} \cap (\mathrm{T} - \mathrm{N})] \cup [\mathrm{S}' \cap (\mathrm{T} - \mathrm{N})] \\ &= \mathrm{S} \cap (\mathrm{T} - \mathrm{N}) = \mathrm{S}' \cap (\mathrm{T} - \mathrm{N})\end{aligned}$$

one has $g = f = g'$, therefore g and g' can differ only on the locally negligible set $(\mathrm{S} \cup \mathrm{S}') \cap \mathrm{N}$. Finally, with respect to θ,

$$\begin{aligned} f|\mathrm{S} \text{ is measurable} \quad &\Leftrightarrow \quad g \text{ is measurable} \\ &\Leftrightarrow \quad g' \text{ is measurable} \\ &\Leftrightarrow \quad f|\mathrm{S}' \text{ is measurable}\end{aligned}$$

(the 1st and 3rd equivalences, by Ch. IV, §5, No. 10, Prop. 15, more precisely, criterion d') in the Note for IV.79, ℓ. 7; the 2nd, by Ch. IV, §5, No. 2, Prop. 6), that is, $(*)$ holds.

V.43, ℓ. -3.

"For, $u\mathbf{f}$ is the extension by 0 of $(u\mathbf{f})|\mathrm{S}$ to T."

By assumption, $u : \mathrm{T} \to \mathbf{C}$ and $\mathbf{f} : \mathrm{T} \to \mathrm{G}$, where $\mathrm{G} = \mathrm{F}$ (a Banach space) or $\mathrm{G} = \overline{\mathbf{R}}$. What does $u\mathbf{f}$ mean? The interpretation $u\mathbf{f} : t \mapsto u(t)\mathbf{f}(t)$ $(t \in \mathrm{T})$ has consequences:

When $\mathrm{G} = \mathrm{F}$ is a real Banach space, then $u : \mathrm{T} \to \mathbf{R}$.

When $\mathrm{G} = \mathrm{F}$ is a Banach space over $\mathbf{C}$, then $u : \mathrm{T} \to \mathbf{C}$.

When $\mathrm{G} = \overline{\mathbf{R}}$, let us assume that $u : \mathrm{T} \to \mathbf{R}$; for, if $\mathbf{f}(t) = \pm\infty$, permitting $u(t)$ to be a non-real complex number opens the door to too many 'infinities'.

In any case, if $u(t) = 0$ then $u(t)\mathbf{f}(t) = 0$ (even if $\mathrm{G} = \overline{\mathbf{R}}$ and $\mathbf{f}(t) = \pm\infty$), consequently

$$(u\mathbf{f})(t) = \begin{cases} u(t)\mathbf{f}(t) & \text{for } t \in \mathrm{S} \\ 0 & \text{for } t \in \mathrm{T} - \mathrm{S}\,. \end{cases}$$

This means that $u\mathbf{f}$ is the extension by 0 of $(u\mathbf{f})\big|\mathrm{S}$ to T. Since S is θ-measurable, it follows that

$$\text{(i)} \qquad u\mathbf{f} \text{ is } \theta\text{-measurable} \quad \Leftrightarrow \quad u\mathbf{f}\big|\mathrm{S} \text{ is } \theta\text{-measurable}$$

(Ch. IV, §5, No. 10, Prop. 15, more precisely, criterion d') in the Note for IV.79, ℓ. 7). But $u\big|\mathrm{S}$ and $\dfrac{1}{u\big|\mathrm{S}}$ are θ-measurable functions on S (Ch. IV, §5, No. 3, Th. 1, as generalized in the Note for IV.80, ℓ. −17 to −14), so from $\mathbf{f}\big|\mathrm{S} = \dfrac{1}{u\big|\mathrm{S}} \cdot \Big(u\mathbf{f}\big|\mathrm{S}\Big)$ it follows that

$$\text{(ii)} \qquad u\mathbf{f}\big|\mathrm{S} \text{ is } \theta\text{-measurable} \quad \Leftrightarrow \quad \mathbf{f}\big|\mathrm{S} \text{ is } \theta\text{-measurable}$$

(Ch. IV, §5, No. 3, Cor. 5 of Th. 1, as generalized in the Note just cited). On the other hand, by Prop. 4,

$$\text{(iii)} \qquad \mathbf{f}\big|\mathrm{S} \text{ is } \theta\text{-measurable} \quad \Leftrightarrow \quad \mathbf{f} \text{ is } (u\cdot\theta)\text{-measurable}\,.$$

From (i)–(iii) we infer that $u\mathbf{f}$ is θ-measurable if and only if $\mathbf{f}$ is $(u\cdot\theta)$-measurable, as asserted.

Suppose, more generally, that the locally θ-integrable function u is defined only locally θ-almost everywhere in T, and let u' be the extension by 0 of u to T. By the case already considered, $\mathbf{f}$ is $(u'\cdot\theta)$-measurable if and only if $u'\mathbf{f}$ is θ-measurable; but $u\cdot\theta = u'\cdot\theta$, and $u\mathbf{f} = u'\mathbf{f}$ locally θ-almost everywhere, whence again $\mathbf{f}$ is $(u\cdot\theta)$-measurable if and only if $u\mathbf{f}$ is θ-measurable.

V.44, ℓ. 6, 7.

"The case that u and θ are positive follows at once from Th. 2 of §4, No. 4."

Since u is locally θ-integrable, there exists a function $u' \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{T},\theta;\mathbf{C})$ such that $u = u'$ locally θ-almost everywhere. Then $u\cdot\theta = u'\cdot\theta$ and $u\mathbf{f} = u'\mathbf{f}$ locally θ-almost everywhere; and when $\mathbf{f}$ is θ-integrable, $u\mathbf{f} = u'\mathbf{f}$ θ-almost everywhere. We can therefore suppose that $u(t)$ is defined and $\geqslant 0$ at every point $t \in \mathrm{T}$, and, in the third assertion, that $0 < u(t) < +\infty$ for all $t \in \mathrm{T}$. Then (I, u) is θ-adapted by the first paragraph of No. 2, hence all three assertions are immediate from the cited Th. 2, inasmuch as $\eta = u\cdot\theta = \int u(t)\varepsilon_t\, d\theta(t)$ and $\mathbf{f}\circ\mathrm{I} = \mathbf{f}$. {The need for u to be everywhere finite may be traced back to Prop. 2 of §4, No. 2.}

V.44, ℓ. 10.

"Finally, ..."

It remains to verify the second assertion, i.e., that when $\mathbf{f}$ is essentially

η-integrable, the equality (6) holds with $\int$ signifying essential integral (it will hold *a fortiori* when $\mathbf{f}$ is η-integrable and $u(t) \neq 0$ for all $t \in \mathrm{T}$, with $\int$ signifying integral).

V.44, ℓ. 11-13.

"... $\mathbf{f}$ is essentially integrable for each of the measures $\eta_{ij} = g_i \cdot \mu_j$ $(i = 1, 2, 3, 4,\ j = 1, 2, 3, 4)$, because these measures are $\leqslant |\eta|$."

Since u is everywhere finite, it is continuous at a point $t \in \mathrm{T}$ if and only if every g_i is continuous at t; it follows that the θ-measurability of u is equivalent to that of all the g_i (Ch. IV, §5, No. 1, Def. 1). Similarly, the θ-integrability of uf for every $f \in \mathscr{K}(\mathrm{T})$ is equivalent to that of the $g_i f$ for all i (*loc. cit.*, No. 6, Th. 5), which is in turn equivalent to that of the $g_i f$ with respect to μ_j for all i and j (because $0 \leqslant \mu_j^* \leqslant |\theta|^*$; see, e.g., the Note for V.10, ℓ. $13, 14$). Thus the (positive) measures $\eta_{ij} = g_i \cdot \mu_j$ exist for all i and j; moreover, $g_i \cdot \mu_j \leqslant |u| \cdot |\theta|$ (that is, $\eta_{ij} \leqslant |\eta|$) because, for every $f \in \mathscr{K}_+(\mathrm{T})$, one has

$$(g_i \cdot \mu_j)(f) = \int g_i f \, d\mu_j \leqslant \int |u| f \, d\mu_j \leqslant \int |u| f \, d|\theta| = (|u| \cdot |\theta|)(f)$$

by No. 2, Def. 2.

Since $\mathbf{f}$ is essentially integrable for η, and $0 \leqslant \eta_{ij} \leqslant |\eta|$, it follows that $\mathbf{f}$ is essentially integrable for every η_{ij}; for, η-negligible sets are η_{ij}-negligible and so measurability of $\mathbf{f}$ for η implies its measurability for the η_{ij}, and, since $\eta_{ij}^\bullet \leqslant |\eta|^\bullet$, the essential integrability of $\mathbf{f}$ for the η_{ij} follows from $\eta_{ij}^\bullet(|\mathbf{f}|) \leqslant |\eta|^\bullet(|\mathbf{f}|) < +\infty$ (§1, No. 3, Prop. 9 and Def. 3).

V.44, ℓ. 15.

"The formula (6) follows immediately from this."

Let us consider the case of a Banach space F (of which the case for $\overline{\mathbf{R}}$ is a simplification).

It will be convenient to write the decompositions (2) in the form

$$u = \sum_{i=1}^{4} a_i g_i \tag{2a}$$

$$\theta = \sum_{j=1}^{4} a_j \mu_j \tag{2b}$$

where $a_1 = 1$, $a_2 = -1$, $a_3 = i$, $a_4 = -i$ (forgive the double use of "i" as index and as complex number).

By the preceding Note, each g_i is locally integrable for $a_j\mu_j$ for all j; therefore, for every $f \in \mathscr{K}(\mathrm{T})$, fg_i is integrable for $a_j\mu_j$ for all j, and so fg_i is integrable for θ and one has

$$\int f g_i \, d\theta = \sum_{j=1}^{4} a_j \int f g_i \, d\mu_j$$

(by (2b) and the *Theorem* in the Note for V.10, ℓ. 13, 14). Thus each g_i is locally integrable for θ, and

$$g_i \cdot \theta = \sum_{j=1}^{4} a_j (g_i \cdot \mu_j) \,.$$

Similarly, since the $a_i g_i$ are locally integrable for θ, it follows from (2a) and the above-cited *Theorem* that

$$u \cdot \theta = \sum_{i=1}^{4} a_i (g_i \cdot \theta) = \sum_{i,j=1}^{4} a_i a_j (g_i \cdot \mu_j) \,,$$

that is, $\eta = \sum_{i,j=1}^{4} a_i a_j \eta_{ij}$. Since $\mathbf{f}$ is essentially integrable for η and the η_{ij}, it follows from *Corollary* 3 of the above-cited *Theorem* that

$$(*) \qquad \int \mathbf{f} \, d\eta = \sum_{i,j=1}^{4} a_i a_j \int \mathbf{f} \, d\eta_{ij} \,.$$

But, by the 'positive case' already treated, one has the 16 equalities

$$(**) \qquad \int \mathbf{f} \, d\eta_{ij} = \int g_i \mathbf{f} \, d\mu_j \qquad (1 \leqslant i, j \leqslant 4).$$

Finally,

$$\begin{aligned}
\int (u\mathbf{f}) d\theta &= \int \big(\sum_i a_i g_i\big) \mathbf{f} \, d\theta = \sum_i \int a_i g_i \mathbf{f} \, d\theta \\
&= \sum_i a_i \int g_i \mathbf{f} \, d\theta = \sum_i a_i \big(\sum_j a_j \int g_i \mathbf{f} \, d\mu_j \big) \\
&= \sum_{i,j} a_i a_j \int g_i \mathbf{f} \, d\mu_j = \sum_{i,j} a_i a_j \int \mathbf{f} \, d\eta_{ij} \\
&= \int \mathbf{f} \, d\eta
\end{aligned}$$

(the 2nd equality, by the additivity of the essential integral; the 4th, by the above-cited *Corollary* 3; the 6th, by (∗∗); and the last, by (∗).

The case that $\mathrm{F} = \overline{\mathbf{R}}$. Given $\mathbf{f} : \mathrm{T} \to \overline{\mathbf{R}}$ essentially integrable for $\eta = u \cdot \theta$, we know by the first assertion of Th. 1 that $u\mathbf{f}$ is essentially integrable for θ; we are to show that $\int \mathbf{f}\, d\eta = \int (u\mathbf{f})\, d\theta$. By assumption, $u : \mathrm{T} \to \mathbf{R}$ is locally integrable for θ (see the convention in the Note for V.43, ℓ. -3).

By the definition of "$\mathbf{f}$ essentially integrable for η" (V.9, ℓ. 7–9) there exists a function $\mathbf{g} \in \overline{\mathscr{L}}^1_{\mathbf{R}}(\mathrm{T}, \eta)$ such that $\mathbf{f} = \mathbf{g}$ locally almost everywhere for η, and one sets $\int \mathbf{f}\, d\eta = \int \mathbf{g}\, d\eta$.

By the case $\mathrm{F} = \mathbf{R}$ (a simplification of the Banach space case), we know that $u\mathbf{g}$ is essentially integrable for θ and $\int \mathbf{g}\, d\eta = \int u\mathbf{g}\, d\theta$. Thus we need only show that $\int u\mathbf{g}\, d\theta = \int u\mathbf{f}\, d\theta$, and for this it suffices to show that $u\mathbf{f} = u\mathbf{g}$ locally almost everywhere for θ.

Let $\mathrm{N} = \{t \in \mathrm{T} :\ \mathbf{f}(t) \neq \mathbf{g}(t)\}$; we know that $|\eta|^\bullet(\mathrm{N}) = 0$, that is, $\int^\bullet \varphi_\mathrm{N}\, d|\eta| = 0$. Citing (5) of Prop. 3, one has

$$0 = \int \varphi_\mathrm{N}\, d|\eta| = \int^\bullet (\varphi_\mathrm{N}|u|)\, d|\theta|\,,$$

thus $\varphi_\mathrm{N}|u| = 0$ locally almost everywhere for $|\theta|$. Let

$$\begin{aligned} \mathrm{P} &= \{t \in \mathrm{T} :\ (\varphi_\mathrm{N}|u|)(t) \neq 0\} \\ &= \{t \in \mathrm{T} :\ t \in \mathrm{N} \text{ and } u(t) \neq 0\} \\ &= \{t \in \mathrm{T} :\ u(t) \neq 0 \text{ and } \mathbf{f}(t) \neq \mathbf{g}(t)\}\,; \end{aligned}$$

we know that P is locally negligible for $|\theta|$. If $u(t)\mathbf{f}(t) \neq u(t)\mathbf{g}(t)$ then $u(t) \neq 0$ (recall the convention $0 \cdot \pm\infty = 0$) and $\mathbf{f}(t) \neq \mathbf{g}(t)$, therefore $t \in \mathrm{P}$; thus

$$\{t \in \mathrm{T} :\ (u\mathbf{f})(t) \neq (u\mathbf{g})(t)\} \subset \mathrm{P}\,,$$

and since P is locally negligible for $|\theta|$ we conclude that $u\mathbf{f} = u\mathbf{g}$ locally almost everywhere for θ.

V.44, ℓ. 16, 17.

"Corollary. — *For the measure* $u \cdot \theta$ *to be bounded, it is necessary and sufficient that* u *be essentially* θ*-integrable.*"

Taking $\mathbf{f}$ to be the constant function $\mathbf{1}$ (continuous, hence universally measurable) in Th. 1, we know that $u = u\mathbf{1}$ is essentially integrable for θ if and only if $\mathbf{1}$ is essentially integrable for $u \cdot \theta$; but

$$\begin{aligned} \mathbf{1} \text{ is essentially integrable for } u \cdot \theta \quad &\Leftrightarrow \quad |u \cdot \theta|^\bullet(\mathbf{1}) < +\infty \\ &\Leftrightarrow \quad |u \cdot \theta|^*(\mathbf{1}) < +\infty \\ &\Leftrightarrow \quad u \cdot \theta \text{ is bounded} \end{aligned}$$

(the first equivalence, by §1, No. 3, Prop. 9, the second by §1, No. 1, Prop. 4, and the third by Ch. IV, §4, No. 7, Prop. 12), whence the assertion.

V.44, ℓ. 18, 19.

"...for φ_A to be locally μ-integrable, it is necessary and sufficient that A be *μ-measurable*."

As announced at the beginning of the chapter, μ denotes any positive measure on the locally compact space T.

Necessity. If φ_A is locally μ-integrable then it is μ-measurable (No. 1, Prop. 1), equivalently, A is μ-measurable (Ch. IV, §5, No. 1, Def. 2).

Sufficiency. If A is μ-measurable then, for every $f \in \mathscr{K}(T)$, $f\varphi_A$ is μ-integrable (Ch. IV, §5, No. 6, Cor. 3 of Th. 5), thus φ_A is locally μ-integrable.

In particular, if θ is a complex measure on T, and $\mu = |\theta|$, then θ-measurability and μ-measurability are equivalent (Ch. IV, §5, No. 1, Def. 1), as are local θ-integrability and local μ-integrability (No. 1, Def. 1); thus the assertion is true with μ replaced by any complex measure θ.

V.44, ℓ. −10 to −8.

"For a mapping g of T into a topological space G to be ν-measurable, it is necessary and sufficient that the restriction of g to A be μ-measurable."

The assertion holds with μ replaced by any complex measure θ, and ν by $\eta = \varphi_A \cdot \theta$. For, writing $u = \varphi_A$ in Prop. 4, one has S = A, thus a mapping $g : T \to G$ is measurable for $\varphi_A \cdot \theta$ if and only if $g|A$ is measurable for θ. Writing $\mu = |\theta|$, this means that g is measurable for $|\varphi_A \cdot \theta| = \varphi_A \cdot |\theta|$ (No. 2, Prop. 2) if and only if $g|A$ is measurable for $|\theta|$; that is, g is measurable for $\nu = \varphi_A \cdot \mu$ if and only if $g|A$ is measurable for μ.

V.44, ℓ. −3 to **V.45,** ℓ. 1.

"...if two mappings of T into G (resp. F, $\overline{\mathbf{R}}$) coincide on A, then for one of them to be ν-measurable (resp. essentially ν-integrable), it is necessary and sufficient that the other be so."

As in the preceding Note, let θ be any complex measure on T, and let $\mu = |\theta|$, $\eta = \varphi_A \cdot \theta$, $\nu = |\eta| = \varphi_A \cdot \mu$.

Suppose, more generally, that $\mathbf{f}, \mathbf{g} : T \to G$ (resp. F, $\overline{\mathbf{R}}$) are mappings such that $\mathbf{f}(t) = \mathbf{g}(t)$ locally η-almost everywhere in A, i.e., that there exists a locally η-negligible set $N \subset A$ such that $\mathbf{f} = \mathbf{g}$ on $A - N$. Note that the set $T - A$ is locally η-negligible; for,

$$|\eta|^\bullet(T - A) = \int^\bullet \varphi_{T-A}\, d|\eta| = \int^\bullet \varphi_{T-A}\varphi_A\, d|\theta| = 0\,.$$

Then $N \cup (T - A)$ is locally η-negligible, and $\mathbf{f} = \mathbf{g}$ on its complement $A - N$; thus $\mathbf{f} = \mathbf{g}$ locally η-almost everywhere in T.

It follows that if $\mathbf{f}$ is measurable for η, then so is $\mathbf{g}$; and (when $\mathrm{G} = \mathrm{F}$ or $\overline{\mathbf{R}}$) if $\mathbf{f}$ is essentially integrable for η then so is $\mathbf{g}$, and $\int \mathbf{g}\, d\eta = \int \mathbf{f}\, d\eta$. Thus the assertion holds when μ is replaced by θ, and ν by $\eta = \varphi_{\mathrm{A}} \cdot \theta$.

V.45, ℓ. 5, 6.

"... if some extension $\overline{\mathbf{f}}$ to T of the restriction of $\mathbf{f}$ to A is essentially ν-integrable"

Then every extension of $\mathbf{f}|\mathrm{A}$ to T is essentially ν-integrable, and any two extensions are equal locally ν-almost everywhere (see the preceding Note).

V.45, ℓ. 7, 8.

"... one then sets

$$\int_{\mathrm{A}} \mathbf{f}\, d\mu = \int_{\mathrm{A}} \overline{\mathbf{f}}\, d\mu = \int \overline{\mathbf{f}} \varphi_{\mathrm{A}}\, d\mu\text{"}$$

The first two members are being defined to be equal to the third, which is equal to $\int \overline{\mathbf{f}}\, d\nu$.

V.45, ℓ. 10, 11.

"... one defines similarly $\int_{\mathrm{A}}^{*} f\, d\mu$ and $\int_{\mathrm{A}}^{\bullet} f\, d\mu$."

Let $\overline{f}$ be any function $\geqslant 0$ on T such that $\overline{f}|\mathrm{A} = f|\mathrm{A}$, and define

$$\int_{\mathrm{A}}^{\bullet} f\, d\mu = \mu^{\bullet}(\overline{f}\varphi_{\mathrm{A}})\,;$$

if also f' is a function $\geqslant 0$ on T such that $f'|\mathrm{A} = f|\mathrm{A}$, we know that $f' = \overline{f}$ locally almost everywhere for $\nu = \varphi_{\mathrm{A}} \cdot \mu$ (see the Note for V.44, ℓ. -3 to V.45, ℓ. 1), therefore $\int^{\bullet} f'\, d\nu = \int^{\bullet} \overline{f}\, d\nu$, whence $\int^{\bullet} f'\varphi_{\mathrm{A}}\, d\mu = \int^{\bullet} \overline{f}\varphi_{\mathrm{A}}\, d\mu$ by the preceding argument in the *Example*. Thus the symbol $\int_{\mathrm{A}}^{\bullet} f\, d\mu$ is well-defined, i.e., is independent of the choice of $\overline{f}$.

The situation for $\int_{\mathrm{A}}^{*} f\, d\mu$ is more delicate. One would like to define it to be $\mu^{*}(\overline{f}\varphi_{\mathrm{A}})$. Assuming that $\overline{f}$ and f' are any two functions $\geqslant 0$ on T that coincide with f on A, one knows that $\overline{f}\varphi_{\mathrm{A}} = f'\varphi_{\mathrm{A}}$ locally μ-almost everywhere; for $\int_{\mathrm{A}}^{*} f\, d\mu$ to be well-defined, one must arrange that $\overline{f}\varphi_{\mathrm{A}} = f'\varphi_{\mathrm{A}}$ μ-almost everywhere (Ch. IV, §2, No. 3, Prop. 6). This will be the case, for example, if A is μ-moderated (§1, No. 3, *Lemma*). CAUTION: One knows that $\mu^{\bullet}(\overline{f}\varphi_{\mathrm{A}}) = \nu^{\bullet}(\overline{f})$ by Prop. 3, but the corresponding equality for outer measure is not justified. {Problem: If A is μ-moderated, is the measure $\varphi_{\mathrm{A}} \cdot \mu$ moderated? (answered in the Note for V.48, ℓ. 2, 3)}

Remarks. (i) If A is a μ-measurable set such that $\mu^{\bullet}(\mathrm{A}) < +\infty$ — in other words if the function φ_{A} is essentially μ-integrable (§1, No. 3, Prop. 9)—then the measure $\nu = \varphi_{\mathrm{A}} \cdot \mu$ is bounded (Cor. of Th. 1), hence moderated (§1, No. 2, *Remark* 2), and so $\nu^* = \nu^\bullet$ (*loc. cit.*, Cor. 2 of Prop. 7). Then, if f is a function $\geqslant 0$ defined on a set $\mathrm{B} \supset \mathrm{A}$, and if $\overline{f}$ is any function $\geqslant 0$ on T such that $\overline{f}\big|\mathrm{A} = f\big|\mathrm{A}$, by the *Example* one has the equality

$$\nu^*(\overline{f}) = \nu^\bullet(\overline{f}) = \mu^\bullet(\overline{f}\varphi_{\mathrm{A}}),$$

the last member of which is known to be independent of the particular extension $\overline{f}$ of $f\big|\mathrm{A}$.

(ii) If the set A is *μ-integrable*, that is, if A is μ-measurable and $\mu^*(\mathrm{A}) < +\infty$, then A is μ-moderated; as we saw above, the symbol

$$(*) \qquad \int_{\mathrm{A}}^* f\, d\mu = \mu^*(\overline{f}\varphi_{\mathrm{A}}) = \int^* \overline{f}\varphi_{\mathrm{A}}\, d\mu$$

is well-defined (f a function $\geqslant 0$ defined on $\mathrm{B} \supset \mathrm{A}$, and $\overline{f}$ any function $\geqslant 0$ on T such that $\overline{f}\big|\mathrm{A} = f\big|\mathrm{A}$). But $\overline{f}\varphi_{\mathrm{A}}$ is also μ-moderated, therefore (§1, No. 2, Prop. 7)

$$(**) \qquad \mu^*(\overline{f}\varphi_{\mathrm{A}}) = \mu^\bullet(\overline{f}\varphi_{\mathrm{A}}).$$

Moreover, $\mu^\bullet(\mathrm{A}) = \mu^*(\mathrm{A}) < +\infty$, thus A is essentially μ-integrable and

$$(***) \qquad \nu^*(\overline{f}) = \mu^\bullet(\overline{f}\varphi_{\mathrm{A}})$$

by (i). From $(**)$ and $(***)$ we have $\mu^*(\overline{f}\varphi_{\mathrm{A}}) = \nu^*(\overline{f})$ and $(*)$ may then be written

$$\int_{\mathrm{A}}^* f\, d\mu = \int^* \overline{f}\varphi_{\mathrm{A}}\, d\mu = \int^* \overline{f}\, d\nu\,.$$

V.45, ℓ. 13, 14.

"... this is equivalent to saying that, for every compact subset K of T, $\overline{g}\varphi_{\mathrm{K}\cap\mathrm{A}}$ is μ-integrable."

The following conditions are equivalent:

a) $\overline{g}$ is locally ν-integrable;
b) $\overline{g}\varphi_{\mathrm{K}}$ is ν-integrable for every compact $\mathrm{K} \subset \mathrm{T}$;
c) $\overline{g}\varphi_{\mathrm{K}}$ is essentially ν-integrable for every compact $\mathrm{K} \subset \mathrm{T}$;
d) $\overline{g}\varphi_{\mathrm{K}}\varphi_{\mathrm{A}}$ is essentially μ-integrable for every compact $\mathrm{K} \subset \mathrm{T}$;
e) $\overline{g}\varphi_{\mathrm{K}\cap\mathrm{A}}$ is μ-integrable for every compact $\mathrm{K} \subset \mathrm{T}$;

for, *a*) $\Leftrightarrow$ *b*) by No. 1, Def. 1; *b*) $\Leftrightarrow$ *c*) and *d*) $\Leftrightarrow$ *e*) because compact sets are (universally) moderated; and *c*) $\Leftrightarrow$ *d*) by the *Example*.

V.45, ℓ. 17.

"... *admitting in* $\mathscr{M}(\mathrm{T})$ *a supremum* λ."

Review: The criterion for this is that, for every $f \in \mathscr{K}_+(\mathrm{T})$, the (increasing directed) family of real numbers $\lambda_\alpha(f) \geqslant 0$ admit a finite upper bound (Ch. II, §2, No. 2, *Lemma*). That the correspondence $f \mapsto \sup_{\alpha\in\mathrm{A}} \lambda_\alpha(f)$ defines an additive function on $\mathscr{K}_+(\mathrm{T})$ follows from the theorem on monotone limits (GT, Ch. IV, §5, No. 2, Th. 2) and the continuity of addition in $\mathbf{R}$; that it is extendible to a linear form λ on $\mathscr{K}(\mathrm{T})$ then results from the fact that every $f \in \mathscr{K}(\mathrm{T})$ is the difference of two functions in $\mathscr{K}_+(\mathrm{T})$ (Ch. II, §2, No. 1, Prop. 2), and, being a positive linear form on $\mathscr{K}(\mathrm{T})$, λ is a measure on T (Ch. III, §1, No. 5, Th. 1).

This construction—as a result of which, the space $\mathscr{M}(\mathrm{T};\mathbf{R})$ of real measures on T is a fully lattice-ordered Riesz space (*loc. cit.*, Th. 3)—is fundamental for the concept of essential integral (cf. §1, No. 4, Prop. 11).

V.45, ℓ. −10.

"It is clear that the condition is necessary."

Suppose g is locally integrable for λ. Given any $h \in \mathscr{K}_+(\mathrm{T})$, one knows that gh is λ-integrable; since λ is the sum of the positive measures λ_α and $\lambda - \lambda_\alpha$, it follows that gh is integrable for every λ_α and $\lambda_\alpha(gh) \leqslant \lambda(gh) < +\infty$ (see, e.g., the Note for V.10, ℓ. 13, 14, the remark preceding the *Theorem*), consequently g is locally integrable for λ_α and $g\cdot\lambda_\alpha \leqslant g\cdot\lambda$, whence the assertion.

Thus, by the preceding Note, the measure $\sup_{\alpha\in\mathrm{A}} g\cdot\lambda_\alpha$ exists, and it is equal to $g\cdot\lambda$ by the following computation: for every $h \in \mathscr{K}_+(\mathrm{T})$,

$$\big(\sup_{\alpha\in\mathrm{A}} g\cdot\lambda_\alpha\big)(h) = \sup_{\alpha\in\mathrm{A}}(g\cdot\lambda_\alpha)(h) = \sup_{\alpha\in\mathrm{A}} \lambda_\alpha(gh) = \lambda(gh) = (g\cdot\lambda)(h)\,;$$

the first equality, by Ch. II, §2, No. 2, *Lemma*; the third, because gh is (universally) moderated and

$$\lambda(gh) = \lambda^\bullet(gh) = \sup_{\alpha\in\mathrm{A}} \lambda^\bullet_\alpha(gh) = \sup_{\alpha\in\mathrm{A}} \lambda_\alpha(gh)$$

by §1, No. 4, Prop. 11. {A striking application of the essential integral!}

V.46, ℓ. 1–5.

COROLLARY.

Let $\mathfrak{F}$ be the set of all finite subsets J of A, and for each $\mathrm{J} \in \mathfrak{F}$ write $\mu_\mathrm{J} = \sum_{\alpha\in\mathrm{J}} \mu_\alpha$; then $(\mu_\mathrm{J})_{\mathrm{J}\in\mathfrak{F}}$ is an increasing directed family of positive measures such that $\mu = \sup_{\mathrm{J}\in\mathfrak{F}} \mu_{\mathrm{J}}$ (§2, No. 1, *Remark* 1). If $\mathrm{J} \in \mathfrak{F}$ and $h \in \mathscr{K}_+(\mathrm{T})$ one knows (§2, No. 2, Cor. 1 of Prop. 3) that

$$gh \text{ is } \mu_{\mathrm{J}}\text{-integrable} \quad \Leftrightarrow \quad gh \text{ is } \mu_\alpha\text{-integrable for every } \alpha \in \mathrm{J}\,,$$

in which case $\mu_{\mathrm{J}}(gh) = \sum_{\alpha\in\mathrm{J}} \mu_\alpha(gh)$. It follows that g is locally μ_{J}-integrable if and only if it is locally μ_α-integrable for every $\alpha \in \mathrm{J}$, in which case $g \cdot \mu_{\mathrm{J}} = \sum_{\alpha\in\mathrm{J}} g \cdot \mu_\alpha$.

Necessity. Suppose g is locally μ-integrable. Since $\mu_{\mathrm{J}} \leqslant \mu$ for every $\mathrm{J} \in \mathfrak{F}$, it follows that g is locally integrable for the μ_{J} and that $g\cdot\mu_{\mathrm{J}} \leqslant g\cdot\mu$ (see the preceding Note), consequently $g \cdot \mu = \sup_{\mathrm{J}\in\mathfrak{F}} g \cdot \mu_{\mathrm{J}}$ by Prop. 5. By the preceding remarks, g is locally μ_α-integrable for every $\alpha \in \mathrm{A}$, and

$$g \cdot \mu = \sup_{\mathrm{J}\in\mathfrak{F}} \sum_{\alpha\in\mathrm{J}} g \cdot \mu_\alpha\,;$$

that is, for every $h \in \mathscr{K}_+(\mathrm{T})$,

$$(g \cdot \mu)(h) = \sup_{\mathrm{J}\in\mathfrak{F}} \Bigl(\sum_{\alpha\in\mathrm{J}} g \cdot \mu_\alpha\Bigr)(h) = \sup_{\mathrm{J}\in\mathfrak{F}} \sum_{\alpha\in\mathrm{J}} (g \cdot \mu_\alpha)(h) = \sum_{\alpha\in\mathrm{A}} (g \cdot \mu_\alpha)(h)\,,$$

therefore the family $(g \cdot \mu_\alpha)_{\alpha\in\mathrm{A}}$ is summable and $\sum_{\alpha\in\mathrm{A}} g \cdot \mu_\alpha = g \cdot \mu$ (§2, No. 1).

Sufficiency. Suppose g is locally μ_α-integrable for every $\alpha \in \mathrm{A}$ and the family $(g \cdot \mu_\alpha)$ is summable, and let $\rho = \sum_{\alpha\in\mathrm{A}} g \cdot \mu_\alpha$; then, for every $\mathrm{J} \in \mathfrak{F}$, g is locally integrable for μ_{J} and $g \cdot \mu_{\mathrm{J}} = \sum_{\alpha\in\mathrm{J}} g \cdot \mu_\alpha$. Moreover, the family $(g \cdot \mu_{\mathrm{J}})_{\mathrm{J}\in\mathfrak{F}}$ is increasing and

$$\rho = \sup_{\mathrm{J}\in\mathfrak{F}} g \cdot \mu_{\mathrm{J}} = \sup_{\mathrm{J}\in\mathfrak{F}} \sum_{\alpha\in\mathrm{J}} g \cdot \mu_\alpha$$

(*loc. cit.*, *Remark* 1). It then follows from Prop. 5 that g is locally μ-integrable and that $g \cdot \mu = \sup_{\mathrm{J}\in\mathfrak{F}} g \cdot \mu_{\mathrm{J}} = \rho = \sum_{\alpha\in\mathrm{A}} g \cdot \mu_\alpha$.

V.46, ℓ. 8, 9.

"... if the family (S_α) is locally countable (Ch. IV, §5, No. 9)"

The concept of local countability (*loc. cit.*, Def. 7) has been defined only for a *set* of subsets (not a *family* of subsets) of a topological space T.

Guided by the definition of a 'locally finite family' (GT, I, §1, No. 5, Def. 8), the appropriate definition is as follows: call a family $(\mathrm{S}_\alpha)_{\alpha\in\mathrm{A}}$ of subsets of T *locally countable* if, for every $t \in \mathrm{T}$, there exists a neighborhood V of t such that the set of indices $\{\alpha \in \mathrm{A} :\ \mathrm{V} \cap \mathrm{S}_\alpha \neq \varnothing\}$ is countable.

It is obvious that if the family $(\mathrm{S}_\alpha)_{\alpha\in\mathrm{A}}$ is locally countable in the foregoing sense, then the set $\mathfrak{A} = \{\mathrm{S}_\alpha :\ \alpha \in \mathrm{A}\}$ is locally countable in the sense of the cited Def. 7 (Ch. IV, § 5, No. 9). However, the converse is false. {For example, if the index set A is uncountable and $\mathrm{S}_\alpha = \mathrm{T}$ for all $\alpha \in \mathrm{A}$, then the set $\mathfrak{A} = \{\mathrm{S}_\alpha :\ \alpha \in \mathrm{A}\} = \{\mathrm{T}\}$ is trivially locally countable, but the family $(\mathrm{S}_\alpha)_{\alpha\in\mathrm{A}}$ is not.}

V.46, ℓ. 9, 10.

"...this amounts to saying that, for every compact set K in T, the set of $\alpha \in \mathrm{A}$ such that $g_\alpha\big|\mathrm{K}$ is not zero is countable."

Consider the conditions:

a) The family of functions $(g_\alpha)_{\alpha\in\mathrm{A}}$ is locally countable, that is, the family $(\mathrm{S}_\alpha)_{\alpha\in\mathrm{A}}$ is locally countable;

b) for every compact set K in T, the set of $\alpha \in \mathrm{A}$ such that $\mathrm{K}\cap\mathrm{S}_\alpha \neq \varnothing$ is countable;

c) for every compact set K in T, the set of $\alpha \in \mathrm{A}$ such that $g\big|\mathrm{K}_\alpha \neq 0$ is countable.

Since $\mathrm{K} \cap \mathrm{S}_\alpha \neq \varnothing \Leftrightarrow g_\alpha\big|\mathrm{K} \neq 0$, the equivalence *b*) ⇔ *c*) is trivial.

a) ⇒ *b*): The finite covering argument for locally countable sets $\mathfrak{A}$, following Def. 7 of Ch. IV, §5, No. 9, is readily adapted to the case of families.

b) ⇒ *a*): When T is locally compact, every point $t \in \mathrm{T}$ has a compact neighborhood K.

Thus the assertion "amounts to saying" entails the local compactness of T.

V.46, ℓ. 16, 17.

"It is clear that g is μ-measurable (Ch. IV, §5, No. 2, Prop. 4 and No. 4, Cor. 1 of Th. 2)."

Lemma. If f is a function on the locally compact space T such that $f\varphi_\mathrm{K}$ is μ-measurable for every compact set $\mathrm{K} \subset \mathrm{T}$, then f is μ-measurable.

Proof. For every $t \in \mathrm{T}$ let V_t be a compact (hence μ-integrable) neighborhood of T. Since $f\varphi_{\mathrm{V}_t}$ is by assumption μ-measurable, and $f = f\varphi_{\mathrm{V}_t}$ on V_t, f is μ-measurable by the Principle of localization (Ch. IV, §5, No. 2, Prop. 4). ◇

Consider now the given function g. Given any compact set K in T, it will suffice to show that $g\varphi_\mathrm{K}$ is μ-measurable. Let

$$\mathrm{A}_\mathrm{K} = \{\alpha \in \mathrm{A} :\ g_\mathrm{K}\big|\mathrm{K} \neq 0\}$$

(by assumption, a countable set). Since $g_\alpha \varphi_{\mathrm{K}} = 0$ when $\alpha \notin \mathrm{A}_{\mathrm{K}}$, one has

$$g\varphi_{\mathrm{K}} = \sum_{\alpha \in \mathrm{A}_{\mathrm{K}}} g_\alpha \varphi_{\mathrm{K}} \,;$$

for, if $t \notin \mathrm{K}$ then both sides are equal to 0 at t, whereas if $t \in \mathrm{K}$ then

$$g(t)\varphi_{\mathrm{K}}(t) = g(t) = \sum_{\alpha \in \mathrm{A}} g_\alpha(t) = \sum_{\alpha \in \mathrm{A}_{\mathrm{K}}} g_\alpha(t) = \sum_{\alpha \in \mathrm{A}_{\mathrm{K}}} g_\alpha(t)\varphi_{\mathrm{K}}(t)\,.$$

Since the g_α are locally integrable, the $g_\alpha \varphi_{\mathrm{K}}$ are measurable by criterion b) of No. 1, Prop. 1, thus $g\varphi_{\mathrm{K}}$ is the sum of a countable family of measurable functions; as the finite subsums are measurable (Ch. IV, §5, No. 3, Cor. 3 of Th. 1) and $g\varphi_{\mathrm{K}}$ is the supremum of a sequence of such subsums, it follows that $g\varphi_{\mathrm{K}}$ is indeed measurable (*loc. cit.*, No. 4, Cor. 1 of Th. 2).

V.46, ℓ. 17, 18.

"For g to be locally μ-integrable, it is therefore necessary and sufficient that $\mu^\bullet(gf)$ be finite for every $f \in \mathscr{K}_+(\mathrm{T})$."

Since gf is (universally) moderated, this will follows from criterion c) of No. 1, Prop. 1 and §1, No. 3, Cor. of Prop. 9 provided gf is shown to be μ-measurable. We know that the $g_\alpha f$ are μ-measurable (even μ-integrable), but the measurability of gf is not obvious, as application of Th. 1 of Ch. IV, §5, No. 3 is thwarted by the fact that multiplication in $\overline{\mathbf{R}}_+$ is not continuous: $\frac{1}{n} \cdot (+\infty) \not\to 0 \cdot (+\infty)$. The following lemmas settle the matter:

Lemma 1. If $c \in \overline{\mathbf{R}}_+$ and $a_k \in \overline{\mathbf{R}}_+$ $(k = 1, 2, 3, \ldots)$ then $c \sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} ca_k$.

Proof. If $c = 0$ or $c = +\infty$ the equality is obvious. Suppose that $0 < c < +\infty$. Write

$$s_n = \sum_{k=1}^{n} a_k\,, \quad t_n = \sum_{k=1}^{n} ca_k\,, \quad s = \sum_{k=1}^{\infty} a_k = \sup_n s_n\,, \quad t = \sum_{k=1}^{\infty} ca_k = \sup_n t_n\,;$$

we are to show that $cs = t$.

If $a_k = +\infty$ for some k, the assertion reduces to $+\infty = +\infty$. Assume that $a_k < +\infty$ for all k; then $cs_n = t_n$ for all n. If the sequence (s_n) is bounded then so is (t_n) and the assertion reduces to $c \lim s_n = \lim cs_n$; whereas if (s_n) is unbounded then so is (t_n) and the assertion reduces to $+\infty = +\infty$. $\diamondsuit$

Lemma 2. gf is μ-measurable for every $f \in \mathscr{K}_+(\mathrm{T})$.

Proof. Let K be a compact set such that $f = 0$ on $\complement\mathrm{K}$, and let

$$\mathrm{A_K} = \{\alpha \in \mathrm{A} : \; g_\alpha \big| \mathrm{K} \neq 0\}$$

(a countable set). As observed in the preceding Note, $g\varphi_{\mathrm{K}} = \sum_{\alpha \in \mathrm{A_K}} g_\alpha \varphi_{\mathrm{K}}$, therefore

$$(g\varphi_{\mathrm{K}})f = \sum_{\alpha \in \mathrm{A_K}} g_\alpha \varphi_{\mathrm{K}} f$$

by Lemma 1, and since $f\varphi_{\mathrm{K}} = f$ one has

$$(*) \qquad gf = g(f\varphi_{\mathrm{K}}) = (g\varphi_{\mathrm{K}})f = \sum_{\alpha \in \mathrm{A_K}} g_\alpha \varphi_{\mathrm{K}} f = \sum_{\alpha \in \mathrm{A_K}} g_\alpha f\,;$$

thus gf is the sum of a sequence of measurable functions $\geqslant 0$, hence is measurable by the argument of the preceding Note. ◇

V.46, ℓ. 18–20.

"... since the set of $\alpha \in \mathrm{A}$ such that $g_\alpha f \neq 0$ is countable, we have $\mu^\bullet(gf) = \sum_{\alpha \in \mathrm{A}} \mu^\bullet(g_\alpha f)$ (§1, No. 1, Cor. of Prop. 2)."

Applying $\mu^\bullet$ to the equality $(*)$ of the preceding Note, one has

$$(**) \qquad \mu^\bullet(gf) = \sum_{\alpha \in \mathrm{A_K}} \mu^\bullet(g_\alpha f)$$

by the cited Cor. of Prop. 2; as $g_\alpha f = 0$ for $\alpha \notin \mathrm{A_K}$, this may be written as $\mu^\bullet(gf) = \sum_{\alpha \in \mathrm{A}} \mu^\bullet(g_\alpha f)$.

Incidentally, since gf and the $g_\alpha f$ are measurable and moderated, the relation $(**)$ may also be deduced from §1, No. 2, Prop. 7 and Ch. IV, §5, No. 6, Cor. 4 of Th. 5.

V.46, ℓ. −4, −3.

"It suffices to apply Prop. 6 to the functions (positive locally almost everywhere) $g'_n = g_{n+1} - g_n$."

{Misprint: In the statement of the Corollary, for (g_α) read (g_n).}

Regarding the g_n as indexed by $n \in \mathbf{N}$, it is not assumed that $g_0 \cdot \mu \geqslant 0$, i.e., that $g_0 \geqslant 0$ locally μ-almost everywhere (No. 3, Cor. 3 of Prop. 3). {Positivity is relevant for sums (GT, IV, §7, No. 5, *Remark*) but not for suprema of increasing families (*loc. cit.*, § 4 No. 2, Prop. 4).}

To avoid dealing with (undefined) differences such as $(+\infty) - (+\infty)$, we can suppose, on redefining the g_n on a countable union of locally μ-negligible sets, that the g_n are everywhere finite-valued (see the *Proposition* in the Note for V.39, ℓ. 22–24); their supremum g (which may take on infinite values) is then altered at most on a locally μ-negligible set—and the measures $g_n \cdot \mu$ and $g \cdot \mu$, not at all (No. 3, Cor. 2 of Prop. 3). The relations $g_n \cdot \mu \leqslant g_{n+1} \cdot \mu$ then imply that

$$0 \leqslant g_{n+1} \cdot \mu - g_n \cdot \mu = (g_{n+1} - g_n) \cdot \mu$$

(see the Note for V.41, ℓ. 6, 7), whence $g_n \leqslant g_{n+1}$ locally μ-almost everywhere. Let $\mathrm{H} = \bigcup_{n \in \mathbf{N}} \{t \in \mathrm{T} : \ g_n(t) > g_{n+1}(t)\}$ (a locally μ-negligible set); redefining the g_n to be equal to 0 on H, we can suppose that, for every n, $g_n \leqslant g_{n+1}$ at every point of T. Thus, for every $t \in \mathrm{T}$, $g(t)$ is the supremum of the increasing sequence $g_n(t)$ of real numbers.

The functions $g'_n = g_{n+1} - g_n$ $(n \in \mathbf{N})$ are positive, locally μ-integrable, and

$$\sum_{k=0}^{n} g'_k = g_{n+1} - g_0 \quad (n \in \mathbf{N}) .$$

Let $\mathfrak{F}$ be the set of all finite sets $\mathrm{J} \subset \mathbf{N}$. For $\mathrm{J} \in \mathfrak{F}$ write

$$s_{\mathrm{J}} = \sum_{k \in \mathrm{J}} g'_k , \quad \nu_{\mathrm{J}} = \sum_{k \in \mathrm{J}} g'_k \cdot \mu ;$$

one knows that $s_{\mathrm{J}} \cdot \mu = \nu_{\mathrm{J}}$. In particular, writing $\mathrm{J}_n = \{k : \ 0 \leqslant k \leqslant n\}$ for $n \in \mathbf{N}$, one has

$$s_{\mathrm{J}_n} = g_{n+1} - g_0 , \quad \nu_{\mathrm{J}_n} = g_{n+1} \cdot \mu - g_0 \cdot \mu .$$

Since the sets J_n are cofinal in the set $\mathfrak{F}$ ordered by $\subset$, one has

$$\begin{aligned} \sum_{k \in \mathbf{N}} g'_k &= \sup_{\mathrm{J} \in \mathfrak{F}} s_{\mathrm{J}} = \sup_{n \in \mathbf{N}} s_{\mathrm{J}_n} = \sup_{n \in \mathbf{N}} (g_{n+1} - g_0) \\ &= -g_0 + \sup_{n \in \mathbf{N}} g_{n+1} = -g_0 + \sup_{n \in \mathbf{N}} g_n \\ &= -g_0 + g , \end{aligned}$$

thus $g = g_0 + \sum_{k \in \mathbf{N}} g'_k$.

Necessity. Assuming the sequence $(g_n \cdot \mu)$ has an upper bound in $\mathscr{M}(\mathrm{T})$, we are to show that g is locally μ-integrable. Since $\mathscr{M}(\mathrm{T})$ is fully lattice-ordered (Ch. III, §1, No. 5, Th. 3), the measure $\rho = \sup_{n \in \mathbf{N}} (g_n \cdot \mu)$ exists, as does

$$\sup_{n \in \mathbf{N}} (g_n \cdot \mu - g_0 \cdot \mu) = \rho - g_0 \cdot \mu .$$

Then

$$\begin{aligned}\rho - g_0 \cdot \mu &= \sup_{n\in\mathbf{N}} (g_n \cdot \mu) - g_0 \cdot \mu = \sup_{n\in\mathbf{N}} (g_{n+1} \cdot \mu) - g_0 \cdot \mu \\ &= \sup_{n\in\mathbf{N}} (g_{n+1} \cdot \mu - g_0 \cdot \mu) = \sup_{n\in\mathbf{N}} \nu_{\mathrm{J}_n} \\ &= \sup_{\mathrm{J}\in\mathfrak{F}} \nu_{\mathrm{J}} = \sup_{\mathrm{J}\in\mathfrak{F}} \Big(\sum_{k\in\mathrm{J}} g'_k \cdot \mu\Big);\end{aligned}$$

this means that the family $(g'_k \cdot \mu)_{k\in\mathbf{N}}$ is summable, with sum equal to $\rho - g_0 \cdot \mu$. It then follows from Prop. 6 that the function $\sum_{k\in\mathbf{N}} g'_k = g - g_0$ is locally μ-integrable, whence so is g, and

$$(g - g_0) \cdot \mu = \sum_{k\in\mathbf{N}} g'_k \cdot \mu = \rho - g_0 \cdot \mu\,;$$

thus $g \cdot \mu = \rho = \sup_{n\in\mathbf{N}} (g_n \cdot \mu)$.

Sufficiency. Conversely, suppose that g is locally μ-integrable. Then, from the local μ-integrability of $g - g_0 = \sum_{k\in\mathbf{N}} g'_k$, one infers (Prop. 6) that the family of measures $(g'_k \cdot \mu)_{k\in\mathbf{N}}$ is summable and that

$$\sum_{k\in\mathbf{N}} g'_k \cdot \mu = (g - g_0) \cdot \mu\,;$$

this means that

$$(g - g_0) \cdot \mu = \sup_{\mathrm{J}\in\mathfrak{F}} \nu_{\mathrm{J}} = \sup_{n\in\mathbf{N}} \nu_{\mathrm{J}_n} = \sup_{n\in\mathbf{N}} (g_{n+1} \cdot \mu - g_0 \cdot \mu)\,,$$

whence $g_{n+1} \cdot \mu - g_0 \cdot \mu \leqslant (g - g_0) \cdot \mu$, that is, $g_{n+1} \cdot \mu \leqslant g \cdot \mu$ for all n. Thus the sequence $(g_n \cdot \mu)_{n\in\mathbf{N}}$ does indeed have an upper bound in $\mathscr{M}(\mathrm{T})$.

V.47, ℓ. 5.

$$g \cdot \nu = \int (g \cdot \lambda_t)\, d\mu(t)\,. \tag{9}$$

When g is continuous the equality is elementary, X can be any locally compact space and $t \mapsto \lambda_t$ any scalarly essentially μ-integrable mapping of T into $\mathscr{M}_+(\mathrm{X})$. For, the measure $g \cdot \nu$ of No. 2, Def. 2 then coincides with the measure

$$h \mapsto \nu(gh) \quad (h \in \mathscr{K}(\mathrm{T}))$$

defined in Ch. III, §1, No. 4. On the other hand, for $h \in \mathscr{K}(\mathrm{T})$, the function

$$t \mapsto (g \cdot \lambda_t)(h) = \lambda_t(gh)$$

is essentially μ-integrable because $gh \in \mathscr{K}(\mathrm{T})$ and the mapping $t \mapsto \lambda_t$ is scalarly essentially μ-integrable; it follows that the mapping $t \mapsto g \cdot \lambda_t$ is also scalarly essentially μ-integrable, and, for $h \in \mathscr{K}(\mathrm{T})$,

$$\langle h, \int (g{\cdot}\lambda_t)\, d\mu(t) \rangle = \int (g{\cdot}\lambda_t)(h)\, d\mu(t) = \int \lambda_t(gh)\, d\mu(t) = \nu(gh) = (g{\cdot}\nu)(h)\,,$$

whence (9).

The formula is more memorable when written as a 'distributive law'

$$(9') \qquad g \cdot \int \lambda_t\, d\mu(t) = \int (g \cdot \lambda_t)\, d\mu(t)\,,$$

but (9) has the merit that it forces one to check the stringent hypotheses on X and the mapping $t \mapsto \lambda_t \in \mathscr{M}_+(\mathrm{X})$.

V.47, ℓ. 7, 8.

"...to say that g is locally η-integrable is equivalent to saying that $g\varphi_{\mathrm{K}_n}$ is η-integrable for every n."

If g is locally η-integrable, then $g\varphi_{\mathrm{K}}$ is η-integrable for every compact set K in T by condition *b*) of No. 1, Prop. 1.

Conversely, suppose $g\varphi_{\mathrm{K}_n}$ is η-integrable for every n. If K is any compact set in T then $\mathrm{K} \subset \mathrm{K}_n$ for some n (GT, I, §9, No. 9, Cor. 1 of Prop. 15), and the η-integrability of $g\varphi_{\mathrm{K}_n}$ implies that of $g\varphi_{\mathrm{K}} = (g\varphi_{\mathrm{K}_n})\varphi_{\mathrm{K}}$; in particular, $g\varphi_{\mathrm{K}}$ is η-measurable for every K, hence g is η-measurable (*Lemma* in the Note for V.46, ℓ. 16, 17), therefore g satisfies the cited condition *b*), hence is locally η-integrable.

V.47, ℓ. 10.

"...let $\mathrm{H} = \bigcup_n \mathrm{H}_n$ "

By the preceding Note, g is not λ_t-integrable if and only if $g\varphi_{\mathrm{K}_n}$ is not λ_t-integrable for some n, therefore

$$\{t \in \mathrm{T} :\ g \text{ is not } \lambda_t\text{-integrable}\} = \bigcup_n \mathrm{H}_n = \mathrm{H}\,.$$

V.47, ℓ. 10, 11.

"...since H_n is locally μ-negligible for all n (§3, No. 3, Th. 1), the same is true of H, which establishes the first assertion of the statement."

For each n the function $g\varphi_{\mathrm{K}_n}$ is ν-integrable, therefore, by Part a) of the cited Th. 1, the set H_n is locally μ-negligible, the function

$$t \mapsto \lambda_t(g\varphi_{\mathrm{K}_n}) \qquad (t \notin \mathrm{H}_n)$$

is essentially μ-integrable, and

$$\int g\varphi_{\mathrm{K}_n}\, d\nu = \int^\bullet \lambda_t(g\varphi_{\mathrm{K}_n})\, d\mu(t).$$

The local μ-negligibility of $\mathrm{H} = \bigcup_n \mathrm{H}_n$ then follows from Ch. IV, §5, No. 2, Def. 3, and the first assertion of the statement follows from the fact that

$$\{t \in \mathrm{T} :\ g \text{ is not } \lambda_t\text{-integrable}\} = \mathrm{H}$$

(preceding Note).

V.47, ℓ. 14–16.

"... we have, by Prop. 3 and by Prop. 5 of §3, No. 2,

$$\int^\bullet h\, d(g\cdot\nu) = \int^\bullet (gh)\, d\nu = \int^\bullet d\mu(t) \int^\bullet (gh)\, d\lambda_t = \int^\bullet d\mu(t) \int^\bullet h\, d(g\cdot\lambda_t)\,."$$

Here, $\nu = \int \lambda_t\, d\mu(t)$. As the cited Prop. 3 employs the notation ν in another sense, it is helpful to restate it: If η is a positive measure on a locally compact space X and if g is a locally η-integrable function $\geqslant 0$ on X, then

$$\int^\bullet f\, d(g\cdot\eta) = \int^\bullet (fg)\, d\eta$$

for every function $f \geqslant 0$ on X.

The displayed equalities, in slow motion: For the first equality, apply the cited Prop. 3 with $\eta = \nu$, applicable because g is locally ν-integrable; for the third equality, apply Prop. 3 with $\eta = \lambda_t$, applicable because g is locally integrable for every λ_t; the second equality holds by Part a) of the cited Prop. 5, applicable since the mapping

$$t \mapsto \lambda_t \in \mathscr{M}_+(\mathrm{X}) \qquad (t \in \mathrm{T})$$

is μ-adequate, with integral $\nu = \int \lambda_t\, d\mu(t)$, and gh is ν-measurable (because g and h are; see the *Lemma* at the end of this Note) and ν-moderated (because g is ν-integrable; in fact, since X is countable at infinity, all functions on X are moderated for every measure on X).

The cited Prop. 5 shows, in addition, that the function

$$t \mapsto \int^\bullet (gh)\,d\lambda_t = \int^\bullet h\,d(g\cdot\lambda_t) \qquad (t \in \mathrm{T})$$

is μ-measurable; when $h \in \mathscr{K}_+(\mathrm{X})$, the equality

$$\int^\bullet d\mu(t) \int^\bullet h\,d(g\cdot\lambda_t) = \int^\bullet h\,d(g\cdot\nu) = (g\cdot\nu)(h) < +\infty$$

shows that the mapping $t \mapsto g\cdot\lambda_t \in \mathscr{M}_+(\mathrm{X})$ is scalarly essentially μ-integrable, and

$$\begin{aligned}\Big\langle h, \int (g\cdot\lambda_t)\,d\mu(t)\Big\rangle &= \int^\bullet \langle h, g\cdot\lambda_t\rangle\,d\mu(t) = \int^\bullet \langle gh, \lambda_t\rangle\,d\mu(t)\\ &= \int^\bullet d\mu(t)\int^\bullet (gh)\,d\lambda_t\\ &= \int^\bullet h\,d(g\cdot\nu) = \langle h, g\cdot\nu\rangle\,;\end{aligned}$$

thus $\int (g\cdot\lambda_t)\,d\mu(t) = g\cdot\nu$, which is the formula (9).

The following minor point, touched on in the Note for V.46, ℓ. 17, 18, seems not to have been established explicitly (if it has, I have forgotten where):

Lemma. Let μ be a measure on a locally compact space X, and let $f, g : \mathrm{X} \to \overline{\mathbf{R}}$ be functions that are (with respect to μ) measurable and are $\geqslant 0$ locally almost everywhere on X. Then fg is measurable.

Proof. We can suppose that $\mu \geqslant 0$ (Ch. IV, §2, No. 1, Def. 1 and §5, No. 1, Def. 1). Redefining $f(x)$ and $g(x)$ to be 0 on the locally negligible set where either number is < 0, we can suppose that f and g are $\geqslant 0$ everywhere on X (Ch. IV, §5, No. 2, Prop. 6).

A function $h \geqslant 0$ on X is measurable if and only if the set

$$\mathrm{S} = \{x \in \mathrm{X} :\ h(x) > a\}$$

is measurable for every real number $a \geqslant 0$ (*loc. cit.*, No. 5, comment preceding the Cor. of Prop. 8, plus the fact that $\mathrm{S} = \mathrm{X}$ when $a < 0$); and since (for the case $a = 0$)

$$\{x :\ h(x) > 0\} = \bigcup_{n=1}^{\infty} \{x :\ h(x) > 1/n\}\,,$$

h is measurable if and only if S is measurable for every real number $a > 0$ (*loc. cit.*, No. 4, Cor. 2 of Th. 2).

It will therefore suffice to show that the set

$$\mathrm{A} = \{x : \ (fg)(x) > a\}$$

is measurable for every real number $a > 0$. Now, $(fg)(x) > a$ if and only if one of the following three conditions holds:

$f(x) = +\infty$ and $g(x) > 0$;
$f(x) > 0$ and $g(x) = +\infty$;
$0 < f(x) < +\infty$, $0 < g(x) < +\infty$ and $f(x)g(x) > a$.

Thus, writing

$$\mathrm{B} = \{x : \ f(x) = +\infty\} \cap \{x : \ g(x) > 0\}$$
$$\mathrm{C} = \{x : \ f(x) > 0\} \cap \{x : \ g(x) = +\infty\}$$
$$\mathrm{D} = \{x : \ 0 < f(x) < +\infty\}, \ \mathrm{E} = \{x : \ 0 < g(x) < +\infty\},$$

one has

$$\mathrm{A} = \mathrm{B} \cup \mathrm{C} \cup (\mathrm{D} \cap \mathrm{E} \cap \{x : \ f(x)g(x) > a\});$$

the sets $\mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ are known to be measurable, so it suffices to show that the set

$$\mathrm{D} \cap \mathrm{E} \cap \{x : \ f(x)g(x) > a\}$$

is measurable. This set can be written

$$\mathrm{D} \cap \mathrm{E} \cap \{x : \ (\varphi_{\mathrm{D}} f)(x) \, \varphi_{\mathrm{E}} g)(x) > a\},$$

so it will suffice to show that the set

$$\{x : \ (\varphi_{\mathrm{D}} f)(x) \, \varphi_{\mathrm{E}} g)(x) > a\}$$

is measurable for every real number $a > 0$. Now, the finite-valued functions $\varphi_{\mathrm{D}} f$ and $\varphi_{\mathrm{E}} g$ are measurable; for example, the set

$$\{x : \ (\varphi_{\mathrm{D}} g)(x) > a\} = \mathrm{D} \cap \{x : \ g(x) > a\}$$

is measurable for every real number $a > 0$. One is thus reduced to proving that the product of measurable functions with values in $\mathbf{R}$ is measurable, and this is known from the comments following Ch. IV, §5, No. 3, Cor. 5 of Th. 1. ♢

V.47, ℓ. 21, 22.

"... it follows at once from these relations that $t \mapsto g \cdot \lambda_t$ is μ-adequate (§3, No. 1, Def. 1)."

As shown in the preceding Note, the mapping

$$(*) \qquad t \mapsto g \cdot \lambda_t \quad (t \in \mathrm{T})$$

is scalarly essentially μ-integrable, and its integral (in the sense of §3, No. 1) is given by the 'distributive law' (9),

$$\int (g \cdot \lambda_t)\, d\mu(t) = g \cdot \nu\,,$$

where $\nu = \int \lambda_t \, d\mu(t)$. Moreover, for every ν-measurable function $h \geqslant 0$ on X, one has the equality

$$\int^\bullet h\, d(g \cdot \nu) = \int^\bullet d\mu(t) \int^\bullet h\, d(g \cdot \lambda_t)\,;$$

in particular, its validity for every lower semi-continuous function $h \geqslant 0$ on X means that the mapping $(*)$ is μ-pre-adequate (*loc. cit.*, Def. 1).

We are to show that the mapping $(*)$ is μ'-pre-adequate for every positive measure $\mu' \leqslant \mu$ on T. Indeed, the original mapping $t \mapsto \lambda_t$ is *a fortiori* scalarly essentially μ'-integrable and, writing $\nu' = \int \lambda_t \, d\mu'(t)$, obviously $\nu' \leqslant \nu$. It follows that g is also locally ν'-integrable, and the foregoing argument shows that $t \mapsto g \cdot \lambda_t$ is μ'-pre-adequate (with integral $g \cdot \nu'$).

Incidentally, if T and X are any locally compact spaces, μ is a positive measure on T, and $t \mapsto \eta_t \in \mathscr{M}_+(\mathrm{X})$ is a μ-pre-adequate mapping whose integral $\int \eta_t \, d\mu(t)$ is a moderated measure on X, then $t \mapsto \eta_t$ is in fact μ-adequate (see the Note for V.17, ℓ. -6 to -4). {This comment is not intended as a generalization of the present Prop. 7, whose proof makes heavy use of the countable compactness of X in arranging that g be locally integrable for every λ_t.}

V.47, ℓ. $-2, -1$.

"For every function $f \in \mathscr{K}_+(\mathrm{T})$ we have, by Propositions 2 and 3,

$$\int^\bullet |g_2|\, f\, d|\theta_1| = \int^\bullet |g_2|\, f\, |g_1|\, d|\theta| = \int^\bullet |g_2 g_1|\, f\, d|\theta|\,."$$

In slow motion:

$$\int^\bullet |g_2|\, f\, d|\theta_1| = \int^\bullet |g_2|\, f\, d|g_1 \cdot \theta| = \int^\bullet |g_2|\, f\, d(|g_1| \cdot |\theta|)$$
$$= \int^\bullet |g_2|\, f\, |g_1|\, d|\theta| = \int^\bullet |g_2 g_1|\, f\, d|\theta|\,.$$

For these equalities to be valid, g_2 can be any complex function and f any positive function on T; it is only necessary that $g_1 \cdot \theta$ be defined, i.e., that g_1 be locally θ-integrable, equivalently (No. 1, Def. 1) that g_1 be locally $|\theta|$-integrable, equivalently (since g_1 is θ-measurable) that $|g_1|$ be locally $|\theta|$-integrable, equivalently that $|g_1| \cdot |\theta|$ be defined (this is implicit in Prop. 2). The first and last equalities are cosmetic; the 2nd equality holds by Prop. 2, and the 3rd by Prop. 3.

In particular, when f belongs to $\mathscr{K}_+(\mathrm{T})$, hence is universally moderated, the equality of the first and last members may be written (§1, No. 2, Prop. 7)

$$\int^* |g_2|\, f\, d|\theta_1| = \int^* |g_2 g_1|\, f\, d|\theta|\,,$$

so that the two members are simultaneously finite or simultaneously infinite; thus when, in addition, g_2 is θ_1-measurable (equivalently, $g_2 g_1$ is θ-measurable), $g_2 f$ is θ_1-integrable (i.e., the left member is finite) if and only if $g_2 g_1 f$ is θ-integrable (i.e., the right member is finite); in other words, assuming g_1 locally θ-integrable, the following conditions are equivalent:

a) g_2 is locally θ_1-integrable—which implies that g_2 is θ_1-measurable (No. 1, Prop. 1), hence $g_2 g_1$ is θ-measurable;

b) $g_2 g_1$ is locally θ-integrable—which implies that $g_2 g_1$ is θ-measurable, hence g_2 is θ_1-measurable.

V.48, ℓ. 2, 3.

"... by Th. 1 we have"

Since f is universally moderated, in applying Th. 1 (twice) the word "essentially" can be omitted, and $\int^\bullet$ replaced by $\int$.

Addendum: The answer to a question posed in the Note for V.45, ℓ. 10, 11 is as follows:

Example. There exist a locally compact space T, a positive measure μ on T, and a μ-moderated subset A of T such that the measure $\varphi_{\mathrm{A}} \cdot \mu$ is not moderated.

Lemma. If μ a positive measure on a locally compact space T such that $\mu = \sup\limits_n \mu_n$, where (μ_n) is an increasing sequence of positive measures, and if A is a μ-measurable subset of T such that $\operatorname{Supp}\mu_n \subset \mathrm{A}$ for all n (for example $\mathrm{A} = \bigcup\limits_n \operatorname{Supp}\mu_n$, which is a Borel set, hence is universally measurable), then $\varphi_{\mathrm{A}} \cdot \mu = \mu$.

Proof. Write $\mathrm{S}_n = \operatorname{Supp}\mu_n$. Since $\mu_n \leqslant \mu$, one knows that A is μ_n-measurable for all n, therefore φ_{A} is locally μ_n-integrable (No. 3, *Example*). Thus, for every $g \in \mathscr{K}_+(\mathrm{T})$, $\varphi_{\mathrm{A}} g$ is μ_n-integrable and, μ_n-almost everywhere, one has $\varphi_{\mathrm{A}} g = \varphi_{\mathrm{S}_n} \varphi_{\mathrm{A}} g = \varphi_{\mathrm{S}_n} g = g$, whence

$$(\varphi_{\mathrm{A}} \cdot \mu_n)(g) = \mu_n^{\bullet}(\varphi_{\mathrm{A}} g) = \mu_n^{\bullet}(g) = \mu_n(g) ,$$

that is, $\varphi_{\mathrm{A}} \cdot \mu_n = \mu_n$. Since $\mu^{\bullet} = \sup_n \mu_n^{\bullet}$ (§1, No. 4, Prop. 11) it follows that, for all $g \in \mathscr{K}_+(\mathrm{T})$,

$$(\varphi_{\mathrm{A}} \cdot \mu)(g) = \mu^{\bullet}(\varphi_{\mathrm{A}} g) = \sup_n \mu_n^{\bullet}(\varphi_{\mathrm{A}} g) = \sup_n \mu_n^{\bullet}(g) = \mu^{\bullet}(g) = \mu(g) ,$$

thus $\varphi_{\mathrm{A}} \cdot \mu = \mu$. ♢

The example. Let T and μ be as in Ch. V, §2, Exer. 4 (on p. V.95). By part *a*) of the exercise, μ is not moderated but $\mu = \sup_n \mu_n$, where (μ_n) is an increasing sequence of positive measures with finite support. The set $\mathrm{A} = \bigcup_n \mathrm{Supp}\, \mu_n$ is the union of a sequence of compact sets, hence is universally moderated, but $\varphi_{\mathrm{A}} \cdot \mu = \mu$ by the Lemma, thus $\varphi_{\mathrm{A}} \cdot \mu$ is not moderated.

Coda. This is a good place to stop—the Notes are getting too frequent and too long, the document a bloated 586 pages, and the complicated arrangement of the proof of the Lebesgue-Nikodym theorem in No. 5 exacerbates the trend; the reader who has gotten this far will probably find filling in the gaps easier than studying how I filled them in.

As epitome of the formal beauty of Bourbaki's treatment of integration, I nominate the formula

$$|\mu + i\nu| = \sqrt{\mu^2 + \nu^2}$$

of No. 9 (μ and ν real measures) with its audacious right member.